Lecture 4

Flows on the circle

Vector field on the circle

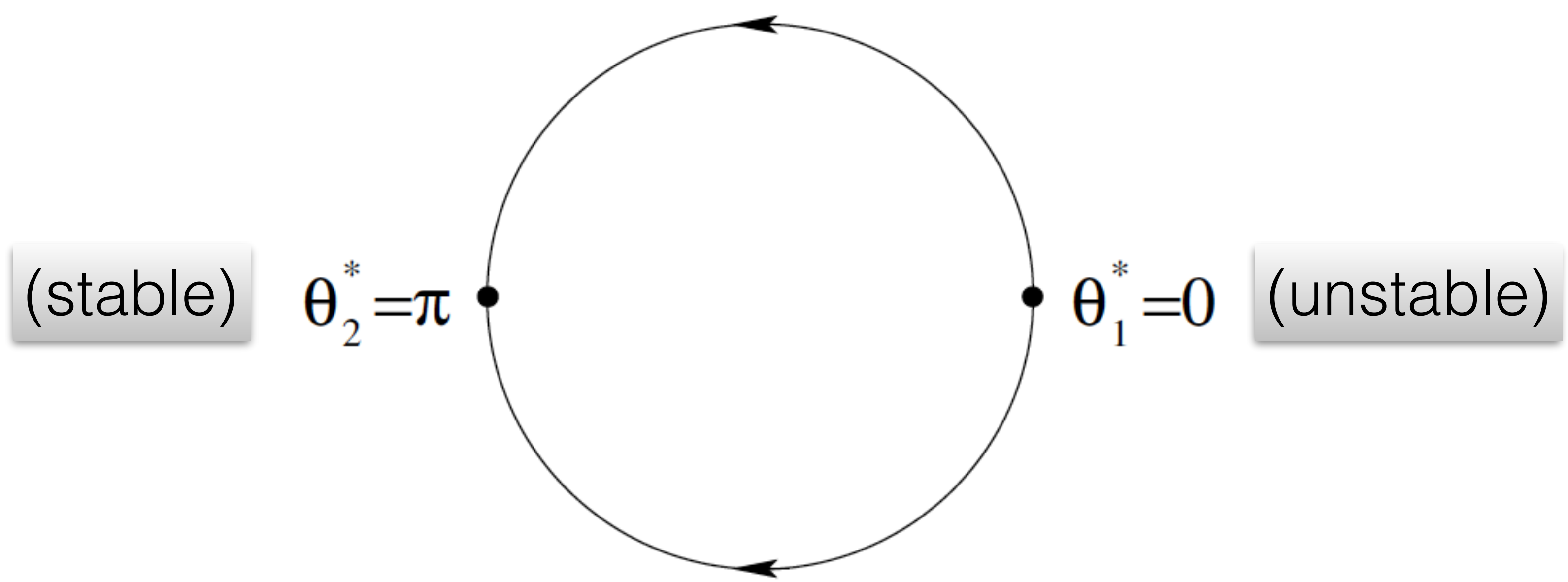
- Consider
$$\dot{\theta} = f(\theta)$$
with $\theta \in S_L$, the circle with circumference(周長) L. $\dot{\theta}$ denotes the rate of change of the angle, or the angular velocity. (Below. we consider $S_L = S_{2\pi}$.)

- The main difference with flows on the line is that now the flow can return to where it was, by going around the circle. Thus, periodic solutions are possible!

An example

- Consider

$$\dot{\theta} = \sin\theta, \quad \theta \in S_{2\pi}$$



- 第二章(Flow on the line)所介紹的平衡點定義與穩定性分析仍適用於此章(Flow on the circle)
- Periodic solutions do not always occur for systems defined on the circle.

Not every system can be considered as a system on the circle

Example:

The system

$$\dot{\theta} = \theta^2$$

cannot be regarded as a system on $S_{2\pi}$.

Exercises

- #4.1.2, #4.1.4, #4.1.8

The uniform oscillator

- The simplest flow on the circle is given by the system:
$$\dot{\theta} = \omega$$
where ω is a real nonzero constant and $\theta \in S_{2\pi}$.

- The solution is .
$$\theta = \omega t + \theta_0$$

- The period of the solution is
$$T = \frac{2\pi}{\omega}$$

An example-beat phenomenon

- Two joggers, Speedy and Pokey, are running at a steady pace around a circular track. It takes Speed T_1seconds to run once around the track, whereas it takes Pokey $T_2 > T_1$ seconds. Of course Speed will periodically overtake Pokey; how long does it take for Speedy to lap Pokey once, assuming that they start together?

- Consider the <u>phase difference:</u> $\phi = \theta_1 - \theta_2$

$$T_{\text{lap}} = (\frac{1}{T_1} - \frac{1}{T_2})^{-1}$$

The nonuniform oscillator

- Consider
$$\dot{\theta} = \omega - a\sin\theta, \quad \theta \in S_{2\pi}$$
where $\omega > 0 \text{ and } a > 0.$

- θ^* is a fixed point if $\omega = a\sin\theta^*$.

- The are three different cases:

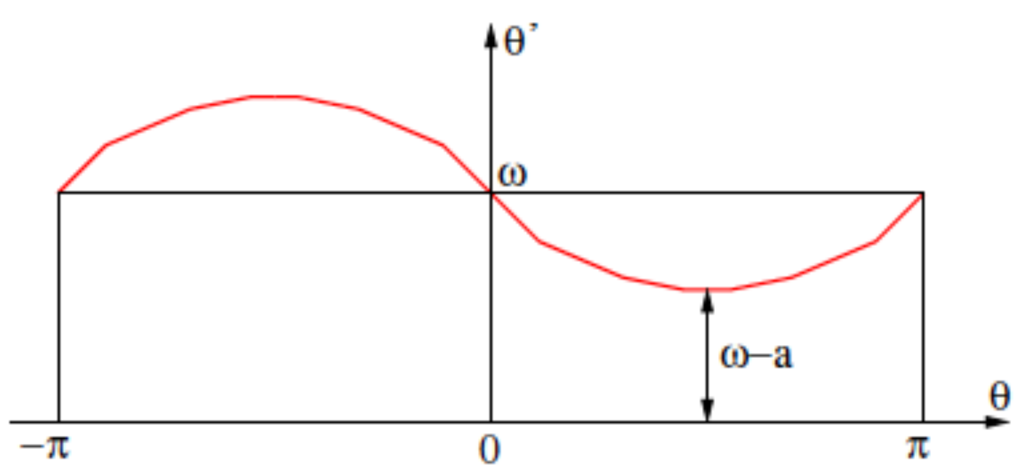
$$a < \omega, \; a = \omega, \; a > \omega.$$

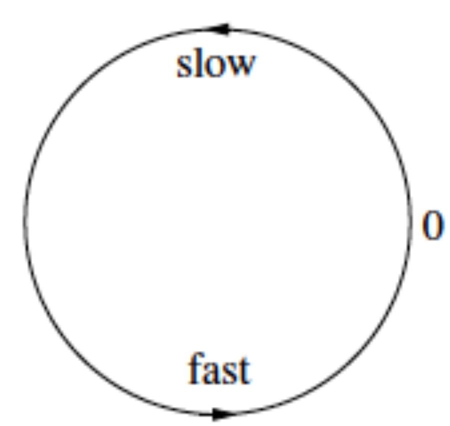
$a < \omega$

- There are no fixed points.

- All solutions are periodic, with period

$$T = \frac{2\pi}{\sqrt{\omega^2 - a^2}}.$$

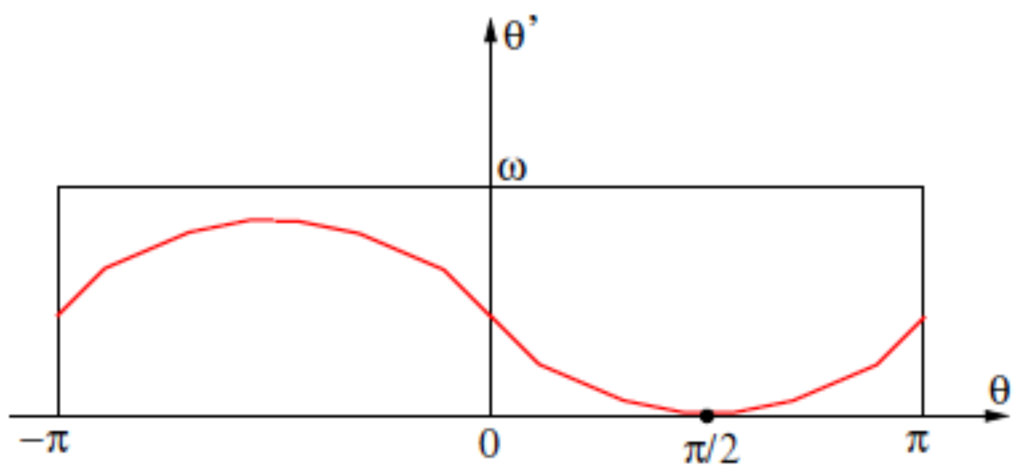


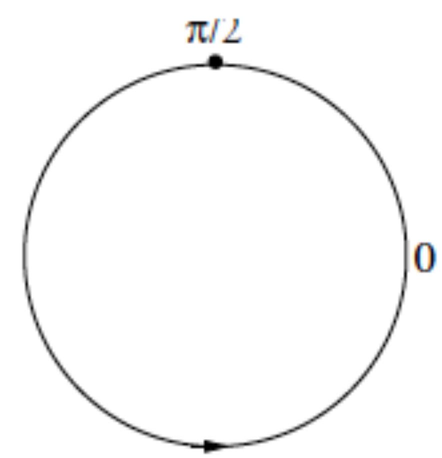


$$a = \omega$$

- There is exactly one fixed point at $\pi/2$, but it is a semi-stable fixed point.

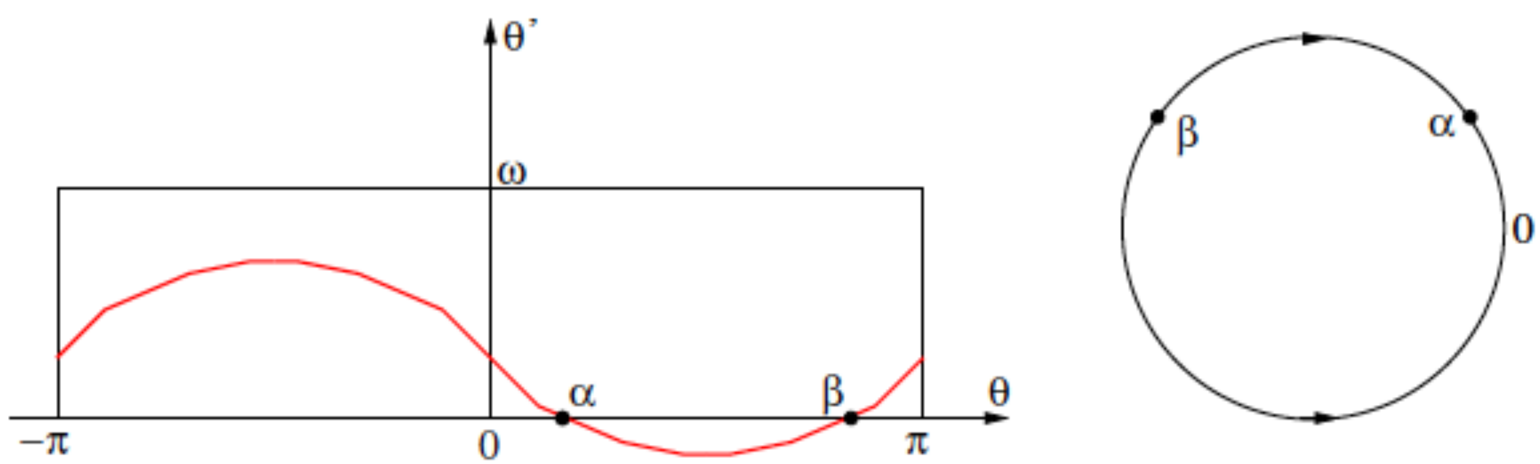
- The fixed point is attracting. This does not imply that the fixed point is stable though.





$a > \omega$

- For $a > \omega$, there are two distinct fixed points, one stable and one unstable. These are born out of a saddle-node bifurcation at the critical value $a = \omega$.



An example

Use linear stability analysis to classify the fixed point of

$$\dot{\theta} = \omega - a\sin\theta, \quad \theta \in S_{2\pi}$$

for $a > \omega$.

Exercises

- #4.3.2, #4.3.7(a bifurcation-like phenomenon, but not a bifurcation, occurs)

Fireflies

- Fireflies provide one of the most spectacular examples of synchronization in nature.

- When one firefly see the flash of another, it slow down or speed up so as to flash more nearly in the phase on the next cycle. Hanson (1978) studies this effect experimentally: For a range of periods close to the firefly's natural period (about 0.9 sec), the firefly <u>was able to match its frequency to the period stimulus</u> (i.e. was *entrained* 跟從 / 附帶在 by the stimulus). However, if the stimulus was too fast or too slow, the entrainment was lost- a kind of beat phenomenon occurred (*phase drift*).

Model

- Model (Ermentrout an Rinzel 1984):

$$\dot{\theta} = \omega + A\sin(\Theta - \theta)$$

where
θ :phase of the firefly's flashing rhythm, where $\theta = 0$ corresponds to the instant when a flash is emitted.
ω: frequency
Θ: periodic stimulus, satisfying $\dot{\Theta} = \Omega$
A: resetting strength

Analysis

- Set $\phi = \Theta - \theta$, we consider

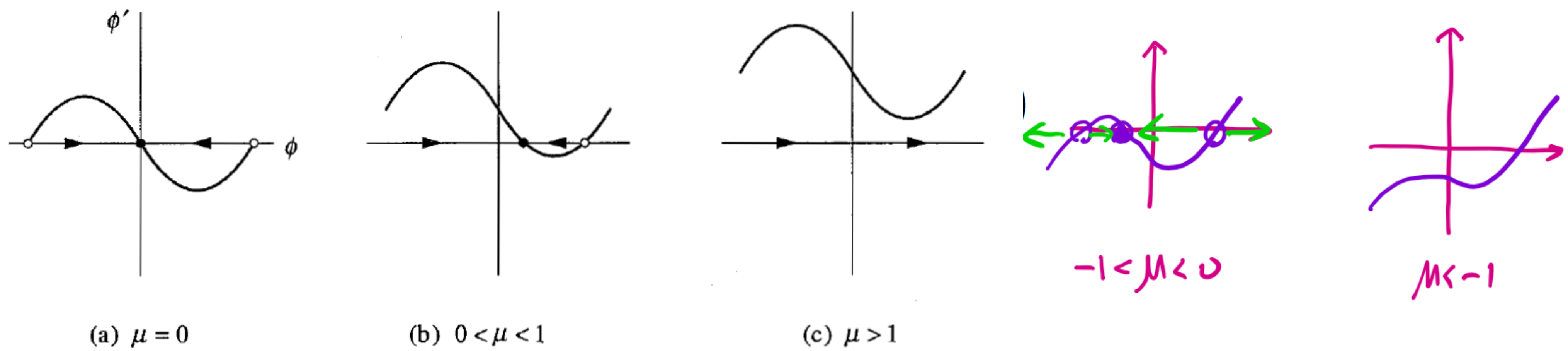
$$\dot{\phi} = \dot{\Theta} - \dot{\theta} = \Omega - \omega + A\sin(\phi)$$

- Set $\tau = At,\ \mu = (\Omega - \omega)/A,$ then

$$\phi' = \mu - \sin(\phi)$$

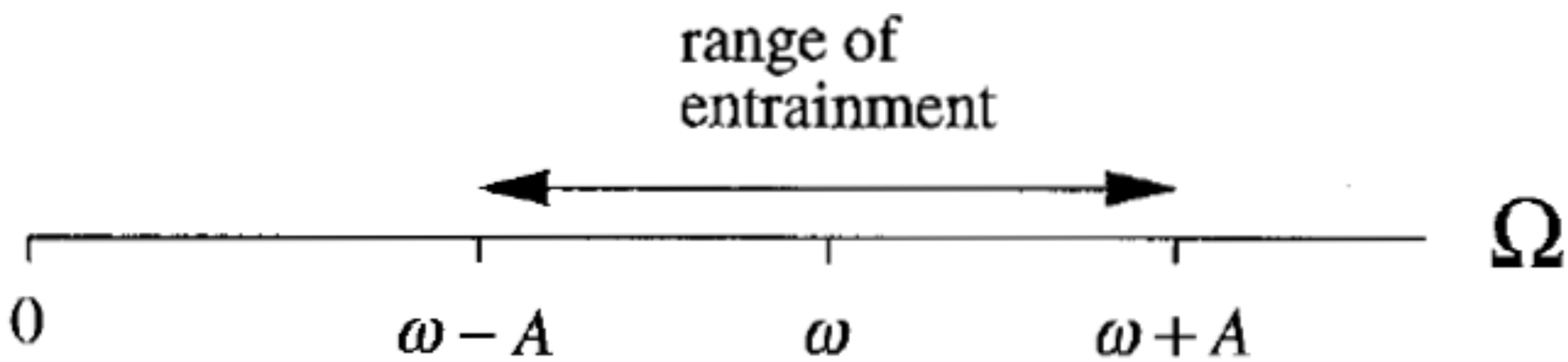
where $\phi' = d\phi/d\tau$.

vector field for $\phi' = \mu - \sin(\phi)$



(a) The firefly eventually entrains(隨附) with zero phase difference
(b) The firefly's rhythm is *phase-locked* to the stimulus.
(c) The phase difference increases (not uniformly)indefinitely. (phase drift)

The range of entrainment



Period of phase drift

For $\mu > 1$, the period of phase drift can be predicted:

$$T_{\mathrm{drift}} = \frac{2\pi}{\sqrt{(\Omega - \omega)^2 - A^2}}$$