SOLUTIONS TO CHAPTER 1

Problem 1.1

(a) Since the growth rate of a variable equals the time derivative of its log, as shown by equation (1.10) in the text, we can write

$$(1)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{d\ln Z(t)}{dt} = \frac{d\ln[X(t)Y(t)]}{dt}.$$

Since the log of the product of two variables equals the sum of their logs, we have

$$(2)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{d[\ln X(t) + \ln Y(t)]}{dt} = \frac{d\ln X(t)}{dt} + \frac{d\ln Y(t)}{dt},$$

or simply

$$(3)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{\dot{X}(t)}{X(t)} + \frac{\dot{Y}(t)}{Y(t)}.$$

(b) Again, since the growth rate of a variable equals the time derivative of its log, we can write

$$(4)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{d\ln Z(t)}{dt} = \frac{d\ln[X(t)/Y(t)]}{dt}.$$

Since the log of the ratio of two variables equals the difference in their logs, we have

$$(5)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{d[\ln X(t) - \ln Y(t)]}{dt} = \frac{d\ln X(t)}{dt} - \frac{d\ln Y(t)}{dt},$$

or simply

$$(6)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{\dot{X}(t)}{X(t)} - \frac{\dot{Y}(t)}{Y(t)}.$$

(c) We have

$$(7)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{d\ln Z(t)}{dt} = \frac{d\ln[X(t)^{\alpha}]}{dt}.$$

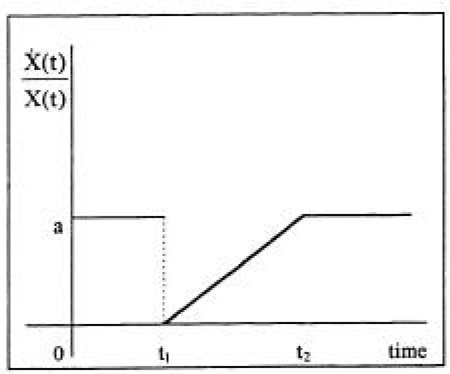
Using the fact that $\ln[X(t)^{\alpha}] = \alpha\ln X(t)$, we have

$$(8)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{d[\alpha\ln X(t)]}{dt} = \alpha\frac{d\ln X(t)}{dt} = \alpha\frac{\dot{X}(t)}{X(t)},$$

where we have used the fact that α is a constant.

Problem 1.2

(a) Using the information provided in the question, the path of the growth rate of X, $\dot{X}(t)/X(t)$, is depicted in the figure at right.



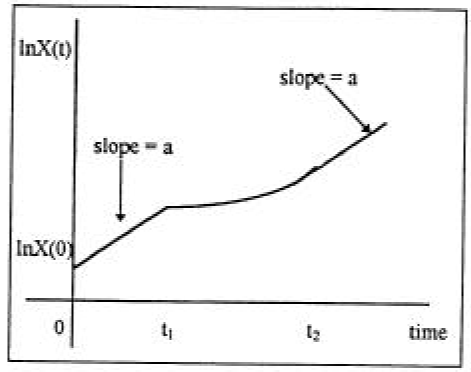
From time 0 to time t_1, the growth rate of X is constant and equal to $a > 0$. At time t_1, the growth rate of X drops to 0. From time t_1 to time t_2, the growth rate of X rises gradually from 0 to a. Note that we have made the assumption that $\dot{X}(t)/X(t)$ rises at a constant rate from t_1 to t_2. Finally, after time t_2, the growth rate of X is constant and equal to a again.

(b) Note that the slope of lnX(t) plotted against time is equal to the growth rate of X(t). That is, we know

$$\frac{d \ln X(t)}{dt} = \frac{\dot{X}(t)}{X(t)}$$

(See equation (1.10) in the text.)

From time 0 to time t_1 the slope of lnX(t) equals $a > 0$. The lnX(t) locus has an inflection point at t_1, when the growth rate of X(t) changes discontinuously from a to 0. Between t_1 and t_2, the slope of lnX(t) rises gradually from 0 to a. After time t_2 the slope of lnX(t) is constant and equal to $a > 0$ again.

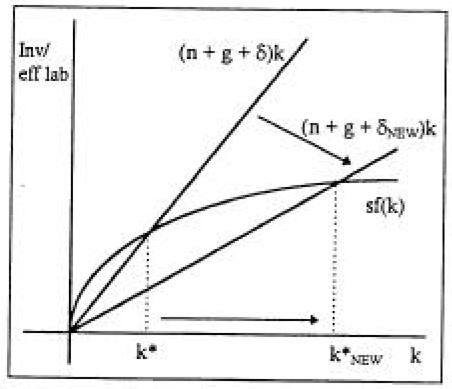


Problem 1.3

(a) The slope of the break-even investment line is given by $(n + g + \delta)$ and thus a fall in the rate of depreciation, δ, decreases the slope of the break-even investment line.

The actual investment curve, sf(k) is unaffected.

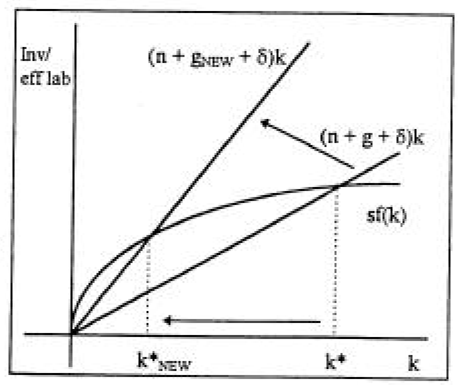
From the figure at right we can see that the balanced-growth-path level of capital per unit of effective labor rises from k^* to k^*_{NEW}.



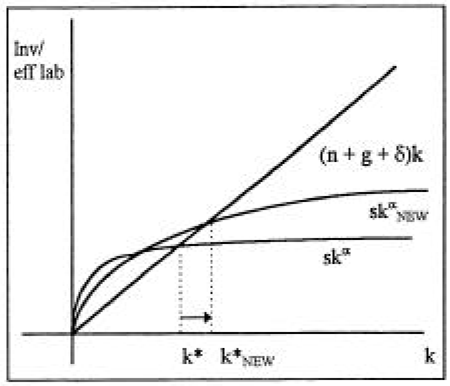
(b) Since the slope of the break-even investment line is given by $(n + g + \delta)$, a rise in the rate of technological progress, g, makes the break-even investment line steeper.

The actual investment curve, sf(k), is unaffected.

From the figure at right we can see that the balanced-growth-path level of capital per unit of effective labor falls from k^* to k^*_{NEW}.



(c) The break-even investment line, $(n + g + \delta)k$, is unaffected by the rise in capital's share, α.



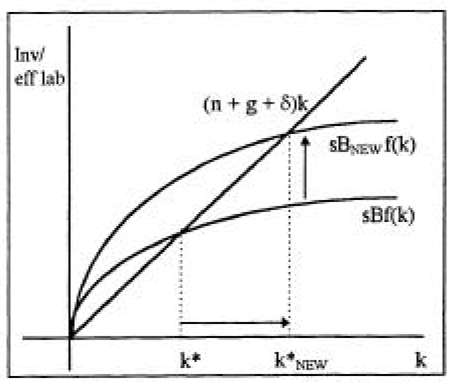
The effect of a change in α on the actual investment curve, sk^{α}, can be determined by examining the derivative $\partial(sk^{\alpha})/\partial\alpha$. It is possible to show that

$$(1) \quad \frac{\partial sk^{\alpha}}{\partial \alpha} = sk^{\alpha} \ln k .$$

For $0 < \alpha < 1$, and for positive values of k, the sign of $\partial(sk^{\alpha})/\partial\alpha$ is determined by the sign of lnk. For $\ln k > 0$, or $k > 1$, $\partial sk^{\alpha}/\partial\alpha > 0$ and so the new actual investment curve lies above the old one. For $\ln k < 0$ or $k < 1$, $\partial sk^{\alpha}/\partial\alpha < 0$ and so the new actual investment curve lies below the old one. At $k = 1$, so that $\ln k = 0$, the new actual investment curve intersects the old one.

In addition, the effect of a rise in α on k^* is ambiguous and depends on the relative magnitudes of s and $(n + g + \delta)$. It is possible to show that a rise in capital's share, α, will cause k^* to rise if $s > (n + g + \delta)$. This is the case depicted in the figure above.

(d) Suppose we modify the intensive form of the production function to include a non-negative constant, B, so that the actual investment curve is given by $sBf(k)$, $B > 0$.



Then workers exerting more effort, so that output per unit of effective labor is higher than before, can be modeled as an increase in B. This increase in B shifts the actual investment curve up.

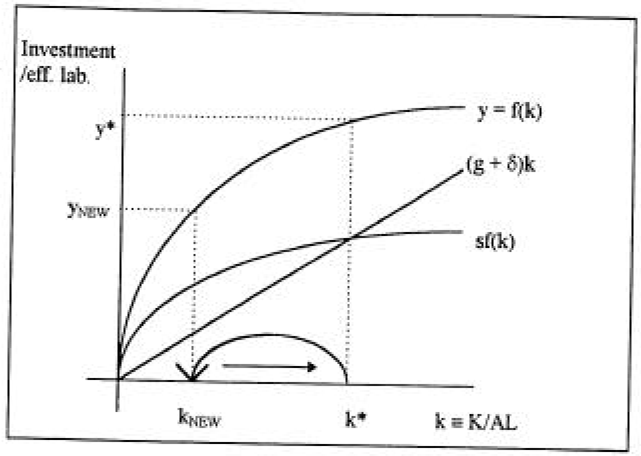
The break-even investment line, $(n + g + \delta)k$, is unaffected.

From the figure at right we can see that the balanced-growth-path level of capital per unit of effective labor rises from k^* to k^*_{NEW}.

Problem 1.4

(a) At some time, call it t_0, there is a discrete upward jump in the number of workers. This reduces the amount of capital per unit of effective labor from k^* to k_{NEW}. We can see this by simply looking at the definition, $k = K/AL$. An increase in L without a jump in K or A causes k to fall. Since $f'(k) > 0$, this fall in the amount of capital per unit of effective labor reduces the amount of output per unit of effective labor as well. In the figure below, y falls from y^* to y_{NEW}.

(b) Now at this lower k_{NEW}, actual investment per unit of effective labor exceeds break-even investment per unit of effective labor. That is, $sf(k_{NEW}) > (g + \delta)k_{NEW}$. The economy is now saving and investing more than enough to offset depreciation and technological progress at this lower k_{NEW}. Thus k begins rising back toward k*. As capital per unit of effective labor begins rising, so does output per unit of effective labor. That is, y begins rising from y_{NEW} back toward y*.



(c) Capital per unit of effective labor will continue to rise until it eventually returns to the original level of k*. At k*, investment per unit of effective labor is again just enough to offset technological progress and depreciation and keep k constant. Since k returns to its original value of k* once the economy again returns to a balanced growth path, output per unit of effective labor also returns to its original value of $y^* = f(k^*)$.

Problem 1.5

(a) The equation describing the evolution of the capital stock per unit of effective labor is given by

(1) $\dot{k} = sf(k) - (n + g + \delta)k$.

Substituting in for the intensive form of the Cobb-Douglas, $f(k) = k^\alpha$, yields

$$\dot{k} = sk^\alpha - (n + g + \delta)k.$$

On the balanced growth path, $\dot{k}$ is zero; investment per unit of effective labor is equal to break-even investment per unit of effective labor and so k remains constant. Denoting the balanced-growth-path value of k as k*, we have $sk^{*\alpha} = (n + g + \delta)k^*$. Rearranging to solve for k* yields

(2) $k^* = [s/(n + g + \delta)]^{1/(1-\alpha)}$.

To get the balanced-growth-path value of output per unit of effective labor, substitute equation (2) into the intensive form of the production function, $y = k^\alpha$:

(3) $y^* = [s/(n + g + \delta)]^{\alpha/(1-\alpha)}$.

Consumption per unit of effective labor on the balanced growth path is given by $c^* = (1 - s)y^*$. Substituting equation (3) into this expression yields

(4) $c^* = (1 - s)[s/(n + g + \delta)]^{\alpha/(1-\alpha)}$.

(b) By definition, the golden-rule level of the capital stock is that level at which consumption per unit of effective labor is maximized. To derive this level of k, take equation (2), which expresses the balanced-growth-path level of k, and rearrange it to solve for s:

(5) $s = (n + g + \delta)k^{*1-\alpha}$.

Now substitute equation (5) into equation (4):

$$c^* = \left[1 - (n + g + \delta)k^{*1-\alpha}\right]\left[(n + g + \delta)k^{*1-\alpha}/(n + g + \delta)\right]^{\alpha/(1-\alpha)}.$$

After some straightforward algebraic manipulation, this simplifies to

(6) $c^* = k^{*\alpha} - (n + g + \delta)k^*$.

Equation (6) can be easily interpreted. Consumption per unit of effective labor is equal to output per unit of effective labor, $k^{*\alpha}$, less actual investment per unit of effective labor, which on the balanced growth path is the same as break-even investment per unit of effective labor, $(n + g + \delta)k^*$.

Now use equation (6) to maximize c^* with respect to k^*. The first-order condition is given by

$$\partial c^*/\partial k^* = \alpha k^{*\alpha-1} - (n + g + \delta) = 0,$$

or simply

(7) $\alpha k^{*\alpha-1} = (n + g + \delta)$.

Note that equation (7) is just a specific form of $f'(k^*) = (n + g + \delta)$, which is the general condition that implicitly defines the golden-rule level of capital per unit of effective labor. Equation (7) has a graphical interpretation: it defines the level of k at which the slope of the intensive form of the production function is equal to the slope of the break-even investment line.

Solving equation (7) for the golden-rule level of k yields

(8) $k^*_{GR} = [\alpha/(n + g + \delta)]^{1/(1-\alpha)}$.

(c) To get the saving rate that will yield the golden-rule level of k, substitute equation (8) into (5):

$$s_{GR} = (n + g + \delta)[\alpha/(n + g + \delta)]^{(1-\alpha)/(1-\alpha)},$$

which simplifies to

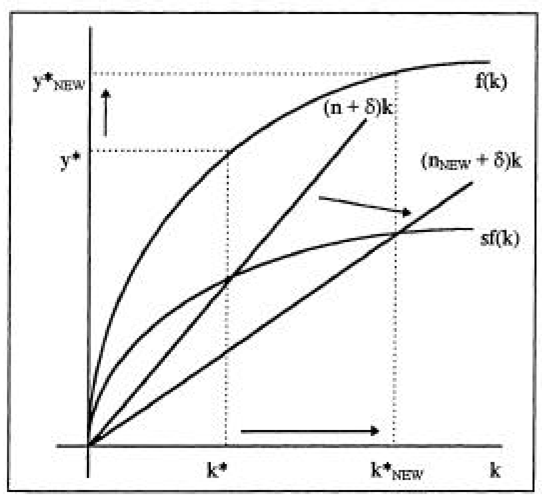
(9) $s_{GR} = \alpha$.

With a Cobb-Douglas production function, the saving rate required to reach the golden rule is equal to the elasticity of output with respect to capital or capital's share in output (if capital earns its marginal product).

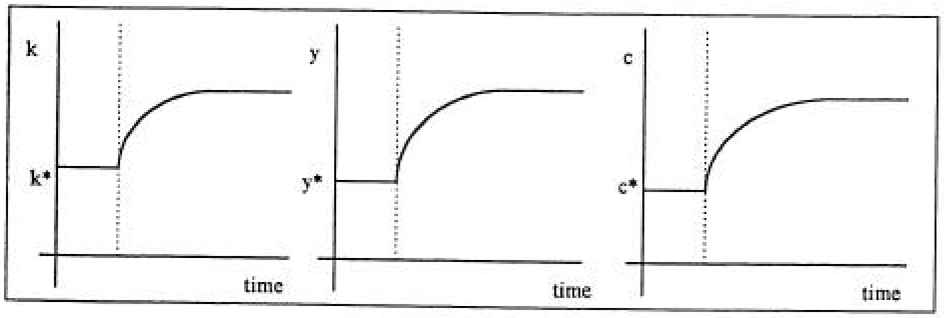
Problem 1.6

(a) Since there is no technological progress, we can carry out the entire analysis in terms of capital and output per worker rather than capital and output per unit of effective labor. With A constant, they behave the same. Thus we can define y = Y/L and k = K/L.

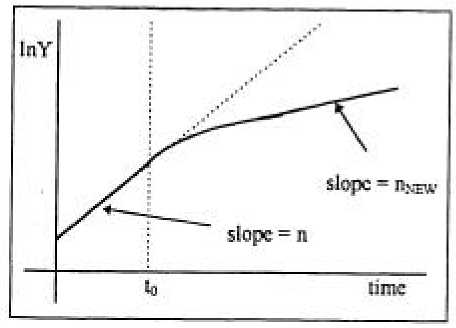
The fall in the population growth rate makes the break-even investment line flatter. In the absence of technological progress, the per unit time change in k, capital per worker, is given by $\dot{k} = sf(k) - (\delta + n)k$. Since $\dot{k}$ was 0 before the decrease in n -- the economy was on a balanced growth path -- the decrease in n causes $\dot{k}$ to become positive. At k^*, actual investment per worker, $sf(k^*)$, now exceeds break-even investment per worker, $(n_{NEW} + \delta)k^*$. Thus k moves to a new higher balanced growth path level. See the figure at right.



As k rises, y -- output per worker -- also rises. Since a constant fraction of output is saved, c -- consumption per worker -- rises as y rises. This is summarized in the figures below.



(b) By definition, output can be written as $Y = Ly$. Thus the growth rate of output is $\dot{Y}/Y = \dot{L}/L + \dot{y}/y$. On the initial balanced growth path, $\dot{y}/y = 0$ -- output per worker is constant -- so $\dot{Y}/Y = \dot{L}/L = n$. On the final balanced growth path, $\dot{y}/y = 0$ again -- output per worker is constant again -- and so $\dot{Y}/Y = \dot{L}/L = n_{NEW} < n$. In the end, output will be growing at a permanently lower rate.



What happens during the transition? Examine the production function $Y = F(K,AL)$. On the initial balanced growth path AL, K and thus Y are all growing at rate n. Then suddenly AL begins growing at some new lower rate n_{NEW}. Thus suddenly Y will be growing at some rate between that of K (which is growing at n) and that of AL (which is growing at n_{NEW}). Thus, during the transition, output grows more rapidly than it will on the new balanced growth path, but less rapidly than it would have without the decrease in population growth. As output growth gradually slows down during the transition, so does capital growth until finally K, AL, and thus Y are all growing at the new lower n_{NEW}.

Problem 1.7

The derivative of $y^* = f(k^*)$ with respect to n is given by

(1) $\partial y^*/\partial n = f'(k^*)[\partial k^*/\partial n]$.

To find $\partial k^*/\partial n$, use the equation for the evolution of the capital stock per unit of effective labor, $\dot{k} = sf(k) - (n + g + \delta)k$. In addition, use the fact that on a balanced growth path, $\dot{k} = 0$, $k = k^*$ and thus $sf(k^*) = (n + g + \delta)k^*$. Taking the derivative of both sides of this expression with respect to n yields

$$sf'(k^*)\frac{\partial k^*}{\partial n} = (n + g + \delta)\frac{\partial k^*}{\partial n} + k^*,$$

and rearranging yields

(2) $$\frac{\partial k^*}{\partial n} = \frac{k^*}{sf'(k^*) - (n + g + \delta)}.$$

Substituting equation (2) into equation (1) gives us

(3) $$\frac{\partial y^*}{\partial n} = f'(k^*)\left[\frac{k^*}{sf'(k^*) - (n+g+\delta)}\right].$$

Rearranging the condition that implicitly defines k^*, $sf(k^*) = (n + g + \delta)k^*$, and solving for s yields

(4) $s = (n + g + \delta)k^*/f(k^*)$.

Substitute equation (4) into equation (3):

(5) $$\frac{\partial y^*}{\partial n} = \frac{f'(k^*)k^*}{[(n+g+\delta)f'(k^*)k^*/f(k^*)] - (n+g+\delta)}.$$

To turn this into the elasticity that we want, multiply both sides of equation (5) by n/y^*:

$$\frac{n}{y^*}\frac{\partial y^*}{\partial n} = \frac{n}{(n+g+\delta)}\frac{f'(k^*)k^*/f(k^*)}{[f'(k^*)k^*/f(k^*)] - 1}.$$

Using the definition that $\alpha_K(k^*) = f'(k^*)k^*/f(k^*)$ gives us

(6) $$\frac{n}{y^*}\frac{\partial y^*}{\partial n} = -\frac{n}{(n+g+\delta)}\left[\frac{\alpha_K(k^*)}{1-\alpha_K(k^*)}\right].$$

Now, with $\alpha_K(k^*) = 1/3$, $g = 2\%$ and $\delta = 3\%$, we need to calculate the effect on y^* of a fall in n from 2% to 1%. Using the midpoint of $n = 0.015$ to calculate the elasticity gives us

$$\frac{n}{y^*}\frac{\partial y^*}{\partial n} = -\frac{0.015}{(0.015+0.02+0.03)}\left(\frac{1/3}{1-1/3}\right) \cong -0.12.$$

So this 50% drop in the population growth rate, from 2% to 1%, will lead to approximately a 6% increase in the level of output per unit of effective labor, since $(-0.50)(-0.12) = 0.06$. This calculation illustrates the point that observed differences in population growth rates across countries are not nearly enough to account for differences in y that we see.

Problem 1.8

(a) A permanent increase in the fraction of output that is devoted to investment from 0.15 to 0.18 represents a 20% increase in the saving rate. From equation (1.27) in the text, the elasticity of output with respect to the saving rate is

(1) $$\frac{s}{y^*}\frac{\partial y^*}{\partial s} = \frac{\alpha_K(k^*)}{1-\alpha_K(k^*)},$$

where $\alpha_K(k^*)$ is the share of income paid to capital (assuming that capital is paid its marginal product).

Substituting the assumption that $\alpha_K(k^*) = 1/3$ into equation (1) gives us

$$\frac{s}{y^*}\frac{\partial y^*}{\partial s} = \frac{\alpha_K(k^*)}{1-\alpha_K(k^*)} = \frac{1/3}{1-1/3} = \frac{1}{2}.$$

Thus the elasticity of output with respect to the saving rate is 1/2. So this 20% increase in the saving rate -- from $s = 0.15$ to $s_{NEW} = 0.18$ -- will cause output to rise relative to what it would have been by about 10%. [Note that the analysis has been carried out in terms of output per unit of effective labor. Since the paths of A and L are not affected, however, if output per unit of effective labor rises by 10%, output itself is also 10% higher than what it would have been.]

(b) Consumption will rise less than output. Although output winds up 10% higher than what it would have been, the fact that the saving rate is higher means that we are now consuming a smaller fraction of output. We can calculate the elasticity of consumption with respect to the saving rate. On the balanced growth path, consumption is given by

(2) $c^* = (1 - s)y^*$.

Taking the derivative with respect to s yields

(3) $\dfrac{\partial c^*}{\partial s} = -y^* + (1-s)\dfrac{\partial y^*}{\partial s}$.

To turn this into an elasticity, multiply both sides of equation (3) by s/c^*:

$$\frac{\partial c^*}{\partial s}\frac{s}{c^*} = \frac{-y^* s}{(1-s)y^*} + (1-s)\frac{\partial y^*}{\partial s}\frac{s}{(1-s)y^*},$$

where we have substituted $c^* = (1 - s)y^*$ on the right-hand side. Simplifying gives us

(4) $\dfrac{\partial c^*}{\partial s}\dfrac{s}{c^*} = \dfrac{-s}{(1-s)} + \dfrac{\partial y^*}{\partial s}\dfrac{s}{(1-s)y^*}$.

From part (a), the second term on the right-hand side of (4), the elasticity of output with respect to the saving rate, equals 1/2. We can use the midpoint between $s = 0.15$ and $s_{NEW} = 0.18$ to calculate the elasticity:

$$\frac{\partial c^*}{\partial s}\frac{s}{c^*} = \frac{-0.165}{(1-0.165)} + 0.5 \cong 0.30.$$

Thus the elasticity of consumption with respect to the saving rate is approximately 0.3. So this 20% increase in the saving rate will cause consumption to be approximately 6% above what it would have been.

(c) The immediate effect of the rise in investment as a fraction of output is that consumption falls. Although y^* does not jump immediately -- it only begins to move toward its new, higher balanced-growth-path level -- we are now saving a greater fraction, and thus consuming a smaller fraction, of this same y^*. At the moment of the rise in s by 3 percentage points -- since $c = (1 - s)y^*$ and y^* is unchanged -- c falls. In fact, the percentage change in c will be the percentage change in $(1 - s)$. Now, $(1 - s)$ falls from 0.85 to 0.82, which is approximately a 3.5% drop. Thus at the moment of the rise in s, consumption falls by about three and a half percent.

We can use some results from the text on the speed of convergence to determine the length of time it takes for consumption to return to what it would have been without the increase in the saving rate. After the initial rise in s, s remains constant throughout. Since $c = (1 - s)y$, this means that consumption will grow at the same rate as y on the way to the new balanced growth path. In the text it is shown that the rate of convergence of k and y, after a linear approximation, is given by $\lambda = (1 - \alpha_K)(n + g + \delta)$. With $(n + g + \delta)$ equal to 6% per year and $\alpha_K = 1/3$, this yields a value for λ of about 4%. This means that k and y move about 4% of the remaining distance toward their balanced-growth-path values of k^* and y^* each year. Since c is proportional to y -- $c = (1 - s)y$ -- it also approaches its new balanced-growth-path value at that same constant rate. That is, analogous to equation (1.31) in the text, we could write

(5) $c(t) - c^* \cong e^{-(1-\alpha_K)(n+g+\delta)t}[c(0) - c^*]$,

or equivalently

(6) $e^{-\lambda t} = \dfrac{c(t) - c^*}{c(0) - c^*}$.

The term on the right-hand side of equation (6) is the fraction of the distance to the balanced growth path that remains to be traveled.

We know that consumption falls initially by 3.5% and eventually will be 6% higher than it would have been. Thus it must change by 9.5% on the way to the balanced growth path. It will therefore be equal to what it would have been about 36.8% (3.5%/9.5% ≅ 36.8%) of the way to the new balanced growth path.

Equivalently, this is when the remaining distance to the new balanced growth path is 63.2% of the original distance. In order to determine the length of time this will take, we need to find a t* that solves

(7) $e^{-\lambda t^*} = 0.632$.

Taking logs of both sides of equation (7) yields

$-\lambda t^* = \ln(0.632)$.

Rearranging to solve for t gives us

$t^* = 0.459/0.04$,

and thus

(8) $t^* \cong 11.5$ years.

It will take a fairly long time -- over a decade -- for consumption to return to what it would have been in the absence of the increase in investment as a fraction of output.

Problem 1.9

(a) Define the marginal product of labor as $w \equiv \partial F(K,AL)/\partial L$. Then write the production function as $Y = ALf(k) = ALf(K/AL)$. Taking the partial derivative of output with respect to L yields

(1) $w \equiv \partial Y/\partial L = ALf'(k)[-K/AL^2] + Af(k) = A[(-K/AL)f'(k) + f(k)] = A[f(k) - kf'(k)]$,

as required.

(b) Define the marginal product of capital as $r \equiv [\partial F(K,AL)/\partial K] - \delta$. Again, writing the production function as $Y = ALf(k) = ALf(K/AL)$ and now taking the partial derivative of output with respect to K yields

(2) $r \equiv [\partial Y/\partial K] - \delta = ALf'(k)[1/AL] - \delta = f'(k) - \delta$.

Substitute equations (1) and (2) into $wL + rK$:

$wL + rK = A[f(k) - kf'(k)]L + [f'(k) - \delta]K = ALf(k) - f'(k)[K/AL]AL + f'(k)K - \delta K$.

Simplifying gives us

(3) $wL + rK = ALf(k) - f'(k)K + f'(k)K - \delta K = Alf(k) - \delta K \equiv ALF(K/AL, 1) - \delta K$.

Finally, since F is constant returns to scale, equation (3) can be rewritten as

(4) $wL + rK = F(ALK/AL, AL) - \delta K = F(K, AL) - \delta K$.

(c) As shown above, $r = f'(k) - \delta$. Since δ is a constant and since k is constant on a balanced growth path, so is $f'(k)$ and thus so is r. In other words, on a balanced growth path, $\dot{r}/r = 0$. Thus the Solow model does exhibit the property that the return to capital is constant over time.

Since capital is paid its marginal product, the share of output going to capital is rK/Y. On a balanced growth path,

(5) $$\frac{\dot{\left(rK/Y\right)}}{\left(rK/Y\right)} = \dot{r}/r + \dot{K}/K - \dot{Y}/Y = 0 + (n+g) - (n+g) = 0.$$

Thus, on a balanced growth path, the share of output going to capital is constant. Since the shares of output going to capital and labor sum to one, this implies that the share of output going to labor is also constant on the balanced growth path.

We need to determine the growth rate of the marginal product of labor, w, on a balanced growth path. As shown above, $w = A[f(k) - kf'(k)]$. Taking the time derivative of the log of this expression yields the growth rate of the marginal product of labor:

(6) $$\frac{\dot{w}}{w} = \frac{\dot{A}}{A} + \frac{\dot{[f(k) - kf'(k)]}}{[f(k) - kf'(k)]} = g + \frac{[f'(k)\dot{k} - \dot{k}f'(k) - kf''(k)\dot{k}]}{f(k) - kf'(k)} = g + \frac{-kf''(k)\dot{k}}{f(k) - kf'(k)}.$$

On a balanced growth path $\dot{k} = 0$ and so $\dot{w}/w = g$. That is, on a balanced growth path, the marginal product of labor rises at the rate of growth of the effectiveness of labor.

(d) As shown in part (c), the growth rate of the marginal product of labor is

$$(6)\quad \frac{\dot{w}}{w} = g + \frac{-kf''(k)\dot{k}}{f(k) - kf'(k)}.$$

If $k < k^*$, then as k moves toward k^*, $\dot{w}/w > g$. This is true because the denominator of the second term on the right-hand side of equation (6) is positive because f(k) is a concave function. The numerator of that same term is positive because k and $\dot{k}$ are positive and $f''(k)$ is negative. Thus, as k rises toward k^*, the marginal product of labor grows faster than on the balanced growth path. Intuitively, the marginal product of labor rises by the rate of growth of the effectiveness of labor on the balanced growth path. As we move from k to k^*, however, the amount of capital per unit of effective labor is also rising which also makes labor more productive and this increases the marginal product of labor even more.

The growth rate of the marginal product of capital, r, is

$$(7)\quad \frac{\dot{r}}{r} = \frac{\dot{[f'(k)]}}{f'(k)} = \frac{f''(k)\dot{k}}{f'(k)}.$$

As k rises toward k^*, this growth rate is negative since $f'(k) > 0$, $f''(k) < 0$ and $\dot{k} > 0$. Thus, as the economy moves from k to k^*, the marginal product of capital falls. That is, it grows at a rate less than on the balanced growth path where its growth rate is 0.

Problem 1.10

(a) By definition a balanced growth path occurs when all the variables of the model are growing at constant rates. Despite the differences between this model and the usual Solow model, it turns out that we can again show that the economy will converge to a balanced growth path by examining the behavior of k = K/AL.

Taking the time derivative of both sides of the definition of k = K/AL gives us

$$(1)\quad \dot{k} = \dot{\left(\frac{K}{AL}\right)} = \frac{\dot{K}(AL) - K[\dot{L}A - \dot{A}L]}{(AL)^2} = \frac{\dot{K}}{AL} - \frac{K}{AL}\left[\frac{\dot{L}A + \dot{A}L}{AL}\right] = \frac{\dot{K}}{AL} - k\left(\frac{\dot{L}}{L} + \frac{\dot{A}}{A}\right).$$

Substituting the capital-accumulation equation, $\dot{K} = [\partial F(K,AL)/\partial K]K - \delta K$, and the constant growth rates of the labor force and technology, $\dot{L}/L = n$ and $\dot{A}/A = g$, into equation (1) yields

$$(2)\quad \dot{k} = \frac{[\partial F(K,AL)/\partial K]K - \delta K}{AL} - (n+g)k = \frac{\partial F(K,AL)}{\partial K}k - \delta k - (n+g)k.$$

Substituting $\partial F(K,AL)/\partial K = f'(k)$ into equation (2) gives us $\dot{k} = f'(k)k - \delta k - (n+g)k$ or simply

$$(3)\quad \dot{k} = [f'(k) - (n+g+\delta)]k.$$

Capital per unit of effective labor will be constant when $\dot{k} = 0$, i.e. when $[f'(k) - (n + g + \delta)]k = 0$. This condition holds if k = 0 (a case we will ignore) or $f'(k) - (n + g + \delta) = 0$. Thus the balanced-growth-path level of the capital stock per unit of effective labor is implicitly defined by $f'(k^*) = (n + g + \delta)$. Since capital per unit of effective labor, k = K/AL, is constant on the balanced growth path, K must grow at the same rate as AL, which grows at rate n + g. Since the production function has constant returns to capital and effective labor, which both grow at rate n + g on the balanced growth path, output must also grow at rate n + g on the balanced growth path. Thus we have found a balanced growth path where all the variables of the model grow at constant rates.

The next step is to show that the economy actually converges to this balanced growth path. At $k = k^*$, $f'(k) = (n + g + \delta)$. If $k > k^*$, $f'(k) < (n + g + \delta)$. This follows from the assumption that $f''(k) < 0$ which means that $f'(k)$ falls as k rises. Thus if $k > k^*$, we have $\dot{k} < 0$ so that k will fall toward its balanced-growth-path value. If $k < k^*$, $f'(k) > (n + g + \delta)$. Again, this follows from the assumption that $f''(k) < 0$ which means that $f'(k)$ rises as k falls. Thus if $k < k^*$, we have $\dot{k} > 0$ so that k will rise toward its balanced-growth-path value. Thus, regardless of the initial value of k (as long as it is not zero), the economy will converge to a balanced growth path at k^*, where all the variables in the model are growing at constant rates.

(b) The golden-rule level of k -- the level of k that maximizes consumption per unit of effective labor -- is defined implicitly by $f'(k^{GR}) = (n + g + \delta)$. Graphically, this occurs when the slope of the production function equals the slope of the break-even investment line. Note that this is exactly the level of k that the economy converges to in this model where all capital income is saved and all labor income is consumed.

In this model, we are saving capital's contribution to output, which is the marginal product of capital times the amount of capital. If that contribution exceeds break-even investment, $(n + g + \delta)k$, then k rises. If it is less than break-even investment, k falls. Thus k settles down to a point where saving, the marginal product of capital times k, equals break-even investment, $(n + g + \delta)k$. That is, the economy settles down to a point where $f'(k)k = (n + g + \delta)k$ or equivalently $f'(k) = (n + g + \delta)$.

Problem 1.11

(a) The production function with capital-augmenting technological progress is given by

(1) $Y(t) = [A(t)K(t)]^{\alpha} L(t)^{1-\alpha}$.

Dividing both sides of equation (1) by $A(t)^{\alpha/(1-\alpha)}L(t)$ yields

$$\frac{Y(t)}{A(t)^{\alpha/(1-\alpha)}L(t)} = \left[\frac{A(t)K(t)}{A(t)^{\alpha/(1-\alpha)}L(t)}\right]^{\alpha}\left[\frac{L(t)}{A(t)^{\alpha/(1-\alpha)}L(t)}\right]^{1-\alpha},$$

and simplifying:

$$\frac{Y(t)}{A(t)^{\alpha/(1-\alpha)}L(t)} = \left[\frac{A(t)^{1-\alpha/(1-\alpha)}K(t)}{L(t)}\right]^{\alpha} A(t)^{-\alpha} = \left[\frac{A(t)^{1-\alpha/(1-\alpha)}A(t)^{-1}K(t)}{L(t)}\right]^{\alpha},$$

and thus finally

$$\frac{Y(t)}{A(t)^{\alpha/(1-\alpha)}L(t)} = \left[\frac{K(t)}{A(t)^{\alpha/(1-\alpha)}L(t)}\right]^{\alpha}.$$

Now, defining $\phi = \alpha/(1 - \alpha)$, $k(t) = K(t)/A(t)^{\phi}L(t)$ and $y(t) = Y(t)/A(t)^{\phi}L(t)$ yields

(2) $y(t) = k(t)^{\alpha}$.

In order to analyze the dynamics of k(t), take the time derivative of both sides of $k(t) = K(t)/A(t)^{\phi}L(t)$:

$$\dot{k}(t) = \frac{\dot{K}(t)\left[A(t)^{\phi}L(t)\right] - K(t)\left[\phi A(t)^{\phi-1}\dot{A}(t)L(t) + \dot{L}(t)A(t)^{\phi}\right]}{\left[A(t)^{\phi}L(t)\right]^{2}},$$

$$\dot{k}(t) = \frac{\dot{K}(t)}{A(t)^{\phi}L(t)} - \frac{K(t)}{A(t)^{\phi}L(t)}\left[\phi\frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)}\right],$$

and then using $k(t) \equiv K(t)/A(t)^{\phi}L(t)$, $\dot{A}(t)/A(t) = \mu$ and $\dot{L}(t)/L(t) = n$ yields

(3) $\dot{k}(t) = \dot{K}(t)/A(t)^{\phi}L(t) - (\phi\mu + n)k(t)$.

The evolution of the total capital stock is given by the usual

(4) $\dot{K}(t) = sY(t) - \delta K(t)$.

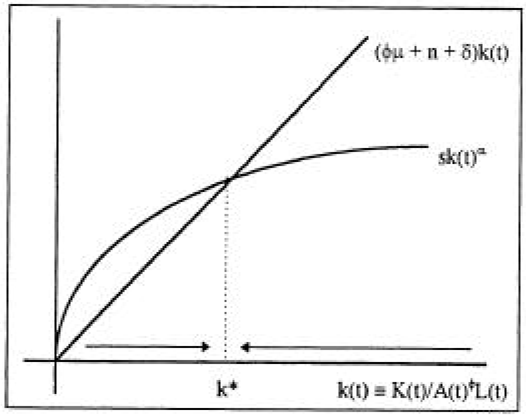
Substituting equation (4) into (3) gives us

$$\dot{k}(t) = sY(t)/A(t)^{\phi}L(t) - \delta K(t)/A(t)^{\phi}L(t) - (\phi\mu + n)k(t) = sy(t) - (\phi\mu + n + \delta)k(t).$$

Finally, using equation (2), $y(t) = k(t)^{\alpha}$, we have

(5) $\dot{k}(t) = sk(t)^{\alpha} - (\phi\mu + n + \delta)k(t)$.

Equation (5) is very similar to the basic equation governing the dynamics of the Solow model with labor-augmenting technological progress. Here, however, we are measuring in units of $A(t)^{\phi}L(t)$ rather than in units of effective labor, $A(t)L(t)$. Using the same graphical technique as with the basic Solow model, we can graph both components of $\dot{k}(t)$. See the figure at right.



When actual investment per unit of $A(t)^{\phi}L(t)$, $sk(t)^{\alpha}$, exceeds break-even investment per unit of $A(t)^{\phi}L(t)$, given by $(\phi\mu + n + \delta)k(t)$, k will rise toward k*. When actual investment per unit of $A(t)^{\phi}L(t)$ falls short of break-even investment per unit of $A(t)^{\phi}L(t)$, k will fall toward k*. Ignoring the case in which the initial level of k is zero, the economy will converge to a situation in which k is constant at k*. Since $y = k^{\alpha}$, y will also be constant when the economy converges to k*.

The total capital stock, K, can be written as $A^{\phi}Lk$. Thus when k is constant, K will be growing at the constant rate of $\phi\mu + n$. Similarly, total output, Y, can be written as $A^{\phi}Ly$. Thus when y is constant, output grows at the constant rate of $\phi\mu + n$ as well. Since L and A grow at constant rates by assumption, we have found a balanced growth path where all the variables of the model grow at constant rates.

(b) The production function is now given by

(6) $Y(t) = J(t)^{\alpha}L(t)^{1-\alpha}$.

Define $\bar{J}(t) \equiv J(t)/A(t)$. The production function can then be written as

(7) $Y(t) = \left[A(t)\bar{J}(t)\right]^{\alpha}L(t)^{1-\alpha}$.

Proceed as in part (a). Divide both sides of equation (7) by $A(t)^{\alpha/(1-\alpha)}L(t)$ and simplify to obtain

$$(8)\quad \frac{Y(t)}{A(t)^{\alpha/(1-\alpha)}L(t)} = \left[\frac{\bar{J}(t)}{A(t)^{\alpha/(1-\alpha)}L(t)}\right]^{\alpha}.$$

Now, defining $\phi = \alpha/(1-\alpha)$, $\bar{j}(t) \equiv \bar{J}(t)/A(t)^{\phi}L(t)$ and $y(t) \equiv Y(t)/A(t)^{\phi}L(t)$ yields

(9) $y(t) = \bar{j}(t)^{\alpha}$.

In order to analyze the dynamics of $\bar{j}(t)$, take the time derivative of both sides of $\bar{j}(t) = \bar{J}(t)/A(t)^{\phi}L(t)$:

$$\dot{\bar{j}} = \frac{\dot{\bar{J}}(t)\left[A(t)^{\phi}L(t)\right] - \bar{J}(t)\left[\phi A(t)^{\phi-1}\dot{A}(t)L(t) + \dot{L}(t)A(t)^{\phi}\right]}{\left[A(t)^{\phi}L(t)\right]^{2}},$$

$$\dot{\bar{j}}(t) = \frac{\dot{\bar{J}}(t)}{A(t)^{\phi}L(t)} - \frac{\bar{J}(t)}{A(t)^{\phi}L(t)}\left[\phi\frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)}\right],$$

and then using $\bar{j}(t) = \bar{J}(t)/A(t)^{\phi}L(t)$, $\dot{A}(t)/A(t) = \mu$ and $\dot{L}(t)/L(t) = n$ yields

(10) $\dot{\bar{j}}(t) = \dot{\bar{J}}(t)/A(t)^{\phi}L(t) - (\phi\mu + n)\bar{j}(t)$.

The next step is to get an expression for $\dot{\bar{J}}(t)$. Take the time derivative of both sides of $\bar{J}(t) = J(t)/A(t)$:

$$\dot{\bar{J}}(t) = \frac{\dot{J}(t)A(t) - J(t)\dot{A}(t)}{A(t)^{2}} = \frac{\dot{J}(t)}{A(t)} - \frac{\dot{A}(t)}{A(t)}\frac{J(t)}{A(t)}.$$

Now use $\bar{J}(t) = J(t)/A(t)$, $\dot{A}(t)/A(t) = \mu$ and $\dot{J}(t) = sA(t)Y(t) - \delta J(t)$ to obtain

$$\dot{\bar{J}}(t) = \frac{sA(t)Y(t)}{A(t)} - \frac{\delta J(t)}{A(t)} - \mu\bar{J}(t),$$

or simply

(11) $\dot{\bar{J}}(t) = sY(t) - (\mu + \delta)\bar{J}(t)$.

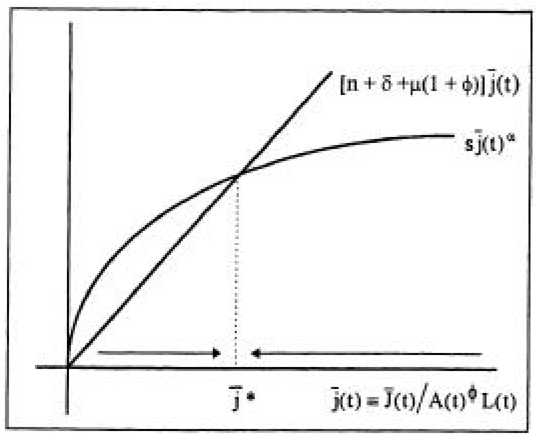
Substitute equation (11) into equation (10):

$$\dot{\bar{j}}(t) = sY(t)/A(t)^{\phi}L(t) - (\mu + \delta)\bar{J}(t)/A(t)^{\phi}L(t) - (\phi\mu + n)\bar{j}(t) = sy(t) - [n + \delta + \mu(1 + \phi)]\bar{j}(t).$$

Finally, using equation (9), $y(t) = \bar{j}(t)^{\alpha}$, we have

(12) $\dot{\bar{j}}(t) = s\bar{j}(t)^{\alpha} - [n + \delta + \mu(1 + \phi)]\bar{j}(t)$.

Using the same graphical technique as in the basic Solow model, we can graph both components of $\dot{\bar{j}}(t)$.



See the figure at right. Ignoring the possibility that the initial value of $\bar{j}$ is zero, the economy will converge to a situation where $\bar{j}$ is constant at $\bar{j}*$. Since $y = \bar{j}^{\alpha}$, y will also be constant when the economy converges to $\bar{j}*$.

The level of total output, Y, can be written as $A^{\phi}Ly$. Thus when y is constant, output grows at the constant rate of $\phi\mu + n$.

By definition, $\bar{J} = A^{\phi}L\bar{j}$. Once the economy converges to the situation where $\bar{j}$ is constant, $\bar{J}$ grows at the constant rate of $\phi\mu + n$. Since $J = \bar{J}A$, the effective capital stock, J, grows at rate $\phi\mu + n + \mu$ or $n + \mu(1 + \phi)$. Thus the economy does converge to a balanced growth path where all the variables of the model are growing at constant rates.

(c) On the balanced growth path, $\dot{\bar{j}}(t) = 0$ and thus from equation (12):

$$s\bar{j}^{\alpha} = [n + \delta + \mu(1+\phi)]\bar{j} \qquad \Rightarrow \qquad \bar{j}^{1-\alpha} = s/[n + \delta + \mu(1+\phi)],$$

and thus

(13) $\bar{j}^{*} = [s/(n + \delta + \mu(1+\phi))]^{1/(1-\alpha)}$.

Substitute equation (13) into equation (9) to get an expression for output per unit of $A(t)^{\phi}L(t)$ on the balanced growth path:

(14) $y^{*} = [s/(n + \delta + \mu(1+\phi))]^{\alpha/(1-\alpha)}$.

Take the derivative of y^* with respect to s:

$$\frac{\partial y^*}{\partial s} = \left[\frac{\alpha}{1-\alpha}\right]\left[\frac{s}{n+\delta+\mu(1+\phi)}\right]^{\alpha/(1-\alpha)-1}\left[\frac{1}{n+\delta+\mu(1+\phi)}\right].$$

In order to turn this into an elasticity, multiply both sides by s/y^* using the expression for y^* from equation (14) on the right-hand side:

$$\frac{\partial y^*}{\partial s}\frac{s}{y^*} = \left[\frac{\alpha}{1-\alpha}\right]\left[\frac{s}{n+\delta+\mu(1+\phi)}\right]^{\alpha/(1-\alpha)-1}\left[\frac{1}{n+\delta+\mu(1+\phi)}\right]s\left[\frac{s}{n+\delta+\mu(1+\phi)}\right]^{-\alpha/(1-\alpha)}.$$

Simplifying yields

$$\frac{\partial y^*}{\partial s}\frac{s}{y^*} = \left[\frac{\alpha}{1-\alpha}\right]\left[\frac{n+\delta+\mu(1+\phi)}{s}\right]\left[\frac{s}{n+\delta+\mu(1+\phi)}\right],$$

and thus finally

(15) $\dfrac{\partial y^*}{\partial s}\dfrac{s}{y^*} = \dfrac{\alpha}{1-\alpha}$.

(d) A first-order Taylor approximation of $\dot{y}$ around the balanced-growth-path value of $y = y^*$ will be of the form

(16) $\dot{y} \cong \partial\dot{y}/\partial y\big|_{y=y^*}[y - y^*]$.

Taking the time derivative of both sides of equation (9) yields

(17) $\dot{y} = \alpha\bar{j}^{\alpha-1}\dot{\bar{j}}$.

Substitute equation (12) into equation (17):

$$\dot{y} = \alpha\bar{j}^{\alpha-1}\left[s\bar{j}^{\alpha} - (n+\delta+\mu(1+\phi))\bar{j}\right],$$

or

(18) $\dot{y} = s\alpha\bar{j}^{2\alpha-1} - \alpha\bar{j}^{\alpha}[n+\delta+\mu(1+\phi)]$.

Equation (18) expresses $\dot{y}$ in terms of $\bar{j}$. We can express $\bar{j}$ in terms of y: since $y = \bar{j}^{\alpha}$, we can write $\bar{j} = y^{1/\alpha}$. Thus $\partial\dot{y}/\partial y$ evaluated at $y = y^*$ is given by

$$\left.\frac{\partial\dot{y}}{\partial y}\right|_{y=y^*} = \left[\left.\frac{\partial\dot{y}}{\partial\bar{j}}\right|_{y=y^*}\right]\left[\left.\frac{\partial\bar{j}}{\partial y}\right|_{y=y^*}\right] = \left[s\alpha(2\alpha-1)\bar{j}^{2(\alpha-1)} - \alpha^2\bar{j}^{\alpha-1}(n+\delta+\mu(1+\phi))\right]\left[\frac{1}{\alpha}y^{(1-\alpha)/\alpha}\right].$$

Now, $y^{(1-\alpha)/\alpha}$ is simply $\bar{j}^{1-\alpha}$ since $y = \bar{j}^{\alpha}$ and thus

$$\left.\frac{\partial\dot{y}}{\partial y}\right|_{y=y^*} = s(2\alpha-1)\bar{j}^{2(\alpha-1)+(1-\alpha)} - \alpha\bar{j}^{\alpha-1+(1-\alpha)}[n+\delta+\mu(1+\phi)] = s(2\alpha-1)\bar{j}^{\alpha-1} - \alpha[n+\delta+\mu(1+\phi)].$$

Finally, substitute out for s by rearranging equation (13) to obtain $s = \bar{j}^{1-\alpha}[n + \delta + \mu(1+\phi)]$ and thus

$$\left.\frac{\partial \dot{y}}{\partial y}\right|_{y=y^*} = \bar{j}^{1-\alpha}[n+\delta+\mu(1+\phi)](2\alpha-1)\bar{j}^{\alpha-1} - \alpha[n+\delta+\mu(1+\phi)],$$

or simply

$$(19)\quad \left.\frac{\partial \dot{y}}{\partial y}\right|_{y=y^*} = -(1-\alpha)[n+\delta+\mu(1+\phi)].$$

Substituting equation (19) into equation (16) gives the first-order Taylor expansion:

$$(20)\quad \dot{y} \equiv -(1-\alpha)[n+\delta+\mu(1+\phi)]\,[y-y^*].$$

Solving this differential equation (as in the text) yields

$$(21)\quad y(t) - y^* = e^{-(1-\alpha)[n+\delta+\mu(1+\phi)]}\,[y(0)-y^*].$$

This means that the economy moves fraction $(1 - \alpha)[n + \delta + \mu(1 + \phi)]$ of the remaining distance toward y^* each year.

(e) The elasticity of output with respect to s is the same in this model as in the basic Solow model. The speed of convergence is faster in this model. In the basic Solow model, the rate of convergence is given by $(1 - \alpha)[n + \delta + \mu]$, which is less than the rate of convergence in this model, $(1 - \alpha)[n + \delta + \mu(1 + \phi)]$, since $\phi \equiv \alpha/(1 - \alpha)$ is positive.

Problem 1.12

(a) The growth-accounting technique of Section 1.7 yields the following expression for the growth rate of output per person:

$$(1)\quad \frac{\dot{Y}(t)}{Y(t)} - \frac{\dot{L}(t)}{L(t)} = \alpha_K(t)\left[\frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)}\right] + R(t),$$

where $\alpha_K(t)$ is the elasticity of output with respect to capital at time t and R(t) is the Solow residual. Now imagine applying this growth-accounting equation to a Solow economy that is on its balanced growth path. On the balanced growth path, the growth rates of output per worker and capital per worker are both equal to g, the growth rate of A. Thus equation (1) implies that growth accounting would attribute a fraction α_K of growth in output per worker to growth in capital per worker. It would attribute the rest -- fraction $1 - \alpha_K$ -- to technological progress, as this is what would be left in the Solow residual. So with our usual estimate of $\alpha_K = 1/3$, growth accounting would attribute about 67% of the growth in output per worker to technological progress and about 33% of the growth in output per worker to growth in capital per worker.

(b) In an accounting sense, the result in part (a) would be true, but in a deeper sense it would not: the <u>reason</u> that the capital-labor ratio grows at rate g on the balanced growth path is because the effectiveness of labor is growing at rate g. That is, the growth in the effectiveness of labor -- the growth in A -- raises output per worker through two channels. First, by directly raising output but also by (for a given saving rate) increasing the resources devoted to capital accumulation and thereby raising the capital-labor ratio. Growth accounting attributes the rise in output per worker through the second channel to growth in the capital-labor ratio, and not to its underlying source. Thus, although growth accounting is often instructive, it is not appropriate to interpret it as shedding light on the underlying determinants of growth.

Problem 1.13

(a) Ordinary least squares (OLS) yields a biased estimate of the slope coefficient of a regression if the explanatory variable is correlated with the error term. We are given that

(1) $\ln\left[(Y/N)_{1979}\right] - \ln\left[(Y/N)_{1870}\right]^{*} = a + b\ln\left[(Y/N)_{1870}\right]^{*} + \varepsilon$, and

(2) $\ln\left[(Y/N)_{1870}\right] = \ln\left[(Y/N)_{1870}\right]^{*} + u$,

where ε and u are assumed to be uncorrelated with each other and with the true unobservable 1870 income per person variable, $\ln[(Y/N)_{1870}]^{*}$.

Substituting equation (2) into (1) and rearranging yields

(3) $\ln\left[(Y/N)_{1979}\right] - \ln\left[(Y/N)_{1870}\right] = a + b\ln\left[(Y/N)_{1870}\right] + \left[\varepsilon - (1+b)u\right]$.

Running an OLS regression on model (3) will yield a biased estimate of b if $\ln[(Y/N)_{1870}]$ is correlated with the error term, $[\varepsilon - (1 + b)u]$. In general, of course, this will be the case since u is the measurement error that helps to determine the value of $\ln[(Y/N)_{1870}]$ that we get to observe. However, in the special case in which the true value of b = -1, the error term in model (3) is simply ε. Thus OLS will be unbiased since the explanatory variable will no longer be correlated with the error term.

(b) Measurement error in the dependent variable will not cause a problem for OLS estimation and is, in fact, one of the justifications for the disturbance term in a regression model. Intuitively, if the measurement error is in 1870 income per capita, the explanatory variable, there will be a bias toward finding convergence. If 1870 income per capita is overstated, growth is understated. This looks like convergence: a "high" initial income country growing slowly. Similarly, if 1870 income per capita is understated, growth is overstated. This also looks like convergence: a "low" initial income country growing quickly.

Suppose instead that it is only 1979 income per capita that is subject to random, mean-zero measurement error. When 1979 income is overstated, so is growth for a given level of 1870 income. When 1979 income is understated, so is growth for a given 1870 income. Either case is equally likely: overstating 1979 income for any given 1870 income is just as likely as understating it (or more precisely, measurement error is on average equal to zero). Thus there is no reason for this to systematically cause us to see more or less convergence than there really is in the data.

Problem 1.14

What is needed for a balanced growth path is that K and Y are each growing at a constant rate. The equation of motion for capital, $\dot{K}(t) = sY(t) - \delta K(t)$, implies the growth rate of K is

(1) $\dfrac{\dot{K}(t)}{K(t)} = s\dfrac{Y(t)}{K(t)} - \delta$.

As in the model in the text, Y/K must be constant in order for the growth rate of K to be constant. That is, the growth rates of Y and K must be equal.

Taking logs of both sides of the production function, $Y(t) = K(t)^{\alpha} R(t)^{\beta} T(t)^{\gamma}\left[A(t)L(t)\right]^{1-\alpha-\beta-\gamma}$, yields

(2) $\ln Y(t) = \alpha\ln K(t) + \beta\ln R(t) + \gamma\ln T(t) + (1 - \alpha - \beta - \gamma)[\ln A(t) + \ln L(t)]$.

Differentiating both sides of (2) with respect to time gives us

(3) $g_Y(t) = \alpha g_K(t) + \beta g_R(t) + \gamma g_T(t) + (1 - \alpha - \beta - \gamma)[g_A(t) + g_L(t)]$.

Substituting in the facts that the growth rates of R, T, and L are all equal to n and the growth rate of A is equal to g gives us

$$g_Y(t) = \alpha g_K(t) + \beta n + \gamma n + (1 - \alpha - \beta - \gamma)(n + g).$$

Simplifying gives us

$$(4)\quad \begin{aligned} g_Y(t) &= \alpha g_K(t) + (\beta + \gamma)n + (1 - \alpha)n - (\beta + \gamma)n + (1 - \alpha - \beta - \gamma)g \\ &= \alpha g_K(t) + (1 - \alpha)n + (1 - \alpha - \beta - \gamma)g \end{aligned}$$

Using the fact that g_Y and g_K must be equal on a balanced growth path leaves us with

$$g_Y = \alpha g_Y + (1 - \alpha)n + (1 - \alpha - \beta - \gamma)g,$$

$$(1 - \alpha)g_Y = (1 - \alpha)n + (1 - \alpha - \beta - \gamma)g,$$

and thus the growth rate of output on the balanced growth path is given by

$$(5)\quad \tilde{g}_Y^{bgp} = \frac{(1 - \alpha)n + (1 - \alpha - \beta - \gamma)g}{1 - \alpha}.$$

The growth rate of output per worker on the balanced growth path is

$$\tilde{g}_{Y/L}^{bgp} = \tilde{g}_Y^{bgp} - \tilde{g}_L^{bgp}.$$

Using equation (5) and the fact that L grows at rate n, we can write

$$\tilde{g}_{Y/L}^{bgp} = \frac{(1 - \alpha)n + (1 - \alpha - \beta - \gamma)g}{1 - \alpha} - n = \frac{(1 - \alpha)n + (1 - \alpha - \beta - \gamma)g - (1 - \alpha)n}{1 - \alpha}.$$

And thus finally

$$(6)\quad \tilde{g}_{Y/L}^{bgp} = \frac{(1 - \alpha - \beta - \gamma)g}{1 - \alpha}.$$

Equation (6) is identical to equation (1.50) in the text.

SOLUTIONS TO CHAPTER 2

Problem 2.1

(a) The firm's problem is to choose the quantities of capital, K, and effective labor, AL, in order to minimize costs, wAL + rK, subject to the production function, Y = ALf(k). Set up the Lagrangian:

$$\mathcal{L} = wAL + rK + \lambda\left[Y - ALf(K/AL)\right].$$

The first-order conditions are given by

$$\frac{\partial \mathcal{L}}{\partial K} = r - \lambda\left[ALf'(K/AL)(1/AL)\right] = 0 \quad \Rightarrow \quad r = \lambda f'(k), \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial (AL)} = w - \lambda\left[f(K/AL) + ALf'(K/AL)(-K)/(AL)^2\right] = 0 \quad \Rightarrow \quad w = \lambda\left[f(k) - kf'(k)\right]. \quad (2)$$

Dividing equation (1) by equation (2) gives us

$$(3) \quad \frac{r}{w} = \frac{f'(k)}{f(k) - kf'(k)}.$$

Equation (3) implicitly defines the cost-minimizing choice of k. Clearly this choice does not depend upon the level of output, Y. Note that equation (3) is the standard cost-minimizing condition: the ratio of the marginal cost of the two inputs, capital and effective labor, must equal the ratio of the marginal products of the two inputs.

(b) Since, as shown in part (a), each firm chooses the same value of k and since we are told that each firm has the same value of A, we can write the total amount produced by the N cost-minimizing firms as

$$\sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} AL_i f(k) = Af(k)\sum_{i=1}^{n} L_i = A\overline{L}f(k),$$

where $\overline{L}$ is the total amount of labor employed.

The single firm also has the same value of A and would choose the same value of k; the choice of k does not depend on Y. Thus if it used all of the labor employed by the N cost-minimizing firms, $\overline{L}$, the single firm would produce $Y = A\overline{L}f(k)$. This is exactly the same amount of output produced in total by the N cost-minimizing firms.

Problem 2.2

(a) The individual's problem is to maximize lifetime utility given by

$$(1) \quad U = \frac{C_1^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_2^{1-\theta}}{1-\theta},$$

subject to the lifetime budget constraint given by

(2) $P_1C_1 + P_2C_2 = W$,

where W represents lifetime income.

Rearrange the budget constraint to solve for C_2:

(3) $C_2 = W/P_2 - C_1P_1/P_2$.

Substitute equation (3) into equation (1):

$$(4) \quad U = \frac{C_1^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{\left[W/P_2 - C_1P_1/P_2\right]^{1-\theta}}{1-\theta}.$$

Now we can solve the unconstrained problem of maximizing utility, as given by equation (4), with respect to first period consumption, C_1. The first-order condition is given by

$$\partial U/\partial C_1 = C_1^{-\theta} + (1/1+\rho)C_2^{-\theta}(-P_1/P_2) = 0 \quad \Rightarrow \quad C_1^{-\theta} = (1/1+\rho)(P_1/P_2)C_2^{-\theta},$$

or simply

(5) $C_1 = (1+\rho)^{1/\theta}(P_2/P_1)^{1/\theta}C_2$.

In order to solve for C_2, substitute equation (5) into equation (3):

$$C_2 = W/P_2 - (1+\rho)^{1/\theta}(P_2/P_1)^{1/\theta}C_2(P_1/P_2) \quad \Rightarrow \quad C_2\left[1+(1+\rho)^{1/\theta}(P_2/P_1)^{(1-\theta)/\theta}\right] = W/P_2,$$

or simply

(6) $C_2 = \dfrac{W/P_2}{\left[1+(1+\rho)^{1/\theta}(P_2/P_1)^{(1-\theta)/\theta}\right]}$.

Finally, to get the optimal choice of C_1, substitute equation (6) into equation (5):

(7) $C_1 = \dfrac{(1+\rho)^{1/\theta}(P_2/P_1)^{1/\theta}(W/P_2)}{\left[1+(1+\rho)^{1/\theta}(P_2/P_1)^{(1-\theta)/\theta}\right]}$.

(b) From equation (5), the optimal ratio of first-period to second-period consumption is

(8) $C_1/C_2 = (1+\rho)^{1/\theta}(P_2/P_1)^{1/\theta}$.

Taking the log of both sides of equation (8) yields

(9) $\ln(C_1/C_2) = (1/\theta)\ln(1+\rho) + (1/\theta)\ln(P_2/P_1)$.

The elasticity of substitution between C_1 and C_2, defined in such a way that it is positive, is given by

$$\frac{\partial(C_1/C_2)}{\partial(P_2/P_1)}\frac{(P_2/P_1)}{(C_1/C_2)} = \frac{\partial[\ln(C_1/C_2)]}{\partial[\ln(P_2/P_1)]} = \frac{1}{\theta},$$

where we have used equation (9) to find the derivative. Thus higher values of θ imply that the individual is less willing to substitute consumption between periods.

Problem 2.3

(a) We can use analysis similar to the intuitive derivation of the Euler equation in Section 2.2 of the text. Think of the household's consumption at two moments of time. Specifically, consider a short (formally infinitesimal) period of time Δt from $(t_0 - \varepsilon)$ to $(t_0 + \varepsilon)$.

Imagine the household reducing consumption per unit of effective labor, c, at $(t_0 - \varepsilon)$ -- an instant before the confiscation of wealth -- by a small (again, infinitesimal) amount Δc. It then invests this additional saving and consumes the proceeds at $(t_0 + \varepsilon)$. If the household is optimizing, the marginal impact of this change on lifetime utility must be zero.

This experiment would have a utility cost of $u'(c_{before})\Delta c$. Ordinarily, since the instantaneous rate of return is r(t), c at time $(t_0 + \varepsilon)$ could be increased by $e^{[r(t)-n-g]\Delta t}\Delta c$. But here, half of that increase will be confiscated. Thus the utility benefit would be $[1/2]u'(c_{after})e^{[r(t)-n-g]\Delta t}\Delta c$. Thus for the path of consumption to be utility-maximizing, it must satisfy

(1) $u'(c_{before})\Delta c = \dfrac{1}{2}u'(c_{after})e^{[r(t)-n-g]\Delta t}\Delta c$.

Rather informally, we can cancel the Δc's and allow $\Delta t \to 0$, leaving us with

(2) $u'(c_{before}) = \frac{1}{2}u'(c_{after})$.

Thus there will be a discontinuous jump in consumption at the time of the confiscation of wealth. Specifically, consumption will jump down. Intuitively, the household's consumption will be high before t_0 because it will have an incentive not to save so as to avoid the wealth confiscation.

(b) In this case, from the viewpoint of an individual household, its actions will not affect the amount of wealth that is confiscated. For an individual household, essentially a predetermined amount of wealth will be confiscated at time t_0 and thus the household's optimization and its choice of consumption path would take this into account. The household would still prefer to smooth consumption over time and there will not be a discontinuous jump in consumption at time t_0.

Problem 2.4

We need to solve the household's problem assuming log utility and in per capita terms rather than in units of effective labor. The household's problem is to maximize lifetime utility subject to the budget constraint. That is, its problem is to maximize

(1) $U = \int_{t=0}^{\infty} e^{-\rho t} \ln C(t) \frac{L(t)}{H} dt$,

subject to

(2) $\int_{t=0}^{\infty} e^{-R(t)} C(t) \frac{L(t)}{H} dt = \frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)} A(t) w(t) \frac{L(t)}{H} dt$.

Now let $W = \frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)} A(t) w(t) \frac{L(t)}{H} dt$.

We can use the informal method, presented in the text, for solving this type of problem. Set up the Lagrangian:

$$\mathcal{L} = \int_{t=0}^{\infty} e^{-\rho t} \ln C(t) \frac{L(t)}{H} dt + \lambda \left[W - \int_{t=0}^{\infty} e^{-R(t)} C(t) \frac{L(t)}{H} dt \right].$$

The first-order condition is given by

$$\frac{\partial \mathcal{L}}{\partial C(t)} = e^{-\rho t} C(t)^{-1} \frac{L(t)}{H} - \lambda e^{-R(t)} \frac{L(t)}{H} = 0.$$

Canceling the L(t)/H yields

(3) $e^{-\rho t} C(t)^{-1} = \lambda e^{-R(t)}$,

which implies

(4) $C(t) = e^{-\rho t} \lambda^{-1} e^{R(t)}$.

Substituting this into the budget constraint, equation (2), gives us

(5) $\int_{t=0}^{\infty} e^{-R(t)} \left[e^{-\rho t} \lambda^{-1} e^{R(t)} \right] \frac{L(t)}{H} dt = W$.

Since $L(t) = e^{nt} L(0)$, this implies

(6) $\lambda^{-1} \frac{L(0)}{H} \int_{t=0}^{\infty} e^{-(\rho - n)t} dt = W$.

As long as $\rho - n > 0$ (which it must be), the integral is equal to $1/(\rho - n)$ and thus λ^{-1} is given by

(7) $\lambda^{-1} = \frac{W}{L(0)/H}(\rho - n)$.

Substituting equation (7) into equation (4) yields

(8) $C(t) = e^{R(t)-\rho t}\left[\frac{W}{L(0)/H}(\rho - n)\right]$.

Initial consumption is therefore

(9) $C(0) = \frac{W}{L(0)/H}(\rho - n)$.

Note that C(0) is consumption per person, W is wealth per household and L(0)/H is the number of people per household. Thus W/[L(0)/H] is wealth per person. This equation says that initial consumption per person is a constant fraction of initial wealth per person, and (ρ - n) can be interpreted as the marginal propensity to consume out of wealth. With logarithmic utility, this propensity to consume is independent of the path of the real interest rate. Also note that the bigger is the household's discount rate ρ -- the more the household discounts the future -- the bigger is the fraction of wealth that it initially consumes.

Problem 2.5

The household's problem is to maximize lifetime utility subject to the budget constraint. That is, its problem is to maximize

(1) $U = \int_{t=0}^{\infty} e^{-\rho t} \frac{C(t)^{1-\theta}}{1-\theta} \frac{L(t)}{H} dt$,

subject to

(2) $\int_{t=0}^{\infty} e^{-rt} C(t) \frac{L(t)}{H} dt = W$,

where W denotes the household's initial wealth plus the present value of its lifetime labor income, i.e. the right-hand side of equation (2.6) in the text. Note that the real interest rate, r, is assumed to be constant.

We can use the informal method, presented in the text, for solving this type of problem. Set up the Lagrangian:

$$\mathcal{L} = \int_{t=0}^{\infty} e^{-\rho t} \frac{C(t)^{1-\theta}}{1-\theta} \frac{L(t)}{H} dt + \lambda\left[W - \int_{t=0}^{\infty} e^{-rt} C(t) \frac{L(t)}{H} dt\right].$$

The first-order condition is given by

$$\frac{\partial \mathcal{L}}{\partial C(t)} = e^{-\rho t} C(t)^{-\theta} \frac{L(t)}{H} - \lambda e^{-rt} \frac{L(t)}{H} = 0.$$

Canceling the L(t)/H yields

(3) $e^{-\rho t} C(t)^{-\theta} = \lambda e^{-rt}$.

Differentiate both sides of equation (3) with respect to time:

$$e^{-\rho t}\left[-\theta C(t)^{-\theta-1} \dot{C}(t)\right] - \rho e^{-\rho t} C(t)^{-\theta} + r\lambda e^{-rt} = 0.$$

This can be rearranged to obtain

(4) $-\theta \frac{\dot{C}(t)}{C(t)} e^{-\rho t} C(t)^{-\theta} - \rho e^{-\rho t} C(t)^{-\theta} + r\lambda e^{-rt} = 0$.

Now substitute the first-order condition, equation (3), into equation (4):

$$-\theta\frac{\dot{C}(t)}{C(t)}\lambda e^{-rt} - \rho\lambda e^{-rt} + r\lambda e^{-rt} = 0.$$

Canceling the λe^{-rt} and solving for the growth rate of consumption, $\dot{C}(t)/C(t)$, yields

(5) $$\frac{\dot{C}(t)}{C(t)} = \frac{r-\rho}{\theta}.$$

Thus with a constant real interest rate, the growth rate of consumption is a constant. If $r > \rho$ -- that is, if the rate that the market pays to defer consumption exceeds the household's discount rate -- consumption will be rising over time. The value of θ determines the magnitude of consumption growth if r exceeds ρ. A lower value of θ -- and thus a higher value of the elasticity of substitution, $1/\theta$ -- means that consumption growth will be higher for any given difference between r and ρ.

We now need to solve for the path of $C(t)$. First, note that equation (5) can be rewritten as

(6) $$\frac{\partial \ln C(t)}{\partial t} = \frac{r-\rho}{\theta}.$$

Integrate equation (6) forward from time $\tau = 0$ to time $\tau = t$:

$$\ln C(t) - \ln C(0) = \left[(r-\rho)/\theta\right]\tau\Big|_{\tau=0}^{\tau=t},$$

which simplifies to

(7) $$\ln\left[C(t)/C(0)\right] = \left[(r-\rho)/\theta\right]t.$$

Taking the exponential function of both sides of equation (7) yields

$$C(t)/C(0) = e^{\left[(r-\rho)/\theta\right]t},$$

and thus

(8) $$C(t) = C(0)e^{\left[(r-\rho)/\theta\right]t}.$$

We can now solve for initial consumption, $C(0)$, by using the fact that it must be chosen to satisfy the household's budget constraint. Substitute equation (8) into equation (2):

$$\int_{t=0}^{\infty} e^{-rt}C(0)e^{\left[(r-\rho)/\theta\right]t}\frac{L(t)}{H}dt = W.$$

Using the fact that $L(t) = L(0)e^{nt}$ yields

(9) $$\frac{C(0)L(0)}{H}\int_{t=0}^{\infty} e^{-\left[\rho-r+\theta(r-n)\right]t/\theta}dt = W.$$

As long as $[\rho - r + \theta(r - n)]/\theta > 0$, we can solve the integral:

(10) $$\int_{t=0}^{\infty} e^{-\left[\rho-r+\theta(r-n)\right]t/\theta}dt = \frac{\theta}{\rho - r + \theta(r-n)}.$$

Substitute equation (10) into equation (9) and solve for $C(0)$:

(11) $$C(0) = \frac{W}{L(0)/H}\left[\frac{(\rho-r)}{\theta} + (r-n)\right].$$

Finally, to get an expression for consumption at each instant in time, substitute equation (11) into equation (8):

(12) $$C(t) = e^{\left[(r-\rho)/\theta\right]t}\frac{W}{L(0)/H}\left[\frac{(\rho-r)}{\theta} + (r-n)\right].$$

Problem 2.6

(a) The equation describing the dynamics of the capital stock per unit of effective labor is

(1) $\dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t)$.

For a given k, the level of c that implies $\dot{k} = 0$ is given by c = f(k) - (n + g)k. Thus a fall in g makes the level of c consistent with $\dot{k} = 0$ higher for a given k. That is, the $\dot{k} = 0$ curve shifts up. Intuitively, a lower g makes break-even investment lower at any given k and thus allows for more resources to be devoted to consumption and still maintain a given k. Since (n + g)k falls proportionately more at higher levels of k, the $\dot{k} = 0$ curve shifts up more at higher levels of k. See the figure.

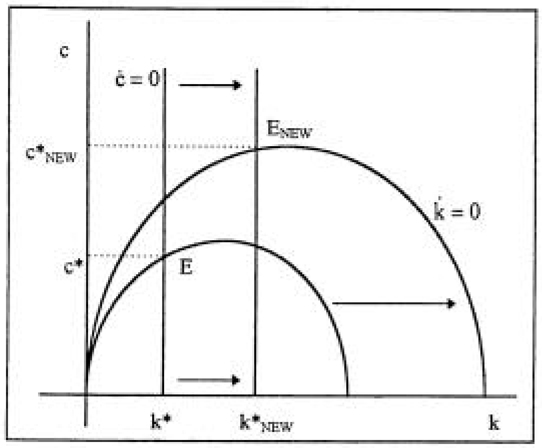
(b) The equation describing the dynamics of consumption per unit of effective labor is given by

(2) $\dfrac{\dot{c}(t)}{c(t)} = \dfrac{f'(k(t)) - \rho - \theta g}{\theta}$.

Thus the condition required for $\dot{c} = 0$ is given by f ' (k) = ρ + θg. After the fall in g, f ' (k) must be lower in order for $\dot{c} = 0$. Since f " (k) is negative this means that the k needed for $\dot{c} = 0$ therefore rises. Thus the $\dot{c} = 0$ curve shifts to the right.

(c) At the time of the change in g, the value of k, the stock of capital per unit of effective labor, is given by the history of the economy, and it cannot change discontinuously. It remains equal to the k* on the old balanced growth path.

In contrast, c, the rate at which households are consuming in units of effective labor, can jump at the time of the shock. In order for the economy to reach the new balanced growth path, c must jump at the instant of the change so that the economy is on the new saddle path.



However, we cannot tell whether the new saddle path passes above or below the original point E. Thus we cannot tell whether c jumps up or down and in fact, if the new saddle path passes right through point E, c might even remain the same at the instant that g falls. Thereafter, c and k rise gradually to their new balanced-growth-path values; these are higher than their values on the original balanced growth path.

(d) On a balanced growth path, the fraction of output that is saved and invested is given by [f(k*) - c*]/f(k*). Since k is constant, or $\dot{k} = 0$ on a balanced growth path then, from equation (1), we can write f(k*) - c* = (n + g)k*. Using this, we can rewrite the fraction of output that is saved on a balanced growth path as

(3) s = [(n + g)k*]/f(k*).

Differentiating both sides of equation (3) with respect to g yields

(4) $\dfrac{\partial s}{\partial g} = \dfrac{f(k^*)[(n+g)(\partial k^*/\partial g) + k^*] - (n+g)k^* f'(k^*)(\partial k^*/\partial g)}{[f(k^*)]^2}$,

which simplifies to

(5) $$\frac{\partial s}{\partial g} = \frac{(n+g)[f(k^*) - k^* f'(k^*)](\partial k^*/\partial g) + f(k^*)k^*}{[f(k^*)]^2}.$$

Since k^* is defined by $f'(k^*) = \rho + \theta g$, differentiating both sides of this expression gives us $f''(k^*)(\partial k^*/\partial g) = \theta$. Solving for $\partial k^*/\partial g$ gives us

(6) $\partial k^*/\partial g = \theta / f''(k^*) < 0$.

Substituting equation (6) into equation (5) yields

(7) $$\frac{\partial s}{\partial g} = \frac{(n+g)[f(k^*) - k^* f'(k^*)]\theta + f(k^*)k^* f''(k^*)}{[f(k^*)]^2 f''(k^*)}.$$

The first term in the numerator is positive, whereas the second is negative and so the sign of $\partial s/\partial g$ is ambiguous. Thus we cannot tell whether the fall in g raises or lowers the saving rate on the new balanced growth path.

(e) When the production function is Cobb-Douglas, $f(k) = k^\alpha$, $f'(k) = \alpha k^{\alpha-1}$ and $f''(k) = \alpha(\alpha - 1)k^{\alpha-2}$. Substituting these facts into equation (7) yields

(8) $$\frac{\partial s}{\partial g} = \frac{(n+g)[k^{*\alpha} - k^* \alpha k^{*\alpha-1}]\theta + k^{*\alpha}\, k^* \alpha(\alpha-1)k^{*\alpha-2}}{k^{*\alpha}\, k^{*\alpha}\, \alpha(\alpha-1)k^{*\alpha-2}},$$

which simplifies to

(9) $$\frac{\partial s}{\partial g} = \frac{(n+g)k^{*\alpha}(1-\alpha)\theta - (1-\alpha)k^{*\alpha}\,\alpha k^{*\alpha-1}}{[-(1-\alpha)k^{*\alpha}(\alpha k^{*\alpha-1})(\alpha k^{*\alpha-1})/\alpha]},$$

which implies

(10) $$\frac{\partial s}{\partial g} = -\alpha \frac{[(n+g)\theta - (\rho + \theta g)]}{(\rho + \theta g)^2}.$$

Thus, finally, we have

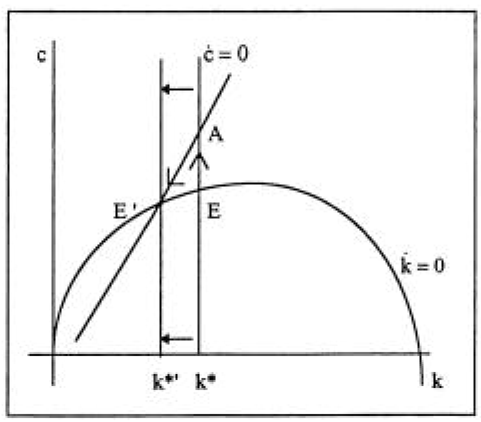
(11) $$\frac{\partial s}{\partial g} = -\alpha \frac{(n\theta - \rho)}{(\rho + \theta g)^2} = \alpha \frac{(\rho - n\theta)}{(\rho + \theta g)^2}.$$

Problem 2.7

The two equations of motion are

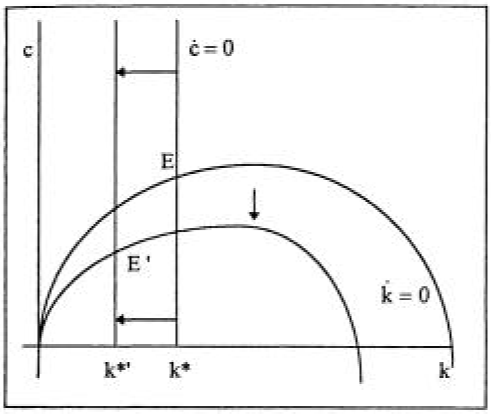
(1) $$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta},$$ and (2) $$\dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t).$$

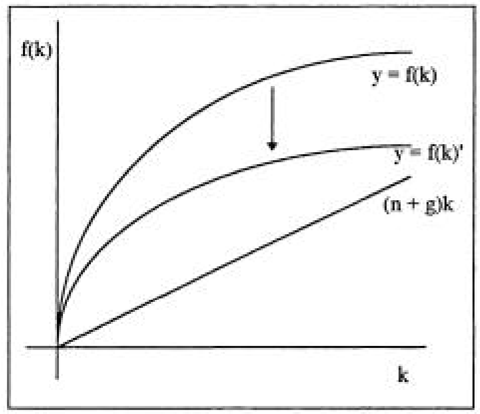
(a) A rise in θ or a fall in the elasticity of substitution, $1/\theta$, means that households become less willing to substitute consumption between periods. It also means that the marginal utility of consumption falls off more rapidly as consumption rises. If the economy is growing, this tends to make households value present consumption more than future consumption.



The capital-accumulation equation is unaffected. The condition required for $\dot{c} = 0$ is given by $f'(k) = \rho + \theta g$. Since $f''(k) < 0$, the $f'(k)$ that makes $\dot{c} = 0$ is now higher. Thus the value of k that satisfies $\dot{c} = 0$ is lower. The $\dot{c} = 0$ locus shifts to the left. The economy moves up to point A on the new saddle path; people consume more now. Movement is then down along the new saddle path until the economy reaches point E '. At that point, c* and k* are lower than their original values.

(b) We can assume that a downward shift of the production function means that for any given k, both f(k) and f ' (k) are lower than before.

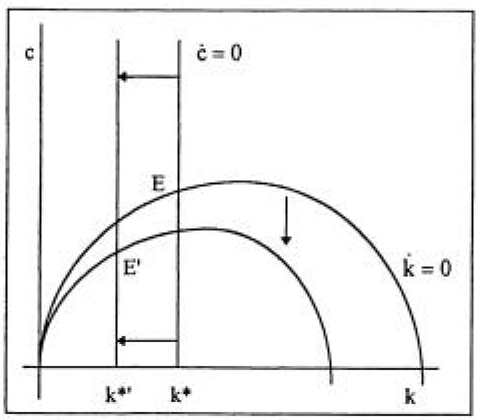


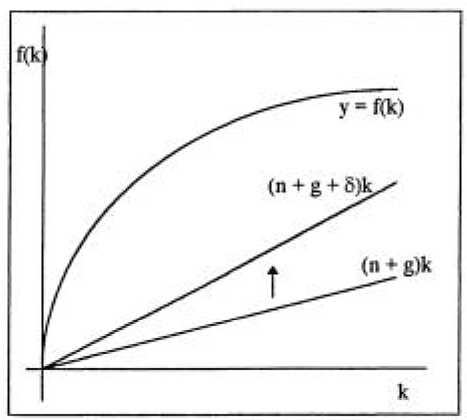


The condition required for $\dot{k} = 0$ is given by $c = f(k) - (n+g)k$. We can see from the figure on the right that the $\dot{k} = 0$ locus will shift down more at higher levels of k. Also, since for a given k, f ' (k) is lower now, the golden-rule k will be lower than before. Thus the $\dot{k} = 0$ locus shifts as depicted in the figure.

The condition required for $\dot{c} = 0$ is given by $f'(k) = \rho + \theta g$. For a given k, f ' (k) is now lower. Thus we need a lower k to keep f ' (k) the same and satisfy the $\dot{c} = 0$ equation. Thus the $\dot{c} = 0$ locus shifts left. The economy will eventually reach point E ' with lower c* and lower k*. Whether c initially jumps up or down depends upon whether the new saddle path passes above or below point E.

(c) With a positive rate of depreciation, $\delta > 0$, the new capital-accumulation equation is
(3) $\dot{k}(t) = f(k(t)) - c(t) - (n + g + \delta)k(t)$.





The level of saving and investment required just to keep any given k constant is now higher -- and thus the amount of consumption possible is now lower -- than in the case with no depreciation. The level of extra investment required is also higher at higher levels of k. Thus the $\dot{k} = 0$ locus shifts down more at higher levels of k.

In addition, the real return on capital is now $f'(k(t)) - \delta$ and so the household's maximization will yield

(4) $\dfrac{\dot{c}(t)}{c(t)} = \dfrac{f'(k(t)) - \delta - \rho - \theta g}{\theta}$.

The condition required for $\dot{c} = 0$ is $f'(k) = \delta + \rho + \theta g$. Compared to the case with no depreciation, $f'(k)$ must be higher and k lower in order for $\dot{c} = 0$. Thus the $\dot{c} = 0$ locus shifts to the left. The economy will eventually wind up at point E ' with lower levels of c* and k*. Again, whether c jumps up or down initially depends upon whether the new saddle path passes above or below the original equilibrium point of E.

Problem 2.7

With a positive depreciation rate, $\delta > 0$, the Euler equation and the capital-accumulation equation are given by

(1) $\dfrac{\dot{c}(t)}{c(t)} = \dfrac{f'(k(t)) - \delta - \rho - \theta g}{\theta}$, and (2) $\dot{k}(t) = f(k(t)) - c(t) - (n + g + \delta)k(t)$.

We begin by taking first-order Taylor approximations to (1) and (2) around k = k* and c = c*. That is, we can write

(3) $\dot{c} \cong \dfrac{\partial \dot{c}}{\partial k}[k - k^*] + \dfrac{\partial \dot{c}}{\partial c}[c - c^*]$, and (4) $\dot{k} \cong \dfrac{\partial \dot{k}}{\partial k}[k - k^*] + \dfrac{\partial \dot{k}}{\partial c}[c - c^*]$,

where $\partial \dot{c}/\partial k$, $\partial \dot{c}/\partial c$, $\partial \dot{k}/\partial k$ and $\partial \dot{k}/\partial c$ are all evaluated at k = k* and c = c*.

Define $\tilde{c} \equiv c - c^*$ and $\tilde{k} \equiv k - k^*$. Since c* and k* are constants, $\dot{c}$ and $\dot{k}$ are equivalent to $\dot{\tilde{c}}$ and $\dot{\tilde{k}}$ respectively. We can therefore rewrite (3) and (4) as

(5) $\dot{\tilde{c}} \cong \dfrac{\partial \dot{c}}{\partial k}\tilde{k} + \dfrac{\partial \dot{c}}{\partial c}\tilde{c}$, and (6) $\dot{\tilde{k}} \cong \dfrac{\partial \dot{k}}{\partial k}\tilde{k} + \dfrac{\partial \dot{k}}{\partial c}\tilde{c}$.

Using equations (1) and (2) to compute these derivatives yields

(7) $\left.\dfrac{\partial \dot{c}}{\partial k}\right|_{bgp} = \dfrac{f''(k^*)c^*}{\theta}$, (8) $\left.\dfrac{\partial \dot{c}}{\partial c}\right|_{bgp} = \dfrac{f'(k^*) - \delta - \rho - \theta g}{\theta} = 0$,

(9) $\left.\dfrac{\partial \dot{k}}{\partial k}\right|_{bgp} = f'(k^*) - (n + g + \delta)$, (10) $\left.\dfrac{\partial \dot{k}}{\partial c}\right|_{bgp} = -1$.

Substituting equations (7) and (8) into (5) and equations (9) and (10) into (6) yields

(11) $\dot{\tilde{c}} \cong \dfrac{f''(k^*)c^*}{\theta}\tilde{k}$, and

(12)
$$
\begin{aligned}
\dot{\tilde{k}} &\cong \left[f'(k^*) - (n + g + \delta)\right]\tilde{k} - \tilde{c} \\
&\cong \left[(\delta + \rho + \theta g) - (n + g + \delta)\right]\tilde{k} - \tilde{c} \\
&\cong \beta\tilde{k} - \tilde{c}.
\end{aligned}
$$

The second line of equation (12) uses the fact that (1) implies that $f'(k^*) = \delta + \rho + \theta g$. The third line uses the definition of $\beta = \rho - n - (1 - \theta)g$.

Dividing equation (11) by $\tilde{c}$ and dividing equation (12) by $\tilde{k}$ yields

(13) $\dfrac{\dot{\tilde{c}}}{\tilde{c}} \cong \dfrac{f''(k^*)c^*}{\theta}\dfrac{\tilde{k}}{\tilde{c}}$, and (14) $\dfrac{\dot{\tilde{k}}}{\tilde{k}} \cong \beta - \dfrac{\tilde{c}}{\tilde{k}}$.

Note that these are exactly the same as equations (2.31) and (2.32) in the text; adding a positive depreciation rate does not alter the expressions for the growth rates of $\tilde{c}$ and $\tilde{k}$. Thus equation (2.36), the expression for μ, the constant growth rate of both $\tilde{c}$ and $\tilde{k}$ as the economy moves toward the balanced growth path, is still valid. Thus

(15) $\mu_1 = \dfrac{\beta - \sqrt{\beta^2 - 4f''(k^*)c^*/\theta}}{2}$,

where we have chosen the negative growth rate so that c and k are moving toward c^* and k^*, not away from them.

Now consider the Cobb-Douglas production function, $f(k) = k^\alpha$. Thus

(16) $f'(k^*) = \alpha k^{*\alpha-1} = r^* + \delta$, and (17) $f''(k^*) = \alpha(\alpha-1)k^{*\alpha-2}$.

Squaring both sides of equation (16) gives us

(18) $(r^* + \delta)^2 = \alpha^2 k^{*2\alpha-2}$,

and so equation (17) can be rewritten as

(19) $f''(k^*) = \dfrac{(r^*+\delta)^2(\alpha-1)}{\alpha k^{*\alpha}} = \dfrac{\alpha-1}{\alpha}\dfrac{(r^*+\delta)^2}{f(k^*)}$.

In addition, defining s^* to be the saving rate on the balanced growth path, we can write the balanced-growth-path level of consumption as

(20) $c^* = (1 - s^*)f(k^*)$.

Substituting equations (19) and (20) into (15) yields

$$\mu_1 = \frac{\beta - \sqrt{\beta^2 - 4\left(\frac{\alpha-1}{\alpha}\right)\frac{(r^*+\delta)^2}{f(k^*)\theta}(1-s^*)f(k^*)}}{2}.$$

Canceling the $f(k^*)$ and multiplying through by the minus sign yields

(21) $\mu_1 = \dfrac{\beta - \sqrt{\beta^2 + \frac{4}{\theta}\left(\frac{1-\alpha}{\alpha}\right)(r^*+\delta)^2(1-s^*)}}{2}$.

On the balanced growth path, the condition required for $\dot{c} = 0$ is given by $r^* = \rho + \theta g$ and thus

(22) $r^* + \delta = \rho + \theta g + \delta$.

In addition, actual saving, $s^*f(k^*)$, equals break-even investment, $(n + g + \delta)k^*$, and thus

(23) $s^* = \dfrac{(n+g+\delta)k^*}{f(k^*)} = \dfrac{(n+g+\delta)}{k^{*\alpha-1}} = \dfrac{\alpha(n+g+\delta)}{(r^*+\delta)}$,

where we have used equation (16), $r^* + \delta = \alpha k^{*\alpha-1}$. From equation (23), we can write

(24) $(1-s^*) = \dfrac{(r^*+\delta) - \alpha(n+g+\delta)}{(r^*+\delta)}$.

Substituting equations (22) and (24) into equation (21) yields

$$(25)\quad \mu_1 = \frac{\beta - \sqrt{\beta^2 + \frac{4}{\theta}\left(\frac{1-\alpha}{\alpha}\right)(\rho+\theta g+\delta)\left[\rho+\theta g+\delta-\alpha(n+g+\delta)\right]}}{2}.$$

Equation (25) is analogous to equation (2.38) in the text. It expresses the rate of adjustment in terms of the underlying parameters of the model. Keeping the values in the text -- $\alpha = 1/3$, $\rho = 4\%$, $n = 2\%$, $g = 1\%$ and $\theta = 1$ -- and using $\delta = 3\%$ yields a value for μ_1 of approximately - 8.8%. This is faster convergence than the -5.4% obtained with no depreciation.

Problem 2.9

(a) The real after-tax rate of return on capital is now given by $(1 - \tau)f'(k(t))$. Thus the household's maximization would now yield the following expression describing the dynamics of consumption per unit of effective labor:

$$(1)\quad \frac{\dot{c}(t)}{c(t)} = \frac{\left[(1-\tau)f'(k(t)) - \rho - \theta g\right]}{\theta}.$$

The condition required for $\dot{c} = 0$ is given by $(1 - \tau)f'(k) = \rho + \theta g$. The after-tax rate of return must equal $\rho + \theta g$. Compared to the case without a tax on capital, $f'(k)$, the pre-tax rate of return on capital, must be higher and thus k must be lower in order for $\dot{c} = 0$. Thus the $\dot{c} = 0$ locus shifts to the left.

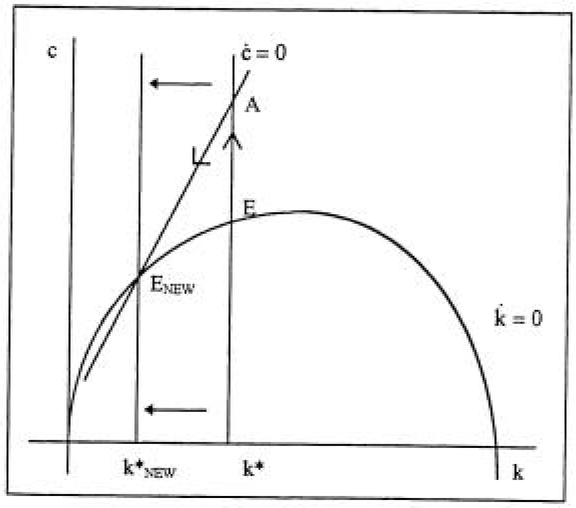
The equation describing the dynamics of the capital stock per unit of effective labor is still given by

$$(2)\quad \dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t).$$

For a given k, the level of c that implies $\dot{k} = 0$ is given by $c(t) = f(k) - (n + g)k$. Since the tax is rebated to households in the form of lump-sum transfers, this $\dot{k} = 0$ locus is unaffected.

(b) At time 0, when the tax is put in place, the value of k, the stock of capital per unit of effective labor, is given by the history of the economy, and it cannot change discontinuously. It remains equal to the k* on the old balanced growth path.

In contrast, c, the rate at which households are consuming in units of effective labor, can jump at the time that the tax is introduced. This jump in c is not inconsistent with the consumption-smoothing behavior implied by the household's optimization problem since the tax was unexpected and could not be prepared for.



In order for the economy to reach the new balanced growth path, it should be clear what must occur. At time 0, c jumps up so that the economy is on the new saddle path. In the figure, the economy jumps from point E to a point such as A. Since the return to saving and accumulating capital is now lower than before, people switch away from saving and into consumption.

After time 0, the economy will gradually move down the new saddle path until it eventually reaches the new balanced growth path at E_{NEW} .

(c) On the new balanced growth path at E_{NEW} , the distortionary tax on investment income has caused the economy to have a lower level of capital per unit of effective labor as well as a lower level of consumption per unit of effective labor.

(d) (i) From the analysis above, we know that the higher is the tax rate on investment income, τ, the lower will be the balanced-growth-path level of k^*, all else equal. In terms of the above story, the higher is τ the more that the $\dot{c} = 0$ locus shifts to the left and hence the more that k^* falls. Thus $\partial k^*/\partial\tau < 0$.

On a balanced growth path, the fraction of output that is saved and invested, the saving rate, is given by $[f(k^*) - c^*]/f(k^*)$. Since k is constant, or $\dot{k} = 0$, on a balanced growth path then from $\dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t)$ we can write $f(k^*) - c^* = (n + g)k^*$. Using this we can rewrite the fraction of output that is saved on a balanced growth path as

(3) $s = [(n + g)k^*]/f(k^*)$.

Use equation (3) to take the derivative of the saving rate with respect to the tax rate, τ:

(4) $$\frac{\partial s}{\partial \tau} = \frac{(n+g)(\partial k^*/\partial\tau)f(k^*) - (n+g)k^* f'(k^*)(\partial k^*/\partial\tau)}{f(k^*)^2}.$$

Simplifying yields

$$\frac{\partial s}{\partial \tau} = \frac{(n+g)}{f(k^*)}\frac{\partial k^*}{\partial\tau} - \frac{(n+g)}{f(k^*)}\frac{k^* f'(k^*)}{f(k^*)}\frac{\partial k^*}{\partial\tau} = \frac{(n+g)}{f(k^*)}\frac{\partial k^*}{\partial\tau}\left[1 - \frac{k^* f'(k^*)}{f(k^*)}\right].$$

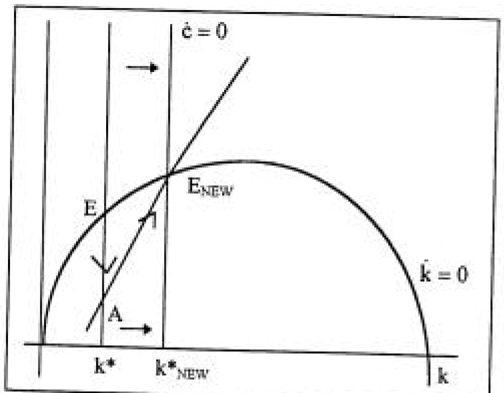
Recall that $k^*f'(k^*)/f(k^*) = \alpha_K(k^*)$ is capital's (pre-tax) share in income, which must be less than one. Since $\partial k^*/\partial\tau < 0$ we can write

(5) $$\frac{\partial s}{\partial \tau} = \frac{(n+g)}{f(k^*)}\frac{\partial k^*}{\partial\tau}\left[1 - \alpha_K(k^*)\right] < 0$$

Thus the saving rate on the balanced growth path is decreasing in τ.

(d) (ii) Citizens in low-τ, high-k^*, high-saving countries do not have the incentive to invest in low-saving countries. From part (a), the condition required for $\dot{c} = 0$ is $(1 - \tau)f'(k) = \rho + \theta g$.. That is, the after-tax rate of return must equal $\rho + \theta g$. Assuming preferences and technology are the same across countries so that ρ, θ and g are the same across countries, the after-tax rate of return will be the same across countries. Since the after-tax rate of return is thus the same in low-saving countries as it is in high-saving countries, there is no incentive to shift saving from a high-saving to a low-saving country.

(e) Should the government subsidize investment instead and fund this with a lump-sum tax? This would lead to the opposite result from above and the economy would have higher c and k on the new balanced growth path.



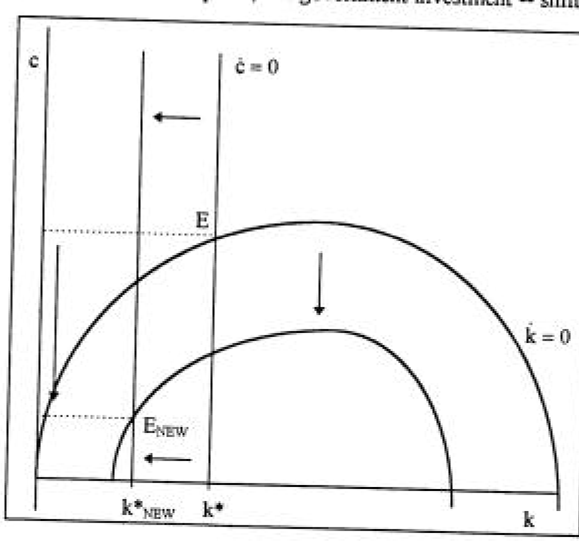
The answer is no. The original market outcome is already the one that would be chosen by a central planner attempting to maximize the lifetime utility of a representative household subject to the capital-accumulation equation. It therefore gives the household the highest possible lifetime utility.

Starting at point E, the implementation of the subsidy would lead to a short-term drop in consumption at point A, but would eventually result in permanently higher consumption at point E_{NEW}. It would turn out that the utility lost from the short-term sacrifice would outweigh the utility gained in the long-term (all in present value terms, appropriately discounted). This is the same type of argument used to explain the reason that households do not choose to consume at the golden-rule level. See Section 2.4 for a more complete description of the welfare implications of this model.

(f) Suppose the government does not rebate the tax revenue to households but instead uses it to make government purchases. Let G(t) represent government purchases per unit of effective labor. The equation describing the dynamics of the capital stock per unit of effective labor is now given by

(2 ') $\dot{k}(t) = f(k(t)) - c(t) - G(t) - (n+g)k(t)$.

The fact that the government is making purchases that do not add to the capital stock -- it is assumed to be government consumption, not government investment -- shifts down the $\dot{k} = 0$ locus.



After the imposition of the tax, the $\dot{c} = 0$ locus shifts to the left, just as it did in the case in which the government rebated the tax to households. In the end, k* falls to k^*_{NEW} just as in the case where the government rebated the tax. Consumption per unit of effective labor on the new balanced growth path at E_{NEW} is lower than in the case where the tax is rebated by the amount of the government purchases, which is $\tau f'(k)k$.

Finally, whether the level of c jumps up or down initially depends upon whether the new saddle path passes above or below the original balanced-growth-path point of E.

Problem 2.10

(a) - (c) Before the tax is put in place, i.e. until time t_1, the equations governing the dynamics of the economy are

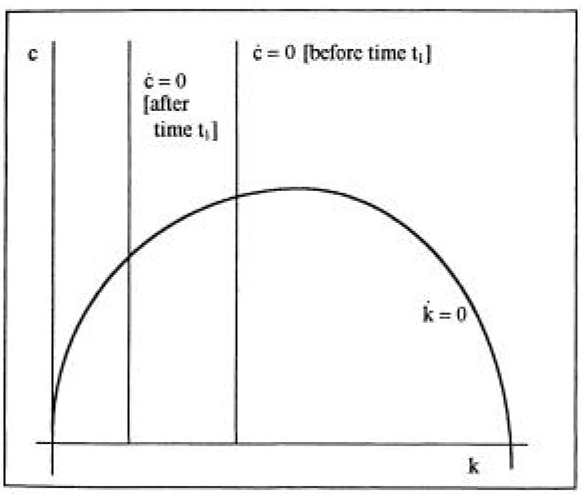
(1) $\dfrac{\dot{c}(t)}{c(t)} = \dfrac{f'(k(t)) - \rho - \theta g}{\theta}$, and (2) $\dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t)$.

The condition required for $\dot{c} = 0$ is given by $f'(k) = \rho + \theta g$. The capital-accumulation equation is not affected when the tax is put in place at time t_1 since we are assuming that the government is rebating the tax, not spending it.

Since the real after-tax rate of return on capital is now $(1 - \tau)f'(k(t))$, the household's maximization yields the following growth rate of consumption:

(3) $\dfrac{\dot{c}(t)}{c(t)} = \dfrac{(1-\tau)f'(k(t)) - \rho - \theta g}{\theta}$.

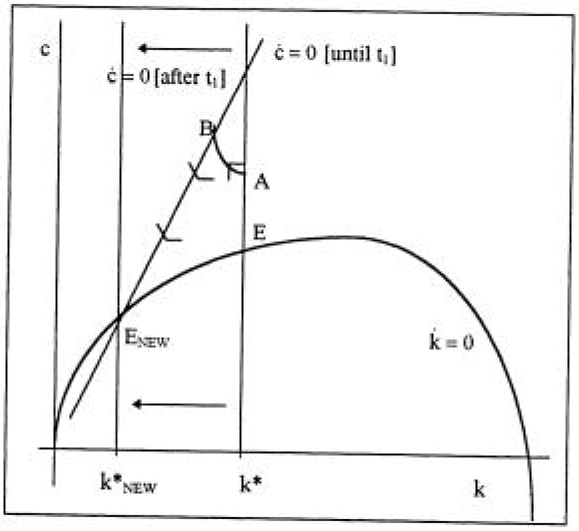
The condition required for $\dot{c} = 0$ is now given by $(1 - \tau)f'(k) = \rho + \theta g$. The after-tax rate of return on capital must equal $\rho + \theta g$. Thus the pre-tax rate of return, $f'(k)$, must be higher and thus k must be lower in order for $\dot{c} = 0$. Thus at time t_1, the $\dot{c} = 0$ locus shifts to the left.



The important point is that the dynamics of the economy are still governed by the original equations of motion until the tax is actually put in place. Between the time of the announcement and the time the tax is actually put in place, it is the original $\dot{c} = 0$ locus that is relevant.

When the tax is put in place at time t_1, c cannot jump discontinuously because households know ahead of time that the tax will be implemented then. A discontinuous jump in c would be inconsistent with the consumption smoothing implied by the household's intertemporal optimization. The household would not want c to be low, and thus marginal utility to be high, a moment before t_1 knowing that c will jump up and be high, and thus marginal utility will be low, a moment after t_1. The household would like to smooth consumption between the two instants in time.

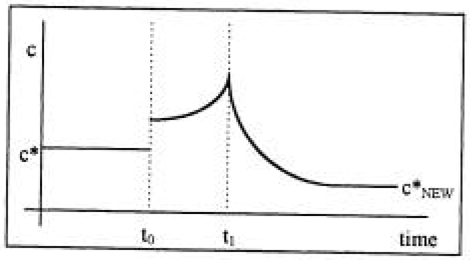
(d) We know that c cannot jump at time t_1. We also know that if the economy is to reach the new balanced growth path at point E_{NEW}, it must be right on the new saddle path at the time that the tax is put in place. Thus when the tax is announced at time t_0, c must jump up to a point such as A. Point A lies between the original balanced growth path at E and the new saddle path.

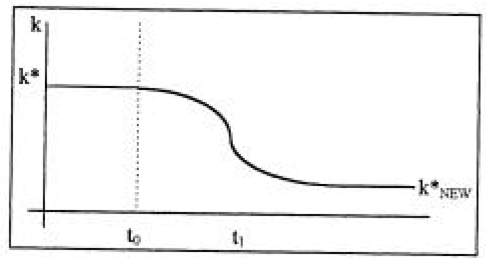


At A, c is too high to maintain the capital stock at k* and so k begins falling. Between t_0 and t_1, the dynamics of the system are still governed by the original $\dot{c} = 0$ locus. The economy is thus to the left of the $\dot{c} = 0$ locus and so consumption begins rising.

The economy moves off to the northwest until at t_1, it is right at point B on the new saddle path. The tax is then put in place and the system is governed by the new $\dot{c} = 0$ locus. Thus c begins falling. The economy moves down the new saddle path, eventually reaching point E_{NEW}.

(e) The story in part (d) implies the following time paths for consumption per unit of effective labor and capital per unit of effective labor.





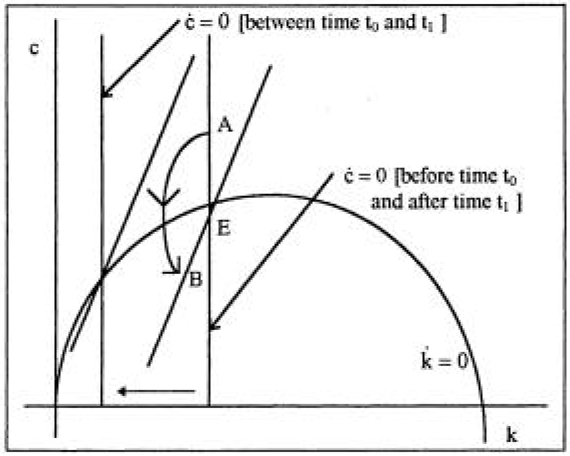
Problem 2.11

(a) The first point is that consumption cannot jump at time t_1. Households know ahead of time that the tax will end then and so a discontinuous jump in c would be inconsistent with the consumption-smoothing behavior implied by the household's intertemporal optimization. Thus, for the economy to return to a balanced growth path, we must be somewhere on the original saddle path right at time t_1.

Before the tax is put in place -- until time t_0 -- and after the tax is removed -- after time t_1 -- the equations governing the dynamics of the economy are

(1) $\dfrac{\dot{c}(t)}{c(t)} = \dfrac{f'(k(t)) - \rho - \theta g}{\theta}$, and (2) $\dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t)$.

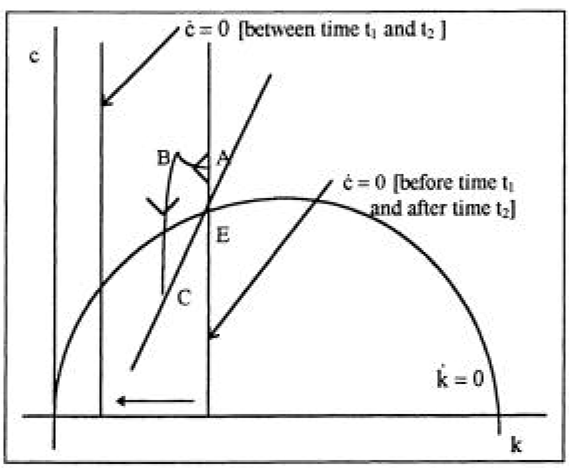
The condition required for $\dot{c} = 0$ is given by $f'(k) = \rho + \theta g$. The capital-accumulation equation, and thus the $\dot{k} = 0$ locus, is not affected by the tax. The $\dot{c} = 0$ locus is affected, however. Between time t_0 and time t_1, the condition required for $\dot{c} = 0$ is that the after-tax rate of return on capital equal $\rho + \theta g$ so that $(1 - \tau)f'(k) = \rho + \theta g$. Thus between t_0 and t_1, $f'(k)$ must be higher and so k must be lower in order for $\dot{c} = 0$. That is, between time t_0 and time t_1, the $\dot{c} = 0$ locus lies to the left of its original position.



At time t_0, the tax is put in place. At point E, the economy is still on the $\dot{k} = 0$ locus but is now to the right of the new $\dot{c} = 0$ locus. Thus if c did not jump up to a point like A, c would begin falling. The economy would then be below the $\dot{k} = 0$ locus and so k would start rising. The economy would drift away from point E in the direction of the southeast and could not be on the original saddle path right at time t_1.

Thus at time t_0, c must jump up so that the economy is at a point like A. Thus, k and c begin falling. Eventually the economy crosses the $\dot{k} = 0$ locus and so k begins rising. This can be interpreted as households anticipating the removal of the tax on capital and thus being willing to accumulate capital again. Point A must be such that given the dynamics of the system, the economy is right at a point like B, on the original saddle path, at time t_1 when the tax is removed. After t_1, the original $\dot{c} = 0$ locus governs the dynamics of the system again. Thus the economy moves up the original saddle path, eventually returning to the original balanced growth path at point E.

(b) The first point is that consumption cannot jump at either time t_1 or time t_2. Households know ahead of time that the tax will be implemented at t_1 and removed at t_2. Thus a discontinuous jump in c at either date would be inconsistent with the consumption-smoothing behavior implied by the household's intertemporal optimization. In order for the economy to return to a balanced growth path, the economy must be somewhere on the original saddle path right at time t_2.



Before the tax is put in place -- until time t_1 -- and after the tax is removed -- after time t_2 -- equations (1) and (2) govern the dynamics of the system. An important point is that even during the time between the announcement and the implementation of the tax -- that is, between time t_0 and time t_1 -- the original $\dot{c} = 0$ locus governs the dynamics of the system.

At time t_0, the tax is announced. Consumption must jump up so that the economy is at a point like A. At A, the economy is still on the $\dot{c} = 0$ locus but is above the $\dot{k} = 0$ locus and so k starts falling. The economy is then to the left of the $\dot{c} = 0$ locus and so c starts rising. The economy drifts off to the northwest.

At time t_1, the tax is implemented, the $\dot{c} = 0$ locus shifts to the left and the economy is at a point like B. The economy is still above the $\dot{k} = 0$ locus but is now to the right of the relevant $\dot{c} = 0$ locus; k continues to fall and c stops rising and begins to fall.

Eventually the economy crosses the $\dot{k} = 0$ locus and k begins rising. Households begin accumulating capital again before the actual removal of the tax on capital income. Point A must be chosen so that given the dynamics of the system, the economy is right at a point like C, on the original saddle path, at time t_2 when the tax is removed. After t_2, the original $\dot{c} = 0$ locus governs the dynamics of the system again. Thus the economy moves up the original saddle path, eventually returning to the original balanced growth path at point E.

Problem 2.12

With government purchases in the model, the capital-accumulation equation is given by

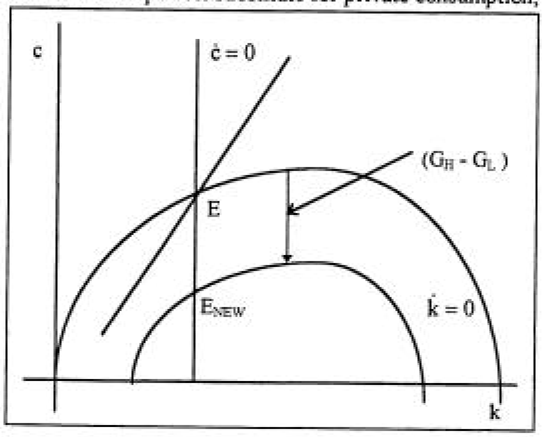
(1) $\dot{k}(t) = f(k(t)) - c(t) - G(t) - (n+g)k(t)$,

where G(t) represents government purchases in units of effective labor at time t.

Intuitively, since government purchases are assumed to be a perfect substitute for private consumption, changes in G will simply be offset one-for-one with changes in c. Suppose that G(t) is initially constant at some level G_L. The household's maximization yields

(2) $$\frac{\dot{c}(t)}{c(t) + G_L} = \frac{f'(k(t)) - \rho - \theta g}{\theta}.$$

Thus the condition for constant consumption is still given by $f'(k) = \rho + \theta g$. Changes in the level of G_L will affect the level of c, but will not shift the $\dot{c} = 0$ locus.



Suppose the economy starts on a balanced growth path at point E. At some time t_0, G unexpectedly increases to G_H and households know this is temporary; households know that at some future time t_1, government purchases will return to G_L. At time t_0, the $\dot{k} = 0$ locus shifts down; at each level of k, the government is using more resources leaving less available for consumption. In particular, the $\dot{k} = 0$ locus shifts down by the amount of the increase in purchases, which is $(G_H - G_L)$.

The difference between this case, in which c and G are perfect substitutes, and the case in which G does not affect private utility, is that c <u>can</u> jump at time t_1 when G returns to its original value. In fact, at t_1, when G jumps down by the amount $(G_H - G_L)$, c <u>must</u> jump up by that exact same amount. If it did not, there would be a discontinuous jump in marginal utility that could not be optimal for households. Thus at t_1, c must jump up by $(G_H - G_L)$ <u>and</u> this must put the economy somewhere on the original saddle path. If it did not, the economy would not return to a balanced growth path. What must happen is that at time t_0, c falls by the amount $(G_H - G_L)$ and the economy jumps to point E_{NEW}. It then stays there until time t_1. At t_1, c jumps back up by the amount $(G_H - G_L)$ and so the economy jumps back to point E and stays there.

Why can't c jump down by less than (G_H - G_L) at t_0 ? If it did, the economy would be above the new $\dot{k} = 0$ locus, k would start falling putting the economy to the left of the $\dot{c} = 0$ locus. Thus c would start rising and so the economy would drift off to the northwest. There would be no way for c to jump up by (G_H - G_L) at t_1 and still put the economy on the original saddle path.

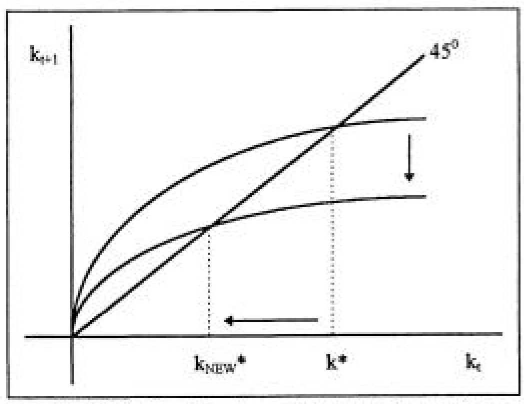
Why can't c jump down by more that (G_H - G_L) at t_1 ? If it did, the economy would be below the new $\dot{k} = 0$ locus, k would start rising putting the economy to the right of the $\dot{c} = 0$ locus. Thus c would start falling and so the economy would drift off to the southeast. Again, there would be no way for c to jump up by (G_H - G_L) at t_1 and still put the economy on the original saddle path.

In summary, the capital stock and the real interest rate are unaffected by the temporary increase in G. At the instant that G rises, consumption falls by an equal amount. It remains constant at that level while G remains high. At the instant that G falls to its initial value, consumption jumps back up to its original value and stays there.

Problem 2.13

Equation (2.59) in the text describes the relationship between k_{t+1} and k_t in the special case of logarithmic utility and Cobb-Douglas production:

(2.59) $$k_{t+1} = \frac{1}{(1+n)(1+g)}\frac{1}{2+\rho}(1-\alpha)k_t^{\alpha}.$$



(a) A rise in n shifts the k_{t+1} function down. From equation (2.59), a higher n means a smaller k_{t+1} for a given k_t . Since the fraction of their labor income that the young save does not depend on n, a given amount of capital per unit of effective labor and thus output per unit of effective labor in time t yields the same amount of saving in period t. Thus it yields the same amount of capital in period t + 1. However, the number of individuals increases more from period t to period t + 1 than it used to. So that capital is spread out among more individuals than it would have been in the absence of the increase in population growth and thus capital per unit of effective labor in period t + 1 is lower for a given k_t .

(b) With the parameter B added to the Cobb-Douglas production function, $f(k) = Bk^{\alpha}$, equation (2.59) becomes

(1) $$k_{t+1} = \frac{1}{(1+n)(1+g)}\frac{1}{2+\rho}(1-\alpha)Bk_t^{\alpha}.$$

This fall in B causes the k_{t+1} function to shift down. See the figure from part (a). A lower B means that a given amount of capital per unit of effective labor in period t now produces less output per unit of effective labor in period t. Since the fraction of their labor income that the young save does not depend on B, this leads to less total saving and a lower capital stock per unit of effective labor in period t + 1 for a given k_t .

(c) We need to determine the effect on k_{t+1} for a given k_t, of a change in α. From equation (2.59):

(2) $$\frac{\partial k_{t+1}}{\partial \alpha} = \frac{1}{(1+n)(1+g)}\frac{1}{2+\rho}\left[-k_t^{\alpha} + (1-\alpha)\frac{\partial k_t^{\alpha}}{\partial \alpha}\right].$$

We need to determine $\partial k_t^{\alpha}/\partial\alpha$. Define $f(\alpha) \equiv k_t^{\alpha}$ and note that $\ln f(\alpha) = \alpha \ln k_t$. Thus

(3) $\partial \ln f(\alpha)/\partial\alpha = \ln k_t$.

Now note that we can write

(4) $$\frac{\partial f(\alpha)}{\partial \alpha} = \frac{\partial f(\alpha)}{\partial \ln f(\alpha)}\frac{\partial \ln f(\alpha)}{\partial \alpha} = \frac{1}{\left[\partial \ln f(\alpha)/\partial f(\alpha)\right]}\frac{\partial \ln f(\alpha)}{\partial \alpha},$$

and thus finally

(5) $\partial f(\alpha)/\partial\alpha = f(\alpha)\ln k_t$.

Therefore, we have $\partial k_t^{\alpha}/\partial\alpha = k_t^{\alpha}\ln k_t$. Substituting this fact into equation (2) yields

(6) $$\frac{\partial k_{t+1}}{\partial \alpha} = \frac{1}{(1+n)(1+g)}\frac{1}{2+\rho}\left[-k_t^{\alpha} + (1-\alpha)k_t^{\alpha}\ln k_t\right],$$

or simply

(7) $$\frac{\partial k_{t+1}}{\partial \alpha} = \frac{1}{(1+n)(1+g)}\frac{1}{2+\rho}\left\{k_t^{\alpha}\left[(1-\alpha)\ln k_t - 1\right]\right\}.$$

Thus, for $(1-\alpha)\ln k_t - 1 > 0$, or $\ln k_t > 1/(1-\alpha)$, an increase in α means a higher k_{t+1} for a given k_t and thus the k_{t+1} function shifts up over this range of k_t's. However, for $\ln k_t < 1/(1-\alpha)$, an increase in α means a lower k_{t+1} for a given k_t. Thus the k_{t+1} function shifts down over this range of k_t's. Finally, right at $\ln k_t = 1/(1-\alpha)$, the old and new k_{t+1} functions intersect.

Problem 2.14

(a) We need to find an expression for k_{t+1} as a function of k_t. Next period's capital stock is equal to this period's capital stock, plus any investment done this period, less any depreciation that occurs. Thus

(1) $K_{t+1} = K_t + sY_t - \delta K_t$.

To convert this into units of effective labor, divide both sides of equation (1) by $A_{t+1}L_{t+1}$:

$$\frac{K_{t+1}}{A_{t+1}L_{t+1}} = \frac{K_t(1-\delta) + sY_t}{A_{t+1}L_{t+1}} = \frac{K_t(1-\delta) + sY_t}{(1+n)(1+g)A_tL_t} = \frac{k_t(1-\delta) + sf(k_t)}{(1+n)(1+g)},$$

which simplifies to

(2) $$k_{t+1} = \left[\frac{1-\delta}{(1+n)(1+g)}\right]k_t + \left[\frac{s}{(1+n)(1+g)}\right]f(k_t).$$

(b) We need to sketch k_{t+1} as a function of k_t.

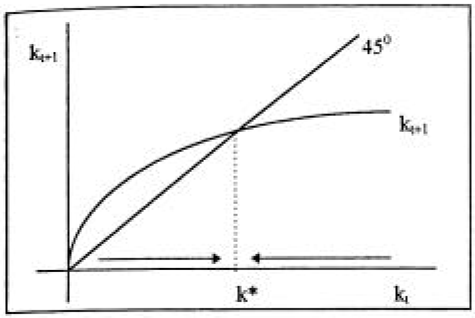
Note that

$$\frac{\partial k_{t+1}}{\partial k_t} = \frac{1-\delta}{(1+n)(1+g)} + \left[\frac{s}{(1+n)(1+g)}\right]f'(k_t) > 0, \quad \text{and} \quad \frac{\partial^2 k_{t+1}}{\partial k_t^2} = \frac{sf''(k_t)}{(1+n)(1+g)} < 0,$$

and using the Inada conditions

$$\lim_{k\to 0}\frac{\partial k_{t+1}}{\partial k_t} = \infty \qquad \lim_{k\to\infty}\frac{\partial k_{t+1}}{\partial k_t} = \frac{1-\delta}{(1+n)(1+g)} < 1.$$

Thus the function eventually has a slope of less than one and will therefore cross the 45 degree line at some point. Also, the function is well-behaved and will cross the 45 degree line only once.



As long as k starts out at some value other than 0, the economy will converge to k^*. For example, if k starts out below k^*, we see that k_{t+1} will be greater than k_t and the economy will move toward k^*. Similarly, if k starts out above k^*, we see that k_{t+1} will be below k_t and again the economy will move toward k^*. At k^*, $y^* = f(k^*)$ is also constant and we have a balanced growth path.

(c) On a balanced growth path, $k_{t+1} = k_t = k^*$ and thus from equation (2):

$$k^* = \left[\frac{1-\delta}{(1+n)(1+g)}\right]k^* + \left[\frac{s}{(1+n)(1+g)}\right]f(k^*),$$

which simplifies to

$$k^*\left[\frac{1+n+g+ng-1+\delta}{(1+n)(1+g)}\right] = \left[\frac{s}{(1+n)(1+g)}\right]f(k^*).$$

Thus on a balanced growth path:

(3) $k^*(n + g + ng + \delta) = sf(k^*)$.

Rearranging equation (3) to get an expression for s on the balanced growth path yields

(4) $s = (n + g + ng + \delta)k^*/f(k^*)$.

Consumption per unit of effective labor on the balanced growth path is given by

(5) $c^* = (1 - s)f(k^*)$.

Substitute equation (4) into equation (5):

$$c^* = \left[1 - \frac{(n+g+ng+\delta)k^*}{f(k^*)}\right]f(k^*) = \left[\frac{f(k^*) - k^*(n+g+ng+\delta)}{f(k^*)}\right]f(k^*).$$

Canceling the $f(k^*)$ yields

(6) $c^* = f(k^*) - (n + g + ng + \delta)k^*$.

To get an expression for the $f'(k^*)$ that maximizes consumption per unit of effective labor on the balanced growth path, we need to maximize c^* with respect to k^*. The first-order condition is given by

$$\partial c^*/\partial k^* = f'(k^*) - (n+g+ng+\delta) = 0.$$

Thus the golden-rule capital stock is defined implicitly by

(7) $f'(k_{GR}) = (n + g + ng + \delta)$.

(d) (i) Substitute a Cobb-Douglas production function, $f(k_t) = k_t^{\alpha}$, into equation (2):

$$(8)\quad k_{t+1} = \left[\frac{1-\delta}{(1+n)(1+g)}\right]k_t + \left[\frac{s}{(1+n)(1+g)}\right]k_t^{\alpha}.$$

(d) (ii) On a balanced growth path, $k_{t+1} = k_t = k^*$. Thus from equation (8):

$$k^* = \left[\frac{1-\delta}{(1+n)(1+g)}\right]k^* + \left[\frac{s}{(1+n)(1+g)}\right]k^{*\alpha}.$$

Simplifying yields

$$\left[\frac{(1+n)(1+g)-(1-\delta)}{(1+n)(1+g)}\right]k^* = \left[\frac{s}{(1+n)(1+g)}\right]k^{*\alpha} \quad \Rightarrow \quad k^{*1-\alpha} = s/(n+g+ng+\delta),$$

and thus finally

(9) $k^* = [s/(n+g+ng+\delta)]^{1/(1-\alpha)}$.

(d) (iii) Using equation (8):

$$(10)\quad \left.\frac{dk_{t+1}}{dk_t}\right|_{k_t=k^*} = \frac{1-\delta}{(1+n)(1+g)} + \frac{\alpha s}{(1+n)(1+g)}k^{*\alpha-1}.$$

Substituting the balanced-growth-path value of k^* -- equation (9) -- into equation (10) yields

$$\left.\frac{dk_{t+1}}{dk_t}\right|_{k_t=k^*} = \frac{1-\delta}{(1+n)(1+g)} + \frac{\alpha s}{(1+n)(1+g)}\left[\frac{(n+g+ng+\delta)}{s}\right].$$

It will be useful to write $(n + g + ng + \delta)$ as $(1 + n)(1 + g) - (1 - \delta)$:

$$\left.\frac{dk_{t+1}}{dk_t}\right|_{k_t=k^*} = \frac{(1-\delta)+\alpha\left[(1+n)(1+g)-(1-\delta)\right]}{(1+n)(1+g)}.$$

Simplifying further yields

$$(11)\quad \left.\frac{dk_{t+1}}{dk_t}\right|_{k_t=k^*} = \alpha + \frac{(1-\delta)(1-\alpha)}{(1+n)(1+g)}.$$

Replacing equation (8) by its first-order Taylor approximation around $k = k^*$ therefore gives us

(12) $k_{t+1} \cong k^* + \left[\alpha + (1-\delta)(1-\alpha)/(1+n)(1+g)\right]\left[k_t - k^*\right]$.

Since we can write this simply as

$$k_{t+1} - k^* \cong \left[\alpha + (1-\delta)(1-\alpha)/(1+n)(1+g)\right]\left[k_t - k^*\right],$$

equation (12) implies

(13) $k_t - k^* \cong \left[\alpha + (1-\delta)(1-\alpha)/(1+n)(1+g)\right]^t\left[k_0 - k^*\right]$.

Thus the economy moves fraction $1-\left[\alpha + (1-\delta)(1-\alpha)/(1+n)(1+g)\right]$ of the way to the balanced growth path each period. Some simple algebra simplifies the expression for this rate of convergence to $(1-\alpha)(n+g+ng+\delta)/(1+n)(1+g)$. With $\alpha = 1/3$, $n = 1\%$, $g = 2\%$ and $\delta = 3\%$, this yields a rate of convergence of about 3.9%. This is slower than the rate of convergence found in the continuous-time Solow model.

Problem 2.15

(a) The individual's optimization problem is not affected by the depreciation which means that $r_t = f'(k_t) - \delta$. The household's problem is still to maximize utility as given by

$$(1)\quad U_t = \frac{C_{1,t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2,t+1}^{1-\theta}}{1-\theta},$$

subject to the budget constraint

$$(2)\quad C_{1,t} + \frac{1}{1+r_{t+1}}C_{2,t+1} = A_t w_t.$$

As in the text, with no depreciation, the fraction of income saved, $s(r_{t+1}) = (1-C_{1,t})A_t w_t$, is given by

$$(3)\quad s(r_{t+1}) = \frac{1}{1+(1+\rho)^{1/\theta}(1+r_{t+1})^{(\theta-1)/\theta}}.$$

Thus the way in which the fraction of income saved depends on the real interest rate, r_{t+1}, is unchanged. The only difference is that the real interest rate itself is now $f'(k_{t+1}) - \delta$, rather than just $f'(k_{t+1})$.

The capital stock in period t + 1 equals the amount saved by young individuals in period t. Thus

(4) $K_{t+1} = S_t L_t$,

where S_t is the amount of saving done by a young person in period t. Note that $S_t = s(r_{t+1})A_t w_t$; the amount of saving done is equal to the fraction of income saved times the amount of income. Thus equation (4) can be rewritten as

(5) $K_{t+1} = L_t s(r_{t+1})A_t w_t$.

To get this into units of time t + 1 effective labor, divide both sides of equation (5) by $A_{t+1}L_{t+1}$:

(6) $$\frac{K_{t+1}}{A_{t+1}L_{t+1}} = \frac{A_t L_t}{A_{t+1}L_{t+1}}\left[s(r_{t+1})w_t\right].$$

Since $A_{t+1} = (1 + g)A_t$, we have $A_t/A_{t+1} = 1/(1 + g)$. Similarly, $L_t/L_{t+1} = 1/(1 + n)$. In addition, $K_{t+1}/A_{t+1}L_{t+1} = k_{t+1}$. Thus

(7) $$k_{t+1} = \frac{1}{(1+n)(1+g)}\left[s(r_{t+1})w_t\right].$$

Finally, substitute for $r_{t+1} = f'(k_{t+1}) - \delta$ and $w_t = f(k_t) - k_t f'(k_t)$:

(8) $$k_{t+1} = \frac{1}{(1+n)(1+g)}\left[s\left(f'(k_{t+1}) - \delta\right)\right]\left[f(k_t) - k_t f'(k_t)\right].$$

This should be compared with equation (2.58) in the text, the analogous expression with no depreciation, which is

$$k_{t+1} = \frac{1}{(1+n)(1+g)}\left[s\left(f'(k_{t+1})\right)\right]\left[f(k_t) - k_t f'(k_t)\right].$$

Thus adding depreciation does alter the relationship between k_{t+1} and k_t. Whether k_{t+1} will be higher or lower for a given k_t depends on the way in which saving varies with r_{t+1}.

(b) With logarithmic utility, the fraction of income saved does not depend upon the rate of return on saving and in fact

(9) $s(r_{t+1}) = 1/(2 + \rho)$.

In addition, with Cobb-Douglas production, $y_t = k_t^{\alpha}$, the real wage is $w_t = k_t^{\alpha} - k_t \alpha k_t^{\alpha-1} = (1 - \alpha)k_t^{\alpha}$. Thus equation (8) becomes

(10) $$k_{t+1} = \frac{1}{(1+n)(1+g)}\left[\frac{1}{2+\rho}(1-\alpha)k_t^{\alpha}\right].$$

We need to compare this with equation (2) in the solution to Problem 2.14, the analogous expression in the discrete-time Solow model, with the additional assumption of 100% depreciation (i.e. $\delta = 1$).

The saving rate in this economy is total saving divided by total output. Note that this is not the same as $s(r_{t+1})$, which is simply the fraction of their labor income that the young save. Denote the economy's total saving rate as $\hat{s}$. Then $\hat{s}$ will equal the saving of the young plus the dissaving of the old, all divided by total output and in addition, all variables are measured in units of effective labor.

The saving of the young is $\left[1/(2+\rho)\right](1-\alpha)k_t^{\alpha}$. Since there is 100% depreciation, the old do not get to dissave by the amount of the capital stock; there is no dissaving by the old. Thus

$$\hat{s} = \frac{\left[1/(2+\rho)\right](1-\alpha)k_t^{\alpha}}{k_t^{\alpha}} = \frac{1}{2+\rho}(1-\alpha).$$

Thus equation (10) can be rewritten as

(11) $k_{t+1} = \dfrac{1}{(1+n)(1+g)} \hat{s} k_t{}^{\alpha} = \left[\dfrac{\hat{s}}{(1+n)(1+g)}\right] f(k_t)$.

Note that this is exactly the same as the expression for k_{t+1} as a function of k_t in the discrete-time Solow model with $\delta = 1$. That is, it is equivalent to equation (2) in the solution to Problem 2.14 with δ set to one. Thus that version of the Solow model does have some microeconomic foundations, although the assumption of 100% depreciation is quite unrealistic.

Problem 2.16

(a) (i) The utility function is given by

(1) $\ln C_{1,t} + [1/(1+\rho)] \ln C_{2,t+1}$.

With the social security tax of T per person, the individual faces the following constraints (with g, the growth rate of technology, equal to 0, A is simply a constant throughout):

(2) $C_{1,t} + S_t = Aw_t - T$, and

(3) $C_{2,t+1} = (1 + r_{t+1})S_t + (1+n)T$,

where S_t represents the individual's saving in the first period. As far as the individual is concerned, the rate of return on social security is $(1 + n)$; in general this will not be equal to the return on private saving which is $(1 + r_{t+1})$. From equation (3), $(1 + r_{t+1})S_t = C_{2,t+1} - (1+n)T$. Solving for S_t yields

(4) $S_t = \dfrac{C_{2,t+1}}{1+r_{t+1}} - \dfrac{(1+n)}{(1+r_{t+1})}T$.

Now substitute equation (4) into equation (2):

$$C_{1,t} + \frac{C_{2,t+1}}{1+r_{t+1}} = Aw_t - T + \frac{(1+n)}{(1+r_{t+1})}T.$$

Rearranging, we get the intertemporal budget constraint:

(5) $C_{1,t} + \dfrac{C_{2,t+1}}{1+r_{t+1}} = Aw_t - \dfrac{(r_{t+1}-n)}{(1+r_{t+1})}T$.

We know that with logarithmic utility, the individual will consume fraction $(1 + \rho)/(2 + \rho)$ of her lifetime wealth in the first period. Thus

(6) $C_{1,t} = \left(\dfrac{1+\rho}{2+\rho}\right)\left[Aw_t - \left(\dfrac{r_{t+1}-n}{1+r_{t+1}}\right)T\right]$.

To solve for saving per person, substitute equation (6) into equation (2):

$$S_t = Aw_t - \left(\frac{1+\rho}{2+\rho}\right)\left[Aw_t - \left(\frac{r_{t+1}-n}{1+r_{t+1}}\right)T\right] - T \;\Rightarrow\; S_t = \left[1 - \left(\frac{1+\rho}{2+\rho}\right)\right]Aw_t - \left[1 - \left(\frac{1+\rho}{2+\rho}\right)\left(\frac{r_{t+1}-n}{1+r_{t+1}}\right)\right]T,$$

(7) $S_t = [1/(2+\rho)]Aw_t - \left[\dfrac{(2+\rho)(1+r_{t+1}) - (1+\rho)(r_{t+1}-n)}{(2+\rho)(1+r_{t+1})}\right]T$

Note that if $r_{t+1} = n$, saving is reduced one-for-one by the social security tax. If $r_{t+1} > n$, saving falls less than one-for-one. Finally, if $r_{t+1} < n$, saving falls more than one-for-one.

Denote $Z_t = [(2+\rho)(1+r_{t+1}) - (1+\rho)(r_{t+1}-n)]/(2+\rho)(1+r_{t+1})$ and thus equation (7) becomes

(8) $S_t = [1/(2+\rho)]Aw_t - Z_t T$.

It is still true that the capital stock in period t+1 will be equal to the total saving of the young in period t, hence

(9) $K_{t+1} = S_t L_t$.

Converting this into units of effective labor by dividing both sides of (9) by AL_{t+1} and using equation (8) yields

(10) $k_{t+1} = [1/(1+n)]\left[(1/(2+\rho))w_t - Z_t T/A\right]$.

With a Cobb-Douglas production function, the real wage is given by

(11) $w_t = (1-\alpha)k_t^{\alpha}$.

Substituting (11) into (10) gives the new relationship between capital in period t + 1 and capital in period t, all in units of effective labor:

(12) $k_{t+1} = [1/(1+n)]\left[(1/(2+\rho))(1-\alpha)k_t^{\alpha} - Z_t T/A\right]$.

(a) (ii) To see what effect the introduction of the social security system has on the balanced-growth-path value of k, we must determine the sign of Z_t. If it is positive, the introduction of the tax, T, shifts down the k_{t+1} curve and reduces the balanced-growth-path value of k. We have

$$Z_t = \frac{(2+\rho)(1+r_{t+1}) - (1+\rho)(r_{t+1}-n)}{(2+\rho)(1+r_{t+1})} = \frac{(1+1+\rho)(1+r_{t+1}) - (1+\rho)(r_{t+1}-n)}{(2+\rho)(1+r_{t+1})},$$

and simplifying further allows us to sign Z_t:

$$Z_t = \frac{(1+r_{t+1}) + (1+\rho)\left[(1+r_{t+1}) - (r_{t+1}-n)\right]}{(2+\rho)(1+r_{t+1})} = \frac{(1+r_{t+1}) + (1+\rho)(1+n)}{(2+\rho)(1+r_{t+1})} > 0.$$

Thus, the k_{t+1} curve shifts down, relative to the case without the social security, and k* is reduced.

(a) (iii) If the economy was initially dynamically efficient, a marginal increase in T would result in a gain to the old generation that would receive the extra benefits. However, it would reduce k* further below k_{GR} and thus leave future generations worse off, with lower consumption possibilities. If the economy was initially dynamically inefficient, so that $k^* > k_{GR}$, the old generation would again gain due to the extra benefits. In this case, the reduction in k* would actually allow for higher consumption for future generations and would be welfare-improving. The introduction of the tax in this case would reduce or possibly eliminate the dynamic inefficiency caused by the over-accumulation of capital.

(b) (i) Equation (3) becomes

(13) $C_{2,t+1} = (1+r_{t+1})S_t + (1+r_{t+1})T$.

As far as the individual is concerned, the rate of return on social security is the same as that on private saving. We can now derive the intertemporal budget constraint. From equation (16),

(14) $S_t = C_{2,t+1}/(1+r_{t+1}) - T$.

Substituting equation (14) into equation (2) yields

$$C_{1,t} + \frac{C_{2,t+1}}{1+r_{t+1}} = Aw_t - T + T,$$

or simply

(15) $C_{1,t} + \dfrac{C_{2,t+1}}{1+r_{t+1}} = Aw_t$.

This is just the usual intertemporal budget constraint in the Diamond model. Solving the individual's maximization problem yields the usual Euler equation:

$$C_{2,t+1} = [1/(1+\rho)]\,(1+r_{t+1})C_{1,t}$$

Substituting this into the budget constraint, equation (15), yields

(16) $C_{1,t} = [(1+\rho)/(2+\rho)]Aw_t$.

To get saving per person, substitute equation (16) into equation (2):

$S_t = Aw_t - [(1+\rho)/(2+\rho)]Aw_t - T$,

or simply

(17) $S_t = [1/(2+\rho)]Aw_t - T$.

The social security tax causes a one-for-one reduction in private saving.

The capital stock in period t +1 will be equal to the sum of total private saving of the young plus the total amount invested by the government. Hence

(18) $K_{t+1} = S_t L_t + TL_t$.

Dividing both sides of (18) by AL_{t+1} to convert this into units of effective labor, and using equation (17) yields

$$k_{t+1} = \left(\frac{1}{1+n}\right)\left[\left(\frac{1}{2+\rho}\right)w_t - \frac{T}{A}\right] + \left(\frac{1}{1+n}\right)\frac{T}{A},$$

which simplifies to

$k_{t+1} = [1/(1+n)][1/(2+\rho)]w_t$.

Using equation (11) to substitute for the wage yields

(19) $k_{t+1} = [1/(1+n)][1/(2+\rho)](1-\alpha)k_t^{\alpha}$.

Thus the fully-funded social security system has no effect on the relationship between the capital stock in successive periods.

(b) (ii) Since there is no effect on the relationship between k_{t+1} and k_t, the balanced-growth-path value of k is the same as it was before the introduction of the fully-funded social security system. (Note that we have been assuming that the amount of the tax is not greater than the amount of saving each individual would have done in the absence of the tax). The basic idea is that total investment and saving is still the same each period; the government is simply doing some of the saving for the young. Since social security pays the same rate of return as private saving, individuals are indifferent as to who does the saving. Thus individuals offset one-for-one any saving that the government does for them.

Problem 2.17

(a) In the decentralized equilibrium, there will be no intergenerational trade. Even if the young would like to trade goods this period for goods next period, the only people around to trade with are the old. Unfortunately, the old will be dead -- and thus in no position to complete the trade -- next period.

The individual's utility function is given by

(1) $\ln C_{1,t} + \ln C_{2,t+1}$.

The constraints are

(2) $C_{1,t} + F_t = A$, and (3) $C_{2,t+1} = xF_t$,

where F_t is the amount stored by the individual.

Substituting equation (3) into (2) yields the intertemporal budget constraint:

(4) $C_{1,t} + C_{2,t+1}/x = A$.

The individual's problem is to maximize lifetime utility, as given by equation (1), subject to the intertemporal budget constraint, as given by equation (4). Set up the Lagrangian:

$\mathcal{L} = \ln C_{1,t} + \ln C_{2,t+1} + \lambda[A - C_{1,t} - C_{2,t+1}/x]$.

The first-order conditions are given by

$\partial\mathcal{L}/\partial C_{1,t} = 1/C_{1,t} - \lambda = 0 \Rightarrow 1/C_{1,t} = \lambda$, and (5)

$\partial\mathcal{L}/\partial C_{2,t+1} = 1/C_{2,t+1} - \lambda/x = 0 \Rightarrow 1/C_{2,t+1} = \lambda/x$. (6)

Substitute equation (5) into equation (6) and rearrange to obtain

(7) $C_{2,t+1} = xC_{1,t}$.

Substitute equation (7) into the intertemporal budget constraint, equation (4), to obtain

$$C_{1,t} + xC_{1,t}/x = A,$$

or simply

(8) $C_{1,t} = A/2$.

To obtain an expression for second-period consumption, substitute equation (8) into equation (7):

(9) $C_{2,t+1} = xA/2$.

When young, each individual consumes half of her endowment and stores the other half; that is, $f_t = 1/2$. This allows her to consume xA/2 when old. Note that with log utility, the fraction of her endowment that the individual stores does not depend upon the return to storage.

(b) What is consumption per unit of effective labor at time t? First, calculate total consumption at time t:

$$C_t = C_{1,t}L_t + C_{2,t}L_{t-1},$$

where there are L_t young and L_{t-1} old individuals alive at time t. Each young person consumes the fraction of her endowment that she does not store, (1 - f)A, and each old person gets to consume the gross return on the fraction of her endowment that she stored, fxA. Thus

$$C_t = (1-f)AL_t + fxAL_{t-1}.$$

To convert this into units of time t effective labor, divide both sides by AL_t to get

$$C_t/AL_t = (1-f) + f\left[x/(1+n)\right].$$

Thus consumption per unit of time t effective labor is a weighted average of one and something less than one, since $x < (1 + n)$. It will therefore be maximized when the weight on one is one; that is, when $f = 0$. (We could also carry out this analysis on consumption per person alive at time t which would not change the result here).

The decentralized equilibrium, with f = 1/2, is not Pareto efficient. Since intergenerational trade is not possible, individuals are "forced" into storage because that is the only way they can save and consume in old age. They must do this even if the return on storage, x, is low. However, at any point in time, a social planner could take one unit from each young person and give (1 + n) units to each old person since there are fewer of them. With (1 + n) > x, this gives a better return than storage. Therefore, the social planner could raise welfare by taking the half of each generation's endowment that it was going to store and instead give it to the old. The planner could then do this each period. This allows individuals to consume A/2 units when young -- the same as in the decentralized equilibrium -- but now they get to consume (1 + n)A/2 units when old. This is greater than the xA/2 units of consumption when old that they would have had in the decentralized equilibrium with storage.

Problem 2.18

(a) The individual has a utility function given by

(1) $\ln C_{1,t} + \ln C_{2,t+1}$,

and constraints, expressed in units of money, given by

(2) $P_tC_{1,t} = P_tA - P_tF_t - M_t^d$, and

(3) $P_{t+1}C_{2,t+1} = P_{t+1}xF_t + M_t^d$,

where M_t^d is nominal money demand and F_t is the amount stored.

One way of thinking about the problem is the following. The individual has two decisions to make. The first is to decide how much of her endowment to consume and how much to "save". Then she must decide the way in which to save, through storage, by holding money or a combination of both. With log utility, we can separate the two decisions since the rate of return on "saving" will not affect the fraction of the first-period endowment that is saved. From the solution to Problem 2.17, we know she will consume half of the endowment in the first period, regardless of the rate of return on money or storage. Thus

(4) $C_{1,t} = A/2$.

What does she do with the other half? That will depend upon the gross rate of return on storage, x, relative to the gross rate of return on money, which is P_t/P_{t+1}. The gross rate of return on money is P_t/P_{t+1} since the individual can sell one unit of consumption in period t and get P_t units of money. In period t + 1, one unit of consumption costs P_{t+1} units of money and thus one unit of money will buy $1/P_{t+1}$ units of consumption. Thus the individual's P_t units of money will buy P_t/P_{t+1} units of consumption in period t+1.

CASE 1 : $x > P_t/P_{t+1}$

She will consume half of her endowment, store the rest and not hold any money since the rate of return on money is less than the rate of return on storage. Thus

$$C_{1,t} = A/2 \qquad F_t = A/2 \qquad M_t^d/P_t = 0 \qquad C_{2,t+1} = xA/2.$$

CASE 2 : $x < P_t/P_{t+1}$

Now storage is dominated by holding money. She will consume half of her endowment and then sell the rest for money:

$$C_{1,t} = A/2 \qquad F_t = 0 \qquad M_t^d/P_t = A/2 \qquad C_{2,t+1} = [P_t/P_{t+1}][A/2].$$

CASE 3 : $x = P_t/P_{t+1}$

Money and storage pay the same rate of return. She will consume half of her endowment and is then indifferent as to how much of the other half to store and how much of it to sell for money. Let $\alpha \in [0,1]$ be the fraction of saving that is in the form of money. Thus

$$C_{1,t} = A/2 \qquad F_t = (1-\alpha)A/2 \qquad M_t^d/P_t = \alpha A/2 \qquad C_{2,t+1} = xA/2 = [P_t/P_{t+1}][A/2].$$

(b) Equilibrium requires that aggregate real money demand equal aggregate real money supply. We can derive expressions for both real money demand and supply in period t:

aggregate real money demand = $L_t[A/2]$, and

aggregate real money supply = $[L_0/(1+n)]M/P_t = [L_t/(1+n)^{t+1}]M/P_t$.

The expression for aggregate real money supply uses the fact that in period 0, each old person, and there are $[L_0/(1+n)]$ of them, receives M units of money. The last step then uses the fact that since population grows at rate n, $L_t = (1+n)^t L_0$ and thus $L_0 = L_t/(1+n)^t$.

We can then use the equilibrium condition to solve for P_t:

$$L_t[A/2] = [L_t/(1+n)^{t+1}]M/P_t \quad \Rightarrow \quad P_t = 2M/[A(1+n)^{t+1}]. \quad (5)$$

We can similarly derive expressions for real money demand and supply in period t + 1:

aggregate real money demand = $L_{t+1}[A/2] = (1+n)L_t[A/2]$, and

aggregate real money supply = $[L_t/(1+n)^{t+1}]M/P_{t+1}$.

We can then use the equilibrium condition to solve for P_{t+1}:

$$(1+n)L_0[A/2] = [L_t/(1+n)^{t+1}]M/P_{t+1} \quad \Rightarrow \quad P_{t+1} = 2M/[A(1+n)^{t+2}]. \quad (6)$$

Dividing equation (6) by equation (5) yields

$$P_{t+1}/P_t = 1/(1+n) \quad \Rightarrow \quad P_{t+1} = P_t/(1+n).$$

This analysis holds for all time periods $t \geq 0$ and so $P_{t+1} = P_t/(1+n)$ is an equilibrium. This shows that if money is introduced into a dynamically inefficient economy, storage will not be used. The monetary equilibrium will thus result in attainment of the "golden-rule" level of storage. See the solution to part (b) of Problem 2.17 for an explanation of the reason that zero storage maximizes consumption per unit of effective labor.

(c) This is the situation where $P_t/P_{t+1} = x$; the return on money is equal to the return on storage. In this case, individuals are indifferent as to how much of their saving to store and how much to hold in the form of money. Let $\alpha_t \in [0,1]$ be the fraction of saving held in the form of money in period t. We can again derive expressions for aggregate real money demand and supply in period t:

aggregate real money demand = $L_t\alpha_t[A/2]$, and

aggregate real money supply = $[L_0/(1+n)]M/P_t = [L_t/(1+n)^{t+1}]M/P_t$.

We can then use the equilibrium condition to solve for P_t:

$$L_t\alpha_t[A/2] = [L_t/(1+n)^{t+1}]M/P_t \quad \Rightarrow \quad P_t = 2M/[\alpha_t A(1+n)^{t+1}]. \quad (7)$$

We can similarly derive expressions for real money demand and supply in period t + 1:

aggregate real money demand = $L_{t+1}\alpha_{t+1}[A/2] = (1+n)L_t\alpha_{t+1}[A/2]$, and

aggregate real money supply = $[L_t/(1+n)^{t+1}]M/P_{t+1}$.

We can then use the equilibrium condition to solve for P_{t+1}:

$$(1+n)L_0\alpha_{t+1}[A/2] = [L_t/(1+n)^{t+1}]M/P_{t+1} \quad \Rightarrow \quad P_{t+1} = 2M/[\alpha_{t+1}A(1+n)^{t+2}]. \quad (8)$$

Dividing equation (8) by (7) yields

$$P_{t+1}/P_t = [\alpha_t/\alpha_{t+1}][1/(1+n)].$$

For $P_{t+1}/P_t = 1/x$, we need

$$[\alpha_t/\alpha_{t+1}][1/(1+n)] = 1/x \quad \Rightarrow \quad [\alpha_{t+1}/\alpha_t] = [x/(1+n)] < 1.$$

Thus for all $t \geq 0$, $P_{t+1} = P_t/x$ will be an equilibrium for any path of α's that satisfies $\alpha_{t+1}/\alpha_t = x/(1+n)$.

(d) $P_t = \infty$ -- money is worthless -- is also an equilibrium. This occurs if the young generation at time 0 does not believe that money will be valued in the next period and thus that the generation one individuals will not accept money for goods. In that case, in period 0, the young simply consume half of their endowment and store the rest, and the old have some useless pieces of paper to go along with their endowment. This is an equilibrium with real money demand equal to zero and real money supply equal to zero as well. If no one believes the next generation will accept money for goods, this equilibrium continues for all future time periods.

This will be the only equilibrium if the economy ends at some date T. The young at date T will not want to sell any of their endowment. They will maximize the utility of their one-period life by consuming all of their endowment in period T. Thus, if the old at date T held any money, they would be stuck with it and it would be useless to them. Thus when they are young, in period T - 1, they will not sell any of their endowment for money, knowing that the money will be of no use to them when old. Thus, if the old at date T - 1 held any money, they would be stuck with it and it would be useless to them. Thus the old at

T - 1 will not want any money when they are young and so on. Working backward, no one would ever want to sell goods for money and money would not be valued.

Problem 2.19

(a) (i) The individual has a utility function given by

(1) $U = \ln C_{1,t} + \ln C_{2,t+1}$,

and a lifetime budget constraint given by

(2) $Q_t C_{1,t} + Q_{t+1} C_{2,t+1} = Q_t (A - S_t) + Q_{t+1} xS_t$.

From Problem 2.17, we know that with log utility, the individual wants to consume A/2 in the first period. The way in which the individual accomplishes this depends on the gross rate of return on storage, x, relative to the gross rate of return on trading.

The individual can sell one unit of the good in period t for Q_t. In period t + 1, it costs Q_{t+1} to obtain one unit of the good or equivalently, it costs one to obtain $1/Q_{t+1}$ units of the good. Thus for Q_t, it is possible to obtain Q_t/Q_{t+1} units of the good. Thus selling a unit of the good in period t allows the individual to buy Q_t/Q_{t+1} units of the good in period t + 1. Thus the gross rate of return on trading is Q_t/Q_{t+1}.

Now, $Q_{t+1} = Q_t/x$ for all t > 0 is equivalent to $x = Q_t/Q_{t+1}$ for all t > 0. In other words, the rate of return on storage is equal to the rate of return on trading and hence the individual is indifferent as to the amount to store and the amount to trade. Let $\alpha_t \in [0, 1]$ represent the fraction of "saving", A/2, that the individual sells in period t. That is, the individual sells α_t (A/2) in period t. This allows the individual to buy the amount $\alpha_t (Q_t/Q_{t+1})(A/2)$ when she is old in period t + 1. The individual stores a fraction $(1 - \alpha_t)$ of her "saving". Thus

(3) $S_t = (1 - \alpha_t)(A/2)$.

Consumption in period t + 1 will be equal to the amount the individual buys plus the amount she has through storage. Thus

(4) $C_{2,t+1} = \alpha_t (Q_t/Q_{t+1})(A/2) + (1 - \alpha_t)x(A/2)$.

Since we are considering a case in which $Q_t/Q_{t+1} = x$, equation (4) can be rewritten as

(5) $C_{2,t+1} = \alpha_t x(A/2) + (1 - \alpha_t)x(A/2) = x(A/2)$.

Consider some period t + 1 and let L represent the total number of individuals born each period, which is constant. Aggregate supply in period t + 1 is equal to the total number of young individuals, L, multiplied by the amount that each young individual wishes to sell, α_{t+1} (A/2). Thus

(6) Aggregate $\text{Supply}_{t+1} = L\alpha_{t+1} (A/2)$.

Aggregate demand in period t + 1 is equal to the total number of old individuals, L, multiplied by the amount each old individual wishes to buy, $(Q_t/Q_{t+1})\alpha_t (A/2)$. Thus

(7) Aggregate $\text{Demand}_{t+1} = L(Q_t/Q_{t+1})\alpha_t (A/2)$.

For the market to clear, aggregate supply must equal aggregate demand or

$L\alpha_{t+1} (A/2) = L(Q_t/Q_{t+1})\alpha_t (A/2)$,

or simply

(8) $\alpha_{t+1} = (Q_t/Q_{t+1})\alpha_t$.

Since the proposed price path has $Q_{t+1} = Q_t/x$, the equilibrium condition given by equation (8) can also be written as

(9) $\alpha_{t+1} = x\alpha_t$.

Now consider the situation in period 0. The old individuals simply consume their endowment. Thus we must have α_0 equal to zero in order for the market to clear in period 0. Thus equation (9) implies that we must have $\alpha_t = 0$ for all $t \geq 0$.

The resulting equilibrium is the same as that in part (a) of Problem 2.17. The individual consumes half of her endowment in the first period of life, stores the rest and consumes $xA/2$ in the second period of life. Note that with $x < 1 + n$ here (since $n = 0$ and $x < 1$), this equilibrium is dynamically inefficient. Thus eliminating incomplete markets by allowing individuals to trade before the start of time does not eliminate dynamic inefficiency.

(a) (ii) Suppose the auctioneer announces $Q_{t+1} < Q_t /x$ or equivalently $x < (Q_t /Q_{t+1})$ for some date t. This means that trading dominates storage for the young at date t. This means that the young at date t will want to sell all of their saving -- $\alpha_t = 1$ so that they want to sell A/2 -- and not store anything. Thus aggregate supply in period t is equal to L(A/2). For the old at date t, Q_{t+1} is irrelevant. They based their decision of how much to buy when old on Q_t / Q_{t-1} which was equal to x. Thus as described in part (a) (i), old individuals were not planning to buy anything. Thus aggregate demand in period t is zero. Thus aggregate demand will be less than aggregate supply and the market for the good will not clear. Thus the proposed price path cannot be an equilibrium.

Suppose instead that the auctioneer announces $Q_{t+1} > Q_t /x$ or equivalently $x > (Q_t /Q_{t+1})$ for some date t. This means that storage dominates trading for the young at date t. This means that the young at date t will want to store their entire endowment and will want to buy A/2. For the old at date t, Q_{t+1} is irrelevant. They based their decision of how much to trade when old on Q_t /Q_{t-1} which was equal to x. Thus each old individual was not planning to buy or sell anything. Thus aggregate demand exceeds aggregate supply and the market for the good will not clear. Thus the proposed price path cannot be an equilibrium.

(b) Consider the social planner's problem. The planner can divide the resources available for consumption between the young and the old in any matter. The planner can take, for example, one unit of each young person's endowment and transfer it to the old. Since there are the same number of old and young people in this model, this increases the consumption of each old person by one. With $x < 1$, this method of transferring from the young to the old provides a better return than storage. _If_ the economy did not end at some date T, the planner could prevent this change from making anyone worse off by requiring the next generation of young to make the same transfer in the following period. However, if the economy ends at some date T, the planner cannot do this. Taking anything from the young at date T would make them worse off since the planner cannot give them anything in return the next period; there is no next period. Thus the planner cannot make some generations better off without making another generation worse off. Thus the decentralized equilibrium is Pareto-efficient.

(c) It is infinite duration that is the source of the dynamic inefficiency. Allowing individuals to trade before the start of time requires a price path that results in an equilibrium which is equivalent to the situation where such a market does not exist. This equilibrium is not Pareto-efficient; a social planner could raise welfare by doing the procedure described in part (b). However, removing infinite duration also removes the social planner's ability to Pareto improve the decentralized equilibrium, as explained in part (b).

Problem 2.20

(a) The individual has utility function given by

$$(1)\quad \frac{C_{1,t}{}^{1-\theta}}{1-\theta}+\frac{C_{2,t+1}{}^{1-\theta}}{1-\theta} \qquad \theta<1.$$

The constraints expressed in units of money are

$$(2)\ P_tC_{1,t} = P_tA - M_t^d, \qquad \text{and} \qquad (3)\ P_{t+1}C_{2,t+1} = M_t^d.$$

Combining equations (2) and (3) yields the lifetime budget constraint:

(4) $P_t C_{1,t} + P_{t+1} C_{2,t+1} = P_t A$.

Note that $\theta < 1$ means that the elasticity of substitution, $1/\theta$, is greater than one. Thus when the rate of return on saving increases, the substitution effect dominates; the individual will consume less now and save more. This is essentially what we will be showing here. As the rate of return on holding money, which is P_t/P_{t+1}, rises, the individual wishes to hold more money.

The individual's problem is to maximize (1) subject to (4). Set up the Lagrangian:

$$\mathcal{L} = \frac{C_{1,t}{}^{1-\theta}}{1-\theta} + \frac{C_{2,t+1}{}^{1-\theta}}{1-\theta} + \lambda\left[P_t A - P_t C_{1,t} - P_{t+1} C_{2,t+1}\right]$$

The first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial C_{1,t}} = C_{1,t}{}^{-\theta} - \lambda P_t = 0 \quad \Rightarrow \quad \lambda = \frac{C_{1,t}{}^{-\theta}}{P_t}, \text{ and} \qquad (5)$$

$$\frac{\partial \mathcal{L}}{\partial C_{2,t+1}} = C_{2,t+1}{}^{-\theta} - \lambda P_{t+1} = 0 \quad \Rightarrow \quad C_{2,t+1}{}^{-\theta} = \lambda P_{t+1}. \qquad (6)$$

Substitute (5) into (6) to obtain

$$C_{2,t+1}{}^{-\theta} = C_{1,t}{}^{-\theta} P_{t+1}/P_t \quad \Rightarrow \quad \left(C_{2,t+1}/C_{1,t}\right)^{\theta} = P_t/P_{t+1} \quad \Rightarrow \quad C_{2,t+1}/C_{1,t} = \left(P_t/P_{t+1}\right)^{1/\theta},$$

or simply

(7) $C_{2,t+1} = \left(P_t/P_{t+1}\right)^{1/\theta} C_{1,t}$.

This is the Euler equation, which can now be substituted into the budget constraint (4):

$$P_t C_{1,t} + P_{t+1}\left(P_t/P_{t+1}\right)^{1/\theta} C_{1,t} = P_t A.$$

Dividing by P_t yields

$$C_{1,t} + \left(P_{t+1}/P_t\right)\left(P_t/P_{t+1}\right)^{1/\theta} C_{1,t} = A,$$

and simplifying yields

$$C_{1,t} + \left(P_t/P_{t+1}\right)^{(1-\theta)/\theta} C_{1,t} = A \quad \Rightarrow \quad C_{1,t}\left[1 + \left(P_t/P_{t+1}\right)^{(1-\theta)/\theta}\right] = A.$$

Thus consumption when young is given by

(8) $C_{1,t} = \dfrac{A}{1 + \left(P_t/P_{t+1}\right)^{(1-\theta)/\theta}}$.

To get the amount of her endowment that the individual sells for money (in real terms), we can use equation (2), expressed in real terms

(2') $M_t^d/P_t = A - C_{1,t}$.

Substitute equation (8) into (2') to obtain

$$\frac{M_t^d}{P_t} = A - \frac{A}{1 + \left(P_t/P_{t+1}\right)^{(1-\theta)/\theta}} \quad \Rightarrow \quad \frac{M_t^d}{P_t} = A\left[1 - \frac{1}{1 + \left(P_t/P_{t+1}\right)^{(1-\theta)/\theta}}\right].$$

Simplifying by getting a common denominator yields

(9) $\dfrac{M_t^d}{P_t} = A\dfrac{\left(P_t/P_{t+1}\right)^{(1-\theta)/\theta}}{1 + \left(P_t/P_{t+1}\right)^{(1-\theta)/\theta}}$.

Dividing the top and bottom of the right-hand side of equation (9) by $\left(P_t/P_{t+1}\right)^{(1-\theta)/\theta}$ yields

$$(10)\quad \frac{M_t^d}{P_t} = \frac{A}{\left(P_t/P_{t+1}\right)^{(\theta-1)/\theta}+1}.$$

Thus the fraction of her endowment that the individual sells for money is

$$(11)\quad h_t = \frac{1}{\left(P_t/P_{t+1}\right)^{(\theta-1)/\theta}+1}.$$

It is straightforward to show that the fraction of her endowment that the agent sells for money is an increasing function of the rate of return on holding money:

$$\frac{\partial h_t}{\partial\left(P_t/P_{t+1}\right)} = \frac{-\left[(\theta-1)/\theta\right]\left(P_t/P_{t+1}\right)^{\left[(\theta-1)/\theta\right]-1}}{\left[\left(P_t/P_{t+1}\right)^{(\theta-1)/\theta}+1\right]^2} > 0 \ \text{ for } \theta < 1.$$

We can also show that as the rate of return on money goes to zero, the amount of her endowment that the individual sells for money goes to zero. Use equation (9) to rewrite h_t as

$$h_t = \frac{\left(P_t/P_{t+1}\right)^{(1-\theta)/\theta}}{\left[1+\left(P_t/P_{t+1}\right)^{(1-\theta)/\theta}\right]},$$

and thus

$$\lim_{\left(P_t/P_{t+1}\right)\to 0} h_t = 0/(1+0) = 0.$$

(b) The constraints expressed in real terms are

(2') $C_{1,t} = A - M_t^d/P_t$, and (3') $C_{2,t+1} = M_t^d/P_{t+1}$.

Since there is no population growth, we can normalize the population to one without loss of generality. From (3'), a generation born at time t plans to buy M^d_t/P_{t+1} units of the good when it is old. Thus, the generation born at time 0 plans to buy M^d_0/P_1 units when it is old (in period 1). Use equation (10) to find M^d_0, substituting t = 0:

$$M_0^d = \frac{P_0 A}{\left(P_0/P_1\right)^{(\theta-1)/\theta}+1} \quad\Rightarrow\quad \frac{M_0^d}{P_1} = \frac{\left[P_0/P_1\right]A}{\left(P_0/P_1\right)^{(\theta-1)/\theta}+1}.\ (12)$$

From equation (2'), a generation born at time t plans to sell M^d_t/P_t units of the good for money. Thus the generation born at time 1 plans to sell M^d_1/P_1 units of the good. Substituting t = 1 into equation (10) gives

$$(13)\quad \frac{M_1^d}{P_1} = \frac{A}{\left(P_1/P_2\right)^{(\theta-1)/\theta}+1}.$$

In order for the amount of the consumption good that generation 0 wishes to buy with its money, given by equation (12), to be equal to the amount of the consumption good that generation 1 wishes to sell for money, given by equation (13), we need

$$\frac{\left[P_0/P_1\right]A}{\left(P_0/P_1\right)^{(\theta-1)/\theta}+1} = \frac{A}{\left(P_1/P_2\right)^{(\theta-1)/\theta}+1} \quad\Rightarrow\quad \frac{\left(P_0/P_1\right)^{(\theta-1)/\theta}+1}{\left(P_1/P_2\right)^{(\theta-1)/\theta}+1} = \frac{P_0}{P_1}.$$

Now with $P_0/P_1 < 1$, we need

$$\left(P_0/P_1\right)^{(\theta-1)/\theta}+1 < \left(P_1/P_2\right)^{(\theta-1)/\theta}+1 \quad\Rightarrow\quad \left(P_0/P_1\right)^{(\theta-1)/\theta} < \left(P_1/P_2\right)^{(\theta-1)/\theta},$$

and since $(\theta - 1)/\theta$ is negative, this implies

$$\frac{P_1}{P_2} < \frac{P_0}{P_1} < 1.$$

(c) Iterating this reasoning forward, the rate of return on money will have to be falling over time. That is,

$$1 > \frac{P_0}{P_1} > \frac{P_1}{P_2} > \frac{P_2}{P_3} > \ldots \qquad \text{and so } \frac{P_t}{P_{t+1}} \to 0.$$

As shown in part (a), this means that the fraction of the endowment that is sold for money will also go to zero. The economy approaches the situation where individuals consume their entire endowment in the first period. This is an equilibrium path in the sense that every time period, markets will clear. Each period, the real money demand by the young will be equal to real money supplied by the old and they will both go to zero as t gets large.

(d) If $P_0 / P_1 > 1$, we obtain the opposite result for the path of prices. That is, P_t / P_{t+1} will rise over time:

$$1 < \frac{P_0}{P_1} < \frac{P_1}{P_2} < \frac{P_2}{P_3} < \ldots \qquad \text{and so } \frac{P_t}{P_{t+1}} \to \infty.$$

From equation (11) we can see that this means that the fraction of the endowment sold for money will go to one. In other words, the economy approaches the situation where no one consumes anything in the first period and individuals sell their entire endowment for money. Thus, total real money demand will go to A, the endowment of the young (we have normalized the population to one). But with this path of prices, the price level goes to zero, which means that real money supplied by the old goes to infinity. Thus this cannot represent an equilibrium path for the economy because there will be a time period when real money supply will exceed real money demand and the market will not clear.

SOLUTIONS TO CHAPTER 3

Problem 3.1

The production functions for output and new knowledge are given by

(1) $Y(t) = A(t)(1 - a_L)L(t)$, and (2) $\dot{A}(t) = Ba_L{}^{\gamma}L(t)^{\gamma}A(t)^{\theta}$ $\theta < 1$.

(a) On a balanced growth path, $\dot{A}(t)/A(t) = g_A{}^* = \gamma n/(1 - \theta)$. (3)

Dividing both sides of equation (2) by A(t) yields

(4) $\dot{A}(t)/A(t) = Ba_L{}^{\gamma}L(t)^{\gamma}A(t)^{\theta-1}$.

Equating (3) and (4) yields

$$Ba_L{}^{\gamma}L(t)^{\gamma}A(t)^{\theta-1} = \gamma n/(1-\theta) \quad \Rightarrow \quad A(t)^{\theta-1} = \gamma n\big/(1-\theta)Ba_L{}^{\gamma}L(t)^{\gamma}.$$

Simplifying and solving for A(t) yields

(5) $A(t) = \left[(1-\theta)Ba_L{}^{\gamma}L(t)^{\gamma}\big/\gamma n\right]^{1/(1-\theta)}$.

(b) Substitute equation (5) into equation (1):

$$Y(t) = \left[(1-\theta)Ba_L{}^{\gamma}L(t)^{\gamma}\big/\gamma n\right]^{1/(1-\theta)}(1-a_L)L(t) = \left[(1-\theta)B/\gamma n\right]^{1/(1-\theta)}a_L{}^{\gamma/(1-\theta)}(1-a_L)L(t)^{[\gamma/(1-\theta)]+1}.$$

We can maximize the log of output with respect to a_L or maximize

$$\ln Y(t) = \left[1/(1-\theta)\right]\ln\left[(1-\theta)B/\gamma n\right] + \left[\gamma/(1-\theta)\right]\ln a_L + \ln(1-a_L) + \left[\left(\gamma/(1-\theta)\right)+1\right]\ln L(t).$$

The first-order condition is given by

$$\frac{\partial \ln Y(t)}{\partial a_L} = \frac{\gamma}{(1-\theta)}\frac{1}{a_L} - \frac{1}{1-a_L} = 0.$$

Some simple algebra yields an expression for $a_L{}^*$:

(6) $a_L{}^* = \dfrac{\gamma}{(1-\theta)+\gamma}$.

The higher is θ, the importance of knowledge in the production of new knowledge, and the higher is γ, the importance of labor in the production of new knowledge, the more of the labor force that should be employed in the knowledge sector.

Problem 3.2

Substituting the production function, $Y_i(t) = K_i(t)^{\theta}$, into the capital-accumulation equation, $\dot{K}_i(t) = s_iY_i(t)$, yields

(1) $\dot{K}_i(t) = s_iK_i(t)^{\theta}$, $\theta > 1$.

Dividing both sides of equation (1) by $K_i(t)$ gives an expression for the growth rate of the capital stock, $g_{K,i}$:

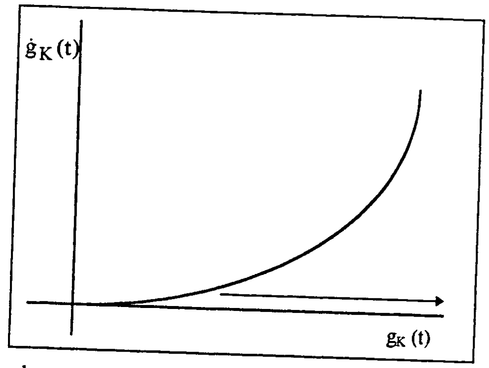
(2) $g_{K,i}(t) \equiv \dot{K}_i(t)/K_i(t) = s_iK_i(t)^{\theta-1}$.

Taking the time derivative of the log of equation (2) yields an expression for the growth rate of the growth rate of capital:

(3) $\dot{g}_{K,i}(t)/g_{K,i}(t) = (\theta-1)g_{K,i}(t)$,

and thus

(4) $\dot{g}_{K,i}(t) = (\theta-1)g_{K,i}(t)^2$.



Equation (4) is plotted at right. With $\theta > 1$, $g_{K,i}$ will be always increasing. The initial value of $g_{K,i}$ is determined by the initial capital stock and the saving rate; see equation (2). Since both economies have the same K(0) but one has a higher saving rate, then from equation (2), the economy with the higher s will have the higher initial $g_{K,i}(0)$. From equation (3), the growth rate of $g_{K,i}$ is increasing in $g_{K,i}$. Thus the growth rate of the capital stock in the high-saving economy will always exceed the growth rate of the capital stock in the low-saving economy. That is, we have $g_{K,1}(t) > g_{K,2}(t)$ for all $t \geq 0$. In fact, the gap between the two growth rates will be increasing over time.

More formally, using the production function, we can write the ratio of output in the high-saving country, country 1, to output in the low-saving country, country 2, as

(5) $Y_1(t)/Y_2(t) = [K_1(t)/K_2(t)]^{\theta}$.

Taking the time derivative of the log of equation (5) yields an expression for the growth rate of the ratio of output in the high-saving economy to output in the low-saving economy:

(6) $$\frac{\dot{[Y_1(t)/Y_2(t)]}}{[Y_1(t)/Y_2(t)]} = \theta\left[\frac{\dot{K}_1(t)}{K_1(t)} - \frac{\dot{K}_2(t)}{K_2(t)}\right] = \theta\left[g_{K,1}(t) - g_{K,2}(t)\right] > 0.$$

As explained above, $g_{K,1}(t)$ will exceed $g_{K,2}(t)$ for all $t \geq 0$. In fact, the gap between the two will be increasing over time. Thus the growth rate of the output ratio will be positive and increasing over time. That is, the ratio of output in the high-saving economy to output in the low-saving economy will be continually rising, and rising at an increasing rate.

Problem 3.3

The equations of the $\dot{g}_K = 0$ and $\dot{g}_A = 0$ lines are given by

(1) $\dot{g}_K = 0 \quad \Rightarrow \quad g_K = g_A + n$,

and

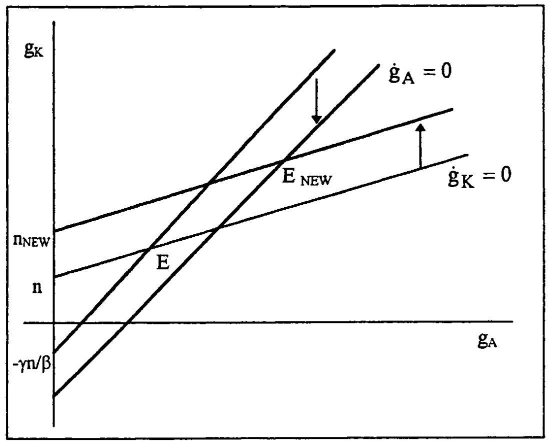
(2) $\dot{g}_A = 0 \quad \Rightarrow \quad g_K = \dfrac{(1-\theta)g_A - \gamma n}{\beta}$.

The expressions for the growth rates of capital and knowledge are

(3) $g_K(t) = c_K[A(t)L(t)/K(t)]^{1-\alpha} \qquad c_K \equiv s(1-a_K)^{\alpha}(1-a_L)^{1-\alpha}$

(4) $g_A(t) = c_A K(t)^{\beta} L(t)^{\gamma} A(t)^{\theta-1} \qquad c_A \equiv Ba_K{}^{\beta}a_L{}^{\gamma}$.

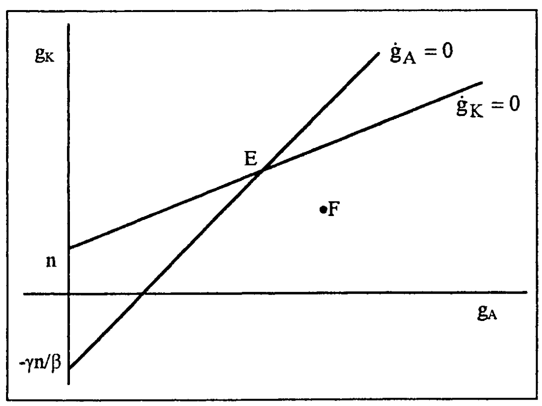
(a) From equation (1), for a given g_A, the value of g_K that satisfies $\dot{g}_K = 0$ is now higher as a result of the rise in population growth from n to n_{NEW}. Thus the $\dot{g}_K = 0$ locus shifts up. From equation (2), for a given g_A, the value of g_K that satisfies $\dot{g}_A = 0$ is now lower. Thus the $\dot{g}_A = 0$ locus shifts down.



Since n does not appear in equation (3), there is no jump in the value of g_K at the moment of the increase in population growth.

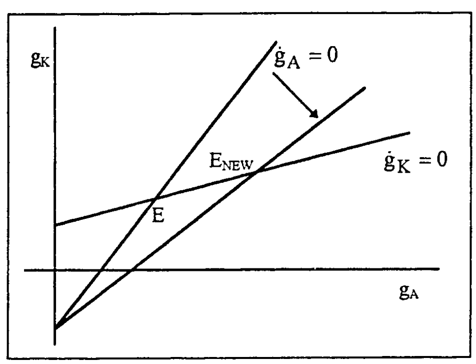
Similarly, since n does not appear in equation (4), there is no jump in the value of g_A at the moment of the rise in population growth.

(b) Note that a_K does not appear in equation (1), the $\dot{g}_K = 0$ line, or in equation (2), the $\dot{g}_A = 0$ line. Thus neither the $\dot{g}_K = 0$ nor the $\dot{g}_A = 0$ line shifts as a result of the increase in the fraction of the capital stock used in the knowledge sector from a_K to a_K^{NEW}.



From equation (3), the rise in a_K causes the growth rate of capital, g_K, to jump down. From equation (4), the growth rate of knowledge, g_A, jumps up at the instant of the rise in a_K. Thus the economy moves to a point such as F in the figure.

(c) Since θ does not appear in equation (1), there is no shift of the $\dot{g}_K = 0$ locus as a result of the rise in θ, the coefficient on knowledge in the knowledge production function. From equation (2), the $\dot{g}_A = 0$ locus has slope $(1 - \theta)/\beta$ and therefore becomes flatter after the rise in θ. See the figure.



Since θ does not appear in equation (3), the growth rate of capital, g_K, does not jump at the time of the rise in θ. θ does appear in equation (4) and thus we need to determine the effect that the rise in θ has on the growth rate of knowledge. It turns out that g_A

may jump up, jump down or stay the same at the instant of the change in θ. Taking the log of both sides of equation (4) gives us

$\ln g_A(t) = \ln c_A + \beta \ln K(t) + \gamma \ln L(t) + (\theta - 1)\ln A(t)$.

Taking the derivative of both sides of this expression with respect to θ yields

(5) $\partial \ln g_A(t)/\partial\theta = \ln A(t)$.

So if A(t) is less than one, so that $\ln A(t) < 0$, the growth rate of knowledge jumps down at the instant of the rise in θ. However, if A(t) is greater than one, so that $\ln A(t) > 0$, the growth rate of knowledge jumps up at the instant of the rise in θ. Finally, if A(t) is equal to one at the time of the change in θ, there is no initial jump in g_A. This means the dynamics of the adjustment to E_{NEW} may differ depending on the value of g_A at the time of the change in θ, but the end result is the same.

Problem 3.4

The equations of the $\dot{g}_K = 0$ and $\dot{g}_A = 0$ loci are

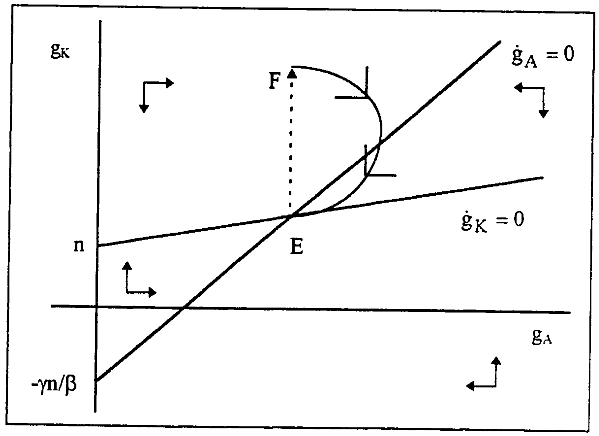
(1) $\dot{g}_K = 0 \;\Rightarrow\; g_K = g_A + n$, and (2) $\dot{g}_A = 0 \;\Rightarrow\; g_K = \dfrac{(1-\theta)g_A - \gamma n}{\beta}$.

The equations defining the growth rates of capital and knowledge at any point in time are

(3) $g_K(t) = c_K\left[A(t)L(t)/K(t)\right]^{1-\alpha}$ $\quad c_K \equiv s(1-a_K)^{\alpha}(1-a_L)^{1-\alpha}$

(4) $g_A(t) = c_A K(t)^{\beta} L(t)^{\gamma} A(t)^{\theta-1}$ $\quad c_A \equiv B a_K{}^{\beta} a_L{}^{\gamma}$.

(a) Since the saving rate, s, does not appear in equations (1) or (2), neither the $\dot{g}_K = 0$ nor the $\dot{g}_A = 0$ locus shifts when s increases. From equation (4), the growth rate of knowledge, g_A, does not change at the moment that s increases. However, from equation (3), a rise in s causes an upward jump in the growth rate of capital, g_K. In the figure, the economy jumps from its balanced growth path at E to a point such as F at the moment that s increases.



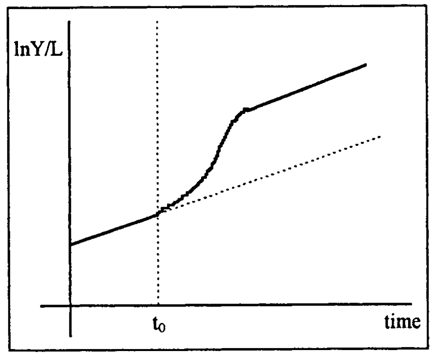
(b) At point F, the economy is above the $\dot{g}_A = 0$ locus and thus g_A is rising. Due to the increase in s, the growth rate of capital is higher than it would have been -- the amount of capital going into the production of knowledge is higher than it would have been -- and so the growth rate of knowledge begins to rise above what it would have been. Also at point F, the economy is above the $\dot{g}_K = 0$ locus and so g_K is falling. The economy drifts to the southeast and eventually crosses the $\dot{g}_A = 0$ locus at which point g_A begins to fall as well. Since there are decreasing returns to capital and knowledge in the production of new knowledge -- $\theta + \beta < 1$ -- the increase in s does not have a permanent effect on the growth rates of K and A. The economy eventually returns to point E.

The production function is given by

(5) $Y(t) = \left[(1-a_K)K(t)\right]^{\alpha}\left[A(t)(1-a_L)L(t)\right]^{1-\alpha}$.

Taking the time derivative of the log of equation (5) will yield the growth rate of total output:

$$(6)\quad \frac{\dot{Y}(t)}{Y(t)} = \alpha g_K(t) + (1-\alpha)\left[g_A(t) + n\right].$$



On the initial balanced growth path, from equation (1), $g_K{}^* = g_A{}^* + n$. From equation (6), this means that total output is also growing at rate $g_K{}^* = g_A{}^* + n$ on the initial balanced growth path. Thus output per person, Y(t)/L(t), is initially growing at rate $g_A{}^*$. During the transition period, both g_K and g_A are growing at a higher rate than on the balanced growth path and so output per worker must also be growing at a rate greater than its balanced-growth-path value of $g_A{}^*$. Whether the growth rate of output per worker is rising or falling will depend, among other things, on the value of α since there is a period of time when g_K is falling and g_A is rising. The figure shows the growth rate of output per worker initially rising and then falling, but the important point is that during the entire transition, the growth rate itself is higher than its balanced-growth-path value of $g_A{}^*$. In the end, once the economy returns to point E, output per worker is again growing at rate $g_A{}^*$, which has not changed.

(c) Note that the effects of an increase in s in this model are qualitatively similar to the effects in the Solow model. Since there are net decreasing returns to the produced factors of production here -- $\theta + \beta < 1$ -- the increase in s has only a level effect on output per worker. The path of output per worker lies above the path it would have taken but there is no permanent effect on the growth rate of output per worker, which on the balanced growth path is equal to the growth rate of knowledge. This is the same effect that a rise in s has in the Solow model in which there are diminishing returns to the produced factor, capital. Quantitatively, the effect is larger than in the Solow model (for a given set of parameters). This is due to the fact that, here, A rises above the path it would have taken whereas that is not true in the Solow model.

Problem 3.5

(a) From equations (3.14) and (3.16) in the text, the growth rates of capital and knowledge are given by

(1) $g_K(t) \equiv \dot{K}(t)/K(t) = c_K\left[A(t)L(t)/K(t)\right]^{1-\alpha}$, where $c_K \equiv s[1 - a_K]^{\alpha}[1 - a_L]^{1-\alpha}$, and

(2) $g_A(t) \equiv \dot{A}(t)/A(t) = c_A K(t)^{\beta} L(t)^{\gamma} A(t)^{\theta-1}$, where $c_A \equiv B a_K{}^{\beta} a_L{}^{\gamma}$.

With the assumptions of $\beta + \theta = 1$ and $n = 0$, these equations simplify to

(3) $g_K(t) = [c_K L^{1-\alpha}][A(t)/K(t)]^{1-\alpha}$, and (4) $g_A(t) = [c_A L^{\gamma}][K(t)/A(t)]^{\beta}$.

Thus given the parameters of the model and the population (which is constant), the ratio A/K determines both growth rates. The two growth rates, g_K and g_A, will be equal when

$$[c_K L^{1-\alpha}][A(t)/K(t)]^{1-\alpha} = [c_A L^{\gamma}][K(t)/A(t)]^{\beta} \quad \Rightarrow \quad [A(t)/K(t)]^{1-\alpha+\beta} = [c_A/c_K]L^{\gamma-(1-\alpha)}.$$

Thus the value of A/K that yields equal growth rates of capital and knowledge is given by

$$(5)\quad A(t)/K(t) = \left[(c_A/c_K)L^{\gamma-(1-\alpha)}\right]^{1/(1-\alpha+\beta)}.$$

(b) In order to find the growth rate of A and K when $g_K = g_A \equiv g^*$, substitute equation (5) into (3):

$$g^* = \left[c_K L^{1-\alpha}\right]\left[(c_A/c_K)L^{\gamma-(1-\alpha)}\right]^{(1-\alpha)/(1-\alpha+\beta)}.$$

Simplifying the exponents yields

$$g^* = \left[c_K^{(1-\alpha+\beta)-(1-\alpha)} c_A^{1-\alpha} L^{(1-\alpha)+\gamma(1-\alpha)-(1-\alpha)^2}\right]^{1/(1-\alpha+\beta)},$$

or simply

$$(6)\quad g^* = \left[c_K^{\beta} c_A^{1-\alpha} L^{(1-\alpha)(\gamma+\alpha)}\right]^{1/(1-\alpha+\beta)}.$$

(c) In order to see the way in which an increase in s affects the long-run growth rate of the economy, substitute the definitions of c_K and c_A into equation (6):

$$(7)\quad g^* = \left[s^{\beta}(1-a_K)^{\alpha\beta}(1-a_L)^{(1-\alpha)\beta} B^{1-\alpha} a_K^{\beta(1-\alpha)} a_L^{\gamma(1-\alpha)} L^{(1-\alpha)\cdot(\gamma+\alpha)}\right]^{1/(1-\alpha+\beta)}.$$

Taking the log of both sides of equation (7) gives us

$$(8)\quad \ln g^* = \left[1/(1-\alpha+\beta)\right]\{\beta \ln s + (1-\alpha)(\gamma+\alpha)\ln L + (1-\alpha)\ln B + \beta[\alpha\ln(1-a_K) + (1-\alpha)\ln a_K] + (1-\alpha)[\beta\ln(1-a_L) + \gamma\ln a_L]\}.$$

Using equation (8), the elasticity of the long-run growth rate of the economy with respect to the saving rate is

$$(9)\quad \partial \ln g^*/\partial \ln s = \beta/(1-\alpha+\beta) > 0.$$

Thus an increase in the saving rate increases the long-run growth rate of the economy This is essentially because it increases the resources devoted to physical capital accumulation and in this model, we have constant returns to the produced factors of production.

(d) We can maximize $\ln g^*$ with respect to a_K to determine the fraction of the capital stock that should be employed in the R&D sector in order to maximize the long-run growth rate of the economy. The first-order condition is

$$\frac{\partial \ln g^*}{\partial a_K} = \frac{\beta}{(1-\alpha+\beta)}\left[\frac{-\alpha}{(1-a_K)} + \frac{(1-\alpha)}{a_K}\right] = 0.$$

Solving for the optimal a_K^* yields

$$\alpha/(1-a_K) = (1-\alpha)/a_K \quad \Rightarrow \quad \alpha a_K = 1 - a_K + \alpha a_K - \alpha \quad \Rightarrow \quad 0 = 1 - a_K - \alpha,$$

and thus

$$(10)\quad a_K^* = (1-\alpha).$$

Thus the optimal fraction of the capital stock to employ in the R&D sector is equal to effective labor's share in the production of output. Note that β, capital's share in the production function for new knowledge, does not affect the optimal allocation of capital to the R&D sector. The reason for this is that an increase in β has two effects. It makes capital more important in the R&D sector, thereby tending to raise the a_K that maximizes g^*. A rise in β also makes the production of new capital more valuable, and new capital is produced when there is more output to be saved and invested. This tends to lower the a_K that maximizes g^* since it implies that more resources should be devoted to the production of output rather than knowledge. In the case we are considering, these two effects exactly cancel each other out.

Problem 3.6

(a) Substituting the assumption that $x(i) = K/A$ for $0 \le i \le A$ into the production function gives us

$$(1)\quad Y = \left[(1-a_L)L\right]^{1-\alpha} \int_{i=0}^{A} [K/A]^{\alpha}\, di.$$

Since $(K/A)^{\alpha}$ is independent of i, this leaves us with

$$(2)\quad Y = \left[(1-a_L)L\right]^{1-\alpha} [K/A]^{\alpha} \int_{i=0}^{A} di.$$

Since $\int_{i=0}^{A} di = [A - 0] = A$, we have

(3) $Y = [(1 - a_L)L]^{1-\alpha}K^{\alpha}A^{1-\alpha}$,

or, rearranging,

(4) $Y = [(1 - a_L)AL]^{1-\alpha}K^{\alpha}$.

(b) (i) For ease of notation, define $L_Y \equiv$ amount of labor employed by the firm. The firm's problem is to choose the quantities of labor and each capital good, x(i), in order to minimize cost, given by

(5) $wL_Y + \int_{i=0}^{A} x(i)p(i)di$,

subject to output being equal to one unit, or

(6) $L_Y{}^{1-\alpha} \int_{i=0}^{A} x(i)^{\alpha} di = 1$.

Thus the Lagrangian for the firm's cost-minimization problem is

(7) $\mathcal{L} = wL_Y + \int_{i=0}^{A} x(i)p(i)di + \lambda\left[1 - L_Y{}^{1-\alpha} \int_{i=0}^{A} x(i)^{\alpha} di\right]$.

(b) (ii) The first-order conditions are given by

(8) $\dfrac{\partial \mathcal{L}}{\partial L_Y} = w - \lambda(1-\alpha)L_Y{}^{-\alpha} \int_{i=0}^{A} x(i)^{\alpha} di = 0$,

and

(9) $\dfrac{\partial \mathcal{L}}{\partial x(i)} = p(i) - \lambda L_Y{}^{1-\alpha}\alpha x(i)^{\alpha-1} = 0$.

(b) (iii) Since there will be full employment, we can find the demand for capital good i, x(i), with L_Y and the p(i)'s taken as given. Dividing equation (8) by (9) gives us

(10) $\dfrac{w}{p(i)} = \dfrac{\lambda(1-\alpha)L_Y{}^{-\alpha} \int_{i=0}^{A} x(i)^{\alpha} di}{\lambda L_Y{}^{1-\alpha}\alpha x(i)^{\alpha-1}} = \dfrac{(1-\alpha)}{\alpha} \dfrac{L_Y{}^{-1}L_Y{}^{1-\alpha} \int_{i=0}^{A} x(i)^{\alpha} di}{L_Y{}^{1-\alpha}x(i)^{\alpha-1}}$.

Using the cost-minimization constraint, equation (6), this simplifies to

(11) $\dfrac{w}{p(i)} = \dfrac{(1-\alpha)}{\alpha} \dfrac{1}{L_Y{}^{2-\alpha}x(i)^{\alpha-1}} = \dfrac{(1-\alpha)}{\alpha} \dfrac{x(i)^{1-\alpha}}{L_Y{}^{2-\alpha}}$.

We can now solve for an expression for the demand for capital good x(i). Rearranging equation (11) yields

(12) $x(i)^{1-\alpha} = \dfrac{w}{p(i)} \dfrac{\alpha}{1-\alpha} L_Y{}^{2-\alpha}$.

Taking both sides of equation (12) to the exponent $1/(1 - \alpha)$ leaves us with

(13) $x(i) = \left[\dfrac{\alpha}{1-\alpha} \dfrac{w}{p(i)} L_Y{}^{2-\alpha}\right]^{\frac{1}{1-\alpha}}$.

Using the fact that labor in the goods-producing sector is paid its marginal product, or

$$w = \frac{\partial Y}{\partial L_Y} = (1-\alpha)L_Y^{-\alpha}\int_{i=0}^{A} x(i)^{\alpha}di,$$

we can rewrite equation (13) as

$$(14)\quad x(i) = \left[\frac{\alpha}{1-\alpha}\frac{(1-\alpha)L_Y^{-\alpha}\int_{i=0}^{A} x(i)^{\alpha}di}{p(i)}L_Y^{2-\alpha}\right]^{\frac{1}{1-\alpha}} = \left[\alpha\frac{L_Y^{1-\alpha}\int_{i=0}^{A} x(i)^{\alpha}di}{p(i)}L_Y^{1-\alpha}\right]^{\frac{1}{1-\alpha}}.$$

Substituting the cost-minimization constraint, equation (6), into (14) yields

$$(15)\quad x(i) = \left[\frac{\alpha L_Y^{1-\alpha}}{p(i)}\right]^{\frac{1}{1-\alpha}} = \left[\frac{\alpha}{p(i)}\right]^{\frac{1}{1-\alpha}}L_Y.$$

Note that we can write the elasticity of demand for capital good i as

$$(16)\quad \frac{\partial x(i)}{\partial p(i)}\frac{p(i)}{x(i)} = \frac{\partial \ln x(i)}{\partial \ln p(i)}.$$

Taking the natural log of both sides of equation (15) gives us

$$(17)\quad \ln x(i) = \frac{1}{1-\alpha}\left[\ln\alpha - \ln p(i)\right] + \ln L_Y.$$

And thus the elasticity of demand is given by

$$(18)\quad \frac{\partial x(i)}{\partial p(i)}\frac{p(i)}{x(i)} = \frac{\partial \ln x(i)}{\partial \ln p(i)} = -\frac{1}{1-\alpha} \equiv -\eta,$$

as required.

To see why this implies that the profit of a monopolistic supplier of capital good i, at the profit-maximizing price, is $(1 - \alpha)p(i)x(i)$, note that profit for a producer of capital good i is given by

$$\pi = [p(i) - c(i)]x(i),$$

where $c(i)$ is the unit cost of producing capital good i. The firm chooses quantity to maximize profit, so the first-order condition is

$$(19)\quad \frac{\partial \pi(i)}{\partial x(i)} = \frac{\partial p(i)}{\partial x(i)}x(i) + p(i) - c(i) = 0.$$

Dividing both sides of equation (19) by $p(i)$ and using equation (18) to substitute for the inverse of the elasticity of demand for capital good i gives us

$$(20)\quad \frac{-1}{\eta} + 1 - \frac{c(i)}{p(i)} = 0.$$

Solving equation (20) for $p(i)$ gives us

$$\frac{c(i)}{p(i)} = \frac{\eta - 1}{\eta},$$

or simply

$$(21)\quad p(i) = \frac{\eta}{\eta - 1}c(i).$$

This expression illustrates the fact that the price of the monopolist is $\eta/(\eta - 1)$, times cost. Substituting the definition of η, which is $\eta \equiv 1/(1 - \alpha)$, into equation (21) yields

$$w = \frac{\partial Y}{\partial L_Y} = (1-\alpha) L_Y^{-\alpha} \int_{i=0}^{A} x(i)^{\alpha} di,$$

we can rewrite equation (13) as

$$(14)\quad x(i) = \left[\frac{\alpha}{1-\alpha} \frac{(1-\alpha) L_Y^{-\alpha} \int_{i=0}^{A} x(i)^{\alpha} di}{p(i)} L_Y^{2-\alpha} \right]^{\frac{1}{1-\alpha}} = \left[\alpha \frac{L_Y^{1-\alpha} \int_{i=0}^{A} x(i)^{\alpha} di}{p(i)} L_Y^{1-\alpha} \right]^{\frac{1}{1-\alpha}}.$$

Substituting the cost-minimization constraint, equation (6), into (14) yields

$$(15)\quad x(i) = \left[\frac{\alpha L_Y^{1-\alpha}}{p(i)} \right]^{\frac{1}{1-\alpha}} = \left[\frac{\alpha}{p(i)} \right]^{\frac{1}{1-\alpha}} L_Y.$$

Note that we can write the elasticity of demand for capital good i as

$$(16)\quad \frac{\partial x(i)}{\partial p(i)} \frac{p(i)}{x(i)} = \frac{\partial \ln x(i)}{\partial \ln p(i)}.$$

Taking the natural log of both sides of equation (15) gives us

$$(17)\quad \ln x(i) = \frac{1}{1-\alpha} \left[\ln \alpha - \ln p(i) \right] + \ln L_Y.$$

And thus the elasticity of demand is given by

$$(18)\quad \frac{\partial x(i)}{\partial p(i)} \frac{p(i)}{x(i)} = \frac{\partial \ln x(i)}{\partial \ln p(i)} = -\frac{1}{1-\alpha} \equiv -\eta,$$

as required.

To see why this implies that the profit of a monopolistic supplier of capital good i, at the profit-maximizing price, is $(1 - \alpha)p(i)x(i)$, note that profit for a producer of capital good i is given by

$$\pi = [p(i) - c(i)]x(i),$$

where $c(i)$ is the unit cost of producing capital good i. The firm chooses quantity to maximize profit, so the first-order condition is

$$(19)\quad \frac{\partial \pi(i)}{\partial x(i)} = \frac{\partial p(i)}{\partial x(i)} x(i) + p(i) - c(i) = 0.$$

Dividing both sides of equation (19) by $p(i)$ and using equation (18) to substitute for the inverse of the elasticity of demand for capital good i gives us

$$(20)\quad \frac{-1}{\eta} + 1 - \frac{c(i)}{p(i)} = 0.$$

Solving equation (20) for $p(i)$ gives us

$$\frac{c(i)}{p(i)} = \frac{\eta - 1}{\eta},$$

or simply

$$(21)\quad p(i) = \frac{\eta}{\eta - 1} c(i).$$

This expression illustrates the fact that the price of the monopolist is $\eta/(\eta - 1)$, times cost. Substituting the definition of η, which is $\eta \equiv 1/(1 - \alpha)$, into equation (21) yields

(22) $p(i) = \left[\dfrac{1/(1-\alpha)}{[1/(1-\alpha)]-1}\right]c(i) = [1/\alpha]c(i)$,

and so $c(i) = \alpha p(i)$. Substituting for c(i) in the expression for profit gives us

$\pi = [p(i) - \alpha p(i)]x(i)$,

or simply

(23) $\pi = (1 - \alpha)p(i)x(i)$.

Problem 3.7

(a) The present discounted value of the profit from renting out a capital good at time t is

(1) $\pi^{PDV}(t) = \int\limits_{\tau=t}^{\infty} e^{-\int_{s=t}^{\tau} r(s)ds}\pi(\tau)d\tau$.

From equation (23) in the solution to Problem 3.6, profit at any point in time is $\pi = (1 - \alpha)p(i)x(i)$. We are examining a balanced growth path where x(i) and p(i) are independent of i and constant over time and where $\overline{p}$ and $\overline{x} = K/A$ are the balanced-growth-path price and quantity of each capital good. Thus equation (1) becomes

(2) $\pi^{PDV}(t) = \int\limits_{\tau=t}^{\infty} e^{-\int_{s=t}^{\tau} r(s)ds}(1-\alpha)\overline{p}\,\overline{x}d\tau$.

In addition, since the real interest rate is constant, $\exp\left(-\int\limits_{s=t}^{\tau} r(s)ds\right) = \exp\left(-r\int\limits_{s=t}^{\tau} ds\right) = \exp(-r(\tau - t))$, and so we have

(3) $\pi^{PDV}(t) = \int\limits_{\tau=t}^{\infty} e^{-r(\tau-t)}(1-\alpha)\overline{p}\overline{x}d\tau = (1-\alpha)\overline{p}\overline{x}\int\limits_{\tau=t}^{\infty} e^{-r(\tau-t)}d\tau$.

Solving the integral in equation (3) yields

$\pi^{PDV}(t) = (1-\alpha)\overline{p}\,\overline{x}\left[-\frac{1}{r}e^{-r(\tau-t)}\Big|_{\tau=t}^{\infty}\right] = (1-\alpha)\overline{p}\,\overline{x}\left[-\frac{1}{r}(0-1)\right]$,

and thus

(4) $\pi^{PDV}(t) = \dfrac{(1-\alpha)\overline{p}\,\overline{x}}{r}$.

(b) The wage of a worker in the knowledge-producing sector will equal the marginal product of labor in the knowledge sector multiplied by the price of the good produced by the knowledge sector or the price of knowledge. More concretely, the price of knowledge can be interpreted as the price of a design for a new capital good. This price will be bid up until it equals the present discounted value of the profit that a monopolistic supplier of the new capital good can extract. Using equation (4) and denoting P_A as the price of knowledge gives us

(5) $P_A = \dfrac{(1-\alpha)\overline{p}\,\overline{x}}{r}$.

From $\dot{A} = Ba_L LA$, the marginal product of labor in the knowledge-producing sector is

(6) $\dfrac{\partial \dot{A}}{\partial a_L L} = BA$.

Thus the wage of a worker in the knowledge-producing sector, denoted W_A, is

(7) $W_A = \dfrac{(1-\alpha)\bar{p}\bar{x}BA}{r}$.

From equation (15) in the solution to Problem 3.6, the demand for capital good i is

(8) $x(i) = \left[\dfrac{\alpha}{p(i)}\right]^{\frac{1}{1-\alpha}} L_Y$.

Taking both sides of equation (8) to the exponent (1 - α), and on a balanced growth path, we can write

(9) $\bar{x}^{1-\alpha} = \dfrac{\alpha}{\bar{p}} L_Y{}^{1-\alpha}$.

Solving for $\bar{p}$ gives us

(10) $\bar{p} = \alpha L_Y{}^{1-\alpha}\bar{x}^{-(1-\alpha)}$.

Substituting equation (10) into equation (7) yields

(11) $W_A = \dfrac{(1-\alpha)\alpha L_Y{}^{1-\alpha}\bar{x}^{-(1-\alpha)}\bar{x}BA}{r}$,

which simplifies to

(12) $W_A = \dfrac{\alpha(1-\alpha)L_Y{}^{1-\alpha}\bar{x}^{\alpha}\bar{x}BA}{r}$.

(c) As in the solution to Problem 3.6, we can define $L_Y \equiv (1 - a_L)L$ as the amount of labor employed in the goods-producing sector. From the production function, $Y = L_Y{}^{1-\alpha}\int_{i=0}^{A} x(i)^{\alpha}di$, on the balanced growth path we have

(13) $Y = L_Y{}^{1-\alpha}\int_{i=0}^{A} x(i)^{\alpha}di = L_Y{}^{(1-\alpha)}\bar{x}^{\alpha}A$.

Thus the marginal product of labor in the goods-producing sector is

(14) $\dfrac{\partial Y}{\partial L_Y} = (1-\alpha)L_Y{}^{-\alpha}\bar{x}^{\alpha}A$.

(d) Note that we want an expression for the marginal product of an extra unit of capital, evaluated on the balanced growth path. This is distinct from the concept of the marginal product of an increase in the balanced-growth-path value of each capital good. That is, we want to find $\partial Y/\partial K$ on the balanced growth path, not $\partial Y/\partial\bar{x} = \partial Y/\partial(K/A)$.

From equation (4) in the solution to Problem 3.6, output when x(i) = K/A is given by

(15) $Y = [(1 - a_L)AL]^{1-\alpha}K^{\alpha} = L_Y{}^{1-\alpha}A^{1-\alpha}K^{\alpha}$.

Thus the marginal product of capital is

(16) $\dfrac{\partial Y}{\partial K} = \alpha L_Y{}^{1-\alpha}A^{1-\alpha}K^{\alpha-1} = \alpha L_Y{}^{1-\alpha}\left(\dfrac{K}{A}\right)^{\alpha-1}$.

Since $\bar{x} = K/A$, we can write

(17) $\dfrac{\partial Y}{\partial K} = \alpha L_Y{}^{1-\alpha}\bar{x}^{\alpha-1}$.

(e) Since labor is mobile between the goods- and knowledge-producing sectors, the wage in both must be equal or $w = W_A$. Since labor will be paid its marginal product in the goods-producing sector (the price of the output good is normalized to one), using equations (12) and (14) we require

(18) $$\frac{\alpha(1-\alpha)L_Y{}^{1-\alpha}\bar{x}^{\alpha}BA}{r} = (1-\alpha)L_Y{}^{-\alpha}\bar{x}^{\alpha}A,$$

which simplifies to

(19) $\alpha L_Y B = r$.

Thus the amount of labor employed in the goods-producing sector is given by

(20) $L_Y \equiv (1 - a_L)L = r/\alpha B$.

(f) Since $\dot{A} = Ba_L LA$, the growth rate of knowledge is given by

(21) $$\frac{\dot{A}}{A} = Ba_L L.$$

On the balanced growth path, K, A, Y, and C all grow at the same rate, which we will denote g, and so

(22) $g = Ba_L L$.

(g) We know that on the balanced growth path consumption grows at rate g, thus

(23) $$\frac{\dot{C}}{C} = \frac{r-\rho}{\theta} = Ba_L L.$$

From equation (20), $a_L L = L - (r/\alpha B)$, and so (23) becomes

(24) $$\frac{r-\rho}{\theta} = BL - \frac{r}{\alpha},$$

or

(25) $\alpha r - \alpha\rho = \alpha\theta BL - \theta r$.

Collecting the terms in the interest rate, r, yields

(26) $r(\alpha + \theta) = \alpha(\rho + \theta BL)$,

and thus the interest rate on the balanced growth path is

(27) $$r = \frac{\alpha(\rho + \theta BL)}{\alpha + \theta}.$$

Note that r is a decreasing function of individuals' patience. The more patient are individuals -- the smaller is ρ, the rate at which the future is discounted -- the lower is the balanced-growth-path value of r.

Next we can solve for the balanced-growth-path value of a_L, the fraction of the labor force employed in the knowledge sector. From equation (20), $(1 - a_L)L = r/\alpha B$, we can write

(28) $$a_L = 1 - \frac{r}{\alpha BL}.$$

Substituting equation (27) into (28) yields

(29) $$a_L = 1 - \frac{\alpha(\rho + \theta BL)}{(\alpha+\theta)\alpha BL} = \frac{(\alpha+\theta)\alpha BL - \alpha\rho - \alpha\theta BL}{(\alpha+\theta)\alpha BL},$$

which simplifies to

(30) $$a_L = \frac{\alpha(\alpha BL - \rho)}{(\alpha+\theta)\alpha BL},$$

or simply

(31) $$a_L = \frac{\alpha BL - \rho}{(\alpha+\theta)BL}.$$

Finally, we can solve for the growth rate of the economy on the balanced growth path. Substituting equation (31) into $g = Ba_L L$ gives us

(32) $g = BL\left[\dfrac{\alpha BL - \rho}{(\alpha + \theta)BL}\right]$,

or simply

(33) $g = \dfrac{\alpha BL - \rho}{\alpha + \theta}$.

(h) We need to examine whether or not a_L can be greater than one given our assumptions about the parameters. Now $a_L > 1$, from equation (31), is equivalent to

$$\frac{\alpha BL - \rho}{(\alpha + \theta)BL} > 1 \quad \Leftrightarrow \quad \alpha BL - \rho > (\alpha + \theta)BL \quad \Leftrightarrow \quad -\rho > \theta BL.$$

Thus, as long as ρ, θ, B, and L are all positive, a_L cannot be greater than one.

However, from equation (31), a_L can be negative if $\rho > \alpha BL$. Intuitively, this would mean that individuals are so impatient -- ρ, the rate at which they discount the future, is so high -- that the future gains from extra knowledge are of no value relative to current consumption.

It is also necessary to examine whether a negative value of a_L would be consistent with the optimization problem for individuals. From equation (2.2) in the text, with the assumption of $n = 0$, we require

(34) $\rho - (1 - \theta)g > 0$

to ensure that lifetime utility does not diverge. Using equation (33) for g, this requires

(35) $(1 - \theta)\left[\dfrac{\alpha BL - \rho}{\alpha + \theta}\right] < \rho$.

This simplifies to

(36) $(1 - \theta)\alpha BL - \rho + \rho\theta < \alpha\rho + \rho\theta$,

or

(37) $(1 - \theta)\alpha BL < \rho(1 + \alpha)$,

which is equivalent to

(38) $\dfrac{\alpha BL}{\rho} < \dfrac{1 + \alpha}{1 - \theta}$.

Recall that a_L will be negative if $\alpha BL < \rho$ or $\alpha BL/\rho < 1$. Since for positive α and θ, $(1 + \alpha)/(1 - \theta) > 1$, it can be the case that

(39) $\dfrac{\alpha BL}{\rho} < 1 < \dfrac{1 + \alpha}{1 - \theta}$.

That is, under the restrictions on the parameters that ensures that lifetime utility is finite, it can still be the case that the balanced-growth-path value of a_L is negative.

Since the fraction of the labor force employed in the knowledge-producing sector cannot actually be negative, we will have a corner solution with $a_L = 0$ in this case. The growth rate of the economy on the balanced growth path will be zero since $g = Ba_L L$.

Problem 3.8

From equation (31) in the solution to Problem 3.7, the balanced-growth-path fraction of the labor force that is employed in the knowledge-producing sector is given by

(1) $a_L = \dfrac{\alpha BL - \rho}{(\alpha + \theta)BL}$.

(a) From equation (1),

(2) $\dfrac{\partial a_L}{\partial \rho} = \dfrac{-1}{(\alpha + \theta)BL} < 0$.

Thus a fall in ρ – a decrease in the rate at which individuals discount the future – raises the balanced-growth-path value of a_L. If individuals become more patient, the future gains from research will be valued more, relative to current consumption. Thus more resources will be devoted to the knowledge-producing sector and balanced-growth-path growth will be higher.

(b) From equation (1),

(3) $\dfrac{\partial a_L}{\partial B} = \dfrac{\alpha L[(\alpha + \theta)]BL - [\alpha BL - \rho](\alpha + \theta)L}{[(\alpha + \theta)BL]^2}$,

which simplifies to

(4) $\dfrac{\partial a_L}{\partial B} = \dfrac{\alpha BL^2(\alpha + \theta) - \alpha BL^2(\alpha + \theta) + \rho(\alpha + \theta)L}{[(\alpha + \theta)BL]^2}$.

Thus the sign of $\partial a_L/\partial B$ is determined by the sign of $\rho(\alpha + \theta)L$, which is positive under our assumptions about ρ, α, θ, and L. Intuitively, an increase in B represents an increase in the productivity of labor in the knowledge sector. Thus the wage in the knowledge sector initially rises. The knowledge sector attracts more workers until the wage there is once again equalized with the wage in the goods-producing sector.

(c) From equation (1),

(5) $\dfrac{\partial a_L}{\partial L} = \dfrac{\alpha B[(\alpha + \theta)]BL - [\alpha BL - \rho](\alpha + \theta)B}{[(\alpha + \theta)BL]^2}$,

which simplifies to

(6) $\dfrac{\partial a_L}{\partial L} = \dfrac{\alpha B^2L(\alpha + \theta) - \alpha B^2L(\alpha + \theta) + \rho(\alpha + \theta)B}{[(\alpha + \theta)BL]^2}$.

Thus the sign of $\partial a_L/\partial L$ is determined by the sign of $\rho(\alpha + \theta)B$, which is positive under our assumptions about ρ, α, θ, and B. An increase in the overall labor force will lead to a higher fraction of the labor force being employed in the knowledge-producing sector.

From equations (12) and (14) in the solution to Problem 3.7, we can see that initially, at the original a_L, a rise in L increases the wage in the knowledge sector and decreases it in the goods sector. This causes movement of labor from the goods sector to the knowledge sector. That is, a_L rises until the wage is once again equal in the two sectors.

Problem 3.9

The relevant equations are

(1) $Y(t) = K(t)^{\alpha}A(t)^{1-\alpha}$, (2) $\dot{K}(t) = sY(t)$, and (3) $\dot{A}(t) = BY(t)$.

(a) Substituting equation (1) into equation (2) yields $\dot{K}(t) = sK(t)^{\alpha}A(t)^{1-\alpha}$. Dividing both sides by K(t) allows us to obtain the following expression for the growth rate of capital, $g_K(t)$:

(4) $g_K(t) \equiv \dot{K}(t)/K(t) = sK(t)^{\alpha-1}A(t)^{1-\alpha}$.

Substituting equation (1) into (3) gives us $\dot{A}(t) = BK(t)^{\alpha} A(t)^{1-\alpha}$. Dividing both sides by A(t) allows us to obtain the following expression for the growth rate of knowledge, $g_A(t)$:

(5) $g_A(t) \equiv \dot{A}(t)/A(t) = BK(t)^{\alpha} A(t)^{-\alpha}$.

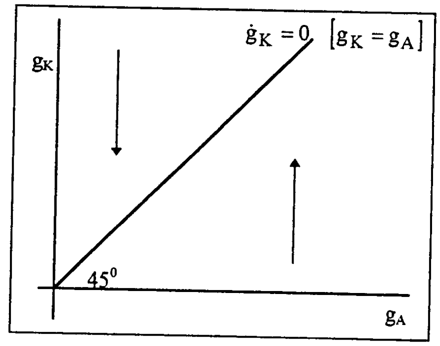
(b) Capital

Taking the time derivative of equation (4) yields the growth rate of the growth rate of capital:

$$\frac{\dot{g}_K(t)}{g_K(t)} = (\alpha - 1)\frac{\dot{K}(t)}{K(t)} + (1-\alpha)\frac{\dot{A}(t)}{A(t)},$$

or

(6) $\dot{g}_K(t)/g_K(t) = (1-\alpha)\left[g_A(t) - g_K(t)\right]$.



From equation (6), g_K will be constant when $g_A = g_K$. Thus the $\dot{g}_K = 0$ locus is a 45^0 line in (g_A, g_K) space. Also, g_K will be rising when $g_A > g_K$. Thus g_K is rising below the $\dot{g}_K = 0$ line. Lastly, g_K will fall when $g_A < g_K$. Thus g_K is falling above the $\dot{g}_K = 0$ line.

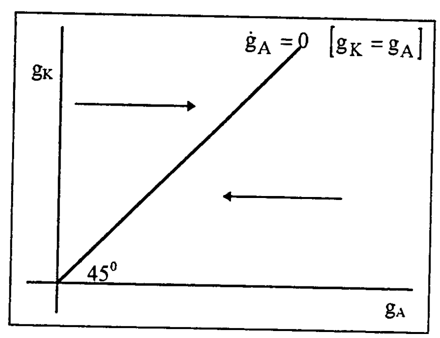
Knowledge

Taking the time derivative of the log of equation (5) yields the growth rate of the growth rate of knowledge:

$$\frac{\dot{g}_A(t)}{g_A(t)} = \alpha\frac{\dot{K}(t)}{K(t)} - \alpha\frac{\dot{A}(t)}{A(t)},$$

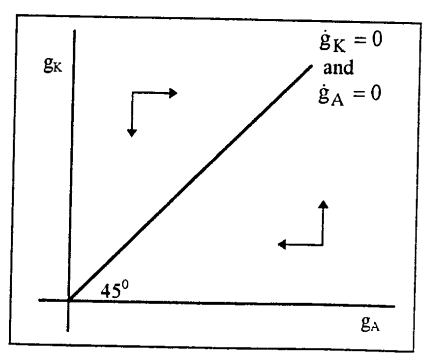
or

(7) $\dot{g}_A(t)/g_A(t) = \alpha\left[g_K(t) - g_A(t)\right]$.



From equation (7), g_A will be constant when $g_K = g_A$. Thus the $\dot{g}_A = 0$ locus is also a 45^0 line in (g_A, g_K) space. Also, g_A will be rising when $g_K > g_A$. Thus above the $\dot{g}_A = 0$ line, g_A will be rising. Finally, g_A will be falling when $g_K < g_A$. Thus below the $\dot{g}_A = 0$ line, g_A will be falling.

(c) We can put the $\dot{g}_K = 0$ and $\dot{g}_A = 0$ loci into one diagram.



Although we can see that the economy will eventually arrive at a situation where $g_K = g_A$ and they are constant, we still do not have enough information to determine the unique balanced growth path. Rewriting equations (4) and (5) gives us

(4) $g_K(t) = sK(t)^{\alpha-1} A(t)^{1-\alpha} = s\left[A(t)/K(t)\right]^{1-\alpha}$,

and

(5) $g_A(t) = BK(t)^{\alpha} A(t)^{-\alpha} = B\left[A(t)/K(t)\right]^{-\alpha}$.

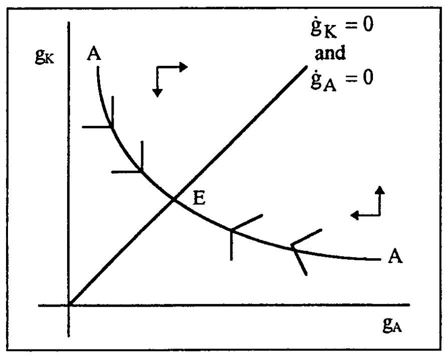
At any point in time, the growth rates of capital and knowledge are linked because they both depend on the ratio of knowledge to capital at that point in time. It is therefore possible to write one growth rate as a function of the other.

From equation (5), $[A(t)/K(t)]^{\alpha} = B/g_A(t)$ or simply

(8) $A(t)/K(t) = [B/g_A(t)]^{1/\alpha}$.

Substituting equation (8) into equation (4) gives us

(9) $g_K(t) = s[B/g_A(t)]^{(1-\alpha)/\alpha}$.



It must be the case that g_K and g_A lie on the locus satisfying equation (9), which is labeled AA in the figure. Regardless of the initial ratio of A/K the economy starts somewhere on this locus and then moves along it to point E. Thus the economy does converge to a unique balanced growth path at E.

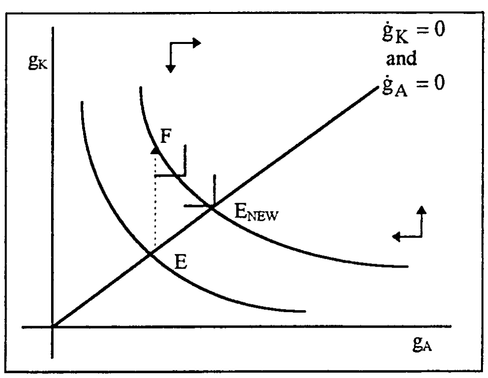
To calculate the growth rates of capital and knowledge on the balanced growth path, note that at point E we are on the $\dot{g}_K = 0$ and $\dot{g}_A = 0$ loci where $g_K = g_A$. Letting g^* denote this common growth rate, then from equation (9), $g^* = s[B/g^*]^{(1-\alpha)/\alpha}$.
Rearranging to solve for g^* yields

(10) $g^* = s^{\alpha}B^{1-\alpha}$.

Taking the time derivative of the log of the production function, equation (1), yields the growth rate of real output, $\dot{Y}(t)/Y(t) = \alpha g_K(t) + (1-\alpha)g_A(t)$. On the balanced growth path, $g_K = g_A \equiv g^*$, and thus

(11) $\dot{Y}(t)/Y(t) = \alpha g^* + (1-\alpha)g^* = g^* \equiv s^{\alpha}B^{1-\alpha}$.

On the balanced growth path, capital, knowledge and output all grow at rate g^*.



(d) Clearly, from equation (10), a rise in the saving rate, s, raises g^* and thus raises the long-run growth rates of capital, knowledge and output.

From equations (6) and (7), neither the $\dot{g}_K = 0$ nor the $\dot{g}_A = 0$ lines shift when s changes since s does not appear in either equation. From equation (4), a rise in s causes g_K to jump up. Also, the locus given by equation (9) shifts out. So at the moment that s rises, the economy moves from its balanced growth path at point E to a point such as F. It then moves down along the AA locus given by equation (9) until it reaches a new balanced growth path at point E_{NEW}.

Problem 3.10

(a) Taking the partial derivative of output at firm i -- $Y_i = K_i^{\alpha} L_i^{1-\alpha} [K^{\phi} L^{-\phi}]$ -- with respect to K_i, treating the aggregate capital stock as given yields

(1) $$r = \frac{\partial Y_i}{\partial K_i} = \alpha K_i^{\alpha-1} L_i^{1-\alpha}\left[K^{\phi} L^{-\phi}\right] = \alpha\left(\frac{K_i}{L_i}\right)^{\alpha-1}\left(\frac{K}{L}\right)^{\phi}.$$

In equilibrium, the capital-labor ratio is equated across firms. Thus K_i/L_i must equal the economy-wide capital-labor ratio, which is K/L. Substituting this fact into equation (1) gives us the private marginal product of capital,

(2) $$r = \alpha\left(\frac{K}{L}\right)^{-(1-\alpha-\phi)}.$$

(b) We can employ the technique used to solve for the balanced growth path in the Solow model. Since K_i/L_i is the same across firms and the production function has constant returns, the aggregate production function is given by $Y = K^{\alpha} L^{1-\alpha} [K^{\phi} L^{-\phi}]$ or simply

(3) $Y = K^{(\alpha+\phi)} L^{1-\alpha-\phi}$

Define $k \equiv K/L$ and $y \equiv Y/L$. Dividing both sides of equation (3) by L gives us

$$\frac{Y}{L} = \left(\frac{K}{L}\right)^{\alpha+\phi}\left(\frac{L}{L}\right)^{1-\alpha-\phi},$$

and thus output per worker is given by

(4) $y = k^{\alpha+\phi}$.

Taking the time derivative of both sides of the definition of $k \equiv K/L$ yields

(5) $$\dot{k} = \frac{\dot{K}L - K\dot{L}}{L^2} = \frac{\dot{K}}{L} - \left(\frac{K}{L}\right)\frac{\dot{L}}{L}.$$

Substituting the capital-accumulation equation, $\dot{K} = sY$, and the assumption that the labor force grows at rate n, $\dot{L}/L = n$, into equation (5) gives us

(6) $\dot{k} = sY/L - nk = sy - nk$.

Substituting equation (4) for output per worker into equation (6) gives us

(7) $\dot{k} = sk^{\alpha+\phi} - nk$.

Just as in the Solow model, the economy will converge to a situation in which actual investment per worker, $sk^{\alpha+\phi}$, is equal to break-even investment per worker, nk. Thus on a balanced growth path, capital per worker will be constant. Setting $\dot{k} = 0$ gives us

$$sk^{\alpha+\phi} = nk \quad \Rightarrow \quad k^{1-\alpha-\phi} = s/n,$$

or simply

(8) $k^* = [s/n]^{1/(1-\alpha-\phi)}$.

Substituting equation (8) into equation (2) yields

$$r = \alpha[s/n]^{-(1-\alpha-\phi)/(1-\alpha-\phi)} = \alpha[s/n]^{-1},$$

and thus the marginal product of capital on the balanced growth path is

(9) $r^* = \alpha n/s$.

(c) The analysis above does not support the claim. The value of ϕ does not affect the steady-state value of the private marginal product of capital, r^*. In addition, ϕ does not affect the way in which this value of r^* changes when the saving rate changes. From equation (9), $\partial r^*/\partial s = -(\alpha n)/s^2$, which does not depend on ϕ. That is, positive externalities from capital do not mitigate the decline in the marginal product of capital caused by a rise in the saving rate. Why? It is true, as the claim asserts, that a higher ϕ means that r responds less to changes in the capital-labor ratio, K/L; see equation (2). However, it is also true that a higher ϕ means that the capital-labor ratio itself responds more to the change in the saving rate; see equation (8). In this case, the effects cancel each other out.

Problem 3.11

The production functions, after the normalization of T = 1, are given by

(1) $C(t) = K_C(t)^{\alpha}$, and (2) $\dot{K}(t) = BK_K(t)$.

(a) The return to employing an additional unit of capital in the capital-producing sector is given by $\partial\dot{K}(t)/\partial K_K(t) = B$. This has value $P_K(t)B$ in units of consumption goods. The return from employing an additional unit of capital in the consumption-producing sector is $\partial C(t)/\partial[K_C(t)] = \alpha[K_C(t)]^{\alpha-1}$. Equating these returns gives us

(3) $P_K(t)B = \alpha[K_C(t)]^{\alpha-1}$.

Taking the time derivative of the log of equation (3) yields the growth rate of the price of capital goods relative to consumption goods,

$$\frac{\dot{P}_K(t)}{P_K(t)} + \frac{\dot{B}}{B} = \frac{\dot{\alpha}}{\alpha} + (\alpha - 1)\left[\frac{\dot{K}_C(t)}{K_C(t)}\right] \Rightarrow \frac{\dot{P}_K(t)}{P_K(t)} = (\alpha - 1)\frac{\dot{K}_C(t)}{K_C(t)}.$$

The last step uses the fact that B and α are constants. Now since $K_C(t)$ is growing at rate $g_K(t)$ and denoting the growth rate of $P_K(t)$ as $g_P(t)$, we have

(4) $g_P(t) = (\alpha - 1)g_K(t)$.

(b) (i) The growth rate of consumption is given by

(5) $g_C(t) \equiv \dot{C}(t)/C(t) = [r(t) - \rho]/\sigma = [B + g_P(t) - \rho]/\sigma = [B + (\alpha - 1)g_K(t) - \rho]/\sigma$,

where we have used equation (4) to substitute for $g_P(t)$.

(b) (ii) Taking the time derivative of the log of the consumption production function, equation(1), yields

(6) $g_C(t) \equiv \dot{C}(t)/C(t) = \alpha[\dot{K}_C(t)/K_C(t)] = \alpha g_K(t)$.

Equating the two expressions for the growth rate of consumption, equations (5) and (6), yields

$$\alpha g_K(t) = [B + (\alpha - 1)g_K(t) - \rho]/\sigma \Rightarrow \alpha\sigma g_K(t) + (1-\alpha)g_K(t) = B - \rho.$$

Thus in order for C to be growing at rate $g_C(t)$, $K_C(t)$ must be growing at the following rate:

(7) $g_K(t) = (B - \rho)/[\alpha\sigma + (1-\alpha)]$.

(b) (iii) We have already solved for $g_K(t)$ in terms of the underlying parameters. To solve for $g_C(t)$, substitute equation (7) into equation (6):

(8) $g_C(t) = \alpha(B - \rho)/[\alpha\sigma + (1-\alpha)]$.

(c) The real interest rate is now $(1 - \tau)(B + g_P)$. Thus equation (5) becomes

(9) $$g_C(t) = \frac{(1-\tau)[B + g_P(t)] - \rho}{\sigma} = \frac{(1-\tau)[B + (\alpha - 1)g_K(t)] - \rho}{\sigma},$$

where we have used equation (4) -- which is unaffected by the imposition of the tax -- to substitute for $g_P(t)$. Equating the two expressions for the growth rate of consumption, equations (9) and (6), yields

$$\alpha g_K(t) = \frac{(1-\tau)[B + (\alpha - 1)g_K(t)] - \rho}{\sigma} \Rightarrow \alpha\sigma g_K(t) + (1-\tau)(1-\alpha)g_K(t) = (1-\tau)B - \rho,$$

and thus

(10) $g_K(t) = \dfrac{(1-\tau)B - \rho}{[\alpha\sigma + (1-\tau)(1-\alpha)]}$.

Substituting equation (10) into equation (6) yields an expression for the growth rate of consumption as a function of the underlying parameters of the model:

(11) $g_C(t) = \alpha\left[\dfrac{(1-\tau)B - \rho}{\alpha\sigma + (1-\tau)(1-\alpha)}\right]$.

In order to see the effects of the tax, take the derivative of $g_C(t)$ with respect to τ:

$$\frac{\partial g_C(t)}{\partial \tau} = -\alpha\left\{\frac{B[\alpha\sigma + (1-\tau)(1-\alpha)] - [(1-\tau)B - \rho](1-\alpha)}{[\alpha\sigma + (1-\tau)(1-\alpha)]^2}\right\} = -\alpha\left\{\frac{B\alpha\sigma + \rho(1-\alpha)}{[\alpha\sigma + (1-\tau)(1-\alpha)]^2}\right\} < 0$$

Thus an increase in the tax rate τ causes the growth rate of consumption to fall.

Problem 3.12

(a) Note that the model of the northern economy is simply the Solow model with a constant growth rate of technology equal to $g = Ba_{LN}L_N$. From our analysis of the Solow model in Chapter 1, we know that the long-run growth rate of northern output per worker will be equal to that constant growth rate of technology.

(b) Taking the time derivative of both sides of the definition, $Z(t) \equiv A_S(t)/A_N(t)$, yields

(1) $\dot{Z}(t) = \dfrac{A_N(t)\dot{A}_S(t) - A_S(t)\dot{A}_N(t)}{A_N(t)^2}$.

Substituting the expressions for $\dot{A}_S(t)$ and $\dot{A}_N(t)$ into equation (1) gives us

$$\dot{Z}(t) = \frac{A_N(t)[\mu a_{LS}L_S(A_N(t) - A_S(t))] - A_S(t)[Ba_{LN}L_N A_N(t)]}{A_N(t)^2}.$$

Simplifying yields

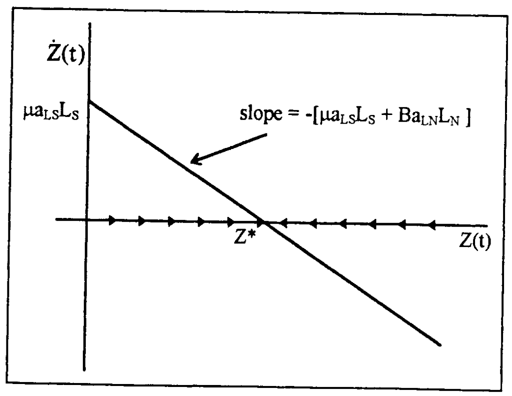
(2) $\dot{Z}(t) = [\mu a_{LS}L_S(1 - A_S(t)/A_N(t))] - [A_S(t)/A_N(t)][Ba_{LN}L_N]$.

Substituting the definition of $Z(t) \equiv A_S(t)/A_N(t)$ into equation (2) gives us

$\dot{Z}(t) = \mu a_{LS}L_S - \mu a_{LS}L_S Z(t) - Ba_{LN}L_N Z(t)$.

Collecting terms yields

(3) $\dot{Z}(t) = \mu a_{LS}L_S - [\mu a_{LS}L_S + Ba_{LN}L_N]Z(t)$.



The phase diagram implied by equation (3) is depicted at right. Note that equation (3) and the accompanying phase diagram do not apply for the case of $Z \geq 1$, since $\dot{A}_S(t) = 0$ for $A_S(t) \geq A_N(t)$.

The relationship between $\dot{Z}(t)$ and $Z(t)$ is linear with slope equal to $-[\mu a_{LS}L_S + Ba_{LN}L_N] < 0$. From the phase diagram, if $Z < Z^*$, then $\dot{Z}(t) > 0$. Thus if Z begins to the left of Z*, it rises toward Z* over time. Similarly if $Z > Z^*$, then $\dot{Z}(t) < 0$. Thus if

Z begins to the right of Z*, it falls toward Z* over time. Thus Z, the ratio of technology in the south to technology in the north, does converge to a stable value. To solve for Z*, set $\dot{Z}(t) = 0$:

$$0 = \mu a_{LS}L_S - [\mu a_{LS}L_S + Ba_{LN}L_N]Z^*.$$

Solving for Z* yields

$$(4)\quad Z^* = \frac{\mu a_{LS}L_S}{\mu a_{LS}L_S + Ba_{LN}L_N}.$$

The next step is to determine the long-run growth rate of southern output per worker. We have just shown that $Z(t) \equiv A_S(t)/A_N(t)$ converges to a constant. Thus in the long-run, $A_S(t)$ must be growing at the same rate as $A_N(t)$. In the long-run, then, the south is a Solow economy with a growth rate of technology equal to $Ba_{LN}L_N$. Thus the long-run growth rate of southern output per worker is equal to that growth rate.

Note that in the long-run, the growth rate of southern output per worker is the same as that in the north. This means that a_{LS}, the fraction of the south's labor force that is engaged in learning the technology of the north, does not affect the south's long-run growth rate. That growth rate is entirely determined by the number of people the north has working to produce new technology.

(c) Dividing the northern production function, $Y_N(t) = K_N(t)^\alpha[A_N(t)(1 - a_{LN})L_N]^{1-\alpha}$, by the quantity of effective labor, $A_N(t)L_N$, yields

$$(5)\quad \frac{Y_N(t)}{A_N(t)L_N} = \left[\frac{K_N(t)}{A_N(t)L_N}\right]^\alpha \left[\frac{A_N(t)(1-a_{LN})L_N}{A_N(t)L_N}\right]^{1-\alpha}.$$

Defining output and capital per unit of effective labor as $y_N(t) \equiv Y_N(t)/A_N(t)L_N$ and $k_N(t) \equiv K_N(t)/A_N(t)L_N$ respectively, we can rewrite equation (5) as

$$(6)\quad y_N(t) = k_N(t)^\alpha(1 - a_{LN})^{1-\alpha}.$$

Now we can use the technique employed to solve the Solow model to show that on the balanced growth path, $k_S^* = k_N^*$. Taking the time derivative of both sides of the definition of $k_N(t) \equiv K_N(t)/A_N(t)L_N$ yields

$$(7)\quad \dot{k}_N(t) = \frac{\dot{K}_N(t)}{A_N(t)L_N} - \frac{K_N(t)}{A_N(t)L_N}\frac{\dot{A}_N(t)}{A_N(t)}.$$

Substituting the capital-accumulation equation, $\dot{K}_N(t) = s_N Y_N(t)$, into equation (7) gives us

$$(8)\quad \dot{k}_N(t) = \frac{s_N Y_N(t)}{A_N(t)L_N} - \frac{\dot{A}_N(t)}{A_N(t)}\frac{K_N(t)}{A_N(t)L_N} = s_N y_N(t) - Ba_{LN}L_N k_N(t),$$

where we have used the definitions of $y_N(t)$ and $k_N(t)$ and have substituted for the growth rate of northern technology. Finally, using equation (8) to substitute for $y_N(t)$ yields

$$(9)\quad \dot{k}_N(t) = s_N k_N(t)^\alpha(1-a_{LN})^{1-\alpha} - Ba_{LN}L_N k_N(t).$$

An analogous derivation for the south would yield

$$(10)\quad \dot{k}_S(t) = s_S k_S(t)^\alpha(1-a_{LS})^{1-\alpha} - Ba_{LN}L_N k_S(t),$$

where we have used the fact that in the long-run, the growth rate of technology in the south is $Ba_{LN}L_N$.

Using the facts that $s_N = s_S$ and $a_{LN} = a_{LS}$, we can see that the equations for the dynamics of k are the same for the two economies. Thus we know that the balanced-growth-path values of k and y will be the same for the two economies. That is, we know that $k_S^* = k_N^*$ and $y_S^* = y_N^*$. This implies

$$(11)\quad y_S^*/y_N^* = 1.$$

Using the definitions of y_S and y_N, this implies that

$$\frac{Y_S/A_S L_S}{Y_N/A_N L_N} = 1 \quad \Rightarrow \quad \frac{Y_S/L_S}{Y_N/L_N} = \frac{A_S}{A_N}. \quad (12)$$

Equation (12) states that, on the balanced growth path, the ratio of output per worker in the south to output per worker in the north is equal to the ratio of technology in the south to technology in the north. From part (b), we know that A_S/A_N converges to Z^* in the long-run. Using equation (4) for Z^* to substitute into equation (12) leaves us with

$$(13)\quad \frac{Y_S/L_S}{Y_N/L_N} = \frac{\mu a_{LS} L_S}{\mu a_{LS} L_S + B a_{LN} L_N}.$$

Note that with $Ba_{LN} L_N > 0$, this ratio must be less than one; output per person in the south will be lower than output per person in the north. Also note that on the balanced growth path, the ratio of output per person in the south to that in the north does depend on a_{LS}, the fraction of southern workers engaged in learning the north's technology. In fact, the higher is a_{LS}, the closer will be the path of output per person in the south to that in the north.

Problem 3.13

(a) We need to find a value of τ such that $[Y_N(t)/L_N]/[Y_S(t)/L_S]$, the ratio of output per worker in the north to that in the south, is equal to 10. From the northern production function,

(1) $Y_N(t)/L_N = A_N(t)(1 - a_L)$.

Taking the time derivative of the natural log of equation (1) yields an expression for the growth rate of northern output per worker:

$$(2)\quad \frac{\dot{[Y_N(t)/L_N]}}{Y_N(t)/L_N} = \frac{\dot{A}_N(t)}{A_N(t)} = 0.03,$$

where we have used the information given in the problem that the growth rate of northern output per worker, and thus of northern knowledge, is 3% per year. Since $\dot{A}_N(t)/A_N(t) = 0.03$ then

(3) $A_N(t) = e^{0.03\tau} A_N(t - \tau)$.

From the southern production function,

(4) $Y_S(t)/L_S = A_S(t)$.

Dividing equation (3) by equation (4) yields an expression for the ratio of output per worker in the north to that in the south:

$$(5)\quad \frac{Y_N(t)/L_N}{Y_S(t)/L_S} = \frac{A_N(t)(1-a_L)}{A_S(t)} \approx \frac{A_N(t)}{A_N(t-\tau)} = e^{0.03\tau},$$

where we have used the fact that $a_L \approx 0$, that $A_S(t) = A_N(t - \tau)$, and equation (3).

For output per person in the north to exceed that in the south by a factor of 10, we need a τ such that

$e^{0.03\tau} = 10$, or

$0.03\tau = \ln(10)$,

which implies that τ must be approximately 76.8 years. Thus, attributing realistic cross-country differences in income per person to slow transmission of knowledge to poor countries requires the transmission to be very slow. Poor countries would need to be using technology that the rich countries developed in the 1920s in order to explain a 10-fold difference in income per person.

(b) (i) Recall that in the Solow model, the balanced-growth-path value of $k \equiv K/AL$ is defined implicitly by the condition that actual investment, $sf(k^*)$, equal break-even investment, $(n + g + \delta)k^*$. Thus for the north, k_N^* is implicitly defined by

(6) $sf(k_N^*) = (n + g + \delta)k_N^*$,

where $g = \dot{A}_N(t)/A_N(t)$.

We are told that s, n, δ and the function f(•) are the same for the north and the south. The only possible source of difference is the growth rate of southern knowledge. However, it is straightforward to show that $\dot{A}_S(t)/A_S(t) = g$ also.

We are told that the knowledge used in the south at time t is the knowledge that was used in the north at time t - τ. That is,

(7) $A_S(t) = A_N(t - \tau)$.

Taking the time derivative of equation (7) yields

(8) $\dot{A}_S(t) = \dot{A}_N(t - \tau)$.

Dividing equation (8) by equation (7) yields

(9) $\dfrac{\dot{A}_S(t)}{A_S(t)} = \dfrac{\dot{A}_N(t-\tau)}{A_N(t-\tau)}$.

The growth rate of northern knowledge is constant and equal to g at all points in time and thus

(10) $\dot{A}_S(t)/A_S(t) = g$.

Therefore, for the south, k_S^* is implicitly defined by

(11) $sf(k_S^*) = (n + g + \delta)k_S^*$.

Since k_N^* and k_S^* are implicitly defined by the same equation, they must be equal.

(b) (ii) Introducing capital will not change the answer to part (a). Since $k_N^* = k_S^*$, output per unit of effective labor on the balanced growth path will also be equal in the north and the south. That is, $y_N^* = y_S^*$ where $y_i^* = [Y_i/A_iL_i]^*$. We can write the balanced-growth-path value of output per worker in the north as

(12) $Y_N(t)/L_N(t) \equiv A_N(t)y_N^*$.

Similarly, the balanced-growth-path value of output per worker in the south is

(13) $Y_S(t)/L_S(t) \equiv A_S(t)y_S^*$.

Dividing equation (12) by equation (13) yields

(14) $\dfrac{Y_N(t)/L_N(t)}{Y_S(t)/L_S(t)} = \dfrac{A_N(t)y_N^*}{A_S(t)y_S^*} = \dfrac{A_N(t)}{A_S(t)} = \dfrac{A_N(t)}{A_N(t-\tau)}$.

The second-to-last step uses the fact that $y_N^* = y_S^*$. The last step uses $A_S(t) = A_N(t - \tau)$. Using equation (3), we again have

$$\frac{Y_N(t)/L_N(t)}{Y_S(t)/L_S(t)} = \frac{A_N(t)}{A_N(t-\tau)} = e^{0.03\tau}.$$

The same calculation as in part (a) would yield a value of τ = 76.8 years in order for $[Y_N(t)/L_N]/[Y_S(t)/L_S] = 10$.

Problem 3.14

(a) Differentiating both sides of the definition of $k(t) \equiv K(t)/A(t)L(t)$ with respect to time yields

(1) $\dot{k}(t) = \dfrac{\dot{K}(t)A(t)L(t) - K(t)[\dot{A}(t)L(t) + A(t)\dot{L}(t)]}{[A(t)L(t)]^2}$.

Using the definition of $k(t) \equiv K(t)/A(t)L(t)$, equation (1) can be rewritten as

(2) $\dot{k}(t) = \dfrac{\dot{K}(t)}{A(t)L(t)} - \left[\dfrac{\dot{A}(t)}{A(t)} + \dfrac{\dot{L}(t)}{L(t)}\right]k(t)$.

Substituting the capital-accumulation equation, $\dot{K}(t) = sY(t) - \delta_K K(t)$, as well as the constant growth rates of knowledge and labor into equation (2) gives us

(3) $\dot{k}(t) = \dfrac{sY(t) - \delta_K K(t)}{A(t)L(t)} - (n + g)k(t)$.

Substituting the production function, $Y(t) = [(1 - a_K)K(t)]^{\alpha}[(1 - a_H)H(t)]^{1-\alpha}$, into equation (3) yields

(4) $\dot{k}(t) = s\left[\dfrac{(1 - a_K)K(t)}{A(t)L(t)}\right]^{\alpha}\left[\dfrac{(1 - a_H)H(t)}{A(t)L(t)}\right]^{1-\alpha} - (n + g + \delta_K)k(t)$.

Finally, defining $c_K \equiv s(1 - a_K)^{\alpha}(1 - a_H)^{1-\alpha}$ and using $k(t) \equiv K(t)/A(t)L(t)$ as well as $h(t) \equiv H(t)/A(t)L(t)$, equation (4) can be rewritten as

(5) $\dot{k}(t) = c_K k(t)^{\alpha} h(t)^{1-\alpha} - (n + g + \delta_K)k(t)$.

Differentiating both sides of the definition of $h(t) \equiv H(t)/A(t)L(t)$ with respect to time yields

(6) $\dot{h}(t) = \dfrac{\dot{H}(t)A(t)L(t) - H(t)\left[\dot{A}(t)L(t) + A(t)\dot{L}(t)\right]}{[A(t)L(t)]^2}$.

Equation (6) can be simplified to

(7) $\dot{h}(t) = \dfrac{\dot{H}(t)}{A(t)L(t)} - \left[\dfrac{\dot{A}(t)}{A(t)} + \dfrac{\dot{L}(t)}{L(t)}\right]h(t)$.

Substituting $\dot{H}(t) = B[a_K K(t)]^{\gamma}[a_H H(t)]^{\phi}[A(t)L(t)]^{1-\gamma-\phi} - \delta_H H(t)$, the human-capital-accumulation equation, as well as the constant growth rates of knowledge and labor into equation (7) gives us

(8) $\dot{h}(t) = B\left[\dfrac{a_K K(t)}{A(t)L(t)}\right]^{\gamma}\left[\dfrac{a_H H(t)}{A(t)L(t)}\right]^{\phi}\left[\dfrac{A(t)L(t)}{A(t)L(t)}\right]^{1-\gamma-\phi} - (n + g + \delta_H)k(t)$.

Finally, defining $c_H \equiv Ba_K^{\gamma}a_H^{\phi}$ allows us to rewrite equation (8) as

(9) $\dot{h}(t) = c_H k(t)^{\gamma} h(t)^{\phi} - (n + g + \delta_H)h(t)$.

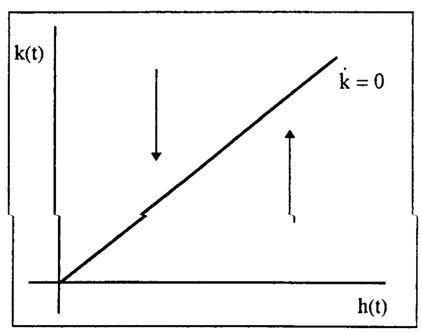
(b) To find the combinations of h and k such that $\dot{k} = 0$, set the right-hand side of equation (5) equal to zero and solve for k as a function of h:

$c_K k(t)^{\alpha}h(t)^{1-\alpha} = (n + g + \delta_K)k(t) \quad \Rightarrow \quad k(t)^{1-\alpha} = c_K h(t)^{1-\alpha}/(n + g + \delta_K)$,

and thus finally

(10) $k(t) = [c_K/(n + g + \delta_K)]^{1/(1-\alpha)} h(t)$.

The $\dot{k} = 0$ locus, as defined by equation (10), is a straight line with slope $[c_K/(n + g + \delta_K)]^{1/(1-\alpha)} > 0$ that passes through the origin. See the figure at right. From equation (5), we can see that $\dot{k}(t)$ is increasing in h(t). Thus to the right of the $\dot{k} = 0$ locus, $\dot{k} > 0$ and so k(t) is rising. To the left of the $\dot{k} = 0$ locus, $\dot{k} < 0$ and so k(t) is falling.



To find the combinations of h and k such that $\dot{h} = 0$, set the right-hand side of equation (9) equal to zero and solve for k as a function of h:

$c_H k(t)^{\gamma} h(t)^{\phi} = (n + g + \delta_H)h(t) \quad \Rightarrow \quad k(t)^{\gamma} = [(n + g + \delta_H)/c_H]h(t)^{1-\phi}$,

and thus finally

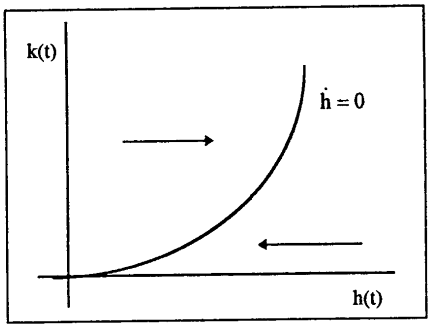
(11) $k(t) = [c_H/(n + g + \delta_H)]^{\gamma} h(t)^{(1-\phi)/\gamma}$.

The following derivatives will be useful:

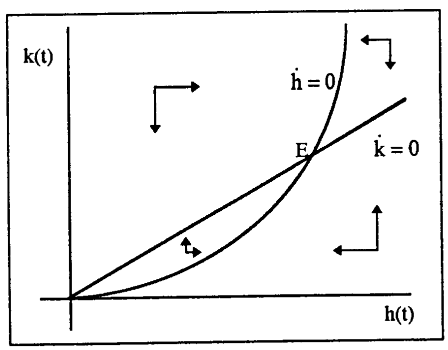
$$dk(t)/dh(t)\big|_{\dot{h}=0} = [(1-\phi)/\gamma][c_H/(n+g+\delta_H)]^{\gamma} h(t)^{(1-\phi-\gamma)/\gamma} > 0, \text{ and}$$

$$d^2k(t)/dh(t)^2\big|_{\dot{h}=0} = [(1-\phi-\gamma)/\gamma][(1-\phi)/\gamma][c_H/(n+g+\delta_H)]^{\gamma} h(t)^{(1-\phi-2\gamma)/\gamma} > 0.$$

The $\dot{h} = 0$ locus, as defined by equation (11), is upward-sloping with a positive second derivative. See the figure at right. From equation (9), we can see that $\dot{h}(t)$ is increasing in k(t). Therefore, above the $\dot{h} = 0$ locus, $\dot{h} > 0$ and so h(t) is increasing. Below the $\dot{h} = 0$ locus, $\dot{h} < 0$ and so h(t) is falling.



(c) Putting the $\dot{k} = 0$ and $\dot{h} = 0$ loci together, we can see that the economy will converge to a stable balanced growth path at point E. This stable balanced growth path is unique (as long as we ignore the origin with $k = h = 0$).



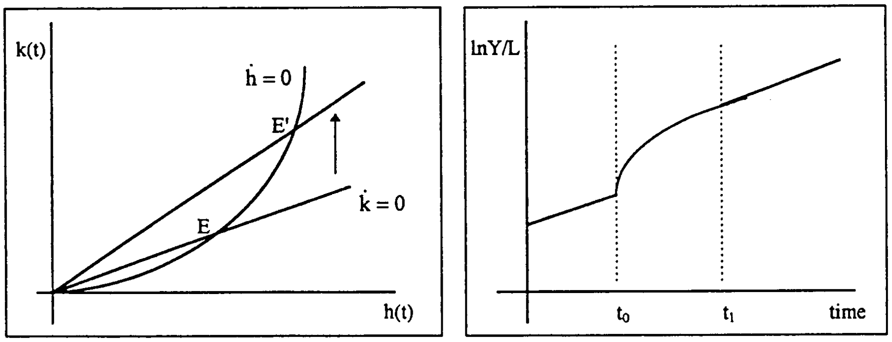
From the figure, physical capital per unit of effective labor, $k(t) \equiv K(t)/A(t)L(t)$, is constant on a balanced growth path. Thus physical capital per person, $K(t)/L(t) \equiv k(t)A(t)$, must grow at the same rate as knowledge, which is g. Similarly, human capital per unit of effective labor, $h(t) \equiv H(t)/A(t)L(t)$, is constant on the balanced growth path. Thus human capital per person, $H(t)/L(t) \equiv H(t)A(t)$, must also grow at the same rate as knowledge, which is g.

Dividing the production function by L(t) gives us an expression for output per person:

(12) $Y(t)/L(t) = [(1 - a_K)K(t)/L(t)]^{\alpha} [(1 - a_H)H(t)/L(t)]^{1-\alpha}$.

Since K(t)/L(t) and H(t)/L(t) both grow at rate g on the balanced growth path and since the production function is constant returns to scale, output per person also grows at rate g on the balanced growth path.

(d) From equation (10), the slope of the $\dot{k} = 0$ locus is $[c_K/(n + g + \delta_K)]^{1/(1-\alpha)}$ where we have defined $c_K \equiv s(1 - a_K)^{\alpha}(1 - a_H)^{1-\alpha}$. Thus a rise in s will make the $\dot{k} = 0$ locus steeper. Since s does not appear in equation (11), the $\dot{h} = 0$ locus is unaffected. See the figure on the left. The economy will move from its old balanced growth path at E to a new balanced growth path at E '.



Output per person grows at rate g until the time that s rises (denoted time t_0 in the figure on the right). During the transition from E to E ', both h(t) and k(t) are rising. Thus human capital per person and physical capital per person grow at a rate greater than g during the transition. From equation (12), this means that output per person grows at a rate greater than g during the transition as well. Once the economy reaches the new balanced growth path (at time t_1 in the diagram), h(t) and k(t) are constant again. Thus human and physical capital per person grow at rate g again. Thus output per person grows at rate g again on the new balanced growth path. A permanent rise in the saving rate has only a level effect on output per person, not a permanent growth rate effect.

Problem 3.15

The relevant equations are

(1) $Y(t) = K(t)^{\alpha}\left[\left(1-a_H\right)H(t)\right]^{\beta}$, (2) $\dot{H}(t) = Ba_H H(t)$, and (3) $\dot{K}(t) = sY(t)$,

where $0 < \alpha < 1$, $0 < \beta < 1$, and $\alpha + \beta > 1$.

(a) To get the growth rate of human capital -- which turns out to be constant -- divide equation (2) by H(t):

(4) $g_H \equiv \dot{H}(t)/H(t) = Ba_H$.

(b) Substitute the production function, equation (1), into the expression for the evolution of the physical capital stock, equation (3), to obtain

(5) $\dot{K}(t) = sK(t)^{\alpha}\left[\left(1-a_H\right)H(t)\right]^{\beta}$.

To get the growth rate of physical capital, divide equation (5) by K(t):

(6) $g_K(t) \equiv \dot{K}(t)/K(t) = sK(t)^{\alpha-1}\left[\left(1-a_H\right)H(t)\right]^{\beta}$.

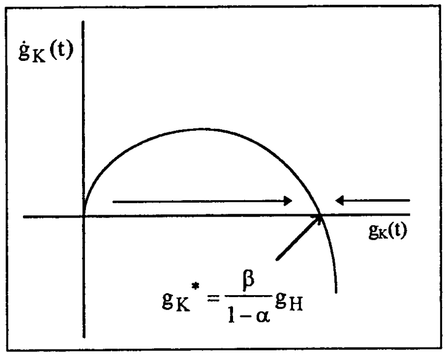
We need to examine the dynamics of the growth rate of physical capital. Taking the time derivative of the log of equation (6) yields the following growth rate of the growth rate of physical capital:

(7) $\dot{g}_K(t)/g_K(t) = (\alpha - 1)\dot{K}(t)/K(t) + \beta\dot{H}(t)/H(t) = (\alpha - 1)g_K(t) + \beta g_H$.

Now we can plot the change in the growth rate of capital, $\dot{g}_K(t)$, as a function of the growth rate of capital itself, $g_K(t)$. Multiplying both sides of equation (7) by $g_K(t)$ gives us

(8) $\dot{g}_K(t) = (\alpha - 1)g_K(t)^2 + \beta g_H g_K(t)$.

Note that we are assuming that $\alpha < 1$ which means that there are decreasing returns to physical capital alone. The phase diagram implied by equation (8) is depicted in the figure at right. Note that $g_K(t)$ is constant when $\dot{g}_K(t) = 0$ or when $(\alpha - 1)g_K(t) + \beta g_H = 0$. Solving this expression for $g_K(t)$ yields $g_K^* = [\beta/(1 - \alpha)]g_H$.



Note that $g_K^* > g_H$ since $\alpha + \beta > 1$ or $\beta > 1 - \alpha$. To the left of g_K^*, from the phase diagram, $\dot{g}_K(t) > 0$ and so $g_K(t)$ rises toward g_K^*.

Similarly, to the right of g_K^*, $\dot{g}_K(t) < 0$ and so $g_K(t)$ falls toward g_K^*. Thus the growth rate of capital converges to a constant value of g_K^* and a balanced growth path exists.

Taking the time derivative of the log of equation (1) yields the growth rate of output:

(9) $\dot{Y}(t)/Y(t) = \alpha \dot{K}(t)/K(t) + \beta \dot{H}(t)/H(t) = \alpha g_K(t) + \beta g_H$.

On the balanced growth path, $g_K(t) = g_K^* = [\beta/(1 - \alpha)]g_H$ and so

(10) $$\frac{\dot{Y}(t)}{Y(t)} = \frac{\alpha\beta}{(1-\alpha)} g_H + \beta g_H = \frac{\alpha\beta + \beta - \alpha\beta}{(1-\alpha)} g_H = \frac{\beta}{(1-\alpha)} g_H = g_K^*.$$

On the balanced growth path, output grows at the same rate as physical capital, which in turn is greater than the constant growth rate of human capital, $g_H = Ba_H$.

<u>Problem 3.16</u>

(a) From equation (3.55) in the text, output per person on the balanced growth path with the assumption that $G(E) = e^{\phi E}$ is given by

(1) $$\left(\frac{Y}{N}\right)^{bgp} = y^* A(t) e^{\phi E} \frac{e^{-nE} - e^{-nT}}{1 - e^{-nT}},$$

where $y^* = f(k^*)$ which is output per unit of effective labor services on the balanced growth path. We can maximize the natural log of $(Y/N)^{bgp}$ with respect to E, noting that y^* and $A(t)$ are not functions of E. The log of output per person on the balanced growth path is

(2) $$\ln\left(\frac{Y}{N}\right)^{bgp} = \ln y^* + \ln A(t) + \phi E + \ln\left[e^{-nE} - e^{-nT}\right] - \ln\left[1 - e^{-nT}\right],$$

and so the first-order condition is given by

(3) $$\frac{\partial \ln(Y/N)^{bgp}}{\partial E} = \phi + \frac{1}{e^{-nE} - e^{-nT}} e^{-nE}(-n) = 0,$$

or

(4) $$\phi\left(e^{-nE} - e^{-nT}\right) = ne^{-nE}.$$

Collecting the terms in e^{-nE} gives us

(5) $(\phi - n)e^{-nE} = \phi e^{-nT}$,

or simply

(6) $$e^{-nE} = \frac{\phi}{\phi - n} e^{-nT}.$$

Taking the natural log of both sides of equation (6) yields

(7) $-nE = [\ln\phi - \ln(\phi - n)] - nT$.

Multiplying both sides of (7) by $-1/n$ gives us the following golden-rule level of education:

(8) $E^* = T - \dfrac{1}{n}\ln\left[\dfrac{\phi}{\phi - n}\right]$.

(b) (i) Taking the derivative of E* with respect to T gives us

(9) $\dfrac{\partial E^*}{\partial T} = 1$.

So a rise in T -- an increase in lifespan -- raises the golden-rule level of education one for one.

(b) (ii) Showing that a fall in n increases the golden-rule level of education is somewhat complicated. From equation (6), we can write

(10) $e^{-n(T-E^*)} = \dfrac{\phi - n}{\phi}$,

or

(11) $1 - e^{-n(T-E^*)} = \dfrac{n}{\phi}$.

Multiplying both sides of equation (11) by ϕ/n gives us

(12) $\dfrac{\phi}{n}[1 - e^{-n(T-E^*)}] = 1$.

Now note that the left-hand side of equation (12) is equivalent to

(13) $V \equiv \phi \displaystyle\int_{s=0}^{T-E^*} e^{-ns} ds$.

Thus, totally differentiating equation (12) gives us

(14) $\dfrac{\partial V}{\partial n}dn + \dfrac{\partial V}{\partial E^*}dE^* = 0$,

and so

(15) $\dfrac{dE^*}{dn} = -\dfrac{\partial V/\partial n}{\partial V/\partial E^*}$.

Now note that

(16) $\dfrac{\partial V}{\partial n} = \phi \displaystyle\int_{s=0}^{T-E^*} -se^{-ns} ds < 0$,

and

(17) $\dfrac{\partial V}{\partial E^*} = -\phi e^{-n(T-E^*)} < 0$.

Thus $dE^*/dn < 0$ and so a fall in n raises the golden-rule level of education.

Problem 3.17

(a) In general, the present discounted value, at time zero, of the worker's lifetime earnings is

(1) $Y = \displaystyle\int_{t=E}^{T} e^{-\bar{r}t} w(t)L(t)dt$.

We can normalize L(t) to one and we are assuming that $w(t) = be^{gt}e^{\phi E}$. Thus (1) becomes

(2) $Y = \int_{t=E}^{T} e^{-\bar{r}t} b e^{gt} e^{\phi E} dt = b e^{\phi E} \int_{t=E}^{T} e^{-(\bar{r}-g)t} dt$.

Solving the integral in (2) gives us

(3) $Y = b e^{\phi E} \left[\frac{-1}{(\bar{r}-g)} e^{-(\bar{r}-g)t} \Big|_{t=E}^{T} \right] = \frac{b e^{\phi E}}{\bar{r}-g} \left[-e^{-(\bar{r}-g)T} + e^{-(\bar{r}-g)E} \right]$,

which can be rewritten as

(4) $Y = \frac{b}{\bar{r}-g} \left[-e^{\phi E-(\bar{r}-g)T} + e^{[\phi-(\bar{r}-g)]E} \right]$.

(b) The first-order condition for the choice of E is given by

(5) $\frac{\partial Y}{\partial E} = \frac{b}{\bar{r}-g} \left[-\phi e^{\phi E-(\bar{r}-g)T} + [\phi - (\bar{r}-g)] e^{[\phi-(\bar{r}-g)]E} \right] = 0$.

This can be rewritten as

(6) $[\phi - (\bar{r}-g)] e^{[\phi-(\bar{r}-g)]E} = \phi e^{\phi E-(\bar{r}-g)T}$.

Dividing both sides by $e^{\phi E}$ and rearranging yields

(7) $e^{-(\bar{r}-g)(E-T)} = \frac{\phi}{\phi - (\bar{r}-g)}$.

Taking the natural log of both sides of equation (7) gives us

(8) $-(\bar{r}-g)(E-T) = \ln\left[\frac{\phi}{\phi - (\bar{r}-g)} \right]$.

Dividing both sides of (8) by $-(\bar{r}-g)$ and then adding T to both sides of the resulting expression gives us

(9) $E^* = T - \frac{1}{\bar{r}-g} \ln\left[\frac{\phi}{\phi - (\bar{r}-g)} \right]$.

(c) (i) From equation (9),

(10) $\frac{\partial E^*}{\partial T} = 1$.

Thus an increase in lifespan increases the optimal amount of education. Intuitively, a longer lifespan provides a longer working period over which to receive the higher wages yielded by more education.

(c) (ii) & (iii) The interest rate, $\bar{r}$, and the growth rate, g, enter the optimal choice of education through their difference, $(\bar{r}-g)$. Intuitively, it should be clear that a rise in $\bar{r}$, and thus a rise in $(\bar{r}-g)$, will cause the individual to choose less education. Getting marginally more education foregoes current earnings for higher future earnings. A higher interest rate means that the higher future wages due to increased education will be worth less in present-value terms and hence the individual chooses less education.

Showing this formally is somewhat complicated, however. Taking the inverse of both sides of equation (7) gives us

(11) $e^{-(\bar{r}-g)(T-E^*)} = \frac{\phi - (\bar{r}-g)}{\phi}$,

or

(12) $1 - e^{-(\bar{r}-g)(T-E^*)} = \frac{(\bar{r}-g)}{\phi}$.

Multiplying both sides of equation (12) by $\phi/(\bar{r}-g)$ gives us

(13) $$\frac{\phi}{(\bar{r}-g)}[1-e^{-(\bar{r}-g)(T-E^*)}]=1.$$

Now note that the left-hand side of equation (13) is equivalent to

(14) $$V \equiv \phi \int_{s=0}^{T-E^*} e^{-(\bar{r}-g)s} ds.$$

Thus, totally differentiating equation (13) gives us

(15) $$\frac{\partial V}{\partial(\bar{r}-g)}d(\bar{r}-g)+\frac{\partial V}{\partial E^*}dE^*=0,$$

and so

(16) $$\frac{dE^*}{d(\bar{r}-g)}=-\frac{\partial V/\partial(\bar{r}-g)}{\partial V/\partial E^*}.$$

Now note that

(17) $$\frac{\partial V}{\partial(\bar{r}-g)}=\phi \int_{s=0}^{T-E^*} -se^{-(\bar{r}-g)s} ds<0,$$

and

(17) $$\frac{\partial V}{\partial E^*}=-\phi e^{-(\bar{r}-g)(T-E^*)}<0.$$

Thus $dE^*/d(\bar{r}-g)<0$. So a rise in $\bar{r}$ decreases the optimal choice of education; a rise in g increases the optimal choice of education.

__Problem 3.18__

(a) The representative producer's problem is to choose f, the fraction of time devoted to protection, to maximize output, which is given by $[1 - L(f, R)](1 - f)B$. The first-order condition is

(1) $-B[1 - L(f, R)] - (1 - f)BL_f(f, R) = 0.$

This can be rearranged to obtain

(2) $$\frac{1}{1-f}=\frac{-L_f(f,R)}{1-L(f,R)}.$$

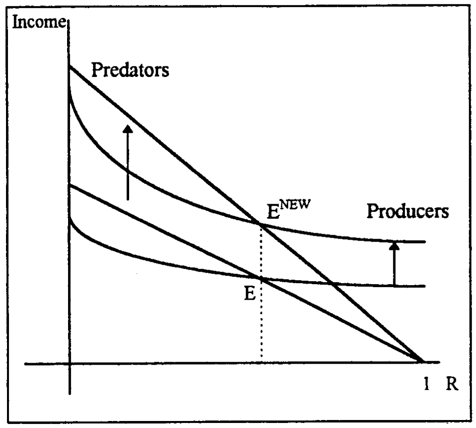
This is identical to equation (3.63) in the text. Thus the value of B does not affect the producer's allocation of time between producing output and protecting it from rent-seekers. The optimal choice of f is still implicitly defined by the same condition as in the model in the text.

(b) Equilibrium requires that income per producer and income per predator be equal. Thus equilibrium requires

(3) $$[1-L(f(R),R)][(1-f(R))B]=\frac{1-R}{R}[(1-f(R))B]L(f(R),R).$$

Since f does not depend on B, we can see that at a given R, an increase in B will increase producers' income and predators' income by the percentage increase in B. Thus an increase in B shifts the producers' and predators' income curves up proportionately. See the figure below.

(c) Since the curves showing producers' and predators' incomes as functions of R both shift up proportionately, they will still intersect at the original equilibrium value of R. In the figure at right, the original equilibrium was at point E; the new equilibrium is at point E^{NEW}. The fraction of the population engaged in rent-seeking does not change and the incomes of producers and predators both rise by the same percentage amount as the increase in B.



Problem 3.19

(a) (i) We have

$$(1)\quad \frac{dy_i(t)}{dt} = -\lambda[y_i(t) - y^*].$$

Since y^* is a constant, the derivative of $y_i(t)$ with respect to time is the same as the derivative of $y_i(t) - y^*$ with respect to time and so equation (1) is equivalent to

$$(2)\quad \frac{d[y_i(t) - y^*]}{dt} = -\lambda[y_i(t) - y^*],$$

which implies that $y_i(t) - y^*$ grows at rate $-\lambda$. Thus

$$(3)\quad y_i(t) - y^* = e^{-\lambda t}[y_i(0) - y^*].$$

Rearranging equation (3) to solve for $y_i(t)$ gives us

$$(4)\quad y_i(t) = (1 - e^{-\lambda t})y^* + e^{-\lambda t}y_i(0).$$

(a) (ii) Adding a mean-zero, random disturbance to $y_i(t)$ gives us

$$(5)\quad y_i(t) = (1 - e^{-\lambda t})y^* + e^{-\lambda t}y_i(0) + u_i(t).$$

Consider the cross-country growth regression given by

$$(6)\quad y_i(t) - y_i(0) = \alpha + \beta y_i(0) + \varepsilon_i.$$

Using the hint in the question, the coefficient on $y_i(0)$ in this regression equals the covariance of $y_i(t) - y_i(0)$ and $y_i(0)$ divided by the variance of $y_i(0)$. Thus the estimate of β is given by

$$(7)\quad \beta = \frac{\text{cov}[y_i(t) - y_i(0), y_i(0)]}{\text{var}[y_i(0)]}.$$

(If the sample size is large enough, we can treat sample parameters as equivalent to their population counterparts.) Now, use the fact that for any two random variables, X and Y, cov[(X - Y), Y] = cov[X, Y] - var[Y] and so

$$(8)\quad \beta = \frac{\text{cov}[y_i(t), y_i(0)] - \text{var}[y_i(0)]}{\text{var}[y_i(0)]} = \frac{\text{cov}[y_i(t), y_i(0)]}{\text{var}[y_i(0)]} - 1.$$

Using equation (5),

$$(9)\quad \text{cov}[y_i(t), y_i(0)] = \text{cov}[(1 - e^{-\lambda t})y^* + e^{-\lambda t}y_i(0) + u_i(t), y_i(0)].$$

Since y^* is a constant, and $u_i(t)$ and $y_i(0)$ are assumed to be uncorrelated, we have

(10) $\text{cov}[y_i(t), y_i(0)] = e^{-\lambda t}\,\text{var}[y_i(0)]$.
Substituting equation (10) into equation (8) gives us

(11) $\beta = \dfrac{e^{-\lambda t}\,\text{var}[y_i(0)]}{\text{var}[y_i(0)]} - 1,$

or

(12) $e^{-\lambda t} = 1 + \beta$.

Taking the natural log of both sides of equation (12) and solving for λ gives us

(13) $\lambda = -\dfrac{\ln(1+\beta)}{t}$.

Thus, given an estimate of β, equation (13) could be used to calculate an estimate of the rate of convergence, λ.

(a) (iii) From equation (5), the variance of $y_i(t)$ is given by

(14) $\text{var}[y_i(t)] = e^{-2\lambda t}\,\text{var}[y_i(0)] + \text{var}[u_i(t)]$.

From equation (13), if $\beta < 0$ then $\lambda > 0$. This does not, however, ensure that $\text{var}[y_i(t)] < \text{var}[y_i(0)]$, so that the variance of cross-country income is falling. The is due to the variance of the random shocks to output, represented by the $\text{var}[u_i(t)]$ term in equation (14). Thus the effect of $\beta < 0$ or $\lambda > 0$, which tends to reduce the dispersion of income, can be offset by the random shocks to output, which tend to raise income dispersion.

If $\beta > 0$ then $\lambda < 0$. From equation (14), we can see that this means $\text{var}[y_i(t)]$ will be greater than $\text{var}[y_i(0)]$. In this case, the effect of $\beta < 0$ or $\lambda > 0$ is to increase income dispersion, and thus this works in the same direction as the random shocks which also tend to increase income dispersion.

(b) (i) Since y_i^* is time-invariant, analysis equivalent to that in part (a) (i) would yield

(15) $y_i(t) = (1 - e^{-\lambda t})y_i^* + e^{-\lambda t}y_i(0)$.

(b) (ii) We will determine the value of λ implied by an estimate of β in this model and compare it to the value implied by using the formula from part (a) (ii). In the cross-country growth regression given by

(16) $y_i(t) - y_i(0) = \alpha + \beta y_i(0) + \varepsilon_i$,

again we have

(17) $\beta = \dfrac{\text{cov}[y_i(t), y_i(0)] - \text{var}[y_i(0)]}{\text{var}[y_i(0)]} = \dfrac{\text{cov}[y_i(t), y_i(0)]}{\text{var}[y_i(0)]} - 1.$

Then, since

(18) $y_i(t) = (1 - e^{-\lambda t})y_i^* + e^{-\lambda t}y_i(0) + e_i$,

we have

(19) $\text{cov}[y_i(t), y_i(0)] = (1 - e^{-\lambda t})\,\text{cov}[y_i^*, y_i(0)] + e^{-\lambda t}\,\text{var}[y_i(0)]$.

Since

(20) $y_i(0) = y_i^* + u_i = a + bX_i + u_i$,

we have

(21) $\text{var}[y_i(0)] = b^2\,\text{var}[X_i] + \text{var}[u_i]$,

and

(22) $\text{cov}[y_i^*, y_i(0)] = \text{cov}[a + bX_i, a + bX_i + u_i] = b^2\,\text{var}[X_i]$,

since X_i and u_i are assumed to be uncorrelated. Substituting equations (21) and (22) into equation (19) gives us

(23) $\operatorname{cov}[y_i(t), y_i(0)] = (1-e^{-\lambda t})b^2 \operatorname{var}[X_i] + b^2 e^{-\lambda t} \operatorname{var}[X_i] + e^{-\lambda t} \operatorname{var}[u_i]$,
or simply

(24) $\operatorname{cov}[y_i(t), y_i(0)] = b^2 \operatorname{var}[X_i] + e^{-\lambda t} \operatorname{var}[u_i]$.

Substituting equations (21) and (24) into (17) gives us

(25) $$\beta = \frac{b^2 \operatorname{var}[X_i] + e^{-\lambda t} \operatorname{var}[u_i]}{b^2 \operatorname{var}[X_i] + \operatorname{var}[u_i]} - 1 = \frac{-(1-e^{-\lambda t}) \operatorname{var}[u_i]}{b^2 \operatorname{var}[X_i] + \operatorname{var}[u_i]}.$$

We can now solve for the value of λ implied by equation (25) and compare it to the one we would calculate if we used equation (13). Equation (25) implies

(26) $$e^{-\lambda t} = 1 + \frac{b^2 \operatorname{var}[X_i] + \operatorname{var}[u_i]}{\operatorname{var}[u_i]} \beta.$$

Taking the natural log of both sides of (26) and solving for λ gives us

(27) $$\lambda = \frac{-\ln\left[1 + \dfrac{b^2 \operatorname{var}[X_i] + \operatorname{var}[u_i]}{\operatorname{var}[u_i]}\beta\right]}{t}.$$

Since $(b^2 \operatorname{var}[X_i] + \operatorname{var}[u_i])/\operatorname{var}[u_i] > 1$, using the formula given by equation (13) would lead us to calculate an estimate for λ that is too small in absolute value. That is, if $\lambda > 0$, using the method of part (a) (ii) would yield an underestimate of the rate of convergence.

(b) (iii) Subtracting $y_i(0)$ from both sides of equation (18) gives us

(28) $y_i(t) - y_i(0) = (1-e^{-\lambda t})y_i^* - (1-e^{-\lambda t})y_i(0) + e_i$.

Substituting equation (20) into (28) yields

(29) $y_i(t) - y_i(0) = (1-e^{-\lambda t})y_i^* - (1-e^{-\lambda t})[y_i^* + u_i] + e_i$,

which simplifies to

(30) $y_i(t) - y_i(0) = (e^{-\lambda t} - 1)u_i + e_i$.

Defining $Q \equiv (e^{-\lambda t} - 1)$, we can see that the regression given by

(31) $y_i(t) - y_i(0) = \alpha + \beta y_i(0) + \gamma X_i + \varepsilon_i$

is equivalent to projecting $Qu_i + e_i$ on a constant, $y_i(0)$, and X_i, where e_i is simply a mean-zero, random error that is uncorrelated with the right-hand side variables. Rearranging $y_i(0) = a + bX_i + u_i$ to solve for u_i gives us

(32) $u_i = -a + y_i(0) - bX_i$,

and so

(33) $Qu_i = -Qa + Qy_i(0) - QbX_i$.

Thus, in the regression given by (31), an estimate of β provides an estimate of Q and an estimate of γ provides an estimate of -Qb. Thus, we can construct an estimate of b by taking the negative of the estimate of γ, divided by the estimate of β, or

(34) $$-\frac{\gamma}{\beta} = -\frac{-Qb}{Q} = b.$$

SOLUTIONS TO CHAPTER 4

Problem 4.3

(a) The equations describing the evolution of technology are given by

(1) $\ln A_t = \overline{A} + gt + \tilde{A}_t$, and (2) $\tilde{A}_t = \rho_A \tilde{A}_{t-1} + \varepsilon_{A,t}$, $-1 < \rho_A < 1$.

From equation (1) and letting $\ln A_0$ denote the value of lnA in period 0, we have $\ln A_0 = \overline{A} + g(0) + \tilde{A}_0$.

Rearranging to solve for $\tilde{A}_0$ gives us

(3) $\tilde{A}_0 = \ln A_0 - \overline{A}$.

In period 1, using equations (1) and (2), we have

(4) $\ln A_1 = \overline{A} + g + \tilde{A}_1$, and (5) $\tilde{A}_1 = \rho_A \tilde{A}_0 + \varepsilon_{A,1}$.

Substituting equation (3) into equation (5) yields

(6) $\tilde{A}_1 = \rho_A(\ln A_0 - \overline{A}) + \varepsilon_{A,1}$.

Finally, substituting equation (6) into equation (4) gives us

(7) $\ln A_1 = \overline{A} + g + \rho_A(\ln A_0 - \overline{A}) + \varepsilon_{A,1}$.

In period 2, using equations (1) and (2), we have

(8) $\ln A_2 = \overline{A} + 2g + \tilde{A}_2$, and (9) $\tilde{A}_2 = \rho_A \tilde{A}_1 + \varepsilon_{A,2}$.

Substituting equation (6) into equation (9) yields

(10) $\tilde{A}_2 = \rho_A\left[\rho_A(\ln A_0 - \overline{A}) + \varepsilon_{A,1}\right] + \varepsilon_{A,2} = \rho_A^2(\ln A_0 - \overline{A}) + \rho_A\varepsilon_{A,1} + \varepsilon_{A,2}$.

Finally, substituting equation (10) into equation (8) gives us

(11) $\ln A_2 = \overline{A} + 2g + \rho_A^2(\ln A_0 - \overline{A}) + \rho_A\varepsilon_{A,1} + \varepsilon_{A,2}$.

In period 3, using equations (1) and (2), we have

(12) $\ln A_3 = \overline{A} + 3g + \tilde{A}_3$, and (13) $\tilde{A}_3 = \rho_A \tilde{A}_2 + \varepsilon_{A,3}$.

Substituting equation (10) into equation (13) yields

(14) $\tilde{A}_3 = \rho_A\left[\rho_A^2(\ln A_0 - \overline{A}) + \rho_A\varepsilon_{A,1} + \varepsilon_{A,2}\right] + \varepsilon_{A,3} = \rho_A^3(\ln A_0 - \overline{A}) + \rho_A^2\varepsilon_{A,1} + \rho_A\varepsilon_{A,2} + \varepsilon_{A,3}$.

Finally, substituting equation (14) into equation (12) gives us

(15) $\ln A_3 = \overline{A} + 3g + \rho_A^3(\ln A_0 - \overline{A}) + \rho_A^2\varepsilon_{A,1} + \rho_A\varepsilon_{A,2} + \varepsilon_{A,3}$.

(b) Using equation (7) to find the expected value of $\ln A_1$ yields

$$E[\ln A_1] = \overline{A} + g + \rho_A(\ln A_0 - \overline{A}),$$

since $E[\varepsilon_{A,1}] = 0$.

Using equation (11) to find the expected value of $\ln A_2$ yields

$$E[\ln A_2] = \overline{A} + 2g + \rho_A^2(\ln A_0 - \overline{A}),$$

since $E[\rho_A\varepsilon_{A,1}] = \rho_A E[\varepsilon_{A,1}] = 0$, $E[\varepsilon_{A,2}] = 0$.

Using equation (15) to find the expected value of $\ln A_3$ yields

$$E[\ln A_3] = \overline{A} + 3g + \rho_A^3(\ln A_0 - \overline{A})$$

since $E[\rho_A^2\varepsilon_{A,1}] = \rho_A^2 E[\varepsilon_{A,1}] = 0$, $E[\rho_A\varepsilon_{A,2}] = \rho_A E[\varepsilon_{A,2}] = 0$, $E[\varepsilon_{A,3}] = 0$.

Problem 4.4

(a) We need to solve the household's one period problem assuming no initial wealth and normalizing the size of the household to one. Thus the problem is given by

$\max_{c,\ell}\ \ln c + b(1 - \ell)^{1-\gamma}/(1 - \gamma)$, subject to the budget constraint $c = w\ell$.

Set up the Lagrangian:

$\mathcal{L} = \ln c + b(1 - \ell)^{1-\gamma}/(1 - \gamma) + \lambda[w\ell - c]$.

The first-order conditions are

(1) $\partial\mathcal{L}/\partial c = (1/c) - \lambda = 0$, and (2) $\partial\mathcal{L}/\partial\ell = -b(1 - \ell)^{-\gamma} + \lambda w = 0$.

Substituting the budget constraint into equation (1) yields

(3) $\lambda = 1/c = 1/(w\ell)$.

Substituting equation (3) into equation (2) yields

$-b(1 - \ell)^{-\gamma} + w/(w\ell) = 0$,

and simplifying slightly gives us

(4) $1/\ell = b/(1 - \ell)^{\gamma}$.

Although labor supply, ℓ, is only implicitly defined by equation (4), we can see that it will not depend upon the real wage.

(b) We want a formula for relative leisure in the two periods. That is, a formula for $(1 - \ell_1)/(1 - \ell_2)$. Assume that the household lives for two periods, has no initial wealth, has size $N_t/H = 1$ for both periods and finally that there is no uncertainty. Thus the problem can be formalized as

$$\max\ \ln c_1 + b\frac{(1-\ell_1)^{1-\gamma}}{1-\gamma} + e^{-\rho}\ln c_2 + e^{-\rho}b\frac{(1-\ell_2)^{1-\gamma}}{1-\gamma},$$

subject to the intertemporal budget constraint given by

$$c_1 + \frac{c_2}{1+r} = w_1\ell_1 + \frac{w_2\ell_2}{1+r}.$$

Set up the Lagrangian:

$$\mathcal{L} = \ln c_1 + b\frac{(1-\ell_1)^{1-\gamma}}{1-\gamma} + e^{-\rho}\ln c_2 + e^{-\rho}b\frac{(1-\ell_2)^{1-\gamma}}{1-\gamma} + \lambda\left[w_1\ell_1 + \frac{w_2\ell_2}{1+r} - c_1 - \frac{c_2}{1+r}\right].$$

There will be four first-order conditions:

(5) $\partial\mathcal{L}/\partial c_1 = (1/c_1) - \lambda = 0$, and (6) $\partial\mathcal{L}/\partial c_2 = (e^{-\rho}/c_2) - [\lambda/(1 + r)] = 0$,

(7) $\partial\mathcal{L}/\partial\ell_1 = -b(1 - \ell_1)^{-\gamma} + \lambda\ell_1 = 0$, and (8) $\partial\mathcal{L}/\partial\ell_2 = -e^{-\rho}b(1 - \ell_2)^{-\gamma} + [\lambda\ell_2/(1 + r)] = 0$.

Rearranging equation (7) yields one expression for λ : $\lambda = b(1 - \ell_1)^{-\gamma}/\ell_1$.

Rearranging (8) yields another expression for λ : $\lambda = [e^{-\rho}b(1 - \ell_2)^{-\gamma}(1 + r)]/\ell_2$.

Equating these two expressions for λ yields

$$\frac{e^{-\rho}b(1+r)}{(1-\ell_2)^{\gamma}w_2} = \frac{b}{(1-\ell_1)^{\gamma}w_1} \Rightarrow \frac{(1-\ell_1)^{\gamma}}{(1-\ell_2)^{\gamma}} = \frac{1}{e^{-\rho}(1+r)}\frac{w_2}{w_1},$$

and thus

$$(9)\quad \frac{(1-\ell_1)}{(1-\ell_2)} = \left[\frac{1}{e^{-\rho}(1+r)}\frac{w_2}{w_1}\right]^{1/\gamma}.$$

If w_2/w_1 rises, then $(1-\ell_1)/(1-\ell_2)$ rises. That is, suppose the real wage in the second period rises relative to the real wage in the first period. Then the individual increases first-period leisure relative to second-period leisure, or reduces first-period labor supply relative to second-period labor supply. We can calculate the elasticity, denoting -- for ease of notation only -- $(1-\ell_1)/(1-\ell_2) \equiv \ell^*$ and $w_2/w_1 \equiv w^*$:

$$\frac{\partial \ell^*}{\partial w^*}\frac{w^*}{\ell^*} = \frac{1}{\gamma}\frac{\left[1/e^{-\rho}(1+r)\right]^{1/\gamma} w^{*(1/\gamma)-1}\, w^*}{\ell^*}.$$

Substitute in the denominator for $\ell^* \equiv (1-\ell_1)/(1-\ell_2)$ from equation (9) to yield

$$\frac{\partial \ell^*}{\partial w^*}\frac{w^*}{\ell^*} = \frac{1}{\gamma}\frac{\left[1/e^{-\rho}(1+r)\right]^{1/\gamma} w^{*1/\gamma}}{\left[\left(1/e^{-\rho}(1+r)\right)w^*\right]^{1/\gamma}} = \frac{1}{\gamma}.$$

Thus the smaller is γ -- or the bigger is $1/\gamma$ -- the more the individual will adjust relative labor supply in response to a change in relative real wages.

From equation (9), we can also see that if r rises then $(1-\ell_1)/(1-\ell_2)$ falls. That is, suppose that there is a rise in the real interest rate. Then the individual reduces first-period leisure relative to second-period leisure, or increases first-period labor supply relative to second-period labor supply. It is straightforward to show that

$$\frac{\partial\left[(1-\ell_1)/(1-\ell_2)\right]}{\partial(1+r)}\frac{(1+r)}{\left[(1-\ell_1)/(1-\ell_2)\right]} = -\frac{1}{\gamma}.$$

Thus the smaller is γ -- or the bigger is $1/\gamma$ -- the more the individual will respond to a change in the real interest rate. Note that with log utility, where $\gamma = 1$, this elasticity is equal to one.

Intuitively, a low value of γ means that utility is not very sharply curved in ℓ. This means that ℓ responds a lot to changes in wages and the interest rate.

Problem 4.5

(a) The problem is to maximize utility as given by

(1) $\ln c_1 + b\ln(1-\ell_1) + e^{-\rho}[\ln c_2 + b\ln(1-\ell_2)]$,

subject to the following lifetime budget constraint:

(2) $c_1 + \frac{1}{1+r}c_2 = w_1\ell_1 + \frac{1}{1+r}w_2\ell_2$

Set up the Lagrangian:

(3) $\mathcal{L} = \ln c_1 + b\ln(1-\ell_1) + e^{-\rho}\left[\ln c_2 + b\ln(1-\ell_2)\right] + \lambda\left[w_1\ell_1 + \frac{1}{1+r}w_2\ell_2 - c_1 - \frac{1}{1+r}c_2\right]$.

The four first-order equations and some simple algebra gives us

$$\frac{\partial\mathcal{L}}{\partial c_1} = \frac{1}{c_1} - \lambda = 0 \quad \Rightarrow \quad c_1 = \frac{1}{\lambda}, \quad (4)$$

$$\frac{\partial\mathcal{L}}{\partial c_2} = \frac{e^{-\rho}}{c_2} - \frac{\lambda}{1+r} = 0 \quad \Rightarrow \quad c_2 = \frac{e^{-\rho}(1+r)}{\lambda}, \quad (5)$$

$$\frac{\partial\mathcal{L}}{\partial \ell_1} = \frac{-b}{(1-\ell_1)} + \lambda w_1 = 0 \quad \Rightarrow \quad \ell_1 = 1 - \frac{b}{\lambda w_1}, \quad (6)$$

$$\frac{\partial \mathcal{L}}{\partial \ell_2} = \frac{-e^{-\rho}b}{(1-\ell_2)} + \frac{\lambda w_2}{1+r} = 0 \quad \Rightarrow \quad \ell_2 = 1 - \frac{(1+r)e^{-\rho}b}{\lambda w_2}. \quad (7)$$

Now substitute equations (4) - (7) into the lifetime budget constraint, equation (2), to obtain

$$\frac{1}{\lambda} + \frac{e^{-\rho}(1+r)}{\lambda(1+r)} = w_1\left[1 - \frac{b}{\lambda w_1}\right] + \frac{w_2}{1+r}\left[1 - \frac{(1+r)e^{-\rho}b}{\lambda w_2}\right].$$

Multiplying both sides by λ gives us

$$1 + e^{-\rho} = \lambda w_1\left[\frac{\lambda w_1 - b}{\lambda w_1}\right] + \frac{\lambda w_2}{1+r}\left[\frac{\lambda w_2 - (1+r)e^{-\rho}b}{\lambda w_2}\right].$$

Simplify further to obtain

$$1 + e^{-\rho} = \lambda w_1 - b + \frac{\lambda w_2}{1+r} - e^{-\rho}b.$$

Finally, solving for λ yields

$$(8)\ \lambda = \frac{(1+e^{-\rho})(1+b)}{\left[w_1 + w_2/(1+r)\right]},$$

where we have used the fact that $1 + e^{-\rho} + b + e^{-\rho}b = (1 + e^{-\rho})(1 + b)$.

Now to obtain an expression for first-period labor supply, substitute equation (8) into equation (6) to obtain

$$\ell_1 = 1 - \frac{b}{\dfrac{(1+e^{-\rho})(1+b)}{\left[w_1 + w_2/(1+r)\right]}w_1} = 1 - \frac{b\left[w_1 + w_2/(1+r)\right]}{(1+e^{-\rho})(1+b)w_1}.$$

Finally, dividing the top and bottom of the second term by w_1 yields

$$(9)\ \ell_1 = 1 - \frac{b\left[1 + (w_2/w_1)(1/(1+r))\right]}{(1+e^{-\rho})(1+b)}.$$

Note that ℓ_1 is a function of the relative wage, w_2/w_1. Thus any change in w_1 and w_2 that leaves w_2/w_1 unchanged will leave ℓ_1 unchanged.

To obtain an expression for second-period labor supply, substitute equation (8) into equation (7) to obtain

$$\ell_2 = 1 - \frac{(1+r)e^{-\rho}b}{\dfrac{(1+e^{-\rho})(1+b)}{\left[w_1 + w_2/(1+r)\right]}w_2} = 1 - \frac{(1+r)e^{-\rho}b\left[w_1 + w_2/(1+r)\right]}{(1+e^{-\rho})(1+b)w_2}.$$

Finally, dividing the top and bottom of the second term by w_2 yields

$$(10)\ \ell_2 = 1 - \frac{(1+r)e^{-\rho}b\left[(w_1/w_2) + 1/(1+r)\right]}{(1+e^{-\rho})(1+b)}.$$

Again, note that ℓ_2 is just a function of the relative wage, w_1/w_2. Thus any change in w_1 and w_2 that leaves w_1/w_2 unchanged will leave ℓ_2 unchanged.

(b) (i) The fact that the household has initial wealth of $Z > 0$ will not affect equation (4.23) in the text -- the Euler equation -- which relates consumption in one period to expectations of consumption the following

$$\frac{\partial \mathcal{L}}{\partial \ell_2} = \frac{-e^{-\rho}b}{(1-\ell_2)} + \frac{\lambda w_2}{1+r} = 0 \quad \Rightarrow \quad \ell_2 = 1 - \frac{(1+r)e^{-\rho}b}{\lambda w_2}. \quad (7)$$

Now substitute equations (4) - (7) into the lifetime budget constraint, equation (2), to obtain

$$\frac{1}{\lambda} + \frac{e^{-\rho}(1+r)}{\lambda(1+r)} = w_1\left[1 - \frac{b}{\lambda w_1}\right] + \frac{w_2}{1+r}\left[1 - \frac{(1+r)e^{-\rho}b}{\lambda w_2}\right].$$

Multiplying both sides by λ gives us

$$1 + e^{-\rho} = \lambda w_1\left[\frac{\lambda w_1 - b}{\lambda w_1}\right] + \frac{\lambda w_2}{1+r}\left[\frac{\lambda w_2 - (1+r)e^{-\rho}b}{\lambda w_2}\right].$$

Simplify further to obtain

$$1 + e^{-\rho} = \lambda w_1 - b + \frac{\lambda w_2}{1+r} - e^{-\rho}b.$$

Finally, solving for λ yields

$$(8) \quad \lambda = \frac{(1+e^{-\rho})(1+b)}{\left[w_1 + w_2/(1+r)\right]},$$

where we have used the fact that $1 + e^{-\rho} + b + e^{-\rho}b = (1 + e^{-\rho})(1 + b)$.

Now to obtain an expression for first-period labor supply, substitute equation (8) into equation (6) to obtain

$$\ell_1 = 1 - \frac{b}{\dfrac{(1+e^{-\rho})(1+b)}{\left[w_1 + w_2/(1+r)\right]} w_1} = 1 - \frac{b\left[w_1 + w_2/(1+r)\right]}{(1+e^{-\rho})(1+b)w_1}.$$

Finally, dividing the top and bottom of the second term by w_1 yields

$$(9) \quad \ell_1 = 1 - \frac{b\left[1 + (w_2/w_1)(1/(1+r))\right]}{(1+e^{-\rho})(1+b)}.$$

Note that ℓ_1 is a function of the relative wage, w_2/w_1. Thus any change in w_1 and w_2 that leaves w_2/w_1 unchanged will leave ℓ_1 unchanged.

To obtain an expression for second-period labor supply, substitute equation (8) into equation (7) to obtain

$$\ell_2 = 1 - \frac{(1+r)e^{-\rho}b}{\dfrac{(1+e^{-\rho})(1+b)}{\left[w_1 + w_2/(1+r)\right]} w_2} = 1 - \frac{(1+r)e^{-\rho}b\left[w_1 + w_2/(1+r)\right]}{(1+e^{-\rho})(1+b)w_2}.$$

Finally, dividing the top and bottom of the second term by w_2 yields

$$(10) \quad \ell_2 = 1 - \frac{(1+r)e^{-\rho}b\left[(w_1/w_2) + 1/(1+r)\right]}{(1+e^{-\rho})(1+b)}.$$

Again, note that ℓ_2 is just a function of the relative wage, w_1/w_2. Thus any change in w_1 and w_2 that leaves w_1/w_2 unchanged will leave ℓ_2 unchanged.

(b) (i) The fact that the household has initial wealth of $Z > 0$ will not affect equation (4.23) in the text -- the Euler equation -- which relates consumption in one period to expectations of consumption the following period. The fact that the household has initial wealth does not change the marginal utility lost from reducing current consumption by a small amount today nor does it change the expected marginal utility gained by using the resulting greater wealth to increase consumption next period above what it otherwise would have been. That is, it does not affect the experiment by which we informally derived the Euler equation. The budget constraint, just as in the Ramsey model, only becomes important when determining the level of consumption each period.

(b) (ii) The result in part (a) will not continue to hold if the household has initial wealth. The new lifetime budget constraint is given by

$$(11)\quad c_1 + \frac{1}{1+r}c_2 = Z + w_1\ell_1 + \frac{1}{1+r}w_2\ell_2.$$

Clearly, this addition of a constant to lifetime wealth will not affect the four first-order conditions. Now take those first-order conditions, equations (4) through (7), and substitute them into this new budget constraint:

$$\frac{1}{\lambda} + \frac{e^{-\rho}(1+r)}{\lambda(1+r)} = Z + w_1\left[1 - \frac{b}{\lambda w_1}\right] + \frac{w_2}{1+r}\left[1 - \frac{(1+r)e^{-\rho}b}{\lambda w_2}\right].$$

Following the same algebra steps as in part (a) will now yield

$$(12)\quad \lambda = \frac{(1+e^{-\rho})(1+b)}{\left[Z + w_1 + w_2/(1+r)\right]}.$$

Now to obtain an expression for first-period labor supply, substitute equation (12) into equation (6) to obtain

$$\ell_1 = 1 - \frac{b}{\dfrac{(1+e^{-\rho})(1+b)}{\left[Z + w_1 + w_2/(1+r)\right]}w_1} = 1 - \frac{b\left[Z + w_1 + w_2/(1+r)\right]}{(1+e^{-\rho})(1+b)w_1}.$$

Finally, dividing the top and bottom of the second term by w_1 yields

$$(13)\quad \ell_1 = 1 - \frac{b\left[(Z/w_1) + 1 + (w_2/w_1)(1/(1+r))\right]}{(1+e^{-\rho})(1+b)}.$$

Taking the derivative of ℓ_1 with respect to w_1 -- imposing the condition that w_2/w_1 remains constant -- yields

$$\frac{\partial \ell_1}{\partial w_1} = \frac{bZ/w_1^2}{(1+e^{-\rho})(1+b)} > 0.$$

Thus a change in w_1, even if it is accompanied by a change in w_2 such that relative wages remain constant, does affect first-period labor supply. In fact, a rise in the first-period wage will increase first-period labor supply.

To obtain an expression for second-period labor supply, substitute equation (12) into equation (7) to obtain

$$\ell_2 = 1 - \frac{(1+r)e^{-\rho}b}{\dfrac{(1+e^{-\rho})(1+b)}{\left[Z + w_1 + w_2/(1+r)\right]}w_2} = 1 - \frac{(1+r)e^{-\rho}b\left[Z + w_1 + w_2/(1+r)\right]}{(1+e^{-\rho})(1+b)w_2}.$$

Finally, dividing the top and bottom of the second term by w_2 yields

(14) $\ell_2 = 1 - \dfrac{(1+r)e^{-\rho}b[(Z/w_2) + (w_1/w_2) + 1/(1+r)]}{(1+e^{-\rho})(1+b)}$.

Taking the derivative of ℓ_2 with respect to w_2 -- imposing the condition that w_1/w_2 remains constant -- yields

$$\frac{\partial \ell_2}{\partial w_2} = \frac{(1+r)e^{-\rho}bZ/w_2{}^2}{(1+e^{-\rho})(1+b)} > 0.$$

Thus a change in w_2, even if it is accompanied by a change in w_1 such that relative wages remain constant, does affect second-period labor supply. In fact, a rise in the second-period wage will increase second-period labor supply.

Problem 4.6

(a) The real interest rate is potentially random, so let $r = Er + \varepsilon$ where ε is a mean-zero random error. The individual wants to maximize expected utility as given by

(1) $U = \ln C_1 + E \ln C_2$,

and substituting in for C_2 yields

(2) $U = \ln C_1 + E \ln[(1+Er+\varepsilon)(Y_1 - C_1)]$.

Set the derivative of equation (2) with respect to C_1 equal to zero to obtain the first-order condition:

(3) $\partial U/\partial C_1 = 1/C_1 + E[(-1)(1+Er+\varepsilon)/(1+Er+\varepsilon)(Y_1 - C_1)] = 0$,

or simplifying

$$1/C_1 - E[1/(Y_1 - C_1)] = 0.$$

Since $1/(Y_1 - C_1)$ is not random, it is true that $E[1/(Y_1 - C_1)] = 1/(Y_1 - C_1)$ and thus after some simple algebra we have

(4) $C_1 = Y_1/2$.

In this case, the choice of C_1 is not affected by whether r is certain or not. Even if r is random, the individual simply consumes half of first-period income and saves the rest.

(b) Now the individual does not receive any first-period income but receives income Y_2 in period 2. So the individual's problem is to maximize expected utility as given by equation (1), subject to

(5) $C_1 = B_1$, and (6) $C_2 = Y_2 - (1+Er+\varepsilon)B_1 = Y_2 - (1+Er+\varepsilon)C_1$,

where B_1 represents the amount of borrowing the individual does in the first period. Substituting (6) into the expected utility function (1) yields

(7) $U = \ln C_1 + E \ln[Y_2 - (1+Er+\varepsilon)C_1]$.

Set the derivative of equation (7) with respect to C_1 equal to zero to find the first-order condition:

(8) $\partial U/\partial C_1 = 1/C_1 - E[(1+Er+\varepsilon)/C_2] = 0$.

Use the formula for the expected value of the product of 2 random variables --$E[XY] = E[X]E[Y] + \text{cov}(X,Y)$ -- to obtain

(9) $1/C_1 = (1+Er)\,E[1/C_2] + \text{cov}(1+Er+\varepsilon, 1/C_2)$.

The covariance term is positive. Intuitively, a higher ε means the individual has to pay more interest on her borrowing which forces her to have lower C_2 and thus higher $1/C_2$.

If r is not random -- so that $r = Er$ because $\varepsilon = 0$ always -- we have from equation (9)

$$1/C_1 = (1+Er)(1/C_2) = (1+Er)/[Y_2 - (1+Er)C_1] \;\Rightarrow\; Y_2 - (1+Er)C_1 = (1+Er)C_1,$$

and thus solving for C_1 yields

(10) $C_1 = Y_2/2(1+Er)$.

Now, from equation (9), in the case where r is random, we still have

$1/C_1 = E[1 + Er + \varepsilon]\, E[1/C_2] + \mathrm{cov}(1 + Er + \varepsilon, 1/C_2)$.

Since $1/C_2$ is a convex function of C_2, then by Jensen's inequality we have $E[1/C_2] > 1/E[C_2]$. In addition, because the covariance term is positive, we can write

$1/C_1 = (1 + Er)E[1/C_2] + \mathrm{cov}(1 + Er + \varepsilon, 1/C_2) > (1 + Er)[1/E[C_2]]$.

Substituting into this inequality the fact that $E[C_2] = Y_2 - (1 + Er)C_1$ yields

$1/C_1 > (1 + Er)/[Y_2 - (1 + Er)C_1] \;\Rightarrow\; Y_2 - (1 + Er)C_1 > (1 + Er)C_1 \;\Rightarrow\; 2(1 + Er)C_1 < Y_2$,

or simply

(11) $C_1 < Y_2/2(1 + Er)$.

Note from equation (10) that the right-hand side of (11) is the optimal choice of C_1 under certainty. Thus we have shown that if r becomes random with no change in the expected value of r, the optimal choice of C_1 becomes smaller. Essentially, if there is some uncertainty about how much interest the individual will have to pay in the second period, she is more cautious in her decision as to how much to borrow and consume in the first period.

Problem 4.7

(a) Imagine the household increasing its labor supply per member in period t by a small amount $\Delta\ell$. Suppose it then uses the resulting greater wealth to allow less labor supply per member in the next period and allowing for consumption per member to be the same in both periods as it otherwise would have been. If the household is behaving optimally, a marginal change of this type must leave expected lifetime utility unchanged.

Household utility and the instantaneous utility function of the representative member of the household are given by

(1) $U = \sum_{t=0}^{t=\infty} e^{-\rho t} u(c_t, 1 - \ell_t) N_t/H$, and (2) $u_t = \ln c_t + b\ln(1 - \ell_t)$.

From equations (1) and (2), the marginal disutility of working in period t is given by

(3) $-\partial U/\partial \ell_t = e^{-\rho t}(N_t/H)[b/(1 - \ell_t)]$.

Thus increasing labor supply per member by $\Delta\ell$ has a utility cost for the household of

Utility Cost $= e^{-\rho t}(N_t/H)[b/(1 - \ell_t)]\Delta\ell$

This change raises income per member in period t by $w_t\Delta\ell$. Note that the household has e^n times as many members in period t + 1 as in period t. Thus the increase in wealth per member in period t + 1 is $e^{-n}[(1 + r_{t+1})w_t\Delta\ell]$.

We need to determine how much this will allow labor supply per member in period t + 1 to fall, if the path of consumption is to be unaffected. In period t + 1, giving up one unit of labor per member costs w_{t+1} in lost income per member. Thus giving up $1/w_{t+1}$ units of labor per member means lost income of one per member. Or, giving up $[e^{-n}(1 + r_{t+1})w_t\Delta\ell]/w_{t+1}$ units of labor results in lost income per member of $e^{-n}(1 + r_{t+1})w_t\Delta\ell$, which is exactly equal to the extra wealth per member the household has from working more last period. Thus we have determined that labor supply per member can fall below what it otherwise would have been by the amount $[e^{-n}(1 + r_{t+1})w_t\Delta\ell]/w_{t+1}$ while still allowing consumption to be the same as it otherwise would have been. The expected utility benefit, as of period t, from this allowable drop in labor supply per member is

$$\text{Expected Utility Benefit} = E_t\left[e^{-\rho(t+1)}\frac{N_{t+1}}{H}\frac{b}{(1-\ell_{t+1})}\frac{e^{-n}(1+r_{t+1})w_t\Delta\ell}{w_{t+1}}\right].$$

Equating the costs and expected benefits yields

$$e^{-\rho t}\frac{N_t}{H}\frac{b}{(1-\ell_t)}\Delta\ell = E_t\left[e^{-\rho(t+1)}\frac{N_{t+1}}{H}\frac{b}{(1-\ell_{t+1})}\frac{e^{-n}(1+r_{t+1})w_t\Delta\ell}{w_{t+1}}\right].$$

Since $e^{-\rho(t+1)}(N_{t+1}/H)e^{-n}$ is not uncertain and since $N_{t+1} = N_t e^n$, this simplifies to

$$(4)\quad \frac{b}{(1-\ell_t)} = e^{-\rho}E_t\left[\frac{b(1+r_{t+1})w_t}{(1-\ell_{t+1})w_{t+1}}\right].$$

(b) Consider the household in period t. Suppose it reduces its current consumption per member by a small amount Δc and then uses the resulting greater wealth to increase consumption per member in the next period above what it otherwise would have been. The following equation, (4.23) in the text, gives the condition this experiment implies, assuming the household is behaving optimally:

$$(4.23)\quad \frac{1}{c_t} = e^{-\rho}E_t\left[\frac{1}{c_{t+1}}(1+r_{t+1})\right].$$

Now imagine the household increasing its labor supply per member in period t by a small amount $\Delta\ell$ and using the resulting income to increase its consumption in that period. The following equation, (4.26) in the text, gives the condition that this experiment implies, assuming that the household is behaving optimally:

$$(4.26)\quad \frac{c_t}{1-\ell_t} = \frac{w_t}{b}.$$

Solving for $1/c_t$ gives us

$$(4.26')\quad \frac{1}{c_t} = \frac{b}{(1-\ell_t)w_t}.$$

Note that equations (4.26) and (4.26') hold in every period. Thus for period t + 1, we can write

$$(4.26'')\quad \frac{1}{c_{t+1}} = \frac{b}{(1-\ell_{t+1})w_{t+1}}.$$

Substituting equations (4.26') and (4.26'') into equation (4.23) yields

$$\frac{b}{(1-\ell_t)w_t} = e^{-\rho}E_t\left[\frac{b(1+r_{t+1})}{(1-\ell_{t+1})w_{t+1}}\right].$$

Multiplying both sides by w_t, and since $E_t[w_t] = w_t$, we have

$$\frac{b}{(1-\ell_t)} = e^{-\rho}E_t\left[\frac{b(1+r_{t+1})w_t}{(1-\ell_{t+1})w_{t+1}}\right].$$

This is the same condition obtained from the experiment in part (a).

Problem 4.8

(a) To obtain the first-order condition or Euler equation, we can use the informal perturbation method. The experiment is to suppose the individual reduces period-t consumption by ΔC. She then uses the resulting greater wealth in period t + 1 to increase consumption above what it otherwise would have been. The utility cost in period t of doing so is given by

$$\text{Utility Cost} = [1/(1+\rho)]^t u'(C_t)\Delta C = [1/(1+\rho)]^t[1-2\theta C_t]\Delta C,$$

where we have used the instantaneous utility function, $u(C_t) = C_t - \theta C_t^2$, to calculate $u'(C_t)$.

The expected utility gain in period t + 1 from the above experiment is

$$\text{Exp. Utility Gain} = E_t\left[(1/(1+\rho))^{t+1} u'(C_{t+1})(1+A)\Delta C\right] = [1/(1+\rho)]^{t+1} E_t[1 - 2\theta C_{t+1}](1+A)\Delta C,$$

where A is the real interest rate. Finally, this simplifies to

$$\text{Exp. Utility Gain} = [1/(1+\rho)]^{t+1}[1 - 2\theta E_t[C_{t+1}]](1+A)\Delta C.$$

If the individual is optimizing, the utility cost from this perturbation must equal the expected utility gain:

$$[1/(1+\rho)]^t[1 - 2\theta C_t]\Delta C = [1/(1+\rho)]^{t+1}[1 - 2\theta E_t[C_{t+1}]](1+A)\Delta C,$$

or simply

$$1 - 2\theta C_t = [1/(1+\rho)](1+A)[1 - 2\theta E_t[C_{t+1}]].$$

Using the fact that $\rho = A$ and simplifying yields

(1) $C_t = E_t[C_{t+1}]$.

Consumption follows a random walk. The expected value of consumption next period is simply equal to today's actual realization of consumption.

(b) We will guess that consumption takes the form:

(2) $C_t = \alpha + \beta K_t + \gamma e_t$

Substitute equation (2) and the production function, $Y_t = AK_t + e_t$, into the capital-accumulation equation, $K_{t+1} = K_t + Y_t - C_t$, to obtain

$$K_{t+1} = K_t + AK_t + e_t - \alpha - \beta K_t - \gamma e_t,$$

or simply

(3) $K_{t+1} = -\alpha + (1 + A - \beta)K_t + (1-\gamma)e_t$.

(c) Substitute equation (2) and equation (2) lagged forward one period into the first-order condition, equation (1):

(4) $\alpha + \beta K_t + \gamma e_t = E_t[\alpha + \beta K_{t+1} + \gamma e_{t+1}]$.

Substituting equation (3) into equation (4) yields

$$\alpha + \beta K_t + \gamma e_t = \alpha + \beta E_t[-\alpha + (1 + A - \beta)K_t + (1-\gamma)e_t] + \gamma E_t[e_{t+1}].$$

Noting that $E_t[e_{t+1}] = E_t[\phi e_t + \varepsilon_{t+1}] = \phi e_t$, we can collect terms to obtain

(5) $\alpha + \beta K_t + \gamma e_t = \alpha(1-\beta) + \beta(1 + A - \beta)K_t + [\beta + \gamma(\phi - \beta)]e_t$.

In order for equation (5) to hold, we need the coefficients on K_t and e_t, as well as the constant term, to be the same on both sides. Equating the coefficients on K_t gives us

$$\beta = \beta(1 + A - \beta) \quad \Rightarrow \quad 1 = 1 + A - \beta,$$

or simply

(6) $\beta = A$.

Equating the coefficients on e_t gives us

$$\gamma = \beta + \gamma(\phi - \beta).$$

Using equation (6) and simplifying yields

$$\gamma(1 - \phi + A) = A,$$

or simply

(7) $\gamma = \dfrac{A}{1 - \phi + A}$.

Finally, equating the constant terms yields

$\alpha = \alpha(1-\beta)$.

Unless $\beta = A = 1$, this requires

(8) $\alpha = 0$.

Note that we are also ignoring the case of $\beta = 0$, $\gamma = 0$ and no restriction on α.

(d) Substituting equations (6) through (8) into the guess for consumption, equation (2), and the capital-accumulation equation, equation (3), yields

(9) $C_t = AK_t + \left(\dfrac{A}{1-\phi+A}\right)e_t$, and (10) $K_{t+1} = K_t + \left(\dfrac{1-\phi}{1-\phi+A}\right)e_t$.

To keep the analysis simple, and without loss of generality, we can assume that ε, and thus e, both equal 0 until some period t. In period t, there is a one-time, positive realization of $\varepsilon_t = 1-\phi+A$. From period t + 1 forward, $\varepsilon = 0$ again. In what follows, the change in a variable refers to the difference between its actual value and the value it would have had in the absence of the one-time shock (i.e. if ε and e had remained at 0 forever).

In period t, K_t is unaffected. From equation (10), we can see that K_t is determined by last period's capital stock and last period's realization of e. From the production function, $Y_t = AK_t + e_t$, we have

$\Delta Y_t = A\Delta K_t + \Delta e_t = 0 + (1-\phi+A)$.

Thus output in the period of the shock is higher by $(1-\phi+A)$. From equation (9), the change in consumption is given by

$$\Delta C_t = A\Delta K_t + \left(\frac{A}{1-\phi+A}\right)\Delta e_t = 0 + \left(\frac{A}{1-\phi+A}\right)(1-\phi+A) = A.$$

Thus consumption in the period of the shock is higher by A.

In period t + 1, even though ε_{t+1} is assumed to be 0 again, e_{t+1} is different than it would have been in the absence of the one-time shock due to the autoregressive form of the e's. More precisely

$\Delta e_{t+1} = \phi\Delta e_t = \phi(1-\phi+A)$.

From equation (10), the change in the capital stock is given by

$$\Delta K_{t+1} = \Delta K_t + \left(\frac{1-\phi}{1-\phi+A}\right)\Delta e_t = 0 + \left(\frac{1-\phi}{1-\phi+A}\right)(1-\phi+A) = (1-\phi).$$

Intuitively, last period, output rose by $(1-\phi+A)$ but consumption rose only by A. The rest of the increase in output -- $(1-\phi)$ -- was devoted to investment and hence the rise in this period's capital stock by an equal amount (we are assuming no depreciation). From the production function, $Y_{t+1} = AK_{t+1} + e_{t+1}$, the change in output is

$$\Delta Y_{t+1} = A\Delta K_{t+1} + \Delta e_{t+1} = A(1-\phi) + \phi(1-\phi) + A = A - \phi A + \phi + \phi A - \phi^2 = A + \phi(1-\phi).$$

From equation (9), the change in consumption is given by

$$\Delta C_{t+1} = A\Delta K_{t+1} + \left(\frac{A}{1-\phi+A}\right)\Delta e_{t+1} = A(1-\phi) + \left(\frac{A}{1-\phi+A}\right)\phi(1-\phi+A) = A - \phi A + \phi A = A.$$

Thus there are no further dynamics for consumption. It remains A higher than it would have been in the absence of the shock.

Similarly, we can calculate these changes for period t+2:

$\Delta e_{t+2} = \phi\Delta e_{t+1} = \phi^2(1-\phi+A)$,

$$\Delta K_{t+2} = \Delta K_{t+1} + \left(\frac{1-\phi}{1-\phi+A}\right)\Delta e_{t+1} = (1-\phi) + \left[\frac{(1-\phi)\phi(1-\phi+A)}{1-\phi+A}\right] = (1-\phi)+\phi(1-\phi) = 1-\phi^2,$$

$$\Delta Y_{t+2} = A\Delta K_{t+2} + \Delta e_{t+2} = A(1-\phi) + A\phi(1-\phi) + \phi^2(1-\phi+A) = A + \phi^2(1-\phi), \text{ and}$$

$$\Delta C_{t+2} = A\Delta K_{t+2} + \left(\frac{A}{1-\phi+A}\right)\Delta e_{t+2} = A(1-\phi)^2 + \left(\frac{A}{1-\phi+A}\right)\phi^2(1-\phi+A) = A - \phi^2 A + \phi^2 A = A.$$

The pattern can now be inferred. Suppose there is a one-time shock of $\varepsilon_t = 1-\phi+A$. In the period of the shock, consumption rises by A and permanently stays at that new level with no further dynamics. In addition, n periods after the shock, the change in output is

$$\Delta Y_{t+n} = A + \phi^n(1-\phi),$$

and the change in the capital stock is

$$\Delta K_{t+n} = 1-\phi^n.$$

The nature of the dynamics of Y and K depends upon the value of ϕ. In the special case in which it is equal to 0, so that there is no persistence in the technology shock, there are no further dynamics after period t + 1. The period after the shock, and in all those thereafter, capital is higher by one and output is higher by A.

For the case of $0 < \phi < 1$, the capital stock rises by $(1-\phi)$ the period after the shock. It then increases more each period until it asymptotically approaches its new long-run level that is one higher than it would have been in the absence of the shock. Output rises by $(1-\phi+A)$ the period of the shock. It then decreases each period until it asymptotically approaches its new long-run level that is A higher than it would have been in the absence of the shock.

For the case of $-1 < \phi < 0$, capital and output oscillate -- alternating above and below their new long-run levels in successive periods -- and gradually settle down to be one and A higher, respectively.

Problem 4.9

(a) To obtain the first-order condition or Euler equation, we can use the informal perturbation method. The experiment is to suppose the individual reduces period-t consumption by ΔC. She then uses the resulting greater wealth in period t + 1 to increase consumption above what it otherwise would have been. The utility cost in period t of doing so is given by

$$\text{Utility Cost} = [1/(1+\rho)]^t u'(C_t)\Delta C = [1/(1+\rho)]^t[1-2\theta(C_t+v_t)]\Delta C,$$

where the last step uses the instantaneous utility function, $u(C_t) = C_t - \theta(C_t + v_t)^2$, to calculate $u'(C_t)$.

The expected utility gain in period t + 1 from the above experiment is

$$\text{Exp. Gain} = E_t\left[(1/(1+\rho))^{t+1} u'(C_{t+1})(1+A)\Delta C\right] = [1/(1+\rho)]^{t+1} E_t[1-2\theta(C_{t+1}+v_{t+1})](1+A)\Delta C,$$

where A is the real interest rate. Now, since v is white noise, $E_t[v_{t+1}] = 0$ and thus

$$\text{Exp. Utility Gain} = [1/(1+\rho)]^{t+1}[1-2\theta E_t[C_{t+1}]](1+A)\Delta C.$$

If the individual is optimizing, the utility cost from this perturbation must equal the expected utility gain:

$$[1/(1+\rho)]^t[1-2\theta(C_t+v_t)]\Delta C = [1/(1+\rho)]^{t+1}[1-2\theta E_t[C_{t+1}]](1+A)\Delta C,$$

or simply

$$1-2\theta\left(C_t+v_t\right)=[1/(1+\rho)](1+A)[1-2\theta E_t[C_{t+1}]].$$

Using the fact that $\rho = A$ and simplifying yields

(1) $C_t + v_t = E_t[C_{t+1}]$.

(b) We will guess that consumption takes the form

(2) $C_t = \alpha + \beta K_t + \gamma v_t$.

Substitute equation (2) and the production function, $Y_t = AK_t$, into the capital-accumulation equation, $K_{t+1} = K_t + Y_t - C_t$, to obtain

$$K_{t+1} = K_t + AK_t - \alpha - \beta K_t - \gamma v_t,$$

or simply

(3) $K_{t+1} = -\alpha + (1 + A - \beta)K_t - \gamma v_t$.

(c) Substitute equation (2) and equation (2) lagged forward one period into the first-order condition, equation (1):

$$\alpha + \beta K_t + \gamma v_t + v_t = E_t[\alpha + \beta K_{t+1} + \gamma v_{t+1}].$$

Noting that $E_t[V_{t+1}] = 0$, we have

(4) $\alpha + \beta K_t + (\gamma + 1)v_t = \alpha + \beta E_t[K_{t+1}]$.

Substitute equation (3) into equation (4). Note that we can drop the expectations operator since K_{t+1} is a function of K_t and v_t which are both known at time t and thus we have

$$\alpha + \beta K_t + (\gamma + 1)v_t = \alpha + \beta[-\alpha + (1 + A - \beta)K_t - \gamma v_t].$$

Simplifying yields

(5) $\alpha + \beta K_t + (\gamma + 1)v_t = \alpha(1 - \beta) + \beta(1 + A - \beta)K_t - \beta\gamma v_t$.

Clearly, in order for equation (5) to hold, we need the coefficients on K_t, v_t ,and the constant term to be the same on both sides. That is, we need

$$\beta = \beta(1 + A - \beta) \;\Rightarrow\; 1 = 1 + A - \beta \;\Rightarrow\; \beta = A, \; (6)$$

$$\gamma + 1 = -\beta\gamma \;\Rightarrow\; \gamma(1 + \beta) = -1 \;\Rightarrow\; \gamma = -1/(1 + \beta) \;\Rightarrow\; \gamma = -1/(1 + A), \; (7)$$

$$\alpha(1 - \beta) = \alpha \;\Rightarrow\; \alpha(1 - A) = \alpha \;\Rightarrow\; \alpha = 0. \; (8)$$

There is another set of parameter values that satisfies equation (5) which is $\beta = 0$, $\gamma = -1$, and no restriction on α. This second solution is economically unappealing, however, since $\beta = 0$ implies that consumption does not depend on the capital stock. This is not realistic since consumption depends on output which in turn is determined by the capital stock. Thus we can, on economic grounds, ignore this second solution.

(d) Substituting equations (6), (7) and (8) into the guess for consumption, equation (2), and the capital-accumulation equation, equation (3), yields

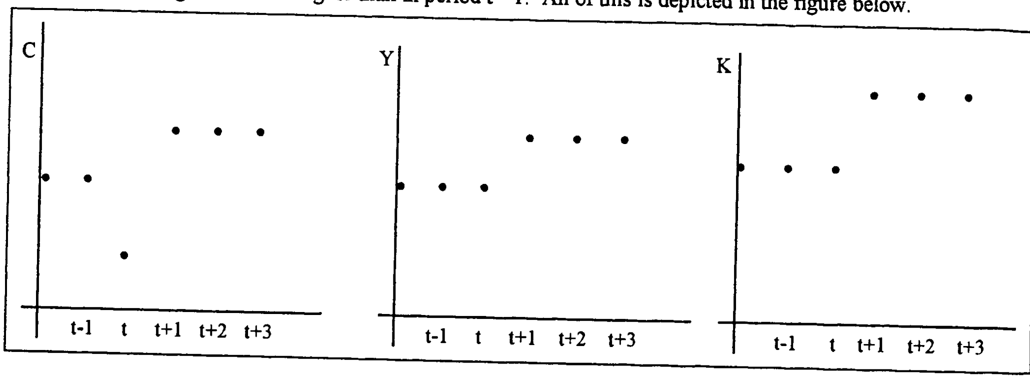
(9) $C_t = AK_t - [1/(1 + A)]v_t$, and (10) $K_{t+1} = K_t + [1/(1 + A)]v_t$.

Without loss of generality, we can assume that $v = 0$ until some period t when there is a one-time positive realization of v_t. To keep the analysis simple, assume that $v_t = (1 + A)$. From period t + 1 forward, $v = 0$ again.

In period t, K_t is unaffected. It is determined by last period's capital stock and last period's saving. From the production function, $Y_t = AK_t$, Y_t is unaffected since K_t is unaffected. From equation (9), we can see that consumption in period t, C_t, is lower by $[1/(1 + A)]v_t = [1/(1 + A)](1 + A) = 1$.

In period t + 1, we can see from equation (10) that K_{t+1} is higher by $[1/(1+A)]v_t = [1/(1+A)](1+A) = 1$. Intuitively, last period's drop in consumption by one, with unchanged output, meant an increase in saving of one. This in turn means an increase in this period's capital stock by one. Through the production function, since K_{t+1} is higher by one, output is higher by A. Finally, with v_{t+1} assumed to be 0, then since K_{t+1} is higher by one, C_{t+1} must be higher than it was in period t - 1 (before the shock) by A. This last fact can be seen from equation (9).

From period t + 2 forward, assuming v = 0 forever, there will be no further dynamics. K stays at its new higher level: one higher than in period t - 1. Y stays at its new higher level: A higher than in period t - 1. C stays at its new higher level: A higher than in period t - 1. All of this is depicted in the figure below.



Problem 4.10

(a) From the Solow, Ramsey and Diamond models it is clear that on the balanced growth path without shocks, the growth rates of Y, K and C are all equal to n + g. In addition, the growth rate of w is g, the growth rate of L is n, and the growth rates of ℓ and r are zero. Note that given the logarithmic structure here, "growth rate" means the change in the logarithm of the variable. That is, the fact that the growth rate of K is n + g means that $\ln(K_{t+1}) - \ln(K_t) = n + g$.

Dividing both sides of the production function, $Y_t = K_t^{\alpha}[A_tL_t]^{1-\alpha}$, by A_tL_t yields

$Y_t/A_tL_t = K_t^{\alpha}[A_tL_t]^{-\alpha} = [K_t/A_tL_t]^{\alpha}$.

Since y* and k* are the balanced-growth-path values of Y/AL and K/AL respectively, we have

(1) $y^* = k^{*\alpha}$.

Similarly, dividing both sides of the capital-accumulation equation, $K_{t+1} = K_t + Y_t - C_t - G_t - \delta K_t$, by A_tL_t gives us

$$\frac{K_{t+1}}{A_tL_t} = \frac{K_t}{A_tL_t} + \frac{Y_t}{A_tL_t} - \frac{C_t}{A_tL_t} - \frac{G_t}{A_tL_t} - \frac{\delta K_t}{A_tL_t}.$$

Using the fact that $K_{t+1} = e^n e^g K_t$ on the balanced growth path and thus that $K_{t+1}/A_tL_t = e^n e^g K_t/A_tL_t$, as well as the notation given in the question yields

(2) $e^n e^g k^* = k^* + y^* - c^* - G^* - \delta k^*$

Dividing both sides of the equation giving the real wage, $w_t = (1-\alpha)[K_t/A_tL_t]^{\alpha}A_t$, by A_t yields

$w_t/A_t = (1-\alpha)[K_t/A_tL_t]^{\alpha}$.

Denoting the value of w/A on the balanced growth path as w* gives us

(3) $w^* = (1-\alpha)k^{*\alpha}$.

From equation (4.4) in the text giving the real interest rate, we have on a balanced growth path

(4) $r^* = \alpha k^{*-(1-\alpha)} - \delta$.

We need to transform textbook equation (4.26), which relates the trade-off between current consumption and current labor supply, into an expression concerning the balanced growth path without shocks. Note that in equation (4.26), $c_t/(1 - \ell_t) = w_t/b$, c is consumption per person, C/N. We are interested in c* which is consumption per unit of effective labor, C/AL. Since C/N = (C/AL)(L/N)A, on the balanced growth path it is true that $c = c^*\ell^*A$. Using this fact and dividing both sides of equation (4.26) by A, we obtain

$$\frac{c^*\ell^*A/A}{(1-\ell^*)} = \frac{w/A}{b}.$$

Since $w^* = W/A$, we have

(5) $$\frac{c^*\ell^*}{(1-\ell^*)} = \frac{w^*}{b}.$$

Finally, to transform textbook equation (4.23), which relates the tradeoff between current and future consumption, first eliminate the expectations term since there is no uncertainty without any shocks. Then multiply both sides of equation (4.23), $1/c_t = e^{-\rho}E_t\,[(1 + r_{t+1})/c_{t+1}]$, by c_{t+1}:

$$c_{t+1}/c_t = e^{-\rho}(1 + r_{t+1}).$$

On the balanced growth path, consumption per person grows at rate g and thus $c_{t+1} = c_t e^g$ or $c_{t+1}/c_t = e^g$. Thus we have

(6) $1 + r^* = e^{\rho+g}$.

Equations (1) - (6) are six equations in the following six variables: y*, k*, c*, w*, ℓ^*, and r*.

(b) We need to assume the following parameter values: $\alpha = 1/3$, $g = 0.005$, $n = 0.0025$, $\delta = 0.025$, $r^* = 0.015$, and $\ell^* = 1/3$. Note that these are quarterly values for n, g and r*.

From equation (4), we can obtain an expression for capital per unit of effective labor on the balanced growth path, k*:

$$k^* = [\alpha/(r^* + \delta)]^{1/(1-\alpha)}.$$

Substituting for the values given yields

$$k^* = [(1/3)/(0.015 + 0.025)]^{1/(1-1/3)} \quad \Rightarrow \quad k^* = 24.0563.$$

Substituting this value for k* into equation (1) gives us a value for quarterly output per unit of effective labor on the balanced growth path:

$$y^* = k^{*\alpha} = (24.0563)^{1/3} \quad \Rightarrow \quad y^* = 2.8868.$$

We are told that the ratio of government purchases to output on the balanced growth path is (G/Y)* = 0.2. This means that:

$$[G/AL]/[Y/AL] = 0.2 \quad \Rightarrow \quad G/AL \equiv G^* = (0.2)(2.8868) \quad \Rightarrow \quad G^* = 0.5774.$$

From equation (2), we can solve for consumption per unit of effective labor on the balanced growth path, c*:

$$c^* = k^* + y^* - G^* - \delta k^* - e^n e^g k^* = (1 - \delta - e^n e^g)k^* + y^* - G^*.$$

Substituting in for the values given yields:

$$c^* = (1 - 0.025 - e^{0.0025}e^{0.005})(24.0563) + 2.8868 - 0.5774 \quad \Rightarrow \quad c^* = 1.5269.$$

It is then straightforward to use these values for c* and y* to solve for the share of output devoted to consumption on the balanced growth path:

$$C/Y = [C/AL]/[Y/AL] \equiv c^*/y^* = 1.5269/2.8868 = 0.5289,$$

and thus consumption's share in output is approximately 53%. Since output is devoted to consumption, investment, or government purchases, we know that

$I/Y = 1 - C/Y - G/Y = 1 - 0.5289 - 0.2 = 0.2711,$

and thus investment's share in output is roughly 27%. Compared to actual figures for the U.S. this is giving slightly too much weight to investment and slightly too little weight to consumption. Finally, the implied ratio of capital to annual output on the balanced growth path is

$K/4Y = [K/AL]/[4Y/AL] \equiv k^*/4y^* = 24.0563/[(4)(2.8868)] = 2.083.$

Problem 4.11

Before invoking the simplifying assumptions, the model here is the RBC model with no government and 100% depreciation, given by

(1) $Y_t = K_t^{\alpha}[A_t L_t]^{1-\alpha}$ (2) $K_{t+1} = Y_t - C_t$ (3) $\ln A_t = \overline{A} + gt + \widetilde{A}_t$

(4) $\widetilde{A}_t = \rho_A \widetilde{A}_{t-1} + \varepsilon_{A,t}$ (5) $\ln N_t = \overline{N} + nt$ (6) $u_t = \ln c_t + b\ln(1 - \ell_t)$.

In this question, we are simplifying by assuming $n = g = \overline{A} = \overline{N} = 0$. This results in the following adjustments to the model. Population is given by

(5 ') $\ln N_t = 0 \quad \Rightarrow \quad N_t = 1.$

We have normalized the population to one and thus ℓ_t, labor supply per person, will be the same as total labor supply, L_t. Thus we can rewrite the production function as

(1 ') $Y_t = K_t^{\alpha}[A_t \ell_t]^{1-\alpha}$.

Finally, with respect to technology, since g and $\overline{A}$ are equal to 0, we have $\ln A_t = \widetilde{A}_t$ and using equation (4) to rewrite this yields

(3 ') $\ln A_t = \rho_A \ln A_{t-1} + \varepsilon_{A,t}$.

(a) Define the value function at time t as

(7) $$V_t = \max E_t\left[\sum_{s=t}^{\infty} e^{-\rho(s-t)}[\ln C_s + b\ln(1-\ell_s)]\right].$$

Since we are solving the social planner's problem, the maximization is subject to the production function, equation (1 '), the capital-accumulation equation (2) and the technology equation (3 '). Thus the value function at time t is the expected present value of lifetime utility, from time t forward, evaluated at all the optimal choices of consumption and labor supply. The technique here is that we can reduce what looks like a complicated multiperiod problem down to a two-period problem. This is due to the fact that the value function must satisfy Bellman's Equation, which is given by

(8) $$V_t(K_t, A_t) = \max_{C,\ell}\{\ln C_t + b\ln(1-\ell_t) + e^{-\rho}E_t[V_{t+1}(K_{t+1}, A_{t+1})]\}.$$

Equation (8) says that the value function at time t is equal to utility at time t, evaluated at the optimal C_t and ℓ_t, plus the discounted expected value as of time t of next period's value function. That is, the expected value of maximized lifetime utility is maximized lifetime utility "today" plus "today's" expectation of maximized lifetime utility from "tomorrow" on, appropriately discounted.

(b) We will guess that the value function is of the form

(9) $V_t(K_t, A_t) = \beta_0 + \beta_K \ln K_t + \beta_A \ln A_t$.

Substituting this guess into equation (8), the Bellman equation, yields

(10) $$V_t(K_t, A_t) = \max_{C,\ell}\{\ln C_t + b\ln(1-\ell_t) + e^{-\rho}E_t[\beta_0 + \beta_K \ln K_{t+1} + \beta_A \ln A_{t+1}]\}.$$

Taking logs and then expectations of both sides of equation (2), the capital-accumulation equation, yields

(11) $E_t[\ln K_{t+1}] = E_t[\ln(Y_t - C_t)] = \ln(Y_t - C_t)$,
where we have used the fact that Y_t and C_t are both known as of time t. Taking expectations of both sides of equation (3 ') yields
(12) $E_t[\ln A_{t+1}] = \rho_A \ln A_t$,
where we have used the fact that the ε shocks have mean zero. Substituting equations (11) and (12) into equation (10) yields

$$(13)\quad V_t(K_t, A_t) = \max_{C,\ell}\left\{\ln C_t + b\ln(1-\ell_t) + e^{-\rho}\left[\beta_0 + \beta_K \ln(Y_t - C_t) + \beta_A \rho_A \ln A_t\right]\right\}.$$

The first-order condition for C_t is

$$0 = 1/C_t + e^{-\rho}\beta_K(-1)/Y_t - C_t \quad \Rightarrow \quad 1/C_t = e^{-\rho}\beta_K\big/(Y_t - C_t),$$

$$(14)\quad Y_t - C_t = e^{-\rho}\beta_K C_t \quad \Rightarrow \quad C_t(1 + e^{-\rho}\beta_K) = Y_t \quad \Rightarrow \quad C_t = \left[1\big/(1 + e^{-\rho}\beta_K)\right]Y_t. \quad (14\,')$$

Thus the ratio of consumption to output is given by
(15) $C_t/Y_t = 1/(1 + e^{-\rho}\beta_K)$.
So clearly, the ratio of consumption to output does not depend on K_t or A_t.

(c) The first-order condition for ℓ_t (noting that $L_t = \ell_t$) is

$$0 = -b/(1 - \ell_t) + [e^{-\rho}\beta_K/(Y_t - C_t)](1 - \alpha)K_t^{\alpha} A_t^{1-\alpha} \ell_t^{-\alpha}.$$

Simplifying yields
(16) $b/(1 - \ell_t) = [e^{-\rho}\beta_K/(Y_t - C_t)](1 - \alpha)(Y_t/\ell_t)$.
Substituting equation (14) into equation (16) yields
(17) $b/(1 - \ell_t) = [e^{-\rho}\beta_K/e^{-\rho}\beta_K C_t](1 - \alpha)(Y_t/\ell_t) = (Y_t/C_t)[(1 - \alpha)/\ell_t]$.
Substituting equation (15) into equation (17) yields

$$b/(1 - \ell_t) = (1 - \alpha)(1 + e^{-\rho}\beta_K)/\ell_t \quad \Rightarrow \quad b\ell_t = (1 - \ell_t)(1 - \alpha)(1 + e^{-\rho}\beta_K).$$

Further simplification allows us to obtain

$$\ell_t[(1 - \alpha)(1 + e^{-\rho}\beta_K) + b] = (1 - \alpha)(1 + e^{-\rho}\beta_K),$$

and thus

$$(18)\quad \ell_t = \frac{(1-\alpha)}{(1-\alpha) + \left[b/(1 + e^{-\rho}\beta_K)\right]}.$$

Thus ℓ_t, labor supply per person, does not depend on K_t or A_t either. In addition, with some simple algebra, it is possible to solve for optimal leisure, an expression which will be useful later on:

$$(19)\quad (1 - \ell_t) = b\big/\left[(1-\alpha)(1 + e^{-\rho}\beta_K) + b\right].$$

(d) Now take these optimal choices of consumption and leisure, as well as the production function, and substitute them all into the value function. It will turn out that the original guess that the value function is loglinear in capital and technology is valid.

Formally, substitute equations (14), (14 ') and (19) into equation (13) to obtain

$$(20)\quad \begin{aligned} V_t(K_t, A_t) = {} & \ln\left[Y_t\big/(1 + e^{-\rho}\beta_K)\right] + b\ln\left\{b\big/\left[(1-\alpha)(1 + e^{-\rho}\beta_K) + b\right]\right\} + \\ & e^{-\rho}\left\{\beta_0 + \beta_K \ln\left[e^{-\rho}\beta_K Y_t\big/(1 + e^{-\rho}\beta_K)\right] + \beta_A \rho_A \ln A_t\right\}. \end{aligned}$$

Substituting equation (1 '), the production function, into equation (20) and expanding some of the logarithms yields

$$
\begin{aligned}
V_t(K_t, A_t) = {} & \alpha \ln K_t + (1-\alpha)\ln A_t + (1-\alpha)\ln \ell_t - \ln(1+e^{-\rho}\beta_K) + b\ln\left\{b\big/\left[(1-\alpha)(1+e^{-\rho}\beta_K)+b\right]\right\} \\
(21) \qquad & + e^{-\rho}\beta_0 + e^{-\rho}\beta_K\left\{\ln(e^{-\rho}\beta_K) - \ln(1+e^{-\rho}\beta_K) + \alpha\ln K_t + (1-\alpha)\ln A_t + (1-\alpha)\ln \ell_t\right\} \\
& + e^{-\rho}\beta_A \rho_A \ln A_t .
\end{aligned}
$$

There is no need to substitute in for ℓ_t since we already know that it does not depend on K_t or A_t and it is really the coefficients on $\ln K_t$ and $\ln A_t$ that we are interested in. It is possible to rewrite equation (21) as

(22) $V_t(K_t, A_t) = \beta_0' + \beta_K' \ln K_t + \beta_A' \ln A_t$,

where $\beta_0' \equiv$ terms that do not depend upon K_t or A_t, $\beta_K' \equiv \alpha(1 + e^{-\rho}\beta_K)$ and $\beta_A' \equiv (1 - \alpha)(1 + e^{-\rho}\beta_K) + e^{-\rho}\beta_A \rho_A$.

(e) In order for our original guess to be correct, we need the coefficient on $\ln K_t$ in equation (22) to be equal to β_K. That is, we need $\beta_K = \alpha(1 + e^{-\rho}\beta_K)$. Solving for β_K yields

(23) $\beta_K = \alpha/(1 - \alpha e^{-\rho})$.

We also need the coefficient on $\ln A_t$ in equation (22) to be equal to β_A. That is, we need

(24) $\beta_A = (1 - \alpha)(1 + e^{-\rho}\beta_K) + e^{-\rho}\beta_A \rho_A$.

Substituting the expression for β_K, equation (23), into equation (24) yields

$\beta_A = (1 - \alpha)\{1 + [\alpha e^{-\rho}/(1 - \alpha e^{-\rho})]\} + e^{-\rho}\beta_A \rho_A$.

Collecting the terms in β_A and simplifying yields

$\beta_A(1 - e^{-\rho}\rho_A) = (1 - \alpha)/(1 - \alpha e^{-\rho})$,

and thus finally

(25) $\beta_A = (1 - \alpha)/[(1 - \alpha e^{-\rho})(1 - \rho_A e^{-\rho})]$.

(f) Substitute the value of β_K that was derived above into the earlier solutions for Y_t/C_t and ℓ_t. That is, substitute equation (23) into equation (15):

$$\frac{C_t}{Y_t} = \frac{1}{1+\left[\alpha e^{-\rho}\big/\left(1-\alpha e^{-\rho}\right)\right]} = \frac{1}{\left[\left(1-\alpha e^{-\rho}+\alpha e^{-\rho}\right)\big/\left(1-\alpha e^{-\rho}\right)\right]},$$

or simply

(26) $C_t/Y_t = 1 - \alpha e^{-\rho}$.

This is the same ratio of consumption to output that was obtained by deriving the competitive solution to this model, with the additional assumption of $n = 0$ incorporated.

Similarly for labor supply, substitute equation (23) into equation (18):

$$\ell_t = \frac{(1-\alpha)}{(1-\alpha) + b\big/\left[1+\left(\alpha e^{-\rho}/(1-\alpha e^{-\rho})\right)\right]}.$$

We have already worked on an expression like the one in the denominator just above and thus

(27) $\ell_t = (1 - \alpha)/[(1-\alpha) + b(1 - \alpha e^{-\rho})]$.

This expression for labor supply is the same as the one that was obtained when deriving the competitive solution to the model, with the additional assumption of $n = 0$ incorporated.

Problem 4.12

The derivation of a constant saving rate, $s_t = \hat{s}$, and a constant labor supply per person, $\ell_t = \hat{\ell}$, did not depend on the behavior of technology. As the text points out, it is the combination of logarithmic utility, Cobb-Douglas production, and 100% depreciation that causes movements in both technology and capital to have offsetting income and substitution effects on saving. These assumptions allowed us to come up with an expression for the saving rate that did not depend on technology which is equation (4.31) in the text. Once a constant saving rate is established, technology plays no role in the derivation that labor supply is also constant. That relied on the form of the utility function and on Cobb-Douglas production. The latter was necessary so that labor's share in income was a constant.

Problem 4.13

(a) Imagine the household increasing its labor supply per member in period t by a small amount $\Delta\ell$ and using the resulting income to increase its consumption in that period. Household utility and the instantaneous utility function are given by

(1) $U = \sum_{t=0}^{t=\infty} e^{-\rho t} u(c_t, 1-\ell_t) N_t / H$, and (2) $u_t = \ln c_t + b(1-\ell_t)^{1-\gamma}/(1-\gamma)$.

From equations (1) and (2), the marginal disutility of working in period t is given by

(3) $-\partial U/\partial\ell_t = e^{-\rho t}(N_t/H)b(1-\ell_t)^{-\gamma}$.

Thus increasing labor supply per member by $\Delta\ell$ has a utility cost for the household of

Utility Cost $= e^{-\rho t}(N_t/H)b(1-\ell_t)^{-\gamma}\Delta\ell$.

Since the change raises consumption by $w_t\Delta\ell$, it has a utility benefit for the household of

Utility Benefit $= e^{-\rho t}(N_t/H)(1/c_t)w_t\Delta\ell$.

If the household is behaving optimally, a marginal change of this type must leave expected lifetime utility unchanged. Thus the utility cost must equal the utility benefit; equating these two expressions gives us

$$e^{-\rho t}\frac{N_t}{H}\frac{b}{(1-\ell_t)^{\gamma}}\Delta\ell = e^{-\rho t}\frac{N_t}{H}\frac{1}{c_t}w_t\Delta\ell,$$

or simply

$$(4)\quad \frac{c_t}{(1-\ell_t)^{\gamma}} = \frac{w_t}{b}.$$

Equation (4) relates current leisure and consumption given the wage.

(b) With this change to the model, the saving rate will still be constant. The derivation of a constant saving rate began from the condition relating current consumption to expectations of future consumption, $1/c_t = e^{-\rho}E_t[(1+r_{t+1})/c_{t+1}]$, which is not affected by this change to the instantaneous utility function. The rest of the derivation did depend on Cobb-Douglas production and 100% depreciation but not on how utility was affected by leisure. Thus equation (4.33) in the text, $\hat{s} = \alpha e^{n-\rho}$, continues to hold.

(c) Leisure per person is still constant as well. Note that c_t in equation (4) is consumption per person. It can be written as $c_t \equiv C_t/N_t = (1-\hat{s})Y_t/N_t$ where $\hat{s}$ is the constant saving rate. Taking logs of equation (4) and substituting for c_t yields

(5) $\ln[(1-\hat{s})Y_t/N_t] - \gamma\ln(1-\ell_t) = \ln w_t - \ln b$.

Since the production function is Cobb-Douglas, labor's share of output is $(1-\alpha)$ and thus

$w_t \ell_t N_t = (1 - \alpha)Y_t$. Note that we have used $L_t \equiv \ell_t N_t$; the total amount of labor, L_t, is equal to labor supply per person, ℓ_t, multiplied by the number of people, N_t. Rearranging, we have $w_t = (1 - \alpha)Y_t / \ell_t N_t$. Substituting this fact into equation (5) yields

$$\ln(1 - \hat{s}) + \ln Y_t - \ln N_t - \gamma \ln(1 - \ell_t) = \ln(1 - \alpha) + \ln Y_t - \ln \ell_t - \ln N_t - \ln b.$$

Canceling terms and rearranging gives us

(6) $\ln \ell_t - \gamma \ln(1 - \ell_t) = \ln(1 - \alpha) - \ln(1 - \hat{s}) - \ln b$.

Taking the exponential function of both sides of equation (6) yields

(7) $$\frac{\ell_t}{(1-\ell_t)^{\gamma}} = \frac{(1-\alpha)}{b(1-\hat{s})}.$$

Equation (7) implicitly defines leisure per person as a function of the constants γ, α, b, and $\hat{s}$. Thus leisure per person is also a constant.

Problem 4.14

(a) Taking logs of the production function, $Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$, gives us

(1) $\ln Y_t = \alpha \ln K_t + (1 - \alpha)(\ln A_t + \ln L_t)$.

In the model of Section 4.5 it was shown that labor supply and the saving rate were constant so that $L_t = \hat{\ell} N_t$ and $K_t = \hat{s} Y_{t-1}$. Thus we can write

(2) $\ln Y_t = \alpha \ln \hat{s} + \alpha \ln Y_{t-1} + (1 - \alpha)(\ln A_t + \ln \hat{\ell} + \ln N_t)$.

Finally we can use the equations for the evolution of technology and population, $\ln A_t = \overline{A} + gt + \widetilde{A}_t$ and $\ln N_t = \overline{N} + nt$, to obtain

(3) $\ln Y_t = \alpha \ln \hat{s} + \alpha \ln Y_{t-1} + (1-\alpha)(\overline{A} + gt) + (1-\alpha)\widetilde{A}_t + (1-\alpha)[\ln \hat{\ell} + \overline{N} + nt]$.

We need to solve for the path that log output would settle down to if there were no technology shocks. Start by subtracting (n + g)t from both sides of equation (3):

(4) $\ln Y_t - (n+g)t = \alpha \ln \hat{s} + \alpha \ln Y_{t-1} - \alpha(n+g)t + (1-\alpha)[\overline{A} + \ln \hat{\ell} + \overline{N} + \widetilde{A}_t]$.

Add and subtract $\alpha(n + g)$ to the right-hand side of equation (4) to yield

(5) $\ln Y_t - (n+g)t = \alpha \ln \hat{s} + (1-\alpha)[\overline{A} + \ln \hat{\ell} + \overline{N}] - \alpha(n+g) + \alpha[\ln Y_{t-1} - (n+g)(t-1)] + (1-\alpha)\widetilde{A}_t$.

Now define $Q \equiv \alpha \ln \hat{s} + (1-\alpha)[\overline{A} + \ln \hat{\ell} + \overline{N}] - \alpha(n+g)$ and use this to rewrite equation (5) as

(6) $\ln Y_t - (n+g)t = Q + \alpha[\ln Y_{t-1} - (n+g)(t-1)] + (1-\alpha)\widetilde{A}_t$.

On a balanced growth path with no shocks to technology, the $\widetilde{A}$'s are uniformly 0. In addition, we know that output will simply grow at rate n + g. With this logarithmic structure that means $\ln Y_t - \ln Y_{t-1} = n + g$ or $\ln Y_{t-1} = \ln Y_t - (n + g)$. Substituting these facts into equation (6) yields

$$\ln Y_t - (n + g)t = Q + \alpha[\ln Y_t - (n + g) - (n + g)(t - 1)] = Q + \alpha[\ln Y_t - (n + g)t].$$

Further simplification yields

$$[\ln Y_t - (n + g)t](1 - \alpha) = Q,$$

or simply

(7) $\ln Y_t^* = Q/(1 - \alpha) + (n + g)t$.

Equation (7) gives an expression for $\ln Y_t^*$, the path that log output would settle down to if there were never any technology shocks.

(b) By definition, $\widetilde{Y}_t \equiv \ln Y_t - \ln Y_t^*$, where $\ln Y_t^*$ is the path found in part (a). Thus $\widetilde{Y}_t$ gives us the difference between what log output actually is in any period and what it would have been in the complete absence of any technology shocks. Substituting for $\ln Y_t^*$ from equation (7) yields

(8) $\tilde{Y}_t = \ln Y_t - Q/(1 - \alpha) - (n + g)t$.

Note that equation (8) holds every period and so we can write

(9) $\tilde{Y}_{t-1} = \ln Y_{t-1} - Q/(1 - \alpha) - (n + g)(t - 1)$.

Multiplying both sides of equation (9) by α and solving for $\alpha\ln Y_{t-1}$ yields

(10) $\alpha\ln Y_{t-1} = \alpha\tilde{Y}_{t-1} + [\alpha/(1 - \alpha)]Q + \alpha(n + g)(t - 1)$.

Substituting equation (10) into equation (3) and then substituting the resulting expression into equation (8) yields

$$\text{(11)}\quad \begin{aligned} \tilde{Y}_t &= \alpha\ln\hat{s} + \alpha\tilde{Y}_{t-1} + [\alpha/(1-\alpha)]Q + \alpha(n+g)(t-1) \\ &\quad + (1-\alpha)[\bar{A} + gt + \tilde{A}_t + \ln\hat{\ell} + \bar{N} + nt] - Q/(1-\alpha) - (n+g)t. \end{aligned}$$

Simplification yields

(12) $\tilde{Y}_t = \alpha(n + g)t + (1 - \alpha)(n + g)t - (n + g)t + \alpha\tilde{Y}_{t-1} + (1 - \alpha)\tilde{A}_t$,

and thus finally

(13) $\tilde{Y}_t = \alpha\tilde{Y}_{t-1} + (1 - \alpha)\tilde{A}_t$.

Equation (13) is identical to equation (4.40) in the text.

Problem 4.15

(a) (i) The equation of motion for capital is given by

(1) $K_{t+1} = K_t + Y_t - C_t - G_t - \delta K_t$,

or substituting the production function into equation (1), we have

(2) $K_{t+1} = K_t + K_t^{\alpha}(A_t L_t)^{1-\alpha} - C_t - G_t - \delta K_t$.

Using equation (1), $\partial\ln K_{t+1}/\partial\ln K_t$ (holding A_t, L_t, C_t, and G_t fixed) is

$$\frac{\partial\ln K_{t+1}}{\partial\ln K_t} = \frac{\partial K_{t+1}}{\partial K_t}\frac{K_t}{K_{t+1}} = \left[1 + \frac{\partial Y_t}{\partial K_t} - \delta\right]\frac{K_t}{K_{t+1}}.$$

By definition, since factors are paid their marginal products, the real interest rate is $r_t = \partial Y_t/\partial K_t - \delta$ and thus

$$\text{(3)}\quad \frac{\partial\ln K_{t+1}}{\partial\ln K_t} = (1 + r_t)\frac{K_t}{K_{t+1}}.$$

(a) (ii) On the balanced growth path without shocks, capital grows at rate n + g, so that $K_{t+1} = e^{n+g}K_t$. In addition, using r* to denote the balanced-growth-path value of the real interest rate, equation (3) can be rewritten as

$$\left.\frac{\partial\ln K_{t+1}}{\partial\ln K_t}\right|_{bgp} = (1 + r^*)\frac{K_t}{e^{n+g}K_t} = \frac{1 + r^*}{e^{n+g}}.$$

(b) Using equation (2), $\partial\ln K_{t+1}/\partial\ln A_t$ (holding K_t, C_t, G_t and L_t fixed) is

$$\text{(4)}\quad \frac{\partial\ln K_{t+1}}{\partial\ln A_t} = \frac{\partial K_{t+1}}{\partial A_t}\frac{A_t}{K_{t+1}} = (1-\alpha)K_t^{\alpha}A_t^{-\alpha}L_t^{1-\alpha}\left(\frac{A_t}{K_{t+1}}\right) = \frac{(1-\alpha)K_t^{\alpha}(A_t L_t)^{1-\alpha}}{K_{t+1}}.$$

Using $Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$, equation (4) becomes

$$\text{(5)}\quad \frac{\partial\ln K_{t+1}}{\partial\ln A_t} = \frac{(1-\alpha)Y_t}{K_{t+1}}.$$

On the balanced growth path without shocks, $K_{t+1} = e^{n+g}K_t$. Since the production function is Cobb-Douglas, the amount of income going to capital -- which is the marginal product of capital multiplied by the amount of capital, $(r^* + \delta)K_t$ -- is equal to αY_t. Thus $Y_t = (r^* + \delta)K_t/\alpha$. Substituting these two facts into equation (5) yields

$$(6)\quad \left.\frac{\partial \ln K_{t+1}}{\partial \ln A_t}\right|_{bgp} = \frac{(1-\alpha)(r^*+\delta)K_t}{\alpha e^{n+g}K_t} = \frac{(1-\alpha)(r^*+\delta)}{\alpha e^{n+g}}.$$

Using equation (2), $\partial \ln K_{t+1}/\partial \ln L_t$ (holding K_t, C_t, G_t and A_t fixed) is

$$(7)\quad \frac{\partial \ln K_{t+1}}{\partial \ln L_t} = \frac{\partial K_{t+1}}{\partial L_t}\frac{L_t}{K_{t+1}} = (1-\alpha)K_t^{\alpha}A_t^{1-\alpha}L_t^{-\alpha}\left(\frac{L_t}{K_{t+1}}\right) = \frac{(1-\alpha)K_t^{\alpha}(A_tL_t)^{1-\alpha}}{K_{t+1}}.$$

Comparing equation (7) to equation (4), we can see that $\partial \ln K_{t+1}/\partial \ln L_t = \partial \ln K_{t+1}/\partial \ln A_t$. Thus by performing the same manipulations as above, we can write

$$(8)\quad \left.\frac{\partial \ln K_{t+1}}{\partial \ln L_t}\right|_{bgp} = \frac{(1-\alpha)(r^*+\delta)}{\alpha e^{n+g}}.$$

Using equation (2), $\partial \ln K_{t+1}/\partial \ln G_t$ (holding K_t, C_t, L_t and A_t fixed) is

$$(9)\quad \frac{\partial \ln K_{t+1}}{\partial \ln G_t} = \frac{\partial K_{t+1}}{\partial G_t}\frac{G_t}{K_{t+1}} = -\frac{G_t}{K_{t+1}}.$$

Multiplying and dividing the right-hand side of equation (9) by Y_t gives us

$$(10)\quad \frac{\partial \ln K_{t+1}}{\partial \ln G_t} = -\frac{Y_t(G_t/Y_t)}{K_{t+1}}.$$

As explained above, on the balanced growth path without shocks, $K_{t+1} = e^{n+g}K_t$ and $Y_t = (r^* + \delta)K_t/\alpha$. Substituting these two facts into equation (10) and using (G^*/Y) to denote the ratio of G to Y on the balanced growth path yields

$$(11)\quad \left.\frac{\partial \ln K_{t+1}}{\partial \ln G_t}\right|_{bgp} = -\frac{(r^*+\delta)K_t(G^*/Y)}{\alpha e^{n+g}K_t} = -\frac{(r^*+\delta)(G^*/Y)}{\alpha e^{n+g}}.$$

Using equation (2), $\partial \ln K_{t+1}/\partial \ln C_t$ (holding K_t, G_t, L_t and A_t fixed) is

$$(12)\quad \frac{\partial \ln K_{t+1}}{\partial \ln C_t} = \frac{\partial K_{t+1}}{\partial C_t}\frac{C_t}{K_{t+1}} = -\frac{C_t}{K_{t+1}}.$$

Using hint (2), we can write this derivative evaluated at the balanced growth path as

$$(13)\quad \left.\frac{\partial \ln K_{t+1}}{\partial \ln C_t}\right|_{bgp} = -\frac{\left[Y_t - G^* - \delta K_t - (e^{n+g}-1)K_t\right]}{K_{t+1}}.$$

Defining $\lambda_1 \equiv (1 + r^*)/e^{n+g}$, $\lambda_2 \equiv (1 - \alpha)(r^* + \delta)/(\alpha e^{n+g})$ and $\lambda_3 \equiv -(r^* + \delta)(G^*/Y)/(\alpha e^{n+g})$, we need to show that $\partial \ln K_{t+1}/\partial \ln C_t$ evaluated at the balanced growth path is equal to $1 - \lambda_1 - \lambda_2 - \lambda_3$. By definition,

$$1-\lambda_1-\lambda_2-\lambda_3 = 1 - \frac{1+r^*}{e^{n+g}} - \frac{(1-\alpha)(r^*+\delta)}{\alpha e^{n+g}} + \frac{(r^*+\delta)(G^*/Y)}{\alpha e^{n+g}},$$

or simply

$$(14)\quad 1-\lambda_1-\lambda_2-\lambda_3 = 1 - \frac{1+r^*}{e^{n+g}} + \frac{(r^*+\delta)\left[(G^*/Y)-(1-\alpha)\right]}{\alpha e^{n+g}}.$$

Note that we can write $(r^* + \delta)$ as

(15) $(r^* + \delta) = \partial Y_t/\partial K_t = \alpha K_t^{\alpha-1}(A_tL_t)^{1-\alpha} = \alpha Y_t/K_t$.

Substituting equation (15) into equation (14) gives us

(16) $1-\lambda_1-\lambda_2-\lambda_3=1-\frac{1+r^*}{e^{n+g}}+\frac{\alpha Y_t[(G^*/Y_t)-(1-\alpha)]}{\alpha K_t e^{n+g}}=1-\frac{1+r^*}{e^{n+g}}+\frac{G^*-(1-\alpha)Y_t}{e^{n+g}K_t}$

Obtaining a common denominator and using the fact that on the balanced growth path without shocks, $K_{t+1}=e^{n+g}K_t$, we have

(17) $1-\lambda_1-\lambda_2-\lambda_3=\frac{e^{n+g}K_t-(1+r^*)K_t+G^*-Y_t+\alpha Y_t}{K_{t+1}}.$

From equation (15), $\alpha Y_t=(r^*+\delta)K_t$. Substituting this into equation (17) yields

(18) $1-\lambda_1-\lambda_2-\lambda_3=\frac{e^{n+g}K_t-K_t-r^*K_t+G^*-Y_t+r^*K_t+\delta K_t}{K_{t+1}}.$

Collecting terms yields

(19) $1-\lambda_1-\lambda_2-\lambda_3=-\frac{\left[Y_t-G^*-\delta K_t-(e^{n+g}-1)K_t\right]}{K_{t+1}}.$

Comparing equations (13) and (19), we have shown that

(20) $\left.\frac{\partial\ln K_{t+1}}{\partial\ln C_t}\right|_{bgp}=1-\lambda_1-\lambda_2-\lambda_3.$

The log linearization is of the form

$$\tilde{K}_{t+1}\cong\left[\left.\frac{\partial\ln K_{t+1}}{\partial\ln K_t}\right|_{bgp}\right]\tilde{K}_t+\left[\left.\frac{\partial\ln K_{t+1}}{\partial\ln A_t}\right|_{bgp}\right]\tilde{A}_t+\left[\left.\frac{\partial\ln K_{t+1}}{\partial\ln L_t}\right|_{bgp}\right]\tilde{L}_t+\left[\left.\frac{\partial\ln K_{t+1}}{\partial\ln G_t}\right|_{bgp}\right]\tilde{G}_t+\left[\left.\frac{\partial\ln K_{t+1}}{\partial\ln C_t}\right|_{bgp}\right]\tilde{C}_t$$

Using equations (3), (6), (8), (11) and (20) as well as the definitions of λ_1, λ_2 and λ_3 to substitute for the derivatives, we have

(21) $\tilde{K}_{t+1}\cong\lambda_1\tilde{K}_t+\lambda_2(\tilde{A}_t+\tilde{L}_t)+\lambda_3\tilde{G}_t+(1-\lambda_1-\lambda_2-\lambda_3)\tilde{C}_t.$

(c) Substituting equation (4.43), $\tilde{C}_t\cong a_{CK}\tilde{K}_t+a_{CA}\tilde{A}_t+a_{CG}\tilde{G}_t$, and equation (4.44), $\tilde{L}_t\cong a_{LK}\tilde{K}_t+a_{LA}\tilde{A}_t+a_{LG}\tilde{G}_t$, into equation (21) yields

(22)
$$\begin{aligned}\tilde{K}_{t+1}\cong{}&\lambda_1\tilde{K}_t+\lambda_2\tilde{A}_t+\lambda_2(a_{LK}\tilde{K}_t+a_{LA}\tilde{A}_t+a_{LG}\tilde{G}_t)+\lambda_3\tilde{G}_t\\&+(1-\lambda_1-\lambda_2-\lambda_3)(a_{CK}\tilde{K}_t+a_{CA}\tilde{A}_t+a_{CG}\tilde{G}_t).\end{aligned}$$

Collecting terms gives us

(23)
$$\begin{aligned}\tilde{K}_{t+1}\cong{}&\left[\lambda_1+\lambda_2a_{LK}+(1-\lambda_1-\lambda_2-\lambda_3)a_{CK}\right]\tilde{K}_t+\left[\lambda_2+\lambda_2a_{LA}+(1-\lambda_1-\lambda_2-\lambda_3)a_{CA}\right]\tilde{A}_t\\&+\left[\lambda_2a_{LG}+\lambda_3+(1-\lambda_1-\lambda_2-\lambda_3)a_{CG}\right]\tilde{G}_t.\end{aligned}$$

Defining $b_{KK}\equiv\lambda_1+\lambda_2a_{LK}+(1-\lambda_1-\lambda_2-\lambda_3)a_{CK}$, $b_{KA}\equiv\lambda_2(1+a_{LA})+(1-\lambda_1-\lambda_2-\lambda_3)a_{CA}$ and $b_{KG}\equiv\lambda_2a_{LG}+\lambda_3+(1-\lambda_1-\lambda_2-\lambda_3)a_{CG}$, equation (23) can be rewritten as

(24) $\tilde{K}_{t+1}\cong b_{KK}\tilde{K}_t+b_{KA}\tilde{A}_t+b_{KG}\tilde{G}_t.$

Equation (24) is identical to equation (4.52) in the text.

Problem 4.16

(a) Suppose the true model is given by

(1) $\Delta\ln Y_t=\varepsilon_t$,

where the ε's are independent, mean-zero errors. This implies that log output is a random walk; that is, $\ln Y_t-\ln Y_{t-1}=\varepsilon_t$, or

(2) $\ln Y_t = \ln Y_{t-1} + \varepsilon_t$.

What happens if the true model is given by equation (1) and we run an Ordinary Least Squares (OLS) regression of the change in log output on a constant and log output lagged one period? That is, what if we run

(3) $\Delta \ln Y_t = a' + b \ln Y_{t-1} + \varepsilon_t$.

In part (a) of the question, we will assume that the sample size is three and the initial value is $\ln Y_0 = 0$. Also, we will assume that there are only two possible values of ε -- +1 and -1 -- and each one occurs with probability 1/2. This means that there are eight possible realizations of $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. For each possible realization, we need to calculate the OLS estimate of b.

For example, look at $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1,1,-1)$. We can generate the left-hand-side and right-hand-side variables for our regression:

LHS Variables	RHS Variables
	$\ln Y_0 = 0$
$\Delta \ln Y_1 = \varepsilon_1 = 1$	$\ln Y_1 = \ln Y_0 + \varepsilon_1 = 0 + 1 = 1$
$\Delta \ln Y_2 = \varepsilon_2 = 1$	$\ln Y_2 = \ln Y_1 + \varepsilon_2 = 1 + 1 = 2$
$\Delta \ln Y_3 = \varepsilon_3 = -1$	

The averages of each are

$\overline{\Delta \ln Y_t} = 1/3$ $\qquad$ $\overline{\ln Y_{t-1}} = 1$

The formula for the OLS estimator is given by

$$\hat{b} = \frac{\sum_{t=1}^{3}\left(\ln Y_{t-1} - \overline{\ln Y_{t-1}}\right)\left(\Delta \ln Y_t - \overline{\Delta \ln Y_t}\right)}{\sum_{t=1}^{3}\left(\ln Y_{t-1} - \overline{\ln Y_{t-1}}\right)^2}.$$

Substituting the values given yields

$$\hat{b} = \frac{(-1)(2/3) + (0)(2/3) + (1)(-4/3)}{(-1)^2 + (0)^2 + (1)^2} = \frac{-6/3}{2} = -1.$$

Similarly for the other 7 possible realizations of the ε's, we can obtain the following:

$(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1,1,1) \Rightarrow \hat{b} = 0$

$(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (-1,-1,-1) \Rightarrow \hat{b} = 0$

$(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1,-1,1) \Rightarrow \hat{b} = -2$

$(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1,-1,-1) \Rightarrow \hat{b} = -1.$

$(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (-1,1,1) \Rightarrow \hat{b} = -1$

$(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (-1,-1,1) \Rightarrow \hat{b} = -1$

$(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (-1,1,-1) \Rightarrow \hat{b} = -2$

The average of the eight OLS estimates turns out to be -1, even though we know for a fact that the data were created with a true value of b = 0. One of the sources of bias in OLS here is the fact that the right-hand-side variable in the regression is not uncorrelated with all leads and lags of the error term. In fact each $\ln Y_{t-1}$ is correlated with all past values of the random shocks.

(b) See the following printout of the Monte Carlo experiment using TSP. Although the data were generated with a true b = 0, the average of the 500 OLS estimates is about -0.029 and 96.6% of the estimates of b are negative. Again, OLS is biased. If we were to plot a histogram of the OLS estimates, their distribution would be highly skewed and non-normal, even though we know for a fact that the errors are normally distributed.

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                     USA


 PROGRAM
LINE ********************************************************************
supres smpl;
set numtrial = 500;
set bhatneg = 0;
regopt(noprint) @logl, @coef;

 do t = 1, numtrial;
   smpl 1,201;
   eps = 0;
   lny = 0;
   smpl 2, 201;
   random eps;
   genr dlny = eps;
   genr lny = lny(-1) + eps;
   olsq(silent) dlny c lny(-1);
   smpl 1,500;
   set bhat(t) = @coef(2);
   if bhat(t) < 0; then;
     set bhatneg = bhatneg + 1;
 enddo;

 smpl 1, 500;
 msd bhat;
 set pbhatneg = bhatneg/numtrial;
 print pbhatneg;
```

SOLUTIONS TO CHAPTER 5

Problem 5.1

Differentiate the LM equation, M/P = L(i, Y) or equivalently M = PL(i, Y), with respect to M, holding P constant:

$$(1)\quad 1 = P\left[L_i \frac{di}{dM} + L_Y \frac{dY}{dM}\right].$$

Rearranging equation (1) to solve for dY/dM yields

$$(2)\quad \frac{dY}{dM} = \frac{1}{PL_Y} - \frac{L_i}{L_Y}\frac{di}{dM}.$$

Differentiate the IS equation, $Y = E(Y, i - \pi^e, G, T)$, with respect to M holding π^e, G and T constant:

$$(3)\quad \frac{dY}{dM} = E_Y \frac{dY}{dM} + E_{i-\pi^e}\frac{di}{dM}.$$

Rearranging equation (3) to solve for di/dM yields

$$(4)\quad \frac{di}{dM} = \frac{(1-E_Y)}{E_{i-\pi^e}}\frac{dY}{dM}.$$

Substitute equation (4) into equation (2):

$$\frac{dY}{dM} = \frac{1}{PL_Y} - \frac{L_i}{L_Y}\frac{(1-E_Y)}{E_{i-\pi^e}}\frac{dY}{dM}.$$

Collecting the terms in dY/dM and obtaining a common denominator leaves us with

$$\frac{dY}{dM}\left[\frac{L_Y E_{i-\pi^e} + L_i(1-E_Y)}{L_Y E_{i-\pi^e}}\right] = \frac{1}{PL_Y}.$$

Solving for dY/dM yields

$$(5)\quad \frac{dY}{dM} = \frac{E_{i-\pi^e}}{P\left[L_Y E_{i-\pi^e} + L_i(1-E_Y)\right]}.$$

Dividing the top and bottom of equation (5) by $E_{i-\pi^e}$ yields

$$(6)\quad \frac{dY}{dM} = \frac{1}{P\left[L_Y + L_i(1-E_Y)/E_{i-\pi^e}\right]}.$$

To obtain di/dM, substitute equation (5) into equation (4):

$$(7)\quad \frac{di}{dM} = \frac{(1-E_Y)}{E_{i-\pi^e}}\frac{E_{i-\pi^e}}{P\left[L_Y E_{i-\pi^e} + L_i(1-E_Y)\right]} = \frac{(1-E_Y)}{P\left[L_Y E_{i-\pi^e} + L_i(1-E_Y)\right]}.$$

Dividing the top and bottom of equation (7) by $(1 - E_Y)$ gives us

$$(8)\quad \frac{di}{dM} = \frac{1}{P\left[L_Y E_{i-\pi^e}/(1-E_Y) + L_i\right]} < 0.$$

Looking at equation (6), since $L_Y > 0$, $L_i < 0$, $E_Y < 1$ and $E_{i-\pi^e} < 0$, we have dY/dM > 0. In addition, note that the bigger is $E_{i-\pi^e}$ (in absolute value) -- the more that planned expenditure responds to changes in the real interest rate -- and the smaller is L_i -- the less responsive is real money demand to changes in the nominal interest rate -- the bigger is dY/dM. Why? Suppose there is an increase in M. To re-establish money-market equilibrium, we need a lower nominal interest rate at a given level of Y. All else equal, the smaller is L_i, the bigger is the drop in i required to re-establish this equilibrium. The drop in i -- which also means a drop in $i - \pi^e$, since π^e is assumed fixed -- then increases planned expenditure. All else equal, the bigger is $E_{i-\pi^e}$ (in absolute value), the more that planned expenditure will rise. Thus the more that output will end up needing to rise to re-establish the equilibrium condition that output equals planned expenditure or Y = E.

Problem 5.2

(a) The equilibrium condition in the money market is M/P = L(i,Y). Now suppose that the central bank has a target interest rate of $i = \bar{i}$. At a given level of Y, demand for real money balances at the target interest rate would be $L(\bar{i}, Y)$. To ensure money market equilibrium, the central bank will simply have to adjust the nominal money supply, M, to ensure that $M/P = L(\bar{i}, Y)$. Thus the "LM curve" -- the set of all combinations of i and Y that cause real money demand and supply to be equal -- will be horizontal at the central bank's target level of the interest rate, $\bar{i}$.

(b) The AD curve will now be vertical. First of all, why is AD downward-sloping when the central bank targets M rather than i? At a lower P, M/P is higher and thus the LM curve shifts down to the right. Thus the level of Y that clears the money market and equates planned and actual expenditure is now higher. The AD curve _is_ all the combinations of P and Y that clear the money market and equate planned and actual expenditure. Hence lower levels of P are associated with higher levels of Y along the AD curve so that the AD is downward-sloping.

Now suppose the central bank is targeting the interest rate at $i = \bar{i}$. At a lower P, the real money supply would be higher. But the horizontal "LM curve" will not shift. The central bank will simply lower M so that at the given combination of income and $\bar{i}$, $M/P = L(\bar{i}, Y)$ again. Thus a lower P will not require a higher level of Y to clear the money market and equate planned and actual expenditure. That is, regardless of the price level, there is one unique level of Y that clears the money market and makes planned and actual expenditure equal -- the level of Y where the horizontal "LM curve" and IS intersect -- and thus the AD curve is vertical.

Problem 5.3

(a) (i) How does an equal increase in G and T -- a "balanced budget" increase in government purchases -- affect the position of the IS curve? We need to determine the effect on Y for a given i to examine the extent of the horizontal shift of the IS curve.

Differentiate the IS curve, $Y = C(Y - T) + I(i - \pi^e) + G$, with respect to G:

$$\frac{dY}{dG} = \frac{\partial C}{\partial (Y - T)}\left[\frac{dY}{dG} - \frac{dT}{dG}\right] + \frac{\partial I}{\partial (i - \pi^e)}\frac{d(i - \pi^e)}{dG} + 1.$$

Although continuing to use total derivative notation, keep in mind that we are now holding π^e and i constant. In addition, we are assuming that dT = dG and so

$$\frac{dY}{dG} = C_{Y-T}\left[\frac{dY}{dG} - 1\right] + 1 \quad \Rightarrow \quad \frac{dY}{dG}\left(1 - C_{Y-T}\right) = 1 - C_{Y-T},$$

or simply

$$\left.\frac{dY}{dG}\right|_{i\,\text{fixed}} = 1.$$

This can be interpreted as meaning the change in Y for a given i is simply equal to the change in government purchases. Thus a change in government purchases, when accompanied by an equal change in taxes, causes the IS curve to shift by a horizontal amount equal to that change in government purchases.

(a) (ii) Now we have to allow for a variable interest rate. We want to know the extent of the horizontal shift of the AD curve. That is, we want to know the effect on Y for a given P, but allowing for the effects of a variable interest rate.

Differentiate the IS curve equation, $Y = C(Y - T) + I(i - \pi^e) + G$, with respect to G:

$$\frac{dY}{dG} = \frac{\partial C}{\partial (Y - T)}\left[\frac{dY}{dG} - \frac{dT}{dG}\right] + \frac{\partial I}{\partial (i - \pi^e)}\left[\frac{di}{dG} - \frac{d\pi^e}{dG}\right] + 1.$$

We are holding π^e (and P) constant, and assuming $dT = dG$ so we have

$$(1)\quad \frac{dY}{dG} = C_{Y-T}\frac{dY}{dG} - C_{Y-T} + I_{i-\pi^e}\frac{di}{dG} + 1.$$

Differentiate the LM equation, $M/P = L(i,Y)$, with respect to G, holding M and P constant:

$$0 = L_i\frac{di}{dG} + L_Y\frac{dY}{dG},$$

and rearranging to solve for di/dG yields

$$(2)\quad \frac{di}{dG} = -\frac{L_Y}{L_i}\frac{dY}{dG}.$$

Substituting equation (2) into equation (1) yields

$$\frac{dY}{dG} = C_{Y-T}\frac{dY}{dG} - C_{Y-T} - \frac{I_{i-\pi^e}L_Y}{L_i}\frac{dY}{dG} + 1.$$

Solving for dY/dG, we have

$$(3)\quad \left.\frac{dY}{dG}\right|_{P\text{ fixed}} = \frac{1 - C_{Y-T}}{1 - C_{Y-T} + \left(I_{i-\pi^e}L_Y/L_i\right)} < 1.$$

The horizontal shift of the AD curve is thus $\left[\left(1 - C_{Y-T}\right)/\left(1 - C_{Y-T} + I_{i-\pi^e}L_Y/L_i\right)\right]dG < dG$. The AD curve shifts less than the IS curve. This is because the shift of the AD takes into account the rising interest rate caused by the upward-sloping LM.

(b) (i) We are now assuming that tax revenues are a function of income; $T = T(Y)$. In addition, $T'(Y) > 0$ so that tax revenues rise when income rises. To find the slope of the IS curve, differentiate $Y = E(Y, i - \pi^e, G, T(Y))$ with respect to i, holding everything else constant:

$$\frac{dY}{di} = E_Y\frac{dY}{di} + E_{i-\pi^e} + E_T T'(Y)\frac{dY}{di} \quad \Rightarrow \quad \frac{dY}{di}\left[1 - E_Y - E_T T'(Y)\right] = E_{i-\pi^e},$$

or simply

$$\frac{dY}{di} = \frac{E_{i-\pi^e}}{1 - E_Y - E_T T'(Y)}.$$

Inverting the above expression yields the slope of the IS curve,

$$(4)\quad \left.\frac{di}{dY}\right|_{IS} = \frac{1 - E_Y - E_T T'(Y)}{E_{i-\pi^e}} < 0.$$

To see how an increase in $T'(Y)$ affects the slope, take the derivative of the expression for the slope with respect to $T'(Y)$:

(5) $$\frac{\partial\left(di/dY|_{IS}\right)}{\partial T'(Y)} = -\frac{E_T}{E_{i-\pi^e}} < 0 \quad \text{since} \quad E_T, E_{i-\pi^e} < 0$$

Thus an increase in T ' (Y) causes the slope of the IS curve to become even more negative. That is, it causes the IS curve to become steeper.

(b) (ii) We need to determine how an increase in T ' (Y) affects the impact on output, for a given price, of a change in G. First, we need the effect of a change in G on Y, for a given P. Differentiate the IS equation, $Y = E(Y, i - \pi^e, G, T(Y))$, with respect to G, holding π^e (and P) constant:

(6) $$\frac{dY}{dG} = E_Y\frac{dY}{dG} + E_{i-\pi^e}\frac{di}{dG} + 1 + E_T T'(Y)\frac{dY}{dG}.$$

Differentiate the LM equation, M/P = L(i,Y), with respect to G, holding P and M constant:

$$0 = L_i\frac{di}{dG} + L_Y\frac{dY}{dG} \Rightarrow \frac{di}{dG} = -\frac{L_Y}{L_i}\frac{dY}{dG}. \quad (7)$$

Substitute equation (7) into equation (6):

$$\frac{dY}{dG} = E_Y\frac{dY}{dG} - \frac{E_{i-\pi^e}L_Y}{L_i}\frac{dY}{dG} + E_G + E_T T'(Y)\frac{dY}{dG}.$$

Solving for dY/dG yields

(8) $$\left.\frac{dY}{dG}\right|_{P\text{ fixed}} = \frac{E_G}{1 - E_Y + \left(E_{i-\pi^e}L_Y/L_i\right) - E_T T'(Y)} > 0.$$

Now to see how the impact of a change in government purchases on output, for a given price level, is affected by a rise in T ' (Y):

(9) $$\frac{\partial\left(dY/dG|_{P\text{ fixed}}\right)}{\partial T'(Y)} = \frac{-E_G}{\left[1 - E_Y + \left(E_{i-\pi^e}L_Y/L_i\right) - E_T T'(Y)\right]^2}(-E_T) < 0 \quad \text{since} \quad E_G > 0, E_T < 0.$$

When T ' (Y) is higher, the increase in Y, for a given P, due to an increase in G is smaller. Thus we have shown that the horizontal shift of the AD curve due to a change in government purchases will be smaller, the larger is T ' (Y).

Now we need to determine how an increase in T ' (Y) affects the impact on output, for a given price, of a change in M. First, we need the effect of a change in M on Y, for a given P. Differentiate the IS equation, $Y = E(Y, i - \pi^e, G, T(Y))$, with respect to M, holding G, π^e (and P) constant:

(10) $$\frac{dY}{dM} = E_Y\frac{dY}{dM} + E_{i-\pi^e}\frac{di}{dM} + E_T T'(Y)\frac{dY}{dM}.$$

Differentiate the LM equation, M/P = L(i,Y), with respect to M, holding P constant:

$$\frac{1}{P} = L_i\frac{di}{dM} + L_Y\frac{dY}{dM} \Rightarrow \frac{di}{dM} = \frac{1}{PL_i} - \frac{L_Y}{L_i}\frac{dY}{dM} \quad (11)$$

Substitute equation (11) into equation (10):

$$\frac{dY}{dM} = E_Y\frac{dY}{dM} + \frac{E_{i-\pi^e}}{PL_i} - \frac{E_{i-\pi^e}L_Y}{L_i}\frac{dY}{dM} + E_T T'(Y)\frac{dY}{dM}.$$

Solving for dY/dM yields

$$(9)\quad \frac{dY}{dM}\bigg|_{P\text{ fixed}} = \frac{E_{i-\pi^e}}{PL_i\left[1 - E_Y + \left(E_{i-\pi^e} L_Y / L_i\right) - E_T T'(Y)\right]} > 0.$$

To see how the impact of a change in M on Y, for a given P, is affected by a rise in T ' (Y), examine the following derivative:

$$(10)\quad \frac{\partial\left(dY/dM\big|_{P\text{ fixed}}\right)}{\partial T'(Y)} = \frac{-E_{i-\pi^e}\left(-PL_i E_T\right)}{\left(PL_i\right)^2\left[1 - E_Y + \left(E_{i-\pi^e} L_Y / L_i\right) - E_T T'(Y)\right]^2} < 0.$$

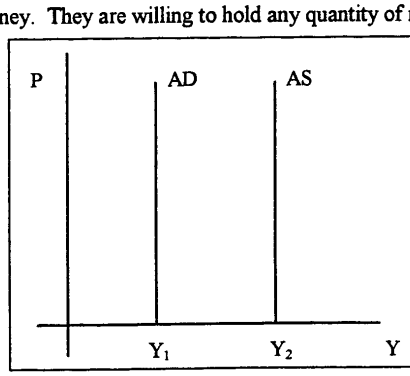
This derivative is negative since $E_{i-\pi^e}$, L_i, and E_T are all negative. When T ' (Y) is higher, the increase in Y, for a given P, due to an increase in M is smaller. Thus the horizontal shift of the AD curve due to a change in the nominal quantity of money will be smaller when T ' (Y) is higher.

This illustrates the concept of income taxes as automatic stabilizers. We have shown that the more that taxes increase with income, the smaller will be the impact on output from shocks to the IS and LM curves such as changes in government purchases or changes in the money supply.

Problem 5.4

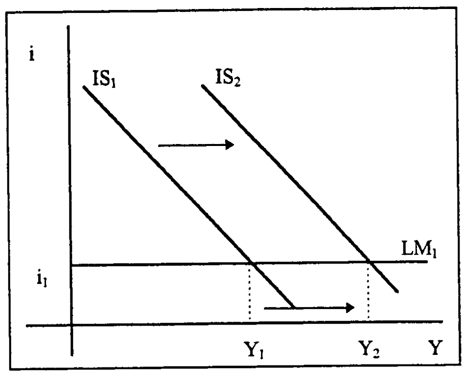
(a) In a liquidity trap, the LM curve is horizontal at the low nominal interest rate that prevails. Intuitively, at a given interest rate, individuals are willing to hold any quantity of real money balances. Thus at higher levels of income, Y, individuals are willing to change their real money balances without a change in the interest rate. Thus higher levels of Y no longer require higher interest rates to clear the money market. That is, LM is horizontal rather than upward sloping. This means that the AD curve will be vertical. This can be seen by going through the exercise of deriving the AD curve. A decrease in P increases the supply of real money balances. Ordinarily, a lower interest rate is needed to clear the money market for a given level of income and so the LM curve would shift down. As a result, i falls and Y rises. Thus the level of output at the intersection of the IS and LM curves is ordinarily a decreasing function of the price level. That is, the AD curve is usually downward sloping; lower levels of P require higher levels of Y to clear the money market and make planned expenditure equal actual expenditure.

In the case of a liquidity trap, the decrease in P which increases the supply of real money balances, M/P, does not shift the horizontal LM curve. At a given level of income, we do not require a lower nominal interest rate to get individuals to hold the extra money. They are willing to hold any quantity of real money balances at a given level of Y. Thus LM does not shift down. The same level of Y continues to clear the money market and make planned expenditure equal actual expenditure, regardless of the price level. In other words, the level of output at the intersection of the IS and LM curves is no longer a function of the price level; AD is vertical.



Now, even if prices are fully flexible (so that the AS curve is vertical), aggregate demand can play a role in determining output. Suppose the AD curve lies to the left of the AS curve as is depicted at right. Then output bought and sold in the economy will be at Y_1, not Y_2. We have a situation of excess supply that cannot be eliminated by a change in price, since price does not affect AD.

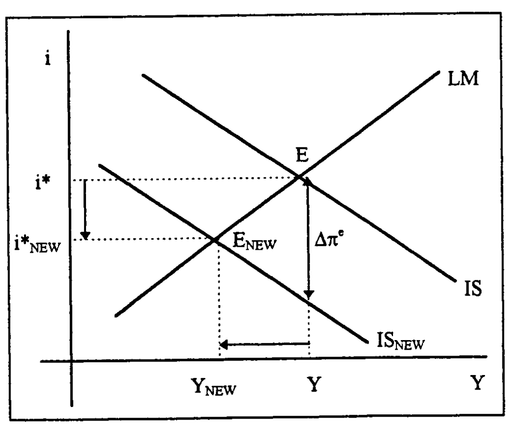
(b) We can now write planned expenditure as
(1) $E = E(Y, i - \pi^e, G, T, M/P)$, $E_{M/P} > 0$.
The important point is that even if there is a liquidity trap so that LM is horizontal, the AD curve will once again be downward-sloping.



Again, we can see this by going through the exercise of deriving the AD curve. Suppose IS_1 and LM_1 correspond to some price level P_1. Now consider a lower price level P_2. As explained in part (a), the horizontal LM will not shift. But the IS curve will. The fall in P raises M/P which is a component of wealth. With $E_{M/P} > 0$, this increase in wealth increases planned expenditure at a given i. In a "Keynesian Cross" diagram, the planned expenditure line would shift up. Thus the level of Y required to equate planned and actual expenditure is now higher at a given i. That is, the IS curve shifts to the right to IS_2. This IS/LM diagram then tells us that at lower P's, the level of Y that clears the money market and equates planned and actual expenditure is now higher. Thus the AD curve, which <u>is</u> all the combinations of P and Y such that the money market clears and planned and actual expenditure are equal, is once again downward-sloping. In this case, a liquidity trap combined with real money holdings affecting planned expenditure, a vertical AS curve once again means that AD is irrelevant to output. Essentially, we have re-introduced a way in which prices affect AD. Thus a situation of excess supply can be eliminated by a falling price level.

<u>Problem 5.5</u>

A fall in expected inflation shifts the IS curve to the left to IS_{NEW}. At a given nominal interest rate, i, the real interest rate, $i - \pi^e$, is now higher and thus planned expenditure is lower. In a "Keynesian Cross" diagram, the planned expenditure line would shift down. Thus the level of Y required to equate planned and actual expenditure is now lower at a given i. That is, IS shifts left.



From the figure at right, we can see that the fall in π^e causes both output and the nominal interest rate to fall. What about the real interest rate, $i - \pi^e$? It rises; that is, i falls less than π^e does. Graphically, think of the IS curve as shifting down. What is the magnitude of the downward shift in IS? It must be the change in expected inflation, $\Delta\pi^e$. Intuitively, the same level of Y would continue to equate planned and actual expenditure if i were to fall by the same amount as π^e so that $i - \pi^e$ (and thus planned expenditure) remained unchanged. Thus on the new IS curve, a given level of Y is now associated with a nominal interest rate that is lower by the amount $\Delta\pi^e$ or in other words, IS shifts down by $\Delta\pi^e$. We can see from the IS-LM diagram, that because of the upward-sloping LM curve, i does not fall by the full extent of the downward shift in IS. Thus i falls by less than π^e does and so the real interest rate, $i - \pi^e$, rises.

More formally, the IS and LM equations are given by
(1) $Y = E(Y, i - \pi^e, G, T)$, and (2) $M/P = L(i, Y)$.
Differentiate the IS equation with respect to π^e, holding G and T constant:

$$\frac{dY}{d\pi^e} = E_Y \frac{dY}{d\pi^e} + E_{i-\pi^e}\left[\frac{di}{d\pi^e} - 1\right]$$

Rearranging to solve for $di/d\pi^e$ gives us

$$E_{i-\pi^e}\frac{di}{d\pi^e} = (1 - E_Y)\frac{dY}{d\pi^e} \quad \Rightarrow \quad \frac{di}{d\pi^e} = \frac{(1-E_Y)}{E_{i-\pi^e}}\frac{dY}{d\pi^e} + 1. \quad (3)$$

Differentiate the LM equation with respect to π^e, holding M and P constant:

$$0 = L_i\frac{di}{d\pi^e} + L_Y\frac{dY}{d\pi^e} \quad \Rightarrow \quad \frac{dY}{d\pi^e} = \frac{-L_i}{L_Y}\frac{di}{d\pi^e}. \quad (4)$$

Substitute equation (4) into equation (3):

$$\frac{di}{d\pi^e} = \frac{(1-E_Y)}{E_{i-\pi^e}}\frac{-L_i}{L_Y}\frac{di}{d\pi^e} + 1 \quad \Rightarrow \quad \frac{di}{d\pi^e}\left[\frac{E_{i-\pi^e}L_Y + (1-E_Y)L_i}{E_{i-\pi^e}L_Y}\right] = 1,$$

and thus

$$(5) \quad \frac{di}{d\pi^e} = \frac{E_{i-\pi^e}L_Y}{E_{i-\pi^e}L_Y + (1-E_Y)L_i} > 0$$

Note that $di/d\pi^e < 1$ which confirms the graphical and intuitive analysis above. The nominal interest rate falls less than expected inflation does and so the real interest rate, $i - \pi^e$, rises. Substituting equation (5) into equation (4) gives us the following expression for the change in output due to the change in expected inflation:

$$(6) \quad \frac{dY}{d\pi^e} = \frac{-L_i}{L_Y}\frac{E_{i-\pi^e}L_Y}{E_{i-\pi^e}L_Y + (1-E_Y)L_i} = \frac{-L_i E_{i-\pi^e}}{E_{i-\pi^e}L_Y + (1-E_Y)L_i} > 0.$$

Since $dY/d\pi^e > 0$, a fall in expected inflation causes output to fall. This confirms the graphical analysis above.

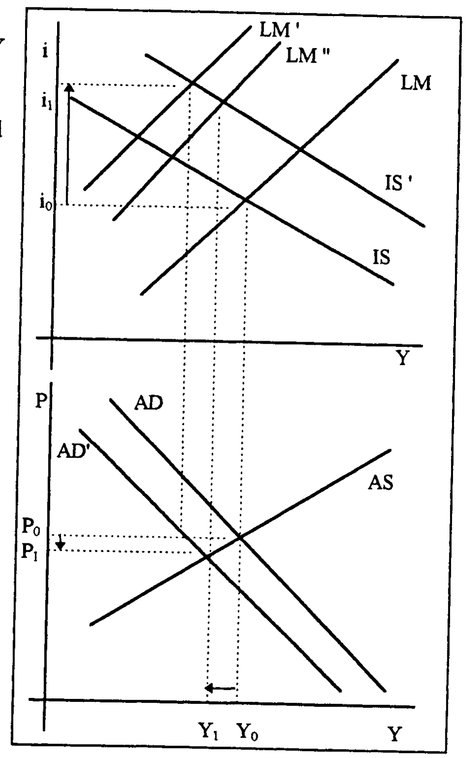
Problem 5.6
(a) Substituting the consumption function, $C_t = a + bY_{t-1}$, and the assumption about investment, $I_t = K_t^* - cY_{t-2}$, into the equation for output, $Y_t = C_t + I_t + G_t$, yields
(1) $Y_t = a + bY_{t-1} + K_t^* - cY_{t-2} + G_t$.
Substituting for the desired capital stock, $K_t^* = cY_{t-1}$, and for the constant level of government purchases, $G_t = \overline{G}$, yields
(2) $Y_t = a + bY_{t-1} + cY_{t-1} - cY_{t-2} + \overline{G}$.
Collecting terms in Y_{t-1} gives us output in period t as a function of Y_{t-1}, Y_{t-2} and the parameters of the model:
(3) $Y_t = a + (b + c)Y_{t-1} - cY_{t-2} + \overline{G}$.

(b) With the assumptions of b = 0.9 and c = 0.5, output in period t is given by
(4) $Y_t = a + 1.4Y_{t-1} - 0.5Y_{t-2} + \overline{G}$.
Throughout the following, the change in a variable represents the change from the path that variable would have taken if G had simply remained constant at $\overline{G}$.

In period t,
$Y_t = a + 1.4Y_{t-1} - 0.5Y_{t-2} + \overline{G} + 1$,
and thus the change in output from the path it would have taken is given by
$\Delta Y_t = +1$.
In period t + 1, using the fact that equation (4) will hold in all future periods,
$\Delta Y_{t+1} = 1.4\Delta Y_t - 0.5\Delta Y_{t-1} = 1.4(+1) - 0.5(0) = +1.4$.
In period t + 2,
$\Delta Y_{t+2} = 1.4\Delta Y_{t+1} - 0.5\Delta Y_t = 1.4(+1.4) - 0.5(+1) = +1.46$.
In period t + 3,
$\Delta Y_{t+3} = 1.4\Delta Y_{t+2} - 0.5\Delta Y_{t+1} = 1.4(+1.46) - 0.5(+1.4) = +1.344$.
With similar calculations, one can show that $\Delta Y_{t+4} = +1.15$, $\Delta Y_{t+5} = 0.938$ and so on. Thus output follows a "hump-shaped" response to the one-time increase in government purchases of one. The maximum effect is felt two periods after the increase in G and the effect then goes to 0 over time.

Problem 5.7

A decrease in taxes means that at a given i, planned expenditure is higher since $E_T < 0$. Thus at a given i, Y must be higher in order for Y = E. Thus the IS curve shifts to the right to IS '. Ordinarily, that would be the whole story. The AD curve would shift to the right and the tax cut would increase the level of output. Here, there is also an effect on the LM curve. A decrease in taxes means that at a given Y, disposable income, (Y - T), is higher. Thus so is real money demand. At a given Y, this requires a higher i than before to have real money demand equal real money supply. So the LM curve shifts up to LM '.



There is an ambiguity. The LM curve <u>could</u> shift up so much that -- as in the case depicted in the figure at right -- the intersection of IS ' and LM ' occurs to the left of the original Y_0. That is, Y for a given P falls and so the AD curve would actually shift to the left in this case. Thus the tax cut could end up reducing output. [Note that the fall in P increases the real money supply and thus shifts LM down to LM " to complete the story].

Determining whether output rises or falls is equivalent to determining whether the AD curve shifts to the left or to the right. So we can see what happens to Y, for a given P, and that will answer the question of whether output rises or falls in the end.

Differentiate the LM relationship, M/P = L(i, Y - T), with respect to T, holding M and P constant:

$$0 = L_i \frac{di}{dT} + L_{Y-T}\left(\frac{dY}{dT} - 1\right).$$

Rearranging to solve for di/dT yields

$$(1)\quad \frac{di}{dT} = \frac{L_{Y-T}}{L_i} - \frac{L_{Y-T}}{L_i}\frac{dY}{dT}.$$

Differentiate the IS relationship, $Y = E(Y, i - \pi^e, G, T)$, with respect to T, holding π^e, G (and P) constant:

$$(2)\quad \frac{dY}{dT} = E_Y \frac{dY}{dT} + E_{i-\pi^e}\frac{di}{dT} + E_T.$$

Substitute equation (1) into equation (2):

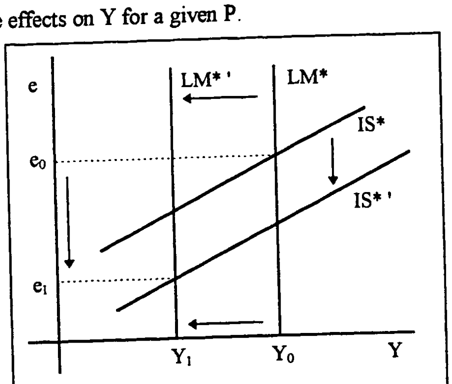
$$\frac{dY}{dT} = E_Y \frac{dY}{dT} + E_{i-\pi^e}\left(\frac{L_{Y-T}}{L_i} - \frac{L_{Y-T}}{L_i}\frac{dY}{dT}\right) + E_T.$$

Solving for dY/dT yields

$$(3)\quad \left.\frac{dY}{dT}\right|_{P\ \text{fixed}} = \frac{\left(E_{i-\pi^e} L_{Y-T}/L_i\right) + E_T}{1 - E_Y + \left(E_{i-\pi^e} L_{Y-T}/L_i\right)}.$$

The denominator of equation (3) is positive since $E_{i-\pi^e} < 0$, $L_{Y\text{-}T} > 0$, $L_i < 0$, and $E_Y < 1$. The sign of the numerator is indeterminate. The first term is negative and is basically capturing the money-market effects of the cut in T. The second term is positive and is basically capturing the effects in the goods market of the cut in T. If the former effect is stronger then dY/dT for a given P will be positive, meaning a cut in T reduces Y for a given P. This is equivalent to saying that the AD curve shifts to the left.

(b) We will again restrict ourselves to looking at the effects on Y for a given P.



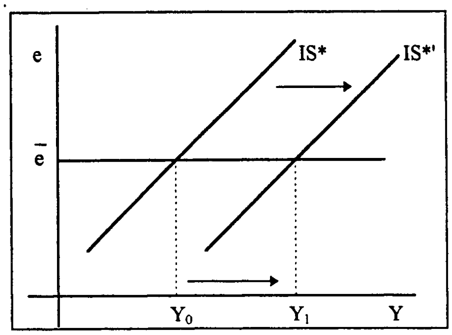
Due to the tax cut, at a given e, planned expenditure is higher since $E_T < 0$ and therefore we need a higher Y at a given e to satisfy the equilibrium condition that Y = E. Thus IS* shifts to the right to IS* '.

Ordinarily that would be the whole story. There would be an appreciation causing a reduction in planned expenditure by an amount equal to the rise in planned expenditure caused by the tax cut. Thus on net, planned expenditure and thus Y would not change. But here, at the original Y_0, Y_0 - T is now higher than before due to the tax cut. Therefore, at the original Y_0, real money demand would be higher, throwing the money market out of equilibrium. In order to clear the money market -- since i* cannot change -- we need a lower Y so that Y - T and thus real money demand is unchanged and still equal to the given M/P. Thus, LM* shifts to the left to LM* '. The end result is that output falls from Y_0 to Y_1. A tax cut actually reduces the level of output for a given P. Formally, differentiate the LM* relationship, M/P = L(i*, Y - T), with respect to T, holding M, P and i* constant:

$$0 = L_{Y-T}\left(\frac{dY}{dT} - 1\right) \Rightarrow \left.\frac{dY}{dT}\right|_{P\ \text{fixed}} = 1.$$

Output is determined entirely in the money market in this model. In response to a disturbance, since i*, M and P are constant, it must be Y that adjusts to ensure equilibrium. As T falls, Y must fall by an equal amount.

(c) Again we will look at the effects on Y for a given P. We are assuming the central bank will ensure money-market equilibrium. It will do so by meeting money demand with the appropriate money supply to ensure that the nominal exchange rate remains fixed. Thus our alteration to the money demand function does not mean that the nominal exchange rate will be affected by tax changes; it simply means that the central bank will have to act to offset the effects of tax changes on money demand.



After T falls, IS* shifts to the right for the same reason as described above. Any increase in money demand that is caused by T falling is simply met with an increase in the money supply by the central bank; M is no longer exogenous. We no longer need a drop in Y to maintain money-market equilibrium. The end result is that Y rises to Y_1. We get the "usual" result that a tax cut increases output for a given P.

Problem 5.8

Planned expenditure is given by

(1) $E = C(Y - T) + I(i - \pi^e) + G + NX(\varepsilon P^*/P)$.

For a floating exchange rate and perfect capital mobility, we have

(2) $Y = C(Y - T) + I(i^* - \pi^e) + G + NX(\varepsilon P^*/P)$ IS* Curve,

(3) $M/P = L(i^*, Y)$ LM* Curve.

For a fixed exchange rate and perfect capital mobility, we have equation (2) for the IS* curve and the following exchange rate equation:

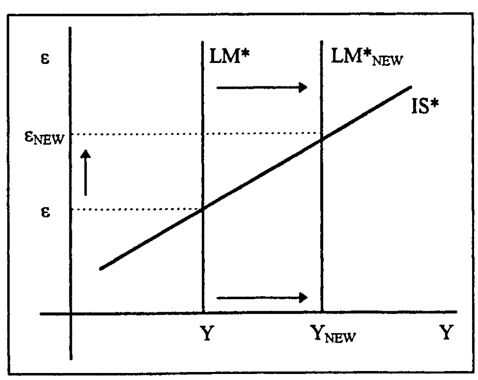
(4) $\varepsilon = \bar{\varepsilon}$.

For a floating exchange rate and imperfect capital mobility, we have

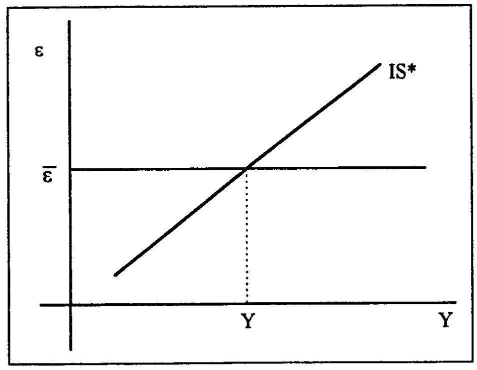
(5) $Y = C(Y - T) + I(i - \pi^e) + G - CF(i - i^*)$ IS** Curve,

(6) $M/P = L(i, Y)$ LM Curve.

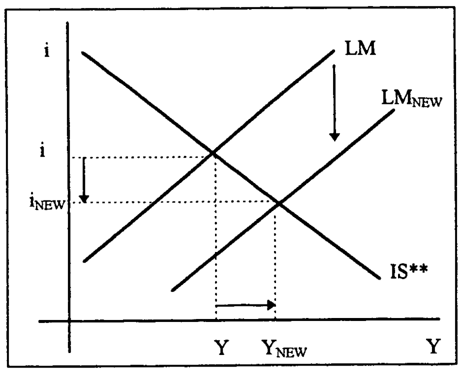
(a) (i) From equation (3), with M/P unchanged, $L(i^*, Y)$ must be unchanged as well for the money market to remain in equilibrium. Since i* does not change and because $L_Y > 0$, Y must rise so that money demand returns to its original value at that given i*. Thus the LM* curve shifts to the right to LM_{NEW}. The IS* curve is unaffected; money demand does not appear in equation (2). Thus for a given P, income rises to Y_{NEW} and ε rises to ε_{NEW} (the domestic currency depreciates). Finally, since $NX_{\varepsilon P^*/P} > 0$, the rise in ε increases net exports for a given P.



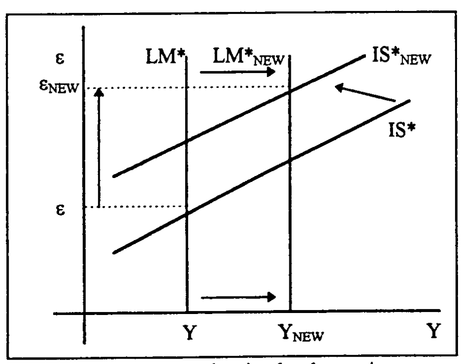
(a) (ii) Again, the drop in the demand for money does not affect the IS* curve. In addition, the nominal exchange rate remains pegged at $\bar{\varepsilon}$. The central bank simply reduces the money supply to match the decrease in money demand. Thus for a given P, income, the exchange rate and net exports are all unaffected. With a fixed exchange rate, disturbances in the money market have no impact on Y for a given P.



(a) (iii) From equation (5), the IS** curve is unaffected. The LM curve shifts down. At a given level of income, the interest rate must be lower in order to drive money demand back up and keep the money market in equilibrium. From the figure, income for a given P rises. Since the domestic interest rate is now lower and since $CF'(i - i^*) > 0$, capital flows are lower. Thus NX must be higher in order for the balance of payments to be 0. Since $NX_{\varepsilon P^*/P} > 0$, ε must be higher. That is, the domestic currency must have depreciated.

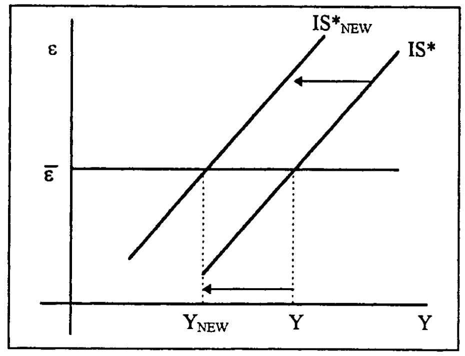


(b) (i) The foreign interest rate rises. Since $L_{i^*} < 0$, the rise in i* tends to reduce real money demand. To offset this and keep the money market in equilibrium, Y must be higher. Since $L_Y > 0$, a higher Y will keep real money demand unchanged and equal to the unchanged real money supply. Thus the LM* curve shifts to the right.

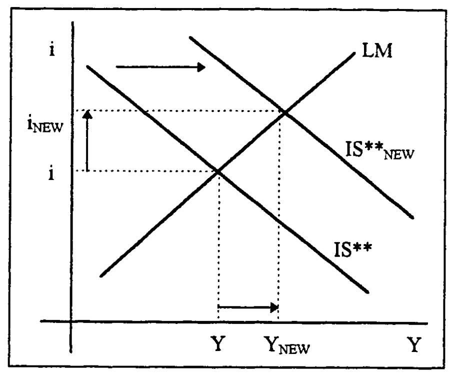


From equation (2), the rise in i* reduces planned expenditure at a given ε since $I_{i-\pi^e} < 0$. This reduces the level of Y that equates planned and actual expenditure at a given ε. In other words, the IS* curve shifts to the left. Thus income for a given P rises. The exchange rate rises; that is, the domestic currency depreciates. Finally, since $NX_{\varepsilon P^*/P} > 0$, net exports rise due to the depreciation.

(b) (ii) Again, the IS* curve shifts to the left due to the rise in the foreign interest rate. The nominal exchange rate remains pegged at $\bar{\varepsilon}$. The central bank is offsetting the effects on money demand with changes in the nominal money supply. Thus for a given P, income falls. Since the exchange rate does not change, neither do net exports.



(b) (iii) From equation (6), the LM curve is unaffected by the rise in i*. At a given i, the rise in i* reduces capital flows and thus, in order for the balance of payments to equal 0, must increase net exports. Thus at a given i, planned expenditure is now higher. Thus at a given i, the level of Y that equates planned and actual expenditure is higher. The IS** curve shifts to the right. At a given P, the level of income rises.



One way to see the effects on NX and ε for a given P is the following. Differentiate both sides of equation (6), the LM curve, with respect to i*, holding M and P constant:

$$0 = L_i \frac{di}{di^*} + L_Y \frac{dY}{di^*}.$$

Solving for dY/di* yields

$$(7) \quad \frac{dY}{di^*} = -\frac{L_i}{L_Y}\frac{di}{di^*}.$$

Differentiate both sides of equation (5), the IS** curve, with respect to i*, holding T, π^e and G constant:

$$(8) \quad \frac{dY}{di^*} = C_{Y-T}\frac{dY}{di^*} + I_{i-\pi^e}\frac{di}{di^*} - CF'(i - i^*)\left[\frac{di}{di^*} - 1\right].$$

Substitute equation (7) into equation (8):

$$-\frac{L_i}{L_Y}\frac{di}{di^*} = -C_{Y-T}\frac{L_i}{L_Y}\frac{di}{di^*} + I_{i-\pi^e}\frac{di}{di^*} - CF'(i - i^*)\frac{di}{di^*} + CF'(i - i^*).$$

Collecting the terms in di/di* gives us

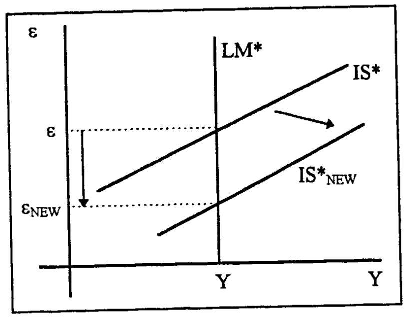
$$\frac{di}{di^*}\left\{-\left[1 - C_{Y-T}\right]\left(L_i/L_Y\right) - I_{i-\pi^e} + CF'(i - i^*)\right\} = CF'(i - i^*),$$

and thus the change in the domestic interest rate due to a change in the foreign interest rate is given by

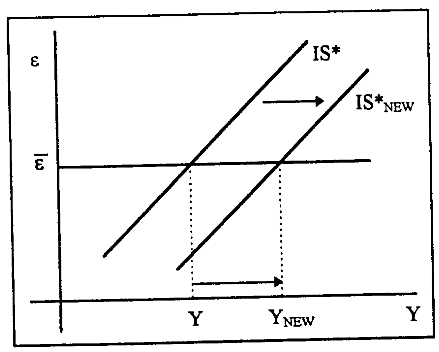
$$(9) \quad \frac{di}{di^*} = \frac{CF'(i - i^*)}{CF'(i - i^*) - I_{i-\pi^e} - \left[1 - C_{Y-T}\right]\left(L_i/L_Y\right)}.$$

Note that $CF'(\bullet) > 0$, $I_{i-\pi^e} < 0$, $[1 - C_{Y-T}] > 0$, $L_i < 0$ and $L_Y > 0$. Thus $di/di^* < 1$. So i rises less than i*; that is, i - i* falls, and thus capital flows decrease. This means that NX must increase at a given P to ensure a balance of payments equal to 0. Finally, since $NX_{\varepsilon P^*/P} > 0$, if NX rises it must be the case that ε rises at a given P. That is, the domestic currency depreciates.

(c) The country adopts protectionist policies so that net exports at a given real exchange rate are higher than before.



(c) (i) From equation (3), the LM* curve is unaffected. Since net exports are higher at a given ε, planned expenditure is higher at a given ε. Thus the level of Y that equates planned and actual expenditure at a given ε is higher. The IS* curve shifts to the right. For a given P, the level of income is unchanged; it is determined entirely in the money market here. All that happens is a drop in ε -- an appreciation of the domestic currency -- which keeps NX unchanged at a given P. Protectionist policies do not improve the trade balance in this model.



(c) (ii) Again, the IS* curve shifts to the right. The nominal exchange rate remains pegged at $\bar{\varepsilon}$. The central bank adjusts the nominal money stock to ensure that the exchange rate does not change. For a given P, income rises.

Even though the exchange rate is unchanged, net exports at a given P wind up higher in the end due to the protectionist policies. Thus unlike with a floating exchange rate, protectionist policies do work with a fixed exchange rate regime.

(c) (iii) The LM curve is unaffected by the protectionist policies. In addition, the IS** curve is unaffected; see equation (5), where NX does not appear. Since CF(i - i*) is not affected by this policy, NX cannot change in the end either. Thus income for a given P does not change, nor do net exports. What must happen is that the domestic currency appreciates -- ε falls -- which offsets the effect of the protectionist policies on net exports. This is the same result obtained with perfect capital mobility and flexible exchange rates.

Problem 5.9

(a) With this foreign exchange market intervention, the balance of payments equation is

(1) $CF(i - i^*) + NX(Y, i - \pi^e, G, T, \varepsilon P^*/P) = a$, where $a > 0$.

Thus net exports are given by

(2) $NX(Y, i - \pi^e, G, T, \varepsilon P^*/P) = a - CF(i - i^*)$.

Substituting this expression for net exports into equation (5.22) in the text, $Y = E^D(Y, i - \pi^e, G, T) + NX(Y, i - \pi^e, G, T, \varepsilon P^*/P)$, yields the IS** curve:

(3) $Y = E^D(Y, i - \pi^e, G, T) + a - CF(i - I^*)$.

Note that CF ' (•) > 0, $I_{i-\pi^e} < 0$, $[1 - C_{Y-T}] > 0$, $L_i < 0$ and $L_Y > 0$. Thus di/di* < 1. So i rises less than i*; that is, i - i* falls, and thus capital flows decrease. This means that NX must increase at a given P to ensure a balance of payments equal to 0. Finally, since $NX_{\varepsilon P^*/P} > 0$, if NX rises it must be the case that ε rises at a given P. That is, the domestic currency depreciates.

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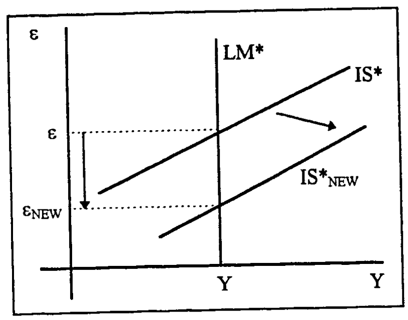

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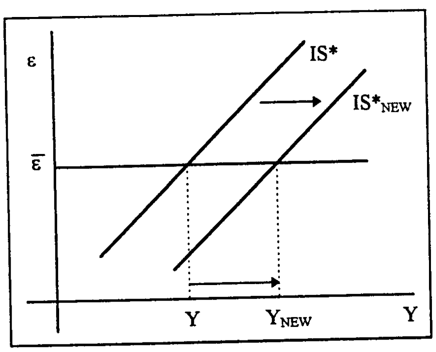

(c) (iii) The LM curve is unaffected by the protectionist policies. In addition, the IS** curve is unaffected; see equation (5), where NX does not appear. Since CF(i - i*) is not affected by this policy, NX cannot change in the end either. Thus income for a given P does not change, nor do net exports. What must happen is that the domestic currency appreciates -- ε falls -- which offsets the effect of the protectionist policies on net exports. This is the same result obtained with perfect capital mobility and flexible exchange rates.

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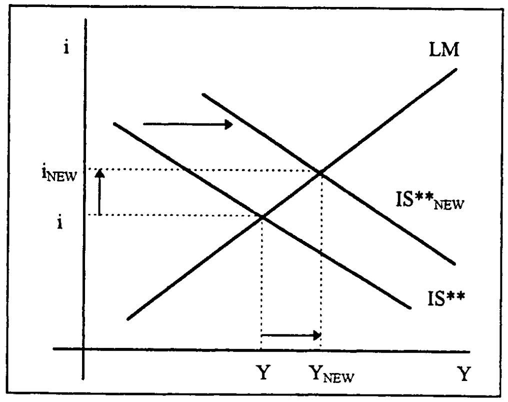
(3) $Y = E^D(Y, i - \pi^e, G, T) + a - CF(i - I^*)$.

Compared to the situation of a = 0, this raises the level of Y for a given interest rate and thus shifts the IS** curve to the right. This tells us that Y for a given P rises and hence the AD curve shifts to the right. As long as the AS curve is upward-sloping, both output and the price level rise in the end. The intervention, in which the central bank sells domestic currency, causes the domestic currency to depreciate. That is, ε rises.



(b) If capital is perfectly mobile, sterilized intervention has no effects. The IS** curve is horizontal under perfect capital mobility. Thus a "shift to the right" of the IS** curve does not actually affect its position.

Problem 5.10

(a) From the equation describing the behavior of inflation, $\dot{p} = \theta y$, when all prices have adjusted ($\dot{p} = 0$), y must equal 0. Substituting $\dot{p} = 0$, y = 0 and $i = \dot{\varepsilon}$ into the IS equation, $y = b(\varepsilon - p) - a(i - \dot{p})$, yields

(1) $0 = b(\varepsilon - p) - a\dot{\varepsilon}$,

and thus the change in the log exchange rate is

(2) $\dot{\varepsilon} = (b/a)(\varepsilon - p)$.

Since p is constant, if ε were greater than p, $\dot{\varepsilon} > 0$ and so ε would rise without bound. If ε were less than p, $\dot{\varepsilon} < 0$ and so ε would continually fall. Thus when all prices have adjusted, we must have $\varepsilon = p$ for ε to remain constant. Substituting y = 0 and $i = \dot{\varepsilon} = 0$ into the LM equation, $m - p = hy - ki$, yields

(3) $m = p$.

Thus once prices have fully adjusted, $y = i = 0$ and $m = p = \varepsilon$.

(b) If ε is to jump to exactly m and then remain constant, we need an equilibrium where $\varepsilon = m$ and $\dot{\varepsilon} = 0$. Since $i = \dot{\varepsilon}$, we must have i remain equal to its fully adjusted level of 0. Thus with i constant, y must adjust to ensure money-market equilibrium. Substituting $\dot{p} = \theta y$, $\varepsilon = m$ and i = 0 into the IS equation yields

(4) $y = b(m - p) + a\theta y$.

Solving equation (4) for y yields

(5) $y = [b/(1 - a\theta)](m - p)$.

Substituting the assumption of i = 0 into the LM equation gives us

(6) $m - p = hy$.

Substituting equation (6) into equation (5) yields

(7) $y = [b/(1 - a\theta)]hy$.

Thus we need the parameters to satisfy

(8) $1 - a\theta = bh$,

or equivalently

(9) $a\theta + bh = 1$.

Problem 5.11

(a) Taking the derivative of both sides of equation (5.12), $Y = E(Y, i - \pi^e, G, T, \varepsilon P^*/P)$, with respect to Y, holding π^e, G, T, P* and P constant yields

$$(1)\quad 1 = E_Y + E_{i-\pi^e}\frac{di}{dY} + E_{\varepsilon P^*/P}\frac{d\varepsilon}{dY}.$$

Taking the derivative of both sides of the balance of payments equation (5.21), $CF(i - i^*) + NX(Y, i - \pi^e, G, T, \varepsilon P^*/P) = 0$, with respect to Y, holding i^*, π^e, G, T, P* and P constant gives us

$$(2)\quad CF'(i)\frac{di}{dY} + NX_Y + NX_{i-\pi^e}\frac{di}{dY} + NX_{\varepsilon P^*/P}\frac{d\varepsilon}{dY} = 0.$$

Rearranging equation (1) to solve for $d\varepsilon/dY$ yields

$$(3)\quad \frac{d\varepsilon}{dY} = \frac{(1 - E_Y)}{E_{\varepsilon P^*/P}} - \frac{E_{i-\pi^e}}{E_{\varepsilon P^*/P}}\frac{di}{dY}.$$

Substituting equation (3) into equation (2) leaves us with

$$CF'(i)\frac{di}{dY} + NX_Y + NX_{i-\pi^e}\frac{di}{dY} + \frac{NX_{\varepsilon P^*/P}(1 - E_Y)}{E_{\varepsilon P^*/P}} - \frac{NX_{\varepsilon P^*/P}E_{i-\pi^e}}{E_{\varepsilon P^*/P}}\frac{di}{dY} = 0.$$

Collecting terms yields

$$\frac{di}{dY}\left[CF'(i) + NX_{i-\pi^e} - \frac{NX_{\varepsilon P^*/P}E_{i-\pi^e}}{E_{\varepsilon P^*/P}}\right] = -\left[NX_Y + \frac{NX_{\varepsilon P^*/P}}{E_{\varepsilon P^*/P}}(1 - E_Y)\right].$$

Thus the slope of the IS** curve is given by

$$(4)\quad \left.\frac{di}{dY}\right|_{IS^{**}} = \frac{-\left[NX_Y + \dfrac{NX_{\varepsilon P^*/P}}{E_{\varepsilon P^*/P}}(1 - E_Y)\right]}{CF'(i) + NX_{i-\pi^e} - \dfrac{NX_{i-\pi^e}E_{i-\pi^e}}{E_{\varepsilon P^*/P}}} < 0.$$

We are told that $NX_Y + (1 - E_Y) > 0$. Since $NX_{\varepsilon P^*/P}/E_{\varepsilon P^*/P} \geq 1$, the term in brackets in the numerator is also positive and thus the numerator itself is negative. It is straightforward to verify that the denominator is positive and thus IS** remains downward-sloping.

(b) Rearranging equation (2) to solve for $d\varepsilon/dY|_{IS^{**}}$ yields

$$(5)\quad \left.\frac{d\varepsilon}{dY}\right|_{IS^{**}} = \frac{-\left[CF'(i) + NX_{i-\pi^e}\right]\left.\dfrac{di}{dY}\right|_{IS^{**}} - NX_Y}{E_{\varepsilon P^*/P}} > 0.$$

$[CF'(i) + NX_{i-\pi}]$ is positive and $di/dY|_{IS^{**}}$ is negative. Since NX_Y is negative, the numerator is positive. The denominator is positive . Thus as we move down the IS** curve and Y rises, ε rises as well. That is, as we move down the IS** curve, the domestic currency depreciates.

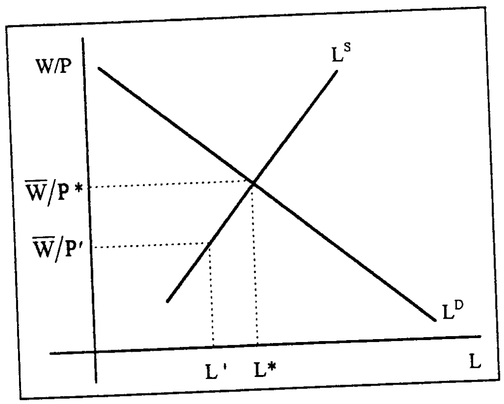
(c) In order to see the effect of an increase in capital mobility, use equation (4) to take the derivative of the slope of the IS** curve with respect to CF ' (i):

$$\frac{\partial\left(\left.\frac{di}{dY}\right|_{IS^{**}}\right)}{\partial CF'(i)} = \frac{NX_Y + \frac{NX_{\varepsilon P^*/P}}{E_{\varepsilon P^*/P}}\left(1 - E_Y\right)}{\left[CF'(i) + NX_{i-\pi^e} - \frac{NX_{\varepsilon P^*/P}\, E_{i-\pi^e}}{E_{\varepsilon P^*/P}}\right]^2} > 0.$$

Again, we are told that $NX_Y + (1 - E_Y) > 0$. Since $NX_{\varepsilon P^*/P} / E_{\varepsilon P^*/P} \geq 1$, the numerator is positive. Thus a rise in CF ' (i) -- an increase in the degree of capital mobility -- causes the slope of the IS** curve to rise or become less negative. That is, the IS** curve become flatter.

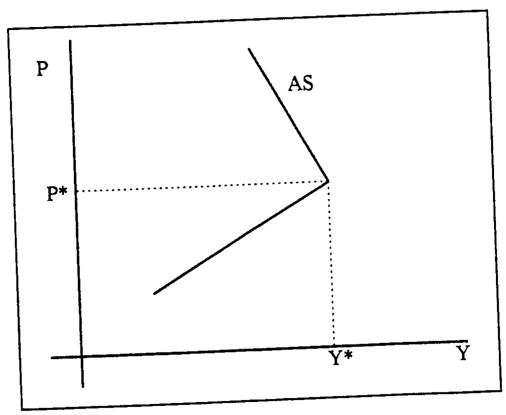
Problem 5.12

(a) (i) At price level P*, and thus at real wage $\overline{W}/P^*$, employment and output are at their maximum possible levels. This is where the labor demand curve and the labor supply curve intersect and there is no unemployment.



(a) (ii) At a higher price, say P ' > P*, and thus at a lower real wage $\overline{W}/P' < \overline{W}/P^*$, labor demand exceeds labor supply. Given the "short-side" rule, this means that employment is determined by labor supply and is at L ' in the figure. Since F ' (L) > 0, output is lower at L ' (and P ') than it is at L* (and P*).

(b) As the price level rises toward P*, employment is determined by labor demand. Thus employment and output rise as the economy moves down the labor demand curve.

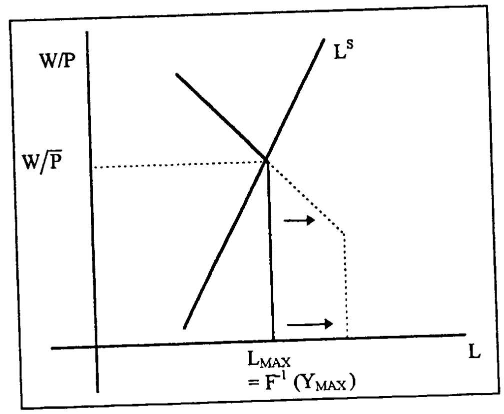


At price P*, output is at Y*, its maximum possible value.

As the price level rises above P*, employment is determined by labor supply. Thus employment and output fall as the economy moves down the labor supply curve. See the figure for what this implies about the shape of the aggregate supply curve. At prices above P*, the AS curve is backward-bending under the "short-side" assumption.

Problem 5.13

For this to be the case, the effective labor demand curve must intersect the labor supply curve right at the point at which it becomes vertical. Any further rightward shift in the effective labor demand curve would not cause the point of intersection of the two curves to change. See the figure at right.



Thus even if the effective labor demand curve shifts out, employment does not rise. Thus output cannot rise above Y_{MAX}.

Problem 5.14

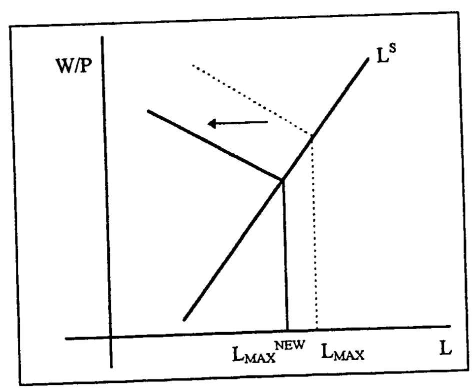
In Model 1, Keynes's Model, firms hire labor up to the point at which the marginal product of labor equals the real wage. That is, L is determined by

(1) $AF'(L) = \overline{W}/P$.

At a given price, and thus at a given real wage, the level of L that equates the marginal product with the real wage is now lower after the fall in A. The labor demand curve shifts to the left. Firms reduce the amount of labor they demand at a given price. So output supplied by firms at a given price level is now lower for two reasons. A given amount of labor now produces less output and firms choose to hire less labor at a given P. Thus the AS curve shifts to the left.

In Model 2 -- sticky prices, flexible wages and a competitive labor market -- the AS curve is still horizontal at $P = \overline{P}$ out to Y_{MAX}, where Y_{MAX} is the level of output at which marginal cost just equals the fixed price level. However, Y_{MAX} itself will now change.

As described in the solution to Problem 5.13, at Y_{MAX}, the effective labor demand curve intersects the labor supply curve right at the point where it becomes vertical. Now with A lower, the downward-sloping portion of the labor demand curve shifts to the left; at a given real wage, the level of L that equates the marginal product of labor and the real wage is lower. This means that there is a new lower L_{MAX}. The maximum amount of labor that it is profitable to hire to meet demand at the fixed price is now lower.



Therefore Y_{MAX} is lower for two reasons. The maximum amount of labor that it is profitable to hire at the fixed price is lower and a given amount of labor now produces less output.

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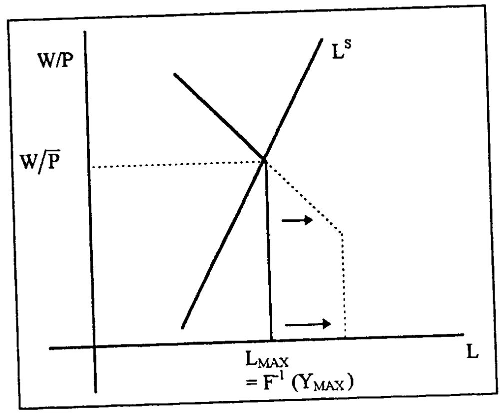

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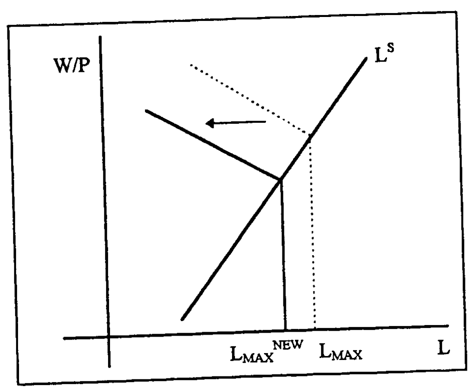


Therefore Y_{MAX} is lower for two reasons. The maximum amount of labor that it is profitable to hire at the fixed price is lower and a given amount of labor now produces less output.

In Model 3 -- sticky prices, flexible wages and real labor market imperfections -- we get essentially the same result as in Model 2. The level of output at which marginal cost just equals the fixed price is lower after the fall in A. Thus the AS curve is horizontal but out to a lower Y_{MAX}.

In Model 4 -- sticky wages, flexible prices and imperfect competition -- the price is set as a markup over marginal cost:

(2) $P = \mu(L)\dfrac{W}{AF'(L)}$.

A fall in A raises marginal cost at a given L. Assuming the markup does not depend on A, the price level rises for a given L. The AS curve shifts up, regardless of whether it is downward-sloping, upward-sloping or horizontal.

Problem 5.15

(a) (i) We can solve for output and the interest rate. Begin by substituting the expression for inflation, $\dot{p} = \theta y$, into the goods-market-equilibrium relationship, $y = -a(i - \dot{p})$, to obtain

(1) $y(t) = -a[i(t) - \theta y(t)]$.

Rearranging the money-market-equilibrium condition, $m - p = -ki$, yields

(2) $i(t) = [p(t) - m(t)]/k$.

Substitute equation (2) into equation (1) to obtain

$$y(t) = -a\big[\big(p(t) - m(t)\big)/k - \theta y(t)\big] \;\Rightarrow\; y(t) - a\theta y(t) = (a/k)[m(t) - p(t)].$$

Simplifying and solving for output yields

(3) $y(t) = \dfrac{a[m(t) - p(t)]}{k(1 - a\theta)}$.

Substituting $p(0) = 0$ and $m(0) = m'$ into equation (3) yields

(4) $y(0) = \dfrac{am'}{k(1 - a\theta)} < 0$, since $a\theta < 1$, $a > 0$, $k > 0$, and $m' < 0$.

Substituting $p(0) = 0$ and $m(0) = m'$ into equation (2) yields

(5) $i(0) = -m'/k > 0$.

Thus the level of output falls from its previous value of 0 and the nominal interest rate rises from its previous value of 0.

In order to see how an increase in θ, the speed of price adjustment, affects the value of y(0), take the derivative of output at time 0 with respect to θ:

$$\frac{\partial y(0)}{\partial \theta} = \frac{-am'(-a)}{k(1 - a\theta)^2} = \frac{a^2 m'}{k(1 - a\theta)^2} < 0, \quad \text{since } m' < 0.$$

Thus y(0) falls even more from its initial value of 0 if θ is higher. The intuition is that a bigger θ, from $\dot{p} = \theta y$, implies that as y falls there will be an even bigger drop in inflation. In turn, this means a bigger increase in the real interest rate. This reduces planned expenditure, and thus equilibrium output, even further.

(a) (ii) Take the time derivative of equation (3), noting that $m(t) = m'$ for all $t \geq 0$:

(6) $\dot{y}(t) = \dfrac{-a\dot{p}(t)}{k(1 - a\theta)}$.

Substituting the expression for inflation, $\dot{p} = \theta y$, into equation (6) yields

In Model 3 -- sticky prices, flexible wages and real labor market imperfections -- we get essentially the same result as in Model 2. The level of output at which marginal cost just equals the fixed price is lower after the fall in A. Thus the AS curve is horizontal but out to a lower Y_{MAX}.

In Model 4 -- sticky wages, flexible prices and imperfect competition -- the price is set as a markup over marginal cost:

(2) $P = \mu(L)\dfrac{W}{AF'(L)}$.

A fall in A raises marginal cost at a given L. Assuming the markup does not depend on A, the price level rises for a given L. The AS curve shifts up, regardless of whether it is downward-sloping, upward-sloping or horizontal.

<u>**Problem 5.15**</u>

(a) (i) We can solve for output and the interest rate. Begin by substituting the expression for inflation, $\dot{p} = \theta y$, into the goods-market-equilibrium relationship, $y = -a(i - \dot{p})$, to obtain

(1) $y(t) = -a[i(t) - \theta y(t)]$.

Rearranging the money-market-equilibrium condition, $m - p = -ki$, yields

(2) $i(t) = [p(t) - m(t)]/k$.

Substitute equation (2) into equation (1) to obtain

$$y(t) = -a\big[\big(p(t) - m(t)\big)/k - \theta y(t)\big] \quad \Rightarrow \quad y(t) - a\theta y(t) = (a/k)\big[m(t) - p(t)\big].$$

Simplifying and solving for output yields

(3) $y(t) = \dfrac{a\big[m(t) - p(t)\big]}{k(1 - a\theta)}$.

Substituting $p(0) = 0$ and $m(0) = m'$ into equation (3) yields

(4) $y(0) = \dfrac{am'}{k(1 - a\theta)} < 0$, since $a\theta < 1$, $a > 0$, $k > 0$, and $m' < 0$.

Substituting $p(0) = 0$ and $m(0) = m'$ into equation (2) yields

(5) $i(0) = -m'/k > 0$.

Thus the level of output falls from its previous value of 0 and the nominal interest rate rises from its previous value of 0.

In order to see how an increase in θ, the speed of price adjustment, affects the value of $y(0)$, take the derivative of output at time 0 with respect to θ:

$$\frac{\partial y(0)}{\partial \theta} = \frac{-am'(-a)}{k(1 - a\theta)^2} = \frac{a^2 m'}{k(1 - a\theta)^2} < 0, \quad \text{since } m' < 0.$$

Thus $y(0)$ falls even more from its initial value of 0 if θ is higher. The intuition is that a bigger θ, from $\dot{p} = \theta y$, implies that as y falls there will be an even bigger drop in inflation. In turn, this means a bigger increase in the real interest rate. This reduces planned expenditure, and thus equilibrium output, even further.

(a) (ii) Take the time derivative of equation (3), noting that $m(t) = m'$ for all $t \geq 0$:

(6) $\dot{y}(t) = \dfrac{-a\dot{p}(t)}{k(1 - a\theta)}$.

Substituting the expression for inflation, $\dot{p} = \theta y$, into equation (6) yields

(7) $\dot{y}(t) = \dfrac{-a\theta}{k(1-a\theta)} y(t)$.

So log output will exponentially approach its original value of 0. More formally, solving the differential equation (7) yields

(8) $y(t) = e^{[-a\theta/(1-a\theta)k]t} y(0)$.

Substituting equation (4) into equation (8) yields

(9) $y(t) = e^{[-a\theta/(1-a\theta)k]t} \dfrac{am'}{k(1-a\theta)}$.

(b) Substituting the expression for y(t) from equation (9) into the definition of the amount of output volatility caused by a disturbance, $V = \int_{t=0}^{\infty} y(t)^2\,dt$, yields

$$V = \int_{t=0}^{\infty} e^{[-2a\theta/(1-a\theta)k]t} \left[\frac{am'}{k(1-a\theta)}\right]^2 dt \;\Rightarrow\; V = \left[\frac{am'}{k(1-a\theta)}\right]^2 \int_{t=0}^{\infty} e^{[-2a\theta/(1-a\theta)k]t} dt.$$

Solving the integral is straightforward and doing so yields

$$V = \left[\frac{am'}{k(1-a\theta)}\right]^2 \left[\frac{(1-a\theta)k}{2a\theta}\right].$$

Simplifying this expression allows us to obtain

(10) $V = \dfrac{am'^2}{2k(\theta - a\theta^2)}$.

To see how a change in the speed of price adjustment, θ, affects volatility, take the derivative of V with respect to θ:

(11) $\dfrac{\partial V}{\partial \theta} = \dfrac{-am'^2}{2k(\theta - a\theta^2)^2}(1-2a\theta)$.

The sign of this derivative will be determined by the sign of (1 - 2aθ).

If (1 - 2aθ) > 0 or θ < 1/2a then ∂V/∂θ < 0. Thus for "small" values of θ, a marginal increase in price flexibility will reduce output volatility. If (1 - 2aθ) < 0 or 1/a > θ > 1/2a then ∂V/∂θ > 0. Thus for "large" values of θ, a marginal increase in price flexibility will actually increase output volatility. [Note that we assumed from the outset that aθ < 1 or equivalently θ < 1/a.] Finally, note that the higher is a -- where a captures how responsive planned expenditure is to changes in the real interest rate -- the smaller is the range of θ's over which increased price flexibility reduces output volatility.

SOLUTIONS TO CHAPTER 6

Problem 6.1

(a) The individual's problem is to choose labor supply, L_i, to maximize expected utility, conditional on the realization of P_i. That is, the problem is

$$\max_{L_i} E\left[\left(C_i - (1/\gamma)L_i^{\gamma}\right) \middle| P_i\right].$$

Substituting $C_i = P_iQ_i/P$ and $Q_i = L_i$ gives us

$$\max_{L_i} E\left[\left(\frac{P_iL_i}{P} - \frac{1}{\gamma}L_i^{\gamma}\right) \middle| P_i\right].$$

Since only P is uncertain, this can be rewritten as

$$\max_{L_i} E\left[\left(P_i/P\right) \middle| P_i\right] L_i - (1/\gamma)L_i^{\gamma}.$$

The first-order condition is given by

(1) $E[(P_i/P)|P_i] - L_i^{\gamma-1} = 0,$

or

$$L_i^{\gamma-1} = E[(P_i/P)|P_i].$$

Thus optimal labor supply is given by

(2) $L_i = \left\{E\left[(P_i/P) \mid P_i\right]\right\}^{1/(\gamma-1)}.$

Taking the log of both sides of equation (2) and defining $\ell_i \equiv \ln L_i$ yields

(3) $\ell_i = [1/(\gamma - 1)]\ln E[(P_i/P)|P_i].$

(b) The amount of labor the individual supplies if she follows the certainty-equivalence rule is given by (in logs)

(4) $\ell_i = [1/(\gamma - 1)]E[\ln(P_i/P)|P_i].$

Since $\ln(P_i/P)$ is a concave function of (P_i/P), then by Jensen's inequality $\ln E[(P_i/P)|P_i] > E[\ln(P_i/P)|P_i]$. Thus the amount of labor the individual supplies if she follows the certainty-equivalence rule is <u>less</u> than the optimal amount derived in part (a).

(c) We are given that

(5) $\ln(P_i/P) = E[\ln(P_i/P)|P_i] + u_i, \qquad u_i \sim N(0,V_u).$

Taking the exponential function of both sides of equation (5) yields

(6) $P_i/P = e^{E\left[\ln(P_i/P)|P_i\right]}e^{u_i}.$

Now take the expected value, conditional on P_i, of both sides of equation (6):

(7) $E\left[\left(P_i/P\right) \middle| P_i\right] = e^{E\left[\ln(P_i/P)|P_i\right]}E\left[e^{u_i} \mid P_i\right].$

Taking the natural log of both sides of equation (7) yields

(8) $\ln E\left[\left(P_i/P\right) \middle| P_i\right] = E\left[\ln(P_i/P)|P_i\right] + \ln E\left[e^{u_i} \mid P_i\right].$

Note that $\ln E\left[e^{u_i} \mid P_i\right]$ is just a constant that is independent of P_i. Substituting equation (8) into equation (3), the expression for the optimal amount of (log) labor supply, gives us

$$\ell_i = [1/(\gamma - 1)]\left[E\left[\ln(P_i/P)|P_i\right] + \ln E\left[e^{u_i} \mid P_i\right]\right],$$

or simply

(9) $\ell_i = [1/(\gamma - 1)]\,E[\ln(P_i/P)|P_i] + [1/(\gamma - 1)]\left[\ln E\left[e^{u_i}\,|\,P_i\right]\right]$.

The first term on the right-hand side of equation (9), $[1/(\gamma - 1)]E[\ln(P_i/P)|P_i]$, is the certainty-equivalence choice of (log) labor supply and the second term is a constant. Thus the ℓ_i that maximizes expected utility differs from the certainty-equivalence rule only by a constant.

Problem 6.2

(a) The individual's problem is to maximize $C_i = \left[\int_{j=0}^{1} Z_j^{1/\eta} C_{ij}^{(\eta-1)/\eta} dj\right]^{\eta/(\eta-1)}$ subject to the budget constraint, $\int_{j=0}^{1} P_j C_{ij} dj = Y_i$. Set up the Lagrangian:

$$(1)\quad \mathcal{L} = \left[\int_{j=0}^{1} Z_j^{1/\eta} C_{ij}^{(\eta-1)/\eta} dj\right]^{\eta/(\eta-1)} + \lambda\left[Y_i - \int_{j=0}^{1} P_j C_{ij} dj\right].$$

The first-order condition for a representative good, C_{ij}, is

$$\frac{\partial \mathcal{L}}{\partial C_{ij}} = \left(\frac{\eta}{\eta-1}\right)\left[\int_{j=0}^{1} Z_j^{1/\eta} C_{ij}^{(\eta-1)/\eta} dj\right]^{[\eta/(\eta-1)]-1}\left(\frac{\eta-1}{\eta}\right) Z_j^{1/\eta} C_{ij}^{[(\eta-1)/\eta]-1} - \lambda P_j = 0.$$

We can simplify the exponents since $[\eta/(\eta - 1)] - 1 = 1/(\eta - 1)$ and $[(\eta - 1)/\eta] - 1 = -1/\eta$. Thus the preceding expression implies

$$(2)\quad C_{ij}^{-1/\eta} = \frac{\lambda P_j}{Z_j^{1/\eta}\left[\int_{j=0}^{1} Z_j^{1/\eta} C_{ij}^{(\eta-1)/\eta} dj\right]^{1/(\eta-1)}}.$$

Taking both sides of equation (2) to the exponent $-\eta$ yields

$$(3)\quad C_{ij} = \frac{Z_j\left[\int_{j=0}^{1} Z_j^{1/\eta} C_{ij}^{(\eta-1)/\eta} dj\right]^{\eta/(\eta-1)}}{\left(\lambda P_j\right)^{\eta}}.$$

(b) For each good on the unit interval, there will be an equation like (3). Thus for some other good, C_{ik}, we can write

$$(4)\quad C_{ik} = \frac{Z_k\left[\int_{j=0}^{1} Z_j^{1/\eta} C_{ij}^{(\eta-1)/\eta} dj\right]^{\eta/(\eta-1)}}{\left(\lambda P_k\right)^{\eta}}.$$

Dividing equation (3) by equation (4) yields

$$\frac{C_{ij}}{C_{ik}} = \frac{Z_j\left[\int_{j=0}^{1} Z_j^{1/\eta} C_{ij}^{(\eta-1)/\eta} dj\right]^{\eta/(\eta-1)}}{Z_k\left[\int_{j=0}^{1} Z_j^{1/\eta} C_{ij}^{(\eta-1)/\eta} dj\right]^{\eta/(\eta-1)}}\,\frac{\left(\lambda P_k\right)^{\eta}}{\left(\lambda P_j\right)^{\eta}}.$$

Simplifying yields

(5) $C_{ij}/C_{ik} = \left(P_k/P_j\right)^{\eta}\left(Z_j/Z_k\right)$.

Writing C_{ij} in terms of C_{ik}, we have

(6) $C_{ij} = \left(P_k/P_j\right)^{\eta}\left(Z_j/Z_k\right)C_{ik}$.

Substituting equation (6) into the budget constraint, $\int_{j=0}^{1} P_j C_{ij}dj = Y_i$, yields

$$\int_{j=0}^{1} P_j\left(P_k/P_j\right)^{\eta}\left(Z_j/Z_k\right)C_{ik}\,dj = Y_i.$$

Pulling the terms not indexed by j out of the integral sign leaves us with

$$\frac{C_{ik}P_k^{\eta}}{Z_k}\int_{j=0}^{1} Z_j P_j^{1-\eta}dj = Y_i,$$

and then solving for C_{ik} yields

(7) $C_{ik} = \dfrac{Y_i Z_k}{P_k^{\eta}\left[\int_{j=0}^{1} Z_j P_j^{1-\eta}dj\right]}$.

Equation (7) holds for all goods. Thus returning to the notation of C_{ij} as our representative good gives us

(8) $C_{ij} = \dfrac{Y_i Z_j}{P_j^{\eta}\left[\int_{j=0}^{1} Z_j P_j^{1-\eta}dj\right]}$.

(c) Substituting equation (8) into the expression for C_i, $C_i = \left[\int_{j=0}^{1} Z_j^{1/\eta}C_{ij}^{(\eta-1)/\eta}dj\right]^{\eta/(\eta-1)}$, yields

$$C_i = \left[\int_{j=0}^{1} \frac{Z_j^{1/\eta}Y_i^{(\eta-1)/\eta}Z_j^{(\eta-1)/\eta}}{P_j^{(\eta-1)}\left[\int_{j=0}^{1} Z_j P_j^{1-\eta}dj\right]^{(\eta-1)/\eta}}dj\right]^{\eta/(\eta-1)}.$$

Pulling the terms not indexed by j out of the integral gives us

$$C_i = \left[\frac{Y_i^{(\eta-1)/\eta}}{\left[\int_{j=0}^{1} Z_j P_j^{1-\eta}dj\right]^{(\eta-1)/\eta}}\left(\int_{j=0}^{1} Z_j P_j^{1-\eta}dj\right)\right]^{\eta/(\eta-1)}.$$

This simplifies to

(9) $C_i = Y_i\left[\int_{j=0}^{1} Z_j P_j^{1-\eta}dj\right]^{[\eta/(\eta-1)]-1}$.

Since $[\eta/(\eta - 1)] - 1 = 1/(\eta - 1) = -1/(1 - \eta)$, equation (9) can be rewritten as

$$(10)\ \ C_i = \frac{Y_i}{\left[\int_{j=0}^{1} Z_j P_j^{1-\eta} dj\right]^{1/(1-\eta)}}.$$

Defining the price index as

$$(11)\ \ P \equiv \left[\int_{j=0}^{1} Z_j P_j^{1-\eta} dj\right]^{1/(1-\eta)},$$

we have

$$(12)\ \ C_i = Y_i / P.$$

(d) Note that $\int_{j=0}^{1} Z_j P_j^{1-\eta} dj \equiv P^{(1-\eta)}$. Using this fact, equation (8), the expression giving individual i's demand for good j, can be rewritten as

$$(13)\ \ C_{ij} = \frac{Y_i Z_j}{P_j^{\eta} P^{1-\eta}}.$$

This can be rearranged to yield

$$(14)\ \ C_{ij} = Z_j \left(P_j / P\right)^{-\eta} \left(Y_i / P\right).$$

Equation (14) gives individual i's demand for good j as a function of the taste shock for good j, good j's relative price, and individual i's real income.

(e) Taking the log of both sides of equation (14) yields

$$(15)\ \ c_{ij} = z_j - \eta(p_j - p) + (y_i - p).$$

This resembles equation (6.7) in the text. However, the price index given in equation (11) is not simply the average of the individual p's, as it is in equation (6.9).

Problem 6.3

(a) Model (i) is given by

$$(1)\ \ y_t = a' z_{t-1} + be_t + v_t.$$

This model says that only the unexpected component of money, e_t, affects output. Model (ii) is given by

$$(2)\ \ y_t = \alpha' z_{t-1} + \beta m_t + v_t.$$

This model says that all money matters for output.

Substituting the assumption about monetary policy, $m_t = c' z_{t-1} + e_t$, into equation (2) yields

$$(3)\ \ y_t = \alpha' z_{t-1} + \beta[c' z_{t-1} + e_t] + v_t,$$

and collecting terms in z_{t-1} gives us

$$(4)\ \ y_t = (\alpha' + \beta c')z_{t-1} + \beta e_t + v_t.$$

The models given by equations (1) and (4) cannot be distinguished from one another. Given some a' and b, $\alpha' = a' - \beta c'$ and $\beta = b$ have the same predictions. Intuitively, it is not possible to separate the direct effect of the z's on output from any possible indirect effect they may have through monetary policy. So it could be the case that only unexpected money matters and the effect of the z's on output that we observe is simply their direct effect. However, it could also be the case that the expected component of money affects output and thus the effect of the z's that we observe consists of both the direct and indirect effects.

(b) Substituting the new assumption about monetary policy, $m_t = c' z_{t-1} + \gamma' w_{t-1} + e_t$, into model (ii) yields

(5) $y_t = \alpha' z_{t-1} + \beta[c' z_{t-1} + \gamma' w_{t-1} + e_t] + v_t$,

or collecting the z_{t-1} terms gives us

(6) $y_t = (\alpha' + \beta c')z_{t-1} + \beta\gamma' w_{t-1} + \beta e_t + v_t$.

In this case, it is possible to distinguish between the two theories. Model (i), only unexpected money matters, predicts that the coefficients on the w's should be zero. Model (ii), all money matters, does not predict this. Intuitively, since the w's do not directly affect output, if they are correlated with output it must be due to their indirect effect through their impact on the money supply.

__Problem 6.4__

Using the correct price index does not alter the analysis of the individual's behavior. That is, equation (6.40) in the text, which defines the optimal relative price of individual i's good as

(1) $\dfrac{P_i}{P} = \dfrac{\eta}{\eta - 1}\dfrac{W}{P}$,

still holds. Similarly, individual i's optimal choice of labor supply is unaffected. It is still given by equation (6.42) in the text, or

(2) $L_i = \left(\dfrac{W}{P}\right)^{1/(\gamma-1)}$.

We need to solve for equilibrium output, Y, and the price level, P, using the fact that total spending in the economy equals M, or

(3) $\displaystyle\int_{i=0}^{1} P_i Q_i di = M$,

where Q_i equals output of good i. Since the production function is $Q_i = L_i$, output of good i is

(4) $Q_i = \left(\dfrac{W}{P}\right)^{1/(\gamma-1)}$.

Multiplying both sides of equation (1) by P gives us

(5) $P_i = \dfrac{\eta}{\eta - 1}W$.

From equation (11) in the solution to Problem 6.2, the price index with all the Z_j's equal to zero is given by

(6) $P \equiv \left[\displaystyle\int_{i=0}^{\infty} P_i^{1-\eta} di\right]^{1/(1-\eta)}$.

Substituting equation (5) into the price index given by equation (6) yields

(7) $P \equiv \left[\displaystyle\int_{i=0}^{\infty} \left(\dfrac{\eta}{\eta-1}W\right)^{1-\eta} di\right]^{1/(1-\eta)} = \left[\left(\dfrac{\eta}{\eta-1}W\right)^{1-\eta} \displaystyle\int_{i=0}^{1} di\right]^{1/(1-\eta)}$,

which implies

(8) $P = \dfrac{\eta}{\eta - 1}W$.

Rearranging equation (8) gives us the equilibrium real wage:

(9) $\dfrac{W}{P} = \dfrac{\eta - 1}{\eta}$.

Substituting equation (9) into equation (4) gives individual i's output:

$$(10)\quad Q_i = \left(\frac{\eta-1}{\eta}\right)^{1/(\gamma-1)}.$$

Substituting $P_i = P$ and equation (10) into equation (3) yields

$$(11)\quad \int_{i=0}^{1} P\left(\frac{\eta-1}{\eta}\right)^{1/(\gamma-1)} di = M.$$

Since nothing inside the integral is indexed by i, we have

$$(12)\quad P\left(\frac{\eta-1}{\eta}\right)^{1/(\gamma-1)} \int_{i=0}^{1} di = M.$$

Solving equation (12) for P gives us

$$(13)\quad P = \left(\frac{\eta-1}{\eta}\right)^{-1/(\gamma-1)} M = \frac{M}{[(\eta-1)/\eta]^{1/(\gamma-1)}}.$$

Equation (13) is identical to equation (6.47) in the text.

Finally, since aggregate demand is given by Y = M/P, equilibrium output is

$$(14)\quad Y = \frac{M}{M/[(\eta-1)/\eta]^{1/(\gamma-1)}} = \left(\frac{\eta-1}{\eta}\right)^{1/(\gamma-1)}.$$

Equation (14) is identical to equation (6.46) in the text. Thus using the correct price index does not affect the expressions for equilibrium price and output.

Problem 6.5

(a) Substituting the expression for the nominal wage, $w = \theta p$, into the aggregate price equation, $p = w + (1 - \alpha)\ell - s$, yields $p = \theta p + (1 - \alpha)\ell - s$. Solving for p yields

(1) $p = [(1 - \alpha)\ell - s]/(1 - \theta)$.

Substituting the aggregate output equation, $y = s + \alpha\ell$, and equation (1) for the price level into the aggregate demand equation, $y = m - p$, yields

$$s + \alpha\ell = m - [(1 - \alpha)\ell - s]/(1 - \theta).$$

Collecting the terms in ℓ leaves us with

$$\alpha\ell + [(1 - \alpha)\ell/(1 - \theta)] = m + [s/(1 - \theta)] - s.$$

Obtaining a common denominator and simplifying gives us

$$[\alpha(1 - \theta) + (1 - \alpha)]\ell/(1 - \theta) = m + [1 - (1 - \theta)]s/(1 - \theta),$$

or

$$(1 - \alpha\theta)\ell/(1 - \theta) = m + [\theta s/(1 - \theta)],$$

and thus finally, employment is given by

$$(2)\quad \ell = \frac{(1-\theta)m + \theta s}{(1-\alpha\theta)}.$$

Substituting equation (2) into equation (1) yields

$$p = \frac{(1-\alpha)[(1-\theta)m + \theta s]}{(1-\theta)(1-\alpha\theta)} - \frac{s}{(1-\theta)}.$$

Simplifying gives us

$$p = \frac{(1-\alpha)(1-\theta)m + (1-\alpha)\theta s - (1-\alpha\theta)s}{(1-\theta)(1-\alpha\theta)} = \frac{(1-\alpha)(1-\theta)m - (1-\theta)s}{(1-\theta)(1-\alpha\theta)}.$$

Thus, the aggregate price level is given by

$$(3)\quad p = \frac{(1-\alpha)m - s}{(1-\alpha\theta)}.$$

Substituting equation (2) into the aggregate output equation, $y = s + \alpha\ell$, and simplifying yields

$$y = s + \frac{\alpha(1-\theta)m + \alpha\theta s}{(1-\alpha\theta)} = \frac{s - \alpha s + \alpha(1-\theta)m + \alpha\theta s}{(1-\alpha\theta)}.$$

And therefore, output is given by

$$(4)\quad y = \frac{s + \alpha(1-\theta)m}{(1-\alpha\theta)}.$$

Finally, to get an expression for the nominal wage, substitute equation (3) into $w = \theta p$:

$$(5)\quad w = \frac{\theta\left[(1-\alpha)m - s\right]}{(1-\alpha\theta)}.$$

The next step is to see how the degree of indexation affects the responsiveness of employment to monetary shocks. First, use equation (2) to find how employment varies with m:

$$(6)\quad \frac{\partial \ell}{\partial m} = \frac{(1-\theta)m + \theta s}{(1-\alpha\theta)}.$$

Taking the derivative of both sides of equation (6) with respect to θ gives us

$$(7)\quad \frac{\partial\left[\partial \ell / \partial m\right]}{\partial \theta} = \frac{(-1)[1-\alpha\theta] - (1-\theta)(-\alpha)}{(1-\alpha\theta)^2} = \frac{(\alpha - 1)}{(1-\alpha\theta)^2} < 0.$$

Thus an increase in the degree of indexation, θ, reduces the amount that employment will change due to a given monetary shock.

The next step is to see how the degree of indexation affects the responsiveness of employment to supply shocks. First, use equation (2) to find how employment varies with s:

$$(8)\quad \frac{\partial \ell}{\partial s} = \frac{\theta}{(1-\alpha\theta)}.$$

Taking the derivative of both sides of equation (8) with respect to θ gives us

$$(9)\quad \frac{\partial\left[\partial \ell / \partial s\right]}{\partial \theta} = \frac{(1)[1-\alpha\theta] - (\theta)(-\alpha)}{(1-\alpha\theta)^2} = \frac{1}{(1-\alpha\theta)^2} > 0.$$

Thus an increase in the degree of wage indexation, θ, increases the amount that employment will change due to a given supply shock.

(b) From equation (2), the variance of employment is given by

$$(10)\quad V_\ell = \left[\frac{(1-\theta)}{(1-\alpha\theta)}\right]^2 V_m + \left[\frac{\theta}{(1-\alpha\theta)}\right]^2 V_s,$$

where we have used the fact that m and s are independent random variables with variances V_m and V_s. We need to find the value of θ that minimizes this variance of employment. The first-order condition for this minimization is

$$(11)\quad \frac{\partial V_\ell}{\partial \theta} = 2\left[\frac{(1-\theta)}{(1-\alpha\theta)}\right]\left[\frac{(\alpha-1)}{(1-\alpha\theta)^2}\right]V_m + 2\left[\frac{\theta}{(1-\alpha\theta)}\right]\left[\frac{1}{(1-\alpha\theta)^2}\right]V_s = 0.$$

Equation (11) simplifies to

$$(1-\theta)(\alpha-1)V_m + \theta V_s = 0.$$

Collecting the terms in θ gives us

$$\theta[(1-\alpha)V_m + V_s] = (1-\alpha)V_m.$$

Thus the optimal degree of wage indexation is

$$(12)\quad \theta = \frac{(1-\alpha)V_m}{(1-\alpha)V_m + V_s}$$

Given the result in part (a) -- that indexation reduces the impact on employment of monetary shocks but increases the impact from supply shocks -- equation (12) is intuitive. First of all, if $V_s = 0$ -- so that there are no supply shocks -- the optimal degree of indexation is one. In addition, the larger is the variance of the supply shocks relative to the variance of the monetary shocks, the lower is the optimal degree of indexation.

(c) (i) As stated in the problem:

$$(13)\quad y_i = y - \phi(w_i - w),$$

where $\phi \equiv \alpha\eta/[\alpha + (1-\alpha)\eta]$. Since $w = \theta p$ and $w_i = \theta_i p$, equation (13) becomes

$$(14)\quad y_i = y - \phi(\theta_i p - \theta p) = y - (\theta_i - \theta)\phi p.$$

From the production function, $y_i = s + \alpha\ell_i$ and $y = s + \alpha\ell$ and thus we can write

$$(15)\quad y_i - y = \alpha(\ell_i - \ell).$$

Solving equation (15) for employment at firm i yields

$$(16)\quad \ell_i = \ell + (1/\alpha)(y_i - y).$$

Substituting equation (14) for $y_i - y$ into equation (16) gives us

$$(17)\quad \ell_i = \ell - (1/\alpha)(\theta_i - \theta)\phi p.$$

Substituting equation (2) for aggregate employment and equation (3) for the price level into equation (17) gives us

$$(18)\quad \ell_i = \frac{(1-\theta)m + \theta s}{(1-\alpha\theta)} - \frac{(\theta_i - \theta)\phi[(1-\alpha)m - s]}{\alpha(1-\alpha\theta)} = \frac{1}{\alpha(1-\alpha\theta)}\left[\alpha(1-\theta)m + \alpha\theta s - (\theta_i - \theta)\phi[(1-\alpha)m - s]\right],$$

which implies

$$(19)\quad \ell_i = \frac{1}{\alpha(1-\alpha\theta)}\left\{m\left[\alpha(1-\theta) - (\theta_i - \theta)\phi(1-\alpha)\right] + s\left[\alpha\theta + (\theta_i - \theta)\phi\right]\right\}.$$

(c) (ii) From equation (19), the variance of employment at firm i is given by

$$(20)\quad \mathrm{Var}(\ell_i) = \left[\frac{\alpha(1-\theta) - (\theta_i - \theta)\phi(1-\alpha)}{\alpha(1-\alpha\theta)}\right]^2 V_m + \left[\frac{\alpha\theta + (\theta_i - \theta)\phi}{\alpha(1-\alpha\theta)}\right]^2 V_s.$$

The first-order condition for the value of the degree of wage indexation at firm i, θ_i, that minimizes the variance of employment at firm i is

$$(21)\quad \frac{\partial \mathrm{Var}(\ell_i)}{\partial \theta_i} = 2\left[\frac{\alpha(1-\theta) - (\theta_i - \theta)\phi(1-\alpha)}{\alpha(1-\alpha\theta)}\right]\left[-\phi(1-\alpha)\right]V_m + 2\left[\frac{\alpha\theta + (\theta_i - \theta)\phi}{\alpha(1-\alpha\theta)}\right]\phi V_s = 0.$$

Equation (21) simplifies to

$$(22)\quad \{\alpha(1-\theta) - \theta_i[\phi(1-\alpha)] + \theta\phi(1-\alpha)\}\phi(1-\alpha)V_m = (\alpha\theta + \theta_i\phi - \theta\phi)\phi V_s,$$

which implies

$$(23)\quad \theta_i\phi^2 V_s + \theta_i[\phi(1-\alpha)]^2 V_m = [\alpha(1-\theta) + \theta\phi(1-\alpha)]\phi(1-\alpha)V_m - (\alpha\theta - \theta\phi)\phi V_s.$$

Thus θ_i is given by

(24) $$\theta_i = \frac{[\alpha(1-\theta)+\theta\phi(1-\alpha)]\phi(1-\alpha)V_m - [\theta(\phi-\alpha)]\phi V_s}{\phi^2 V_s + [\phi(1-\alpha)]^2 V_m}.$$

(c) (iii) We need to find a value of θ such that the first-order condition given by equation (22) holds when $\theta_i = \theta$. That is, we need to find a value of θ such that if economy-wide indexation is given by θ, the representative firm, in order to minimize its employment fluctuations, wishes to choose θ as well. Setting $\theta_i = \theta$ in equation (22) gives us

(25) $\alpha(1 - \theta)\phi(1 - \alpha)V_m = \alpha\theta\phi V_s$,

which implies

(26) $\theta[V_s + (1 - \alpha)V_m] = (1 - \alpha)V_m$.

Thus the Nash-equilibrium value of θ is

(27) $$\theta^{EQ} = \frac{(1-\alpha)V_m}{(1-\alpha)V_m + V_s}.$$

This is exactly the same value of θ we found in part (b); see equation (12). The value of θ that minimizes the variance of aggregate fluctuations in employment is also a Nash equilibrium. Given that other firms are choosing θ^{EQ} as their degree of wage indexation, it is optimal for any individual firm to choose θ^{EQ} as well.

Problem 6.6

(a) The representative individual will set her price equal to the average of the optimal price for t and the expected optimal price for t + 1. Thus

(1) $x_t = (p_{it}^* + E_t p_{it+1}^*)/2$.

Since $p_{it}^* = \phi m_t + (1 - \phi)p_t$ and this holds for all periods, we have

(2) $x_t = [(\phi m_t + (1 - \phi)p_t) + (\phi E_t m_{t+1} + (1 - \phi)E_t p_{t+1})]/2$.

(b) With synchronization, $p_t = x_t$ and $p_{t+1} = x_t$ and thus

(3) $x_t = [(\phi m_t + (1 - \phi)x_t) + (\phi E_t m_{t+1} + (1 - \phi)x_t]/2$.

Simplifying yields

$x_t = [2(1 - \phi)x_t + \phi(m_t + E_t m_{t+1})]/2 \quad \Rightarrow \quad [1 - (1 - \phi)]x_t = \phi(m_t + E_t m_{t+1})/2$,

and thus

(4) $x_t = (m_t + E_t m_{t+1})/2$.

Firms set their price equal to the average of this period's value of m and the expected value of next period's value of m.

(c) Substitute $p_t = x_t = (m_t + E_t m_{t+1})/2$ into the aggregate demand equation to obtain

$y_t = m_t - (m_t + E_t m_{t+1})/2$.

Simplifying gives us

(5) $y_t = (m_t - E_t m_{t+1})/2$.

Assuming that m follows a random walk so that $E_t m_{t+1} = m_t$, we have

$y_t = (m_t - m_t)/2$,

or simply

(6) $y_t = 0$.

Now, substituting $p_{t+1} = x_t = (m_t + E_t m_{t+1})/2$ into the aggregate demand equation for period t + 1 gives us

$y_{t+1} = m_{t+1} - [(m_t + E_t m_{t+1})/2]$.

Assuming that m follows a random walk so that $E_t m_{t+1} = m_t$, we have

$y_{t+1} = m_{t+1} - (m_t + m_t)/2$,

or simply

(7) $y_{t+1} = m_{t+1} - m_t$.

The central result of the Taylor model does not continue to hold. Nominal disturbances that occur in periods when firms are not setting prices feed through one-for-one into output; see equation (7). However, once firms set prices again, output returns to its normal value of 0; see equation (6).

Intuitively, we have removed the mechanism by which the Taylor model generates long-lasting effects of nominal disturbances. In the Taylor model, once firms get to adjust price, they do not adjust fully to a nominal disturbance because they know that not all firms are adjusting at that time. Thus fully adjusting will change their relative price, which they are reluctant to do. But here, firms know that all firms are also adjusting their price at the same time. Thus firms' real prices will not change if they fully adjust and thus they do so.

Problem 6.7

When t is even, the price level is given by

(1) $p_t = fp_t^1 + (1 - f)p_t^2$,

where p_t^1 denotes the price set for t by individuals who set their prices in t - 1 -- which was an odd period and hence fraction f of firms set p_t^1 -- and p_t^2 denotes the price set for t by individuals setting prices in t - 2 -- which was an even period and hence fraction (1 - f) of firms set p_t^2. Now, p_t^1 equals the expectation as of period t - 1 of p_{it}^* and thus

(2) $p_t^1 = E_{t-1}\, p_{it}^* = E_{t-1}\, [\phi m_t + (1 - \phi)p_t]$.

Substituting equation (1) into equation (2) and using the fact that p_t^2 is already known when $p_1^{\prime}$ is set and is thus not uncertain, yields

(3) $p_t^1 = \phi E_{t-1}\, m_t + (1 - \phi)\, [fp_t^1 + (1 - f)p_t^2]$.

Some simple algebra allows us to solve for p_t^1:

$$(4)\quad p_t^1 = \frac{\phi}{1-(1-\phi)f}E_{t-1}m_t + \frac{(1-\phi)(1-f)}{1-(1-\phi)f}p_t^2.$$

Now, p_t^2 equals the expectation as of period t - 2 of p_{it}^* and thus

(5) $p_t^2 = E_{t-2}\, p_{it}^* = E_{t-2}\, [\phi m_t + (1 - \phi)p_t]$.

Substituting equation (1) into equation (5) yields

(6) $p_t^2 = \phi E_{t-2}\, m_t + (1 - \phi)\, [fE_{t-2}\, p_t^1 + (1 - f)p_t^2]$.

We need to find $E_{t-2}\, p_t^1$. Since the left- and right-hand sides of equation (4) are equal and since expectations are rational, the expectation as of period t - 2 of these two expressions must be equal. That is, we have

$$(7)\quad E_{t-2}p_t^1 = \frac{\phi}{1-(1-\phi)f}E_{t-2}m_t + \frac{(1-\phi)(1-f)}{1-(1-\phi)f}p_t^2,$$

where we have used the law of iterated projections so that $E_{t-2}\, E_{t-1}\, m_t = E_{t-2}\, m_t$. Substituting equation (7) into equation (6) yields

$$p_t^2 = \phi E_{t-2}m_t + (1-\phi)\left[\frac{\phi f}{1-(1-\phi)f}E_{t-2}m_t + \frac{(1-\phi)(1-f)f}{1-(1-\phi)f}p_t^2 + (1-f)p_t^2\right].$$

Collecting terms gives us

$$p_t^2 = \left[\frac{\phi - \phi(1-\phi)f + (1-\phi)\phi f}{1-(1-\phi)f}\right]E_{t-2}m_t + (1-\phi)\left[\frac{(1-\phi)(1-f)f + (1-f) - (1-\phi)(1-f)f}{1-(1-\phi)f}\right]p_t^2,$$

which simplifies to

$$p_t^2 = \frac{\phi}{1-(1-\phi)f}E_{t-2}m_t + \frac{(1-\phi)(1-f)}{1-(1-\phi)f}p_t^2.$$

Collecting the terms in p_t^2 gives us

$$\left[\frac{1-(1-\phi)f-(1-\phi)(1-f)}{1-(1-\phi)f}\right]p_t^2 = \frac{\phi}{1-(1-\phi)f}E_{t-2}m_t,$$

or simply

$$\frac{\phi}{1-(1-\phi)f}p_t^2 = \frac{\phi}{1-(1-\phi)f}E_{t-2}m_t.$$

And thus the price set in period t - 2 is given by

(8) $p_t^2 = E_{t-2}\,m_t$.

Now to solve for the price set for t by those setting price in t - 1, substitute equation (8) into equation (4):

(9) $$p_t^1 = \frac{\phi}{1-(1-\phi)f}E_{t-1}m_t + \frac{(1-\phi)(1-f)}{1-(1-\phi)f}E_{t-2}m_t.$$

Adding and subtracting $E_{t-2}\,m_t$ to the right-hand side of equation (9) yields

(10) $$p_t^1 = E_{t-2}m_t + \frac{\phi}{1-(1-\phi)f}E_{t-1}m_t + \left[\frac{(1-\phi)(1-f)-1+(1-\phi)f}{1-(1-\phi)f}\right]E_{t-2}m_t.$$

Since $(1 - \phi)(1 - f) - 1 + (1 - \phi)f = -(1 - \phi)f + (1 - \phi) - 1 + (1 - \phi)f = -\phi$, equation (10) can be rewritten as

(11) $$p_t^1 = E_{t-2}m_t + \frac{\phi}{1-(1-\phi)f}\left(E_{t-1}m_t - E_{t-2}m_t\right).$$

To get an expression for the aggregate price level, substitute the formulas for p_t^1 and p_t^2, equations (11) and (8), into equation (1):

$$p_t = f\left[E_{t-2}m_t + \frac{\phi}{1-(1-\phi)f}\left(E_{t-1}m_t - E_{t-2}m_t\right)\right] + (1-f)E_{t-2}m_t.$$

Simplifying yields

(12) $$p_t = E_{t-2}m_t + \frac{\phi f}{1-(1-\phi)f}\left(E_{t-1}m_t - E_{t-2}m_t\right).$$

To solve for output in period t, substitute equation (12) into the expression for aggregate demand, $y_t = m_t - p_t$:

(13) $$y_t = m_t - E_{t-2}m_t - \frac{\phi f}{1-(1-\phi)f}\left(E_{t-1}m_t - E_{t-2}m_t\right).$$

Collecting the terms in $E_{t-2}\,m_t$ as well as adding and subtracting $E_{t-1}\,m_t$ to the right-hand side of equation (13) gives us

(14) $$y_t = m_t - E_{t-1}m_t + \left[\frac{\phi f - 1 + (1-\phi)f}{1-(1-\phi)f}\right]E_{t-2}m_t + \left[\frac{1-(1-\phi)f-\phi f}{1-(1-\phi)f}\right]E_{t-1}m_t.$$

Since $\phi f - 1 + (1 - \phi)f = -(1 - f)$ and $1 - (1 - \phi)f - \phi f = (1 - f)$, equation (14) can be rewritten as

(15) $$y_t = \frac{(1-f)}{1-(1-\phi)f}\left(E_{t-1}m_t - E_{t-2}m_t\right) + \left(m_t - E_{t-1}m_t\right).$$

Equations (12) and (15) give equilibrium price and output for an even period.

The analysis for the case of t odd is identical to the preceding analysis, except that the roles of f and (1 - f) are reversed. Thus derivations analogous to those used to obtain (12) and (15) yield

(16) $p_t = E_{t-2}m_t + \dfrac{\phi(1-f)}{1-(1-\phi)(1-f)}\left(E_{t-1}m_t - E_{t-2}m_t\right),$

and

(17) $y_t = \dfrac{f}{1-(1-\phi)(1-f)}\left(E_{t-1}m_t - E_{t-2}m_t\right) + \left(m_t - E_{t-1}m_t\right).$

Equations (16) and (17) give equilibrium price and output for an odd period.

Problem 6.8

We will deal first with firms that set price in odd periods. Suppose that period t is an even period. Then p_{it} is the price set for an even period by a firm that sets prices in odd periods and p_{it+1} is the price set for an odd period by a firm that sets prices in odd periods. From equation (11) in the solution to Problem 6.7, p_{it} is given by

(1) $p_{it} = E_{t-2}m_t + \dfrac{\phi}{1-(1-\phi)f}\left(E_{t-1}m_t - E_{t-2}m_t\right).$

With the assumption that m is a random walk, we have $E_{t-2}\, m_t = m_{t-2}$ and $E_{t-1}\, m_t = m_{t-1}$. Thus

(2) $p_{it} = m_{t-2} + \dfrac{\phi}{1-(1-\phi)f}\left(m_{t-1} - m_{t-2}\right).$

As usual, the optimal price for period t is given by

(3) $p_{it}^* = \phi m_t + (1-\phi)p_t$.

From equation (12) in the solution to Problem 6.7, the aggregate price level in an even period is

(4) $p_t = E_{t-2}m_t + \dfrac{\phi f}{1-(1-\phi)f}\left(E_{t-1}m_t - E_{t-2}m_t\right).$

Again, since m follows a random walk, this is equivalent to

(5) $p_t = m_{t-2} + \dfrac{\phi f}{1-(1-\phi)f}\left(m_{t-1} - m_{t-2}\right).$

Substituting equation (5) into equation (3) yields the optimal price in period t:

(6) $p_{it}^* = \phi m_t + (1-\phi)m_{t-2} + \dfrac{(1-\phi)\phi f}{1-(1-\phi)f}\left(m_{t-1} - m_{t-2}\right).$

Thus the amount of profit a firm expects to lose in period t is proportional to

$$E_t(p_{it} - p_{it}^*)^2 = E_t\left[m_{t-2} + \frac{\phi}{1-(1-\phi)f}\left(m_{t-1} - m_{t-2}\right) - \phi m_t - (1-\phi)m_{t-2} + \frac{(1-\phi)\phi f}{1-(1-\phi)f}\left(m_{t-1} - m_{t-2}\right)\right]^2$$

Collecting terms yields

(7) $E_t(p_{it} - p_{it}^*)^2 = E_t\left[-\phi m_t + \dfrac{\phi[1-(1-\phi)f]}{1-(1-\phi)f}\left(m_{t-1} - m_{t-2}\right) + \phi m_{t-2}\right]^2.$

Simplifying gives us

$E_t\,(p_{it} - p_{it}^*)^2 = E_t\,[-\phi m_t + \phi m_{t-1} - \phi m_{t-2} + \phi m_{t-2}\,]^2,$

or

(8) $E_t\,(p_{it} - p_{it}^*)^2 = \phi^2\, E_t\,(m_{t-1} - m_t\,)^2.$

Now, the price set for an odd period by a firm that sets prices in odd periods is given by

(9) $p_{it+1} = E_{t-1}\, m_{t+1}$.

Since m follows a random walk, $E_{t-1}\, m_{t+1} = m_{t-1}$ and thus

(10) $p_{it+1} = m_{t-1}$.

The optimal price for period t + 1 is given by

(11) $p_{it+1}^* = \phi m_{t+1} + (1 - \phi)p_{t+1}$.

From equation (16) in the solution to Problem 6.7, the aggregate price level in an odd period is

(12) $p_{t+1} = E_{t-1}m_{t+1} + \dfrac{\phi(1-f)}{1-(1-\phi)(1-f)}\left(E_t m_{t+1} - E_{t-1}m_{t+1}\right).$

Since $E_{t-1}\, m_{t+1} = m_{t-1}$ and $E_t\, m_{t+1} = m_t$, we have

(13) $p_{t+1} = m_{t-1} + \dfrac{\phi(1-f)}{1-(1-\phi)(1-f)}\left(m_t - m_{t-1}\right).$

Substituting equation (13) into equation (11) yields the optimal price in period t + 1:

(14) $p_{it+1}^* = \phi m_{t+1} + (1-\phi)m_{t-1} + \dfrac{(1-\phi)\phi(1-f)}{1-(1-\phi)(1-f)}\left(m_t - m_{t-1}\right).$

Thus the amount of profit a firm expects to lose in period t +1 is proportional to

(15) $E_t(p_{it+1} - p_{it+1}^*)^2 = E_t\left[m_{t-1} - \phi m_{t+1} - (1-\phi)m_{t-1} - \dfrac{(1-\phi)\phi(1-f)}{1-(1-\phi)(1-f)}\left(m_t - m_{t-1}\right)\right]^2.$

Collecting terms gives us

(16) $E_t(p_{it+1} - p_{it+1}^*)^2 = E_t\left[\phi(m_{t-1} - m_{t+1}) + \dfrac{(1-\phi)\phi(1-f)}{1-(1-\phi)(1-f)}\left(m_{t-1} - m_t\right)\right]^2.$

Expanding the right-hand side of equation (16) yields

(17)
$$E_t(p_{it+1} - p_{it+1}^*)^2 = \phi^2 E_t(m_{t-1} - m_{t+1})^2 + \frac{2\phi(1-\phi)\phi(1-f)}{1-(1-\phi)(1-f)}E_t(m_{t-1} - m_{t+1})(m_{t-1} - m_t) +$$
$$\left[\frac{(1-\phi)\phi(1-f)}{1-(1-\phi)(1-f)}\right]^2 E_t(m_{t-1} - m_t)^2.$$

Note that since m follows a random walk, we have

$E_t\,(m_{t-1} - m_{t+1})(m_{t-1} - m_t) = E_t\,(m_{t-1} - m_{t+1})E_t\,(m_{t-1} - m_t) = (m_{t-1} - m_t)(m_{t-1} - m_{t-1}) = 0.$

Thus the second term on the right-hand side of equation (17) is equal to 0. Using this fact, we can add equations (8) and (17). Thus the total amount of profit a firm setting prices in odd periods expects to lose is proportional to $E_t\,(p_{it} - p_{it}^*)^2 + E_t\,(p_{it+1} - p_{it+1}^*)^2$ or

(18) $\text{Exp. Loss}_{odd} = \phi^2 E_t(m_{t-1} - m_t)^2 + \phi^2 E_t(m_{t-1} - m_{t+1})^2 + \left[\dfrac{(1-\phi)\phi(1-f)}{1-(1-\phi)(1-f)}\right]^2 E_t\left(m_{t-1} - m_t\right)^2.$

Now we will deal with firms that set price in even periods. When period t is an odd period, analysis parallel to that used to derive (8) shows that the amount of profit a firm expects to lose in period t + 1 is proportional to

(19) $E_t\,(p_{it} - p_{it}^*)^2 = \phi^2\, E_t\,(m_{t-1} - m_t)^2.$

Analysis parallel to that used to derive (17) shows that the amount of profit a firm expects to lose in period t + 1 is proportional to

(20)
$$E_t(p_{it+1} - p_{it+1}^*)^2 = \phi^2 E_t(m_{t-1} - m_{t+1})^2 + \frac{2\phi(1-\phi)\phi f}{1-(1-\phi)f}E_t(m_{t-1} - m_{t+1})(m_{t-1} - m_t) +$$
$$\left[\frac{(1-\phi)\phi f}{1-(1-\phi)f}\right]^2 E_t(m_{t-1} - m_t)^2.$$

Note that this is identical to (17) except that the roles of f and (1 - f) are reversed. Proceeding as above, we note that since m follows a random walk, we have

$E_t(m_{t-1} - m_{t+1})(m_{t-1} - m_t) = E_t(m_{t-1} - m_{t+1})E_t(m_{t-1} - m_t) = (m_{t-1} - m_t)(m_{t-1} - m_{t-1}) = 0.$

Thus the second term on the right-hand side of equation (20) is equal to 0. Using this fact, we can add equations (19) and (20). Thus the total amount of profit a firm setting prices in even periods expects to lose is proportional to $E_t(p_{it} - p_{it}^*)^2 + E_t(p_{it+1} - p_{it+1}^*)^2$ or

(21) $\text{Exp. Loss}_{even} = \phi^2 E_t(m_{t-1} - m_t)^2 + \phi^2 E_t(m_{t-1} - m_{t+1})^2 + \left[\dfrac{(1-\phi)\phi f}{1-(1-\phi)f}\right]^2 E_t(m_{t-1} - m_t)^2.$

We need to compare the right-hand sides of equations (18) and (21). Recall that f is the fraction of firms that set prices in odd periods. Note that with $f < 1/2$ -- more firms setting prices in even periods than in odd periods -- we have $(1 - f) > f$. Using this and the fact that $\phi < 1$, we can say that

(22) $\left[\dfrac{(1-\phi)\phi(1-f)}{1-(1-\phi)(1-f)}\right]^2 > \left[\dfrac{(1-\phi)\phi f}{1-(1-\phi)f}\right]^2,$

since $(1 - \phi)\phi(1 - f) > (1 - \phi)\phi f$ and $1 - (1 - \phi)(1 - f) < 1 - (1 - \phi)f$.

Thus the right-hand side of equation (18) is greater than the right-hand side of equation (21). This means that the profit a firm expects to lose if it sets prices in odd periods exceeds the profit a firm expects to lose if it sets prices in even periods. Thus it is not optimal to set prices in odd periods and firms would like to switch to setting prices in even periods. This means that with $\phi < 1$, if we start with $f < 1/2$, we would expect to see f go to zero; no firms will set price in odd periods.

By reasoning analogous to that above, we could show that if $f > 1/2$, the inequality in (22) is reversed. Firms setting prices in even periods expect to lose more than firms setting prices in odd periods. Thus it is not optimal to set price in even periods and firms would like to switch to setting prices in odd periods. This means that with $\phi < 1$, if we start with $f > 1/2$, we would expect to see f go to one; all firms will set price in odd periods.

Thus if $\phi < 1$, staggered price setting with $f = 1/2$ is not a stable equilibrium. If the economy starts with anything other than $f = 1/2$, staggered price setting will not prevail. The economy will move to a situation in which all firms set price in the same period.

Problem 6.9

The price set by firms in period t is

(1) $x_t = (p_{it}^* + E_t p_{i,t+1}^*)/2 = [(\phi m_t + (1 - \phi)p_t) + (\phi E_t m_{t+1} + (1 - \phi)E_t p_{t+1})]/2,$

where we have used the fact that $p^* = \phi m + (1 - \phi)p$. Since $p_t = (x_t + x_{t-1})/2$ and $E_t m_{t+1} = 0$, equation (1) can be rewritten as

(2) $x_t = \phi m_t + \dfrac{(1-\phi)(x_t + x_{t-1})}{2} + \dfrac{[(1-\phi)(x_t + E_t x_{t+1})/2]}{2},$

which simplifies to

(3) $x_t = \dfrac{\phi m_t}{2} + \dfrac{(1-\phi)(x_{t-1} + 2x_t + E_t x_{t+1})}{4}.$

Solving for x_t yields

(4) $x_t = A(x_{t-1} + E_t x_{t+1}) + [(1 - 2A)/2]m_t,$

where $A \equiv \dfrac{1}{2}\dfrac{1-\phi}{1+\phi}.$

We need to eliminate $E_t\, x_{t+1}$ from the expression in (4). As in the text, it is reasonable to guess that x_t is a linear function of x_{t-1} and m_t, or

(5) $x_t = \mu + \lambda x_{t-1} + v m_t$.

We need to determine whether there are actually values of μ, λ, and v that solve the model. As explained in the text, the flexible-price equilibrium involves each price equaling m. In light of this, consider a situation where x_{t-1} and m_t are both equal to zero. If period-t price-setters also set their prices to $m_t = 0$, the economy is at its flexible-price equilibrium. In addition, since m is white noise, the period-t price-setters have no reason to expect m_{t+1} to be on average either more or less than zero, and hence no reason to expect x_{t+1} to depart on average from zero. Thus in this situation, p_{it}^* and $E_t\, p_{it+1}^*$ are both equal to zero and so price-setters would choose $x_t = 0$. In summary, it is reasonable to guess that when $x_{t-1} = m_t = 0$, $x_t = 0$. In terms of equation (5), this condition is

(6) $0 = \mu + \lambda(0) + v(0)$,

or simply $\mu = 0$. Thus equation (5) becomes

(7) $x_t = \lambda x_{t-1} + v m_t$.

Our goal is to find values of λ and v that solve the model. Since equation (7) holds each period, it implies that $x_{t+1} = \lambda x_t + v m_{t+1}$. Thus the expectation as of time t of x_{t+1} is λx_t since $E_t\, m_{t+1} = 0$. Using equation (7) to substitute for x_t yields

(8) $E_t\, x_{t+1} = \lambda^2 x_{t-1} + \lambda v m_t$.

Substituting equation (8) into equation (4) gives us

(9) $x_t = A(x_{t-1} + \lambda^2 x_{t-1} + \lambda v m_t) + [(1 - 2A)/2]m_t$,

which implies

(10) $x_t = (A + A\lambda^2)x_{t-1} + \{A\lambda v + [(1 - 2A)/2]\}m_t$.

The coefficients on x_{t-1} and m_t must be the same in equations (7) and (10). This requires

(11) $A + A\lambda^2 = \lambda$,

and

(12) $A\lambda v + (1 - 2A)/2 = v$.

Equation (11) is the same as equation (6.68) in the text for the version of the model where m follows a random walk. The solution to this quadratic is thus given by

(13) $\lambda = \dfrac{1-\sqrt{\phi}}{1+\sqrt{\phi}}$.

Solving for v in equation (12) yields

(14) $v(1 - A\lambda) = (1 - 2A)/2$.

From equation (11), dividing through by λ and rearranging, gives us $1 - A\lambda = A\lambda$. Substituting this expression into equation (14) gives us

(15) $v = \dfrac{1-2A}{2}\dfrac{\lambda}{A}$.

Substituting equation (15) into equation (7) yields

(16) $x_t = \lambda x_{t-1} + \dfrac{1-2A}{2}\dfrac{\lambda}{A}m_t$.

Thus equation (16) with λ given by equation (13) solves the model.

We can now describe the behavior of output. Using the definitions of A and λ, some simple algebra allows us to rewrite equation (16) as

(17) $x_t = \lambda x_{t-1} + \dfrac{2\phi}{(1+\sqrt{\phi})^2} m_t = \lambda x_{t-1} + Cm_t$,

where we have defined $C \equiv \dfrac{2\phi}{(1+\sqrt{\phi})^2}$.

Since $y_t = m_t - p_t$ and $p_t = (x_t + x_{t-1})/2$ we have
(18) $y_t = m_t - [(x_t + x_{t-1})]/2$.
Substituting equation (17), and equation (17) lagged one period, into equation (18) yields
(19) $y_t = m_t - [(\lambda x_{t-1} + Cm_t + \lambda x_{t-2} + Cm_{t-1})/2]$,
or simply
(20) $y_t = m_t - \lambda p_{t-1} - [(C/2)m_t] - [(C/2)m_{t-1}]$,
where we have used the fact that $p_{t-1} = (x_{t-1} + x_{t-2})/2$. Now since $y_{t-1} = m_{t-1} - p_{t-1}$, this implies
(21) $y_t = m_t + \lambda y_{t-1} - \lambda m_{t-1} - (C/2)m_t - (C/2)m_{t-1}$.
Collecting terms yields

(22) $y_t = \left(1 - \dfrac{C}{2}\right)m_t - \left(\lambda + \dfrac{C}{2}\right)m_{t-1} + \lambda y_{t-1}$.

Finally, since

(23) $1 - \dfrac{C}{2} = 1 - \dfrac{\phi}{(1+\sqrt{\phi})^2} = \dfrac{1 + 2\sqrt{\phi} + \phi - \phi}{(1+\sqrt{\phi})^2} = \dfrac{1+2\sqrt{\phi}}{(1+\sqrt{\phi})^2}$,

and

(24) $\lambda + \dfrac{C}{2} = \dfrac{1-\sqrt{\phi}}{1+\sqrt{\phi}} + \dfrac{\phi}{(1+\sqrt{\phi})^2} = \dfrac{(1-\sqrt{\phi})(1+\sqrt{\phi}) + \phi}{(1+\sqrt{\phi})^2} = \dfrac{1-\phi+\phi}{(1+\sqrt{\phi})^2} = \dfrac{1}{(1+\sqrt{\phi})^2}$.

Using (23) and (24), equation (22) can be rewritten as

(25) $y_t = \lambda y_{t-1} + \dfrac{1+2\sqrt{\phi}}{(1+\sqrt{\phi})^2}\varepsilon_t - \dfrac{1}{(1+\sqrt{\phi})^2}\varepsilon_{t-1}$,

where we have substituted for $m_t = \varepsilon_t$ and $m_{t-1} = \varepsilon_{t-1}$. Thus if the money stock is white noise, output is an ARMA(1,1) process rather than an AR(1) process.

Problem 6.10
We could proceed as in the text and obtain equation (6.80), which holds for a general process for m, and which is given by

(6.80) $x_t = \lambda x_{t-1} + \dfrac{\lambda}{A}\dfrac{1-2A}{2}[m_t + (1+\lambda)(E_t m_{t+1} + \lambda E_t m_{t+2} + \lambda^2 E_t m_{t+3} + \ldots)]$.

When the money stock is white noise, $E_t m_{t+s} = 0$ for all $s > 0$. Thus equation (6.80) simplifies to

(1) $x_t = \lambda x_{t-1} + \dfrac{\lambda}{A}\dfrac{1-2A}{2} m_t$.

Note that equation (1) is identical to equation (16) in the solution to Problem 6.9. As in that solution, we could now proceed to determine the behavior of output.

Problem 6.11
(a) The price set by individuals at time t is

(1) $x(t) = \dfrac{1}{T}\int_{\tau=0}^{T} E_t[m(t+\tau)]dt = \dfrac{1}{T}\int_{\tau=0}^{T}(t+\tau)gd\tau$,

where we have substituted for the fact that $m(t) = gt$ and thus that $E_t[m(t + \tau)] = g(t + \tau)$. Solving the integral in (1) gives us

$$(2)\quad \int_{\tau=0}^{T}(t+\tau)g\,d\tau = \left[gt\tau\Big|_{\tau=0}^{\tau=T}\right] + \left[\frac{1}{2}\tau^2\Big|_{\tau=0}^{\tau=T}\right] = gtT + \frac{1}{2}gT^2.$$

Substituting equation (2) back into equation (1) gives us the price set by individuals at time t, which is
(3) $x(t) = gt + (gT/2)$.

The aggregate price level is the average of the prices set over the last interval of length T. Thus

$$(4)\quad p(t) = \frac{1}{T}\int_{\tau=0}^{T} x(t-\tau)\,d\tau.$$

Substituting equation (3) into equation (4) yields

$$(5)\quad p(t) = \frac{1}{T}\int_{\tau=0}^{T}\left[g(t-\tau) + \frac{1}{2}gT\right]d\tau.$$

Solving the integral in (5) gives us

$$(6)\quad \int_{\tau=0}^{T}\left[g(t-\tau) + \frac{1}{2}gT\right]d\tau = \left[gt\tau - \frac{1}{2}g\tau^2 + \frac{1}{2}gT\tau\right]\Bigg|_{\tau=0}^{\tau=T} = gtT - \frac{1}{2}gT^2 + \frac{1}{2}gT^2 = gtT.$$

Substituting equation (6) back into equation (5) gives us the price level at time t, which is
(7) $p(t) = gt$.

Substituting $m(t) = gt$ and $p(t) = gt$ into $y(t) = m(t) - p(t)$ gives us
(8) $y(t) = 0$.

(b) (i) Suppose that $x(t) = gT/2$ for all $t > 0$. Then for $t > T$, since $p(t)$ is just the average of the x's set over the last interval of length T, $p(t) = gT/2$. Now we know that for $t > T$, $m(t) = gT/2$. Thus for $t > T$, we do have $p(t) = m(t)$. From $y(t) = m(t) - p(t)$, this means that for all $t > T$, $y(t) = 0$, which would have been its value in the absence of the change in policy.

Now consider the situation for some time t between time 0 and time T. From time 0 to time t, we are assuming that individuals set price equal to gT/2. From equation (3), we know that before time 0, individuals set price equal to $gt + (gT/2)$. The aggregate price at time t, which is the average of the prices set by individuals over the past interval of length T, is therefore given by

$$(9)\quad p(t) = \frac{1}{T}\left[\int_{\tau=t-T}^{0}\left(g\tau + \frac{gT}{2}\right)d\tau + \int_{\tau=0}^{t}\left(\frac{gT}{2}\right)d\tau\right] = \frac{1}{T}\left[\int_{\tau=t-T}^{0}\left(g\tau + \frac{gT}{2}\right)d\tau + \frac{gTt}{2}\right].$$

Solving the remaining integral in equation (9) gives us

$$(10)\quad \int_{\tau=t-T}^{0}\left(g\tau + \frac{gT}{2}\right)d\tau = \left[\frac{g\tau^2}{2} + \frac{gT\tau}{2}\right]\Bigg|_{\tau=t-T}^{\tau=0} = -\frac{g(t-T)^2}{2} - \frac{gT(t-T)}{2}.$$

Substituting equation (10) back into equation (9) and expanding yields

$$(11)\quad p(t) = \frac{1}{T}\left[-\frac{g(t-T)^2}{2} - \frac{gT(t-T)}{2} + \frac{gTt}{2}\right] = \frac{1}{T}\left[\frac{-gt^2 + 2gtT - gT^2 - gTt + gT^2 + gTt}{2}\right],$$

which implies

$$(12)\quad p(t) = gt - \frac{gt^2}{2T} = gt\left[1 - \frac{t}{2T}\right].$$

Thus $p(t) = m(t)$ for t between 0 and T as well. Thus if $x(t) = gT/2$ for all $t > 0$, then $p(t) = m(t)$ for all $t > 0$, and thus output is the same as it would be without the change in policy.

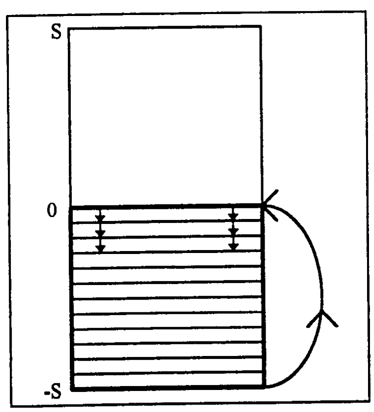
(b) (ii) At time t, individuals set their prices equal to the average of the expected money supply over the next interval of length T. We know that $m(t) = gT/2$ for $t \geq T$, but it is strictly less than $gT/2$ for $t < T$. Thus individuals setting prices at some time t before T are going to set their prices less than $gT/2$. Why? They will be averaging some m's equal to $gT/2$ with some m's less than $gT/2$ and so we must have $x(t) < gT/2$ for $0 < t < T$. For $T \leq t < 2T$, the money supply is expected to be constant at $gT/2$ and thus individuals set prices equal to this constant money supply. Thus $x(t) = gT/2$ for $T \leq t < 2T$.

We have shown in part (b) (i), that <u>if</u> everyone sets prices to $gT/2$, $p(t) = m(t)$ and thus $y(t) = 0$, which is its value in the absence of the change in policy. But as we have just explained, individuals actually set prices less than $gT/2$ for $0 < t < T$. Thus the aggregate price level will be less than $m(t)$ over the interval $0 < t < 2T$. Since $y(t) = m(t) - p(t)$, this means that output will be greater than zero during this interval. Thus this steady reduction in money growth actually causes output to be higher than it would have been in the absence of the policy change.

<u>Problem 6.12</u>

(a) Suppose first that the elevator is not at the top or bottom of the shaft. Now assume that the money supply rises by a small (formally, infinitesimal) amount dm. Since $p_i - p_i^*$ does not equal S or -S for anyone, no prices change. All the $(p_i - p_i^*)$'s fall by dm. The elevator moves down the shaft by dm and stays of height S. Similarly, if m falls, no prices change. The elevator moves up the shaft by dm and stays of height S.

Now suppose the elevator is at the bottom of the shaft. Assume that the money supply rises by dm. Firms that initially have $p_i - p_i^*$ "just above" -S reach the barrier. They therefore raise their price so that $p_i - p_i^* = 0$. Everyone else moves down the shaft by dm. Since the height of the elevator was S, the top of the elevator was initially at zero.



In the figure at right, the horizontal lines represent "slices" of the elevator of infinitesimal height dm. Essentially, the firms at the bottom of the elevator jump up to the top and everyone else moves down by dm. So the elevator does not move or change shape. Thus, with an infinitesimal change in m, the distribution of $p_i - p_i^*$ is unchanged, just as in the Caplin-Spulber model.

The situation where the elevator is at the bottom of the shaft and m falls is similar to the case where the elevator is not at the top or bottom of the shaft. It simply moves up by dm.

Finally, the case where the elevator is at the top of the shaft is the reverse of the case where it starts at the bottom. If m falls, the elevator does not move. If m rises, the elevator moves down by dm.

(b) For an increase in m, average price is unchanged except if the elevator is at the bottom of the shaft. In this case, the average price rises exactly as much as m. Thus, on average, increases in the money supply increase output.

Problem 6.13

(a) Substitute the aggregate price level, $p = qp^r + (1 - q)p^f$, into the expression for the price set by flexible-price firms, $p^f = (1 - \phi)p + \phi m$, to yield

(1) $p^f = (1 - \phi)[qp^r + (1 - q)p^f] + \phi m.$

Solving for p^f yields

(2) $p^f[1 - (1 - \phi)(1 - q)] = (1 - \phi)qp^r + \phi m.$

Since $1 - (1 - \phi)(1 - q) = q + \phi - \phi q = \phi + (1 - \phi)q$, equation (2) can be rewritten as

(3) $p^f[\phi + (1 - \phi)q] = (1 - \phi)qp^r + \phi m,$

and thus finally

(4) $p^f = \dfrac{(1-\phi)q}{\phi+(1-\phi)q}p^r + \dfrac{\phi}{\phi+(1-\phi)q}m = p^r + \dfrac{\phi}{\phi+(1-\phi)q}(m - p^r).$

(b) Since rigid-price firms set $p^r = (1 - \phi)Ep + \phi Em$, we need to solve for Ep, the expectation of the aggregate price level. Taking the expected value of both sides of $p = qp^r + (1 - q)p^f$ gives us

(5) $Ep = qp^r + (1 - q)Ep^f.$

Thus we have

(6) $p^r = (1 - \phi)[qp^r + (1 - q)Ep^f] + \phi Em.$

The rigid-price firms know how the flexible-price firms will set their price. That is, they know that flexible-price firms will use equation (4) to set their prices. Thus the rational expectation of the price set by the flexible-price firms is

(7) $Ep^f = p^r + \dfrac{\phi}{\phi+(1-\phi)q}(Em - p^r).$

Substituting equation (7) into equation (6) yields

(8) $p^r = (1-\phi)\left\{qp^r + (1-q)\left[p^r + \dfrac{\phi}{\phi+(1-\phi)q}(Em - p^r)\right]\right\} + \phi\, Em,$

which implies

(9) $p^r = (1-\phi)p^r + \phi\, Em + \dfrac{(1-\phi)(1-q)\phi}{\phi+(1-\phi)q}(Em - p^r).$

Defining $C \equiv [(1 - \phi)(1 - q)\phi]/[\phi + (1 - \phi)q]$, we can rewrite equation (9) as

(10) $p^r = [1 - (1 - \phi) + C] = (\phi + C)Em,$

or

(11) $p^r(\phi + C) = (\phi + C)Em,$

and thus finally

(12) $p^r = Em.$

Rigid-price firms simply set their prices equal to the expected value of the nominal money stock.

(c) The aggregate price level is given by

(13) $p = qp^r + (1 - q)p^f.$

Substituting equation (4) for p^f into equation (13) yields

(14) $p = qp^r + (1-q)\left[p^r + \dfrac{\phi}{\phi+(1-\phi)q}(m - p^r)\right] = p^r + \dfrac{(1-q)\phi}{\phi+(1-\phi)q}(m - p^r).$

Finally, from equation (12), we know that $p^r = Em$. Thus the aggregate price level is

(15) $p = Em + \dfrac{(1-q)\phi}{\phi+(1-\phi)q}(m - Em).$

We know that $y = m - p$. Adding and subtracting Em on the right-hand side of this expression yields
(16) $y = Em + (m - Em) - p$.
Substituting equation (15) into equation (16) yields

$$(17)\quad y = (m - Em) - \frac{(1-q)\phi}{\phi + (1-\phi)q}(m - Em) = \frac{\phi + (1-\phi)q - (1-q)\phi}{\phi + (1-\phi)q}(m - Em),$$

which simplifies to

$$(18)\quad y = \frac{q}{\phi + (1-\phi)q}(m - Em).$$

(c) (i) From equations (15) and (18), we can see that anticipated changes in m affect only prices. Specifically, consider the effects of an upward shift in the entire distribution of m, with the realization of m - Em held fixed. From equation (18) we can see that this will have no effects on real output. In this case, rigid-price firms get to set their price knowing that m has changed and thus incorporate it into their price-setting decision.

(c) (ii) Unanticipated changes in m affect real output. That is, a higher value of m given its distribution – that is, given Em -- does raise y as we can see from equation (18). In this case, the rigid-price firms do not get to observe the higher realization of m and cannot incorporate it into their price-setting decision and hence the economy does not achieve the flexible-price equilibrium.

In addition, flexible-price firms are reluctant to allow their real prices to change. One can show that

$$\frac{\partial y}{\partial \phi} = \frac{-(1-q)q}{[\phi + (1-\phi)q]^2}[m - Em] < 0 \quad \text{for } m > Em.$$

Thus a lower value of ϕ -- that is, a higher degree of "real rigidity" -- leads to a higher level of output for any given positive realization of m - Em. This means that the impact on real output of an unanticipated increase in aggregate demand is larger the larger is the degree of real rigidity or the more reluctant are flexible-price firms to allow their real prices to vary.

Problem 6.14

(a) $\pi(y_1, r^*(y_1))$ is the profit a firm receives at aggregate output level y_1, if it charges the profit-maximizing real price, $r^*(y_1)$. $\pi(y_1, r^*(y_0))$ is the profit a firm receives at aggregate output level y_1 if it continues to charge a real price of $r^*(y_0)$, which was the optimal price to charge when aggregate output was y_0. Thus $G = \pi(y_1, r^*(y_1)) - \pi(y_1, r^*(y_0))$ is the additional profit a firm would receive if, when aggregate output changes from y_0 to y_1, the firm changes its price to its new profit-maximizing level. This represents, therefore, the firm's incentive to change its price in the face of a change in aggregate real output.

(b) The second-order Taylor approximation will be of the form

$$(1)\quad G \cong G\big|_{y_1=y_0} + \left[\frac{\partial G}{\partial y_1}\bigg|_{y_1=y_0}\right][y_1 - y_0] - \frac{1}{2}\left[\frac{\partial^2 G}{\partial y_1^2}\bigg|_{y_1=y_0}\right][y_1 - y_0]^2.$$

Clearly, G evaluated at $y_1 = y_0$ is equal to zero. In addition
(2) $\partial G/\partial y_1 = \pi_1(y_1, r^*(y_1)) + \pi_2(y_1, r^*(y_1))[r^{*\prime}(y_1)] - \pi_1(y_1, r^*(y_0))$.
Evaluating this derivative at $y_1 = y_0$ gives us
(3) $\partial G/\partial y_1 |_{y_1=y_0} = \pi_1(y_0, r^*(y_0)) + \pi_2(y_0, r^*(y_0))[r^{*\prime}(y_0)] - \pi_1(y_0, r^*(y_0)) = 0$.
Since $r^*(y_0)$ is defined implicitly by $\pi_2(y_0, r^*(y_0)) = 0$, the right-hand side of equation (3) is equal to zero.

Using equation (2) to find the second derivative of G with respect to y_1 gives us

(4) $\partial^2 G/\partial y_1^2 = \pi_{11}(y_1, r^*(y_1)) + \pi_{12}(y_1, r^*(y_1))[r^{*\prime}(y_1)] +$
$[\pi_{21}(y_1, r^*(y_1)) + \pi_{22}(y_1, r^*(y_1))r^{*\prime}(y_1)][r^{*\prime}(y_1)] + \pi_2(y_1, r^*(y_1))[r^{*\prime\prime}(y_1)] -$
$\pi_{11}(y_1, r^*(y_0))$.

Using the fact that $\pi_2(y_1, r^*(y_1)) = 0$ and $\pi_{12}(y_1, r^*(y_1)) = \pi_{21}(y_1, r^*(y_1))$, equation (4) becomes

(5) $\partial^2 G/\partial y_1^2 = \pi_{11}(y_1, r^*(y_1)) + 2\pi_{12}(y_1, r^*(y_1))[r^{*\prime}(y_1)] + \pi_{22}(y_1, r^*(y_1))[r^{*\prime}(y_1)]^2 - \pi_{11}(y_1, r^*(y_0))$.

Evaluating this derivative at $y_1 = y_0$ leaves us with

(6) $\partial^2 G/\partial y_1^2 |_{y_1=y_0} = 2\pi_{12}(y_0, r^*(y_0))[r^{*\prime}(y_0)] + \pi_{22}(y_0, r^*(y_0))[r^{*\prime}(y_0)]^2$.

Now differentiate both sides of the equation that implicitly defines $r^*(y_0)$, $\pi_2(y_0, r^*(y_0)) = 0$, with respect to y_0 to obtain

(7) $\pi_{21}(y_0, r^*(y_0)) + \pi_{22}(y_0, r^*(y_0))[r^{*\prime}(y_0)] = 0$,

and thus

(8) $\pi_{21}(y_0, r^*(y_0)) = -\pi_{22}(y_0, r^*(y_0))[r^{*\prime}(y_0)]$.

Substituting equation (8) into equation (6) yields

(9) $\partial^2 G/\partial y_1^2 |_{y_1=y_0} = -2\pi_{22}(y_0, r^*(y_0))[r^{*\prime}(y_0)]^2 + \pi_{22}(y_0, r^*(y_0))[r^{*\prime}(y_0)]^2 = -\pi_{22}(y_0, r^*(y_0))[r^{*\prime}(y_0)]^2$.

Thus, since G and $\partial G/\partial y_1$ evaluated at $y_1 = y_0$ are both equal to zero, substituting equation (9) into equation (1) gives us the second-order Taylor approximation:

(10) $G \cong -\pi_{22}(y_0, r^*(y_0))[r^{*\prime}(y_0)]^2 [y_1 - y_0]^2/2$.

(c) The $[r^{*\prime}(y_0)]^2$ component reflects the degree of real rigidity. It tells us how much the firm's profit-maximizing real price responds to changes in aggregate real output. The $\pi_{22}(y_0, r^*(y_0))$ component reflects insensitivity of the profit function. It tells us the curvature of the profit function and thus the cost in lost profits from the firm allowing its real price to differ from its profit-maximizing value.

Problem 6.15

(a) Substituting the expression for aggregate demand, $y = m - p$, into the equation that defines the optimal price for firms, $p^* = p + \phi y$, yields $p^* = p + \phi(m - p)$ or simply

(1) $p^* = (1 - \phi)p + \phi m$.

Substituting the aggregate price level, $p = fp^*$, and $m = m'$ into equation (1) yields

$p^* = (1 - \phi)fp^* + \phi m'$.

Solving for p^* gives us

(2) $p^* = \dfrac{\phi}{1-(1-\phi)f} m'$.

Now substitute equation (2) into the expression for the aggregate price level, $p = fp^*$, to obtain

(3) $p = \dfrac{\phi f}{1-(1-\phi)f} m'$.

Substituting equation (3) and $m = m'$ into the expression for aggregate demand, $y = m - p$, yields

$$y = m' - \frac{\phi f}{1-(1-\phi)f} m' = \left[\frac{1 - f + \phi f - \phi f}{1-(1-\phi)f}\right] m',$$

or simply

(4) $y = \dfrac{(1-f)}{1-(1-\phi)f} m'$.

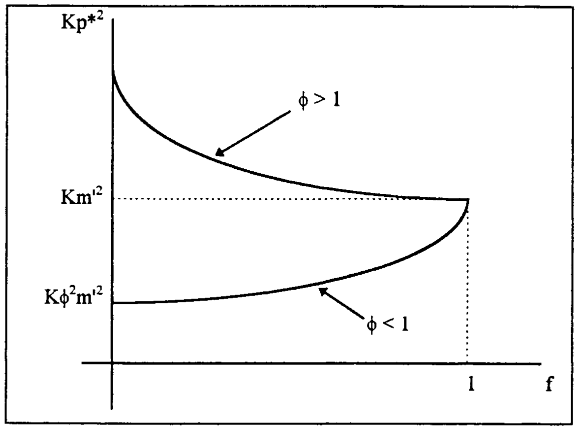
(b) Substituting equation (2), the expression for a firm's optimal price, into the expression describing the firm's incentive to adjust its price, Kp^{*2}, yields

(5) $Kp^{*2} = K\left[\dfrac{\phi m'}{1-(1-\phi)f}\right]^2.$

We need to plot this incentive to change price as a function of f, the fraction of firms that change their price. The following derivatives will be useful:

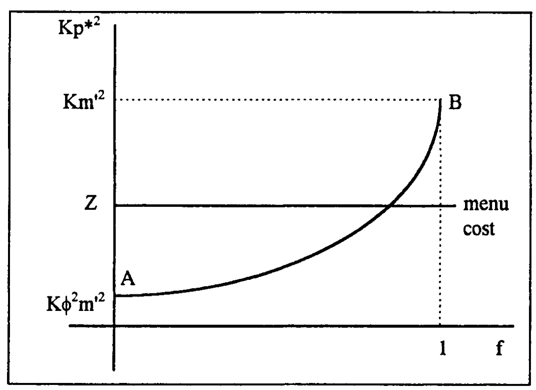
(6) $\dfrac{\partial\left[Kp^{*2}\right]}{\partial f} = \dfrac{2K(1-\phi)(\phi m')^2}{\left[1-(1-\phi)f\right]^3},$ and (7) $\dfrac{\partial^2\left[Kp^{*2}\right]}{\partial f^2} = \dfrac{6K(1-\phi)^2(\phi m')^2}{\left[1-(1-\phi)f\right]^4}.$

When $\phi < 1$, $\partial[Kp^{*2}]/\partial f > 0$ and $\partial^2[Kp^{*2}]/\partial f^2 > 0$. From equation (5), at $f = 1$, $Kp^{*2} = K[\phi m']^2/\phi^2 = Km'^2$. At $f = 0$, $Kp^{*2} = K\phi^2 m'^2 < Km'^2$ when $\phi < 1$. Thus when $\phi < 1$, the incentive for a firm to adjust its price is an increasing function of how many other firms change their price. See the figure at right.



When $\phi > 1$, $\partial[Kp^{*2}]/\partial f < 0$ and $\partial^2[Kp^{*2}]/\partial f^2 > 0$. From equation (5), at $f = 1$, $Kp^{*2} = K[\phi m']^2/\phi^2 = Km'^2$. At $f = 0$, $Kp^{*2} = K\phi^2 m'^2 > Km'^2$ when $\phi > 1$. Thus when $\phi > 1$, the incentive for a firm to adjust its price is a decreasing function of how many other firms change their price. See the figure.

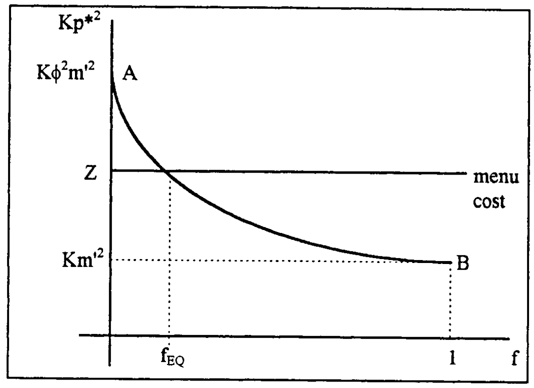
(c) In the case of $\phi < 1$, there can be a situation where both adjustment by all firms and adjustment by no firms are equilibria. See the figure at right where the menu cost, Z, is assumed to be such that $K\phi^2m'^2 < Z < Km'^2$.



Point A is an equilibrium with $f = 0$. Consider the situation of a representative firm at point A. If no one else is changing their price, the profits a firm loses by not changing its price, which are given by $Kp^{*2} = K\phi^2m'^2$, are less than the menu cost of Z. Thus it is optimal for the representative firm not to change its price. This is true for all firms and thus no one changing price is an equilibrium.

Point B is also an equilibrium with $f = 1$. Consider the situation of a representative firm at point B. If everyone else is changing their price, the profits a firm loses by not changing its price, $Kp^{*2} = Km'^2$, exceed the menu cost of Z. Thus it is optimal for the representative firm to change its price. This is true for all firms and thus everyone changing price is also an equilibrium.

In the case of $\phi > 1$, there can be a situation where neither adjustment by all firms nor adjustment by no firms are equilibria. See the figure at right where the menu cost, Z, is assumed to be such that $Km'^2 < Z < K\phi^2 m'^2$.



Consider the situation of f = 0 at point A. If no one else is changing their price, the profits that a representative firm would lose by not changing price, $Kp^{*2} = K\phi^2 m'^2$, exceed the menu cost Z. Thus it is optimal for the firm to change its price. This is true for all firms and thus it cannot be an equilibrium for no one to change their price.

Now consider the situation of f = 1 at point B. If everyone else is changing their price, the profits that a representative firm would lose by not changing its price, $Kp^{*2} = Km'^2$, are less than the menu cost of Z. Thus it is optimal for the representative firm not to change its price. This is true for all firms and thus it cannot be an equilibrium for all firms to change their price.

From this discussion, we can see that the equilibrium in this case is for fraction f_{EQ} of firms to change their price. If fraction f_{EQ} of firms are changing their price, the profit that a representative firm would lose by not changing its price is exactly equal to the menu cost, Z. Thus the representative firm is indifferent and there is no tendency for the economy to move away from this point where fraction f_{EQ} of firms are changing their price.

Problem 6.16

(a) We can use the intuitive reasoning employed to explain equation (9.29) in Chapter 9. Consider an asset that "pays" $-c$ when the individual climbs a palm tree and pays $\bar{u}$ when an individual trades and eats another's coconut. Assume that this asset is being priced by risk-neutral investors with required rate of return equal to r, the individual's discount rate. Since the expected present value of this asset is the same as the individual's expected value of lifetime utility, the asset must have price V_P while the individual is looking for palm trees and price V_C while the individual is looking for other people with coconuts.

For the asset to be held, it must provide an expected rate of return of r. That is, its dividends per unit time plus any expected capital gains or losses per unit time, must equal rV_P. When the individual is looking for palm trees, there are no dividends per unit time. There is a probability b per unit time of a capital "gain" of $(V_C - V_P) - c$; if the individual finds a palm tree and climbs it, the difference in the price of the asset is $V_C - V_P$ and the asset "pays" $-c$ at that time. Thus we have

(1) $rV_P = b(V_C - V_P - c)$.

(b) The asset must have price V_C while the individual is looking for others with coconuts and must provide an expected rate of return of r. Thus its dividends per unit time plus any expected capital gains or losses per unit time, must equal rV_C. When the individual is looking for others with coconuts, there are no dividends per unit time. There is a probability aL per unit time of a capital gain of $(V_P - V_C) + \bar{u}$; if the individual finds someone else with a coconut, trades and eats that coconut, the change in the price of the asset is $(V_P - V_C)$ and the asset pays $\bar{u}$ at that time. Thus we have

(2) $rV_C = aL(V_P - V_C + \bar{u})$.

(c) Solving for V_P in equation (2) gives us

(3) $V_P = (rV_C/aL) + V_C - \bar{u}$.

Substituting equation (3) into equation (1) yields

$$r[(rV_C/aL + V_C - \bar{u}] = b[V_C - (rV_C/aL) - V_C + \bar{u} - c].$$

Collecting terms in V_C gives us

(4) $V_C[(r^2/aL) + r + (br/aL)] = r\bar{u} + b\bar{u} - bc$.

Equation (4) can be rewritten as

$$V_C[r(r + aL + b)]/aL = \bar{u}(r + b) - bc.$$

Thus finally, the value of being in state C is given by

(5) $$V_C = \frac{aL[\bar{u}(r+b) - bc]}{r(r + aL + b)}.$$

Substituting equation (5) into equation (3) yields the following value of being in state P:

(6) $$V_P = \frac{\bar{u}(r+b) - bc}{r + aL + b} + \frac{aL[\bar{u}(r+b) - bc]}{r(r + aL + b)} - \bar{u}.$$

Subtracting equation (6) from equation (5) gives us

$$V_C - V_P = -\left[\frac{\bar{u}(r+b) - bc}{r + aL + b}\right] + \bar{u} = \frac{-\bar{u}r - \bar{u}b + bc + \bar{u}r + \bar{u}aL + \bar{u}b}{r + aL + b},$$

or simply

(7) $$V_C - V_P = \frac{bc + \bar{u}aL}{r + aL + b}.$$

(d) For a steady state -- in which L, the total number of people carrying coconuts, is constant -- the flows out of state C must always equal the flows into state C. That is, the number of people finding a trading partner and eating their coconut per unit time must equal the number of people finding and climbing a tree per unit time.

The number of people leaving state C per unit time is given by the probability of finding a trading partner, aL, multiplied by the number of people with coconuts and looking for a trading partner, L. The number of people entering state C per unit time is given by the probability of finding a tree, b, multiplied by the number of people looking for a tree, (N - L). For a steady state, these two must be equal. That is, a steady state requires

(8) $(aL)L = b(N - L)$.

Rearranging equation (8), we have the following quadratic equation in L:

(9) $aL^2 + bL - bN = 0$.

Using the quadratic formula gives us

(10) $$L = \frac{-b \pm \sqrt{b^2 + 4abN}}{2a} = \frac{-b \pm \sqrt{9b^2}}{2a} = \frac{b}{a},$$

where we have used the given condition that aN = 2b. Also note that we can ignore the solution with $L = -2b/a < 0$.

(e) For such a steady-state equilibrium, the gain to an individual from climbing a tree, $V_C - V_P$ -- going from having the value of being in state P to the value of being in state C -- must be greater than or equal to the cost to the individual of climbing the tree, c. That is, for a steady-state equilibrium where everyone who finds a palm tree actually climbs it, we require

(11) $V_C - V_P \geq c$.

Substituting the steady-state value of $L = b/a$ from equation (10) into the expression for $V_C - V_P$ given in equation (7), we have

(12) $$V_C - V_P = \frac{bc + \bar{u}a(b/a)}{r + a(b/a) + b} = \frac{bc + b\bar{u}}{r + 2b}.$$

Substituting equation (12) into inequality (11) yields

$$\frac{bc + b\bar{u}}{r + 2b} \geq c \quad \Rightarrow \quad bc + b\bar{u} \geq c(r + 2b) \quad \Rightarrow \quad c(r + 2b - b) \leq b\bar{u},$$

and thus the cost of climbing a tree must be such that

(13) $c \leq b\bar{u}/(r + b)$.

Note that the maximum possible cost for which it is optimal to always climb a tree when one is found (as long as everyone else is doing so) is increasing in the utility gained from eating a coconut and decreasing in the individual's discount rate.

(f) The situation in which no one who finds a tree climbs it is a steady-state equilibrium for any $c > 0$. If no one else is climbing a tree when they find one, it is optimal for an individual not to climb a tree when she finds one. If the individual were to climb a tree and pick a coconut, she would lose c units of utility with no hope of ever trading with someone else. If she does not climb the tree, she loses no utility. Thus it is optimal not to climb the tree. The decision process is the same for every individual who comes across a tree. Thus no one climbing a tree -- $L = 0$ -- is a steady-state equilibrium for any $c > 0$. This implies that for $0 < c \leq b\bar{u}/(r + b)$, there is more than one steady-state equilibrium. We have shown two: $L = 0$ and $L = b/a$.

In the situation of multiple equilibria, the one with $L = b/a$ involves higher welfare than the one with $L = 0$. We have shown that in part (e), with $c \leq b\bar{u}/(r + b)$, individuals end up gaining utility each time they climb a tree. That is why they do it. They know that the utility they will eventually receive by trading their coconut outweighs the cost of climbing the tree to obtain their coconut. Thus the equilibrium where people go through a cycle of searching, climbing, searching, trading and eating etc., generates positive utility for the individual. The equilibrium with $L = 0$ means that people never achieve any positive utility since they never trade and obtain the $\bar{u}$ units of utility from eating another person's coconut.

SOLUTIONS TO CHAPTER 7

Problem 7.1

Since transitory income is on average equal to zero, we can interpret average income as average permanent income. Thus we are told that on average, farmers have lower permanent income than nonfarmers do, or $\overline{Y}_F^P < \overline{Y}_{NF}^P$. We can interpret the fact that farmers' incomes fluctuate more from year to year as meaning that the variance of transitory income for farmers is larger than the variance of transitory income for nonfarmers, or $\text{var}(Y^T{}_F) > \text{var}(Y^T{}_{NF})$.

Consider the following regression model:

(1) $C_i = a + bY_i + e_i$,

where C_i is current consumption -- which according to the permanent-income hypothesis is determined entirely by Y^P so that $C = Y^P$ -- and Y_i is current income, which is assumed to be the sum of permanent income and transitory income so that $Y = Y^P + Y^T$. From equation (7.8) in the text, the Ordinary Least Squares (OLS) estimator of b takes the form

(2) $$\hat{b} = \frac{\text{var}(Y^P)}{\text{var}(Y^P) + \text{var}(Y^T)}.$$

As long as $\text{var}(Y^P)$ is the same across the two groups, the fact that $\text{var}(Y^T{}_F) > \text{var}(Y^T{}_{NF})$ means that the estimated slope coefficient should be smaller for farmers than it is for nonfarmers. This means that the estimated impact on consumption of a marginal increase in current income is smaller for farmers than for nonfarmers. According to the permanent-income hypothesis, this is because the increase is much more likely to be due to transitory income for farmers than for nonfarmers. Thus it can be expected to have a smaller impact on consumption for farmers than for nonfarmers.

From equation (7.9) in the text, the OLS estimator of the constant term takes the form

(3) $\hat{a} = (1 - \hat{b})\overline{Y}^P$.

The fact that farmers, on average, have lower permanent incomes than nonfarmers tends to make the estimated constant term smaller for farmers. However, as was just explained, $\hat{b}$ is smaller for farmers than it is for nonfarmers. This tends to make the estimated constant term bigger for farmers than for nonfarmers. Thus the effect on the estimated constant term is ambiguous.

We can, say, however, that at the average level of permanent income for farmers, the estimated consumption function for farmers is expected to lie below the one for nonfarmers. Thus if the two estimated consumption functions do cross, they cross at a level of income less than $\overline{Y}^P{}_F$. Why?

Consider a member of each group whose income equals the average income among farmers. Since there are many more nonfarmers with permanent incomes above this level than there are with permanent incomes below it, the nonfarmer's permanent income is much more likely to be greater than her current income than less. As a result, nonfarmers with this current income have on average higher permanent income; thus on average they consume more than their income. For the farmer, in contrast, her permanent income is about as likely to be more than current income as it is to be less; as a result, farmers with this current income on average have the same permanent income, and thus on average they consume their income. Thus the consumption function for farmers is expected to lie below the one for nonfarmers at the average level of income for farmers.

Problem 7.2

(a) We need to find an expression for $[(C_{t+2} + C_{t+3})/2] - [(C_t + C_{t+1})/2]$. We can write C_{t+1}, C_{t+2} and C_{t+3} in terms of C_t and the e's. Specifically, we can write

(1) $C_{t+1} = C_t + e_{t+1}$,

(2) $C_{t+2} = C_{t+1} + e_{t+2} = C_t + e_{t+1} + e_{t+2}$, and

(3) $C_{t+3} = C_{t+2} + e_{t+3} = C_t + e_{t+1} + e_{t+2} + e_{t+3}$,

where we have used equation (1) in deriving (2) and equation (2) in deriving (3). Using equations (1) through (3), the change in measured consumption from one two-period interval to the next is

(4) $$\frac{C_{t+2} + C_{t+3}}{2} - \frac{C_t + C_{t+1}}{2} = \frac{(C_t + e_{t+1} + e_{t+2}) + (C_t + e_{t+1} + e_{t+2} + e_{t+3})}{2} - \frac{C_t + (C_t + e_{t+1})}{2},$$

which simplifies to

(5) $$\frac{C_{t+2} + C_{t+3}}{2} - \frac{C_t + C_{t+1}}{2} = \frac{e_{t+3} + 2e_{t+2} + e_{t+1}}{2}.$$

(b) Through similar manipulations as in part (a), the previous value of the change in measured consumption would be

(6) $$\frac{C_t + C_{t+1}}{2} - \frac{C_{t-2} + C_{t-1}}{2} = \frac{e_{t+1} + 2e_t + e_{t-1}}{2}.$$

Using equations (5) and (6), the covariance between successive changes in measured consumption is

(7) $$\text{cov}\left[\left(\frac{C_{t+2} + C_{t+3}}{2} - \frac{C_t + C_{t+1}}{2}\right), \left(\frac{C_t + C_{t+1}}{2} - \frac{C_{t-2} + C_{t-1}}{2}\right)\right] =$$
$$\text{cov}\left[\left(\frac{e_{t+3} + 2e_{t+2} + e_{t+1}}{2}\right), \left(\frac{e_{t+1} + 2e_t + e_{t-1}}{2}\right)\right].$$

Since the e's are uncorrelated with each other and since e_{t+1} is the only value of e that appears in both expressions, the covariance reduces to

(8) $$\text{cov}\left[\left(\frac{C_{t+2} + C_{t+3}}{2} - \frac{C_t + C_{t+1}}{2}\right), \left(\frac{C_t + C_{t+1}}{2} - \frac{C_{t-2} + C_{t-1}}{2}\right)\right] = \frac{\sigma_e^2}{4},$$

where σ_e^2 denotes the variance of the e's. So the change in measured consumption is correlated with its previous value. Since the covariance is positive, this means that if measured consumption in the two-period interval (t, t + 1) is greater than measured consumption in the two-period interval (t - 2, t - 1), measured consumption in (t + 2, t + 3) will tend to be greater than measured consumption in (t, t + 1). When a variable follows a random walk, successive changes in the variable are uncorrelated. For example, with actual consumption in this model, we have $C_t - C_{t-1} = e_t$ and $C_{t+1} - C_t = e_{t+1}$. Since e_t and e_{t+1} are uncorrelated, successive changes in actual consumption are uncorrelated. Thus if C_t were bigger than C_{t-1}, it would not mean that C_{t+1} would tend to be higher than C_t. Since successive changes in measured consumption are correlated, measured consumption is not a random walk. The change in measured consumption today does provide us with some information as to what the change in measured consumption is likely to be tomorrow.

(c) From equation (5), the change in measured consumption from (t, t + 1) to (t + 2, t + 3) depends on e_{t+1}, the innovation to consumption in period t + 1. But this is known as of t + 1, which is part of the first two-period interval. Thus the change in consumption from one two-period interval to the next is not uncorrelated with everything known as of the first two-period interval. However, it is uncorrelated with everything known in the two-period interval immediately preceding (t, t + 1). From equation (5), e_{t+3}, e_{t+2} and e_{t+1} are all unknown as of the two-period interval (t - 2, t - 1).

(d) We can write C_{t+3} as a function of C_{t+1} and the e's. Specifically, we can write

(9) $C_{t+3} = C_{t+2} + e_{t+3} = C_{t+1} + e_{t+2} + e_{t+3}$.

Thus the change in measured consumption from one two-period interval to the next is

(10) $C_{t+3} - C_{t+1} = C_{t+1} + e_{t+2} + e_{t+3} - C_{t+1} = e_{t+2} + e_{t+3}$.

The same calculations would yield the previous value of the change in measured consumption,

(11) $C_{t+1} - C_{t-1} = e_t + e_{t+1}$.

And so the covariance between successive changes in measured consumption is

(12) $\text{cov}[(C_{t+3} - C_{t+1}), (C_{t+1} - C_{t-1})] = \text{cov}[(e_{t+2} + e_{t+3}), (e_t + e_{t+1})]$.

Since the e's are uncorrelated with each other, the covariance is zero. Thus measured consumption is a random walk in this case. The amount that C_{t+1} differs from C_{t-1} does not provide any information about what the difference between C_{t+1} and C_{t+3} will be.

Problem 7.3

(a) Consider the usual experiment of a decrease in consumption by a small (formally, infinitesimal) amount dC in period t. With the CRRA utility function given by

(1) $u(C_t) = C_t^{1-\theta}/(1 - \theta)$,

the marginal utility of consumption in period t is $C_t^{-\theta}$. Thus the change has a utility cost of

(2) utility cost $= C_t^{-\theta}\,dC$.

The marginal utility of consumption in period t + 1 is $C_{t+1}^{-\theta}$. With a real interest rate of r, the individual gets to consume an additional (1 + r)dC in period t + 1. This has a discounted expected utility benefit of

(3) expected utility benefit $= \dfrac{1}{1+\rho} E_t\left[C_{t+1}^{-\theta}(1+r)dC\right]$.

If the individual is optimizing, a marginal change of this type does not affect expected utility. This means that the utility cost must equal the expected utility benefit or

(4) $C_t^{-\theta} = \dfrac{1+r}{1+\rho} E_t\left[C_{t+1}^{-\theta}\right]$,

where we have (rather informally) canceled the dC's. Equation (4) is the Euler equation.

(b) For any variable x, $e^{\ln x} = x$, and so we can write

(5) $E_t\left[C_{t+1}^{-\theta}\right] = E_t\left[e^{-\theta \ln C_{t+1}}\right]$.

Using the hint in the question -- if $x \sim N(\mu, V)$ then $E[e^x] = e^{\mu} e^{V/2}$ -- then since the log of consumption is distributed normally, we have

(6)
$$\begin{aligned} E_t\left[C_{t+1}^{-\theta}\right] &= E_t\left[e^{-\theta E_t \ln C_{t+1}} e^{\theta^2\sigma^2/2}\right] \\ &= e^{-\theta E_t \ln C_{t+1}} e^{\theta^2\sigma^2/2}. \end{aligned}$$

In the first line, we have used the fact that conditional on time t information, the variance of log consumption is σ^2. In addition, we have written the mean of log consumption in period t + 1, conditional on time t information, as $E_t \ln C_{t+1}$. Finally, in the last line we have used the fact that $e^{-\theta E_t \ln C_{t+1}} e^{\theta^2\sigma^2/2}$ is simply a constant.

Substituting equation (6) back into equation (2) and taking the log of both sides yields

(7) $-\theta \ln C_t = \ln(1 + r) - \ln(1 + \rho) - \theta E_t \ln C_{t+1} + \theta^2 \sigma^2/2$.

Dividing both sides of equation (7) by $(-\theta)$ leaves us with

(8) $\ln C_t = E_t \ln C_{t+1} + [\ln(1 + \rho) - \ln(1 + r)]/\theta - \theta\sigma^2/2$.

(c) Rearranging equation (8) to solve for $E_t \ln C_{t+1}$ gives us

(9) $E_t \ln C_{t+1} = \ln C_t + [\ln(1 + r) - \ln(1 + \rho)]/\theta + \theta\sigma^2/2$.

Equation (9) implies that consumption is expected to change by the constant amount $[\ln(1 + r) - \ln(1 + \rho)]/\theta + \theta\sigma^2/2$ from one period to the next. Changes in consumption other than this deterministic amount are unpredictable. By the definition of expectations we can write

(10) $E_t \ln C_{t+1} = \ln C_t + [\ln(1 + r) - \ln(1 + \rho)]/\theta + \theta\sigma^2/2 + u_{t+1}$,

where the u's have mean zero and are serially uncorrelated. Thus log consumption follows a random walk with drift where $[\ln(1 + r) - \ln(1 + \rho)]/\theta + \theta\sigma^2/2$ is the drift parameter.

(d) From equation (9), expected consumption growth is

(11) $E_t [\ln C_{t+1} - \ln C_t] = [\ln(1 + r) - \ln(1 + \rho)]/\theta + \theta\sigma^2/2$.

Clearly, a rise in r raises expected consumption growth. We have

$$(12)\quad \frac{\partial E_t\left[\ln C_{t+1} - \ln C_t\right]}{\partial r} = \frac{1}{\theta}\frac{1}{(1+r)} > 0.$$

Note that the smaller is θ -- the bigger is the elasticity of substitution, $1/\theta$ -- the more that consumption growth increases due to a given increase in the real interest rate.

An increase in σ^2 also increases consumption growth since

$$(13)\quad \frac{\partial E_t\left[\ln C_{t+1} - \ln C_t\right]}{\partial \sigma^2} = \frac{\theta}{2} > 0.$$

It is straightforward to verify that the CRRA utility function has a positive third derivative. From equation (1), $u'(C_t) = C_t^{-\theta}$ and $u''(C_t) = -\theta C_t^{-\theta-1}$. Thus

(14) $u'''(C_t) = -\theta(-\theta - 1)C_t^{-\theta-2} = (\theta^2 + \theta)C_t^{-\theta-2} > 0$.

So an individual with a CRRA utility function exhibits the precautionary saving behavior explained in Section 7.6. A rise in uncertainty (as measured by σ^2, the variance of log consumption) increases saving and thus expected consumption growth.

Problem 7.4

(a) Substituting the expression for consumption in period t, which is

$$(1)\quad C_t = \frac{r}{1+r}\left[A_t + \sum_{s=0}^{\infty}\frac{E_t[Y_{t+s}]}{(1+r)^s}\right],$$

into the expression for wealth in period t + 1, which is

(2) $A_{t+1} = (1 + r)[A_t + Y_t - C_t]$,

gives us

$$(3)\quad A_{t+1} = (1+r)\left[A_t + Y_t - \frac{r}{1+r}A_t - \frac{r}{1+r}\left(Y_t + \frac{E_tY_{t+1}}{1+r} + \frac{E_tY_{t+2}}{(1+r)^2} + \ldots\right)\right].$$

Obtaining a common denominator of (1 + r) and then canceling the (1 + r)'s gives us

$$(4)\quad A_{t+1} = A_t + Y_t - r\left(\frac{E_tY_{t+1}}{1+r} + \frac{E_tY_{t+2}}{(1+r)^2} + \ldots\right).$$

Since equation (1) holds in all periods, we can write consumption in period t + 1 as

$$(5)\quad C_{t+1} = \frac{r}{1+r}\left[A_{t+1} + \sum_{s=0}^{\infty}\frac{E_{t+1}[Y_{t+1+s}]}{(1+r)^s}\right].$$

Substituting equation (4) into equation (5) yields

$$(6)\ C_{t+1} = \frac{r}{1+r}\left[A_t + Y_t - r\left(\frac{E_t Y_{t+1}}{1+r} + \frac{E_t Y_{t+2}}{(1+r)^2} + \ldots\right) + \left(E_{t+1} Y_{t+1} + \frac{E_{t+1} Y_{t+2}}{1+r} + \ldots\right)\right].$$

Taking the expectation, conditional on time t information, of both sides of equation (6) gives us

$$(7)\ E_t C_{t+1} = \frac{r}{1+r}\left[A_t + Y_t - r\left(\frac{E_t Y_{t+1}}{1+r} + \frac{E_t Y_{t+2}}{(1+r)^2} + \ldots\right) + \left(E_t Y_{t+1} + \frac{E_t Y_{t+2}}{1+r} + \ldots\right)\right],$$

where we have used the law of iterated projections so that for any variable x, $E_t E_{t+1} x_{t+2} = E_t x_{t+2}$. If this did not hold, individuals would be expecting to revise their estimate either upward or downward and thus their original expectation could not have been rational. Collecting terms in equation (7) gives us

$$(8)\ E_t C_{t+1} = \frac{r}{1+r}\left[A_t + Y_t - \left(1 - \frac{r}{1+r}\right)E_t Y_{t+1} + \left(\frac{1}{1+r} - \frac{r}{(1+r)^2}\right)E_t Y_{t+2} + \ldots\right],$$

which simplifies to

$$(9)\ E_t C_{t+1} = \frac{r}{1+r}\left[A_t + Y_t + \frac{E_t Y_{t+1}}{1+r} + \frac{E_t Y_{t+2}}{(1+r)^2} + \ldots\right].$$

Using summation notation, and noting that $E_t Y_t = Y_t$, we have

$$(10)\ E_t C_{t+1} = \frac{r}{1+r}\left[A_t + \sum_{s=0}^{\infty} \frac{E_t Y_{t+s}}{(1+r)^s}\right].$$

The right-hand sides of equations (1) and (10) are equal and thus

(11) $E_t C_{t+1} = C_t$.

Consumption follows a random walk; changes in consumption are unpredictable.

Since consumption follows a random walk, the best estimate of consumption in any future period is simply the value of consumption in this period. That is, for any $s \geq 0$, we can write

(12) $E_t C_{t+s} = C_t$.

Using equation (12), we can write the present value of the expected path of consumption as

$$(13)\ \sum_{s=0}^{\infty} \frac{E_t[C_{t+s}]}{(1+r)^s} = \sum_{s=0}^{\infty} \frac{C_t}{(1+r)^s} = C_t \sum_{s=0}^{\infty} \frac{1}{(1+r)^s}.$$

Since $1/(1+r) < 1$, the infinite sum on the right-hand side of (13) converges to $1/[1 - 1/(1+r)] = (1+r)/r$ and thus

$$(14)\ \sum_{s=0}^{\infty} \frac{E_t[C_{t+s}]}{(1+r)^s} = \frac{1+r}{r} C_t.$$

Substituting equation (1) for C_t into the right-hand side of equation (14) yields

$$(15)\ \sum_{s=0}^{\infty} \frac{E_t[C_{t+s}]}{(1+r)^s} = \frac{r}{1+r}\left(\frac{1+r}{r}\right)\left[A_t + \sum_{s=0}^{\infty} \frac{E_t[Y_{t+s}]}{(1+r)^s}\right] = A_t + \sum_{s=0}^{\infty} \frac{E_t[Y_{t+s}]}{(1+r)^s}.$$

Equation (15) states that the present value of the expected path of consumption equals initial wealth plus the present value of the expected path of income.

(b) Taking the expected value, as of time t - 1, of both sides of equation (1) yields

$$(16)\ E_{t-1} C_t = \frac{r}{1+r}\left[A_t + \sum_{s=0}^{\infty} \frac{E_{t-1}[Y_{t+s}]}{(1+r)^s}\right],$$

where we have used the fact that $A_t = (1 + r)[A_{t-1} + Y_{t-1} - C_{t-1}]$ is not uncertain as of t - 1. In addition, we have used the law of iterated projections so that $E_{t-1} E_t [Y_{t+s}] = E_{t-1} [Y_{t+s}]$. Subtracting equation (16) from equation (1) gives us the innovation in consumption:

$$(17)\quad C_t - E_{t-1}C_t = \frac{r}{1+r}\left[\sum_{s=0}^{\infty}\frac{E_t[Y_{t+s}]}{(1+r)^s} - \sum_{s=0}^{\infty}\frac{E_{t-1}[Y_{t+s}]}{(1+r)^s}\right] = \frac{r}{1+r}\left[\sum_{s=0}^{\infty}\frac{E_t[Y_{t+s}] - E_{t-1}[Y_{t+s}]}{(1+r)^s}\right].$$

The innovation in consumption will be fraction $r/(1 + r)$ of the present value of the change in expected lifetime income.

The next step is to determine the present value of the change in expected lifetime income. That is, we need to determine

$$(18)\quad \sum_{s=0}^{\infty}\frac{E_t[Y_{t+s}] - E_{t-1}[Y_{t+s}]}{(1+r)^s} = [Y_t - E_{t-1}Y_t] + \left[\frac{E_tY_{t+1} - E_{t-1}Y_{t+1}}{1+r}\right] + \left[\frac{E_tY_{t+2} - E_{t-1}Y_{t+2}}{(1+r)^2}\right] + \dots .$$

In what follows, "expected to be higher" means "expected, as of period t, to be higher than it was, as of period t - 1". We are told that $u_t = 1$ and thus

(19) $Y_t - E_{t-1} Y_t = 1$.

In period t + 1, since $\Delta Y_{t+1} = \phi\Delta Y_t + u_{t+1}$, the change in Y_{t+1} is expected to be $\phi\Delta Y_t = \phi$ higher. Thus the level of Y_{t+1} is expected to be higher by $1 + \phi$. Thus

$$(20)\quad \frac{E_tY_{t+1} - E_{t-1}Y_{t+1}}{1+r} = \frac{1+\phi}{1+r}.$$

In period t + 2, since $\Delta Y_{t+2} = \phi\Delta Y_{t+1} + u_{t+2}$, the change in Y_{t+2} is expected to be higher by $\phi\Delta Y_{t+1} = \phi^2$. Thus the level of Y_{t+2} is expected to be higher by $1 + \phi + \phi^2$. Therefore, we have

$$(21)\quad \frac{E_tY_{t+2} - E_{t-1}Y_{t+2}}{(1+r)^2} = \frac{1+\phi+\phi^2}{(1+r)^2}.$$

The pattern should be clear. We have

$$(22)\quad \sum_{s=0}^{\infty}\frac{E_t[Y_{t+s}] - E_{t-1}[Y_{t+s}]}{(1+r)^s} = 1 + \frac{1+\phi}{1+r} + \frac{1+\phi+\phi^2}{(1+r)^2} + \frac{1+\phi+\phi^2+\phi^3}{(1+r)^3} + \dots .$$

Note that this infinite series can be rewritten as

$$(23)\quad \sum_{s=0}^{\infty}\frac{E_t[Y_{t+s}] - E_{t-1}[Y_{t+s}]}{(1+r)^s} = \left[1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + \dots\right] + \left[\frac{\phi}{1+r} + \frac{\phi}{(1+r)^2} + \frac{\phi}{(1+r)^3}\right] + \left[\frac{\phi^2}{(1+r)^2} + \frac{\phi^2}{(1+r)^3} + \dots\right] + \dots .$$

For ease of notation, define $\gamma \equiv 1/(1 + r)$. Then the first sum on the right-hand side of (23) converges to $1/(1 - \gamma)$. The second sum converges to $\phi\gamma/(1 - \gamma)$. The third sum converges to $\phi^2\gamma^2/(1 - \gamma)$. And so on. Thus equation (23) can be rewritten as

$$(24)\quad \sum_{s=0}^{\infty}\frac{E_t[Y_{t+s}] - E_{t-1}[Y_{t+s}]}{(1+r)^s} = \frac{1}{1-\gamma}\left[1 + \phi\gamma + \phi^2\gamma^2 + \dots\right] = \frac{1}{(1-\gamma)}\frac{1}{(1-\phi\gamma)}.$$

Using the definition of γ to rewrite equation (24) yields

$$(25)\quad \sum_{s=0}^{\infty} \frac{E_t[Y_{t+s}] - E_{t-1}[Y_{t+s}]}{(1+r)^s} = \frac{1}{1-[1/(1+r)]}\,\frac{1}{1-[\phi/(1+r)]} = \frac{(1+r)}{r}\,\frac{(1+r)}{(1+r-\phi)}.$$

Substituting equation (25) into equation (24) gives us the following change in consumption:

$$(26)\quad C_t - E_{t-1}C_t = \frac{r}{(1+r)}\left[\frac{(1+r)}{r}\,\frac{(1+r)}{(1+r-\phi)}\right] = \frac{(1+r)}{(1+r-\phi)}.$$

(c) The variance of the innovation in consumption is

$$(27)\quad \mathrm{var}\left(C_t - E_{t-1}C_t\right) = \mathrm{var}\left[\frac{(1+r)}{(1+r-\phi)}u_t\right] = \left[\frac{(1+r)}{(1+r-\phi)}\right]^2 \mathrm{var}(u_t) > \mathrm{var}(u_t)$$

Since $(1 + r)/(1 + r - \phi) > 1$, the variance of the innovation in consumption is greater than the variance of the innovation in income. Intuitively, an innovation to income means that, on average, the consumer will experience further changes in income in the same direction in future periods.

It is not clear whether consumers use saving and borrowing to smooth consumption relative to income. Income is not stationary, so it is not obvious what it means to smooth it.

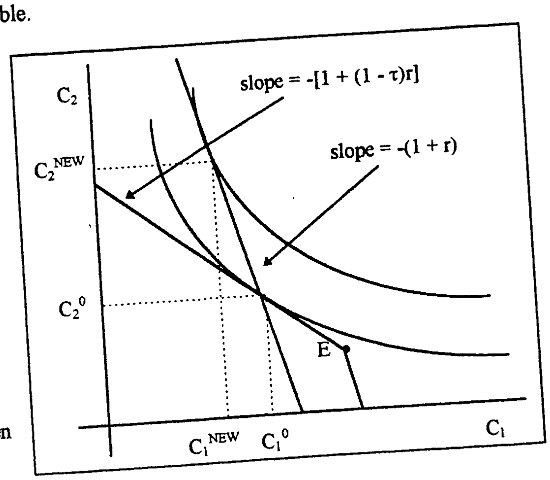
Problem 7.5

(a) The present value of the lump-sum taxes is $T_1 + [T_2/(1 + r)]$. The present value of the tax on interest income is $[r/(1 + r)]\tau(Y_1 - C_1^0)$, where τ is the tax rate on interest income. The government must choose T_1 and T_2 so that these two quantities are equal, or

$$(1)\quad T_1 + \frac{T_2}{1+r} = \frac{r}{1+r}\tau(Y_1 - C_1^0).$$

(b) Suppose the new taxes satisfy condition (1). This means that at the point where the individual consumes C_1^0, she pays the same with the new lump-sum tax as she did with the old tax on interest income. That is, right at C_1^0, the individual's after-tax lifetime income is the same under both tax schemes. Thus at C_1^0, the individual has just enough to consume C_2^0 in the second period under both tax schemes. This means that the new budget line must go through (C_1^0, C_2^0) just as the old one did. Since (C_1^0, C_2^0) lies right on the new budget line, it is just affordable.

(c) First-period consumption must fall. Consider the figure at right. Point E represents the endowment, (Y_1, Y_2). The budget line under the tax on interest income has slope $-[1 + (1 - \tau)r]$ for $C_1 < Y_1$; for $C_1 > Y_1$ there is no positive saving and therefore no tax on interest income so that the slope equals $-(1 + r)$.



As explained in part (b), the budget line with revenue-neutral, lump-sum taxes goes through the initial optimum consumption bundle, (C_1^0, C_2^0). It has slope equal to $-(1 + r)$. With saving no longer taxed, then for any $C_1 < Y_1$, giving up one unit of period-one consumption yields more units of period-two consumption. Specifically, it yields $(1 + r)$ rather than $[1 + (1 - \tau)r]$. From the figure, we can see that the new tangency must involve lower consumption in the first period.

Intuitively, the government has set the tax rate so that there is no income effect from the change in policy, only a substitution effect. Thus, since the rate of return on saving increases, the individual chooses to save more and consume less in the first period.

Problem 7.6

(a) The change in purchases in period t, dE_t, must leave the present value of spending unchanged, so that
(1) $dE_t + dE_{t+1} + dE_{t+2} = 0$.
In addition, it must leave consumption in period t + 2 unchanged, or
(2) $(1 - \delta)^2 dE_t + (1 - \delta)dE_{t+1} + dE_{t+2} = 0$.
To see why equation (2) must hold, note that we can write the change in C_t as $dC_t = dE_t$. The change in C_{t+1} is $dC_{t+1} = (1 - \rho)dC_t + dE_{t+1}$ or substituting for dC_t, we have $dC_{t+1} = (1 - \rho)dE_t + dE_{t+1}$. The change in C_{t+2} is $dC_{t+2} = (1 - \delta)dC_{t+1} + dE_{t+2}$ or substituting for dC_{t+1}, we have $dC_{t+2} = (1 - \delta)^2 dE_t + (1 - \delta)dE_{t+1} + dE_{t+2}$. Thus if C_{t+2} is not to change, equation (2) must hold.

Thus we have two equations in two unknowns. Solving equation (1) for dE_{t+2} gives us
(3) $dE_{t+2} = -dE_t - dE_{t+1}$.
Substituting equation (3) into equation (2) yields
(4) $(1 - \delta)^2 dE_t + (1 - \delta)dE_{t+1} - dE_t - dE_{t+1} = 0$.
Expanding and collecting terms gives us
(5) $dE_t [1 - 2\delta + \delta^2 - 1] + [1 - \delta - 1]dE_{t+1} = 0$,
and thus
(6) $dE_{t+1} = (\delta - 2)dE_t$.
Substituting equation (6) into equation (3) yields
(7) $dE_{t+2} = -dE_t - (\delta - 2)dE_t$,
and thus
(8) $dE_{t+2} = (1 - \delta)dE_t$.

(b) Since $C_t = (1 - \delta)C_{t-1} + E_t$, then
(9) $dC_t = dE_t$.
Since $C_{t+1} = (1 - \delta)C_t + E_{t+1}$, then
(10) $dC_{t+1} = (1 - \delta)dC_t + dE_{t+1}$.
Substituting equations (9) and (6) into equation (10) gives us
(11) $dC_{t+1} = (1 - \delta)dE_t + (\delta - 2)dE_t = -dE_t$.

Since only C_t and C_{t+1} are changed -- C_{t+2} is unchanged by construction -- we only need to look at expected utility in periods t and t + 1. Since instantaneous utility is quadratic, the marginal utility of consumption in period t is $1 - aC_t$. Thus the change in utility in period t is $(1 - aC_t)(dE_t)$. The marginal utility of consumption in period t + 1 is given by $1 - aC_{t+1}$. Since $dC_{t+1} = -dE_t$, the change in expected utility in period t + 1 is the expected value of $(1 - aC_{t+1})(-dE_t)$.

(c) For this change in expected utility to be zero -- as it must be, if the individual is optimizing -- we require
(12) $(1 - aC_t)(dE_t)$ + expected value of $[(1 - aC_{t+1})(-dE_t)] = 0$.
Canceling the dE_t's (which is somewhat informal), subtracting one from both sides and then dividing both sides by (- a) yields

(13) expected value of $C_{t+1} = C_t$.
Thus consumption follows a random walk since changes in consumption are unpredictable. The best estimate of consumption in period t + 1 is simply what consumption equals this period.

(d) Rearranging $C_t = (1 - \delta)C_{t-1} + E_t$ to solve for E_t gives us

(14) $E_t = C_t - (1 - \delta)C_{t-1}$.

Equation (14) holds for all periods and so we can write

(15) $E_{t-1} = C_{t-1} - (1 - \delta)C_{t-2}$.

Subtracting equation (15) from equation (14) gives us

(16) $E_t - E_{t-1} = C_t - (1 - \delta)C_{t-1} - C_{t-1} + (1 - \delta)C_{t-2}$,

which implies

(17) $E_t - E_{t-1} = (C_t - C_{t-1}) - (1 - \delta)(C_{t-1} - C_{t-2})$.

Since consumption is a random walk, we can write

(18) $C_t = C_{t-1} + u_t$,

where u_t is a variable whose expectation as of t - 1 is zero. Using equation (18), and the fact that (18) holds in all periods, equation (17) can be rewritten as

(19) $E_t - E_{t-1} = u_t - (1 - \delta)u_{t-1}$.

Equation (19) states that the change in purchases from t - 1 to t has a predictable component -- a component that is known as of t - 1 -- which is u_{t-1} , the innovation to consumption in period t - 1. Thus purchases of durable goods will not follow a random walk.

As explained in Section 7.2, any change in expected lifetime resources is spread out equally among consumption in each remaining period of the individual's life. Although we are simplifying by using a discount rate of zero, the basic ideas are general.

Now suppose that in period t - 1, the individual's estimate of lifetime resources changes in such a way that C_{t-1} is one unit higher than C_{t-2} , that is, $u_{t-1} = 1$. This also means that expected consumption in all future periods is one unit higher than it used to be. In order to get C_{t-1} up by one, purchases in period t - 1 must be one higher than they were expected to be. But now look at the change in purchases from t - 1 to t. From equation (19), the expectation (as of period t - 1) of the change in purchases from t - 1 to t is -(1 - δ), since u_{t-1} is assumed to equal one.

Intuitively, some of the new goods purchased in period t - 1 will still be around in period t. Thus to keep expected consumption in period t at the new higher path -- one higher than it was before -- it is not expected to be necessary to buy one unit of goods all over again. The individual only has to purchase enough to replace the fraction of the extra t - 1 purchases that depreciated, which is fraction δ. Thus purchases in period t are expected to be less than purchases in t - 1. Specifically, they are expected to be lower by the amount that does not depreciate, which is (1 - δ). Thus, as of period t - 1, part of the change in purchases between t - 1 and t is predictable and thus purchases do not follow a random walk.

Now consider what happens if δ = 0, the case of no depreciation. Then from equation (19), the expectation (as of period t - 1) of the change in purchases from t - 1 to t is -1. Now all of the new goods purchased in period t - 1 will still be around in period t. Thus to keep expected consumption at its new higher path -- one higher than it was before -- it is not expected to be necessary to purchase anything new in period t. Thus purchases are expected to fall by the whole amount of the innovation in purchases the previous period.

Problem 7.7

(a) Consider the following experiment. In period t, the individual reduces consumption by a small (formally, infinitesimal) amount dC and uses the proceeds to purchase stock. Since one unit of stock costs P_t, dC will buy the individual dC/P_t units of stock. This change has a utility cost of dC since utility is linear in consumption.

In period t + 1, the individual will receive D_{t+1} [dC/P_t] in dividends which she can consume. She can then sell the stock, receiving P_{t+1} [dC/P_t] which she can also consume. The discounted expected utility benefit of doing this is $E_t\,[[1/(1 + r)][D_{t+1} + P_{t+1}][dC/P_t]]$. If the individual is optimizing, a marginal change of this type must leave expected utility unchanged. Thus the utility cost must equal the expected utility benefit, or

$$(1)\quad dC = E_t\left[\left(\frac{1}{1+r}\right)(D_{t+1} + P_{t+1})\frac{dC}{P_t}\right].$$

Canceling the dC's, which is somewhat informal, and multiplying both sides of the resulting expression by P_t yields

$$(2)\quad P_t = E_t\left[\frac{D_{t+1} + P_{t+1}}{1+r}\right].$$

(b) Equation (2) holds in all periods and so we can write

$$(3)\quad P_{t+1} = E_{t+1}\left[\frac{D_{t+2} + P_{t+2}}{1+r}\right].$$

Substituting equation (3) into equation (2) gives us

$$(4)\quad P_t = E_t\left[\frac{D_{t+1}}{1+r}\right] + E_t E_{t+1}\left[\frac{D_{t+2} + P_{t+2}}{(1+r)^2}\right].$$

Now we can use the law of iterated projections. For a variable x, $E_t E_{t+1} x_{t+2} = E_t x_{t+2}$. Equation (4) then becomes

$$(5)\quad P_t = E_t\left[\frac{D_{t+1}}{1+r} + \frac{D_{t+2}}{(1+r)^2}\right] + E_t\left[\frac{P_{t+2}}{(1+r)^2}\right].$$

We could now substitute for P_{t+2} and then P_{t+3} and so on. We would have

$$(6)\quad P_t = E_t\left[\frac{D_{t+1}}{1+r} + \frac{D_{t+2}}{(1+r)^2} + \ldots + \frac{D_{t+s}}{(1+r)^s}\right] + E_t\left[\frac{P_{t+s}}{(1+r)^s}\right].$$

Imposing the no-bubbles condition that $\lim_{s\to\infty} E_t\,[P_{t+s}/(1 + r)^s] = 0$, we can write P_t as

$$(7)\quad P_t = \sum_{s=1}^{\infty} E_t\left[\frac{D_{t+s}}{(1+r)^s}\right].$$

Equation (7) says that the price of the stock is the present value of the stream of expected future dividends.

Problem 7.8

(a) (i) With the bubble term, the price of the stock in period t is now

$$(1)\quad P_t = \sum_{s=1}^{\infty} E_t\left[\frac{D_{t+s}}{(1+r)^s}\right] + (1+r)^t b.$$

We need to see if such a price path satisfies the individual's first-order condition, which is given by

$$(2)\quad P_t = E_t\left[\frac{D_{t+1} + P_{t+1}}{1+r}\right].$$

Specifically, then, we need to see if the right-hand sides of equations (1) and (2) are equivalent. Since equation (1) holds every period, we can write the price of the stock in period t + 1 as

$$(3)\quad P_{t+1} = \sum_{s=1}^{\infty} E_{t+1}\left[\frac{D_{t+1+s}}{(1+r)^s}\right] + (1+r)^{t+1} b.$$

Dividing both sides of equation (3) by (1 + r) and then taking the time-t expectation of both sides of the resulting expression gives us

$$(4)\quad E_t\left[\frac{P_{t+1}}{1+r}\right] = \sum_{s=1}^{\infty} E_t\left[\frac{D_{t+1+s}}{(1+r)^{s+1}}\right] + (1+r)^t b,$$

where we have used the law of iterated projections so that $E_t E_{t+1} x_{t+2} = E_t x_{t+2}$ for any variable x. Now add $E_t [D_{t+1}/(1 + r)]$ to both sides of equation (4) to obtain

$$(5)\quad E_t\left[\frac{D_{t+1}+P_{t+1}}{1+r}\right] = E_t\left[\frac{D_{t+1}}{1+r}\right] + \sum_{s=1}^{\infty} E_t\left[\frac{D_{t+1+s}}{(1+r)^{s+1}}\right] + (1+r)^t b = \sum_{s=1}^{\infty} E_t\left[\frac{D_{t+s}}{(1+r)^s}\right] + (1+r)^t b.$$

Thus the right-hand sides of equations (1) and (2) are equivalent and so the proposed price path satisfies the individual's first-order condition. In this case, consumers are willing to pay more than the present value of the stream of expected future dividends. That is because they anticipate the price of the stock will keep rising so that they can enjoy capital gains that exactly offset the premium they are paying.

(a) (ii) If b were negative, then as $t \to \infty$, the bubble term, $(1 + r)^t b$, would go to minus infinity. Thus the price of the stock would eventually become negative and go to minus infinity. But that is not possible. The stock would never sell for a negative price. The strategy of just holding on to the stock and never selling it would avoid the capital loss from selling at a negative price. Or even more simply, an individual could just throw her stock certificate away rather than sell it for a negative price. Thus b cannot be negative.

(b) (i) With this bubble term, the price of the stock in period t is

$$(6)\quad P_t = \sum_{s=1}^{\infty} E_t\left[\frac{D_{t+s}}{(1+r)^s}\right] + q_t,$$

where q_t equals $(1 + r)q_{t-1}/\alpha$ with probability α and equals zero with probability $(1 - \alpha)$. Again, we need to see if the right-hand side of equation (6) is equivalent to the right-hand side of equation (2), the first-order condition. Since equation (6) holds in every period, we can write the price of the stock in period t + 1 as

$$(7)\quad P_{t+1} = \sum_{s=1}^{\infty} E_{t+1}\left[\frac{D_{t+1+s}}{(1+r)^s}\right] + q_{t+1}.$$

Taking the time t expectation of both sides of equation (7) and using the law of iterated projections, we have

$$(8)\quad E_t\left[P_{t+1}\right] = \sum_{s=1}^{\infty} E_t\left[\frac{D_{t+1+s}}{(1+r)^s}\right] + \frac{(1+r)q_t}{\alpha}\alpha + (0)(1-\alpha) = \sum_{s=1}^{\infty} E_t\left[\frac{D_{t+1+s}}{(1+r)^s}\right] + (1+r)q_t.$$

Dividing both sides of equation (8) by (1 + r) and then adding $E_t [D_{t+1}/(1 + r)]$ to both sides of the resulting expression gives us

$$(9)\quad E_t\left[\frac{D_{t+1}+P_{t+1}}{1+r}\right] = E_t\left[\frac{D_{t+1}}{1+r}\right] + \sum_{s=1}^{\infty} E_t\left[\frac{D_{t+1+s}}{(1+r)^{s+1}}\right] + q_t = \sum_{s=1}^{\infty} E_t\left[\frac{D_{t+s}}{(1+r)^s}\right] + q_t.$$

Thus the right-hand sides of equations (6) and (2) are equivalent and so the proposed price path satisfies the individual's first-order condition.

(b) (ii) The probability that the bubble has burst by time $t + s$ is the probability that it bursts in $t + 1$ plus the probability that it bursts in $t + 2$, given that it did not burst in $t + 1$, plus the probability that it bursts in $t + 3$, given that it did not burst in $t + 1$ or $t + 2$ and so on. The probability that the bubble bursts in period $t + 1$ is $(1 - \alpha)$. The probability that the bubble bursts in $t + 2$, given that it did not burst in period $t + 1$ is given by $\alpha(1 - \alpha)$. The probability that the bubble bursts in $t + 3$, given that it did not burst in periods $t + 1$ or $t + 2$ is $\alpha^2 (1 - \alpha)$. And so on, up to the probability that the bubble bursts in period s, given that it has not burst in any previous period, which is $\alpha^{s-1} (1 - \alpha)$. Thus the probability that the bubble has burst by time $t + s$ is given by the sum of all these probabilities, or

(10) $\text{Prob(burst by } t + s) = (1 - \alpha)(1 + \alpha + \alpha^2 + \ldots + \alpha^{s-1})$.

As we allow s to go to infinity, then since $\alpha < 1$, $1 + \alpha + \alpha^2 + \ldots + \alpha^{s-1}$ converges to $1/(1 - \alpha)$. Thus the probability that the bubble has burst by time $t + s$, as s goes to infinity is $(1 - \alpha)/(1 - \alpha)$ or simply one.

(c) (i) The price of the stock in period t, in the absence of bubbles, is given by

$$(11)\quad P_t = \sum_{s=1}^{\infty} E_t\left[\frac{D_{t+s}}{(1+r)^s}\right].$$

If dividends follow a random walk, then $E_t D_{t+s} = D_t$ for any $s \geq 0$. Since changes in dividends are unpredictable, the best estimate of dividends in any future period is what dividends are today. Thus P_t can be written as

$$(12)\quad P_t = \sum_{s=1}^{\infty} \frac{D_t}{(1+r)^s} = D_t \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} = D_t\left[\frac{1}{1+r} + \frac{1}{(1+r)^2} + \ldots\right].$$

With $1/(1 + r) < 1$, we have

$$(13)\quad \frac{1}{1+r} + \frac{1}{(1+r)^2} + \ldots = \frac{1/(1+r)}{1-[1/(1+r)]} = \frac{1/(1+r)}{r/(1+r)} = \frac{1}{r}.$$

Substituting equation (13) into equation (12) gives us the following price of the stock in period t:

(14) $P_t = D_t / r$.

(c) (ii) With the bubble term, the price of the stock in period t is given by

(15) $P_t = (D_t / r) + b_t = (D_t / r) + (1 + r)b_{t-1} + ce_t$.

We need to see if the right-hand side of equation (15) is equivalent to the right-hand side of equation (2), the first-order condition. Since equation (15) holds every period, we can write the price of the stock in period $t + 1$ as

(16) $P_{t+1} = (D_{t+1} / r) + (1 + r)b_t + ce_{t+1} = [(D_t + e_{t+1})/r] + (1 + r)b_t + ce_{t+1}$,

where we have used the fact that $D_{t+1} = D_t + e_{t+1}$. Dividing both sides of equation (16) by $(1 + r)$ and taking the time-t expectation of the resulting expression gives us

$$(17)\quad E_t\left[\frac{P_{t+1}}{1+r}\right] = \frac{D_t}{r(1+r)} + b_t.$$

Adding $E_t [D_{t+1} /(1 + r)]$ to both sides of equation (17) gives us

$$(18)\quad E_t\left[\frac{D_{t+1} + P_{t+1}}{1+r}\right] = E_t\left[\frac{D_{t+1}}{1+r}\right] + \frac{D_t}{r(1+r)} + b_t = \frac{rD_t + D_t}{r(1+r)} + b_t = \frac{(1+r)D_t}{r(1+r)} + b_t,$$

and thus finally

$$(19)\quad E_t\left[\frac{D_{t+1} + P_{t+1}}{1+r}\right] = \frac{D_t}{r} + b_t.$$

Therefore, the right-hand sides of equations (15) and (2) are equivalent and thus the first-order condition is satisfied. With this formulation, the innovation to dividends, e, gets built into the bubble. Thus a positive realization of e does not just raise the expected path of dividends, it also raises the path of the bubble and the current price responds to both of these changes. It is in this sense that the price of the stock overreacts to changes in dividends.

Problem 7.9

(a) Suppose the individual reduces her consumption by a small (formally infinitesimal) amount dC in period t. The utility cost of doing this equals the marginal utility of consumption in period t, $1/C_t$, times dC. Thus we have

(1) utility cost = dC/C_t.

This reduction in consumption allows the individual to purchase dC/P_t trees in period t. In period t + 1, the individual receives the extra output from her additional holdings of trees. She gets to consume an extra $[dC/P_t]Y_{t+1}$. The individual then sells her additional holdings of trees for $[dC/P_t]P_{t+1}$ and consumes the proceeds. Thus her total extra consumption in period t + 1 is given by $[dC/P_t]Y_{t+1} + [dC/P_t]P_{t+1}$. The marginal utility of consumption in period t + 1 is $1/C_{t+1}$. Thus the expected discounted utility benefit from this action is

$$\text{(2) expected utility benefit} = E_t\left[\frac{1}{1+\rho}\frac{1}{C_{t+1}}\left(\frac{dC}{P_t}Y_{t+1} + \frac{dC}{P_t}P_{t+1}\right)\right].$$

If the individual is optimizing, a marginal change of this type must leave expected utility unchanged. This means that the utility cost must equal the expected utility benefit, or

$$\text{(3)}\quad \frac{dC}{C_t} = E_t\left[\frac{1}{1+\rho}\frac{1}{C_{t+1}}\frac{dC}{P_t}\left(Y_{t+1} + P_{t+1}\right)\right].$$

Canceling the dC's (which is somewhat informal) gives us

$$\text{(4)}\quad \frac{1}{C_t} = E_t\left[\frac{1}{1+\rho}\frac{1}{C_{t+1}}\frac{1}{P_t}\left(Y_{t+1} + P_{t+1}\right)\right].$$

We can now solve equation (4) for P_t in terms of Y_t and expectations involving Y_{t+1}, P_{t+1} and C_{t+1}. Note that we can replace C_t with Y_t and that P_t is not uncertain at time t. Using these facts, equation (4) can be rewritten as

$$\text{(5)}\quad \frac{1}{Y_t} = \frac{1}{P_t}E_t\left[\frac{1}{1+\rho}\frac{1}{C_{t+1}}\left(Y_{t+1} + P_{t+1}\right)\right].$$

Solving equation (5) for the price of a tree in period t gives us

$$\text{(6)}\quad P_t = \frac{Y_t}{1+\rho}E_t\left[\frac{Y_{t+1} + P_{t+1}}{C_{t+1}}\right].$$

(b) Since $C_{t+s} = Y_{t+s}$ for all $s \geq 0$, equation (6) can be written as

$$\text{(7)}\quad P_t = \frac{Y_t}{1+\rho}E_t\left[1 + \frac{P_{t+1}}{Y_{t+1}}\right] = \frac{Y_t}{1+\rho} + \frac{Y_t}{1+\rho}E_t\left[\frac{P_{t+1}}{Y_{t+1}}\right].$$

Equation (7) holds for all periods and so we can write the price of a tree in period t + 1 as

$$\text{(8)}\quad P_{t+1} = \frac{Y_{t+1}}{1+\rho} + \frac{Y_{t+1}}{1+\rho}E_{t+1}\left[\frac{P_{t+2}}{Y_{t+2}}\right].$$

Substituting equation (8) into equation (7) yields

(9) $$P_t = \frac{Y_t}{1+\rho} + \frac{Y_t}{1+\rho} E_t\left[\frac{1}{1+\rho} + \frac{1}{1+\rho} E_{t+1}\left(\frac{P_{t+2}}{Y_{t+2}}\right)\right].$$

Now use the law of iterated projections that states that for any variable x, $E_t E_{t+1} x_{t+2} = E_t x_{t+2}$, to obtain

(10) $$P_t = \frac{Y_t}{1+\rho} + \frac{Y_t}{(1+\rho)^2} + \frac{Y_t}{(1+\rho)^2} E_t\left[\frac{P_{t+2}}{Y_{t+2}}\right].$$

After repeated substitutions, we will have

(11) $$P_t = \frac{Y_t}{1+\rho} + \frac{Y_t}{(1+\rho)^2} + \ldots + \frac{Y_t}{(1+\rho)^s} + \frac{Y_t}{(1+\rho)^s} E_t\left[\frac{P_{t+s}}{Y_{t+s}}\right].$$

Imposing the no-bubbles condition that $\lim_{s\to\infty} E_t\,[(P_{t+s}/Y_{t+s})/(1+\rho)^s] = 0$, the price of a tree in period t can be written as

(12) $$P_t = Y_t\left[\frac{1}{1+\rho} + \frac{1}{(1+\rho)^2} + \ldots\right].$$

Since $1/(1+\rho) < 1$, the sum converges and we can write

(13) $$P_t = Y_t\left[\frac{1/(1+\rho)}{1-[1/(1+\rho)]}\right] = Y_t\left[\frac{1/(1+\rho)}{\rho/(1+\rho)}\right].$$

Thus, finally, the price of a tree in period t is

(14) $P_t = Y_t/\rho$.

(c) There are two effects of an increase in the expected value of dividends at some future date. The first is the fact that at a given marginal utility of consumption, the higher expected dividends increase the attractiveness of owning trees. This tends to raise the current price of a tree. However, since consumption equals dividends in this model, higher expected dividends in that future period mean higher consumption and thus lower marginal utility of consumption in that future period. This tends to reduce the attractiveness of owning trees -- the tree is going to pay off more in a time when marginal utility is expected to be low -- and thus tends to lower the current price of a tree. In the case of logarithmic utility, these two forces exactly offset each other, leaving the current price of a tree unchanged in the face of a rise in expected future dividends.

(d) The path of consumption is equivalent to the path of output. Thus if output follows a random walk, so does consumption. But if output does not follow a random walk, then consumption does not either.

Problem 7.10

(a) Suppose the individual reduces her holdings of the good-state asset by a small (formally, infinitesimal) amount dA_G. This change means that if the good state occurs -- which it will, with probability 1/2 -- the individual loses dA_G times the marginal utility of consumption in the good state, which is $U'(1)$. Thus

(1) expected utility loss $= U'(1)dA_G/2$.

Since p represents the relative price of the bad-state asset to the good-state asset, selling dA_G of the good-state asset allows the individual to purchase dA_G/p of the bad-state asset. This means that if the bad state occurs -- which it will, with probability 1/2 -- the individual gains dA_G/p times the expected marginal utility of consumption in the bad state. Fraction λ of the population consumes $1 - (\phi/\lambda)$ in the bad state and fraction $(1 - \lambda)$ consumes one. Thus the expected marginal utility of consumption in the bad state is $\lambda U'(1 - (\phi/\lambda)) + (1 - \lambda)U'(1)$. Putting all of this together, we have

(2) expected utility benefit $= [dA_G/p][\lambda U'(1 - (\phi/\lambda)) + (1 - \lambda)U'(1)]/2$.

If the individual is optimizing, this change in holdings of the two assets must leave expected utility unchanged. Thus the expected utility loss must equal the expected utility gain, or

(3) $U'(1)dA_G/2 = [dA_G/p][\lambda U'(1 - (\phi/\lambda)) + (1 - \lambda)U'(1)]/2$.

(b) From equation (3), canceling the 1/2's and the dA_G's (which is somewhat informal), we have

(4) $U'(1) = [1/p][\lambda U'(1 - (\phi/\lambda)) + (1 - \lambda)U'(1)]$.

Solving equation (4) for p gives us

(5) $$p = \frac{\lambda U'(1-(\phi/\lambda)) + (1-\lambda)U'(1)}{U'(1)}.$$

(c) The change in the equilibrium relative price of the bad-state asset to the good-state asset due to a change in λ is

(6) $$\frac{\partial p}{\partial \lambda} = U'(1-(\phi/\lambda)) + \lambda U''(1-(\phi/\lambda))(\phi/\lambda^2) - U'(1),$$

which simplifies to

(7) $$\frac{\partial p}{\partial \lambda} = U'(1-(\phi/\lambda)) - U'(1) + U''(1-(\phi/\lambda))(\phi/\lambda).$$

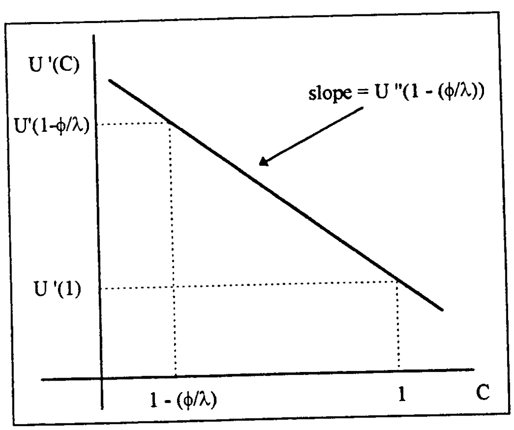
(d) If utility is quadratic, then $U'(C)$ is a linear function of C since $U''(C)$ is a constant. See the figure at right. We can calculate the slope of the $U'(C)$ line as

(8) $$\text{slope} = \frac{U'(1-(\phi/\lambda)) - U'(1)}{1-(\phi/\lambda) - 1},$$

or

(9) $$\text{slope} = \frac{U'(1-(\phi/\lambda)) - U'(1)}{-\phi/\lambda}.$$

We also know that the slope of this line is $U''(C)$ at any value of C and in particular it equals $U''(1 - (\phi/\lambda))$. Equating these two expressions for the slope gives us



(10) $$\frac{U'(1-(\phi/\lambda)) - U'(1)}{-\phi/\lambda} = U''(1-(\phi/\lambda)),$$

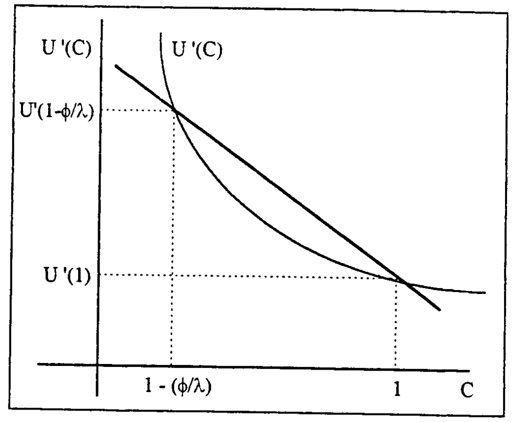
and hence

(11) $U'(1-(\phi/\lambda)) - U'(1) + (\phi/\lambda)U''(1-(\phi/\lambda)) = 0$.

The left-hand side of equation (11) is $\partial p/\partial\lambda$ and thus it equals zero as required. With quadratic utility, a marginal change in the concentration of aggregate shocks has no effect on the relative price of the bad-state asset to the good-state asset.

(e) If $U'''(\bullet) > 0$ everywhere, then $U'(C)$ is a convex function of C; $U''(C)$ rises, or becomes less negative, as C rises. Just as in part (d), we can draw in a straight line that goes through $[1 - \phi/\lambda, U'(1 - \phi/\lambda)]$ as well as $[1, U'(1)]$. Again, the line will have slope equal to

$$(9)\quad \text{slope} = \frac{U'(1-(\phi/\lambda)) - U'(1)}{-\phi/\lambda}.$$



However, from the figure, we can see that the slope of this line is less negative, or greater, than the slope of $U'(C)$ at $1 - (\phi/\lambda)$, which is $U''(1 - (\phi/\lambda))$. That is, we have

$$(12)\quad \frac{U'(1-(\phi/\lambda)) - U'(1)}{-\phi/\lambda} > U''(1-(\phi/\lambda)),$$

or simply

$$(13)\quad U'(1-(\phi/\lambda)) - U'(1) + (\phi/\lambda)U''(1-(\phi/\lambda)) < 0.$$

The left-hand side of equation (13) is $\partial p/\partial\lambda$ and thus $\partial p/\partial\lambda < 0$. If $U'''(\bullet)$ is everywhere positive, a fall in λ causes p to rise. That is, a marginal increase in the concentration of the aggregate shock -- a fall in λ -- causes the relative price of the bad-state asset to the good-state asset to rise.

Problem 7.11

(a) (i) Under commitment, the individual chooses c_1, c_2, and c_3 to maximize

$$(1)\quad U_1 = \ln c_1 + \delta \ln c_2 + \delta \ln c_3,$$

where $0 < \delta < 1$. With a real interest rate of zero and wealth of W, the individual's consumption choices must satisfy $c_1 + c_2 + c_3 = W$. We can solve an unconstrained optimization; $c_3 = W - c_1 - c_2$ can be substituted into equation (1) yielding

$$(2)\quad U_1 = \ln c_1 + \delta \ln c_2 + \delta \ln(W - c_1 - c_2).$$

Thus the individual chooses c_1 and c_2 to maximize (2) with c_3 determined by $c_3 = W - c_1 - c_2$. The first-order conditions are

$$(3)\quad \frac{\partial U_1}{\partial c_1} = \frac{1}{c_1} - \frac{\delta}{W - c_1 - c_2} = 0, \text{ and}$$

$$(4)\quad \frac{\partial U_1}{\partial c_2} = \frac{\delta}{c_2} - \frac{\delta}{W - c_1 - c_2} = 0.$$

From equation (3) we have

$$\delta c_1 = W - c_1 - c_2,$$

or

$$(1+\delta)c_1 = W - c_2,$$

and thus

$$(5)\quad c_1 = \frac{1}{1+\delta}(W - c_2).$$

From equation (4) we have

$$c_2 = W - c_1 - c_2,$$

or simply

(6) $c_2 = (W - c_1)/2$.

Substituting equation (6) into equation (5) yields

$$c_1 = \frac{1}{1+\delta}\left[W - (W - c_1)/2\right] = \frac{1}{2(1+\delta)}(W + c_1).$$

Solving for c_1 gives us

$$c_1\,[2(1+\delta) - 1] = W,$$

or simply

(7) $$c_1 = \frac{1}{1+2\delta}W.$$

Equation (7) gives the individual's optimal choice of first-period consumption under commitment. To solve for second-period consumption, substitute equation (7) into (6):

$$c_2 = \frac{1}{2}\left(W - \frac{1}{1+2\delta}W\right) = \frac{1}{2(1+2\delta)}(W + 2\delta W - W),$$

which simplifies to

(8) $$c_2 = \frac{\delta}{1+2\delta}W.$$

It should be clear from the first-period objective function that c_2 and c_3 will be equal but to verify this, substitute equations (7) and (8) into the constraint, $c_3 = W - c_1 - c_2$, to obtain

$$c_3 = W - \frac{1}{1+2\delta}W - \frac{\delta}{1+2\delta}W = \frac{1+2\delta-1-\delta}{1+2\delta}W,$$

which simplifies to

(9) $$c_3 = \frac{\delta}{1+2\delta}W.$$

(a) (ii) In period 2, the individual chooses c_2 taking her choice of c_1 -- which was made last period -- as given and with the constraint that $c_3 = W - c_1 - c_2$. Thus the individual chooses c_2 to maximize

(10) $U_2 = \ln c_2 + \delta\ln[W - c_1 - c_2]$.

The first-order condition is given by

(11) $$\frac{\partial U_2}{\partial c_2} = \frac{1}{c_2} + \frac{\delta}{W - c_1 - c_2}(-1) = 0.$$

Solving for c_2 as a function of W and c_1 yields

(12) $\delta c_2 = W - c_1 - c_2$,

or simply

(13) $$c_2 = \frac{1}{1+\delta}(W - c_1).$$

This means that third-period consumption as a function of W and the choice of c_1 is given by

(14) $$c_3 = W - c_1 - \frac{1}{1+\delta}(W - c_1) = \frac{(1+\delta)W - (1+\delta)c_1 - W + c_1}{1+\delta},$$

or simply

(15) $$c_3 = \frac{\delta}{1+\delta}(W - c_1).$$

The individual chooses c_1 in period 1 just as she did under commitment since she (wrongly) believes she will choose c_2 in the same way as under commitment. Thus, again we have

(16) $c_1 = \frac{1}{1+2\delta}W.$

Substituting equation (16) into equation (13) yields

(17) $c_2 = \frac{1}{1+\delta}\left[W - \frac{1}{1+2\delta}W\right] = \frac{1}{1+\delta}\left[\frac{1+2\delta-1}{1+2\delta}W\right],$

or simply

(18) $c_2 = \frac{2\delta}{(1+\delta)(1+2\delta)}W.$

Finally, we can obtain third-period consumption by substituting equation (16) into equation (15):

(19) $c_3 = \frac{\delta}{1+\delta}\left[W - \frac{1}{1+2\delta}W\right] = \frac{\delta}{1+\delta}\left[\frac{1+2\delta-1}{1+2\delta}W\right],$

or simply

(20) $c_3 = \frac{2\delta^2}{(1+\delta)(1+2\delta)}W.$

(a) (iii) Now, in period 1, the individual chooses c_1 realizing that her choices of c_2 and c_3 -- which will be functions of her choice of c_1 -- will be given by equations (13) and (15). Thus we can substitute (13) and (15) into the period-1 objective function:

(21) $U_1 = \ln c_1 + \delta \ln\left[\frac{1}{1+\delta}(W - c_1)\right] + \delta \ln\left[\frac{\delta}{1+\delta}(W - c_1)\right].$

The first-order condition for the optimal choice of period-1 consumption is

(22) $\frac{\partial U_1}{\partial c_1} = \frac{1}{c_1} + \frac{\delta}{[1/(1+\delta)](W - c_1)}\left[\frac{(-1)}{1+\delta}\right] + \frac{\delta}{[\delta/(1+\delta)](W - c_1)}\left[\frac{(-\delta)}{1+\delta}\right] = 0,$

which simplifies to

(23) $\frac{1}{c_1} = \frac{2\delta}{W - c_1}.$

Solving for c_1 yields

(24) $2\delta c_1 = W - c_1,$

or simply

(25) $c_1 = \frac{1}{1+2\delta}W.$

Note that the choice of period-1 consumption is the same here as it was under "naiveté". Since c_2 and c_3 will be chosen the same way as under "naiveté", they will be the same also and are once again given by equations (18) and (20).

(b) (i) The individual's preferences are time-inconsistent because the optimal choice of period-2 consumption that is made in the first period is no longer the optimal choice once period 2 actually arrives. This is illustrated by the fact that if the individual does not commit to period-2 consumption in the first period, then when period 2 arrives she chooses

(18) $c_2 = \frac{2\delta}{(1+\delta)(1+2\delta)}W,$

rather than the choice she had originally made in the first period, which was

(8) $c_2 = \frac{\delta}{1+2\delta}W.$

And, in fact, since $2/(1 + \delta) > 1$, she chooses a higher value of period-2 consumption once period 2 actually arrives.

We can see from the period-1 objective function that in the first period the individual is indifferent between period-2 and period-3 consumption; they are both discounted by δ. But when period 2 actually occurs, we can see from the period-2 objective function that the individual then prefers period-2 consumption over period-3 consumption.

(b) (ii) The key to the result that sophistication does not affect behavior is the assumption of log utility. The intuition behind this result is very similar to the intuition behind the version of the Tabellini-Alesina model with logarithmic utility that is presented in Section 11.6.

Think of a sophisticated individual contemplating a marginal decrease in c_1, relative to what a naive individual would do. The naive individual believes she will allocate the increase in saving equally between c_2 and c_3 and that marginal utility will be the same in the two future periods. The sophisticated individual realizes that she will, in fact, devote most of the increase in saving to c_2 and that c_2 will be high. The individual does not particularly value c_2 thus marginal utility in period 2 will be low. This tends to make the increase in saving look relatively less attractive to the sophisticated individual than to the naive individual.

But the sophisticated individual also realizes that some of the increase in saving will be devoted to c_3, which will be low. The individual values c_3 as much as c_2 and thus marginal utility will be high in period 3. This tends to make the increase in saving look relatively more attractive to the sophisticated individual than to the naive individual.

With log utility, these two effects exactly offset each other. With a general utility function, a sophisticated individual can consume either more or less in the first period than a naive individual.

SOLUTIONS TO CHAPTER 8

Problem 8.1

(a) Given K and the fixed quantity demanded, Y, the firm will hire enough labor to meet that demand. Given the production function

(1) $Y = K^{\alpha} L^{1-\alpha}$,

the firm will hire

(2) $L = Y^{1/(1-\alpha)} K^{-\alpha/(1-\alpha)}$.

(b) Substituting equation (2) for the firm's choice of L into the profit function, $\pi = PY - WL - r_K K$, we have

(3) $\pi = PY - W[Y^{1/(1-\alpha)} K^{-\alpha/(1-\alpha)}] - r_K K$.

(c) The first-order condition for the firm's choice of K is

(4) $\dfrac{\partial \pi}{\partial K} = \dfrac{\alpha}{1-\alpha} W Y^{1/(1-\alpha)} K^{[-\alpha/(1-\alpha)]-1} - r_K = 0,$

or simply

(5) $\dfrac{\alpha}{1-\alpha} W Y^{1/(1-\alpha)} K^{-1/(1-\alpha)} = r_K$.

In order for the value of K in equation (5) to be a maximum, we require $\partial^2\pi/\partial K^2$ to be negative. This derivative is

(6) $\dfrac{\partial^2 \pi}{\partial K^2} = \left(\dfrac{-1}{1-\alpha}\right)\dfrac{\alpha}{1-\alpha} W Y^{1/(1-\alpha)} K^{(\alpha-2)/(1-\alpha)} < 0.$

So the second-order condition is satisfied since $\alpha < 1$.

(d) Solving equation (5) for K gives us

(7) $K^{1/(1-\alpha)} = \left(\dfrac{\alpha}{1-\alpha}\right)\dfrac{W Y^{1/(1-\alpha)}}{r_K}$.

Taking both sides of equation (7) to the exponent $(1 - \alpha)$ gives us the firm's choice of K:

(8) $K = Y\left(\dfrac{\alpha}{1-\alpha}\right)^{1-\alpha}\left(\dfrac{W}{r_K}\right)^{(1-\alpha)}$.

Thus changes in the price of the firm's product do not directly affect the profit-maximizing choice of K, although changes in P likely change Y. The elasticity of K with respect to the wage, W, is $(1 - \alpha)$, which is positive. Its elasticity with respect to the rental price of capital, r_K, is $-(1 - \alpha)$, which is negative. Finally, the elasticity of K with respect to the quantity demanded is one.

Problem 8.2

(a) At each point in time from t to t + T, the firm is allowed to deduct P_K/T from its taxable income. This allows it to save $\tau(P_K/T)$ in taxes at each point in time from t to t + T, where τ is the marginal tax rate. If i is the constant interest rate, the present value of the reduction in the firm's corporate tax liabilities, denoted X, is given by the following expression:

(1) $X = \int_{s=t}^{t+T} e^{-i(s-t)} \tau(P_K/T) ds,$

which implies

(2) $X = \tau(P_K/T)\int_{s=t}^{t+T} e^{-i(s-t)}ds = \tau(P_K/T)\left[-\frac{1}{i}e^{-i(s-t)}\Big|_{s=t}^{s=t+T}\right] = \tau(P_K/T)\left[\frac{1-e^{-iT}}{i}\right].$

Since the after-tax price of the capital good, denoted P_K^{AT}, is its pretax price, P_K, minus the present value of the tax saving that results, we have

(3) $P_K^{AT} = P_K - \tau(P_K/T)\left[\frac{1-e^{-iT}}{i}\right] = P_K\left[1-(\tau/T)\left(\frac{1-e^{-iT}}{i}\right)\right].$

(b) An increase in inflation, π, without a change in the real interest rate, r, increases the nominal interest rate, i. From equation (1), the present value of the reduction in the firm's corporate tax liabilities as a result of purchasing the capital good is

(4) $X = \tau(P_K/T)\int_{s=t}^{t+T} e^{-i(s-t)}ds.$

The change in the present value of the reduction in the firm's corporate tax liabilities due to a change in the nominal interest rate is therefore

(5) $\frac{\partial X}{\partial i} = \tau(P_K/T)\int_{s=t}^{t+T} -(s-t)e^{-i(s-t)}ds = -\tau(P_K/T)\int_{s=t}^{t+T}(s-t)e^{-i(s-t)}ds < 0.$

The increase in i reduces the present value of the tax savings from purchasing the capital good. Therefore, it increases the after-tax price of the capital good.

Problem 8.3

From equation (8.4) in the text, the real user cost of capital is

(1) $r_K(t) = [r(t) + \delta - (\dot{p}_K(t)/p_K(t))]p_K(t),$

where r(t) is the relevant real interest rate, δ is the rate of depreciation and $p_K(t)$ is the real price of capital. Here, capital refers to owner-occupied housing. The after-tax real interest rate for owner-occupied housing is $r(t) - \tau i(t)$ where τ is the marginal tax rate. This is due to the fact that nominal interest payments are tax deductible.

Intuitively, if an individual foregoes selling her home, she does lose $r(t)p_K(t)$ -- the interest she could obtain by selling it and saving the proceeds -- but she does get the bonus of deducting her nominal interest payments from her income. Thus she receives τ times $i(t)p_K(t)$ in tax savings by holding on to her home, which reduces the user cost of capital. Thus for owner-occupied housing, equation (1) becomes

(2) $r_K(t) = [r(t) - \tau i(t) + \delta - (\dot{p}_K(t)/p_K(t))]p_K(t).$

Substituting $i(t) = r(t) + \pi(t)$ into equation (2) gives us

(3) $r_K(t) = [r(t) - \tau r(t) - \tau\pi(t) + \delta - (\dot{p}_K(t)/p_K(t))]p_K(t),$

which implies

(4) $r_K(t) = [(1-\tau)r(t) - \tau\pi(t) + \delta - (\dot{p}_K(t)/p_K(t))]p_K(t).$

To see how an increase in inflation for a given real interest rate affects $r_K(t)$, take the derivative of $r_K(t)$ with respect to $\pi(t)$:

(5) $\partial r_K(t)/\partial\pi(t) = -\tau p_K(t) < 0.$

An increase in inflation reduces the user cost of owner-occupied housing since it allows the homeowner more in the way of tax deductions on the nominal interest payments. Thus an increase in inflation increases the desired stock of owner-occupied housing.

Problem 8.4

(a) The planner's problem is to maximize the discounted value of lifetime utility for the representative household, which is given by

$$(1)\quad U = \int_{t=0}^{\infty} e^{-\beta t}\frac{c(t)^{1-\theta}}{1-\theta}dt \qquad \beta \equiv \rho - n - (1-\theta)g,$$

subject to the capital-accumulation equation given by

$$(2)\quad \dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t).$$

The control variable is the variable that can be controlled freely by the planner, which is consumption per unit of effective labor, $c(t)$. The state variable is the variable whose value at any time is determined by past decisions of the planner, which is capital per unit of effective labor, $k(t)$. Finally, the shadow value of the state variable is the costate variable, which we will denote $\mu(t)$.

The current-value Hamiltonian is thus

$$(3)\quad H\big(k(t),c(t)\big) = \frac{c(t)^{1-\theta}}{1-\theta} + \mu(t)[f(k(t)) - c(t) - (n+g)k(t)].$$

(b) The first condition characterizing the optimum is that the derivative of the Hamiltonian with respect to the control variable at each point is zero, or

$$(4)\quad \frac{\partial H\big(k(t),c(t)\big)}{\partial c(t)} = c(t)^{-\theta} - \mu(t) = 0.$$

The second condition is that the derivative of the Hamiltonian with respect to the state variable equals the discount rate times the costate variable minus the derivative of the costate variable with respect to time, or

$$(5)\quad \frac{\partial H\big(k(t),c(t)\big)}{\partial k(t)} = \mu(t)f'(k(t)) - \mu(t)(n+g) = \beta\mu(t) - \dot{\mu}(t).$$

The final condition is the transversality condition. The limit as t goes to infinity of the discounted value of the costate variable times the state variable equals zero, or

$$(6)\quad \lim_{t\to\infty} e^{-\beta t}\mu(t)k(t) = 0.$$

(c) From equation (4) we have

$$(7)\quad \mu(t) = c(t)^{-\theta}.$$

Taking the time derivative of both sides of equation (7) yields

$$(8)\quad \dot{\mu}(t) = -\theta c(t)^{-\theta-1}\dot{c}(t) = -\theta c(t)^{-\theta}\frac{\dot{c}(t)}{c(t)}.$$

From equation (5), we have

$$(9)\quad \dot{\mu}(t) = \mu(t)[\beta - f'(k(t)) + (n+g)].$$

Equating these two expressions for $\dot{\mu}(t)$ gives us

$$(10)\quad -\theta c(t)^{-\theta}\frac{\dot{c}(t)}{c(t)} = \mu(t)[\beta - f'(k(t)) + (n+g)].$$

Substituting equation (7) for $\mu(t)$ and $f'(k(t)) = r(t)$ into equation (10) yields

$$(11)\quad -\theta c(t)^{-\theta}\frac{\dot{c}(t)}{c(t)} = c(t)^{-\theta}[\beta - r(t) + (n+g)].$$

Canceling the $c(t)^{-\theta}$, dividing both sides by $-\theta$, and substituting for $\beta \equiv \rho - n - (1-\theta)g$ gives us

$$(12)\quad \frac{\dot{c}(t)}{c(t)} = \frac{r(t) - (n+g) - \rho + n + (1-\theta)g}{\theta},$$

which simplifies to

(13) $\dfrac{\dot{c}(t)}{c(t)} = \dfrac{r(t) - \rho - \theta g}{\theta}$.

Equation (13) is identical to the Euler equation in the decentralized equilibrium. See equation (2.20) in the text.

(d) Dividing both sides of equation (9) by $\mu(t)$ leaves us with

(14) $\dfrac{\dot{\mu}(t)}{\mu(t)} = \beta + (n + g) - r(t)$,

where we have used $f'(k(t)) = r(t)$. Note that equation (14) can be written as

(15) $\partial \ln\mu(t)/\partial t = \beta + (n + g) - r(t)$.

Integrating both sides of equation (15) from time $\tau = 0$ to time $\tau = t$ gives us

(15) $\ln\mu(t) - \ln\mu(0) = [\beta + (n + g)]\,\tau\Big|_{\tau=0}^{\tau=t} - \int_{\tau=0}^{t} r(\tau)d\tau$.

Using the definition of R(t) and simplifying gives us

(16) $\ln\mu(t) = \ln\mu(0) + \beta t + (n + g)t - R(t)$.

Taking the exponential function of both sides of equation (16) yields

(17) $\mu(t) = \mu(0)e^{\beta t}\, e^{(n+g)t}\, e^{-R(t)}$

Thus $e^{-\beta t}\,\mu(t)$ is proportional to $e^{-R(t)}\, e^{(n+g)t}$.

This implies that the transversality condition, equation (6), is equivalent to

(18) $\lim_{t\to\infty} e^{-R(t)} e^{(n+g)t} k(t) = 0$.

From equation (2.15) in the text, the household's budget constraint, expressed in terms of limiting behavior, is given by

(19) $\lim_{t\to\infty} e^{-R(t)} e^{(n+g)t} k(t) \geq 0$.

Comparing equations (18) and (19), we can see that the transversality condition will hold if and only if the budget constraint is met with equality. Thus we have shown that the solution to the social planner's problem in the Ramsey model is the same as the decentralized equilibrium. Hence that decentralized equilibrium must be Pareto efficient.

Problem 8.5

The equation of motion for the market value of capital, q, is

(1) $\dot{q}(t) = rq(t) - \pi(K(t))$,

where $\pi'(\bullet) < 0$. The condition required for $\dot{q} = 0$ is given by

(2) $q = \pi(K)/r$.

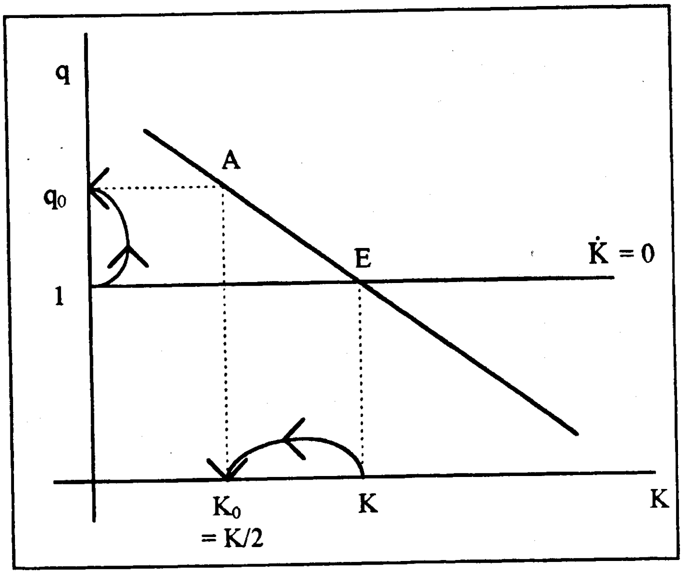
The equation of motion for capital, K, is

(3) $\dot{K}(t) = f(q(t))$,

where $f(q) = NC'^{-1}(q - 1)$ with $f(1) = 0$ and $f'(\bullet) > 0$. The condition required for $\dot{K} = 0$ is given by

(4) $q = 1$.

(a) The destruction of half of the capital stock does not cause either the $\dot{K} = 0$ or the $\dot{q} = 0$ loci to shift. Both of these are already drawn allowing for K to vary. At the time of the destruction, K falls to $K_0 = K/2$.



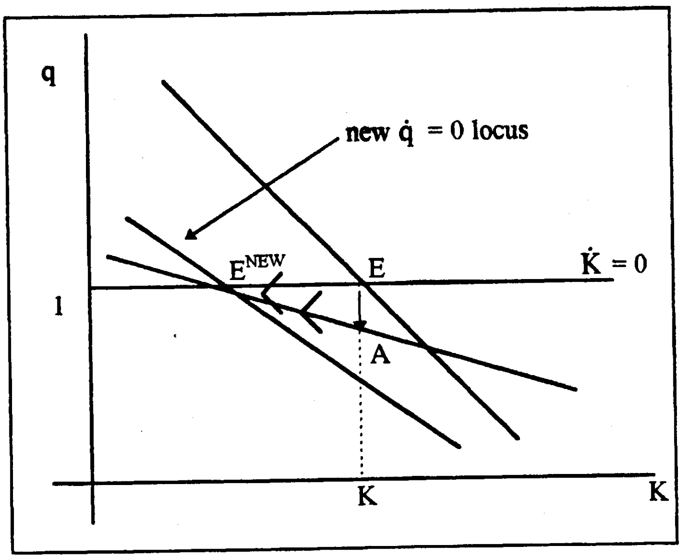
For the economy to return to a stable equilibrium, q must adjust so that the economy is on the saddle path. Thus q must jump up to q_0, putting the economy at point A in the figure at right. Intuitively, since profits are higher at the lower K, the capital that is left is more valuable and so the market value of capital is now higher.

The economy then moves down the saddle path with q falling and K rising. Intuitively, the higher market value of capital attracts investment and so the capital stock begins to build back up. As it does so, profits begin to fall and thus so does the market value of capital. This process continues until the market value of capital returns to its long-run-equilibrium value of one and the capital stock is back at its original level. Hence the economy eventually returns to point E.

(b) Profits at a given K are now $(1 - \tau)\pi(K)$ rather than $\pi(K)$. The condition required for $\dot{q} = 0$ is now given by

(5) $q = (1 - \tau)\pi(K)/r$.

At a given K, the value of q that makes $\dot{q} = 0$ is now lower so the new $\dot{q} = 0$ locus lies below the old one. In addition, the slope of the $\dot{q} = 0$ locus is $\partial q/\partial K = (1 - \tau)\pi'(K)/r$ rather than $\pi'(K)/r$. With $(1 - \tau) < 1$, this new slope is less negative and so the $\dot{q} = 0$ locus becomes flatter. The $\dot{K} = 0$ locus is unaffected. See the figure at right.



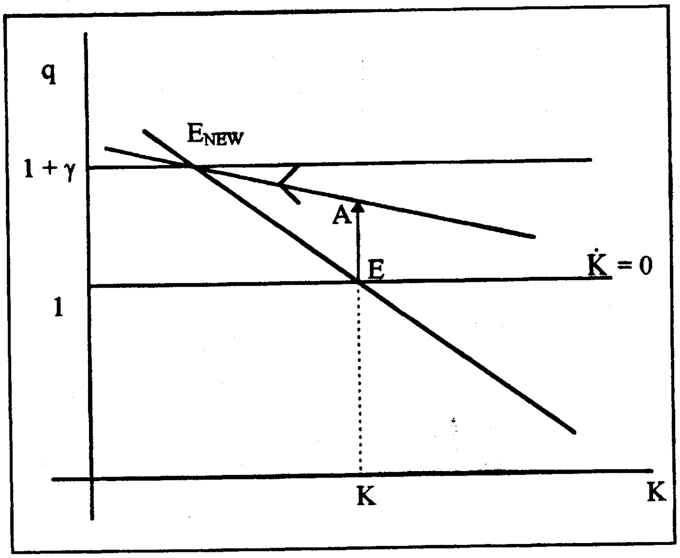
K, the stock of capital, cannot jump at the time of the implementation of the tax. Thus q must jump down so that the economy is on the new saddle path at point A. Intuitively, since the government is now taking a fraction of profits, existing capital is less valuable and so the market value of capital falls. The economy then moves up the new saddle path with K falling and q rising. Intuitively, the lower market value of capital discourages investment and so the capital stock begins falling. As it does so, profits begin to rise back up and thus so does the market value of capital. This process continues until the market value of capital returns to its long-run-equilibrium value of one and the capital stock is at a permanently lower level. The economy winds up at point E_{NEW} in the diagram. The lower capital and thus higher pretax profits offset the fact that the government takes a fraction of those profits.

(c) One of the conditions required for optimization is that the firm invests to the point where the cost of acquiring capital equals the value of that capital, q. With this tax on investment, the cost of acquiring a unit of capital is the purchase price (which is fixed at one) plus the tax, γ, plus the marginal adjustment cost, $C'(I)$. Thus analogous to equation (8.18) in the text, we now have

(6) $1 + \gamma + C'(I(t)) = q(t)$.

Since $C'(0)$ is zero, equation (6) implies that $I(t)$ is zero (and thus $\dot{K} = 0$) when $q(t) = 1 + \gamma$. So the equation of the $\dot{K} = 0$ locus is now

(7) $q = 1 + \gamma$.



Thus an investment tax of γ shifts the $\dot{K} = 0$ locus up by γ. The $\dot{q} = 0$ locus is unaffected. See the figure.

K, the stock of capital, cannot jump at the time of the implementation of the tax. Thus q must jump up so that the economy is on the new saddle path at point A. Intuitively, because the tax will reduce investment, it means that the industry's profits (neglecting the tax) will eventually be higher, and thus that existing capital is more valuable. The economy then moves up the new saddle path until it reaches point E_{NEW}. The capital stock is permanently lower and the pretax market value of capital is equal to $1 + \gamma$; the after-tax market value is again equal to one.

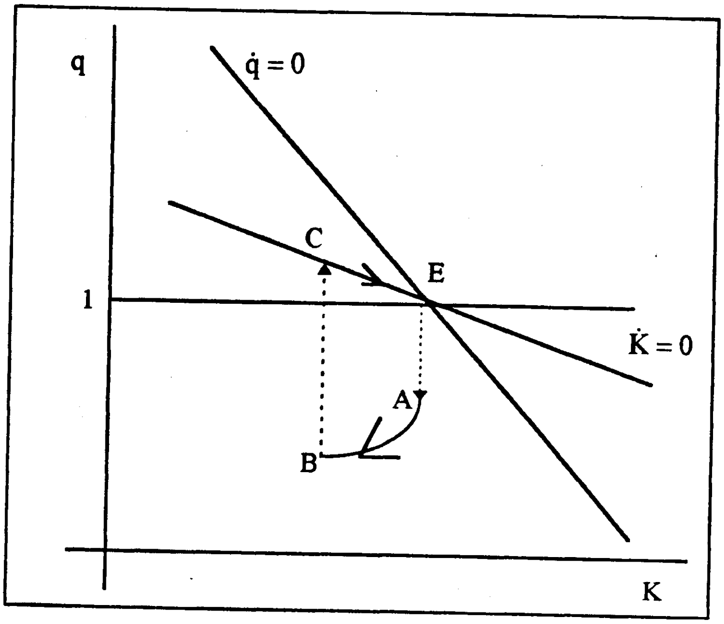
Problem 8.6

The important point is that q is anticipated to jump up discontinuously at the time of the capital levy, time T. Consider what is required, if there is a market for shares in firms, for individuals to be willing to hold those shares through the interval where the one-time tax on capital holdings is imposed. Consider the market value of capital an instant, ε, before the levy and an instant after the levy and then look at what happens as ε goes to zero. The key point is that the market value of capital an instant before the levy, $q(T - \varepsilon)$, must equal $(1 - f)$ times the market value of capital an instant after the levy. If it did not, -- in light of the levy -- holders of shares in firms would be expecting capital losses that they could avoid. Therefore, $q(T - \varepsilon)$ must equal $(1 - f)q(T + \varepsilon)$ or

(1) $q(T - \varepsilon)/q(T + \varepsilon) = (1 - f)$.

For example, if $f = 0.10$ or ten percent, then the value of q an instant before the levy must equal 90 percent of its value an instant after the levy. Thus at time T, q jumps up to close that 10 percent gap. In addition, that jump must put the economy somewhere on the saddle path in order for the economy to return to a stable equilibrium.

Thus at the time of the news, q must jump down, putting the economy at a point such as A in the figure at right. The economy is then in a region where both q and K are falling. Thus between the time of the news and the time the levy is imposed, the market value of capital and the capital stock are falling. Intuitively, firms begin decumulating capital in anticipation of the one-time levy.



Point A must be chosen so that at the time of the levy, q can jump up by the required amount discussed above and that required jump must put the economy right on the saddle path. The stock of capital does not jump at the time of the levy. Thus at time T, the economy jumps from a point such as B to a point such as C where $q_B / q_C = (1 - f)$.

After the time of the levy, the economy moves down the saddle path, eventually returning to the original equilibrium at point E. Intuitively, once the one-time tax is over with, since K is lower, profits are higher and so investment is attractive once again. Thus the capital stock begins rising back to its initial level.

Problem 8.7

(a) The evolution of the stock of housing is given by

(1) $\dot{H} = I(p_H) - \delta H$.

Thus the condition required for $\dot{H} = 0$ is given by $I(p_H) = \delta H$. That is, in order for the stock of housing to remain constant, new investment in housing (which is an increasing function of the real price of housing) must exactly offset depreciation of the existing housing stock. Differentiating both sides of this expression with respect to H gives us the following slope of the $\dot{H} = 0$ locus:

(2) $I'(p_H) dp_H / dH = \delta$,

or

(3) $dp_H / dH = \delta / I'(p_H) > 0$.

Since $I'(p_H) > 0$, the $\dot{H} = 0$ locus is upward-sloping in (H, p_H) space.

Rental income plus capital gains must equal the exogenous rate of return, r, or

(4) $$\frac{R(H) + \dot{p}_H}{p_H} = r.$$

Solving equation (4) for $\dot{p}_H$ yields

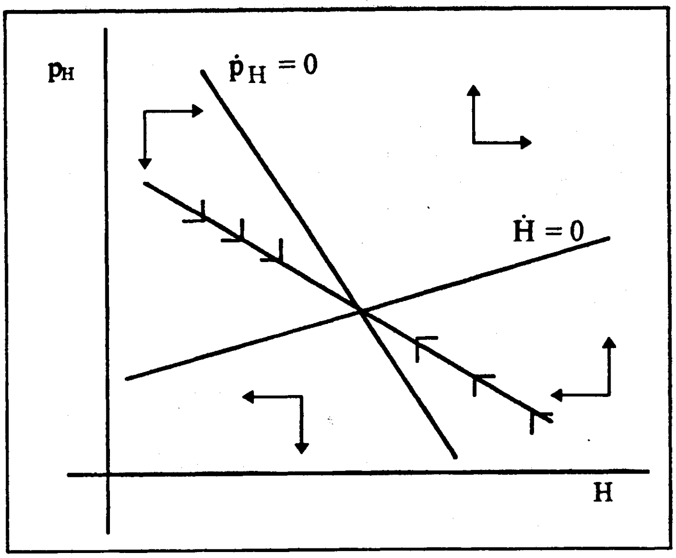
(5) $\dot{p}_H = rp_H - R(H)$.

Therefore the condition required for $\dot{p}_H = 0$ is $rp_H - R(H) = 0$ or $p_H = R(H)/r$. Differentiating both sides of this expression with respect to H gives us the following slope of the $\dot{p}_H = 0$ locus:

(6) $dp_H / dH = R'(H)/r$.

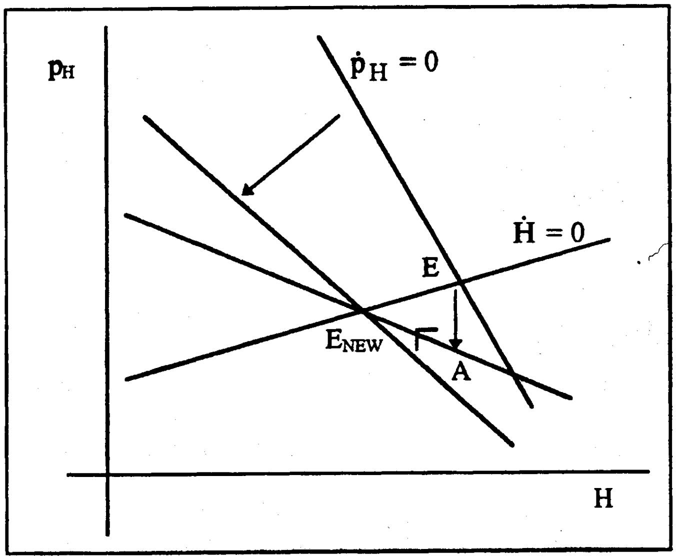
Since $R'(H) < 0$ -- rent is a decreasing function of the stock of housing -- the $\dot{p}_H = 0$ locus is downward-sloping in (H, p_H) space.

(b) Since $I'(p_H) > 0$, then from equation (1), $\dot{H}$ is increasing in p_H. This means that above the $\dot{H} = 0$ locus, $\dot{H} > 0$ and so H is rising. Intuitively, at a given H, if p_H is higher than the price necessary to keep the stock of housing constant, investment (which is an increasing function of p_H) is higher than necessary to offset depreciation. Thus the stock of housing is rising above the $\dot{H} = 0$ locus. Similarly, below the $\dot{H} = 0$ locus, $\dot{H} < 0$ and so H is falling. Intuitively, p_H and thus investment are too low to offset depreciation and keep the stock of housing constant at a given H. Thus the stock of housing is falling below the $\dot{H} = 0$ locus.



Since $R'(H) < 0$, then from equation (5), $\dot{p}_H$ is increasing in H. This means that to the right of the $\dot{p}_H = 0$ locus, $\dot{p}_H > 0$ and so p_H is rising. Intuitively, at a given p_H, if H is higher -- and thus rent lower -- than the level necessary to keep the price of housing constant, this lower rent must be offset by capital gains -- a rising p_H -- if investors are to earn the required exogenous return of r. Similarly, to the left of the $\dot{p}_H = 0$ locus, $\dot{p}_H < 0$ and so p_H is falling. If H is lower -- and thus rent higher -- than the level necessary to keep the price of housing constant, this higher rent must be offset by capital losses in order for investors to earn the rate of return r.

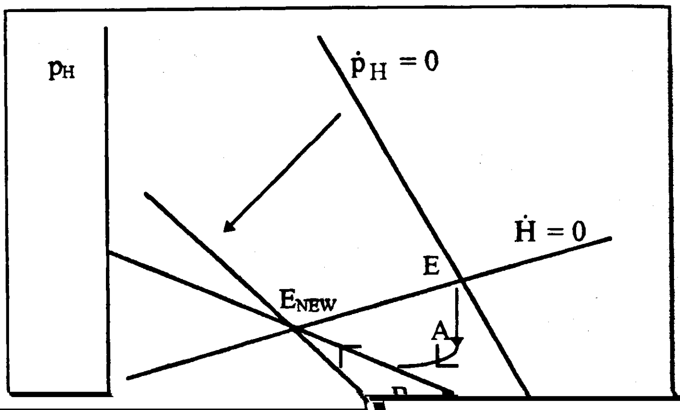
(c) The $\dot{p}_H = 0$ locus is defined by $p_H = R(H)/r$. A rise in r means that the p_H that makes $\dot{p}_H = 0$ is now lower at a given H. Thus the new $\dot{p}_H$ locus lies below the old one. In addition, the slope of the $\dot{p}_H = 0$ locus is $R'(H)/r$ and so the rise in r makes the slope less negative. Thus the new $\dot{p}_H = 0$ locus is flatter than the old one. The $\dot{H} = 0$ locus is defined by $I(p_H) = \delta H$. Since r does not appear in this equation, the $\dot{H} = 0$ locus is unaffected.



At the time of the increase in r, H -- the stock of existing housing -- cannot jump discontinuously. The real price of housing, p_H, must jump down to put the economy on the new saddle path. In the figure, at the time of the rise in r, the economy jumps from point E to point A.

The discontinuous downward jump in p_H causes the amount of investment to jump down. Thus investment is no longer enough to offset depreciation at the initial value of H -- we are below the $\dot{H} = 0$ locus -- and so the stock of housing begins to fall. As H begins to fall, rent begins to rise since $R'(H) < 0$. As the economy moves up the new saddle path, the real price of housing is rising. This means that investment is rising back up since $I'(p_H) > 0$. The economy eventually reaches point E_{NEW} where p_H is constant at a new lower level and thus investment is constant at a new lower level. In addition, the stock of housing is constant at a new lower level. Finally, rent is higher.

(d) An important point is that the dynamics of the system are still governed by the original $\dot{p}_H = 0$ and $\dot{H} = 0$ loci until the actual time of the increase in r. At the time of the change, H cannot jump and, importantly, neither can p_H. If p_H did jump, an instant before the rise in r, people would be expecting capital gains or losses that could be arbitraged away. Thus right at the time of the increase in r, the economy must be somewhere on the new saddle path.



(1) $H(\kappa(t),I(t)) = \pi(K(t))\kappa(t) - I(t) - C\left(\frac{I(t)}{\kappa(t)} - \delta\right)\kappa(t) + q(t)[I(t) - \delta\kappa(t)].$

(b) The first condition characterizing the optimum is that the derivative of the Hamiltonian with respect to the control variable at each point is zero. Here, the control variable is investment and thus

(2) $\frac{\partial H(\kappa(t),I(t))}{\partial I(t)} = -1 - C'\left(\frac{\dot{\kappa}(t)}{\kappa(t)}\right)\frac{1}{\kappa(t)}\kappa(t) + q(t) = 0.$

The second condition is that the derivative of the Hamiltonian with respect to the state variable equals the discount rate times the costate variable minus the derivative of the costate variable with respect to time. Since the state variable is the capital stock, we have

(3) $\frac{\partial H(\kappa(t),I(t))}{\partial \kappa(t)} = \pi(K(t)) - C'\left(\frac{\dot{\kappa}(t)}{\kappa(t)}\right)\left(\frac{-I(t)}{\kappa(t)^2}\right)\kappa(t) - C\left(\frac{\dot{\kappa}(t)}{\kappa(t)}\right) - \delta q(t) = rq(t) - \dot{q}(t).$

The final condition is the transversality condition. The limit as t goes to infinity of the present value of the costate variable times the state variable must be zero. Thus, we have

(4) $\lim_{t\to\infty} e^{-rt}q(t)\kappa(t) = 0.$

(c) Equation (2) states that each firm invests to the point at which the purchase price of capital plus the marginal adjustment cost equals the value of capital: $1 + C'(\dot{\kappa}/\kappa) = q$. Since $C'(\dot{\kappa}/\kappa)$ is increasing in $\dot{\kappa}/\kappa$, this condition implies that $\dot{\kappa}/\kappa$ is increasing in q. And since $C'(0)$ is zero, it also implies that $\dot{\kappa}/\kappa$ is zero when q equals one. Finally, note that since q is the same for all firms, all firms choose the same value of $\dot{\kappa}/\kappa$. Thus the growth rate of the aggregate capital stock, $\dot{K}/K$, is given by the value of $\dot{\kappa}/\kappa$ that satisfies (2). Putting this information together, we can write

(5) $\dot{K}(t)/K(t) = f(q(t)) \qquad f(1) = 0,\ f'(\bullet) > 0,$

where f(q) is the value of $\dot{K}/K$ that satisfies $C'(\dot{K}/K) = q - 1$: $f(q) = C'^{-1}(q - 1)$. Equation (5) implies that K is increasing when $q > 1$, decreasing when $q < 1$ and constant when $q = 1$. Thus the $\dot{K} = 0$ locus is a horizontal line at $q = 1$ when drawn in (K, q) space.

(d) Rearranging equation (3) to solve for $\dot{q}(t)$ yields

(6) $\dot{q}(t) = (r+\delta)q(t) - \left[\pi(K(t)) + C'\left(\frac{\dot{\kappa}(t)}{\kappa(t)}\right)\left(\frac{I(t)}{\kappa(t)}\right) - C\left(\frac{\dot{\kappa}(t)}{\kappa(t)}\right)\right].$

To simplify this expression, note first that I/κ equals $(\dot{\kappa} + \delta\kappa)/\kappa$ or $(\dot{\kappa}/\kappa) + \delta$. In addition, as we just showed, the growth rate of the representative firm's capital stock, $\dot{\kappa}/\kappa$, is the same as the growth rate of the industry-wide capital stock, $\dot{K}/K$. Thus we can rewrite equation (6) as

(7) $\dot{q}(t) = (r+\delta)q(t) - \left[\pi(K(t)) + C'\left(\frac{\dot{K}(t)}{K(t)}\right)\left(\frac{\dot{K}(t)}{K(t)} + \delta\right) - C\left(\frac{\dot{K}(t)}{K(t)}\right)\right].$

We can now use equation (5), $\dot{K}/K = f(q)$, to substitute for $\dot{K}/K$, and then use the fact that the definition of $f(\bullet)$ implies $C'(f(q)) = q - 1$. This yields

(8) $\dot{q}(t) = (r+\delta)q(t) - \left[\pi(K(t)) + [q(t) - 1][f(q(t)) + \delta] - C\big(f(q(t))\big)\right] \equiv G\big(K(t),q(t)\big).$

(e) The condition $G(K, q) = 0$ implicitly defines the locus of points in (K, q) space for which $\dot{q}$ is zero. To see how q varies with K along this locus, we therefore implicitly differentiate this condition with respect to K. This yields

(9) $G_K(K,q)+G_q(K,q)\left.\frac{dq}{dK}\right|_{\dot{q}=0}=0,$

or

(10) $\left.\frac{dq}{dK}\right|_{\dot{q}=0}=\frac{-G_K(K,q)}{G_q(K,q)},$

where subscripts denote partial derivatives and $\left.\frac{dq}{dK}\right|_{\dot{q}=0}$ denotes the derivative of q with respect to K along the $\dot{q}=0$ locus.

Using equation (8) to compute the derivatives in (10) yields

(11) $G_K(K, q) = -\pi'(K),$

and

(12) $G_q(K, q) = (r + \delta) - [(q - 1)f'(q) + (f(q) + \delta) - C'(f(q))f'(q)] = r - f(q),$

where we again use the fact that $C'(f(q)) = q - 1$.

Substituting equations (11) and (12) into equation (10) gives us

(13) $\left.\frac{dq}{dK}\right|_{\dot{q}=0}=\frac{\pi'(K)}{r-f(q)}.$

Note that f(q) is zero when q equals one. Thus the slope of the $\dot{q}=0$ locus at the point where q = 1 is simply $\pi'(K)/r$. Note that at this particular point, this slope is exactly the same as the slope of the $\dot{q}=0$ locus in the version of the model in the text where adjustment costs took the form $C(\dot{\kappa})$.

Problem 8.9

(a) One of the conditions for optimization is that the marginal revenue product of capital, $\pi(K(t))$, equals its user cost, $rq(t) - \dot{q}(t)$. Rearranging this condition gives us the following equation of motion for q:

(1) $\dot{q}(t) = rq(t) - \pi(K(t))$.

Substituting the profit function, $\pi(K) = a - bK$, into equation (1) gives us

(2) $\dot{q}(t) = rq(t) - a + bK(t)$.

The $\dot{q}=0$ locus is therefore given by

(3) $rq - a + bK = 0$,

or solving for q as a function of K, we have

(4) $q = (a - bK)/r$.

So the $\dot{q}=0$ locus has a constant slope of $-b/r$.

To find the long-run-equilibrium value of K, we need to find the intersection of the $\dot{q}=0$ locus -- as given by equation (4) -- and the $\dot{K}=0$ locus. The $\dot{K}=0$ locus is given by q = 1, which means that we already know that the long-run-equilibrium value of q, q*, is one. Substituting q = 1 into equation (4) and solving for K* gives us

(5) $K^* = (a - r)/b$.

We can now use the method of Section 2.6 to find the slope of the saddle path. We first need to solve for the equation of motion of K(t). One of the conditions for optimization is that each firm invests to the point at which the purchase price of capital (which is fixed at one), plus the marginal adjustment cost, equals the value of capital, q. We are assuming quadratic costs of adjustment, $C(\dot{\kappa}) = \alpha\dot{\kappa}^2/2$, and thus the marginal adjustment cost is

(9) $G_K(K,q) + G_q(K,q)\left.\frac{dq}{dK}\right|_{\dot{q}=0} = 0,$

or

(10) $\left.\frac{dq}{dK}\right|_{\dot{q}=0} = \frac{-G_K(K,q)}{G_q(K,q)},$

where subscripts denote partial derivatives and $\left.\frac{dq}{dK}\right|_{\dot{q}=0}$ denotes the derivative of q with respect to K along the $\dot{q} = 0$ locus.

Using equation (8) to compute the derivatives in (10) yields
(11) $G_K(K, q) = -\pi'(K)$,
and
(12) $G_q(K, q) = (r + \delta) - [(q - 1)f'(q) + (f(q) + \delta) - C'(f(q))f'(q)] = r - f(q)$,
where we again use the fact that $C'(f(q)) = q - 1$.

Substituting equations (11) and (12) into equation (10) gives us

(13) $\left.\frac{dq}{dK}\right|_{\dot{q}=0} = \frac{\pi'(K)}{r - f(q)}.$

Note that f(q) is zero when q equals one. Thus the slope of the $\dot{q} = 0$ locus at the point where $q = 1$ is simply $\pi'(K)/r$. Note that at this particular point, this slope is exactly the same as the slope of the $\dot{q} = 0$ locus in the version of the model in the text where adjustment costs took the form $C(\dot{\kappa})$.

Problem 8.9

(a) One of the conditions for optimization is that the marginal revenue product of capital, $\pi(K(t))$, equals its user cost, $rq(t) - \dot{q}(t)$. Rearranging this condition gives us the following equation of motion for q:
(1) $\dot{q}(t) = rq(t) - \pi(K(t))$.
Substituting the profit function, $\pi(K) = a - bK$, into equation (1) gives us
(2) $\dot{q}(t) = rq(t) - a + bK(t)$.
The $\dot{q} = 0$ locus is therefore given by
(3) $rq - a + bK = 0$,
or solving for q as a function of K, we have
(4) $q = (a - bK)/r$.
So the $\dot{q} = 0$ locus has a constant slope of $-b/r$.

To find the long-run-equilibrium value of K, we need to find the intersection of the $\dot{q} = 0$ locus -- as given by equation (4) -- and the $\dot{K} = 0$ locus. The $\dot{K} = 0$ locus is given by $q = 1$, which means that we already know that the long-run-equilibrium value of q, q^*, is one. Substituting $q = 1$ into equation (4) and solving for K^* gives us
(5) $K^* = (a - r)/b$.

We can now use the method of Section 2.6 to find the slope of the saddle path. We first need to solve for the equation of motion of K(t). One of the conditions for optimization is that each firm invests to the point at which the purchase price of capital (which is fixed at one), plus the marginal adjustment cost, equals the value of capital, q. We are assuming quadratic costs of adjustment, $C(\dot{\kappa}) = \alpha\dot{\kappa}^2/2$, and thus the marginal adjustment cost is

(6) $\partial C(\dot{\kappa})/\partial\dot{\kappa} = \alpha\dot{\kappa}$.
Thus we have $1 + \alpha\dot{\kappa} = q$, which implies
(7) $\dot{\kappa} = (q - 1)/\alpha$.
Since q is the same for all firms, all firms choose the same value of investment, $\dot{\kappa}$. Thus the rate of change of the aggregate capital stock, $\dot{K}$, is given by
(8) $\dot{K} = N(q - 1)/\alpha$,
where N is the number of firms.

Define $\tilde{q} \equiv q - q^*$ and $\tilde{K} \equiv K - K^*$. Since q* and K* are constants, $\dot{q}$ and $\dot{K}$ are equivalent to $\dot{\tilde{q}}$ and $\dot{\tilde{K}}$ respectively. Thus we can rewrite the equations of motion, equations (2) and (8), as
(9) $\dot{\tilde{q}} = rq - a + bK$, and (10) $\dot{\tilde{K}} = N(q - 1)/\alpha$.
Dividing both sides of equation (9) by $\tilde{q}$ gives us

(11) $\dfrac{\dot{\tilde{q}}}{\tilde{q}} = \dfrac{rq - a + bK}{\tilde{q}}$.

From equation (5), we can write
(12) $\tilde{K} = (bK - a + r)/b$,
or rearranging to solve for bK:
(13) $bK = b\tilde{K} + a - r$.
Substituting equation (13) into equation (10) gives us

(14) $\dfrac{\dot{\tilde{q}}}{\tilde{q}} = \dfrac{rq - a + b\tilde{K} + a - r}{\tilde{q}} = \dfrac{r(q-1)}{\tilde{q}} + \dfrac{b\tilde{K}}{\tilde{q}} = r + b\dfrac{\tilde{K}}{\tilde{q}}$,

where we have used the fact that q* = 1 so that $\tilde{q} \equiv q - q^* = q - 1$.

Dividing both sides of equation (10) by $\tilde{K}$ and noting that q* = 1 we have

(15) $\dfrac{\dot{\tilde{K}}}{\tilde{K}} = \dfrac{N}{\alpha}\dfrac{\tilde{q}}{\tilde{K}}$.

Equations (14) and (15) imply that the growth rates of $\tilde{q}$ and $\tilde{K}$ depend only on the ratio of $\tilde{q}$ to $\tilde{K}$. Given this, consider what happens if the values of q and K are such that $\tilde{q}$ and $\tilde{K}$ are falling at the same rate. This implies that the ratio of $\tilde{q}$ to $\tilde{K}$ is not changing, and thus that their growth rates are not changing. Thus $\tilde{q}$ and $\tilde{K}$ continue to fall at equal rates. In terms of a phase diagram, from a point at which $\tilde{q}$ and $\tilde{K}$ are falling at equal rates, the economy simply moves along a straight-line saddle path to (K*, q*) with the distance from (K*, q*) falling at a constant rate.

Let μ denote $\dot{\tilde{K}}/\tilde{K}$. Then equation (15) implies

(16) $\mu = \dfrac{N}{\alpha}\dfrac{\tilde{q}}{\tilde{K}}$,

or solving for the ratio of $\tilde{q}$ to $\tilde{K}$:

(17) $\dfrac{\tilde{q}}{\tilde{K}} = \dfrac{\alpha\mu}{N}$.

From equation (14), the condition that $\dot{\tilde{q}}/\tilde{q}$ always equals $\dot{\tilde{K}}/\tilde{K}$ is thus
(18) $\mu = r + (bN/\alpha\mu)$,
or
(19) $\alpha\mu^2 - \alpha r\mu - bN = 0$.

Using the quadratic formula to solve for μ yields

(20) $$\mu = \frac{\alpha r \pm \sqrt{\alpha^2 r^2 + 4\alpha bN}}{2\alpha} = \frac{r \pm \sqrt{r^2 + (4bN/\alpha)}}{2}.$$

If μ is positive, then $\tilde{q}(t) \equiv q(t) - q^*$ and $\tilde{K}(t) \equiv K(t) - K^*$ are growing. That is, instead of moving along a straight line toward (K^*, q^*), the economy is moving on a straight line away from (K^*, q^*). Thus μ must be negative and hence

(21) $$\mu_1 = \frac{r - \sqrt{r^2 + (4bN/\alpha)}}{2}.$$

Thus equation (17) with $\mu = \mu_1$ tells us how q and K must be related on the saddle path. Substituting equation (21) into equation (17) gives us

(22) $$\frac{\tilde{q}}{\tilde{K}} \equiv \frac{q - q^*}{K - K^*} = \frac{\alpha\left[r - \sqrt{r^2 + (4bN/\alpha)}\right]}{2N},$$

or solving for q as a function of K:

(23) $$q = q^* + \alpha\left[\frac{r - \sqrt{r^2 + (4bN/\alpha)}}{2N}\right](K - K^*).$$

Thus, the slope of the saddle path is

(24) $$\left.\frac{\partial q}{\partial K}\right|_{sp} = \alpha\left[\frac{r - \sqrt{r^2 + (4bN/\alpha)}}{2N}\right] < 0.$$

Problem 8.10

(a) Consider the situation where $a(t + \tau)$ is certain to equal $E_t[a(t + \tau)]$ for all $\tau \geq 0$ so that there is no uncertainty. The value of q at some date $t + \tau$ can then be written as the value of q in period t plus the "sum" of all the changes in q from time t to time $t + \tau$. More formally:

(1) $$\hat{q}(t+\tau, t) = q(t) + \int_{s=t}^{\tau} \dot{\hat{q}}(t+s, t)ds,$$

where $\hat{q}(t+\tau, t)$ denotes the path of q when a is certain to equal its expected value. Since $\dot{q} = rq - \pi(K)$, then with this particular profit function we have $\dot{q} = rq - a + bK$. Thus equation (1) can be written as

(2) $$\hat{q}(t+\tau, t) = q(t) + \int_{s=t}^{\tau} \left[r\hat{q}(t+s, t) - E_t[a(t+s)] + b\hat{K}(t+s, t)\right]ds,$$

where we have substituted in for $a(t + s) = E_t[a(t + s)]$ and where $\hat{K}(\bullet)$ denotes the path of K given that a is certain to equal its expected value.

Now consider the situation where $a(t + \tau)$ is uncertain. Then the expected value, as of time t, of q at some future date $t + \tau$ can be written as the value of q in period t plus the "sum" of all the expected changes in q from time t to time $t + \tau$. More formally:

(3) $$E_t[q(t+\tau)] = q(t) + \int_{s=t}^{\tau} E_t[\dot{q}(t+s)]ds.$$

Since equation (8.28) in the text, $E_t[\dot{q}(t)] = rq(t) - \pi(K(t))$, holds in all periods and since $\pi(K(t)) = a - bK(t)$, we can write

(4) $E_{t+s}[\dot{q}(t+s)] = rq(t+s) - a(t+s) + bK(t+s)$.
Taking the expected value, as of information available at time t, of both sides of equation (4) yields
(5) $E_t[\dot{q}(t+s)] = rE_t[q(t+s)] - E_t[a(t+s)] + bE_t[K(t+s)]$,
where we have used the law of iterated projections so that $E_t E_{t+s}[\dot{q}(t+s)] = E_t[\dot{q}(t+s)]$. Substituting equation (5) into equation (3) gives us

(6) $$E_t[q(t+\tau)] = q(t) + \int_{s=t}^{\tau}\left[rE_t[q(t+s)] - E_t[a(t+s)] + bE_t[K(t+s)]\right]ds.$$

If $E_t[q(t+\tau)] = \hat{q}(t+\tau,t)$ for all $\tau \geq 0$, then the right-hand sides of equations (2) and (6) must be equal for all $\tau \geq 0$. Thus

(7) $$q(t) + \int_{s=t}^{\tau}\left[r\hat{q}(t+s,t) - E_t[a(t+s)] + b\hat{K}(t+s,t)\right]ds =$$
$$q(t) + \int_{s=t}^{\tau}\left[rE_t[q(t+s)] - E_t[a(t+s)] + bE_t[K(t+s)]\right]ds.$$

Again, using $E_t[q(t+\tau)] = \hat{q}(t+\tau,t)$ for all $\tau \geq 0$, this simplifies to

(8) $$b\int_{s=t}^{\tau}\hat{K}(t+s,t)ds = b\int_{s=t}^{\tau}E_t[K(t+s)]ds.$$

Canceling the b's and using Leibniz's rule to take the derivative of both sides of equation (8) with respect to τ gives us
(9) $\hat{K}(t+\tau,t) = E_t[K(t+\tau)]$.
Equation (9) holds for all $\tau \geq 0$.

(b) Consider the situation where $a(t+\tau)$ is certain to equal $E_t[a(t+\tau)]$ for all $\tau \geq 0$ so that there is no uncertainty. Then from equation (8.22) in the text, we can write the market value of capital at time t as the present value of its future marginal revenue products and so

(10) $$\hat{q}(t,t) = \int_{\tau=0}^{\infty}e^{-r\tau}\left[E_t[a(t+\tau)] - b\hat{K}(t+\tau,t)\right]d\tau,$$

where we have used the facts that $\pi(K) = a - bK$ and $a(t+\tau) = E_t[a(t+\tau)]$. Now $\hat{q}(t,t)$ denotes the value of q given that a always equals its expected value and $\hat{K}(t+\tau,t)$ has the same meaning as in part (a). It is the path of K given that a is always certain to equal its expected value.

Now consider the situation where $a(t+\tau)$ is uncertain. Then, using $\pi(K) = a - bK$, equation (8.26) in the text becomes

(11) $$q(t) = \int_{\tau=0}^{\infty}e^{-r\tau}\left[E_t[a(t+\tau)] - bE_t[K(t+\tau)]\right]d\tau.$$

As shown in part (a), if $E_t[q(t+\tau)] = \hat{q}(t+\tau,t)$ for all $\tau \geq 0$, then $E_t[K(t+\tau)] = \hat{K}(t+\tau,t)$ for all $\tau \geq 0$. This means that the right-hand sides of equations (10) and (11) are equal and so $q(t) = \hat{q}(t,t)$. That is, for the case in which π is linear and the uncertainty concerns the intercept of the π function, the market value of capital is the same with the uncertainty as it is if the future values of the π function are certain to equal their expected values.

Even with uncertainty, each firm invests to the point at which the cost of acquiring a unit of new capital equals the market value of capital. That is, investment satisfies
(12) $1 + C'(I(t)) = q(t)$.

Since $C = \alpha I^2/2$, $C'(I) = \alpha I$. In addition, as we have just shown, $q(t) = \hat{q}(t,t)$. Thus equation (12) can be rewritten as

(13) $1 + \alpha I(t) = \hat{q}(t,t)$.

By definition, the change in each firm's capital stock is equal to $I(t)$. Since each firm faces the same $\hat{q}(t,t)$, they choose the same level of investment. Thus the change in the aggregate capital stock is given by $\dot{K}(t) = NI(t)$, where N is the number of firms. Substituting this expression into equation (13) yields

(14) $1 + \alpha\dot{K}(t)/N = \hat{q}(t,t)$.

Solving (14) for $\dot{K}(t)$ gives us

(15) $\dot{K}(t) = N[\hat{q}(t,t) - 1]/\alpha$

Under these special circumstances -- when π is linear, the uncertainty concerns the intercept of the π function, and adjustment costs are quadratic -- investment is the same with the uncertainty as it is if the future values of the π function are certain to be equal to their expected values.

Problem 8.11

(a) If the firm does not undertake the investment, its expected profits are zero. Thus we have

(1) $E[\pi^{NO}] = 0$.

If the firm does undertake the investment, its expected profits are the certain payoff in period 1 plus the expected payoff in period 2 less the cost of undertaking the investment. Thus we have

(2) $E[\pi^{YES}] = \pi_1 + E[\pi_2] - I$.

The firm will undertake the investment if its expected profits from doing so are greater than its expected profits from not investing, or when

(3) $E[\pi^{YES}] > E[\pi^{NO}]$,

or simply when

(4) $\pi_1 + E[\pi_2] - I > 0$.

(b) Suppose the firm does not invest in period 1. Then in period 2, if $\pi_2 > I$, it will invest and earn $\pi_2 - I$. If $\pi_2 < I$, it will not invest in period 2 and will earn zero. Thus the expected profits from not investing in period 1 are

(5) $E[\pi^{NO\ IN\ 1}] = \text{Prob}(\pi_2 > I)E[\pi_2 - I \mid \pi_2 > I]$.

From equation (2), the expected profits from investing in period 1 are

(6) $E[\pi^{YES\ IN\ 1}] = \pi_1 + E[\pi_2] - I$.

Thus the difference in the firm's expected profits between not investing in period 1 and investing in period 1 are

(7) $E[\pi^{NO\ IN\ 1}] - E[\pi^{YES\ IN\ 1}] = \text{Prob}(\pi_2 > I)E[\pi_2 - I \mid \pi_2 > I] - (\pi_1 + E[\pi_2] - I)$.

Even if $\pi_1 + E[\pi_2] - I > 0$, as long as $\text{Prob}(\pi_2 > I)E[\pi_2 - I \mid \pi_2 > I]$ is greater than $\pi_1 + E[\pi_2] - I$, the firm's expected profits are higher if it does not invest in period 1 than if it does.

(c) The cost of waiting is that the firm foregoes any payoff in period 1. That is, it foregoes π_1 and hence

(8) cost of waiting $= \pi_1$.

The benefit of waiting is that the firm can observe π_2, see if it is less than I and decide not to invest and avoid a loss if this is the case. The expected loss that the firm avoids by waiting is equal to the probability that π_2 is less than I, multiplied by the expected loss given that $\pi_2 < I$, which is $E[I - \pi_2 \mid \pi_2 < I]$. Hence

(9) benefit of waiting $= \text{Prob}(\pi_2 < I)\ E[I - \pi_2 \mid \pi_2 < I]$.

Note that by the definition of conditional expected values, we can write

(10) $E[\pi_2 - I] = \text{Prob}(\pi_2 > I)\ E[\pi_2 - I \mid \pi_2 > I] + \text{Prob}(\pi_2 < I)\ E[\pi_2 - I \mid \pi_2 < I]$.

Substituting this into equation (7) yields

(11) $E[\pi^{NO\ IN\ 1}] - E[\pi^{YES\ IN\ 1}] = Prob(\pi_2 > I)\ E[\pi_2 - I \mid \pi_2 > I] - \pi_1 - Prob(\pi_2 > I)\ E[\pi_2 - I \mid \pi_2 > I] - Prob(\pi_2 < I)\ E[\pi_2 - I \mid \pi_2 < I]$.

Note that we can write $Prob(\pi_2 < I) E[\pi_2 - I \mid \pi_2 < I] = - Prob(\pi_2 < I)\ E[I - \pi_2 \mid \pi_2 < I]$. Using this fact, equation (11) becomes

(12) $E[\pi^{NO\ IN\ 1}] - E[\pi^{YES\ IN\ 1}] = - \pi_1 + Prob(\pi_2 < I)\ E[I - \pi_2 \mid \pi_2 < I]$.

Since π_1 is the cost of waiting and $Prob(\pi_2 < I)\ E[I - \pi_2 \mid \pi_2 < I]$ is the benefit of waiting, we do have

(13) $E[\pi^{NO\ IN\ 1}] - E[\pi^{YES\ IN\ 1}]$ = benefit of waiting - cost of waiting.

Problem 8.12

(a) Consider the value of a unit of debt. It pays off one unit of output at time $t + \tau$, for all $\tau \geq 0$. The consumer values this payoff according to the marginal utility of consumption at each time $t + \tau$. Thus the value of having one unit of output at time $t + \tau$ rather than at t is equal to the discounted marginal utility of consumption at time $t + \tau$ relative to the marginal utility of consumption at time t, which is given by $e^{-\rho\tau} u'(C(t + \tau))/u'(C(t))$. Thus the value of a unit of debt at time t is simply the appropriately discounted "sum" of all the future payoffs, or

$$(1)\quad P(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t\left[\frac{u'(C(t+\tau))}{u'(C(t))}\right] d\tau.$$

Equity holders are the residual claimant and thus at time $t + \tau$, $\tau \geq 0$, they receive the additional profit generated by the marginal unit of capital, $\pi(K(t + \tau))$, minus the total amount paid to bond holders, which is b (the total number of outstanding bonds). Again, individuals value this payoff at time $t + \tau$ according to the discounted marginal utility of consumption at time $t + \tau$ relative to the marginal utility of consumption at time t. Thus the value of the equity in the marginal unit of capital is

$$(2)\quad V(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t\left[\frac{u'(C(t+\tau))}{u'(C(t))}\big(\pi(K(t+\tau)) - b\big)\right] d\tau.$$

(b) Adding equation (2) to b times equation (1) gives us the following market value of the claim on the marginal unit of capital:

$$(3)\quad P(t)b + V(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} bE_t\left[\frac{u'(C(t+\tau))}{u'(C(t))}\right] d\tau + \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t\left[\frac{u'(C(t+\tau))}{u'(C(t))}\big(\pi(K(t+\tau)) - b\big)\right] d\tau.$$

Combining the integrals yields

$$(4)\quad P(t)b + V(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t\left[\frac{u'(C(t+\tau))}{u'(C(t))}\big(b + \pi(K(t+\tau)) - b\big)\right] d\tau,$$

and thus

$$(5)\quad P(t)b + V(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t\left[\frac{u'(C(t+\tau))}{u'(C(t))}\pi(K(t+\tau))\right] d\tau.$$

The division of financing between bonds and equity as captured by b, the number of outstanding bonds, does not affect the market value of the claims on the marginal unit of capital. The present discounted value of that unit of capital is determined by its expected effect on the path of profits. Since the division of $\pi(K(t + \tau))$ between bonds and equity does not affect the size of $\pi(K(t + \tau))$, it does not affect the market value of the claim on the unit of capital.

(c) The market value of each of the n assets is given by

$$(6)\quad V_i(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t\left[\frac{u'(C(t+\tau))}{u'(C(t))} d_i(t+\tau)\right] d\tau.$$

There will be n equations of the form of (6). Adding these n equations together gives us the following total value of the n financial instruments:

$$(7)\quad V_1(t)+\ldots+V_n(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t\left[\frac{u'(C(t+\tau))}{u'(C(t))}\left(d_1(t+\tau)+\ldots+d_n(t+\tau)\right)\right] d\tau.$$

Since $d_1(t+\tau) + \ldots + d_n(t+\tau) = \pi(K(t+\tau))$, we can rewrite equation (7) as

$$(8)\quad V_1(t)+\ldots+V_n(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t\left[\frac{u'(C(t+\tau))}{u'(C(t))}\pi(K(t+\tau))\right] d\tau.$$

The total market value of the n financial instruments is determined by the expected effect on the path of profits of the marginal unit of capital. It does not depend upon the individual payoffs to the assets.

(d) The value of a unit of debt continues to be given by equation (1). The value of a unit of equity is now

$$(9)\quad V(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t\left[\frac{u'(C(t+\tau))}{u'(C(t))}\left\{(1-\theta)\left[\pi(K(t+\tau)) - b\right]\right\}\right] d\tau.$$

Adding equation (9) to b times equation (1) gives the following market value of the claims on the marginal unit of capital:

$$(10)\quad P(t)b + V(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} bE_t\left[\frac{u'(C(t+\tau))}{u'(C(t))}\right] d\tau + \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t\left[\frac{u'(C(t+\tau))}{u'(C(t))}\left\{(1-\theta)\left[\pi(K(t+\tau)) - b\right]\right\}\right] d\tau.$$

Combining the integrals yields

$$(11)\quad P(t)b + V(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t\left[\frac{u'(C(t+\tau))}{u'(C(t))}\left[(1-\theta)\pi(K(t+\tau)) + \theta b\right]\right] d\tau.$$

Now the division of the financing between bonds and equity does matter. The number of bonds issued, b, does affect the market value of the claim on the marginal unit of capital. The division of the additional profits between bonds and capital does affect the size of those profits. Specifically, a switch toward debt financing increases profits since interest payments are tax deductible.

SOLUTIONS TO CHAPTER 9

Problem 9.1

(a) Substituting the expression for the average wage, $w_a = fw_u + (1 - f)w_n$, into the expression for the nonunion wage, $w_n = (1 - bu)w_a/(1 - \beta)$, yields

$$(1)\quad w_n = \frac{(1-bu)}{(1-\beta)}\left[fw_u + (1-f)w_n\right].$$

Substituting the union wage, $w_u = (1 + \mu)w_n$, into equation (1) yields

$$(2)\quad w_n = \frac{(1-bu)}{(1-\beta)}\left[f(1+\mu)w_n + (1-f)w_n\right] = \frac{(1-bu)}{(1-\beta)}\left[(1+\mu f)w_n\right].$$

Simplifying gives us

(3) $(1 - bu)(1 + \mu f) = (1 - \beta)$.

Since $(1 - bu)(1 + \mu f) = 1 + \mu f - bu - b\mu fu$, equation (3) can be rewritten as

(4) $-u(b + b\mu f) = -\beta - \mu f$,

and thus the equilibrium unemployment rate is

$$(5)\quad u = \frac{\beta + \mu f}{b(1+\mu f)}.$$

(b) (i) Substituting $\mu = f = 0.15$, $\beta = 0.06$ and $b = 1$ into equation (5) gives us

$$(6)\quad u = \frac{(0.06) + (0.15)(0.15)}{1 + (0.15)(0.15)} = \frac{0.0825}{1.0225} = 0.081.$$

Equilibrium unemployment is approximately 8.1%, which is higher than the 6% obtained with $\beta = 0.06$ and $b = 1$ in the standard version of this model without a union sector.

In order to determine by what proportion the cost of effective labor in the union sector exceeds that in the nonunion sector, we need to calculate the equilibrium effort level in each sector. The union wage as a function of the average wage is

(7) $w_u = (1 + \mu)w_n = (1 + \mu)(1 - bu)w_a/(1 - \beta)$.

Substituting equation (7) and the definition of the index of labor-market conditions, $x = (1 - bu)w_a$, into the expression for effort, $e = [(w - x)/x]^{\beta}$, gives us

$$(8)\quad e_u = \left[\frac{\left[(1+\mu)(1-bu)w_a/(1-\beta)\right] - (1-bu)w_a}{(1-bu)w_a}\right]^{\beta} = \left[\frac{(1+\mu)(1-bu) - (1-\beta)(1-bu)}{(1-\beta)(1-bu)}\right]^{\beta},$$

or simply

$$(9)\quad e_u = \left[\left(\frac{1+\mu}{1-\beta}\right) - 1\right]^{\beta} = \left(\frac{\mu+\beta}{1-\beta}\right)^{\beta}.$$

Substituting $w_n = (1 - bu)w_a/(1 - \beta)$ into the expression for effort yields

$$(10)\quad e_n = \left[\frac{\left[(1-bu)w_a/(1-\beta)\right] - (1-bu)w_a}{(1-bu)w_a}\right]^{\beta} = \left[\frac{(1-bu) - (1-\beta)(1-bu)}{(1-\beta)(1-bu)}\right]^{\beta},$$

or simply

$$(11)\quad e_n = \left[\frac{1}{1-\beta} - 1\right]^{\beta} = \left(\frac{\beta}{1-\beta}\right)^{\beta}.$$

In the union sector, it costs a firm w_u to buy one unit of labor which provides e_u units of effective labor. Thus it costs a firm w_u / e_u to buy one unit of effective labor. Using the fact that $w_u = (1 + \mu)w_n$ and equation (9), we can write

$$(12)\quad \frac{w_u}{e_u} = \frac{(1+\mu)w_n}{[(\mu+\beta)/(1-\beta)]^{\beta}}.$$

Similarly, the cost to a nonunion firm of obtaining one unit of effective labor is w_n / e_n. Using equation (11), we can write

$$(13)\quad \frac{w_n}{e_n} = \frac{w_n}{[\beta/(1-\beta)]^{\beta}}.$$

Dividing equation (12) by equation (13) gives us the following ratio of the cost of effective labor in the union to the nonunion sector:

$$(14)\quad \frac{w_u/e_u}{w_n/e_n} = \frac{(1+\mu)w_n}{[(\mu+\beta)/(1-\beta)]^{\beta}}\,\frac{[\beta/(1-\beta)]^{\beta}}{w_n} = (1+\mu)\left(\frac{\beta}{\mu+\beta}\right)^{\beta}.$$

Substituting $\mu = 0.15$ and $\beta = 0.06$ into equation (14) gives us

$$(15)\quad \frac{w_u/e_u}{w_n/e_n} = (1.15)\left(\frac{0.06}{0.21}\right)^{0.06} = 1.0667.$$

Note that although the cost of labor in the union sector exceeds the cost of labor in the nonunion sector by a factor of $(1 + \mu) = 1.15$, the cost of effective labor is only higher by a factor of about 1.07. This is because union workers exert more effort since they are paid a higher wage.

(b) (ii) Substituting $\mu = f = 0.15$, $\beta = 0.03$ and $b = 0.5$ into equation (5) yields

$$(16)\quad u = \frac{(0.03) + (0.15)(0.15)}{0.5[1 + (0.15)(0.15)]} = \frac{0.0525}{0.51125} = 0.103.$$

Equilibrium unemployment is now higher at about 10.3%. Substituting $\mu = 0.15$ and $\beta = 0.03$ into equation (14) gives us

$$(17)\quad \frac{w_u/e_u}{w_n/e_n} = (1.15)\left(\frac{0.03}{0.18}\right)^{0.03} = 1.0898.$$

With the elasticity of effort with respect to the wage lower at $\beta = 0.03$ and less weight on unemployment in the index of labor-market conditions, the ratio of the cost of effective labor in the union sector to that in the nonunion sector is now higher.

Problem 9.2

(a) (i) With $e = 1$ and taking w as given, the firm's problem is to choose L in order to maximize profits as given by

(1) $\pi = L^{\alpha} / \alpha - wL$.

The first-order condition is

(2) $\partial\pi/\partial L = L^{\alpha-1} - w = 0$,

and thus the firm's choice of employment is

(3) $L = w^{-1/(1-\alpha)}$.

Substituting equation (3) into the expression for profits yields

$$\pi = w^{-1/(1-\alpha)} / \alpha - w^{[(1-\alpha)-1]/(1-\alpha)} = w^{-\alpha/(1-\alpha)}\,[(1/\alpha) - 1],$$

and thus the level of profits is

(4) $\pi = [(1 - \alpha)/\alpha]w^{-\alpha/(1-\alpha)}$.

(a) (ii) Substituting equation (3) for L into the union's objective function, U = (w - x)L, gives us

(5) $U = (w - x)x^{-1/(1-\alpha)}$.

Using equation (5) and equation (4) for profits gives us the following bargaining problem:

$$\max_{w} \quad (w-x)^{\gamma} w^{-\gamma/(1-\alpha)}\left[\left(\frac{1-\alpha}{\alpha}\right) w^{-\alpha/(1-\alpha)}\right]^{1-\gamma}.$$

It will simplify the algebra to maximize the log of $U^{\gamma}\pi^{1-\gamma}$ and so the problem becomes

$$\max_{w} \quad \gamma\ln(w-x) - \frac{\gamma}{1-\alpha}\ln w + (1-\gamma)\ln\left(\frac{1-\alpha}{\alpha}\right) - \frac{\alpha(1-\gamma)}{1-\alpha}\ln w.$$

The first-order condition is

$$(6) \quad \frac{\partial\left[\ln\left(U^{\gamma}\pi^{1-\gamma}\right)\right]}{\partial w} = \gamma\frac{1}{w-x} - \frac{\gamma}{1-\alpha}\frac{1}{w} - \frac{\alpha(1-\gamma)}{(1-\alpha)}\frac{1}{w} = 0.$$

Equation (6) can be rewritten as

$$(7) \quad \gamma\frac{1}{w-x} = \frac{\gamma+\alpha-\alpha\gamma}{1-\alpha}\frac{1}{w}.$$

Cross-multiplying yields

(8) $(1 - \alpha)\gamma w = [\alpha + \gamma(1 - \alpha)](w - x)$.

Subtracting $(1 - \alpha)\gamma w$ from both sides and rearranging yields

(9) $\alpha w = [\alpha + (1 - \alpha)\gamma]x$,

and thus finally, the wage chosen in the bargaining process is

$$(10) \quad w = \frac{\alpha+(1-\alpha)\gamma}{\alpha}x.$$

(b) (i) Substituting $e = [(w - x)/x]^{\beta}$ into the expression for profits allows us to write the firm's problem as

$$(11) \quad \max_{L} \quad \pi = \frac{1}{\alpha}\left(\frac{w-x}{x}\right)^{\alpha\beta} L^{\alpha} - wL.$$

The first-order condition is

$$(12) \quad \frac{\partial\pi}{\partial L} = \left(\frac{w-x}{x}\right)^{\alpha\beta} L^{\alpha-1} - w = 0,$$

and thus the firm's choice of employment is

$$(13) \quad L = \left(\frac{w-x}{x}\right)^{\alpha\beta/(1-\alpha)} w^{-1/(1-\alpha)}.$$

Substituting equation (15) into the expression for profits yields

$$(14) \quad \pi = \frac{1}{\alpha}\left(\frac{w-x}{x}\right)^{\alpha\beta}\left(\frac{w-x}{x}\right)^{\alpha^2\beta/(1-\alpha)} w^{-\alpha/(1-\alpha)} - w^{1-[1/(1-\alpha)]}\left(\frac{w-x}{x}\right)^{\alpha\beta/(1-\alpha)}.$$

Since $\alpha\beta + [\alpha^2\beta/(1 - \alpha)] = [\alpha\beta - \alpha^2\beta + \alpha^2\beta]/(1 - \alpha) = \alpha\beta/(1 - \alpha)$ and $1 - [1/(1 - \alpha)] = [(1 - \alpha - 1)/(1 - \alpha)] = -\alpha/(1 - \alpha)$, equation (14) can be written as

$$(15) \quad \pi = \frac{1}{\alpha}\left(\frac{w-x}{x}\right)^{\alpha\beta/(1-\alpha)} w^{-\alpha/(1-\alpha)} - w^{-\alpha/(1-\alpha)}\left(\frac{w-x}{x}\right)^{\alpha\beta/(1-\alpha)}.$$

Collecting terms and simplifying yields

$$(16) \quad \pi = \frac{1-\alpha}{\alpha}\left(\frac{w-x}{x}\right)^{\alpha\beta/(1-\alpha)} w^{-\alpha/(1-\alpha)}.$$

(b) (ii) Substituting equation (13) for L into the union's objective function, U = (w - x)L, gives us

(17) $U = (w-x)(w-x)^{\frac{\alpha\beta}{1-\alpha}}(1/x)^{\frac{\alpha\beta}{1-\alpha}}w^{\frac{-1}{1-\alpha}}$,

which simplifies to

(18) $U = (w-x)^{\frac{1-\alpha(1-\beta)}{1-\alpha}}(1/x)^{\frac{\alpha\beta}{1-\alpha}}w^{\frac{-1}{1-\alpha}}$.

Using equation (18) and equation (16) for profits gives us the following bargaining problem:

$$\max_{w} \quad (w-x)^{\frac{[1-\alpha(1-\beta)]\gamma}{1-\alpha}}(1/x)^{\frac{\alpha\beta\gamma}{1-\alpha}}w^{\frac{-\gamma}{1-\alpha}}\left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha}(w-x)^{\frac{\alpha\beta(1-\gamma)}{1-\alpha}}(1/x)^{\frac{\alpha\beta(1-\gamma)}{1-\alpha}}w^{\frac{-\alpha(1-\gamma)}{1-\alpha}}.$$

We will again maximize the log of $U^{\gamma}\pi^{1-\gamma}$ and so, ignoring the terms not involving w, we have the following bargaining problem:

$$\max_{w} \quad \frac{[1-\alpha(1-\beta)]\gamma}{1-\alpha}\ln(w-x) - \frac{\gamma}{1-\alpha}\ln w + \frac{\alpha\beta(1-\gamma)}{1-\alpha}\ln(w-x) - \frac{\alpha(1-\gamma)}{1-\alpha}\ln w.$$

The first-order condition is

(19) $\dfrac{\partial\left[\ln\left(U^{\gamma}\pi^{1-\gamma}\right)\right]}{\partial w} = \dfrac{[1-\alpha(1-\beta)]\gamma}{1-\alpha}\dfrac{1}{w-x} - \dfrac{\gamma}{1-\alpha}\dfrac{1}{w} + \dfrac{\alpha\beta(1-\gamma)}{1-\alpha}\dfrac{1}{w-x} - \dfrac{\alpha(1-\gamma)}{1-\alpha}\dfrac{1}{w} = 0,$

which can be rewritten as

(20) $\dfrac{1}{1-\alpha}[\gamma - \alpha\gamma(1-\beta) + \alpha\beta - \alpha\beta\gamma]\dfrac{1}{w-x} = \dfrac{1}{1-\alpha}[\gamma + \alpha - \alpha\gamma]\dfrac{1}{w}.$

Multiplying both sides of (20) by $(1 - \alpha)$ and simplifying yields

(21) $[\gamma - \alpha(\gamma - \beta)]\dfrac{1}{w-x} = [\alpha + (1-\alpha)\gamma]\dfrac{1}{w}.$

Cross-multiplying gives us

(22) $[\gamma - \alpha(\gamma - \beta)]w = [\alpha + (1 - \alpha)\gamma](w - x)$.

Subtracting γw from both sides of (22) and simplifying yields

(23) $-\alpha(\gamma - \beta)w = \alpha(1 - \gamma)w - [\alpha + (1 - \alpha)\gamma]x$.

Collecting the terms in w gives us

(24) $[-\alpha\gamma + \alpha\beta - \alpha + \alpha\gamma]w = - [\alpha + (1 - \alpha)\gamma]x$,

which simplifies to

(25) $- [\alpha(1 - \beta)]w = - [\alpha + (1 - \alpha)\gamma]x$,

and thus finally, the wage chosen in the bargaining process is

(26) $w = \dfrac{\alpha + (1-\alpha)\gamma}{\alpha(1-\beta)}x.$

Note that in the case of $\beta = 0$, equation (26) does simplify to equation (10).

(b) (iii) The proportional impact of workers' bargaining power on wages can be measured by the elasticity $\partial[\ln w]/\partial\gamma$. In the absence of efficiency wages, the wage chosen in the bargaining process is given by equation (10). Comparing equation (10) to equation (26) we can see that efficiency-wage considerations simply raise the wage by a multiplicative factor of $1/(1 - \beta)$. Thus the presence of efficiency wages does not affect the elasticity given by $\partial[\ln w]/\partial\gamma$. Thus in this model, the proportional impact on wages of workers' bargaining power is not greater with efficiency wages than without and is not greater when efficiency-wage effects are greater.

Problem 9.3

The no-shirking condition (NSC) is given by

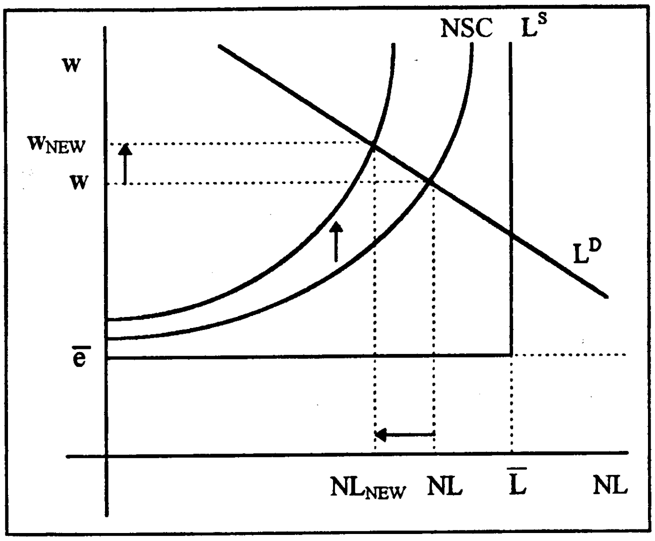
$$(1)\quad w = \bar{e} + \left(\rho + \frac{\bar{L}}{\bar{L} - NL} b\right)\frac{\bar{e}}{q},$$

and the labor demand curve is given by

$$(2)\quad F'(\bar{e}L) = w/\bar{e}.$$

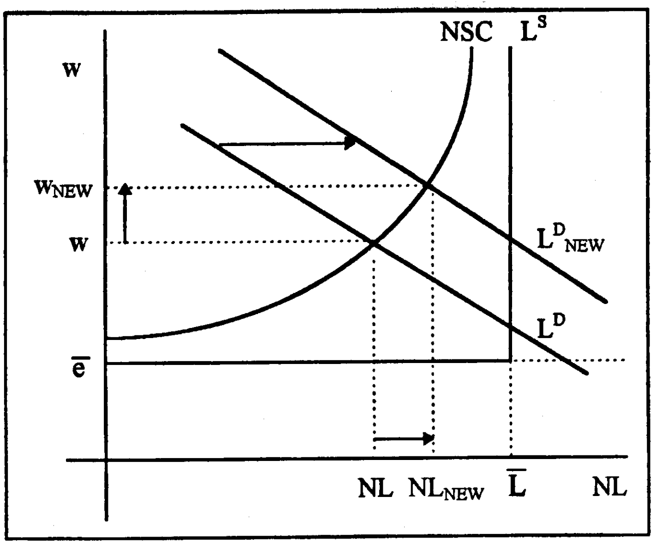
Equation (2) states that firms choose L so that the marginal product of effective labor equals the marginal cost of effective labor, where the wage, w, is set to satisfy (1).

(a) An increase in ρ shifts up the no-shirking locus. From equation (1), for a given NL, the wage needed to get workers to exert effort is now higher. Intuitively, since workers discount the future more, it matters less to them if they are caught shirking, are fired and have to go through a period of unemployment. Thus at a given level of employment, firms must pay a higher wage to deter shirking. The labor demand curve is unaffected. As shown in the figure at right, equilibrium employment falls and the equilibrium wage rises.

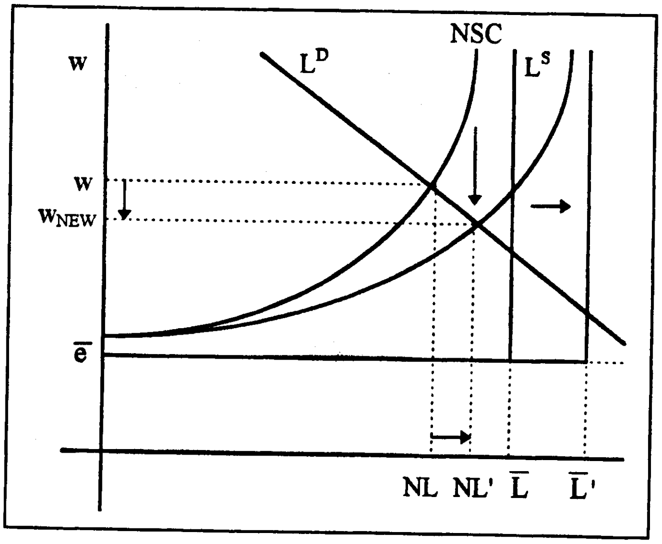


(b) An increase in the job breakup rate, b, shifts up the no-shirking locus. From equation (1), for a given NL, the wage required to get workers to exert effort is now higher. Intuitively, since workers are more likely to lose their job anyway, the value of being employed is lower. Thus workers are not as concerned about being caught shirking and fired. So at a given level of employment, firms must pay a higher wage to deter shirking. The labor demand curve is unaffected. Equilibrium employment falls and the wage rises.

(c) The rise in A shifts the labor demand curve to the right. The no-shirking locus is unaffected. As shown in the figure at right, the equilibrium wage rises as does the level of employment. Note that if efficiency wages were not present, inelastic labor supply would mean that increases in technology would lead only to increases in the wage, not to increases in employment.



(d) The vertical portion of the labor supply curve shifts to the right. The labor demand curve is unaffected. The no-shirking locus shifts down. Intuitively, at a given NL, $\bar{L}$ - NL is now higher. Thus at a given level of employment, if workers become unemployed, they are likely to stay unemployed longer. Thus at a given level of employment, the cost of shirking is greater for a worker and thus firms can get away with paying a lower wage to deter shirking. From the figure at right, the equilibrium wage falls and employment rises.



Problem 9.4

(a) The total number of unemployed workers is $\bar{L}$ - NL. If there is no shirking, the number of workers becoming unemployed per unit time is the number of firms, N, times the number of workers per firm, L, times the rate of job breakup, b. In a steady state, this is also the number of workers becoming employed per unit time. If people who have been unemployed the longest are hired first, the length of time it takes to get a job, which we can denote t*, is equal to the total number of unemployed workers divided by the number of people who get hired per unit time. For example, if there are 1000 unemployed workers and 100 workers become employed per unit time, then the number of units of time it takes to get a job is 1000/100 = 10. Thus in general

(1) $t^* = \dfrac{\bar{L} - NL}{NLb}$.

(b) There is no uncertainty involved when calculating the value of becoming newly unemployed as a function of the value of being employed. When a worker loses her job, she knows that she will be unemployed for $t^* = (\bar{L} - NL)/NLb$ units of time, at which point she will become employed again. Thus the value of becoming newly unemployed is that t* units of time into the future, the individual will have the value of being employed. The discounted value of being employed t* units of time into the future is given by $e^{-\rho t^*} V_E$. Thus

(2) $V_U = e^{-\rho(\bar{L}-NL)/NLb} V_E$.

(c) As in the usual version of the Shapiro-Stiglitz model, the firm chooses a wage so that the value of being employed, V_E, just equals the value of shirking, V_S. From equation (9.35) in the text, this implies

(3) $V_E - V_U = \dfrac{\bar{e}}{q}$.

Substituting equation (2) into equation (3) yields

$$V_E - e^{-\rho t^*} V_E = \frac{\bar{e}}{q}.$$

Solving for V_E gives us

(4) $V_E = \dfrac{\bar{e}}{(1 - e^{-\rho t^*})q}$.

The next step is to determine what the wage must be in order for the value of employment to be given by equation (4). From equation (9.30) in the text for the return from being employed, which is given by $\rho V_E = (w - \bar{e}) - b(V_E - V_U)$, we can solve for w:

(5) $w = \bar{e} + \rho V_E + b(V_E - V_U)$.

Substituting equations (3) and (4) into equation (5) yields

(6) $$w = \bar{e} + \frac{\rho\bar{e}}{(1 - e^{-\rho t^*})q} + \frac{b\bar{e}}{q}.$$

Substituting $t^* = (\bar{L} - NL)/NLb$ into equation (6) gives us

(7) $$w = \bar{e} + \left[\frac{\rho}{1 - e^{-\rho(\bar{L}-NL)/NLb}} + b\right]\frac{\bar{e}}{q}.$$

Equation (7) is the no-shirking condition. Note that as $NL \to \bar{L}$ (as unemployment goes to zero), there is no wage that will deter shirking. This is because as the number of unemployed workers goes to zero, there is no line to stand in and wait for a job. An individual who is caught shirking and is fired will be at the front of the line and will be rehired instantly. As $NL \to 0$, the wage needed to deter shirking goes to $\bar{e} + (\rho + b)\bar{e}/q$. This is exactly the same wage needed to deter shirking as $NL \to 0$ in the standard Shapiro-Stiglitz model.

(d) In order to compare the equilibrium unemployment rate in this model with the equilibrium unemployment rate in the Shapiro-Stiglitz model, we need to compare the two no-shirking loci. If the wage needed to deter shirking for a given level of employment is higher in one of the two models, equilibrium unemployment will be higher in that model.

Intuitively, in both models, the value of being newly unemployed comes from the possibility of becoming employed. For a given level of employment, the expected time to becoming employed is the same in the two models. Here it is certain; in the Shapiro-Stiglitz model it is uncertain. That is, in this model, the newly unemployed worker knows that she will be rehired in t^* units of time; in the Shapiro-Stiglitz model, she has probability $1/t^*$ per unit time of becoming employed again and thus on average will be employed again in t^* units of time.

Now, since $e^{-\rho t}$ is convex in t, the uncertainty about the time it takes to get employed again in the Shapiro-Stiglitz model raises V_U for a given V_E relative to this model. This means that firms must pay a higher wage in the Shapiro-Stiglitz model, for a given level of employment, to deter shirking. Thus equilibrium unemployment is higher in the Shapiro-Stiglitz model.

More formally, our claim is that

(8) NSC wage for a given NL in Shapiro-Stiglitz > NSC wage for a given NL in this model.

From equation (9.39) in the text and equation (7) here, the claim is

(9) $$\bar{e} + \left[\rho + \frac{\bar{L}}{\bar{L} - NL}b\right]\frac{\bar{e}}{q} > \bar{e} + \left[\frac{\rho}{1 - e^{-\rho t^*}} + b\right]\frac{\bar{e}}{q}.$$

Subtracting $\bar{e}$ from both sides and dividing both sides of the resulting expression by $\bar{e}/q$ leaves us with

(10) $$\rho + \frac{\bar{L}}{\bar{L} - NL}b > \frac{\rho}{1 - e^{-\rho t^*}} + b.$$

Now, using the definition of $t^* = (\bar{L} - NL)/NLb$, we can write $NLbt^* = \bar{L} - NL$ or

(11) $NL = \bar{L}/(1 + bt^*)$.

Substituting equation (11) into $[\bar{L}/(\bar{L} - NL)]b$ gives us

(12) $\dfrac{\overline{L}}{\overline{L}-NL}b = \dfrac{\overline{L}}{\overline{L}-[\overline{L}/(1+bt^*)]}b = \dfrac{1+bt^*}{(1+bt^*)-1}b = \dfrac{1+bt^*}{t^*}$.

Substituting equation (12) into our claim gives us

(13) $\rho + \dfrac{1+bt^*}{t^*} > \dfrac{\rho}{1-e^{-\rho t^*}} + b$.

Multiplying both sides of (13) by t* gives us the following equivalent expression:

(14) $\rho t^* + 1 + bt^* > \dfrac{\rho t^*}{1-e^{-\rho t^*}} + bt^*$.

Subtracting bt* from both sides of (14) and then multiplying both sides of the resulting expression by $(1 - e^{-\rho t^*})$ yields

(15) $\rho t^* - \rho t^* e^{-\rho t^*} + 1 - e^{-\rho t^*} > \rho t^*$.

Thus, finally, our original claim is equivalent to

(16) $1 - e^{-\rho t^*} - \rho t^* e^{-\rho t^*} > 0$.

We need to show that (16) actually holds. Note that it takes the form $1 - e^{-x} - xe^{-x}$ where $x \equiv \rho t^*$. This expression is greater than zero for $x > 0$. To formally see this, let f(x) denote the left-hand side of equation (16). Then $f(0) = 0$ and

(17) $f'(x) = e^{-x} + xe^{-x} - e^{-x} = xe^{-x} > 0$ for $x > 0$.

Thus, since f is equal to zero at zero and is increasing for all x greater than zero, it must be positive for all $x > 0$. Therefore our claim holds and equilibrium unemployment is higher in the Shapiro-Stiglitz model than it is in this model.

Problem 9.5

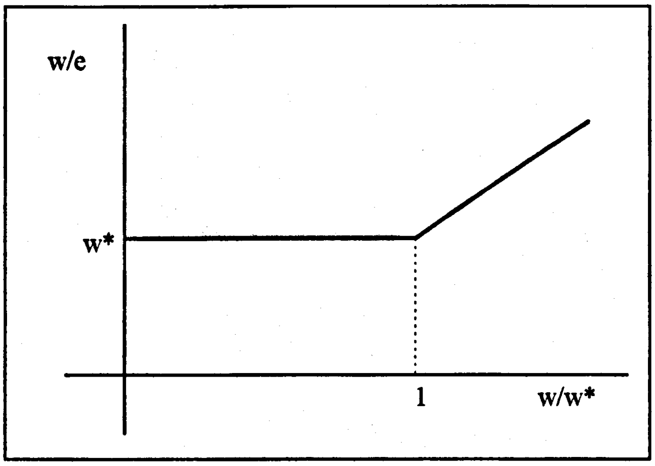
(a) The firm obtains e units of effective labor for a wage cost of w. Thus the cost to the firm of one unit of effective labor is w/e.

For $w/w^* < 1$, $e = w/w^*$ and so we have

(1) $\dfrac{w}{e} = \dfrac{w}{w/w^*} = w^*$.

For $w/w^* \geq 1$, $e = 1$ and so we have

(2) $\dfrac{w}{e} = \dfrac{w}{1} = w$.



See the figure at right, which plots the cost of a unit of effective labor, w/e, as a function of the firm's wage relative to the fair wage, w/w*. We can see that any wage such that $w/w^* \leq 1$ or $w \leq w^*$, minimizes the cost of effective labor.

(b) (i) Given the assumption that the firm pays the highest wage in the range described in part (a), it chooses $w = w^*$; that is, in order to minimize the cost per unit of effective labor, the firm chooses to pay the fair wage. Thus

(3) $w = \overline{w} + a - bu$.

(b) (ii) Assume that there is positive unemployment. If this is the case, the firm is unconstrained in its choice of the wage and therefore pays the fair wage, nothing more. Since this is true for all firms, the average wage, $\overline{w}$, must equal w. Thus we have
(4) $w = w + a - bu$.
Solving equation (4) for the unemployment rate yields
(6) $u = a/b$.
Note that the higher is a -- the higher above the average wage is the perceived fair wage -- the higher is the equilibrium unemployment rate. Also, the lower is b -- the less responsive is the fair wage to unemployment -- the higher is the equilibrium unemployment rate.

Since we have derived this under the assumption of positive unemployment, we need to verify that this is actually the case. If $u = 0$, then the fair wage is $w^* = w + a$, where we have used the fact that $\overline{w} = w$ (all firms pay the same wage). But if $a > 0$, this means that $w < w^*$ and so firms are paying less than the fair wage. But this violates our assumption that firms do not choose to pay below the fair wage. So as long as $a > 0$, there will be positive unemployment in equilibrium.

(b) (iii) From part (b), (ii), we can see that if $a = 0$, equilibrium unemployment will be zero. The fair wage will equal the actual wage. If $a < 0$, the perceived fair wage is always less than the average wage for any value of unemployment. Thus the representative firm, taking the average wage as given, wants to pay less than the average wage. Since workers are willing to work at any positive wage, firms need only pay ε above zero to get workers to be willing to work, even with zero unemployment.

(c) (i) Analysis such as that in part (a) continues to hold for each type of worker. The representative firm attempts to minimize the cost of effective labor for each type of worker. If the firm is unconstrained in its choice of w, this can be accomplished by paying any wage such that $w_1 \leq w_1^*$ for the high-productivity workers and $w_2 \leq w_2^*$ for the low-productivity workers. Assuming the firm pays the highest wage in these ranges, neither type of worker will be paid less than its fair wage.

(c) (ii) Firms will hire each type of worker until the cost of one unit of effective labor is the same for each type of worker. If this were not the case, firms could reduce their costs by hiring more of the workers with lower effective labor costs and fewer of the workers with higher effective labor costs.

For a low-productivity worker, the firm obtains e_2 units of effective labor at a wage cost of w_2. Thus the cost to the firm of one unit of low-productivity effective labor is w_2/e_2. For a high-productivity worker, the firm obtains Ae_1 units of effective labor at a wage cost of w_1. Thus the cost to the firm of one unit of high-productivity effective labor is w_1/Ae_1. Equating these effective labor costs gives us

(7) $$\frac{w_1}{Ae_1} = \frac{w_2}{e_2}.$$

Since both types of workers are paid at least the fair wage, $e_1 = e_2 = 1$ and so equation (7) can be rewritten as
(8) $w_1 = Aw_2$.
The wage for the high-productivity workers exceeds that of the low-productivity workers by a factor of A.

(c) (iii) In equilibrium, there will be no unemployment among the high-productivity workers. Suppose instead that there was. We have just shown that high-productivity workers have a higher wage than low-productivity workers and so w_1 is higher than the average wage or
(9) $w_1 > (w_1 + w_2)/2$,

where we have used the fact that all firms pay the same wage so that $\overline{w}_1 = w_1$ and $\overline{w}_2 = w_2$. Thus the fair wage for high-productivity workers is

(10) $w_1^* = (w_1 + w_2)/2 - bu_1 < w_1$.

But inequality (10) says that the firm is paying a wage higher than the fair wage to a group that has unemployment. But the firm does not need to do this; it is unconstrained in its choice of w_1 if $u_1 > 0$. It could cut the wage down to the fair wage level. Thus there cannot be unemployment among the high-productivity workers.

(c) (iv) In equilibrium, there will be unemployment among low-productivity workers. In part (c), (ii), we explained that the low-productivity workers receive a lower wage than the high-productivity workers. This means that their wage is less than the average wage, or

(11) $w_2 < (w_1 + w_2)/2$.

Now suppose that there was no unemployment among the low-productivity workers. Then the fair wage for them would be

(12) $w_2^* = (w_1 + w_2)/2 > w_2$.

But inequality (12) violates our assumption that firms will not pay a wage below the fair wage. Thus there must be some positive unemployment rate, $u_2 > 0$, such that $w_2 = w_2^*$.

Problem 9.6

(a) Suppose there are N states of the world. Then the firm's expected profits are

$$(1)\quad E(\pi) = \sum_{i=1}^{N} p_i\left[A_i F(L_i) - C_i^E L_i - C_i^U(\overline{L} - L_i)\right].$$

The expected utility of a representative worker is

$$(2)\quad E(u) = \sum_{i=1}^{N} p_i\left\{\left(\frac{L_i}{\overline{L}}\right)\left[U(C_i^E) - K\right] + \left(\frac{\overline{L} - L_i}{\overline{L}}\right)U(C_i^U)\right\}.$$

The firm's problem is to choose the L_i's, C_i^E's and C_i^U's in order to maximize equation (1) subject to equation (2). Thus the Lagrangian is

$$\mathcal{L} = \sum_{i=1}^{N} p_i\left[A_i F(L_i) - C_i^E L_i - C_i^U(\overline{L} - L_i)\right] + \lambda\left[\sum_{i=1}^{N} p_i\left\{\left(\frac{L_i}{\overline{L}}\right)\left[U(C_i^E) - K\right] + \left(\frac{\overline{L} - L_i}{\overline{L}}\right)U(C_i^U)\right\} - u_0\right].$$

(b) The first-order conditions are

$$(3)\quad \frac{\partial \mathcal{L}}{\partial L_i} = p_i A_i F'(L_i) - p_i C_i^E + p_i C_i^U + \lambda p_i\left(\frac{1}{\overline{L}}\right)\left[U(C_i^E) - K\right] - \lambda p_i\left(\frac{1}{\overline{L}}\right)U(C_i^U) = 0,$$

$$(4)\quad \frac{\partial \mathcal{L}}{\partial C_i^E} = -p_i L_i + \lambda p_i\left(\frac{L_i}{\overline{L}}\right)U'(C_i^E) = 0, \text{ and}$$

$$(5)\quad \frac{\partial \mathcal{L}}{\partial C_i^U} = -p_i(\overline{L} - L_i) + \lambda p_i\left(\frac{\overline{L} - L_i}{\overline{L}}\right)U'(C_i^U) = 0.$$

Solving equation (4) for $U'(C_i^E)$ gives us

(6) $U'(C_i^E) = \overline{L}/\lambda$.

Equation (6) implies the marginal utility of consumption for the employed workers is constant across states and thus, with $U''(\bullet) < 0$, consumption of the employed workers is constant across states.

Solving equation (5) for $U'(C_i^U)$ gives us

(7) $U'(C_i^U) = \overline{L}/\lambda$.

Thus consumption of the unemployed workers is also constant across states. Comparing equations (6) and (7), it is also true that the marginal utility of consumption is the same for both employed and unemployed workers. This implies that the level of consumption of both types of workers is the same. That is,
(8) $C^E = C^U$.
So C^E and C^U do not depend on the state and are always equal to each other.

(c) The unemployed workers are actually better off. They consume the same amount as the employed workers and do not suffer the disutility of work, K.

Problem 9.7
(a) We can maximize $2E(\pi) = A_G F(L_G) - wL_G + A_B F(L_B) - wL_B - fB(L_G - L_B)$ subject to $(w - K) + (L_B/L_G)(w - K) + [(L_G - L_B)/L_G]B = 2u_0$. Thus the Lagrangian is
(1) $\mathcal{L} = A_G F(L_G) - wL_G + A_B F(L_B) - wL_B - fB(L_G - L_B) +$
$\lambda\{(w - K) + (L_B/L_G)(w - K) + [(L_G - L_B)/L_G]B - 2u_0\}$.

(b) The first-order conditions are
(2) $\partial\mathcal{L}/\partial w = - L_G - L_B + \lambda + \lambda(L_B/L_G) = 0$,
(3) $\partial\mathcal{L}/\partial L_B = A_B F'(L_B) - w + fB + (\lambda/L_G)(w - K) - (\lambda/L_G)B = 0$, and
(4) $\partial\mathcal{L}/\partial L_G = A_G F'(L_G) - w - fB - (\lambda L_B/L_G^2)(w - K) + (\lambda L_B/L_G^2)B = 0$.

(c) We can solve equation (2) for λ:
$\lambda[(L_G + L_B)/L_G] = L_G + L_B$,
or simply
(5) $\lambda = L_G$.
Substituting equation (5) into equation (3) yields
$A_B F'(L_B) - w + fB + (w - K) - B = 0$,
which implies
(6) $A_B F'(L_B) - K - B(1 - f) = 0$.
Differentiating both sides of equation (6) with respect to f gives us
(7) $A_B F''(L_B)[\partial L_B/\partial f] + B = 0$.
Solving equation (7) for $\partial L_B/\partial f$ yields
(8) $$\frac{\partial L_B}{\partial f} = \frac{-B}{A_B F''(L_B)} > 0,$$
since $B > 0$, $A_B > 0$ and $F''(\bullet) < 0$. Thus a fall in the fraction of the unemployment benefit paid by the firm actually causes the firm to hire fewer workers in the bad state.

Similarly, we can differentiate both sides of equation (6) with respect to B to yield
(9) $A_B F''(L_B)[\partial L_B/\partial B] - (1 - f) = 0$.
Solving equation (9) for $\partial L_B/\partial B$ gives us
(10) $$\frac{\partial L_B}{\partial B} = \frac{(1-f)}{A_B F''(L_B)} < 0,$$
as long as $f < 1$. Thus a rise in the unemployment benefit causes the firm to hire fewer workers in the bad state.

(d) Substituting equation (5) into equation (4) yields
$A_G F'(L_G) - w - fB - (L_B/L_G)(w - K) + (L_B/L_G)B = 0$,
or simply

(11) $A_G F'(L_G) - w - fB - (L_B/L_G)(w - K - B) = 0$.
Equation (11) can be rearranged to obtain
(12) $A_G F'(L_G) - fB = w + (L_B/L_G)(w - K - B)$.
Multiplying both sides of the expected utility constraint by two yields
(13) $(w - K) + (L_B/L_G)(w - K) + [(L_G - L_B)/L_G]B = 2u_0$.
Equation (13) can be rearranged to obtain
(14) $w + (L_B/L_G)(w - K - B) = 2u_0 + K - B$.

Equations (12) and (14) therefore imply
(15) $A_G F'(L_G) - fB = 2u_0 + K - B$,
or simply
(16) $A_G F'(L_G) = (2u_0 + K) - (1 - f)B$.

Differentiating both sides of equation (16) with respect to f yields
(17) $A_G F''(L_G)[\partial L_G/\partial f] = B$.
Solving equation (17) for $\partial L_G/\partial f$ yields

(18) $$\frac{\partial L_G}{\partial f} = \frac{B}{A_G F''(L_G)} < 0,$$

since $B > 0$, $A_G > 0$, and $F''(\bullet) < 0$. Thus a fall in the fraction of the unemployment benefit paid by the firm actually causes the firm to hire more workers in the good state.

Similarly, differentiating both sides of equation (16) with respect to B yields
(19) $A_G F''(L_G)[\partial L_G/\partial B] = -(1 - f)$.
Solving (19) for $\partial L_G/\partial B$ gives us

(20) $$\frac{\partial L_G}{\partial B} = \frac{-(1-f)}{A_G F''(L_G)} > 0,$$

as long as $f < 1$. Thus a rise in the unemployment benefit causes the firm to hire more workers in the good state.

So an increase in unemployment benefits or a reduction in the fraction of those benefits paid by firms will increase employment fluctuations by causing firms to hire less labor in the bad state and more labor in the good state.

Problem 9.8

(a) With efficient contracts, as shown in Section 9.5 of the text, $C = wL$ is constant across states. In addition, employment is increasing in A so that $L_G > L_B$. Given A and given the fact that wL is constant, profit, $\pi = AF(L) - wL$, is increasing in employment. Thus when the true state is A_B, the firm is better off announcing that A is actually A_G and employing L_G. When the true state is A_G, the firm is again better off announcing that A is A_G. Thus when the state is A_G, it is in the firm's interest to announce the true state. However, in the bad state, it is not in the firm's interest to announce the true state and so the efficient contract is not incentive-compatible.

(b) The incentive-compatibility constraint that is binding is that the firm not prefer to claim that $A = A_G$ when in fact $A = A_B$. Assuming that this constraint holds with equality, this requires
(1) $A_B F(L_B) - C_B = A_B F(L_G) - C_G$.
The left-hand side of equation (1) is the firm's profit in the bad state if it announces the bad state whereas the right-hand side of equation (1) is the firm's profit in the bad state if it announces the good state. The other constraint is that workers' expected utility be equal to u_0 or

(2) $[U(C_B) - V(L_B)]/2 + [U(C_G) - V(L_G)]/2 = u_0$.
The firm's expected profit is
(3) $E[\pi] = [A_B F(L_B) - C_B]/2 + [A_G F(L_G) - C_G]/2$.

The problem facing the firm is to choose L_B, C_B, L_G and C_G to maximize expected profits subject to equations (1) and (2). The Lagrangian is
(4) $\mathcal{L} = [A_B F(L_B) - C_B]/2 + [A_G F(L_G) - C_G]/2 + \lambda_1 \{[U(C_B) - V(L_B)]/2 + [U(C_G) - V(L_G)]/2 - u_0\}$
$+\lambda_2 \{[A_B F(L_B) - C_B] - [A_B F(L_G) - C_G]\}$.

The first-order conditions are
(5) $\partial\mathcal{L}/\partial C_B = (-1/2) + (1/2)\lambda_1 U'(C_B) - \lambda_2 = 0$,
(6) $\partial\mathcal{L}/\partial C_G = (-1/2) + (1/2)\lambda_1 U'(C_G) + \lambda_2 = 0$,
(7) $\partial\mathcal{L}/\partial L_B = (1/2)A_B F'(L_B) - (1/2)\lambda_1 V'(L_B) + \lambda_2 A_B F'(L_B) = 0$, and
(8) $\partial\mathcal{L}/\partial L_G = (1/2)A_G F'(L_G) - (1/2)\lambda_1 V'(L_G) - \lambda_2 A_B F'(L_G) = 0$.

(c) Adding equations (5) and (6) yields
$-1 + (1/2)\lambda_1 U'(C_B) + (1/2)\lambda_1 U'(C_G) = 0$.
Solving for λ_1 gives us
(9) $$\lambda_1 = \frac{2}{U'(C_B) + U'(C_G)}.$$
Substituting equation (9) into equation (5) yields
$$2\lambda_2 = -1 + \frac{2U'(C_B)}{U'(C_B) + U'(C_G)} = \frac{-U'(C_B) - U'(C_G) + 2U'(C_B)}{U'(C_B) + U'(C_G)},$$
and thus
(10) $$\lambda_2 = \frac{U'(C_B) - U'(C_G)}{2[U'(C_B) + U'(C_G)]}.$$

Substituting equations (9) and (10) into equation (7) yields
(11) $$\frac{1}{2}A_B F'(L_B) - \frac{V'(L_B)}{U'(C_B) + U'(C_G)} + \frac{A_B F'(L_B)[U'(C_B) - U'(C_G)]}{2[U'(C_B) + U'(C_G)]} = 0.$$
Multiplying both sides of equation (11) by $2[U'(C_B) + U'(C_G)]$ gives us
(12) $A_B F'(L_B)[U'(C_B) + U'(C_G)] - 2V'(L_B) + A_B F'(L_B)[U'(C_B) - U'(C_G)] = 0$.
Simplifying yields
(13) $2A_B F'(L_B) U'(C_B) = 2V'(L_B)$,
and finally
(14) $$A_B F'(L_B) = \frac{V'(L_B)}{U'(C_B)}.$$
Equation (14) states that, in the bad state, the marginal product and the marginal disutility of labor are equated.

(d) Substituting equations (9) and (10) into equation (8) yields
(15) $$\frac{1}{2}A_G F'(L_G) - \frac{V'(L_G)}{U'(C_B) + U'(C_G)} - \frac{A_B F'(L_G)[U'(C_B) - U'(C_G)]}{2[U'(C_B) + U'(C_G)]} = 0.$$
Multiplying both sides of equation (15) by $[U'(C_B) + U'(C_G)]$ and rearranging yields

$$(16)\quad \frac{A_G F'(L_G)\left[U'(C_B)+U'(C_G)\right]}{2}-\frac{A_B F'(L_G)\left[U'(C_B)-U'(C_G)\right]}{2}=V'(L_G).$$

Dividing both sides of equation (16) by $U'(C_G)$ and factoring out an $F'(L_G)$ from the left-hand side leaves us with

$$\left[\frac{A_G}{2}+\frac{A_B}{2}+\frac{A_G U'(C_B)-A_B U'(C_B)}{2U'(C_G)}\right]F'(L_G)=\frac{V'(L_G)}{U'(C_G)}.$$

This is equivalent to

$$\left[A_G-\frac{A_G}{2}+\frac{A_B}{2}+\frac{A_G U'(C_B)-A_B U'(C_B)}{2U'(C_G)}\right]F'(L_G)=\frac{V'(L_G)}{U'(C_G)}.$$

Collecting terms yields

$$\left\{A_G+\frac{A_G}{2}\left[\frac{U'(C_B)}{U'(C_G)}-1\right]-\frac{A_B}{2}\left[\frac{U'(C_B)}{U'(C_G)}-1\right]\right\}F'(L_G)=\frac{V'(L_G)}{U'(C_G)},$$

and thus finally

$$(17)\quad \left\{A_G+\left(\frac{A_G-A_B}{2}\right)\left[\frac{U'(C_B)-U'(C_G)}{U'(C_G)}\right]\right\}F'(L_G)=\frac{V'(L_G)}{U'(C_G)}.$$

The contract must clearly involve $L_G > L_B$ and must therefore have $C_G > C_B$ to be incentive-compatible. [Note that if $L_G = L_B$, then C_G must equal C_B for incentive-compatibility. But then equation (10) implies $\lambda_2 = 0$, which implies that equations (7) and (8) cannot both be satisfied.] Since $C_G > C_B$, it follows that $U'(C_B) > U'(C_G)$ and so the second term in brackets is positive. Thus $V'(L_G)/U'(C_G)$ exceeds $A_G F'(L_G)$. The marginal disutility of work exceeds the marginal product of labor in the good state or in other words, there is overemployment in the good state.

(e) Given the fact that there is no unemployment in the bad state and overemployment in the good state, this model does not appear to be helpful in understanding the high level of average unemployment. But since it is in the good state that the overemployment occurs, the model does suggest a reason that employment might be procyclical and more responsive to shocks than under symmetric information.

Problem 9.9

(a) (i) The firm chooses L in order to maximize profits as given by

(1) $\pi = AL^{\alpha}/\alpha - wL$.

The first-order condition is

(2) $\partial\pi/\partial L = AL^{\alpha-1} - w = 0 \quad\Rightarrow\quad L^{\alpha-1} = w/A \quad\Rightarrow\quad L = (w/A)^{1/(\alpha-1)}$,

and thus the firm's choice of L is

(3) $L = A^{1/(1-\alpha)}\, w^{-1/(1-\alpha)}$.

(a) (ii) Substituting equation (3) for the firm's choice of L into the union's objective function yields

$$U^U = [U(w) - K]A^{1/(1-\alpha)}\, w^{-1/(1-\alpha)} + U(w_u)\,[N - A^{1/(1-\alpha)}\, w^{-1/(1-\alpha)}].$$

Collecting terms gives us

(4) $U^U = A^{1/(1-\alpha)}\, w^{-1/(1-\alpha)}\,[U(w) - K - U(w_u)] + U(w_u)N$.

The union's problem is to choose w in order to maximize its utility as given in equation (4). The first-order condition is

(5) $\partial U^U/\partial w = -[1/(1-\alpha)]\, A^{1/(1-\alpha)}\, w^{[-1/(1-\alpha)]-1}\,[U(w) - K - U(w_u)] + A^{1/(1-\alpha)}\, w^{-1/(1-\alpha)}\,[U'(w)] = 0$.

Equation (5) can be written as

$$[1/(1-\alpha)]\, A^{1/(1-\alpha)}\, w^{[-1/(1-\alpha)]-1}\, [U(w) - K - U(w_u)] = A^{1/(1-\alpha)}\, w^{-1/(1-\alpha)}\, U'(w),$$

which simplifies to

$$[1/(1-\alpha)] w^{-1}\, [U(w) - K - U(w_u)] = U'(w),$$

and thus w is defined implicitly by

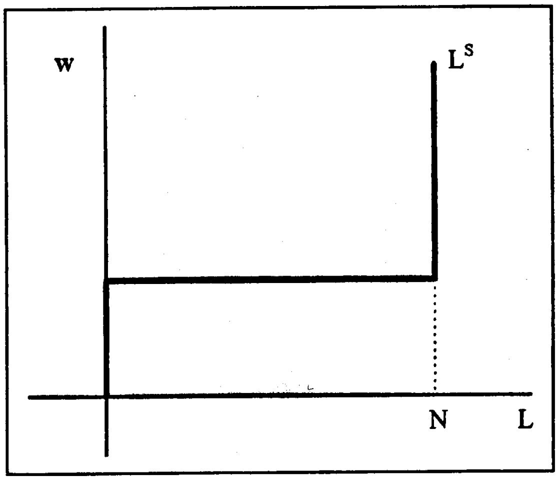
$$(6)\quad w = \frac{1}{1-\alpha}\left[\frac{U(w) - K - U(w_u)}{U'(w)}\right].$$

Although equation (6) only defines the union's choice of the wage implicitly, we can see that this choice does not depend on the shock to labor demand, A. From equation (3), since w does not vary with A, the elasticity of L with respect to A is

$$(7)\quad \frac{\partial L}{\partial A}\frac{A}{L} = \frac{\partial \ln L}{\partial \ln A} = \frac{1}{1-\alpha}.$$

(b) The union's objective function is

$$(8)\quad U^U = [U(w) - K]L + U(w_u)\,[N - L].$$



If the wage is such that $U(w) - K > U(w_u)$, then the union's objective function is maximized by having all of its members employed; that is, by choosing L = N. If the wage is such that that $U(w) - K < U(w_u)$, the union's objective function is maximized by having no one employed. Finally, if the wage is such that that $U(w) - K = U(w_u)$, the union is indifferent as to how many of its members are employed.

The union's labor supply curve under spot markets is depicted in the figure at right. The way in which the wage and employment vary with A will depend on whether the labor demand curve intersects the completely elastic or the completely inelastic portion of the labor supply curve.

So first of all, if the labor demand curve intersects the inelastic portion, L does not vary with A under spot markets; employment does not vary in response to shocks to labor demand. Rearranging equation (3) to solve for w yields

$$(9)\quad w = AL^{-1/(1-\alpha)},$$

where L = N. Thus the elasticity of the wage with respect to A is

$$(10)\quad \frac{\partial w}{\partial A}\frac{A}{w} = \frac{\partial \ln w}{\partial \ln A} = 1.$$

However, if the labor demand curve intersects the elastic portion of the labor supply curve, we get the opposite result. The real wage does not respond at all to changes in A. From equation (3), this means that the elasticity of employment with respect to A is $1/(1-\alpha)$.

(c) (i) With spot markets, the union takes w as given and chooses L to maximize $U^U = wL - [\sigma/(\sigma + 1)]L^{(\sigma+1)/\sigma}$. The first-order condition is

$$(11)\quad \partial U^U/\partial L = w - L^{1/\sigma} = 0,$$

and thus the union's choice of labor supply as a function of the wage is

$$(12)\quad L = w^{\sigma},$$

and so σ represents the elasticity of labor supply with respect to the wage.

Under spot markets, w will adjust to equate labor demand and labor supply. Setting the right-hand sides of equations (3) and (12) equal to each other gives us

(13) $A^{1/(1-\alpha)} w^{-1/(1-\alpha)} = w^{\sigma} \quad \Rightarrow \quad Aw^{-1} = w^{\sigma(1-\alpha)}$,

and thus the spot-market wage is

(14) $w = A^{1/[1+\sigma(1-\alpha)]}$.

The elasticity of the spot-market wage with respect to the labor demand shocks is

(15) $$\frac{\partial w}{\partial A}\frac{A}{w} = \frac{\partial \ln w}{\partial \ln A} = \frac{1}{1+\sigma(1-\alpha)}.$$

Substituting equation (14) into either the demand or supply curve will give us equilibrium employment. Substituting it into equation (12) yields

(16) $L = A^{\sigma/[1+\sigma(1-\alpha)]}$.

The elasticity of employment with respect to the labor demand shocks is

(17) $$\frac{\partial L}{\partial A}\frac{A}{L} = \frac{\partial \ln L}{\partial \ln A} = \frac{\sigma}{1+\sigma(1-\alpha)}.$$

(c) (ii) Substituting equation (3) for the firm's choice of L into the union's objective function yields

(18) $U^{U} = A^{1/(1-\alpha)} w^{-\alpha/(1-\alpha)} - [\sigma/(\sigma+1)] A^{(\sigma+1)/\sigma(1-\alpha)} w^{-(\sigma+1)/\sigma(1-\alpha)}$.

The union's problem is to choose w in order to maximize its utility as given in equation (18). The first-order condition is

(19) $$\frac{\partial U^{U}}{\partial w} = \left(\frac{-\alpha}{1-\alpha}\right) A^{1/(1-\alpha)} w^{-1/(1-\alpha)} + \left(\frac{\sigma}{\sigma+1}\right)\left[\frac{\sigma+1}{\sigma(1-\alpha)}\right] A^{(\sigma+1)/\sigma(1-\alpha)} w^{[-(\sigma+1)-\sigma(1-\alpha)]/\sigma(1-\alpha)} = 0.$$

Equation (19) can be written as

(20) $\alpha A^{1/(1-\alpha)} w^{-1/(1-\alpha)} = A^{(\sigma+1)/\sigma(1-\alpha)} w^{[-(\sigma+1)-\sigma(1-\alpha)]/\sigma(1-\alpha)}$.

Taking both sides of equation (20) to the exponent $\sigma(1 - \alpha)$ yields

(21) $\alpha^{\sigma(1-\alpha)} A^{\sigma} w^{-\sigma} = A^{\sigma+1} w^{-(\sigma+1)-\sigma(1-\alpha)}$,

or simply

(22) $w^{-[1+\sigma(1-\alpha)]} = \alpha^{\sigma(1-\alpha)} A^{-1}$,

and thus the union's choice of w is

(23) $w = \alpha^{\sigma(1-\alpha)/[1-\sigma(1-\alpha)]} A^{1/[1+\sigma(1-\alpha)]}$.

And thus the elasticity of this wage determined by the union with respect to the labor demand shocks is

(24) $$\frac{\partial w}{\partial A}\frac{A}{w} = \frac{\partial \ln w}{\partial \ln A} = \frac{1}{1+\sigma(1-\alpha)}.$$

This is exactly the same elasticity as when the wage was determined by the spot market. Thus with this union objective function, the behavior of the wage in response to labor demand shocks does not depend on whether or not unions get to set the wage or whether the wage is determined by supply and demand in a spot market.

Substituting equation (23) into the firm's choice of L given in equation (3) yields

(25) $L = A^{1/(1-\alpha)} \alpha^{-\sigma(1-\alpha)/(1-\alpha)[1+\sigma(1-\alpha)]} A^{-1/(1-\alpha)[1+\sigma(1-\alpha)]}$.

Combining the terms in A gives us

$$L = \alpha^{-\sigma/[1+\sigma(1-\alpha)]} A^{[1+\sigma(1-\alpha)-1]/(1-\alpha)[1+\sigma(1-\alpha)]},$$

or simply

(26) $L = \alpha^{-\sigma/[1+\sigma(1-\alpha)]} A^{\sigma/[1+\sigma(1-\alpha)]}$.

The elasticity of equilibrium employment with respect to A is

(27) $$\frac{\partial L}{\partial A}\frac{A}{L}=\frac{\partial \ln L}{\partial \ln A}=\frac{\sigma}{1+\sigma(1-\alpha)}.$$

Again, this is exactly the same elasticity as in the spot-market case. With this union objective function, whether there is a spot market or whether the union sets the wage, the behavior of employment in response to labor demand shocks is the same.

Problem 9.10

(a) (i) The union's problem is to choose w and L in order to maximize $[U(w) - K]L + U(w_u)[N - L]$ subject to $AL^{\alpha}/\alpha - wL \geq \pi_0$. Since the union has no reason to give the firm profits greater than π_0, it satisfies the constraint with equality. The Lagrangian is

(1) $\mathcal{L} = [U(w) - K]L + U(w_u)[N - L] + \lambda[AL^{\alpha}/\alpha - wL - \pi_0]$.

(a) (ii) The first-order conditions are

(2) $\partial\mathcal{L}/\partial w = U'(w)L - \lambda L = 0$, and (3) $\partial\mathcal{L}/\partial L = U(w) - K - U(w_u) + \lambda AL^{\alpha-1} - \lambda w = 0$.

(a) (iii) The wage is the means by which the union extracts the rent from being able to reduce the firm's profits to its minimum required level.

(a) (iv) From equation (2), $\lambda = U'(w)$. Substituting this fact into equation (3) yields

(4) $U(w) - K - U(w_u) + U'(w)AL^{\alpha-1} - U'(w)w = 0$.

Rearranging equation (4) yields

(5) $U'(w)AL^{\alpha-1} = U'(w)w - U(w) + K + U(w_u)$,

and thus solving for the union's choice of L gives us

(6) $$L=\left[\frac{U'(w)A}{U'(w)w-U(w)+K+U(w_u)}\right]^{1/(1-\alpha)}.$$

From equation (6), the elasticity of employment with respect to the labor demand shocks is

(7) $$\frac{\partial L}{\partial A}\frac{A}{L}=\frac{\partial \ln L}{\partial \ln A}=\frac{1}{1-\alpha}>0.$$

In part (b) of Problem 9.9, if labor demand intersected the elastic portion of labor supply, the elasticity of employment with respect to the labor demand shocks was the same as it is here.

(b) Now the union's problem is to choose w and L in order to maximize $wL - [\sigma/(\sigma + 1)]L^{(\sigma+1)/\sigma}$ subject to $AL^{\alpha}/\alpha - wL \geq \pi_0$. The Lagrangian is

(8) $\mathcal{L} = wL - [\sigma/(\sigma + 1)]L^{(\sigma+1)/\sigma} + \lambda[AL^{\sigma}/\alpha - wL - \pi_0]$.

The first-order conditions are

(9) $\partial\mathcal{L}/\partial w = L - \lambda L = 0$, and (10) $\partial\mathcal{L}/\partial L = w - L^{1/\sigma} + \lambda AL^{\alpha-1} - \lambda w = 0$.

From equation (9), $\lambda = 1$. Substituting this fact into equation (10) gives us

(11) $w - L^{1/\sigma} + AL^{\alpha-1} - w = 0$.

Equation (11) can be written as

$$L^{1/\sigma} = AL^{\alpha-1} \quad \Rightarrow \quad L^{[1-\sigma(\alpha-1)]/\sigma} = A,$$

and thus the union's choice of L is

(12) $L = A^{\sigma/[1+\sigma(1-\alpha)]}$.

The elasticity of employment with respect to the labor demand shocks is

(13) $\dfrac{\partial L}{\partial A}\dfrac{A}{L} = \dfrac{\partial \ln L}{\partial \ln A} = \dfrac{\sigma}{1+\sigma(1-\alpha)}$.

Note that this is exactly the same elasticity as in the spot market case; see equation (17) in the solution to Problem 9.9. Thus equilibrium employment varies in exactly the same way (due to the labor demand shocks) regardless of whether the union controls the wage and employment or whether the labor market equilibrium is determined by the forces of demand and supply.

Problem 9.11

(a) In equilibrium, the number of people in the primary sector will be equal to the number of employed people, which in turn equals the number of primary sector jobs, N_p, plus the number of unemployed people in the economy, U. Since people are picked at random for jobs, in equilibrium, the probability of obtaining a primary-sector job, q, is equal to the total number of jobs, N_p, divided by the equilibrium pool of primary-sector workers, $N_p + U$. Thus in equilibrium

(1) $q = N_p/(N_p + U)$.

In addition, in equilibrium, the expected utility of choosing the primary sector, $qw_p + (1 - q)b$, must equal the expected utility of choosing the secondary sector, w_s. Thus in equilibrium

(2) $qw_p + (1 - q)b = w_s$.

Solving equation (2) for q gives us

(3) $q = (w_s - b)/(w_p - b)$.

We have two conditions that q must satisfy in equilibrium. Setting the right-hand sides of equations (1) and (3) equal, we have

$$N_p/(N_p + U) = (w_s - b)/(w_p - b) \quad \Rightarrow \quad N_p(w_p - b) = N_p(w_s - b) + (w_s - b)U.$$

Solving for equilibrium unemployment we have

$$(w_s - b)U = (w_p - w_s)N_p,$$

or simply

(4) $U = \left(\dfrac{w_p - w_s}{w_s - b}\right)N_p$.

(b) To see how an increase in the number of primary-sector jobs affects unemployment, take the derivative of U with respect to N_p:

(5) $\dfrac{\partial U}{\partial N_p} = \left(\dfrac{w_p - w_s}{w_s - b}\right) > 0,$

since we are assuming $b < w_s < w_p$. Equation (5) implies that a rise in the number of primary-sector jobs actually increases equilibrium unemployment. The fact that there are more of these jobs does, for a given number of primary-sector workers, increase the likelihood of getting a job. But that very fact encourages more people to choose the primary sector over the secondary sector. And indeed, so many more people choose the primary sector that the number of primary-sector workers who do not get jobs actually rises.

(c) To see the effects of an increase in the level of unemployment benefits, take the derivative of U with respect to b:

(6) $\dfrac{\partial U}{\partial b} = \dfrac{(w_p - w_s)}{(w_s - b)^2}N_p > 0.$

Thus unemployment rises when b rises. Again, the intuition is that higher unemployment benefits make the primary sector more attractive. Thus more people choose the primary sector. Since there are a fixed number of jobs in the primary sector, more people will end up unemployed.

Problem 9.12

(a) Note that $V = E[w - Cn'] = E[w] - CE[n']$ and is the expected value of the wage the worker will eventually accept if she searches more minus the expected cost of further searching. The expected cost of further searching is the expected number of jobs to be sampled multiplied by the (known) cost of sampling each job. Thus V can be interpreted as the expected value of further searching. Clearly, if the worker is offered a job that pays a wage of $\hat{w}$, where $\hat{w}$ exceeds the expected value of further searching, it is optimal to stop searching and take that job. However, if the wage offered is less than the expected value of further searching, it is optimal to reject that job and continue searching.

(b) Note that

(1) $$V = F(V)V + \int_{w=V}^{\infty} wf(w)dw - C$$

can be rewritten as

(2) $$V = \frac{\int_{w=V}^{\infty} wf(w)dw}{1-F(V)} - \frac{C}{1-F(V)}.$$

Consider the first term on the right-hand side of equation (2). The denominator is the probability that a wage drawn from the distribution will be greater than the cutoff, V. Thus this first term represents the expected wage conditional on that wage being greater than the reservation wage of V.

Now consider the second term on the right-hand side of equation (2). Since $1 - F(V)$ is the probability of drawing a wage greater than V, $1/[1 - F(V)]$ is the expected number of jobs that will need to be sampled in order to draw a job with a wage greater than V. For example, if the probability of drawing a wage greater than V is 1/2, then on average it will take two draws in order to obtain a wage greater than V. Therefore, $C/[1 - F(V)]$ is the expected cost of sampling jobs. Thus $V = E[w] - CE[n']$ must satisfy equation (2).

(c) By Leibniz's rule and the chain rule, we have

(3) $$\frac{\partial\left[\int_{w=V(C)}^{\infty} wf(w)dw\right]}{\partial C} = \frac{\partial\left[\int_{w=V(C)}^{\infty} wf(w)dw\right]}{\partial V}\frac{\partial V}{\partial C} = -Vf(V)\frac{\partial V}{\partial C}.$$

Differentiating both sides of equation (1) with respect to C and using the result in equation (3) yields

(4) $$\frac{\partial V}{\partial C} = \left[f(V)\frac{\partial V}{\partial C}\right]V + F(V)\frac{\partial V}{\partial C} - Vf(V)\frac{\partial V}{\partial C} - 1.$$

Collecting terms in equation (4) gives us

(5) $[1 - F(V)]\partial V/\partial C = -1$,

or simply

(6) $$\frac{\partial V}{\partial C} = \frac{-1}{1-F(V)}.$$

With $F(V) < 1$, $\partial V/\partial C < 0$ so that an increase in the cost of sampling jobs reduces the value of the reservation wage.

(d) A searcher will never accept a job that she has previously rejected. From equation (2), V is a constant. If a searcher rejects a job paying some wage $\hat{w}$, it must mean that $\hat{w}$ is less than V and will always be less than V. Thus the worker will never accept the job paying $\hat{w}$.

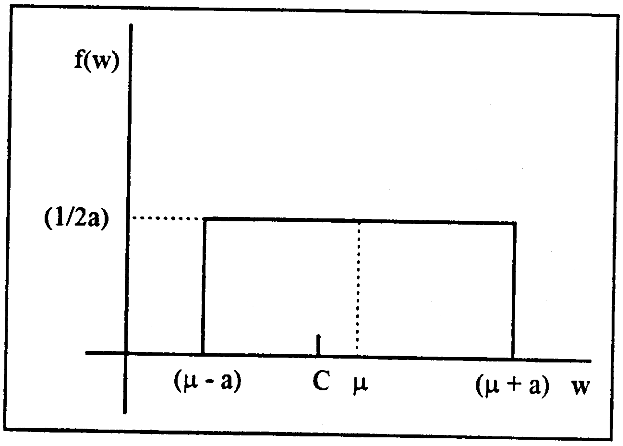
Problem 9.13

(a) The distribution of wages is depicted at right. The cost of sampling a job, C, is also depicted under the assumption that $C < \mu$. Over the interval from $(\mu - a)$ to $(\mu + a)$, this uniform distribution has a probability density function given by

(1) $f(w) = 1/2a$,

and an associated cumulative distribution function given by

(2) $F(w) = \dfrac{w - (\mu - a)}{2a}$.



As explained in the solution to part (b) of Problem 9.12, the cutoff or reservation wage, V, must satisfy

(3) $V = F(V)V + \int_{w=V}^{\mu+a} wf(w)dw - C.$

Note that we can write the upper bound of the integral as $(\mu + a)$ since $f(w) = 0$ for all $w > (\mu + a)$. Substituting equations (1) and (2) into equation (3) yields

(4) $V = \left[\dfrac{V - (\mu - a)}{2a}\right]V + \int_{w=V}^{\mu+a} (w/2a)\,dw - C.$

The value of the integral in equation (4) is

(5) $\int_{w=V}^{\mu+a} (w/2a)\,dw = \dfrac{1}{2a}\left[\dfrac{1}{2}w^2\Big|_{w=V}^{w=\mu+a}\right] = \dfrac{1}{4a}\left[(\mu + a)^2 - V^2\right].$

Substituting equation (5) into equation (4) and multiplying both sides of the resulting expression by 4a gives us

(6) $4aV = 2V^2 - 2(\mu - a)V + (\mu + a)^2 - V^2 - 4aC.$

Collecting terms yields

(7) $V^2 - 2(\mu + a)V + (\mu + a)^2 - 4aC = 0.$

Using the quadratic formula, we have

(8) $V = \dfrac{2(\mu + a) \pm \sqrt{4(\mu + a)^2 - 4(\mu + a)^2 + 16aC}}{2} = \dfrac{2(\mu + a) \pm 4\sqrt{aC}}{2}.$

We can ignore the solution with $V > (\mu + a)$ since $(\mu + a)$ is the highest possible wage. Thus V is given by

(9) $V = (\mu + a) - 2a^{1/2}C^{1/2}$.

Note that if there is no cost to sampling a job so that $C = 0$, then $V = (\mu + a)$ meaning that the worker simply keeps searching until she is offered the highest wage in the distribution. In addition, if $C = a$, then $V = (\mu + a) - 2a$ or $V = (\mu - a)$ meaning that the worker will accept any wage. Finally, if $C > a$, then $V < (\mu + a)$ and so again, the worker accepts any wage that is offered.

To see how V varies with a (which measures the dispersion of wages), use equation (9) to find the derivative of V with respect to a:

(10) $\partial V/\partial a = 1 - a^{-1/2}C^{1/2} = 1 - (C/a)^{1/2}$

With $C < a$, a rise in a increases the reservation wage, V. The fact that there are now more higher paying jobs increases the value of further searching and thus increases the cutoff wage.

Problem 9.14

(a) From equation (9.82) in the text, the rV_V locus is given by

(1) $$rV_V = -C + \frac{\alpha}{a + \alpha + 2b + 2r} A.$$

Thus an increase in b directly reduces rV_V for a given level of employment. However, α and a also depend on b and so we must examine how a rise in b affects them for a given level of employment. From equation (9.83) in the text, a, the rate at which unemployed workers find jobs, is

(2) $$a = \frac{bE}{\bar{L} - E}.$$

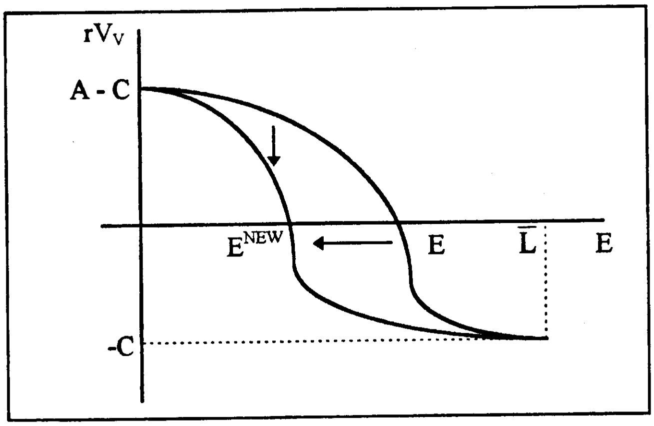
Thus a rise in b increases a for a given level of employment. From equation (1), a rise in a reduces rV_V for a given level of employment. From equation (9.86) in the text, α, the rate at which vacancies are filled, is

(3) $$\alpha = K^{1/\gamma} (bE)^{(\gamma-1)/\gamma} (\bar{L} - E)^{\beta/\gamma}.$$

With $\gamma < 1$, a rise in b reduces α for a given level of employment. From equation (1),

(4) $$\frac{\partial [rV_V]}{\partial \alpha} = \frac{A(a + \alpha + 2b + 2r) - \alpha A}{(a + \alpha + 2b + 2r)^2} = \frac{A(a + 2b + 2r)}{(a + \alpha + 2b + 2r)^2} > 0.$$

Thus a fall in α reduces rV_V for a given level of employment. In summary, all of these effects work in the same direction. The rise in the job breakup rate, b, reduces rV_V for a given level of employment. Thus the rV_V locus shifts down as shown in the figure at right.



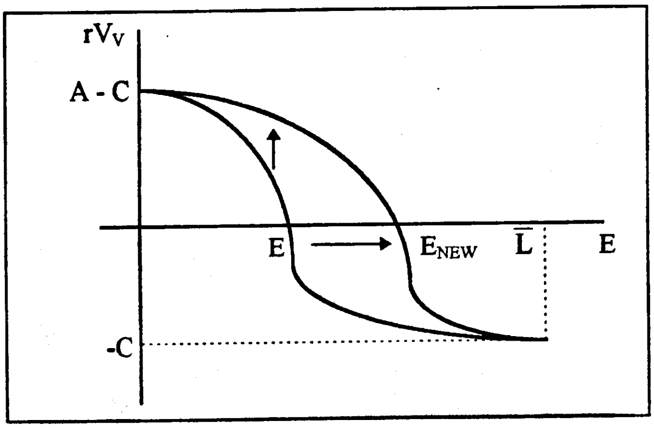
The equilibrium level of employment, which is given by the intersection of the rV_V locus with the free-entry condition that implies $rV_V = 0$, falls from E to E^{NEW}.

(b) We need to determine if the rV_V locus shifts up or down as a result of the increase in r. At a given level of employment, since a and α do not depend on r, we have

(5) $$\frac{\partial [rV_V]}{\partial r} = \frac{-2\alpha A}{(a + \alpha + 2b + 2r)^2} < 0.$$

Thus the rV_V locus shifts down; the equilibrium level of employment falls as a result of the increase in the interest rate. See the figure from part (a).

(c) At a given level of employment, a, which is given by $bE/(\bar{L} - E)$ does not depend upon K. At a given level of employment, α, which is given by $K^{1/\gamma} (bE)^{(\gamma-1)/\gamma} (\bar{L} - E)^{\beta/\gamma}$, is increasing in K. As shown in equation (4), a rise in α causes rV_V to rise for a given level of employment. Thus an increase in the effectiveness of matching, K, shifts the rV_V locus up as depicted in the figure at right. The increase in the effectiveness of matching causes the equilibrium level of employment to rise from E to E_{NEW}.



Problem 9.15

(a) Substituting equation (9.77), $(V_E - V_U) = w/(a + b + r)$, and equation (9.78), which is given by $(V_F - V_V) = (A - w)/(\alpha + b + r)$, into the assumption that fraction f of the surplus goes to the worker whereas fraction (1 - f) goes to the firm yields

$$(1)\quad (1-f)\left[\frac{w}{a+b+r}\right]=f\left[\frac{A-w}{\alpha+b+r}\right].$$

We need to solve for w. Rearranging equation (1) and obtaining a common denominator gives us

$$(2)\quad \frac{w(\alpha+b+r)-w[f(\alpha+b+r)]+w[f(a+b+r)]}{(a+b+r)(\alpha+b+r)}=\frac{fA}{\alpha+b+r},$$

or simply

$$(3)\quad w[\alpha(1 - f) + fa + b + r] = fA(a + b + r),$$

and thus w is given by

$$(4)\quad w=\frac{fA(a+b+r)}{fa+(1-f)\alpha+b+r}.$$

Substituting equation (4) into equation (9.77), $(V_E - V_U) = w/(a + b + r)$, yields

$$(5)\quad V_E-V_U=\frac{fA(a+b+r)}{[fa+(1-f)\alpha+b+r](a+b+r)}=\frac{fA}{fa+(1-f)\alpha+b+r}.$$

Equation (9.75) in the text states that rV_V equals $-C + \alpha(V_F - V_V)$. Our assumption about how the surplus is split implies that $V_F - V_V = [(1 - f)/f][V_E - V_U]$. Thus

$$(6)\quad rV_V = -C + \alpha[(1 - f)/f][V_E - V_U].$$

Substituting equation (5) into equation (6) yields

$$(7)\quad rV_V=-C+\frac{\alpha(1-f)}{f}\frac{fA}{fa+(1-f)\alpha+b+r}=-C+\frac{(1-f)\alpha}{fa+(1-f)\alpha+b+r}A.$$

It is straightforward to verify that equation (7) reduces to equation (9.82) in the text when f = 1/2.

Recognizing the fact that α and a are functions of E, and imposing $rV_V = 0$ gives us

$$(8)\quad -C+\frac{(1-f)\alpha(E)}{fa(E)+(1-f)\alpha(E)+b+r}A=0.$$

Equation (8) is analogous to equation (9.87) in the text.

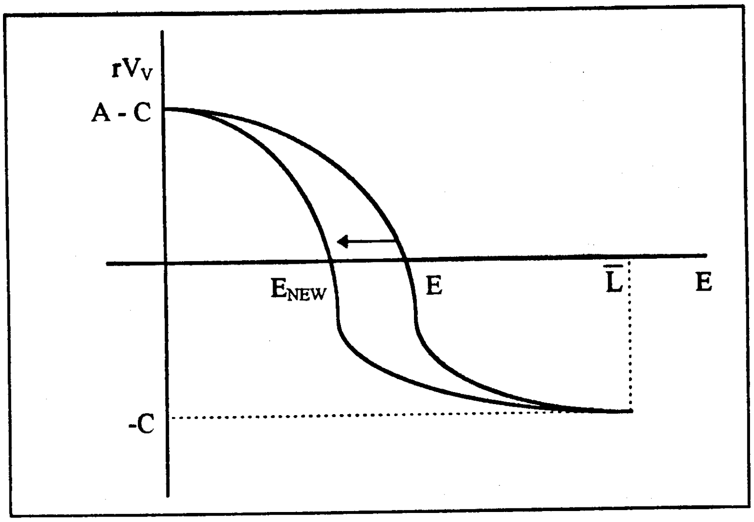
(b) For a given level of E, a and α do not depend on f. Thus we can use equation (7) to examine the effect of a change in f on the rV_V locus. We have, for a given level of employment,

$$(9)\quad \frac{\partial[rV_V]}{\partial f}=\frac{-\alpha A[fa+(1-f)\alpha+b+r]-(1-f)\alpha A(a-\alpha)}{fa+(1-f)\alpha+b+r}.$$

The sign of $\partial[rV_V/\partial f]$ will be determined by the sign of the numerator in equation (9). Simplifying that numerator, we have

$$-\alpha A[fa + (1 - f)\alpha + b + r] - (1 - f)\alpha A(a - \alpha) =$$
$$-f\alpha Aa - (1 - f)\alpha^2 A - \alpha A - \alpha A(b + r) - \alpha Aa + f\alpha Aa + (1 - f)\alpha^2 A = -\alpha A(a + b + r) < 0.$$

Thus an increase in f causes rV_V to be lower for a given level of E. That is, it causes the rV_V locus to shift down as shown in the figure at right. An increase in the fraction of the surplus that goes to the worker reduces the equilibrium level of employment. Intuitively, the only decision that a firm has is whether to enter or not. If f rises, so that the worker gets a bigger fraction of the surplus from a job, entry is less attractive. Thus the equilibrium level of employment falls.



Problem 9.16
After the fall in A, there is no reason for firms whose positions are filled to discharge their workers. Thus employment and unemployment do not change discontinuously at the time of the shock. The reduced attractiveness of hiring does cause the value of a vacancy, V_V, to fall. However, since exiting is not allowed, we do not require $V_V = 0$ and so vacancies do not change. Since employment, unemployment and vacancies are not affected at the time of the fall in A, the number of new matches, $M = KU^\beta V^\gamma$, continues to equal the flows into unemployment, bE. In summary, if we rule out entry and exit, unemployment and vacancies do not respond at all to the fall in A.

Problem 9.17
(a) In a steady state, M(U, V) = bE(N). The number of matches per unit time must equal the number of jobs that end per unit time. In addition, the number of unemployed workers is $U = \bar{L} - E(N)$ and the number of vacancies is V = N - E(N). Substituting these expressions for U and V into the matching function, $M(U, V) = KU^\beta V^\gamma$, and setting it equal to bE(N), we have

(1) $bE(N) = K[\bar{L} - E(N)]^\beta [N - E(N)]^\gamma$.

To find how E varies with N, differentiate both sides of equation (1) with respect to N:

(2) $bE'(N) = K\beta[\bar{L} - E(N)]^{\beta-1} [N - E(N)]^\gamma [-E'(N)] + K[\bar{L} - E(N)]^\beta \gamma[N - E(N)]^{\gamma-1} [1 - E'(N)]$.

Simplifying yields

(3) $$bE'(N) = \frac{K\beta[\bar{L} - E(N)]^\beta [N - E(N)]^\gamma [-E'(N)]}{\bar{L} - E(N)} + \frac{K\gamma[\bar{L} - E(N)]^\beta [N - E(N)]^\gamma [1 - E'(N)]}{N - E(N)}.$$

Using equation (1) for bE(N), equation (3) can be written as

(4) $$bE'(N) = \frac{\beta bE(N)}{\bar{L} - E(N)}[-E'(N)] + \frac{\gamma bE(N)}{N - E(N)}[1 - E'(N)].$$

Collecting the terms in E'(N) and dividing through by b yields

(5) $$E'(N)\left[1 + \frac{\beta E(N)}{\bar{L} - E(N)} + \frac{\gamma E(N)}{N - E(N)}\right] = \frac{\gamma E(N)}{N - E(N)},$$

or simply

$$(6)\ E'(N) = \frac{\dfrac{\gamma E(N)}{N-E(N)}}{1+\dfrac{\beta E(N)}{\bar{L}-E(N)}+\dfrac{\gamma E(N)}{N-E(N)}} = \frac{\gamma E(N)[\bar{L}-E(N)]}{[N-E(N)][\bar{L}-E(N)]+\beta E(N)[N-E(N)]+\gamma E(N)[\bar{L}-E(N)]}.$$

(b) Since welfare is given by $W(N) = AE(N) - NC$, the change in welfare due to a change in the number of jobs is given by

(7) $W'(N) = AE'(N) - C$.

Substituting equation (6) into equation (7) yields

$$(8)\ W'(N) = \frac{\gamma E(N)[\bar{L}-E(N)]}{[N-E(N)][\bar{L}-E(N)]+\beta E(N)[N-E(N)]+\gamma E(N)[\bar{L}-E(N)]}A - C.$$

(c) To simplify the notation, we can drop the "EQ" subscripts on N, recalling that N is the number of jobs in equilibrium. In the case of $r = 0$, equation (9.82) in the text simplifies to

(9) $C = \alpha A/(a + \alpha + 2b)$.

Substituting $a = bE(N)/[\bar{L} - E(N)]$ and $\alpha = bE(N)/V(N)$ into equation (9) gives us

$$(10)\ C = \frac{bE(N)/V(N)}{bE(N)/[\bar{L}-E(N)]+bE(N)/V(N)+2b}A.$$

Multiplying the top and bottom of the right-hand side of equation (10) by $V(N)[\bar{L} - E(N)]/b$ yields

$$(11)\ C = \frac{E(N)[\bar{L}-E(N)]}{E(N)V(N)+E(N)[\bar{L}-E(N)]+2V(N)[\bar{L}-E(N)]}A.$$

Finally, using the definition of $V(N) = N - E(N)$, equation (11) can be written as

$$(12)\ C = \frac{E(N)[\bar{L}-E(N)]}{E(N)[N-E(N)]+E(N)[\bar{L}-E(N)]+2[N-E(N)][\bar{L}-E(N)]}A.$$

(d) Using the definitions of $U(N) = \bar{L} - E(N)$ and $V(N) = N - E(N)$, equation (12) for C can be rewritten as

$$(13)\ C = \frac{E(N)U(N)}{E(N)V(N)+E(N)U(N)+2V(N)U(N)}A.$$

Substituting equation (13) into equation (8) gives us the following expression for how welfare changes with equilibrium employment:

$$(14)\ W'(N) = A\left\{\frac{\gamma E(N)U(N)}{V(N)U(N)+\beta E(N)V(N)+\gamma E(N)U(N)} - \frac{E(N)U(N)}{E(N)V(N)+E(N)U(N)+2V(N)U(N)}\right\}.$$

After obtaining a common denominator, the sign of $W'(N)$ will be determined by the sign of

$$\gamma U(N)[E(N)V(N) + E(N)U(N) + 2V(N)U(N)] - U(N)[V(N)U(N) + \beta E(N)V(N) + \gamma E(N)U(N)],$$

which simplifies to

$$(\gamma - \beta)U(N)E(N)V(N) + (\gamma - \gamma)E(N)U(N)^2 + (2\gamma - 1)V(N)U(N)^2 = U(N)V(N)[(\gamma - \beta)E(N) + (2\gamma - 1)U(N)].$$

Thus the sign of $W'(N)$ will be determined by the sign of $(\gamma - \beta)E(N) + (2\gamma - 1)U(N)$. Finally, using the fact that $U(N) = \bar{L} - E(N)$, the sign of $W'(N)$ will be determined by

(15) $\text{sign}[(\gamma - \beta)E(N) + (2\gamma - 1)\bar{L} - (2\gamma - 1)E(N)] = \text{sign}[(2\gamma - 1)\bar{L} + (1 - \gamma - \beta)E(N)]$.

If $\beta + \gamma = 1$ -- if matching has constant returns -- the sign of $W'(N)$ is determined by the sign of $(2\gamma - 1)$. If $\gamma > 1/2$, $W'(N) > 0$ and thus an increase in the equilibrium number of jobs would raise welfare or in other words, equilibrium unemployment is inefficiently high. But if $\gamma < 1/2$, $W'(N) < 0$ and thus an increase in N reduces welfare and so equilibrium unemployment is inefficiently low.

If $\gamma = 1/2$, the sign of $W'(N)$ is determined by the sign of $(1 - \gamma - \beta) = [(1/2) - \beta]$. Thus if $\beta < 1/2$ so that $\gamma + \beta < 1$ -- matching has decreasing returns -- $W'(N) > 0$ and so equilibrium unemployment is inefficiently high. But if $\beta > 1/2$ so that $\gamma + \beta > 1$ -- matching has increasing returns -- $W'(N) < 0$ and so equilibrium unemployment is inefficiently low. Thus a greater role of unemployment in creating matches (that is, a larger value of β given γ) makes it more likely that the decentralized equilibrium involves too many jobs.

SOLUTIONS TO CHAPTER 10

Problem 10.1

(a) From $m_t - p_t = c - b(E_t\, p_{t+1} - p_t)$, collecting the terms in p_t yields

$p_t\,(1 + b) = m_t - c + bE_t\, p_{t+1}$,

and so p_t is given by

$$(1)\quad p_t = \left(\frac{b}{1+b}\right)E_t p_{t+1} + \left(\frac{1}{1+b}\right)(m_t - c).$$

(b) Equation (1) holds in all periods so that we can write p_{t+1} as

$$(2)\quad p_{t+1} = \left(\frac{b}{1+b}\right)E_{t+1} p_{t+2} + \left(\frac{1}{1+b}\right)(m_{t+1} - c).$$

Taking the expected value, as of time t, of both sides of equation (2) yields

$$(3)\quad E_t p_{t+1} = \left(\frac{b}{1+b}\right)E_t p_{t+2} + \left(\frac{1}{1+b}\right)(E_t m_{t+1} - c),$$

where we have used the law of iterated projections, which states that $E_t\, E_{t+1}\, p_{t+2} = E_t\, p_{t+2}$ .If this did not hold, individuals would be expecting to revise their estimate of p_{t+2} either up or down, which would imply that their original estimate was not rational.

(c) Substituting equation (3) into equation (1) yields

$$(4)\quad p_t = \left(\frac{b}{1+b}\right)^2 E_t p_{t+2} + \left(\frac{1}{1+b}\right)\left[(m_t - c) + \left(\frac{b}{1+b}\right)(E_t m_{t+1} - c)\right].$$

Again using the fact that equation (1) holds in all periods, we can write p_{t+2} as

$$(5)\quad p_{t+2} = \left(\frac{b}{1+b}\right)E_{t+2} p_{t+3} + \left(\frac{1}{1+b}\right)(m_{t+2} - c).$$

Taking the expected value, as of time t, of both sides of equation (5) gives us

$$(6)\quad E_t p_{t+2} = \left(\frac{b}{1+b}\right)E_t p_{t+3} + \left(\frac{1}{1+b}\right)(E_t m_{t+2} - c),$$

where we have again used the law of iterated projections so that $E_t\, E_{t+2}\, p_{t+3} = E_t\, p_{t+3}$. Substituting equation (6) into equation (4) leaves us with

$$(7)\quad p_t = \left(\frac{b}{1+b}\right)^3 E_t p_{t+3} + \left(\frac{1}{1+b}\right)\left[(m_t - c) + \left(\frac{b}{1+b}\right)(E_t m_{t+1} - c) + \left(\frac{b}{1+b}\right)^2 (E_t m_{t+2} - c)\right].$$

The pattern should now be clear. We can write p_t as the following infinite sum:

$$(8)\quad p_t = \left(\frac{1}{1+b}\right)\left[(m_t - c) + \left(\frac{b}{1+b}\right)(E_t m_{t+1} - c) + \left(\frac{b}{1+b}\right)^2 (E_t m_{t+2} - c) + \left(\frac{b}{1+b}\right)^3 (E_t m_{t+3} - c) + \ldots\right].$$

(d) With output and the real interest rate constant, the price level must adjust to clear the money market. If m_{t+i} is higher, p_{t+i} will need to be higher to clear the money market. Thus if individuals expect, in period m_{t+i-1} , that m_{t+i} will be higher they will also expect p_{t+i} to be higher. Thus in period t + i - 1, expected inflation will be higher. This reduces real money demand in period t + i - 1. For a given value of m_{t+i-1} , this means that p_{t+i-1} will need to rise to clear the money market. Now go back one more period. Suppose that individuals expect, in period t + i - 2, that m_{t+i} will be higher. Then they expect, through the reasoning above, that p_{t+i-1} will be higher. Thus expected inflation in t + i - 2 will be higher, real money demand will be lower and thus p_{t+i-2} will be need to be higher to clear the money market. Reasoning backward, as soon as people expect the nominal money supply to rise in some future period, the price level will rise in the current period.

(e) Equation (8) can be written using summation notation as

$$(9)\quad p_t = \frac{1}{1+b}\sum_{i=0}^{\infty}\left(\frac{b}{1+b}\right)^i\left(E_t m_{t+i} - c\right).$$

Substituting the assumption that $E_t m_{t+i} = m_t + gi$ into equation (9) yields

$$(10)\quad p_t = \frac{1}{1+b}\sum_{i=0}^{\infty}\left(\frac{b}{1+b}\right)^i(m_t + gi - c) = \frac{1}{1+b}\left[(m_t - c)\sum_{i=0}^{\infty}\left(\frac{b}{1+b}\right)^i + g\sum_{i=0}^{\infty}i\left(\frac{b}{1+b}\right)^i\right].$$

Now we can use the facts that

$$(11)\quad \sum_{i=0}^{\infty}\left(\frac{b}{1+b}\right)^i = 1 + \left(\frac{b}{1+b}\right) + \left(\frac{b}{1+b}\right)^2 + \ldots = \frac{1}{1-[b/(1+b)]} = 1+b,$$

and

$$(12)\quad \sum_{i=0}^{\infty}i\left(\frac{b}{1+b}\right)^i = \frac{b/(1+b)}{\{1-[b/(1+b)]\}^2} = \frac{b/(1+b)}{1/(1+b)^2} = b(1+b).$$

Equation (12) uses the result that

$$(13)\quad \sum_{i=0}^{\infty}ix^i = \frac{x}{(1-x)^2}.$$

A (not entirely rigorous) way to see why (13) and thus (12) hold is to note that with $x < 1$, we have

$$(14)\quad 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}.$$

Differentiating both sides of equation (14) with respect to x (which means differentiating term by term on the left-hand side) gives us

$$(15)\quad 1 + 2x + 3x^2 + \ldots = \frac{1}{(1-x)^2}.$$

Multiplying both sides of equation (15) by x yields

$$(16)\quad x + 2x^2 + 3x^3 + \ldots = \frac{x}{(1-x)^2}.$$

Note that (16) and (13) are equivalent; the left-hand side of equation (13) is simply the left-hand side of (16) written in summation notation.

Substituting equations (11) and (12) into equation (10) yields

$$(17)\quad p_t = \frac{1}{1+b}\left[(m_t - c)(1+b) + gb(1+b)\right].$$

Thus the price level is given by

$$(18)\quad p_t = (m_t - c) + bg.$$

To see how the price level changes when money growth changes, use equation (18) to take the derivative of p_t with respect to g:

$$(19)\quad \frac{\partial p_t}{\partial g} = b > 0.$$

Thus a rise in money growth, even without a rise in the level of the current period's money supply, causes an upward jump in the current price level.

Problem 10.2

(a) Substituting the normalized, flexible-price level of output, $y_0 = 0$, into the IS equation, $y_0 = c - ar_0$, gives us $0 = c - ar_0$. Solving for the real interest rate in period 0 yields

(1) $r_0 = c/a$.

Since the nominal money stock is expected to be constant, the price level is expected to be constant and thus expected inflation from period 0 to period 1 is

(2) $E_0 [p_1] - p_0 = 0$.

The nominal interest rate in period 0, $i_0 = r_0 + [E_0 [p_1] - p_0]$, is simply equal to the real interest rate:

(3) $i_0 = c/a$.

Finally, substituting the assumptions that $m_0 = 0$ and $y_0 = 0$ as well as equation (3) into the LM curve, $m_0 - p_0 = b + hy_0 - ki_0$, yields $-p_0 = b - (ck/a)$ or simply

(4) $p_0 = (ck/a) - b$.

(b) In period 2, the economy is once again at the flexible-price equilibrium level of output, which is 0. Substituting this fact into the IS equation allows us to solve for the real interest rate in period 2:

(5) $r_2 = c/a$.

Since expected inflation from period 2 to period 3 is equal to g -- the price level is expected to rise by the same amount as the nominal money supply each period -- the nominal interest rate in period 2 is given by

(6) $i_2 = (c/a) + g$.

Since m was equal to 0 in period 0 and then increases by g in each following period, the nominal money supply in period 2 is $m_2 = 2g$. Substituting this fact as well as $y_2 = 0$ and $i_2 = (c/a) + g$ into the LM equation leaves us with

$$2g - p_2 = b - (ck/a) - kg.$$

Solving for p_2 gives us

(7) $p_2 = -b + (ck/a) + (2 + k)g$.

(c) The price level is completely unresponsive to unanticipated monetary shocks for one period. Thus the price level in period 1 does not change from its period 0 value and hence

(8) $p_1 = (ck/a) - b$.

The expectation of inflation from period 1 to period 2, $E_1 [p_2] - p_1$, is therefore

(9) $E_1 [p_2] - p_1 = -b + (ck/a) + (2 + k)g - (ck/a) + b = (2 + k)g$,

where we have used equations (7) and (8) to substitute for p_2 and p_1.

Now substitute the IS equation, $y_1 = c - ar_1$, into the LM equation, $m_1 - p_1 = b + hy_1 - ki_1$, to obtain

(10) $m_1 - p_1 = b + hc - ahr_1 - ki_1$.

By assumption, the nominal money supply in period 1 is g. In addition, $i_1 = r_1 + [E_1 [p_2] - p_1]$, which, using equation (9), is equivalent to $i_1 = r_1 + (2 + k)g$. Substituting these facts as well as equation (8) for the price level into equation (10) gives us

(11) $g - (ck/a) + b = b + hc - ahr_1 - kr_1 - (2 + k)kg$.

Simplifying and collecting the terms in r_1 yields

(12) $r_1 [ah + k] = hc + (ck/a) - g - (2 + k)kg$.

Thus the real interest rate in period 1 is given by

(13) $$r_1 = \frac{hc + (ck/a) - g - (2 + k)kg}{ah + k}.$$

Finally, substituting equations (9) and (13) into $i_1 = r_1 + [E_1 [p_2] - p_1]$ gives us

(14) $$i_1 = \frac{hc + (ck/a) - g - (2 + k)kg}{ah + k} + (2 + k)g = \frac{hc + (ck/a) - g - (2 + k)kg + (2 + k)ahg + (2 + k)kg}{ah + k}.$$

Thus the nominal interest rate in period 1 is given by

$$(15)\ i_1 = \frac{hc + (ck/a) - g + (2+k)ahg}{ah + k}.$$

(d) Using equations (15) and (3), the change in the nominal interest rate from period 0 to period 1 is

$$i_1 - i_0 = \frac{hc + (ck/a) - g + (2+k)ahg}{ah + k} - (c/a) = \frac{hc + (ck/a) - g + (2+k)ahg - hc - (ck/a)}{ah + k}.$$

Simplifying yields

$$(16)\ i_1 - i_0 = \frac{(2+k)ahg - g}{ah + k}.$$

We can determine the condition required of the parameters in order for the nominal interest rate to fall from period 0 to period 1; that is, for $i_1 - i_0 < 0$. From equation (16), this condition is

$$\frac{g[(2+k)ah - 1]}{ah + k} < 0,$$

or simply

(17) $(2 + k)ah < 1$.

The smaller is a (the elasticity of output with respect to changes in the real interest rate), the smaller is h (the income elasticity of real money demand) and the smaller is k (the interest semi-elasticity of real money demand), the more likely it is for the condition in (17) to be satisfied and thus the more likely it is for the nominal interest rate to fall in response to the monetary expansion.

For the nominal interest rate, $i = r + \pi^e$, to fall, we need the liquidity effect to outweigh the expected inflation effect. That is, we need the real interest rate to fall by more than expected inflation rises. With the price level fixed by assumption in period 1, y and i must adjust to ensure money market equilibrium. If k is small, changes in i will not affect real money demand very much. We need y to rise to increase real money demand and get it equal to the new higher real money stock. If h is small, we need y to rise a lot in order to accomplish this. If y is to rise a lot, we need -- from the IS equation -- the real interest rate to fall a lot. If furthermore, a is small, we need r to fall a lot just to generate an increase in output. Thus small values of k, h, and a all work to make the drop in r larger and thus make it more likely that i will fall.

Problem 10.3

(a) Any shock to the nominal money supply in period t + 1 is fully reflected in the price level by period t + 2. That is, the only reason the price level will change from period t + 1 to period t + 2 is if there is a non-zero realization of u in period t + 1. From the law of iterated projections, we have

(1) $E_t\,[E_{t+1}\,[p_{t+2}] - p_{t+1}] = E_t\,[p_{t+2} - p_{t+1}]$.

Since the expected value, as of period t, of u_{t+1} is zero, the price level is not expected to change from period t + 1 to period t + 2. Thus

(2) $E_t\,[E_{t+1}\,[p_{t+2}] - p_{t+1}] = 0$.

Since the LM equation must hold each period, we can write

(3) $m_{t+1} - p_{t+1} = b + hy_{t+1} - kr_{t+1} - k(E_{t+1}\,[p_{t+2}] - p_{t+1})$,

where we have substituted in for $i_{t+1} = r_{t+1} + (E_{t+1}\,[p_{t+2}] - p_{t+1})$. Taking the expected value of both sides of equation (3) yields

(4) $E_t m_{t+1} - E_t p_{t+1} = b + h\bar{y} - k\bar{r}$,

where we have used the result from equation (2) that $E_t\,[E_{t+1}\,[p_{t+2}] - p_{t+1}] = 0$. In addition, since y_{t+1} and r_{t+1} will only depend on the u_{t+1} shock, which is expected to be zero, they are expected to be equal to their average values.

(b) Rearranging equation (4), we have

(5) $E_t p_{t+1} = E_t m_{t+1} - b - h\bar{y} + k\bar{r}$.

Since $m_{t+1} = m_t + u_{t+1}$, $E_t m_{t+1} = m_t$. Using this fact and subtracting p_t from both sides of equation (5) yields

(6) $E_t p_{t+1} - p_t = (m_t - p_t) - b - h\bar{y} + k\bar{r}$.

As explained in part (a), expected inflation is equal to u_t and thus we can write

(7) $u_t = (m_t - p_t) - b - h\bar{y} + k\bar{r}$.

Substituting $m_t = m_{t-1} + u_t$ into equation (7) and rearranging to solve for p_t yields

(8) $p_t = m_{t-1} - b - h\bar{y} + k\bar{r}$.

The next step is to solve for output in period t. Rearranging the LM equation to solve for i_t yields

(9) $i_t = [b + hy_t - (m_t - p_t)]/k$.

From equation (7), we have

(10) $(m_t - p_t) = u_t + b + h\bar{y} - k\bar{r}$.

Substituting equation (10) into equation (9) gives us

(11) $i_t = \dfrac{b + hy_t - u_t - b - h\bar{y} + k\bar{r}}{k} = \dfrac{h(y_t - \bar{y}) + k\bar{r} - u_t}{k}$.

Substituting equation (11) for i_t and using the fact that $\pi_t^e = u_t$, the IS equation becomes

(12) $y_t = c - a\left[\dfrac{h(y_t - \bar{y}) + k\bar{r} - u_t}{k}\right] + au_t$.

Collecting the terms in y_t, we have

(13) $\left[\dfrac{k + ah}{k}\right] y_t = c + \dfrac{ah\bar{y} - ak\bar{r} + au_t}{k} + au_t$,

which implies

(14) $y_t = \dfrac{kc + ah\bar{y} - ak\bar{r} + au_t + kau_t}{k + ah}$,

and thus output in period t is given by

(15) $y_t = \dfrac{kc + a[h\bar{y} - k\bar{r} + (1 + k)u_t]}{k + ah}$.

In order to determine the real interest rate, rearrange the IS equation to obtain

(16) $r_t = (c/a) - (y_t/a)$.

Substituting equation (15) into equation (16) yields

(17) $r_t = \dfrac{c}{a} - \dfrac{kc + a[h\bar{y} - k\bar{r} + (1 + k)u_t]}{a(k + ah)}$,

which implies

(18) $r_t = \dfrac{ck + cah - kc - a[h\bar{y} - k\bar{r} + (1 + k)u_t]}{a(k + ah)} = \dfrac{ch - [h\bar{y} - k\bar{r} + (1 + k)u_t]}{k + ah}$.

Thus the real interest rate in period t is

(19) $r_t = \dfrac{h(c - \bar{y}) + k\bar{r} - (1 + k)u_t}{k + ah}$.

The nominal interest rate is $i_t = r_t + \pi_t^e$, where $\pi_t^e = u_t$:

(20) $i_t = r_t + u_t$.

Substituting equation (19) into equation (20) gives us

(21) $i_t = \dfrac{h(c - \bar{y}) + k\bar{r} - (1 + k)u_t + (k + ah)u_t}{k + ah} = \dfrac{h(c - \bar{y}) + k\bar{r} + (ah - 1)u_t}{k + ah}$.

(c) From equation (21), with $\pi_t^e = u_t$, we have

(22) $$i_t = \frac{h(c-\bar{y}) + k\bar{r}}{k+ah} + \frac{ah-1}{k+ah}\pi_t^e.$$

From equation (22), we can see that changes in expected inflation are not reflected one-for-one in the nominal rate. This is due to the fact that prices are completely unresponsive to the monetary disturbance for one period. This means that, in general, output and the nominal interest rate will adjust to clear the money market. In order for output to change, the real interest rate must change and therefore, in general, the nominal interest rate will not move one-for-one with inflation.

Problem 10.4

(a) Under rational expectations,

(1) $\pi_{t+1} = E_t\, \pi_{t+1} + \varepsilon_{t+1}$,

where ε_{t+1} is a disturbance that is uncorrelated with anything known at t. Now consider the regression:

(2) $i_t = a + b\pi_{t+1} + e_t$.

Using the hint in the question, the OLS estimator of b is given by

(3) $$\hat{b} = \frac{\text{cov}(i_t, \pi_{t+1})}{\text{var}(\pi_{t+1})}.$$

Using $i_t = r_t + E_t\, \pi_{t+1}$ and equation (1), we can write the covariance in the numerator as

(4) $\text{cov}(i_t, \pi_{t+1}) = \text{cov}(r_t + E_t\, \pi_{t+1}, E_t\, \pi_{t+1} + \varepsilon_{t+1})$.

Since r_t and $E_t\, \pi_{t+1}$ are uncorrelated and ε_{t+1} is uncorrelated with anything known at t, this implies

(5) $\text{cov}(i_t, \pi_{t+1}) = \text{var}(E_t\, \pi_{t+1})$.

Again using equation (1), the variance in the denominator of equation (3) can be written as

(6) $\text{var}(\pi_{t+1}) = \text{var}(E_t\, \pi_{t+1} + \varepsilon_{t+1}) = \text{var}(E_t\, \pi_{t+1}) + \text{var}(\varepsilon_{t+1})$,

where we have used the fact that $\text{cov}(E_t\, \pi_{t+1}, \varepsilon_{t+1}) = 0$. Substituting equations (5) and (6) into equation (3) allows us to write the OLS estimator as

(7) $$\hat{b} = \frac{\text{var}(E_t \pi_{t+1})}{\text{var}(E_t \pi_{t+1}) + \text{var}(\varepsilon_{t+1})} < 1.$$

The hypothesis that the real interest rate is constant, so that changes in expected inflation cause one-for-one movements in the nominal interest rate, only predicts that the coefficient on π_{t+1} should be positive and less than one, not that it will take on any specific value.

(b) Now consider a regression of the form

(8) $\pi_{t+1} = a' + b'\, i_t + e_t'$.

The OLS estimator of b' is of the form

(9) $$\hat{b}' = \frac{\text{cov}(i_t, \pi_{t+1})}{\text{var}(i_t)}.$$

The covariance in the numerator of equation (9) is still given by equation (5). Since $i_t = r_t + E_t\, \pi_{t+1}$, we can write the denominator of equation (9) as

(10) $\text{var}(i_t) = \text{var}(r) + \text{var}(E_t\, \pi_{t+1})$,

where we have used the fact that $\text{cov}(r, E_t\, \pi_{t+1}) = 0$. Substituting equations (5) and (10) into equation (9) gives us the following OLS estimator:

(11) $$\hat{b}' = \frac{\text{var}(E_t \pi_{t+1})}{\text{var}(r) + \text{var}(E_t \pi_{t+1})}.$$

The hypothesis that the real interest rate is constant, so that var(r) = 0, predicts a coefficient of one on i_t.

(c) Consider the following regression:
(12) $i_t = a + b_0\,\pi_t + b_1\,\pi_{t-1} + \ldots + b_n\,\pi_{t-n} + \varepsilon_t$.
So, for example, the coefficient b_0 represents the direct impact on i_t of a change in π_t , holding the other π's constant.

Now suppose that the behavior of actual inflation is given by
(13) $\pi_t = \rho\pi_{t-1} + e_t$.
If $i_t = r + E_t\,\pi_{t+1}$, with r constant, changes in expected inflation should cause one-for-one movements in i_t . Thus since $\pi_{t+1} = \rho\pi_t + e_{t+1}$, a change in π_t of $\Delta\pi_t$ will cause $E_t\,\pi_{t+1}$, and thus i_t , to change by $\rho\Delta\pi_t$. So we would expect $b_0 = \rho$ in the above regression.

But now, controlling for π_t , the other variables -- π_{t-1} , ..., π_{t-n} -- provide no new information about π_{t+1} . Any effect that π_{t-1} , say, has on π_{t+1} is already captured indirectly by π_{t-1}'s impact on π_t . Thus we would expect $b_1 = \ldots = b_n = 0$ in the above regression. Thus the claim is incorrect since we would have $b_0 + b_1 + \ldots + b_n = \rho$, not $b_0 + b_1 + \ldots + b_n = 1$.

Problem 10.5
(a) We have $\pi_t = p_t - p_{t-1}$ and $\pi_t^e = p_t^e - p_{t-1}$. Thus $\pi_t - \pi_t^e = (p_t - p_{t-1}) - (p_t^e - p_{t-1}) = p_t - p_t^e$. We can therefore write the Lucas supply function as
(1) $y_t = \bar{y} + b(p_t - p_t^e)$.
Setting aggregate supply equal to aggregate demand (which is given by $y_t = m_t - p_t$) gives us
(2) $m_t - p_t = \bar{y} + b(p_t - p_t^e)$.
Solving equation (2) for p_t yields
(3) $p_t = \dfrac{1}{1+b}m_t + \dfrac{b}{1+b}p_t^e - \dfrac{1}{1+b}\bar{y}$.
With rational expectations, the expected value of both sides of equation (3) must be equal. Hence
(4) $p_t^e = \dfrac{1}{1+b}\left(m_{t-1} + a\right) + \dfrac{b}{1+b}p_t^e - \dfrac{1}{1+b}\bar{y}$,
where we have used the fact that the expected value of $m_t = m_{t-1} + a + \varepsilon_t$ is equal to $m_{t-1} + a$ since ε is white noise. Subtracting equation (4) from equation (3) yields
(5) $p_t - p_t^e = \dfrac{1}{1+b}m_t - \dfrac{1}{1+b}\left(m_{t-1} + a\right) = \dfrac{1}{1+b}\left(m_t - m_{t-1} - a\right)$.
Substituting equation (5) into equation (1) gives us
(6) $y_t = \bar{y} + \dfrac{b}{1+b}\left(m_t - m_{t-1} - a\right)$.

(b) From equation (6), we can see that we also need to know a, as well as m_t and m_{t-1} , in order to determine the current level of output. Intuitively, equation (6) says that only unexpected money affects output since the difference between m_t and $(m_{t-1} + a)$ is the random shock, ε_t . However, if we don't know a, we cannot determine how much of the change in the nominal money stock from period t - 1 to period t was due to a (and thus was expected) and how much was due to ε (and thus was unexpected).

(c) Again, it must be true that with rational expectations, the expected value of both sides of equation (3) must be equal. However, the expected value of m_t is now $m_{t-1} + \rho(0) + (1 - \rho)a = m_{t-1} + (1 - \rho)a$ since private agents believe that the probability that $a = 0$ is ρ. Thus
(7) $p_t^e = \dfrac{1}{1+b}\left[m_{t-1} + (1-\rho)a\right] + \dfrac{b}{1+b}p_t^e - \dfrac{1}{1+b}\bar{y}$.

Subtracting equation (7) from equation (3) yields

(8) $p_t - p_t^e = \frac{1}{1+b}\left[m_t - m_{t-1} - (1-\rho)a\right].$

Substituting equation (8) into equation (1) gives us

(9) $y_t = \bar{y} + \frac{b}{1+b}\left[m_t - m_{t-1} - (1-\rho)a\right].$

(d) Equation (6) holds in any period in which there is no regime shift. Thus if there is no regime shift in period t - 1, we can write

(10) $y_{t-1} = \bar{y} + \frac{b}{1+b}\left(m_{t-1} - m_{t-2} - a\right).$

Subtracting equation (10) from equation (6) yields

(11) $y_t - y_{t-1} = \frac{b}{1+b}\left[\left(m_t - m_{t-1}\right) - \left(m_{t-1} - m_{t-2}\right)\right].$

Defining $\Delta y_t \equiv y_t - y_{t-1}$ and $\Delta m_t \equiv m_t - m_{t-1}$, we have

(12) $\Delta y_t = \frac{b}{1+b}\left[\Delta m_t - \Delta m_{t-1}\right].$

Equation (12) states that in the absence of regime shifts, output growth is determined by the change in money growth.

If there is a regime shift in period t, equation (9) holds. Subtracting equation (10) from equation (9) yields

$$y_t - y_{t-1} = \frac{b}{1+b}\left[\left(m_t - m_{t-1}\right) - \left(m_{t-1} - m_{t-2}\right)\right] + \frac{b}{1+b}\left[a - (1-\rho)a\right],$$

or simply

(13) $\Delta y_t = \frac{\rho ab}{1+b} + \frac{b}{1+b}\left[\Delta m_t - \Delta m_{t-1}\right].$

Under the null hypothesis of no credibility of the announcement of the regime shift, $\rho = 0$, the first term on the right-hand side of equation (13) is equal to zero. Thus if the announcement is not believed, equations (13) and (12) are identical. Thus we can run a regression of Δy_t on $[\Delta m_t - \Delta m_{t-1}]$ and a dummy variable that equals one in the period of a regime shift. The coefficient on that dummy variable will reflect the amount of credibility of the policymaker's announcement. In fact, since we will have an estimate of $b/(1 + b)$ and can determine a (the average change in the money stock before the regime shift), we can calculate an estimate of ρ from the coefficient on the dummy variable.

Problem 10.6

(a) (i) The one-period nominal interest rate is given by $i_t^1 = E_t\, \pi_{t+1}$ since the real interest rate is assumed constant at zero. Since $\pi_{t+1} = \Delta m_{t+1}$, we have

(1) $i_t^1 = E_t\, \Delta m_{t+1}$.

Since money growth is given by

(2) $\Delta m_t = k\Delta m_{t-1} + \varepsilon_t$,

and since equation (2) holds in all periods, we can write

(3) $\Delta m_{t+1} = k\Delta m_t + \varepsilon_{t+1}$.

Substituting equation (3) into equation (1), we have

(4) $i_t^1 = E_t\,[k\Delta m_t + \varepsilon_{t+1}] = k\Delta m_t$,

where we have used the fact that Δm_t is known as of time t and $E_t\,[\varepsilon_{t+1}] = 0$.

(a) (ii) The expectation, as of time t, of the nominal interest rate from period t + 1 to t + 2 is
(5) $E_t\, i_{t+1}^1 = E_t\, \pi_{t+2} = E_t\, \Delta m_{t+2}$.
Since equation (2) holds every period, we can write
(6) $\Delta m_{t+2} = k\Delta m_{t+1} + \varepsilon_{t+2}$.
Substituting equation (3) into equation (6) gives us Δm_{t+2} as a function of Δm_t :
(7) $\Delta m_{t+2} = k^2\, \Delta m_t + k\varepsilon_{t+1} + \varepsilon_{t+2}$.
Substituting equation (7) into equation (5) gives us
(8) $E_t\, i_{t+1}^1 = E_t\, [k^2\, \Delta m_t + k\varepsilon_{t+1} + \varepsilon_{t+2}\,] = k^2\, \Delta m_t$,
where we have used the fact that Δm_t is known at t and the ε's are mean-zero disturbances.

(a) (iii) Under the rational-expectations theory of the term structure, the two-period interest rate is
(9) $i_t^2 = [i^1 + E_t\, i_{t+1}^1\,]/2$.
Substituting equation (8) into equation (9), we have
(10) $i_t^2 = [i_t^1 + k^2\, \Delta m_t\,]/2$.
From equation (4), $k\Delta m_t = i_t^1$ and so equation (10) can be rewritten as
(11) $i_t^2 = [i_t^1 + k i_t^1\,]/2 = i_t^1\,(1 + k)/2$.

(a) (iv) From equation (11), a rise in k will increase the two-period interest rate, i_t^2 , for any given one-period rate. For a given level of inflation in period t, expected inflation for period t + 1 will now be higher. Thus for a given one-period interest rate in t, the one-period rate in t + 1 is expected to be higher. Therefore i_t^2 , which is the average of the one-period rate in t and the expected one-period rate in t + 1, will now be higher for a given i_t^1 .

Note that as k goes to one, so that money growth and thus inflation approach a random walk, the two-period interest rate becomes equal to the one-period interest rate. That is because with inflation a random walk, next period's inflation (and thus next period's one-period nominal rate) is expected to be equal to this period's inflation (and thus this period's one-period nominal rate).

(b) (i) Equation (4) holds in all periods and thus the actual one-period interest rate in t + 1 is
(12) $i_{t+1}^1 = k\Delta m_{t+1}$.
Substituting equation (3) into equation (12) yields
(13) $i_{t+1}^1 = k^2\, \Delta m_t + k\varepsilon_{t+1}$.
Thus
(14) $i_{t+1}^1 - i_t^1 = k^2\Delta m_t + k\varepsilon_{t+1} - k\Delta m_t = k(k - 1)\Delta m_t + k\varepsilon_{t+1}$.
From equation (11), we can write
(15) $i_t^2 - i_t^1 = [i_t^1\,(1 + k)/2] - i_t^1 = [i_t^1\,(1 + k - 2)/2]$,
or substituting in for $i_t^1 = k\Delta m_t$, we have

(16) $$i_t^2 - i_t^1 = \frac{k(k-1)\Delta m_t}{2}.$$

Using the hint in the question, the OLS estimator of b in the following regression:
(17) $i_{t+1}^1 - i_t^1 = a + b[i_t^2 - i_t^1\,] + e_{t+1}$,
is given by

(18) $$\hat{b} = \frac{\text{cov}[(i_{t+1}^1 - i_t^1), (i_t^2 - i_t^1)]}{\text{var}(i_t^2 - i_t^1)}.$$

Using equations (14) and (16), the covariance in the numerator of (18) can be written as

(19) $\text{cov}[(i^1_{t+1} - i^1_t), (i^2_t - i^1_t)] = \text{cov}\left[k(k-1)\Delta m_t + k\varepsilon_{t+1}, \dfrac{k(k-1)\Delta m_t}{2}\right].$

Since ε is white noise and $\text{var}(\Delta m_t) = \sigma_\varepsilon^2$, we have

(20) $\text{cov}[(i^1_{t+1} - i^1_t), (i^2_t - i^1_t)] = \dfrac{k^2(k-1)^2}{2}\sigma_\varepsilon{}^2.$

Using equation (16), the variance in the denominator of equation (18) can be written as

(21) $\text{var}(i^2_t - i^1_t) = \dfrac{k^2(k-1)^2}{4}\sigma_\varepsilon{}^2.$

Substituting equations (20) and (21) into equation (18) gives us

(22) $\hat{b} = \dfrac{\dfrac{k^2(k-1)^2}{2}\sigma_\varepsilon{}^2}{\dfrac{k^2(k-1)^2}{4}\sigma_\varepsilon{}^2} = 2.$

(b) (ii) With the time-varying term premium, equation (16) becomes

(23) $i^2_t - i^1_t = \dfrac{k(k-1)\Delta m_t}{2} + \theta_t.$

Using equations (14) and (23), the covariance in the numerator of equation (18) is now given by

(24) $\text{cov}[(i^1_{t+1} - i^1_t), (i^2_t - i^1_t)] = \text{cov}\left[k(k-1)\Delta m_t + k\varepsilon_{t+1}, \dfrac{k(k-1)\Delta m_t}{2} + \theta_t\right].$

Since ε and θ are white noise, this is simply

(25) $\text{cov}[(i^1_{t+1} - i^1_t), (i^2_t - i^1_t)] = \dfrac{k^2(k-1)^2}{2}\sigma_\varepsilon{}^2.$

This covariance is the same as it was without the time-varying term premium. However, the variance of $(i_t^2 - i_t^1)$ will change however. It is now given by

(26) $\text{var}(i^2_t - i^1_t) = \dfrac{k^2(k-1)^2}{4}\sigma_\varepsilon{}^2 + \sigma_\theta{}^2,$

where we have used the fact that the covariance between ε and θ is zero.

Substituting equations (25) and (26) into equation (18) gives us

(27) $\hat{b} = \dfrac{\dfrac{k^2(k-1)^2}{2}\sigma_\varepsilon{}^2}{\left(\dfrac{k^2(k-1)^2}{4}\sigma_\varepsilon{}^2\right) + \sigma_\theta{}^2} = \dfrac{2}{1 + \left(\dfrac{4\sigma_\theta{}^2}{k^2(k-1)^2}\right)}.$

(b) (iii) Since $k^2(k-1)^2$ reaches a maximum at $k = 1/2$, the OLS estimator is highest when $k = 1/2$. For $k > 1/2$, an increase in k (more persistent money growth and inflation), reduces the value of the OLS estimator. As k approaches one, so that money growth, inflation and thus the one-period nominal interest rate all approach random walks, the OLS estimator goes to zero.

Problem 10.7

As described in the text, in equilibrium, output equals $\bar{y}$ and inflation equals $\pi^* + (b/a)(y^* - \bar{y})$. Substituting these values into the loss function given by equation (10.11) in the text, which is given by

$L = (1/2)(y - y^*)^2 + (1/2)a(\pi - \pi^*)^2$, yields the following value of the loss function in equilibrium:

(1) $L^{EQ} = \frac{1}{2}(\bar{y} - y^*)^2 + \frac{1}{2}a\left[\frac{b}{a}(y^* - \bar{y})\right]^2 = \frac{1}{2}(y^* - \bar{y})^2 + \frac{1}{2}\frac{b^2}{a}(y^* - \bar{y})^2$,

or simply

(2) $L^{EQ} = \frac{1}{2}(y^* - \bar{y})^2\left[1 + \frac{b^2}{a}\right]$.

Output equals $\bar{y}$ in equilibrium, regardless of the value of a. Thus to see how the equilibrium loss varies with a, use equation (2) to take the derivative of L^{EQ} with respect to a:

(3) $\frac{\partial L^{EQ}}{\partial a} = \frac{-b^2}{2a^2}(y^* - \bar{y})^2 < 0$.

Equation (3) states that a fall in a increases L^{EQ}. That is, a reduction in the cost of inflation increases the loss to society. It is true that any given deviation in inflation from its optimal level, π^*, has a lower cost to society. However, the problem is that this causes the equilibrium level of inflation itself to be higher. Intuitively, at a given π^e, the marginal cost of additional inflation is now lower for the policymaker. For a given π^e, it then becomes optimal to set a higher inflation rate. But the public knows this and thus the level of π for which $\pi^e = \pi$ is now higher. It turns out that the fact that π^{EQ} exceeds π^* by more than it used to, outweighs the fact that any given deviation in π^{EQ} from π^* has a lower cost to society.

Problem 10.8

(a) Suppose that π differs from $\hat{\pi}$ in some period t_0. Then $\pi^e = b/a$ for all periods after t_0. Substituting this expression for expected inflation into the Lucas supply function, $y_t = \bar{y} + b(\pi_t - \pi_t^e)$, gives us output in each subsequent period:

(1) $y_t = \bar{y} + b(\pi_t - b/a)$ for all $t > t_0$.

With expected inflation now constant into the future, the equilibrium in each period is independent of the policymaker's action in the previous period. Thus we can concentrate on a representative period, t; the equilibrium in all periods will be the same. Substituting equation (1) into the policymaker's objective function for period t, $w_t = y_t - (a\pi^2/2)$, yields

(2) $w_t = \bar{y} + b(\pi_t - b/a) - a\pi_t^2/2$ for all $t > t_0$.

The first-order condition for the choice of inflation is

(3) $\partial w_t/\partial \pi_t = b - a\pi_t = 0$,

and thus the policymaker chooses

(4) $\pi_t = b/a$ for all $t > t_0$.

Since $\pi_t = \pi_t^e = b/a$, then from the Lucas supply function we have

(5) $y_t = \bar{y}$ for all $t > t_0$.

(b) To keep things simple, we can assume that the monetary authority chooses to depart from $\pi = \hat{\pi}$ in period 0. This does not alter the message. Since π has always been equal to $\hat{\pi}$, $\pi_0^e = \hat{\pi}$. Substituting this into the Lucas supply function gives us

(6) $y_0 = \bar{y} + b(\pi_0 - b/a)$.

Given the fact that the policymaker is choosing to depart from $\pi = \hat{\pi}$, the particular choice of π_0 does not affect π^e and thus the equilibrium in future periods. Thus only the current period's objective function matters to the policymaker. She will choose π_0 to maximize

(7) $w_0 = \bar{y} + b(\pi_0 - \hat{\pi}) - (a\pi_0^2/2)$.

The first-order condition for the choice of π_0 is

(8) $\partial w_0 / \partial \pi_0 = b - a\pi_0 = 0$,
and thus the policymaker chooses
(9) $\pi_0 = b/a$.

With this choice of inflation, using the Lucas supply function, output in period 0 is given by
(10) $y_0 = \bar{y} + b[(b/a) - \hat{\pi}]$.
Substituting equations (9) and (10) into the policymaker's objective function, $w_0 = y_0 - (a\pi_0^2/2)$, yields
(11) $w_0 = \bar{y} + (b^2/a) - b\hat{\pi} - (b^2/2a)$,
or simply
(12) $w_0 = \bar{y} + (b^2/2a) - b\hat{\pi}$.

As shown in part (a), in all subsequent periods after the policymaker has deviated, $\pi_t = b/a$ and $y_t = \bar{y}$. Substituting these values of output and inflation into the objective function, $w_t = y_t - (a\pi^2/2)$, gives us
(13) $w_t = \bar{y} - (b^2/2a)$ for all $t > 0$.
Thus the policymaker's lifetime objective function if she deviates is given by

(14) $$W^D = \bar{y} + (b^2/2a) - b\hat{\pi} + \sum_{t=1}^{\infty}\beta^t\left[\bar{y} - (b^2/2a)\right].$$

Pulling the $[\bar{y} - (b^2/2a)]$ out of the summation sign and using the fact that, since $\beta < 1$, we have

(15) $$\sum_{t=1}^{\infty}\beta^t = \beta + \beta^2 + \beta^3 + \ldots = \beta\left(1 + \beta + \beta^2 + \ldots\right) = \beta/(1-\beta),$$

we can write the lifetime objective function as

(16) $$W^D = \bar{y} + \frac{b^2}{2a} - b\hat{\pi} + \left(\frac{\beta}{1-\beta}\right)\left[\bar{y} - \frac{b^2}{2a}\right] = \left(1 + \frac{\beta}{1-\beta}\right)\bar{y} - b\hat{\pi} + \left(1 - \frac{\beta}{1-\beta}\right)\frac{b^2}{2a},$$

or simply

(17) $$W^D = \left(\frac{1}{1-\beta}\right)\bar{y} - b\hat{\pi} + \left(\frac{1-2\beta}{1-\beta}\right)\frac{b^2}{2a}.$$

If the policymaker chooses $\pi = \hat{\pi}$ every period, output will be equal to $\bar{y}$ every period. The policymaker's objective function in each period is therefore given by
(18) $w_t = \bar{y} - (a\hat{\pi}^2/2)$.
Thus the policymaker's lifetime objective function if she does not deviate is given by

(19) $$W^{ND} = \sum_{t=0}^{\infty}\beta^t\left[\bar{y} - (a\hat{\pi}^2/2)\right].$$

Pulling the $[\bar{y} - (a\hat{\pi}^2/2)]$ out of the summation sign and using the fact that, since $\beta < 1$, we have

(20) $$\sum_{t=0}^{\infty}\beta^t = 1 + \beta + \beta^2 + \ldots = 1/(1-\beta),$$

we can write the lifetime objective function as

(21) $$W^{ND} = \left(\frac{1}{1-\beta}\right)\left[\bar{y} - \frac{a\hat{\pi}^2}{2}\right].$$

(c) One way of solving the problem is to calculate the benefit and cost of deviating as a function of $\hat{\pi}$ and the other parameters. We can then examine the range of $\hat{\pi}$'s over which the cost exceeds the benefit and thus the range of $\hat{\pi}$'s over which the policymaker will choose not to deviate from $\pi = \hat{\pi}$.

The benefit of departing from $\pi = \hat{\pi}$ in some period t_0 is that welfare in period t_0 is $\bar{y} + (b^2/2a) - b\hat{\pi}$ [see equation (12)] rather than $\bar{y} - (a\hat{\pi}^2/2)$ [see equation (18)]. Thus the benefit of deviating, B, is

$$B = \bar{y} + (b^2/2a) - b\hat{\pi} - \bar{y} + (a\hat{\pi}^2/2),$$

or simply

(22) $B = (b^2/2a) + (a\hat{\pi}^2/2) - b\hat{\pi}$.

The cost of deviating is that in all periods subsequent to t_0, welfare will be $\bar{y} - (b^2/2a)$ [see equation (13)] rather than $\bar{y} - (a\hat{\pi}^2/2)$. Thus the cost of deviating in each future period is $\bar{y} - (a\hat{\pi}^2/2) - \bar{y} + (b^2/2a)$ or simply $(b^2/2a) - (a\hat{\pi}^2/2)$. The total cost of deviating, discounted to time t_0 is

(23) $C = \sum_{t=t_0+1}^{\infty} \beta^{t-t_0}\left[(b^2/2a) - (a\hat{\pi}^2/2)\right] = \left[(b^2/2a) - (a\hat{\pi}^2/2)\right]\left(\beta + \beta^2 + \beta^3 + \ldots\right).$

Substituting the result in equation (15) into equation (23) gives us the following cost of deviating:

(24) $C = \left(\dfrac{\beta}{1-\beta}\right)\left[\dfrac{b^2}{2a} - \dfrac{a\hat{\pi}^2}{2}\right].$

We can plot the benefit and cost from deviating as a function of $\hat{\pi}$. First, we will deal with the benefit from deviating. From equation (22), we have

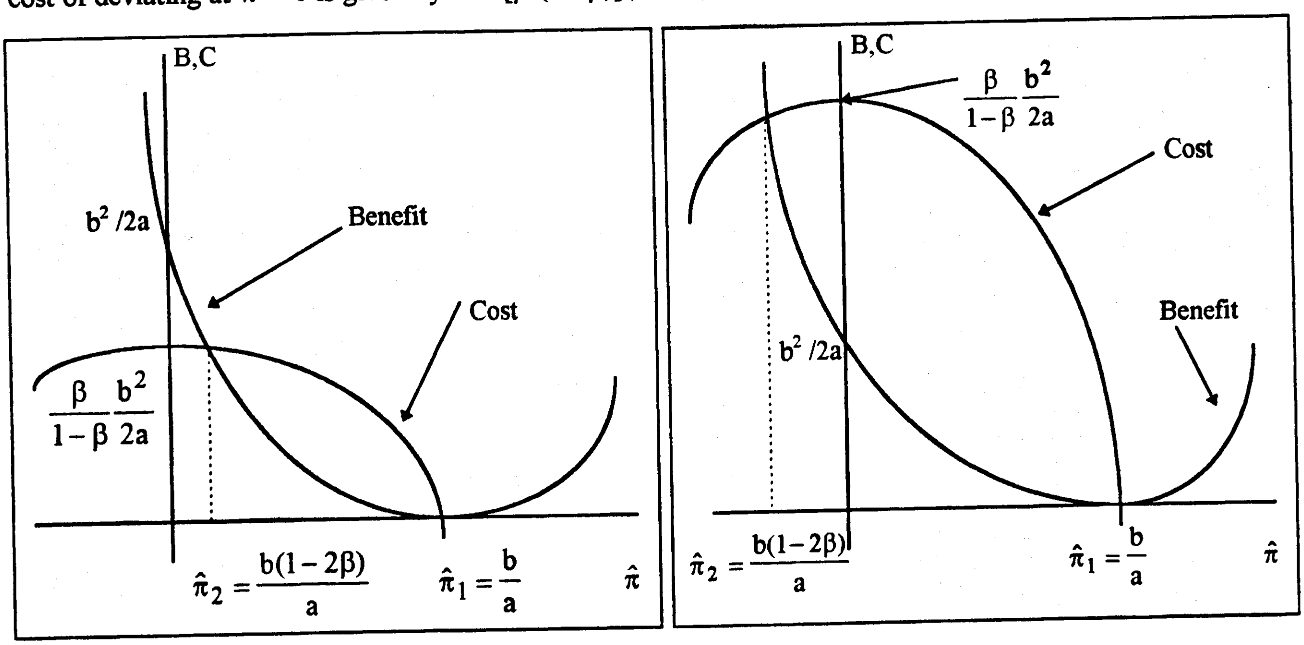
(25) $\partial B/\partial\hat{\pi} = a\hat{\pi} - b$, and (26) $\partial^2 B/\partial\hat{\pi}^2 = a > 0$.

Thus B is a parabola that reaches a minimum at $\hat{\pi} = b/a$. From equation (22), at $\hat{\pi} = 0$, $B = b^2/2a$. Finally, at its minimum at $\hat{\pi} = b/a$, $B = (b^2/2a) + (b^2/2a) - (b^2/a) = 0$. B, the benefit from deviating as a function of $\hat{\pi}$, is plotted in both figures below.

Now dealing with the cost of deviation, we have from equation (24)

(27) $\partial C/\partial\hat{\pi} = -[\beta/(1-\beta)]a\hat{\pi}$, and (28) $\partial^2 C/\partial\hat{\pi}^2 = -[\beta/(1-\beta)]a < 0$.

Thus C is an inverted parabola that reaches a maximum at $\hat{\pi} = 0$. From equation (24), the value of the cost of deviating at $\hat{\pi} = 0$ is given by $C = [\beta/(1-\beta)](b^2/2a)$. In addition, at $\hat{\pi} = b/a$, $C = 0$.



The case of $\beta < 1/2$ -- so that $\beta/(1-\beta) < 1$ -- is depicted in the left figure. The case of $\beta > 1/2$ -- so that $\beta/(1-\beta) > 1$ is depicted in the right figure.

We need to solve for the values of $\hat{\pi}$ where the benefit of deviating equals the cost of deviating. Setting the right-hand sides of equations (22) and (24) equal to each other yields

$$\frac{b^2}{2a} + \frac{a\hat{\pi}^2}{2} - b\hat{\pi} = \frac{\beta}{1-\beta}\left[\frac{b^2}{2a} - \frac{a\hat{\pi}^2}{2}\right] \quad \Rightarrow \quad \left(1 + \frac{\beta}{1-\beta}\right)\frac{a\hat{\pi}^2}{2} - b\hat{\pi} + \left(1 - \frac{\beta}{1-\beta}\right)\frac{b^2}{2a} = 0,$$

or simply

$$(29)\quad \left(\frac{1}{1-\beta}\right)\frac{a\hat{\pi}^2}{2} - b\hat{\pi} + \left(\frac{1-2\beta}{1-\beta}\right)\frac{b^2}{2a} = 0.$$

Multiplying both sides of equation (29) by $(1 - \beta)2a$ gives us an equivalent condition for $B = C$:

$$(30)\quad a^2\hat{\pi}^2 - 2a(1-\beta)b\hat{\pi} + (1-2\beta)b^2 = 0.$$

Using the quadratic formula, we have

$$(31)\quad \hat{\pi} = \frac{2ab(1-\beta) \pm \sqrt{4a^2b^2(1-\beta)^2 - 4a^2b^2(1-2\beta)}}{2a^2} = \frac{2ab(1-\beta) \pm \sqrt{4a^2b^2\left[1 - 2\beta + \beta^2 - 1 + 2\beta\right]}}{2a^2}.$$

Some further algebra yields

$$(32)\quad \hat{\pi} = \frac{2ab(1-\beta) \pm 2ab\beta}{2a^2} = \frac{b(1-\beta) \pm b\beta}{a},$$

and thus finally

$$(33)\quad \hat{\pi}_1 = \frac{b(1-\beta) + b\beta}{a} = \frac{b}{a}, \quad \text{and} \quad (34)\quad \hat{\pi}_2 = \frac{b(1-\beta) - b\beta}{a} = \frac{b(1-2\beta)}{a}.$$

These two values of $\hat{\pi}$ for which the benefit of deviating just equals the cost of deviating are depicted in the figures above. Note that for the case of $\beta > 1/2$ -- the figure on the right -- $\hat{\pi}_2$ is negative and is thus not relevant. We can now interpret the figures.

For the case of $\beta > 1/2$ -- depicted in the figure on the right -- the cost of deviating exceeds the benefit of deviating for any $\hat{\pi}$ such that $0 \le \hat{\pi} < b/a$. With these values of the parameters, the policymaker will choose not to deviate from $\pi = \hat{\pi}$. Right at $\hat{\pi} = b/a$, the policymaker is indifferent and in fact at $\hat{\pi} = b/a$, deviating is actually the same as producing $\pi = \hat{\pi}$. Finally, for any value of $\hat{\pi} > b/a$, the benefit of deviating exceeds the cost of deviating and hence the policymaker will in fact deviate from $\pi = \hat{\pi}$.

For the case of $\beta < 1/2$ -- depicted in the figure on the left -- the cost of deviating exceeds the benefit of deviating for any value of $\hat{\pi}$ such that $[b(1 - 2\beta)]/a < \hat{\pi} < b/a$. With these values of the parameters, the policymaker will choose not to deviate from $\pi = \hat{\pi}$. Right at $\hat{\pi} = b/a$ and $\hat{\pi} = [b(1 - 2\beta)]/a$, the policymaker is indifferent. Finally, for any value of $\hat{\pi} < [b(1 - 2\beta)]/a$ or $\hat{\pi} > b/a$, the benefit of deviating exceeds the cost of deviating and hence the policymaker will in fact deviate from $\pi = \hat{\pi}$.

For the policymaker to actually set $\pi = 0$ if $\hat{\pi} = 0$, we would need the cost of deviating to exceed the benefit of deviating, evaluated at $\hat{\pi} = 0$. From our earlier discussion, we know this will be true as long as $\beta > 1/2$. Thus regardless of the values of a and b, the policymaker will choose to set inflation to zero if $\hat{\pi} = 0$ as long as the discount rate is greater than 1/2.

Problem 10.9

(a) We can use the same technique as in part (c) of the solution to Problem 10.8. We can examine the range of $\hat{\pi}$'s over which the cost of deviating from setting $\pi = \hat{\pi}$ exceeds the benefit of deviating. This gives the range of $\hat{\pi}$'s over which the policymaker chooses $\pi = \hat{\pi}$ each period. The benefit from deviating, B, is the same as it was in Problem 10.8. Thus we have

$$(1)\quad B = (b^2/2a) + (a\hat{\pi}^2/2) - b\hat{\pi}.$$

The cost of deviating in some period is that in the following period, $\pi^e = b/a$ rather than $\pi^e = \hat{\pi}$. As shown in part (a) of the solution to Problem 10.8, when $\pi^e = b/a$, the policymaker chooses $\pi = b/a$. Thus output is equal to $\bar{y}$ in the following period. Note that since the policymaker chooses $\pi = \pi^e$ in the period after deviating, expected inflation reverts to $\pi^e = \hat{\pi}$ in all subsequent periods. Thus there is only a one-period cost to deviating in this setup. Specifically, the cost is that in the period after deviating, the value of the policymaker's objective function is $\bar{y} - (b^2/2a)$ rather than $\bar{y} - (a\hat{\pi}^2/2)$. Discounting that back to the period in which the deviation occurs yields the following cost, C:

(2) $C = \beta[\bar{y} - (a\hat{\pi}^2/2) - \bar{y} + (b^2/2a)] = \beta[(b^2/2a) - (a\hat{\pi}^2/2)]$.

We can now plot the benefit and cost of deviating as a function of $\hat{\pi}$. The benefit from deviating is the same as in Problem 10.8 and so we can concentrate on the cost. From equation (2):

(3) $\partial C/\partial\hat{\pi} = -\beta a\hat{\pi}$, and (4) $\partial^2 C/\partial\hat{\pi}^2 = -\beta a < 0$.

Thus C is an inverted parabola that reaches a maximum at $\hat{\pi} = 0$. From equation (2), the value of the cost of deviating at $\hat{\pi} = 0$ is given by $C = \beta(b^2/2a) < (b^2/2a)$ since $\beta < 1$. The next step is to solve for the values of $\hat{\pi}$ where the benefit of deviating equals the cost of deviating. Setting the right-hand sides of equations (1) and (2) equal to each other yields

$$\frac{b^2}{2a} + \frac{a\hat{\pi}^2}{2} - b\hat{\pi} = \frac{\beta b^2}{2a} - \frac{\beta a\hat{\pi}^2}{2} \quad \Rightarrow \quad \frac{(1+\beta)a}{2}\hat{\pi}^2 - b\hat{\pi} + \frac{(1-\beta)b^2}{2a} = 0. \qquad (5)$$

Multiplying both sides of equation (5) by 2a gives us an equivalent condition for B = C:

(6) $(1 + \beta)a^2\hat{\pi}^2 - 2ab\hat{\pi} + (1 - \beta)b^2 = 0$.

Using the quadratic formula, we have

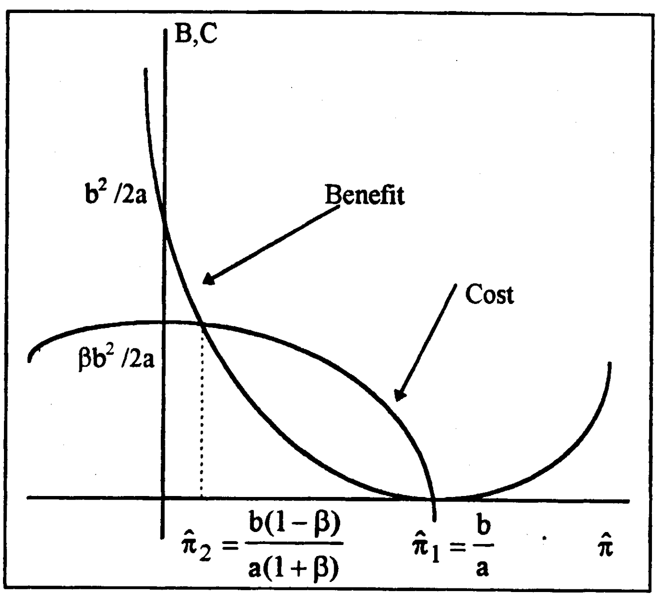
$$(7)\quad \hat{\pi} = \frac{2ab \pm \sqrt{4a^2b^2 - 4a^2b^2(1+\beta)(1-\beta)}}{2a^2(1+\beta)} = \frac{2ab \pm \sqrt{4a^2b^2\left[1 - 1 + \beta^2\right]}}{2a^2(1+\beta)}.$$

Some further algebra yields

$$(8)\quad \hat{\pi} = \frac{2ab \pm 2ab\beta}{2a^2(1+\beta)} = \frac{b(1\pm\beta)}{a(1+\beta)},$$

and thus finally

$$(9)\quad \hat{\pi}_1 = \frac{b(1+\beta)}{a(1+\beta)} = \frac{b}{a}, \qquad \text{and} \qquad (10)\quad \hat{\pi}_2 = \frac{b(1-\beta)}{a(1+\beta)}.$$



From the figure at left, we can see that the cost of deviating from $\pi = \hat{\pi}$ exceeds the benefit from deviating for any $\hat{\pi}$ such that

$b(1 - \beta)/a(1 + \beta) < \hat{\pi} < b/a$.

With these values of the parameters, the policymaker will choose not to deviate. For any value of $\hat{\pi}$ greater than b/a or less than $b(1 - \beta)/a(1 + \beta)$, the benefit from deviating exceeds the cost of deviating and hence the policymaker will in fact deviate from $\pi = \hat{\pi}$.

(b) Again, we will employ the same technique. The benefit from deviating remains the same; it is given by equation (1), $B = (b^2/2a) + (a\hat{\pi}^2/2) - b\hat{\pi}$.

We need to determine the cost of deviating for the policymaker. Suppose the policymaker deviates in some period t. Then in period t + 1, $\pi_{t+1}^e = \pi_0 > b/a$. We can also write this as $\pi_{t+1}^e = b/a + x$, $x > 0$. The variable x will capture the extent of the punishment for deviating. When the policymaker takes expected inflation as given, she chooses to set inflation equal to b/a. Thus, using the Lucas supply function, output in period t + 1, the period after deviating, is

(11) $y_{t+1} = \bar{y} + b[(b/a) - (b/a) - x] = \bar{y} - bx$.

Thus output is below the natural rate the period after deviating. The value of the policymaker's objective function in period t + 1 is

(12) $w_{t+1} = y_{t+1} - (a\pi^2/2) = \bar{y} - bx - (b^2/2a)$.

Thus the cost of deviating in period t + 1 is that welfare is given by (12) rather than being equal to $\bar{y} - (a\hat{\pi}^2/2)$. Discounting this back to period t, we have the cost in period t + 1:

(13) $C_{t+1} = \beta[\bar{y} - (a\hat{\pi}^2/2) - \bar{y} + bx + (b^2/2a)]$,

or simply

(14) $C_{t+1} = \beta[bx - (a\hat{\pi}^2/2) + (b^2/2a)]$.

Now consider the situation in period t + 2, two periods after a deviation. Expected inflation equals b/a. Taking expected inflation as given, the policymaker chooses to set inflation equal to b/a. Thus output is at the natural rate. The value of the policymaker's objective function in t + 2 is

(15) $w_{t+2} = y_{t+2} - (a\pi^2/2) = \bar{y} - (b^2/2a)$.

Thus the cost of deviating in period t + 2 is that welfare is equal to $\bar{y} - (b^2/2a)$ rather than $\bar{y} - (a\hat{\pi}^2/2)$. Discounting this back to period t, we have the cost in period t + 2:

(16) $C_{t+2} = \beta^2[\bar{y} - (a\hat{\pi}^2/2) - \bar{y} + (b^2/2a)] = \beta^2[(b^2/2a) - (a\hat{\pi}^2/2)]$.

In period t + 3, since actual inflation last period was equal to expected inflation last period, expected inflation reverts to $\hat{\pi}$ and there is no further cost to the deviation in period t.

Thus the total cost of the deviation is

(17) $C = \beta[bx - (a\hat{\pi}^2/2) + (b^2/2a)] + \beta^2[(b^2/2a) - (a\hat{\pi}^2/2)]$,

or simply

(18) $C = \beta bx + \beta(1+\beta)[(b^2/2a) - (a\hat{\pi}^2/2)]$.

From equation (18),

(19) $\partial C/\partial\hat{\pi} = \beta(1+\beta)[-2a\hat{\pi}/2] = -\beta(1+\beta)a\hat{\pi}$, and (20) $\partial^2 C/\partial\hat{\pi}^2 = -\beta(1+\beta)a < 0$.

Thus C is an inverted parabola that reaches a maximum at $\hat{\pi} = 0$. The value of the cost of deviating at $\hat{\pi} = 0$ is given by $\beta bx + \beta(1 + \beta)(b^2/2a)$. From earlier analysis, we know that the benefit of deviating at $\hat{\pi} = 0$ is $b^2/2a$. Thus if the value of x, the excess punishment, is high enough, the cost of deviating at $\hat{\pi} = 0$ will exceed the benefit and there can be an equilibrium with zero inflation. Specifically, we need the following condition to hold:

(20) $\beta bx + \beta(1 + \beta)(b^2/2a) > b^2/2a$,

or

(21) $\beta bx > (b^2/2a)[1 - \beta(1 + \beta)]$,

or simply

(22) $x > (b^2/2a)[1 - \beta(1 + \beta)]/\beta b$.

We can determine the value of $\hat{\pi}$ at which C = 0. From equation (18), C = 0 when

(23) $\beta(1+\beta)[(a\hat{\pi}^2/2) - (b^2/2a)] = \beta bx$,

which implies

(24) $(a\hat{\pi}^2/2) - (b^2/2a) = bx/(1+\beta)$,

and thus

(25) $\hat{\pi}^2 = [2bx/a(1+\beta)] + (b^2/a^2)$.

Therefore C = 0 when

(26) $\hat{\pi} = \sqrt{\dfrac{2bx}{a(1+\beta)} + \dfrac{b^2}{a^2}} > \dfrac{b}{a}$.

Thus the cost of deviating is equal to zero at a value of $\hat{\pi}$ greater than the one for which the benefit of deviating is equal to zero (which is $\hat{\pi} = b/a$). The values of $\hat{\pi}$ for which it is an equilibrium for the policymaker not to deviate are those -- just as in part (a) -- where the cost of deviating exceeds the benefit. The basic idea here is that higher values of x lead to a wider range of $\hat{\pi}$'s for which the cost exceeds the benefit and thus a wider range of $\hat{\pi}$'s for which the policymaker does not deviate.

(c) As we have shown previously, if the policymaker takes expected inflation as given, she chooses inflation equal to b/a. Thus if $\pi^e = b/a$, the policymaker chooses $\pi = b/a$, so that the public's expectation is fulfilled and output is at the natural rate. There is no incentive for the policymaker to choose a different inflation rate and there is no incentive for the public to change its expectation of inflation and thus $\pi = \pi^e = b/a$ will be an equilibrium for any a > 0, b > 0.

Problem 10.10

Consider the situation in the last period, denoted T. The policymaker's choice of π has no effect on next period's expected inflation; there is no next period. Thus the policymaker's problem in the final period is to take expected inflation as given and choose π in order to maximize the period T objective function. From previous analysis in the solution to Problem 10.8, we know that the policymaker's choice of inflation in this type of situation is $\pi_T = b/a$. Since the public knows how the policymaker behaves, expected inflation also equals b/a and thus output equals $\bar{y}$.

Now consider the situation in period T - 1. The important point is that the policymaker knows her choice of π_{T-1} will have no bearing on what happens the next and final period. Regardless of the level of π she chooses in period T - 1, expected inflation next period will be b/a, as described above. Since the policymaker's problem has no impact on the future, she chooses π, taking π^e as given, in order to maximize the period T - 1 objective function. Again, the optimal choice is $\pi_{T-1} = b/a$. The public knows this and so $\pi_{T-1}^e = b/a$ and thus output in period T - 1 equals $\bar{y}$.

Working backward, the same thing happens each period. The policymaker knows that expected inflation the following period will be b/a regardless of what she does this period. Thus she acts to maximize the one-period objective function and chooses $\pi = b/a$, which results in output equal to the natural rate. Therefore the unique equilibrium for all periods is $\pi_t^e = \pi_t = b/a$ and $y_t = \bar{y}$.

Problem 10.11

(a) The policymaker wants to choose $\hat{\pi}_1$ and $\hat{\pi}_2$ in order to maximize the objective function given by

(1) $W = E[b(\pi_1 - \pi_1^e) + c\pi_1 - (a\pi_1^2/2) + b(\pi_2 - \pi_2^e) + c\pi_2 - (a\pi_2^2/2)]$.

Substituting $\pi_t = \hat{\pi}_t + \varepsilon_t$ and $\pi_2^e = \alpha + \beta\pi_1$ into the objective function given in equation (1) yields

(2) $W = E\{b(\hat{\pi}_1 + \varepsilon_1 - \pi_1^e) + c(\hat{\pi}_1 + \varepsilon_1) - [a(\hat{\pi}_1 + \varepsilon_1)^2/2] + b[\hat{\pi}_2 + \varepsilon_2 - \alpha - \beta(\hat{\pi}_1 + \varepsilon_1)]$
$+ c(\hat{\pi}_2 + \varepsilon_2) - [a(\hat{\pi}_2 + \varepsilon_2)^2/2]\}$.

The first-order condition for the policymaker's choice of $\hat{\pi}_2$ is

(3) $\partial W/\partial\hat{\pi}_2 = E[b + c - a\hat{\pi}_2] = 0$.

Solving for $\hat{\pi}_2$, noting that c is not uncertain from the policymaker's perspective, we have

(4) $\hat{\pi}_2 = (b + c)/a$.

Substituting equation (4) into the expected value of the policymaker's second-period objective function yields

$$(5)\ \ E[w_2] = E\left[b\left(\frac{b+c}{a} + \varepsilon_2 - \pi_2^e\right) + c\left(\frac{b+c}{a} + \varepsilon_2\right) - \frac{a\left[(b+c)/a + \varepsilon_2\right]^2}{2}\right].$$

Since this expectation is being taken with respect to the policymaker's information set, c is not random. Thus equation (5) can be rewritten as

$$(6)\ \ E[w_2] = \frac{b(b+c)}{a} - b\pi_2^e + \frac{c(b+c)}{a} - \frac{a}{2}\left[\frac{(b+c)^2}{a^2} + \frac{2(b+c)E[\varepsilon_2]}{a} + E[\varepsilon_2^{\,2}]\right].$$

Note that $E[\varepsilon_2] = 0$ and $E[\varepsilon_2^2] = \sigma_\varepsilon^2$. Thus equation (6) simplifies to

$$(7)\ \ E[w_2] = \frac{(b+c)^2}{a} - b\pi_2^e - \frac{(b+c)^2}{2a} - \frac{a\sigma_\varepsilon^{\,2}}{2},$$

and thus finally

$$(8)\ \ E[w_2] = \frac{(b+c)^2}{2a} - \frac{a\sigma_\varepsilon^{\,2}}{2} - b\pi_2^e.$$

(b) Using equation (2), the first-order condition for the policymaker's choice of $\hat{\pi}_1$ is

(9) $\partial W/\partial\hat{\pi}_1 = E[b + c - a\hat{\pi}_1 - b\beta] = 0$.

Since nothing on the right-hand side of equation (9) is uncertain for the policymaker, solving for $\hat{\pi}_1$ gives us

(10) $\hat{\pi}_1 = [b(1 - \beta) + c]/a$.

(c) π_1 and π_2 are linear functions of c and ε, which are normal random variables. Thus π_1 and π_2 are also normal random variables. Therefore we can use the following formula for the conditional expectation of a normal:

$$(11)\ \ E\left[\pi_2 \middle| \pi_1\right] = E[\pi_2] + \frac{\text{cov}(\pi_2, \pi_1)}{\text{var}(\pi_1)}\left[\pi_1 - E[\pi_1]\right].$$

Equation (11) is intuitive. Suppose that π_1 and π_2 have a positive covariance which means that π_2 tends to be above its mean when π_1 is above its mean. Then if we observe a realization of π_1 greater than its expected value, the second term on the right-hand side of equation (11) is positive. This means that given this realization of π_1, we should expect the realization of π_2 to be greater than its unconditional mean of $E[\pi_2]$.

We need to solve for $E[\pi_1]$, $E[\pi_2]$, $\text{cov}(\pi_2, \pi_1)$ and $\text{var}(\pi_1)$. We know that $\pi_1 = \hat{\pi}_1 + \varepsilon_1$ and thus

(12) $E[\pi_1] = E[\hat{\pi}_1] + E[\varepsilon_1] = [b(1 - \beta) + \overline{c}]/a$.

We have used the fact that the public knows that the policymaker will choose $\hat{\pi}_1$ according to equation (10). Thus with rational expectations, the expected value of $\hat{\pi}_1$ must equal the expected value of the right-hand side of equation (10). In addition, we have used the fact that $E[\varepsilon_1] = 0$.
Since $\pi_2 = \hat{\pi}_2 + \varepsilon_2$, we have
(13) $E[\pi_2] = E[\hat{\pi}_2] + E[\varepsilon_2] = (b + \bar{c})/a$.
Again, we have used the fact that the public knows that the policymaker will choose $\hat{\pi}_2$ according to equation (4). Thus with rational expectations, the expected value of $\hat{\pi}_2$ must equal the expected value of the right-hand side of equation (4). In addition, we have used the fact that $E[\varepsilon_2] = 0$.

Now we need to find the variance of inflation in period 1:
(14) $\text{var}(\pi_1) = \text{var}(\hat{\pi}_1 + \varepsilon_1) = \text{var}([b(1 - \beta) + c]/a + \varepsilon_1)$.
Since c and ε_1 are independent, we have
(15) $\text{var}(\pi_1) = (1/a^2)\sigma_C^2 + \sigma_\varepsilon^2$.
Finally, the covariance between π_1 and π_2 is given by
(16) $\text{cov}(\pi_1, \pi_2) = \text{cov}([b(1 - \beta) + c]/a + \varepsilon_1, (b + c)/a + \varepsilon_2)$.
Since ε_1, ε_2 and c are independent, this covariance is equal to
(17) $\text{cov}(\pi_1, \pi_2) = \text{cov}(c/a, c/a) = \text{var}(c/a) = (1/a^2)\sigma_C^2$.
Substituting equations (12), (13), (15) and (17) into equation (11) yields

$$(18)\quad E\left[\pi_2 \middle| \pi_1\right] = \frac{(b+\bar{c})}{a} + \frac{(1/a^2)\sigma_C^2}{(1/a^2)\sigma_C^2 + \sigma_\varepsilon^2}\left[\pi_1 - \frac{b(1-\beta)+\bar{c}}{a}\right],$$

and thus β is given by

$$(19)\quad \beta = \frac{(1/a^2)\sigma_C^2}{(1/a^2)\sigma_C^2 + \sigma_\varepsilon^2}.$$

The intuition behind equation (18) is as follows. The public wants to form its expectation of inflation in period 2, given its observation of inflation in period 1. In order to do so, the public would like to know for sure what the policymaker's taste for inflation, c, is. The problem is that actual inflation in period 1 does depend on what the true c is, but it also depends upon the random, unobservable ε shock. Now, if the public sees a π_1 greater than its expected value of $[b(1 - \beta) + \bar{c}]/a$, it knows this could be due to a policymaker with a higher than average c. If this is the case, the public should revise upward its estimate of π_2 from $[b + \bar{c}]/a$, its unconditional mean. However, the fact that π_1 is greater than its expected value could also be due to a positive realization of ε_1. If this is the case, it should have no bearing on the public's estimate of π_2. Equation (18) says that if the variance of the policymaker's taste for inflation, σ_C^2, is very large relative to the variance of the random shocks, σ_ε^2, β will be close to one. The public will attribute most of the above average realization of π_1 to a policymaker with a higher than average c and raise the expectation of π_2 accordingly.

(d) The policymaker knows that her choice of $\hat{\pi}_1$ will affect the public's expectation of inflation in the second period, π_2^e. When π_1 turns out to be high, the public attributes some of this to a policymaker with a high c and accordingly raise π_2^e. From equation (8), we can see that a higher value of π_2^e reduces the expected value of the policymaker's second period objective function. Thus the policymaker chooses a lower $\hat{\pi}_1$ to try and establish a "good reputation" as someone with a low c in order to keep π_2^e down. In the second period, however, there is no future period. Thus there is no need to worry about the effects that this period's inflation will have on future expected inflation.

Problem 10.12
(a) The policymaker wants to choose inflation in order to maximize her objective function, which is given by $W = c\gamma y - (a\pi^2/2)$, subject to output being given by the Lucas Supply function, $y = \bar{y} + b(\pi - \pi^e)$. Thus the policymaker's problem is

$$\max_{\pi} \; W = c\gamma[\bar{y} + b(\pi - \pi^e)] - (a\pi^2/2).$$

The first-order condition is
(1) $\partial W/\partial\pi = bc\gamma - a\pi = 0.$
Thus the policymaker's choice of π is
(2) $\pi = bc\gamma/a.$

(b) The public knows the policymaker will set inflation according to equation (2). Thus with rational expectations, expected inflation must equal the expectation of the right-hand side of equation (2):
(3) $\pi^e = E[bc\gamma/a] = bcE[\gamma]/a = bc\bar{\gamma}/a.$

(c) The true social welfare function is given by $W^{SOC} = \gamma y - (a\pi^2/2)$. Taking the expectation of both sides of this expression with respect to the public's information set, so that γ is random, gives us
(4) $E[W^{SOC}] = E[\gamma(\bar{y} + b(\pi - \pi^e)) - (a\pi^2/2)],$
where we have substituted for $y = \bar{y} + b(\pi - \pi^e)$. Now substitute the policymaker's choice of π, equation (2), and the public's expectation of inflation, equation (3), into equation (4):

$$(5)\quad E[W^{SOC}] = E\left[\gamma\left[\bar{y} + b\left(\frac{bc\gamma}{a} - \frac{bc\bar{\gamma}}{a}\right)\right] - \frac{ab^2c^2\gamma^2}{2a^2}\right].$$

Simplifying yields

$$(6)\quad E[W^{SOC}] = \bar{y}E[\gamma] + \frac{b^2cE[\gamma^2]}{a} - \frac{b^2c\bar{\gamma}E[\gamma]}{a} - \frac{b^2c^2E[\gamma^2]}{2a}.$$

Since $E[\gamma] = \bar{\gamma}$, equation (6) becomes

$$(7)\quad E[W^{SOC}] = \bar{y}\bar{\gamma} + \frac{b^2c}{a}\left[E[\gamma^2] - \bar{\gamma}^2\right] - \frac{b^2c^2E[\gamma^2]}{2a}.$$

Now use the facts that for a random variable X:
(8) $\mathrm{var}(X) = E[X^2] - (E[X])^2$, and (9) $E[X^2] = \mathrm{var}(X) + (E[X])^2$.
Here, this means that we can write
(10) $\sigma_\gamma^2 = E[\gamma^2] - \bar{\gamma}^2$, and (11) $E[\gamma^2] = \sigma_\gamma^2 + \bar{\gamma}^2$.
Substituting equations (10) and (11) into equation (7) gives us the following expected value of the true social welfare function:

$$(12)\quad E[W^{SOC}] = \bar{y}\bar{\gamma} + \frac{b^2c}{a}\sigma_\gamma^2 - \frac{b^2c^2}{2a}\left(\sigma_\gamma^2 + \bar{\gamma}^2\right).$$

(d) To find the first-order condition for the maximization, use equation (12) to set the derivative of the expected value of the social welfare function with respect to c equal to zero:

$$(13)\quad \frac{\partial E[W^{SOC}]}{\partial c} = \frac{b^2}{a}\sigma_\gamma^2 - \frac{b^2c}{a}\left(\sigma_\gamma^2 + \bar{\gamma}^2\right) = 0.$$

Solving for c yields

(14) $c = \dfrac{\sigma_\gamma^2}{\sigma_\gamma^2 + \bar{\gamma}^2}$.

There is a tradeoff here. From equation (2), we can see that choosing a more "conservative" policymaker, that is, one with a low c, produces a better performance in terms of average inflation. However, such a policymaker would not respond well to the shocks. Thus there is some optimal level of "conservatism" that balances these two forces.

The value of c that maximizes the expected value of true social welfare is decreasing in the mean of γ. Since we know that π^e will equal π on average (since γ will equal $\bar{\gamma}$ on average), output will equal full employment output on average, regardless of the values of c or $\bar{\gamma}$. From equation (2), we can see that if γ is higher on average, inflation will also be higher on average, for a given c. Thus it will be welfare-improving to offset this and keep inflation lower on average by having a policymaker with a lower c; that is, having a more "conservative" policymaker.

However, the value of c that maximizes expected social welfare is increasing in the variance of the γ shock. The more variable is the shock, the less "conservative" the central banker should be. Since the policymaker can act after γ is realized, she can choose to offset any deviation in γ from its expected value, which will raise welfare. The policymaker will do this only to the extent that she cares about the shock's effect. Thus the more that γ varies, the better it is to have a policymaker who cares about the shock's effect and will act to offset it.

Problem 10.13

(a) Social welfare is higher when the policymaker turns out to be a Type-1, the type that shares the public's preferences concerning output and inflation. The choice of setting $\pi = 0$ in both periods -- as the Type-2 policymaker does -- is a choice available to the Type-1 policymaker. She chooses not to do this; in order to maximize social welfare, she decides to choose another pair of inflation rates. Since she is attempting to maximize social welfare, welfare must be higher under the choices made by the Type-1 policymaker. For example, as explained in the text, if $\beta < 1/2$, it is optimal for the Type-1 policymaker to choose $\pi_1 = b/a$ and $\pi_2 = b/a$. That must be because it achieves higher welfare than choosing $\pi_1 = 0$, $\pi_2 = 0$.

(b) Expected inflation, π^e, is determined by the public's beliefs. So both the " a' " policymaker and the "a" policymaker face the same π^e, since in either case, the public believes it is facing an " a' " policymaker. Thus both policymakers have the same choice set. The "a" policymaker makes her choice to maximize true social welfare, whereas the " a' " policymaker makes her choice to maximize something else. Thus social welfare must be higher with the "a" policymaker.

Problem 10.14

(a) When the policymaker fixes i, the LM curve is irrelevant. Equilibrium output is determined by the IS curve and the fixed nominal interest rate, $\bar{i}$. Substituting $\bar{i}$ into the IS curve yields

(1) $y = c - a\bar{i} + \varepsilon_{IS}$.

The variance of y is simply

(2) $\text{var}(y) = \text{var}(\varepsilon_{IS}) = \sigma_{IS}^2$.

(b) When the policymaker fixes m, the equilibrium level of output is determined by the intersection of the IS and LM curves. Rearranging the IS curve to solve for i gives us

(3) $i = (c + \varepsilon_{IS} - y)/a$.

Substituting equation (3) and the assumption that $m = \overline{m}$ into the LM curve, $m - p = hy - ki + \varepsilon_{LM}$, gives us

$$\overline{m} - p = hy - [k(c + \varepsilon_{IS} - y)/a] + \varepsilon_{LM} = [h + (k/a)]y - (kc/a) - (k/a)\varepsilon_{IS} + \varepsilon_{LM}.$$

Solving for y yields

$$(4)\quad y = \frac{\overline{m} - p + (kc/a) + (k/a)\varepsilon_{IS} + \varepsilon_{LM}}{h + (k/a)} = \frac{a(\overline{m} - p) + kc + k\varepsilon_{IS} + a\varepsilon_{LM}}{ah + k}.$$

The variance of y is

$$(5)\quad \text{var}(y) = \left(\frac{k}{ah + k}\right)^2 \sigma_{IS}^2 + \left(\frac{a}{ah + k}\right)^2 \sigma_{LM}^2.$$

(c) If $\sigma_{IS}^2 = 0$ -- if there are only LM shocks -- then from equations (2) and (5):

$$(6)\quad \text{var}(y)\big|_{i=\bar{i}} = 0, \quad \text{and} \quad (7)\quad \text{var}(y)\big|_{m=\overline{m}} = \left(\frac{a}{ah + k}\right)^2 \sigma_{LM}^2 > 0.$$

Thus interest-rate targeting leads to a lower variance of output than money-stock targeting. In fact, output in constant under interest targeting.

(d) If $\sigma_{LM}^2 = 0$ -- if there are only IS shocks -- then from equations (2) and (5):

$$(8)\quad \text{var}(y)\big|_{i=\bar{i}} = \sigma_{IS}^2, \quad \text{and} \quad (9)\quad \text{var}(y)\big|_{m=\overline{m}} = \left(\frac{k}{ah + k}\right)^2 \sigma_{IS}^2 < \sigma_{IS}^2.$$

Thus money-stock targeting leads to a lower variance of output than interest-rate targeting.

(e) Consider the situation in part (c) in which there are only LM shocks. If the policymaker targets the nominal money stock, the LM shocks cause the LM curve to shift around and equilibrium output in the economy is determined by the intersection of that shifting LM curve with the stable IS curve. If the policymaker targets the nominal interest rate, it ensures that i remains constant in the face of any LM shock. Since i is not allowed to change, planned expenditure does not change and thus the level of output that equates planned and actual expenditure does not change in the face of an LM shock.

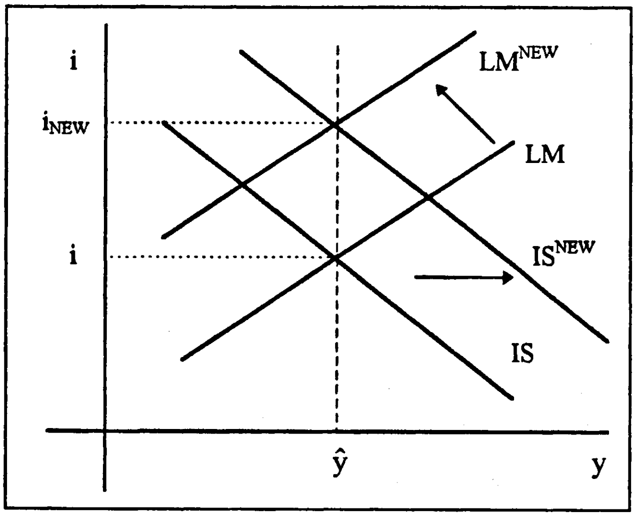
Consider the situation in part (d) in which there are only IS shocks. If the policymaker targets the nominal interest rate, equilibrium output changes by the full extent of the shift in the IS curve caused by a shock to the IS curve. Now consider the case in which the policymaker targets the nominal money stock. A positive IS shock shifts the IS curve to the right. With m fixed, as Y rises to equate planned and actual expenditure, i rises as well in order for the money market to remain in equilibrium. This rise in i reduces planned expenditure and thus mitigates some of the positive shock. Therefore Y does not end up rising as much.

The same idea is true in the opposite direction. A negative IS shock shifts IS to the left. If the policymaker targets m, i will fall along with Y in order to keep the money market in equilibrium. This fall in i raises planned expenditure and helps to offset the original negative shock to planned expenditure. Thus Y does not fall as much as if the policymaker had kept i constant.

(f) If there are only IS shocks, it is possible to keep y constant at some target level $\hat{y}$. By rearranging the LM curve with y set to $\hat{y}$, the nominal money supply must be such that

$$(10)\quad m = p + h\hat{y} - ki.$$

The policymaker knows the fixed p, has picked $\hat{y}$ herself and can observe i. Thus when i changes -- and since there are only IS shocks, we know this must be due to a shift of the IS curve -- the policymaker must change m accordingly. As i rises, for example, the policymaker must reduce m.



In the figure at right, as i rises due to the rightward shift of the IS curve, the policymaker reduces m which shifts up the LM curve and increases i more. The policymaker can stop reducing m when m and i are such that equation (10) is satisfied. At that point, LM^{NEW} will intersect IS^{NEW} right at the target level of $\hat{y}$.

Problem 10.15

(a) Using the fact that for a random variable X, $\text{var}(X) = E[X^2] - (E[X])^2$ or $E[X^2] = \text{var}(X) + (E[X])^2$, we have

(1) $E[(y - y^*)^2] = \text{var}(y - y^*) + (E[y - y^*])^2$.

Substituting the expression for output, $y = x + (k + \varepsilon_k)z + u$, into $\text{var}(y - y^*)$ and simplifying yields

(2) $\text{var}(y - y^*) = \text{var}(x + kz + \varepsilon_k z + u - y^*) = z^2 \sigma_k^2 + \sigma_u^2$.

Substituting for output in $(E[y - y^*])^2$ and simplifying yields

(3) $(E[y - y^*])^2 = (E[x + kz + \varepsilon_k z + u - y^*])^2 = (x + kz - y^*)^2$,

where we have used the fact that ε_k and u both have mean zero. Substituting equations (2) and (3) into equation (1) gives us

(4) $E[(y - y^*)^2] = z^2 \sigma_k^2 + \sigma_u^2 + (x + kz - y^*)^2$.

(b) The policymaker wants to choose z in order to minimize $E[(y - y^*)^2]$. Using equation (4), the first-order condition is

(5) $\partial(E[(y - y^*)^2])/\partial z = 2z\sigma_k^2 + 2k(x + kz - y^*) = 0$.

Collecting the terms in z yields

$z(\sigma_k^2 + k^2) = (y^* - x)k$,

and thus the optimal choice of z is

(6) $$z = \frac{(y^* - x)k}{\sigma_k^2 + k^2}.$$

(c) To see how policy should respond to shocks (i.e. changes in x), use equation (6) to take the derivative of z with respect to x:

(7) $$\frac{\partial z}{\partial x} = -\frac{k}{\sigma_k^2 + k^2} < 0.$$

The fact that the derivative in (7) is negative implies that higher values of x should be offset with lower values of z in order to keep output from varying as much.

Note that $\partial z/\partial x$ does not depend upon σ_u^2, which represents uncertainty about the state of the economy. Thus in this model, the optimal degree of "fine-tuning" does not depend upon the amount of uncertainty about the state of the economy.

(d) In contrast, $\partial z/\partial x$ does depend upon σ_k^2, which represents uncertainty about the effects of the policy instrument. In fact, we have

(8) $$\frac{\partial[\partial z/\partial x]}{\partial \sigma_k^2} = \frac{k}{(\sigma_k^2 + k^2)^2} > 0.$$

Since $\partial z/\partial x$ is negative to begin with, a rise in σ_k^2 makes it less negative. That is, higher values of σ_k^2 -- more uncertainty about the effects of the policy instrument -- reduces the amount that policy should respond to shocks or in other words, reduces the amount of "fine-tuning" that should be done.

Problem 10.16

We can focus on a situation in which g_M, π, i, and r are constant and in which $\pi^e = \pi$. Although not technically correct -- since Y and thus M/P are growing -- such a situation will be referred to as a steady state in what follows. Under these assumptions, it is therefore reasonable to assume that output, and the real interest rate are unaffected by the rate of money growth and that actual and expected inflation are equal. Taking the exponential function of both sides of the money demand function, which is given by $\ln(M(t)/P(t)) = a - bi + \ln Y(t)$, yields

(1) $M(t)/P(t) = e^{a-bi}Y(t)$.

The nominal interest rate is given by $i = r + \pi^e$. In steady state, π^e and r are constant and thus so is the nominal interest rate. Thus in steady state, the quantity of real balances must grow at the same rate as Y(t). In other words, $\dot{M}(t)/M(t) - \dot{P}(t)/P(t) = g_Y$. Solving for inflation yields

(2) $\pi = g_M - g_Y$,

where g_M is the growth rate of the nominal money stock. This means that the nominal interest rate in steady state is given by

(3) $i = \bar{r} + \pi = \bar{r} + g_M - g_Y$,

where we have used the fact that actual and expected inflation are equal. Substituting equation (3) into equation (1) gives steady-state real balances:

(4) $M(t)/P(t) = e^a e^{-b(\bar{r}+g_M-g_Y)}Y(t)$.

Seignorage is given by

(5) $$S(t) = \frac{\dot{M}(t)}{P(t)} = \frac{\dot{M}(t)}{M(t)}\frac{M(t)}{P(t)} = g_M \frac{M(t)}{P(t)}.$$

Substituting equation (4) into equation (5) gives steady-state seignorage:

(6) $S(t) = g_M e^a e^{-b(\bar{r}+g_M-g_Y)}Y(t) = Cg_M e^{-bg_M}Y(t)$,

where $C \equiv e^a e^{-b(\bar{r}-g_Y)}$. We need to find the choice of nominal money growth, g_M, that maximizes steady-state seignorage. Again, we are assuming that output is unaffected by money growth. The first-order condition is

(7) $\partial S(t)/\partial g_M = Ce^{-bg_M}Y(t) - bCg_M e^{-bg_M}Y(t) = 0$,

which simplifies to

(8) $1 - bg_M = 0$.

Thus seignorage is maximized when money growth is given by

(9) $g_M = 1/b$.

From equation (2), we know that $\pi = g_M - g_Y$ and thus the rate of inflation that maximizes seignorage is

(10) $\pi = (1/b) - g_Y$.
Equation (10) implies that the higher is the growth rate of real output, the lower is the rate of inflation that maximizes steady-state seignorage.

Problem 10.17
(a) Desired real money holdings are given by

(1) $m(t) = Ce^{-b\pi^e(t)}$.

The assumption is that expected inflation adjusts gradually toward actual inflation. Specifically, our assumption is

(2) $\dot{\pi}^e(t) = \beta[\pi(t) - \pi^e(t)]$.

As usual, seignorage is given by $\dot{M}(t)/P(t)$ or equivalently $[\dot{M}(t)/M(t)][M(t)/P(t)]$. Assuming that the nominal money supply is growing at rate $g_M(t)$, we can write seignorage as

(3) $S(t) = g_M(t)m(t)$.

To see the dynamics of inflation and money holdings formally, note that the growth rate of real money, $\dot{m}(t)/m(t)$, equals the growth rate of nominal money, $g_M(t)$, minus the rate of inflation, $\pi(t)$. Rewriting this as an equation for inflation gives us

(4) $\pi(t) = g_M(t) - [\dot{m}(t)/m(t)]$.

Define G as the amount of real purchases that the government needs to finance with seignorage. Thus from equation (3), we have

(5) $g_M(t) = G/m(t)$.

Taking the time derivative of both sides of equation (1) yields

(6) $\dot{m}(t) = -b\dot{\pi}^e(t)Ce^{-b\pi^e(t)}$.

Dividing both sides of equation (6) by m(t) gives us

(7) $\dot{m}(t)/m(t) = -b\dot{\pi}^e(t)$.

Substituting equations (5) and (7) into equation (4) yields

(8) $\pi(t) = \dfrac{G}{m(t)} + b\dot{\pi}^e(t)$.

Substituting equation (8) into equation (2) gives us

(9) $\dot{\pi}^e(t) = \beta\left[\dfrac{G}{m(t)} + b\dot{\pi}^e(t) - \pi^e(t)\right]$.

Collecting the terms in $\dot{\pi}^e(t)$ yields

$$\dot{\pi}^e(t)[1 - \beta b] = \beta\left[\frac{G}{m(t)} - \pi^e(t)\right],$$

and thus

(10) $\dot{\pi}^e(t) = \dfrac{\beta}{1-\beta b}\left[\dfrac{G - \pi^e(t)m(t)}{m(t)}\right]$.

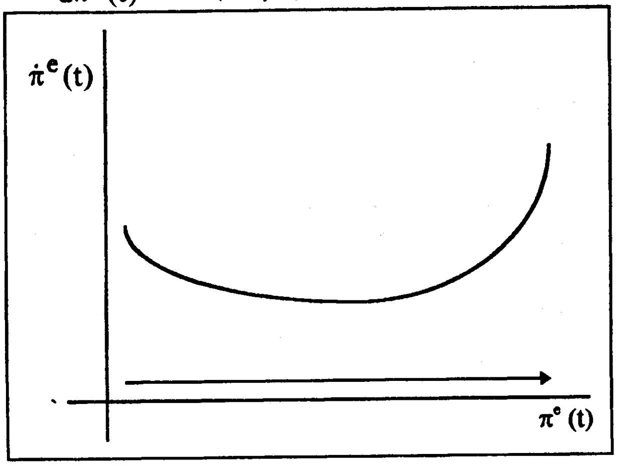
(b) The assumption that $G > S^*$ (where S^* represents the maximum steady-state value of seignorage) is equivalent to $G > \pi^e m$ for all possible values of π^e. Thus since $\beta b < 1$, the right-hand side of equation (10) is everywhere positive: regardless of where it starts, expected inflation grows without bound. To examine the shape of the phase diagram, substitute $m(t) = Ce^{-b\pi^e(t)}$ into equation (10):

(11) $\dot{\pi}^e(t) = \dfrac{\beta G}{(1-\beta b)Ce^{-b\pi^e(t)}} - \dfrac{\beta}{1-\beta b}\pi^e(t).$

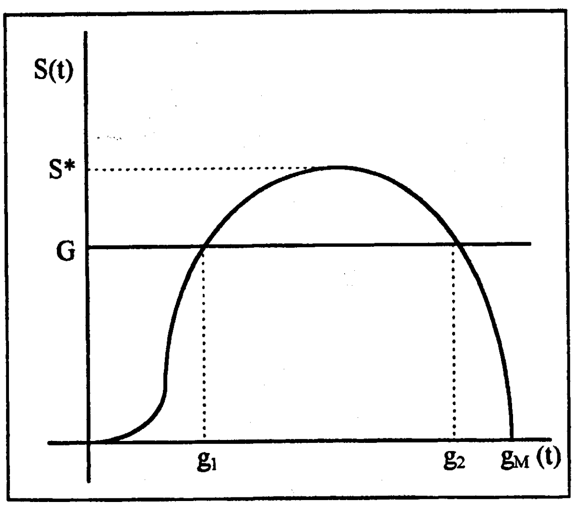
The following derivatives will be useful:

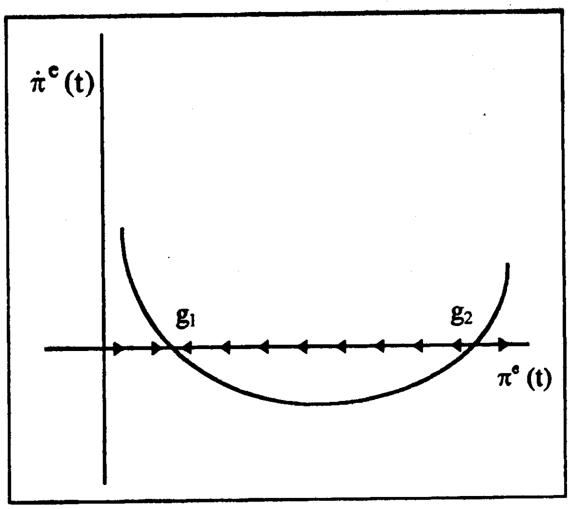
(12) $\dfrac{d\dot{\pi}^e(t)}{d\pi^e(t)} = \dfrac{\beta b G e^{b\pi^e(t)}}{(1-\beta b)C} - \dfrac{\beta}{1-\beta b}$, and (13) $\dfrac{d^2\dot{\pi}^e(t)}{d\pi^e(t)^2} = \dfrac{\beta b^2 G e^{b\pi^e(t)}}{(1-\beta b)C} > 0.$

By setting the right-hand side of equation (12) equal to zero, it is straightforward to show that $\dot{\pi}^e(t)$ reaches a minimum at $\pi^e(t) = [\ln(C/bG)]/b$. Thus the phase diagram has the shape depicted in the figure at right. From equation (1), and since π^e rises without bound, the real money stock is continually falling. If m(t) is continually falling, then from equation (3), it must be the case that the growth rate of the nominal money supply, $g_M(t)$, is continually rising if the government is to obtain G in seignorage.



(c) Now consider the case of G < S*. The left-hand figure below reproduces Figure 10.8 from the text. It gives the amount of seignorage the government can obtain in steady state as a function of the growth rate of the nominal money supply. In the case of G < S*, there are two possible growth rates of the nominal money supply, labeled g_1 and g_2 in the figure, consistent with raising the amount G in seignorage. Recall that in a steady state, expected inflation equals actual inflation which in turn equals the constant growth rate of the nominal money supply. Thus, by assumption, $\pi^e(t)m(t) = G$ at $\pi^e(t) = g_1$ and $\pi^e(t) = g_2$. From equation (10) then, $\dot{\pi}^e(t) = 0$ at $\pi^e(t) = g_1$ and $\pi^e(t) = g_2$. From the figure on the left, when $g_1 < \pi^e(t) < g_2$, we have $\pi^e(t)m(t) > G$ and thus $\dot{\pi}^e(t) < 0$. Otherwise, $\pi^e(t)m(t) < G$ and thus $\dot{\pi}^e(t) < 0$. Putting all of this information together gives us the phase diagram depicted on the right. The low-inflation steady state with $\pi^e(t) = \pi(t) = g_1$ is stable and the high-inflation steady state with $\pi^e(t) = \pi(t) = g_2$ is unstable.





SOLUTIONS TO CHAPTER 11

Problem 11.1

(a) (i) Taking the time derivative of d(t) = D(t)/Y(t) gives us

(1) $\dot{d}(t) = \dfrac{\dot{D}(t)Y(t) - D(t)\dot{Y}(t)}{Y(t)^2}$.

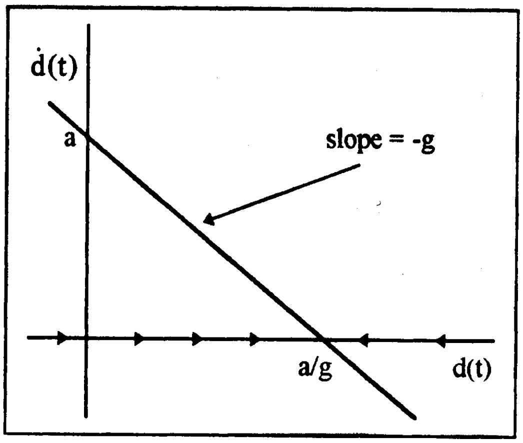
Substituting $\dot{D}(t) = \delta(t)$ -- the rate of change of the amount of debt outstanding equals the budget deficit -- and the fact that $\dot{Y}(t) = Y(t)g$, which follows from the fact that output grows at rate g, and simplifying gives us

(2) $\dot{d}(t) = \dfrac{\delta(t)}{Y(t)} - \dfrac{D(t)g}{Y(t)}$.

Substituting the assumption that the deficit-to-output ratio is constant -- $\delta(t)/Y(t) = a$ -- and using the definition of d(t) = D(t)/Y(t), yields

(3) $\dot{d}(t) = a - gd(t)$.

(a) (ii) The phase diagram for the ratio of debt to output is depicted in the figure at right.



In (d, $\dot{d}$) space, equation (3) is a line with slope equal to -g. We can see that the system is stable. If the debt-to-output ratio is less than a/g, $\dot{d}(t) > 0$ and so d(t) rises toward a/g.

Similarly, if the debt-to-output ratio is greater than a/g, $\dot{d}(t) < 0$ and so d(t) falls toward a/g.

Note that the value of the debt-to-output ratio to which the economy converges is increasing in the deficit-to-output ratio, a, and decreasing in the growth rate of output, g.

(b) (i) Once again, taking the time derivative of d(t) = D(t)/Y(t) gives us

(4) $\dot{d}(t) = \dfrac{\dot{D}(t)Y(t) - D(t)\dot{Y}(t)}{Y(t)^2}$.

Substituting $\dot{D}(t) = \delta(t) = aY(t) + r(d(t))D(t)$ and the fact that $\dot{Y}(t) = Y(t)g$ into equation (4) and simplifying gives us

(5) $\dot{d}(t) = a + \dfrac{r(d(t))D(t)}{Y(t)} - \dfrac{D(t)g}{Y(t)}$.

Using the definition of d(t) = D(t)/Y(t), yields

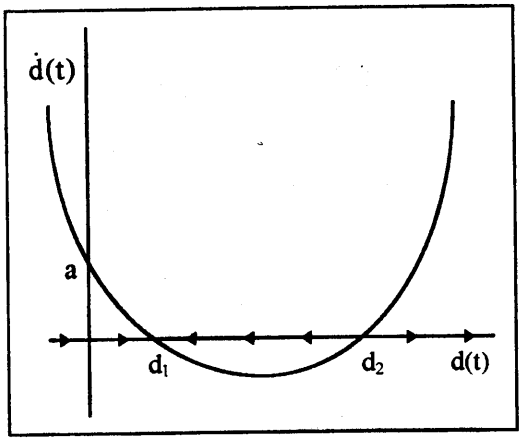
(6) $\dot{d}(t) = a + [r(d(t)) - g]d(t)$.

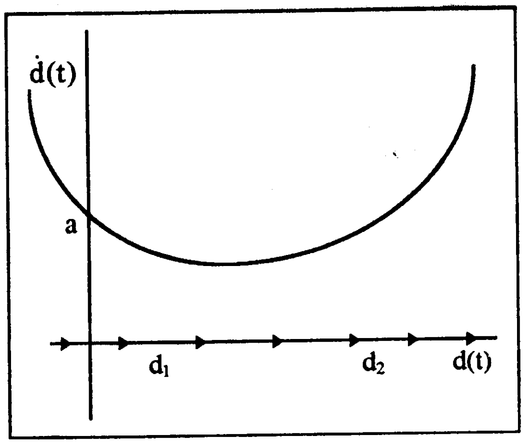
(b) (ii) When plotted in (d, $\dot{d}$) space, the slope of the locus given by equation (6) equals r(d(t)) - g. Given the assumptions about the behavior of r, this slope is negative for large negative values of d(t) and increases as d increases and so equation (6) defines a convex function as depicted in the figures below.

Once again, $\dot{d}(t) = a > 0$ when d(t) equals zero. And $\dot{d}(t) = 0$ if $d(t) = \dfrac{a}{g - r(d(t))}$.

The case in which a is sufficiently small that $\dot{d}$ is negative for some values of d is depicted on the left-hand side. In this case, d_1 is a stable equilibrium whereas d_2 is not. If the debt-to-output ratio starts off less than d_2, it converges to d_1. If the debt-to-output ratio starts off greater than d_2, it rises without bound.

The case in which a is sufficiently large that $\dot{d}$ is positive for all values of d is depicted in the figure on the right-hand side below. In this case, d will always be rising and there is no stable equilibrium.





Problem 11.2

Throughout, we will assume $U'(\bullet) > 0$ and $U''(\bullet) < 0$. In addition, since the expected value of Y_2 is equal to Y_1, we can write $Y_2 = Y_1 + \varepsilon$ with $E[\varepsilon] = 0$.

(a) The individual's problem is to choose C_1 and C_2 in order to maximize $U(C_1) + E[U(C_2)]$ subject to

(1) $C_2 = (1 - \tau_1)Y_1 - C_1 + (1 - \tau_2)(Y_1 + \varepsilon)$.

We can substitute for C_2 and solve the unconstrained problem of choosing C_1:

$$\max\ U(C_1) + E\left[U\left((1-\tau_1)Y_1 - C_1 + (1-\tau_2)(Y_1+\varepsilon)\right)\right].$$

The first-order condition is given by

$$U'(C_1) + E\left[U'(C_2)(-1)\right] = 0,$$

or simply

(2) $U'(C_1) = E\left[U'(C_2)\right]$.

If the individual is optimizing, the marginal utility of consumption in period one must equal the expected marginal utility of consumption in period two.

(b) If Y_2 is not random, the first-order condition reduces to $U'(C_1) = U'(C_2)$. With $U''(\bullet) < 0$ everywhere, this implies that $C_1 = C_2$. If utility is quadratic then $U'(C_2)$ is a linear function of C_2 and so $E[U'(C_2)] = U'(E[C_2])$. Thus the first-order condition given by equation (2) can be rewritten as $U'(C_1) = U'(E[C_2])$. Since $U''(\bullet) < 0$ everywhere, this implies that $C_1 = E[C_2]$.

(c) Now, $U'(\bullet) > 0$, $U''(\bullet) < 0$ and $U'''(\bullet) > 0$. Marginal utility is now a convex function of consumption and so by Jensen's inequality $E[U'(C_2)] > U'(E[C_2])$. Combining this with the first-order condition, $U'(C_1) = E\left[U'(C_2)\right]$, yields $U'(C_1) > U'(E[C_2])$. Since $U'(\bullet) > 0$ and $U''(\bullet) < 0$ we have $C_1 < E[C_2]$. The individual plans in such a way that if second-period income turns out to be equal to its expected value, C_2 would turn out to be higher than C_1. Thus, in the face of uncertainty and with $U'''(\bullet) > 0$, the individual undertakes "precautionary saving".

(d) The government is cutting first-period taxes, τ_1, and raising second-period taxes, τ_2, in such a way that expected tax revenue remains unchanged. Expected tax revenue, $\overline{R}$, can be expressed as $\tau_1 Y_1 + \tau_2 E[Y_1 + \varepsilon] = \overline{R}$. Using the fact that $E[\varepsilon] = 0$, we can solve for τ_2:

$$\tau_1 Y_1 + \tau_2 Y_1 = \overline{R} \quad \Rightarrow \quad \tau_2 = \overline{R}/Y_1 - \tau_1 .$$

In order to keep $\overline{R}$ constant, the change in taxes must satisfy

(3) $\partial\tau_2/\partial\tau_1 = -1$.

The question is whether or not this change in the timing of taxes alters the individual's consumption behavior. Substitute equation (1) into the first-order condition, equation (2), to obtain

(4) $U'(C_1) = E\left[U'\left((1-\tau_1)Y_1 - C_1 + (1-\tau_2)(Y_1+\varepsilon)\right)\right]$.

Differentiating both sides of this equation with respect to τ_1 yields

$$U''(C_1)\,\partial C_1/\partial\tau_1 = E\left[U''(C_2)\left\{-Y_1 - \partial C_1/\partial\tau_1 - (Y_1+\varepsilon)\,\partial\tau_2/\partial\tau_1\right\}\right],$$

and now using equation (3), $\partial\tau_2/\partial\tau_1 = -1$, we have

$$U''(C_1)\,\partial C_1/\partial\tau_1 = E\left[U''(C_2)\left(-Y_1 - \partial C_1/\partial\tau_1 + Y_1 + \varepsilon\right)\right],$$

$$U''(C_1)\,\partial C_1/\partial\tau_1 = E\left[U''(C_2)\left(-\partial C_1/\partial\tau_1\right)\right] + E\left[U''(C_2)\varepsilon\right],$$

or

$$\left[U''(C_1) + E\left[U''(C_2)\right]\right]\partial C_1/\partial\tau_1 = E\left[U''(C_2)\varepsilon\right].$$

Now use the fact that for any two random variables X and Y, $E[XY] = E[X]E[Y] + \text{cov}[X,Y]$:

$$\left[U''(C_1) + E\left[U''(C_2)\right]\right]\partial C_1/\partial\tau_1 = E\left[U''(C_2)\right]E[\varepsilon] + \text{cov}\left[U''(C_2),\varepsilon\right].$$

Finally, $E[\varepsilon] = 0$ and thus the change in first-period consumption due to this change in the timing of taxes is given by

(5) $$\frac{\partial C_1}{\partial\tau_1} = \frac{\text{cov}\left[U''(C_2),\varepsilon\right]}{U''(C_1) + E\left[U''(C_2)\right]}.$$

(e) If Y_2 is not random then $\varepsilon = 0$ always and thus the covariance in the numerator of equation (5) is 0. In addition, if utility is quadratic, then $U''(\bullet)$ is a constant and again the covariance is 0. In both of these cases, $\partial C_1/\partial\tau_1 = 0$ and thus first-period consumption does not change in response to the tax cut.

(f) In the case of $U'''(\bullet) > 0$ we need to show that $\partial C_1/\partial\tau_1 < 0$. That is, we need to show that C_1 rises in response to the reduction in τ_1. Intuitively, the higher is ε, the higher will be C_2. The individual simply consumes any extra random income in the second period. If $U'''(\bullet) > 0$, then as C_2 rises so will $U''(C_2)$, and thus it will be the case that $\text{cov}[U''(C_2),\varepsilon] > 0$. The denominator of equation (5) will be negative since $U''(\bullet) < 0$ and thus $\partial C_1/\partial\tau_1 < 0$ as required. The intuition is that the change in the timing of taxes leaves the individual with the same expected after-tax lifetime income, but more of it comes with certainty in the first period. If the individual is undertaking precautionary saving -- and if $U'''(\bullet) > 0$ she is -- the amount of such saving will be reduced and she will consume more in the first period.

Problem 11.3

In the Barro tax-smoothing model in which output and the real interest rate are constant, the government finds it optimal to set taxes equal to a constant such that its budget constraint is satisfied with equality.

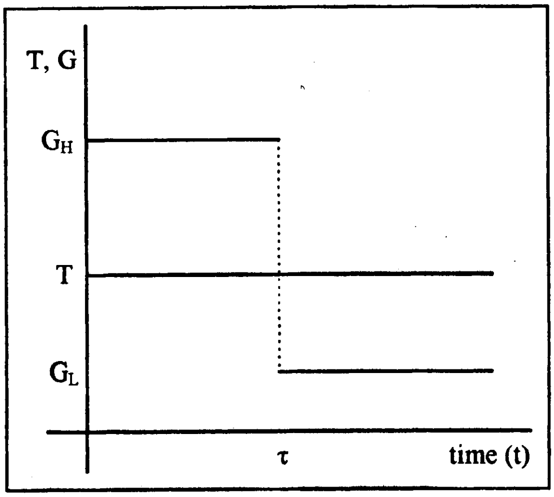
Thus the government will find it optimal to set taxes such that $T < G_H$. That is, during the war from time

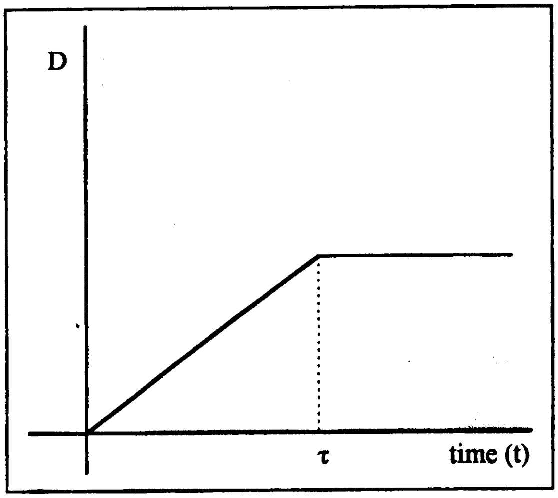
$0 \le t \le \tau$, $T < G_H$ and so the government runs a deficit and its debt will be growing over time. The deficit, which equals the rate of change of debt, will be

(1) $\dot{D}(t) = G_H - T + rD(t) > 0$.

Even though the primary deficit, $G_H - T$, is constant, the total deficit will be rising over time since the government debt outstanding and thus interest payments on that debt, $rD(t)$, are rising.

At time τ, government debt will be positive: $D(\tau) > 0$. To satisfy its budget constraint at that time, taxes must have been set so that $T = G_L + rD(\tau) > G_L$. Thus for $t > \tau$, the budget will balance so that the deficit is zero and the debt will then be constant at its level as of time τ. See the figures below.





Problem 11.4

(a) First, we will use dynamic programming (see Section 9.4, the Shapiro-Stiglitz model, for more information regarding this technique) to find an expression for the expected present value of the revenue the government must raise when $G = G_H$, denoted $V_H(\Delta t)$. This expression is given by

(1) $$V_H(\Delta t) = \int_{t=0}^{\Delta t} e^{-rt} e^{-at}(G_H + rD)dt + e^{-r\Delta t}[e^{-a\Delta t}V_H(\Delta t) + (1 - e^{-a\Delta t})V_L(\Delta t)].$$

The first term on the right-hand side of (1) reflects the revenue the government must raise during the interval $(0, \Delta t)$. The probability that government spending is still high at time t is e^{-at}, in which case the government must raise $G_H + rD$. The e^{-rt} term discounts this using the constant interest rate, r. The second term reflects revenue needs after Δt. At time Δt, government purchases are still high with probability $e^{-a\Delta t}$, and have switched to being low with probability $(1 - e^{-a\Delta t})$. V_H and V_L denote the expected present value of the revenue the government must raise in each case. And this is then discounted by the $e^{-r\Delta t}$ term.

The integral in (1) can be solved as follows:

(2) $$\int_{t=0}^{\Delta t} e^{-(a+r)t}(G_H + rD)dt = (G_H + rD)\left[\frac{-1}{(a+r)} e^{-(a+r)t}\Bigg|_{t=0}^{\Delta t}\right],$$

which simplifies to

(3) $$\int_{t=0}^{\Delta t} e^{-(a+r)t}(G_H + rD)dt = \frac{G_H + rD}{a+r}\left[1 - e^{-(a+r)\Delta t}\right].$$

Substituting equation (3) into (1) yields

$$(4)\quad V_H(\Delta t) = \frac{G_H + rD}{a+r}[1 - e^{-(a+r)\Delta t}] + e^{-(a+r)\Delta t} V_H(\Delta t) + e^{-r\Delta t}(1 - e^{-a\Delta t})V_L(\Delta t),$$

and collecting terms in $V_H(\Delta t)$ gives us

$$(5)\quad V_H(\Delta t)[1 - e^{-(a+r)\Delta t}] = \frac{G_H + rD}{a+r}[1 - e^{-(a+r)\Delta t}] + e^{-r\Delta t}(1 - e^{-a\Delta t})V_L(\Delta t),$$

or simply

$$(6)\quad V_H(\Delta t) = \frac{G_H + rD}{a+r} + \frac{e^{-r\Delta t}(1 - e^{-a\Delta t})}{1 - e^{-(a+r)\Delta t}} V_L(\Delta t).$$

As described in Section 9.4, we now take the limit of the expression in (6) as the interval of time goes to zero. This requires using l'Hopital's rule. The derivative with respect to Δt of the numerator of the second term on the right-hand side of (6) is $-re^{-r\Delta t} + (a+r)e^{-(a+r)\Delta t}$. The limit of this as $\Delta t \to 0$ is a. The derivative of the denominator of that same term is $(a+r)e^{-(a+r)\Delta t}$ which goes to $(a + r)$ as $\Delta t \to 0$. Thus, as $\Delta t \to 0$, we have

$$(7)\quad V_H = \frac{G_H + rD}{a+r} + \frac{a}{a+r} V_L = \frac{G_H + rD + aV_L}{a+r}.$$

Rearranging (7) gives an expression that can be interpreted as an asset-pricing condition:

$$(8)\quad rV_H = (G_H + rD) - a(V_H - V_L).$$

Similar analysis to the above would yield the following expression for V_L, the expected present value of the revenue the government must raise when $G = G_L$:

$$(9)\quad rV_L = (G_L + rD) - b(V_L - V_H),$$

or

$$(10)\quad V_L = \frac{G_L + rD + bV_H}{b+r}.$$

We can now solve (7) and (10) for V_H and V_L. Substituting equation (7) into equation (9) yields

$$(11)\quad rV_L = G_L + rD - b\left[V_L - \left(\frac{G_H + rD + aV_L}{a+r}\right)\right] = G_L + rD - b\left(\frac{rV_L - G_H - rD}{a+r}\right).$$

Collecting the terms in V_L yields

$$(12)\quad \left(r + \frac{br}{a+r}\right)V_L = G_L + rD + \frac{b}{a+r}(G_H + rD),$$

which simplifies to

$$(13)\quad \left[\frac{r(a+b+r)}{a+r}\right]V_L = G_L + \frac{b}{a+r}G_H + \frac{r(a+b+r)}{a+r}D,$$

and thus we have

$$(14)\quad V_L = \frac{(a+r)G_L + bG_H}{r(a+b+r)} + D.$$

Substituting (14) into (7) yields

$$(15)\quad V_H = \frac{G_H + rD + a\left[\frac{(a+r)G_L + bG_H}{r(a+b+r)} + D\right]}{a+r},$$

which implies

(16) $V_H = \dfrac{aG_L}{r(a+b+r)} + \dfrac{[r(a+b+r)+ab]G_H}{(a+r)r(a+b+r)} + D.$

Note that $r(a + b + r) + ab = br + r(a + r) + ab = b(a + r) + r(a + r) = (a + r)(b + r)$, and so

(17) $V_H = \dfrac{aG_L + (b+r)G_H}{r(a+b+r)} + D.$

Equations (14) and (17) give the expected present value of the revenue the government must raise as a function of its expenditures, the amount of debt outstanding, and the parameters of the model. With quadratic distortion costs and constant output, the optimal policy is for taxes to be expected to be constant also. Thus, when government spending is high, the government expects to impose a tax, T_H, such that

(18) $\int\limits_{t=0}^{\infty} e^{-rt}T_H dt = V_H.$

Solving the integral and using equation (17) for V_H yields

(19) $\dfrac{1}{r}T_H = \dfrac{aG_L + (b+r)G_H}{r(a+b+r)} + D,$

or simply

(20) $T_H = \dfrac{aG_L + (b+r)G_H}{(a+b+r)} + rD.$

Similar analysis would show that when $G = G_L$, the government sets taxes, T_L, equal to

(21) $T_L = \dfrac{(a+r)G_L + bG_H}{(a+b+r)} + rD.$

(b) From equations (20) and (21) we can see that the path of taxes during an interval in which G is constant is driven by the path of outstanding debt, D. In general, the change in debt -- or the budget deficit -- is given by

(22) $\dot{D} = G - T + rD.$

From equation (20), the path of taxes during an interval in which G equals G_H is given by

(23) $\dot{T}_H = r\dot{D} = r(G_H - T_H + rD).$

Substituting equation (20) for T_H into equation (23) yields

(24) $\dot{T}_H = r\left[G_H - \left(\dfrac{aG_L + (b+r)G_H}{a+b+r} + rD\right) + rD\right],$

which simplifies to

(25) $\dot{T}_H = r\left[\dfrac{(a+b+r)G_H - aG_L - (b+r)G_H}{a+b+r}\right],$

or

(26) $\dot{T}_H = \dfrac{ar(G_H - G_L)}{a+b+r} > 0.$

As long as G equals G_H, the government runs a deficit and taxes are thus increasing over time because of the increased interest on the outstanding debt. Intuitively, the government knows there is a probability that its expenditures will fall in the future and so it runs a deficit in order to smooth taxes over time.

At the moment that G falls to G_L, taxes will drop from T_H to T_L. The path of taxes when G equals G_L is again driven by the path of outstanding debt, so that

(27) $\dot{T}_L = r\dot{D} = r(G_L - T_L + rD).$

Substituting equation (21) for T_L yields

(28) $\dot{T}_L = r\left[G_L - \left(\frac{(a+r)G_L + bG_H}{a+b+r} + rD\right) + rD\right]$,

which simplifies to

(29) $\dot{T}_L = r\left[\frac{(a+b+r)G_L - bG_H - (a+r)G_L}{a+b+r}\right]$,

or

(30) $\dot{T}_L = \frac{br(G_L - G_H)}{a+b+r} < 0.$

As long as G equals G_L, the government runs a surplus and taxes are thus decreasing over time because of the decreased interest on the outstanding debt. Intuitively, the government knows there is a probability that its expenditures will rise in the future and so it runs a surplus in order to smooth taxes over time.

Problem 11.5

It is true that the model has an implication about the long run that is clearly incorrect and undesirable. But as with all models, we need to look at whether this is important for the issues the model was meant to address. This implication about the long run does not mean that the model does not provide a good approximation to actual and/or optimal fiscal policy in the short and medium terms.

The motive for studying tax smoothing was to examine its implications for the behavior of deficits over short and moderate time frames. And in fact, the model does provide interesting implications for the behavior of deficits during such short-run phenomena as wars and recessions. The simplifying assumptions that give rise to the result that the tax rate is a random walk -- and thus that the tax rate would eventually exceed 100 percent or become negative -- should only be considered problematic if they cause the model to give incorrect answers to the questions it was meant to address.

Problem 11.6

Assuming that everyone votes truthfully in each two-way contest, policy A would beat policy B by a vote of two to one and policy B would beat policy C by a vote of two to one. If society's preferences as a whole exhibited transitivity we would then expect policy A should beat policy C. But instead policy C would defeat policy A by a vote of two to one.

Thus the order of the pairwise voting would determine the outcome. If policies A and B were voted on first, C would be the eventual winner whereas, for example, if policies B and C were voted on first, A would be the eventual winner. Thus the voting choice essentially reverts back to the choice of agenda.

Also note that this provides for the incentive not to vote truthfully. For example, consider Voter 2. If the first vote is between policies B and C and Voter 2 votes truthfully (as do the other voters), then A -- Voter 2's least-desired outcome -- would be the eventual winner. If, however, Voter 2 casts her ballot for policy C in the first vote then C wins that vote and goes on to beat A in the second vote. Thus Voter 2 winds up better off with her second-best alternative by misrepresenting her preferences in that first vote.

Problem 11.7

Since the real interest rate is assumed to be zero, the period-1 policymaker has no interest payments on the initial debt, D_0. Thus the period-1 budget constraint remains

(1) $M_1 + N_1 = W + D$,

where W is the economy's endowment and D is the amount of debt the period-1 policymaker issues. In period 2, the policymaker must now pay off the initial debt, D_0, plus whatever was borrowed in the first period. Thus the period-2 constraint is

(2) $M_2 + N_2 = W - (D + D_0)$.

As explained in the text, the period-2 policymaker simply devotes all available resources, which are now given by $W - (D + D_0)$, to the type of government purchases preferred by the period-2 median voter.

Consider the first period and assume the period-1 median voter has $\alpha = 1$. Her expected utility, denoted E[V], as a function of D is given by

(3) $E[V] = U(W + D) + \pi U(W - (D + D_0)) + (1 - \pi)U(0)$.

The first term on the right-hand side of (3) reflects the fact that with $\alpha = 1$ for the median voter, the period-1 policymaker chooses $M_1 = W + D$ and $N_1 = 0$ and thus receives utility $U(W + D)$. With probability π, the period-2 median voter has $\alpha = 1$ and devotes all available resources, $W - (D + D_0)$, to military goods giving utility $U(W - (D + D_0))$ to the period-1 policymaker. Finally, with probability $(1 - \pi)$, the period-2 median voter has $\alpha = 0$ and so all available resources are devoted to non-military goods giving U(0) to the period-1 policymaker.

The first-order condition for the period-1 policymaker's choice of D is

(4) $U'(W + D) - \pi U'(W - (D + D_0)) = 0$.

To see how the first-period deficit, $D = M_1 + N_1 - W$, responds to a change in D_0, implicitly differentiate equation (4) with respect to D_0 to obtain

(5) $$U''(W+D)\frac{\partial D}{\partial D_0} - \pi U''(W-(D+D_0))\left(-\frac{\partial D}{\partial D_0} - 1\right) = 0.$$

Collecting the terms in $\partial D/\partial D_0$ gives us

(6) $$[U''(W+D) + \pi U''(W-(D+D_0))]\frac{\partial D}{\partial D_0} = -\pi U''(W-(D+D_0)),$$

and thus

(7) $$\frac{\partial D}{\partial D_0} = \frac{-\pi U''(W-(D+D_0))}{U''(W+D) + \pi U''(W-(D+D_0))}.$$

Since $U''(\bullet) < 0$ and π is between zero and one, we can see that $-1 < \partial D/\partial D_0 < 0$.

Similar analysis for the case in which the period-1 median voter has $\alpha = 0$ would yield the following expression for the change in the first-period deficit due to a change in D_0:

(8) $$\frac{\partial D}{\partial D_0} = \frac{-(1-\pi)U''(W-(D+D_0))}{U''(W+D) + (1-\pi)U''(W-(D+D_0))},$$

and so again we have $-1 < \partial D/\partial D_0 < 0$.

Thus an increase in initial debt reduces the period-1 deficit; that is, it reduces borrowing by the first-period policymaker. An increase in debt, all else equal, reduces the resources available to the period-2 policymaker since she is the one that has to pay off this initial debt. In this model, the reason there are deficits is that there is a positive probability that the period-2 policymaker will devote the economy's resources to an activity that, in the view of the period-1 policymaker, simply wastes resources. The period-1 policymaker therefore has an incentive to reduce resources available in the second period by transferring resources from the second period to the first period by borrowing.

There is, however, also a chance that the period-2 policymaker will share the same preferences as the period-1 policymaker and devote all resources to the same type of purchases. But since an increase in initial debt reduces the resources available in period two, it reduces the amount the period-2 policymaker can purchase and thus increases the marginal utility of purchases in the second period. Since it is optimal to smooth purchases over time this would mean that the period-1 policymaker would actually have incentive to transfer resources to the second period to the extent that it is possible that the period-2 policymaker shares the same preferences. And this competing incentive increases as initial debt increases. Thus the period-1 policymaker borrows less the higher is the initial level of debt.

Problem 11.8

(a) Consider an individual with $\alpha = 1$; that is, someone who prefers military goods. In period one, with probability π the median voter also has $\alpha = 1$ and so the policymaker purchases all military goods giving the individual utility of $U(W + D)$. With probability $(1 - \pi)$, the median voter has $\alpha = 0$ resulting in the purchase of all non-military goods giving the individual $U(0)$.

Similarly in period two, with probability π the median voter has $\alpha = 1$ and so the policymaker devotes all available resources, $W - D$, to military goods giving utility of $U(W - D)$ to the $\alpha = 1$ individual. With probability $(1 - \pi)$, the median voter has $\alpha = 0$ resulting in the purchase of all non-military goods giving the individual $U(0)$.

Thus the individual with $\alpha = 1$ has expected utility, denoted $E[V]$, given by

(1) $E[V] = \pi U(W + D) + (1 - \pi)U(0) + \pi U(W - D) + (1 - \pi)U(0)$.

(b) The first-order condition for this individual's most preferred value of D is

(2) $\dfrac{\partial E[V]}{\partial D} = \pi U'(W + D) + \pi U'(W - D)(-1) = 0$,

or

(3) $U'(W + D) = U'(W - D)$.

With a well-behaved utility function, for example with $U''(\bullet) < 0$ everywhere, this implies

(4) $W + D = W - D$,

and thus implies

(5) $D = 0$.

The individual prefers a balanced budget so that no debt is issued.

(c) Similarly for someone with $\alpha = 0$ -- someone who prefers all non-military goods -- expected utility is given by

(6) $E[V] = \pi U(0) + (1 - \pi)U(W + D) + \pi U(0) + (1 - \pi)U(W - D)$.

The first-order condition is given by

(7) $\dfrac{\partial E[V]}{\partial D} = (1 - \pi)U'(W + D) + (1 - \pi)U'(W - D)(-1) = 0$,

or

(8) $U'(W + D) = U'(W - D)$,

and thus, again, this implies

(9) $D = 0$.

(d) Since all voters prefer $D = 0$, so does the median voter and thus the policymaker will pursue a balanced budget policy and not issue any debt.

(e) A balanced-budget requirement forces $D = 0$ for everyone. Without a requirement, the period-1 policymaker would choose D freely. Thus, it is possible that an $\alpha = 0$ individual would choose a different

D than an $\alpha = 1$ individual; in fact, unless $\pi = 1/2$, they definitely would choose different values of D. Thus, answering part (d) does not answer the question of whether individuals will support a balanced-budget requirement.

Problem 11.9

(a) Since $\alpha = 1$, the period-1 median voter -- who controls policy in both periods one and two -- purchases all military goods in those two periods giving utility of $U(W + D_1)$ in the first period and $U(W + D_2)$ in the second period, where D_i represents the amount of debt issued in period i. In the third period, with probability π the period-3 median voter has $\alpha = 1$ and devotes all available resources, $W - D_1 - D_2$, to military purchases giving utility of $U(W - D_1 - D_2)$ to the $\alpha = 1$ individual. With probability $(1 - \pi)$, the period-3 median voter purchases all non-military goods giving utility of $U(0)$ to the $\alpha = 1$ individual. Thus expected utility for someone with $\alpha = 1$, denoted E[V], is

(1) $E[V] = U(W + D_1) + U(W + D_2) + \pi U(W - D_1 - D_2) + (1 - \pi)U(0)$.

The period-1 median voter chooses D_1 and D_2. The first-order conditions are

(2) $$\frac{\partial E[V]}{\partial D_1} = U'(W + D_1) - \pi U'(W - D_1 - D_2) = 0,$$

and

(3) $$\frac{\partial E[V]}{\partial D_2} = U'(W + D_2) - \pi U'(W - D_1 - D_2) = 0.$$

Equations (2) and (3) imply

(4) $U'(W + D_1) = U'(W - D_2)$.

With $U''(\bullet) < 0$ everywhere, this implies

(5) $W + D_1 = W + D_2$,

and so

(6) $D_1 = D_2$.

Thus the policymaker issues the same amount of debt in each of the first two periods and so purchases in each of the first two periods, $M_1 = W + D_1$ and $M_2 = W + D_2$, must also be equal.

(b) To see how the amount of debt issued in period two, D_2, varies with π we can implicitly differentiate the first-order condition given by equation (3) with respect to π. Note that we are treating D_1 as given since we are assuming that the change in π occurs after period one and thus after D_1 has been chosen. We have

(7) $$U''(W + D_2)\frac{\partial D_2}{\partial \pi} + (-1)U'(W - D_1 - D_2) + (-\pi)U''(W - D_1 - D_2)\left(-\frac{\partial D_2}{\partial \pi}\right) = 0.$$

Collecting the terms in $\partial D_2/\partial\pi$ gives us

(8) $$[U''(W + D_2) + \pi U''(W - D_1 - D_2)]\frac{\partial D_2}{\partial \pi} = U'(W - D_1 - D_2),$$

and thus

(9) $$\frac{\partial D_2}{\partial \pi} = \frac{U'(W - D_1 - D_2)}{U''(W + D_2) + \pi U''(W - D_1 - D_2)} < 0,$$

since $U'(\bullet) > 0$ and $U''(\bullet) < 0$. Thus a fall in π increases D_2. Thus the policymaker issues more debt and increases purchases in period two after the news that it is less likely that the period-3 median voter also prefers military goods. Intuitively, since it is now more likely that the period-3 median voter will prefer non-military goods, which the period-1 median voter deems wasteful, the period-1 median voter transfers more resources from the third period to the second period by borrowing more and devotes the additional resources with certainty to the type of good she prefers.

Problem 11.10

(a) The period-2 policymaker's objective function is

(1) $F_2 = U + \alpha_2 [V(G_1) + V(G_2)]$.

Substituting for private utility, $U = W - C(T_1) - C(T_2)$, and using the fact that taxes in period two must equal government consumption plus debt, $T_2 = G_2 + D$, gives us

(2) $F_2 = W - C(T_1) - C(G_2 + D) + \alpha_2 [V(G_1) + V(G_2)]$.

The period-2 policymaker takes W, T_1, and D as given and thus the first-order condition is

(3) $$\frac{\partial F_2}{\partial G_2} = -C'(G_2 + D) + \alpha_2 V'(G_2) = 0.$$

(b) Implicitly differentiating equation (3) with respect to D yields

(4) $$-C''(G_2 + D)\left[\frac{\partial G_2}{\partial D} + 1\right] + \alpha_2 V''(G_2)\frac{\partial G_2}{\partial D} = 0,$$

or

(5) $$\left[\alpha_2 V''(G_2) - C''(G_2 + D)\right]\frac{\partial G_2}{\partial D} = C''(G_2 + D).$$

This implies

(6) $$\frac{\partial G_2}{\partial D} = \frac{C''(G_2 + D)}{\alpha_2 V''(G_2) - C''(G_2 + D)} < 0,$$

since $C''(\bullet) > 0$ and $V''(\bullet) < 0$. Thus an increase in debt reduces the period-2 policymaker's choice of government consumption.

(c) The period-1 policymaker's objective function, substituting for private utility, is

(7) $F_1 = W - C(T_1) - C(T_2) + \alpha_1 [V(G_1) + V(G_2)]$.

Note that G_2 is a function of D or $G_2 = G_2(D)$, and that since $D = G_1 - T_1$ we can write $T_1 = G_1 - D$. In addition, $T_2 = G_2 + D$. Thus (7) becomes

(8) $F_1 = W - C(G_1 - D) - C(G_2(D) + D) + \alpha_1 [V(G_1) + V(G_2(D))]$.

The first-order conditions for the choices of G_1 and D are

(9) $$\frac{\partial F_1}{\partial G_1} = -C'(G_1 - D) + \alpha_1 V'(G_1) = 0,$$

and

(10) $$\frac{\partial F_1}{\partial D} = -C'(G_1 - D)(-1) - C'(G_2(D) + D)[G_2'(D) + 1] + \alpha_1 V'(G_2(D))G_2'(D) = 0.$$

(d) Solving equation (3) for $V'(G_2(D))$ gives us

(11) $$V'(G_2(D)) = \frac{C'(G_2(D) + D)}{\alpha_2}.$$

Substituting equation (11) into equation (10) yields

(12) $$C'(G_1 - D) - C'(G_2(D) + D)[G_2'(D) + 1] + \frac{\alpha_1}{\alpha_2}C'(G_2(D) + D)G_2'(D) = 0,$$

which can be rewritten as

(13) $$C'(G_1 - D) - C'(G_2(D) + D) = C'(G_2(D) + D)G_2'(D) - \frac{\alpha_1}{\alpha_2}C'(G_2(D) + D)G_2'(D).$$

Collecting terms on the right-hand side of (13) gives us

(14) $C'(G_1 - D) - C'(G_2(D) + D) = C'(G_2(D) + D)G_2'(D)\left(1 - \frac{\alpha_1}{\alpha_2}\right)$.

As shown in part (b), $G_2'(D) < 0$ and since $C'(\bullet) > 0$, then if $\alpha_1 < \alpha_2$, the right-hand side of (14) is negative. Thus

(15) $C'(G_1 - D) - C'(G_2(D) + D) < 0$,

or

(16) $C'(G_1 - D) < C'(G_2(D) + D)$.

Since $C''(\bullet) > 0$ this implies

(17) $G_1 - D < G_2(D) + D$.

Since $D = G_1 - T_1$ or $T_1 = G_1 - D$ and $T_2 = G_2(D) + D$, this is equivalent to

(18) $T_1 < T_2$.

Intuitively, if $\alpha_1 < \alpha_2$ this means the period-1 policymaker values government consumption less than the period-2 policymaker. Thus the period-1 policymaker attempts to "enforce discipline" on the period-2 policymaker. The lower-α policymaker in period one keeps taxes low and thus passes along a relatively higher level of D in order to force the period-2 policymaker to choose a lower level of government consumption.

(e) Not necessarily. If $\alpha_1 < \alpha_2$, the period-1 policymaker will choose a lower level of government purchases than the period-2 policymaker. To see this, substitute equations (3) and (9) into the first-order condition given by (10):

(19) $\alpha_1 V'(G_1) - \alpha_2 V'(G_2(D))[G_2'(D) + 1] + \alpha_1 V'(G_2(D))G_2'(D) = 0$,

which can be rewritten as

(20) $\alpha_1 V'(G_1) = V'(G_2(D))[-\alpha_1 G_2'(D) + \alpha_2 (G_2'(D) + 1)]$,

which implies

(21) $$\frac{V'(G_1)}{V'(G_2(D))} = \frac{\alpha_2}{\alpha_1} + \left(\frac{\alpha_2 - \alpha_1}{\alpha_1}\right)G_2'(D).$$

Adding and subtracting $(\alpha_2 - \alpha_1)/\alpha_1$ from the right-hand side of (21) yields

(22) $$\frac{V'(G_1)}{V'(G_2(D))} = \frac{\alpha_2}{\alpha_1} - \frac{\alpha_2 - \alpha_1}{\alpha_1} + \left(\frac{\alpha_2 - \alpha_1}{\alpha_1}\right)[G_2'(D) + 1],$$

or simply

(23) $$\frac{V'(G_1)}{V'(G_2(D))} = 1 + \left(\frac{\alpha_2 - \alpha_1}{\alpha_1}\right)[G_2'(D) + 1].$$

From equation (6), we can see that $G_2'(D) > -1$ or $G_2'(D) + 1 > 0$. In addition, our assumption is that $\alpha_2 - \alpha_1 > 0$. Thus

(24) $$\frac{V'(G_1)}{V'(G_2(D))} > 1,$$

or $V'(G_1) > V'(G_2(D))$. Since $V'(\bullet) < 0$, this implies $G_1 < G_2(D)$. Thus, not only does the period-1 policymaker choose a lower level of taxes, she also chooses a lower level of government consumption than the period-2 policymaker. Thus $D = G_1 - T_1 = T_2 - G_2$ can be either positive or negative.

<u>**Problem 11.11**</u>

(a) As T, the amount of taxes that reform requires, falls then $V'(X)$ at $X = A$ also falls since

(1) $$V'(X = A) = \frac{[B - (W - T)] - 2A}{B - A},$$

and we have

$$(2)\quad \frac{\partial V'(X=A)}{\partial T}=\frac{1}{B-A}>0.$$

In the case in which V '(X) at X = A was already negative, it is now more negative and there is no effect on workers' offer or the probability of reform. Workers continue to offer X* = A and the probability of reform, P(X*), continues to equal one.

If initially V '(X) at X = A was positive and the change in T is small enough, V '(X = A) will still be positive. In this case, from equation (11.36) in the text, workers' offer is

$$(3)\quad X^*=\frac{B-(W-T)}{2},$$

and so

$$(4)\quad \frac{\partial X^*}{\partial T}=\frac{1}{2}>0.$$

Thus a fall in T reduces workers' offer. From equation (11.37) in the text, the probability of reform is

$$(5)\quad P(X^*)=\frac{B+(W-T)}{2(B-A)},$$

and so

$$(6)\quad \frac{\partial P(X^*)}{\partial T}=\frac{-1}{2(B-A)}<0.$$

The fall in T increases the probability of reform in this case.

Finally, if V '(X) at X = A was initially positive and the change in T is large enough, it will become negative. In this case, workers' offer will now be X* = A and reform will now occur with certainty.

(b) An increase in B, the upper bound on capitalists' pre-tax payoff from reform, means that V '(X) at X = A also increases since

$$(7)\quad \frac{\partial V'(X=A)}{\partial B}=\frac{(B-A)-[B-(W-T)]+2A}{(B-A)^2}=\frac{(W-T)+A}{(B-A)^2}>0.$$

Thus, if initially V '(X) at X = A was positive, it still will be. Workers' offer continues to be given by equation (3) and

$$(8)\quad \frac{\partial X^*}{\partial B}=\frac{1}{2}>0.$$

Thus workers' offer increases. That is, with an increase in the upper bound on capitalists' payoff, workers ask capitalists to pay a greater share of the costs of reform. Using equation (5) we have

$$(9)\quad \frac{\partial P(X^*)}{\partial B}=\frac{2(B-A)-[B+(W-T)]2}{4(B-A)^2}=\frac{-[A+(W-T)]}{2(B-A)^2}<0.$$

The increase in B causes the probability of reform to fall.

If initially V '(X) at X = A was negative and the rise in B is small enough, it will continue to be negative. There will be no effect on workers' offer, which continues to be X* = A, or on the probability of reform, which continues to be one. If, however, the rise in B is large enough, V '(X) at X = A becomes positive in which case workers' offer will now be greater than A and the probability of reform will fall below one.

(c) An upward shift in the distribution of capitalists' payoff -- an equal increase in A and B -- means that V '(X) at the new X = A' will be lower than V '(X) at the original X = A. We can see this since

$$(10)\ V'(X=A)=\frac{[B-(W-T)]-2A}{B-A}.$$

An equal increase in A and B leaves the denominator unchanged and reduces the numerator.

In the case in which V '(X) at X = A was negative, it is now more negative at the new X = A'. Thus workers' offer rises and equals the new A' and the probability of reform continues to be one.

If initially V '(X) at X = A was positive and the change in A and B is small enough, V ' (X) will still be positive at the new X = A'. Note that A does not enter workers' offer here and so we need only examine the derivative of X* with respect to B:

$$(11)\ \frac{\partial X^*}{\partial B}=\frac{1}{2}>0.$$

Hence workers' offer rises proportionately less than B (or A). From equation (5) which gives the probability of reform, we can see that the probability of reform increases since the numerator rises whereas the denominator is unchanged.

Finally, if V '(X) at X = A was positive and the change in A and B is large enough, V '(X) at the new X = A' will be negative. Thus workers' offer will equal the new A' and the probability of reform becomes one.

Problem 11.12

(a) If the capitalists accept the workers' proposal and reform occurs, their payoff is π - X. If they reject the proposal, their payoff is now -C, C ≥ 0, rather than zero. They therefore accept when π - X > -C, or π > X - C. Since π is distributed uniformly on [A, B] this probability is

$$(1)\ P(X)=\begin{cases}1 & \text{if}\quad X-C\le A \quad\text{or}\quad X\le A+C\\[4pt] \dfrac{B-(X-C)}{B-A} & \text{if}\quad A<X-C<B \quad\text{or}\quad A+C<X<B+C\\[4pt] 0 & \text{if}\quad X-C\ge B \quad\text{or}\quad X\ge B+C,\end{cases}$$

where we have used the fact that for A + C < X < B + C, P(X) = P(π > X - C) = 1 - P(π < X - C) which in turns equals 1 - [(X - C) - A]/(B - A) or simply [B - (X - C)]/(B - A).

The workers receive (W - T) + X if their proposal is accepted and -C if it is rejected. Their expected payoff, V(X), therefore equals P(X)[(W - T) + X] + [1 - P(X)](-C). Using equation (1), this equals

$$(2)\ V(X)=\begin{cases}(W-T)+X & \text{if}\quad X\le A+C\\[4pt] \dfrac{[B-(X-C)][(W-T)+X+F]}{B-A}+\left[\dfrac{1-(B-(X-C))}{B-A}\right](-C) & \text{if}\quad A+C<X<B+C\\[4pt] -C & \text{if}\quad X\ge B+C.\end{cases}$$

As in the model in the text, there are two possibilities. First, the workers may choose a value of X in the interior of [A + C, B + C] so that the probability of the capitalists accepting the proposal is strictly between zero and one. Second, the workers may make the least-generous proposal that they know will be accepted for sure, which is X = A + C.

Using equation (2) to find the derivative of V(X) with respect to X for A + C < X < B + C yields

(3) $V'(X) = \dfrac{B-(W-T)-2X-C+C}{B-A} = \dfrac{[B-(W-T)]-2X}{B-A}.$

Note that V "(X) is negative over the whole range we are considering. Thus if V '(X) is negative at X = A + C, it is negative over all of [A + C, B + C]. In this case, workers propose X = A + C, the least-generous proposal they know will be accepted for sure. This occurs when V '(X = A + C) < 0 or when [B - (W - T)] - 2(A + C) < 0.

The alternative is for V '(X) to be positive at X = A + C. In this case, the optimum is interior to the interval [A + C, B + C] and is defined by V '(X) = 0. From equation (3), this occurs when [B - (W - T)] - 2X = 0. Thus, analogous to equation (11.36) in the text, we have

(4) $X^* = \begin{cases} A+C & \text{if } [B-(W-T)]-2(A+C) \le 0 \\ \dfrac{B-(W-T)}{2} & \text{if } [B-(W-T)]-2(A+C) > 0. \end{cases}$

Thus, using equation (1) and substituting for X*, we have the following expression for the equilibrium probability that the proposal is accepted:

(5) $P(X^*) = \begin{cases} 1 & \text{if } [B-(W-T)]-2(A+C) \le 0 \\ \dfrac{B+(W-T)+2C}{B-A} & \text{if } [B-(W-T)]-2(A+C) > 0. \end{cases}$

Equation (5) is analogous to equation (11.37) in the text.

(b) If, in equilibrium, V '(X) at X = A + C is less than or equal to zero, then workers offer X* = A + C and P(X*) = 1. In this case, workers get (W - T) + (A + C) and capitalists expected payoff is E[π] - (A + C). Thus social welfare, SW(X*), is given by

(6) SW(X*) = (W - T) + (A + C) + E[π] - (A + C) = (W - T) + E[π].

Since π is distributed uniformly on [A, B], E[π] = (A + B)/2 and thus

(7) SW(X*) = (W - T) + (A + B)/2.

From equation (3), we can see that V '(X) evaluated at X = A + C is decreasing in C. Thus if V '(X) is negative initially, it still will be after an increase in C and social welfare will remain unchanged as reform still occurs with probability one. Social welfare is higher with reform than without and so initially if V '(X) at X = A + C is positive and the increase in C is large enough, it becomes negative at the new X = A + C'. The reform now occurs with certainty and social welfare is therefore higher.

Finally, if V '(X) at X = C was initially positive and the rise in C is small enough, V '(X) at the new X = A + C' will still be positive. We need to determine equilibrium social welfare in this case and the change in equilibrium social welfare due to a change in C.

For workers, the expected payoff, denoted V(X*), equals the probability of acceptance times the payoff from acceptance plus the probability of rejection -- which is one minus the probability of acceptance -- times the payoff from rejection, or from equation (2),

(8) $V(X^*) = \dfrac{[B-(X^*-C)][(W-T)+X^*]}{B-A} + \left[1 - \dfrac{B-(X^*-C)}{B-A}\right](-C),$

which can be rewritten as

(9) $V(X^*) = \dfrac{[B-(X^*-C)](W-T)}{B-A} + \dfrac{[B-(X^*-C)]X^*}{B-A} + \dfrac{[A-(X^*-C)]C}{B-A}.$

For capitalists, if π turns out to be less than $X^* - C$, they reject the proposal and receive $-C$. If π turns out to be greater than $X^* - C$, they accept the proposal and receive $\pi - X^*$. Since π is distributed uniformly on [A, B], the probability density function of π over that interval is $f(\pi) = 1/(B - A)$. Thus, capitalists' expected payoff, denoted $K(X^*)$, is given by

$$(10)\quad K(X^*) = \int_{\pi=A}^{X^*-C} \frac{-C}{B-A} d\pi + \int_{\pi=X^*-C}^{B} \frac{\pi - X^*}{B-A} d\pi .$$

The first integral on the right-hand side of equation (10) is given by

$$(11)\quad \int_{\pi=A}^{X^*-C} \frac{-C}{B-A} d\pi = \frac{-C[(X^*-C)-A]}{B-A} = \frac{[A-(X^*-C)]C}{B-A} .$$

The second integral on the right-hand side of (10) is given by

$$(12)\quad \int_{\pi=X^*-C}^{B} \frac{\pi - X^*}{B-A} d\pi = \frac{1}{B-A}\left[\left(\frac{1}{2}\pi^2 - X^*\pi\right)\Bigg|_{\pi=X^*-C}^{B}\right],$$

or

$$(13)\quad \int_{\pi=X^*-C}^{B} \frac{\pi - X^*}{B-A} d\pi = \frac{1}{B-A}\left[\frac{1}{2}B^2 - BX^* - \frac{1}{2}(X^*-C)^2 + (X^*-C)X^*\right],$$

which can be factored as follows:

$$(14)\quad \int_{\pi=X^*-C}^{B} \frac{\pi - X^*}{B-A} d\pi = \frac{1}{2(B-A)}\left[B^2 - (X^*-C)^2\right] - \frac{1}{B-A}\left[B-(X^*-C)\right]X^* .$$

Social welfare, which is the sum of the expected payoffs of workers and capitalists, can be obtained by adding equations (9), (11), and (14):

$$(15)\quad SW(X^*) = \frac{[B-(X^*-C)](W-T)}{B-A} + 2\frac{[A-(X^*-C)]C}{B-A} + \frac{[B^2-(X^*-C)^2]}{2(B-A)} .$$

Since X^* does not depend on C -- see equation (4) -- the change in equilibrium social welfare due to a change in the cost of a crisis, C, is

$$(16)\quad \frac{\partial SW(X^*)}{\partial C} = \frac{(W-T)+2A-2X^*+4C}{B-A} + \frac{2X^*-2C}{2(B-A)} = \frac{2(W-T)+4A+6C-2X^*}{2(B-A)} .$$

Substituting for $X^* = [B - (W - T)]/2$ gives us

$$(17)\quad \frac{\partial SW(X^*)}{\partial C} = \frac{2(W-T)+4A+6C-B+(W-T)}{2(B-A)} = \frac{3(W-T)+4A+6C-B}{2(B-A)} .$$

Thus, depending on the magnitude of B relative to $3(W - T) + 4A + 6C$, an increase in the cost of a crisis can, but does not necessarily, increase social welfare. So, for example, a high value of B -- the upper bound on capitalists' pre-tax payoff from reform -- makes it less likely that an increase in the cost of a crisis will increase social welfare.

Problem 11.13

If the capitalists accept the workers' proposal and reform occurs, their payoff is $\pi + F - X$, where $F > 0$ is the amount of aid they receive from the international agency. If they reject the proposal, they receive zero. They therefore accept when $\pi + F - X > 0$, or $\pi > X - F$. Since π is distributed uniformly on [A, B] this probability is

$$(1)\quad P(X)=\begin{cases}1 & \text{if}\quad X-F\le A \quad\text{or}\quad X\le A+F\\ \dfrac{B-(X-F)}{B-A} & \text{if}\quad A<X-F<B \quad\text{or}\quad A+F<X<B+F\\ 0 & \text{if}\quad X-F\ge B\quad\text{or}\quad X\ge B+F,\end{cases}$$

where we have used the fact that for A + F < X < B + F, P(X) = P(π > X - F) = 1 - P(π < X - F) which in turns equals 1 - [(X - F) - A]/(B - A) or simply [B - (X - F)]/(B - A).

The workers receive (W - T) + X + F if their proposal is accepted and zero if it is rejected. Their expected payoff, V(X), therefore equals P(X)[(W - T) + X + F]. Using equation (1), this equals

$$(2)\quad V(X)=\begin{cases}(W-T)+X+F & \text{if}\quad X\le A+F\\ \dfrac{[B-(X-F)][(W-T)+X+F]}{B-A} & \text{if}\quad A+F<X<B+F\\ 0 & \text{if}\quad X\ge B+F.\end{cases}$$

As in the model in the text, there are two possibilities. First, the workers may choose a value of X in the interior of [A + F, B + F] so that the probability of the capitalists accepting the proposal is strictly between zero and one. Second, the workers may make the least-generous proposal that they know will be accepted for sure, which is X = A + F.

Using equation (2) to find the derivative of V(X) with respect to X for A + F < X < B + F yields

$$(3)\quad V'(X)=\frac{B-(W-T)-2X-F+F}{B-A}=\frac{[B-(W-T)]-2X}{B-A}.$$

Note that V "(X) is negative over the whole range we are considering. Thus if V '(X) is negative at X = A + F, it is negative over all of [A + F, B + F]. In this case, workers propose X = A + F, the least-generous proposal they know will be accepted for sure. This occurs when V '(X = A + F) < 0 or when [B - (W - T)] - 2(A + F) < 0.

The alternative is for V '(X) to be positive at X = A + F. In this case, the optimum is interior to the interval [A + F, B + F] and is defined by V '(X) = 0. From equation (3), this occurs when [B - (W - T)] - 2X = 0. Thus, analogous to equation (11.36) in the text, we have

$$(4)\quad X^*=\begin{cases}A+F & \text{if}\quad [B-(W-T)]-2(A+F)\le 0\\ \dfrac{B-(W-T)}{2} & \text{if}\quad [B-(W-T)]-2(A+F)>0.\end{cases}$$

Thus, using equation (1) and substituting for X*, we have the following expression for the equilibrium probability that the proposal is accepted:

$$(5)\quad P(X^*)=\begin{cases}1 & \text{if}\quad [B-(W-T)]-2(A+F)\le 0\\ \dfrac{B+(W-T)+2F}{B-A} & \text{if}\quad [B-(W-T)]-2(A+F)>0.\end{cases}$$

Comparing equation (5) to equation (11.37) in the text, we can see that the presence of F > 0, the positive amount of aid, increases the probability of reform, if reform did not already occur with certainty. If F is large enough, reform now occurs with probability one since, as discussed above, reform occurs with certainty if [B - (W - T)] - 2(A + F) < 0.

Otherwise, from equation (5), we can see that P(X*) rises as F rises since

$$(6)\quad \frac{\partial P(X^*)}{\partial F} = \frac{2}{B-A} > 0.$$

We now need to determine the impact of the international aid on social welfare, defined as the sum of the expected payoffs of workers and capitalists. If, in equilibrium, V '(X) at X = A + F is less than or equal to zero, then workers offer X* = A + F and P(X*) = 1. In this case, workers get (W - T) + X* + F or simply (W - T) + A + 2F. Capitalists expected payoff is E[π] - X* + F or simply E[π] - A. Thus social welfare, SW(X*), is given by

(7) $SW(X^*) = (W - T) + A + 2F + E[\pi] - A = (W - T) + 2F + E[\pi]$.

Since π is distributed uniformly on [A, B], E[π] = (A + B)/2 and thus

(8) $SW(X^*) = (W - T) + 2F + (A + B)/2$.

From equation (3), we can see that V '(X) evaluated at X = A + F is decreasing in F. Thus if V '(X) is negative initially, it still will be. Here, since reform would have occurred anyway, social welfare simply increases by the total payoff from the international agency, which is 2F, and the entire amount of aid is extracted by workers.

If initially, V '(X) at X = A was positive and F is large enough, the aid causes V '(X) at X = A + F to be negative so that reform now occurs with certainty. Since social welfare is higher with reform, social welfare is higher in this case also.

Finally, if V '(X) at X = A was initially positive and F is small enough, V '(X) at X = A + F will still be positive. We need to determine equilibrium social welfare in this case. Equation (2) describes workers' expected payoff. It equals the probability of acceptance times the payoff from acceptance; the payoff from rejection is zero. Thus, from equation (2),

$$(9)\quad V(X^*) = \frac{[B-(X^*-F)][(W-T)+X^*+F]}{B-A}.$$

Substituting X* = [B - (W - T)]/2 into equation (9) gives us

$$(10)\quad V(X^*) = \frac{\left[B-\left(\frac{B-(W-T)}{2}\right)+F\right]\left[(W-T)+\left(\frac{B-(W-T)}{2}\right)+F\right]}{B-A},$$

which simplifies to

$$(11)\quad V(X^*) = \frac{[2B-B+(W-T)+2F][2(W-T)+B-(W-T)+2F]}{4(B-A)},$$

and thus workers' expected payoff is given by

$$(12)\quad V(X^*) = \frac{[B+(W-T)+2F]^2}{4(B-A)}.$$

For capitalists, if π turns out to be less than X* - F, they reject the proposal and receive zero. If π turns out to be greater than X* - F, they accept the proposal and receive π - X* + F or π - (X* - F). Since π is distributed uniformly on [A, B], the probability density function of π over that interval is f(π) = 1/(B - A). Thus, capitalists' expected payoff, denoted K(X*), is given by

$$(13)\quad K(X^*) = \int_{\pi=X^*-F}^{B} \frac{\pi-(X^*-F)}{B-A}\,d\pi.$$

Solving the integral in equation (13) gives us

(14) $K(X^*) = \frac{1}{B-A}\left[\left(\frac{1}{2}\pi^2 - (X^*-F)\pi\right)\Bigg|_{\pi=X^*-F}^{B}\right]$,

which simplifies to

(15) $K(X^*) = \frac{1}{B-A}\left[\frac{1}{2}B^2 - (X^*-F)B - \frac{1}{2}(X^*-F)^2 + (X^*-F)^2\right]$,

which can be factored as

(16) $K(X^*) = \frac{1}{2(B-A)}\left[B^2 - 2B(X^*-F) + (X^*-F)^2\right] = \frac{1}{2(B-A)}[B-(X^*-F)]^2$.

Substituting $X^* = [B - (W - T)]/2$ into equation (16) gives us

(17) $K(X^*) = \frac{1}{2(B-A)}\left[B - \left(\frac{B-(W-T)}{2}\right) + F)\right]^2 = \frac{1}{8(B-A)}[B+(W-T)+2F]^2$.

Total social welfare is the sum of the expected payoffs for workers and capitalists. Adding equations (12) and (17) gives us

(18) $SW(X^*) = V(X^*) + K(X^*) = \frac{[B+(W-T)+2F]^2}{4(B-A)} + \frac{[B+(W-T)+2F]^2}{8(B-A)}$,

or simply

(19) $SW(X^*) = \frac{3[B+(W-T)+2F]^2}{8(B-A)}$.

From equation (19), we can see that social welfare is increasing in F and so the aid package from the international agency does raise social welfare unambiguously.

Problem 11.14

(a) Of the fraction f of the population that knows its welfare under both policies, fraction α is better off with Policy A. Thus fraction αf of those who know their welfare prefer Policy A.

Ex ante, the individuals in the fraction (1 - f) of the population that does not know its welfare are all identical. Each of these individuals will prefer Policy A if their expected utility from A exceeds that from B. The expected utility from Policy A, relative to that from Policy B, denoted $E[U^A]$, is given by

(1) $E[U^A] = \beta(+1) + (1 - \beta)(-1) = 2\beta - 1$,

since with probability β they will be one unit of utility better off and with probability $(1 - \beta)$ they will be one unit of utility worse off. These individuals will all prefer Policy A if $2\beta - 1 > 0$ or $\beta > 1/2$. If $\beta < 1/2$, all of these individuals prefer Policy B and if $\beta = 1/2$, they are indifferent.

Thus the fraction of the population that prefers Policy A under uncertainty, denoted X_u^A, is given by

(2) $X_u^A = \begin{cases} \alpha f + (1-f) & \text{if } \beta > 1/2 \\ \alpha f & \text{if } \beta < 1/2. \end{cases}$

(Note that if $\beta = 1/2$, the fraction of the population that prefers Policy A would be αf plus (1 - f) times the fraction of those who are indifferent who decide to choose A.)

(b) Of the fraction f that always knows its welfare under both policies, fraction α prefer A. Now, of the fraction (1 - f) that previously did not know its welfare, fraction β find out that they are definitely better off under Policy A. Thus $\beta(1 - f)$ now prefer A. Thus the fraction of the population who prefer Policy A under certainty, denoted X_c^A, is given by

(3) $X_c^A = \alpha f + \beta(1 - f)$.

(c) There are cases when whichever policy is initially in effect is retained. Suppose Policy A is in effect. From equation (2) we can see that a proposal to switch to Policy B will be defeated if, for example, $\beta > 1/2$ and $\alpha f + (1 - f) \geq 1/2$. The sum of the people who know their welfare and are better off with A, αf, plus the entire fraction of the population who are uncertain, $(1 - f)$, vote to retain A in this case. If they constitute at least half of the population, the proposal is defeated.

Suppose Policy B is in effect. We are assuming that no one votes for a switch to Policy A if they know that once everyone's welfare is revealed, the majority would vote to revert back to Policy B. From equation (3), once welfare is revealed, fraction $\alpha f + \beta(1 - f)$ prefer A. If this is less than 1/2, the majority would vote to return to Policy B.

Thus whichever policy is in effect would be retained if $\beta > 1/2$, $\alpha f + (1 - f) \geq 1/2$, and $\alpha f + \beta(1 - f) < 1/2$. Because $\alpha f + (1 - f)$ is greater than $\alpha f + \beta(1 - f)$, it is easy to find parameter values that satisfy these conditions. One example is $f = 0.5$, $\alpha = 0.2$, and $\beta = 0.6$. In this particular example, everyone knows that ex post, Policy B is preferred by the majority. Yet if Policy A is in effect, it is retained. This is driven by the fact that the entire portion of the population that is uncertain about its welfare maximizes its expected utility by voting for A but once that fraction of the population learns its welfare, not enough of them are better under A to constitute a majority when joined with the others who always preferred A.

<u>**Problem 11.15**</u>

(a) The representative from district j will maximize the utility of the representative person in that district, which is given by

(1) $U_j = E + V(G_j) - C(T)$,

subject to the budget constraint given by

(2) $\sum_{i=1}^{M} G_i = MT$,

which can be rewritten as

$$(3)\quad T = \frac{\sum_{i=1}^{M} G_i}{M}.$$

Substituting equation (3) into equation (1) gives us

$$(4)\quad U_j = E + V(G_j) - C\left(\frac{\sum_{i=1}^{M} G_i}{M}\right).$$

The first-order condition is given by

$$(5)\quad \frac{\partial U_j}{\partial G_j} = V'(G_j) - C'\left(\frac{\sum_{i=1}^{M} G_i}{M}\right)\frac{1}{M} = 0,$$

or

$$(6)\quad V'(G_j) = \frac{1}{M}C'\left(\frac{\sum_{i=1}^{M} G_i}{M}\right).$$

(b) We want a value of G, denoted G^N, that is optimal for a representative to choose given that all other representatives are choosing that level. Substituting that common choice of G^N for G_j and all the G_i's in the condition defining the optimal choice of G, equation (6), gives us

$$\text{(7)}\quad V'(G^N) = \frac{1}{M}C'\left(\frac{\sum_{i=1}^{M} G^N}{M}\right) = \frac{1}{M}C'\left(\frac{MG^N}{M}\right),$$

or simply

$$\text{(8)}\quad V'(G^N) = \frac{1}{M}C'(G^N).$$

Representatives choose a level of the local public good, G^N, such that the marginal utility of G^N equals only their district's share of the marginal distortion costs of the taxes needed to finance that good.

(c) To see if the Nash equilibrium is Pareto efficient, we can examine the social planner's problem. A social planner would maximize the sum of the utilities of the representative person in each district, which is given by

$$\text{(9)}\quad \sum_{j=1}^{M} U_j = \sum_{j=1}^{M}\left[E + V(G_j) - C\left(\frac{\sum_{i=1}^{M} G_i}{M}\right)\right],$$

where we have already substituted for the budget constraint using equation (3). Equation (9) simplifies to

$$\text{(10)}\quad \sum_{j=1}^{M} U_j = ME + \sum_{j=1}^{M} V(G_j) - MC\left(\frac{\sum_{i=1}^{M} G_i}{M}\right).$$

The social planner chooses the same level of the public good in each district, which we can denote $\overline{G}$. Thus equation (10) becomes

$$\text{(11)}\quad \sum_{j=1}^{M} U_j = ME + \sum_{j=1}^{M} V(\overline{G}) - MC\left(\frac{M\overline{G}}{M}\right) = ME + MV(\overline{G}) - MC(\overline{G}).$$

The first-order condition for the choice of $\overline{G}$ is

$$\text{(12)}\quad \frac{\partial \sum_{j=1}^{M} U_j}{\partial \overline{G}} = MV'(\overline{G}) - MC'(\overline{G}) = 0,$$

or simply

$$\text{(13)}\quad V'(\overline{G}) = C'(\overline{G}).$$

The social planner equates the marginal utility of the level of the local public good with the total marginal distortion costs of the taxes required to finance that good. Comparing equations (6) and (13), then since $V''(\bullet) < 0$, we can see that the social planner chooses a lower level of G for each district than the representatives do in the Nash equilibrium. That is, the Nash equilibrium involves an inefficiently high level of local public goods.

Intuitively, in the decentralized equilibrium, an increase in the level of a local public good in any given district gives the representative person in that district marginal utility of $V'(G)$. But the marginal cost of the distortion caused by the extra taxation needed to finance that good is borne by all individuals in all districts. Essentially, there is a negative externality from higher government purchases. Since individuals in any given district do not bear all the costs of extra purchases in that district, purchases are inefficiently high.

Problem 11.16

(a) The representative from district j will maximize the utility of the representative person in that district, which is given by

(1) $U_j = E + V(G_j) - C(T)$,

subject to the budget constraint given by

(2) $D + \sum_{i=1}^{M} G_i = MT$,

which can be rewritten as

(3) $T = \dfrac{D}{M} + \dfrac{\sum_{i=1}^{M} G_i}{M}$.

Substituting equation (3) into equation (1) gives us

(4) $U_j = E + V(G_j) - C\left(\dfrac{D}{M} + \dfrac{\sum_{i=1}^{M} G_i}{M}\right)$.

The first-order condition is given by

(5) $\dfrac{\partial U_j}{\partial G_j} = V'(G_j) - C'\left(\dfrac{D}{M} + \dfrac{\sum_{i=1}^{M} G_i}{M}\right)\dfrac{1}{M} = 0.$

The Nash equilibrium value of G, denoted G^N, is the one that is optimal for a representative to choose given that all other representatives are choosing that level. Substituting that common choice of G^N for G_j and all the G_i's into equation (5), the condition defining the optimal choice of G, gives us

(6) $V'(G^N) - \dfrac{1}{M}C'\left(\dfrac{D}{M} + G^N\right) = 0.$

To see how G^N is affected by changes in the initial amount of debt, we can implicitly differentiate equation (6) with respect to D, which yields

(7) $V''(G^N)\dfrac{\partial G^N}{\partial D} - \dfrac{1}{M}C''\left(\dfrac{D}{M} + G^N\right)\left[\dfrac{\partial G^N}{\partial D} + \dfrac{1}{M}\right] = 0.$

Collecting the terms in $\partial G^N/\partial D$ gives us

(8) $\left[V''(G^N) - \dfrac{1}{M}C''\left(\dfrac{D}{M} + G^N\right)\right]\dfrac{\partial G^N}{\partial D} = \dfrac{1}{M^2}C''\left(G^N + \dfrac{D}{M}\right),$

and thus

(9) $\dfrac{\partial G^N}{\partial D} = \dfrac{\dfrac{1}{M^2}C''\left(G^N + \dfrac{D}{M}\right)}{V''(G^N) - \dfrac{1}{M}C''\left(\dfrac{D}{M} + G^N\right)} < 0,$

since $C''(\bullet) > 0$ and $V''(\bullet) < 0$. Thus an increase in initial debt reduces the Nash equilibrium level of the local public good.

(b) As explained in the solution to Problem 11.15, the representatives would choose an inefficiently high level of local public goods in the first period; the distortion costs of the taxation needed to finance those goods would be inefficiently high. As shown in part (a), the representatives know that by having $D > 0$, they can reduce the purchases of public goods and thus the distortion costs of the taxes because a positive value of debt will reduce the inefficiently high level of government purchases in the second period.

(c) If representatives were to choose D before the first-period value of G is determined, the representatives would choose not to issue any debt. It is true that with $D = 0$, there will be distortion in the choice of local public goods each period as shown in Problem 11.5; the level of local public goods will be inefficiently high. Choosing $D > 0$ reduces the choice of local public goods in the second period, as shown in part (a), which at the margin is desirable. But analogous reasoning would show that it would raise the choice of local public goods in the first period, which at the margin is undesirable. Thus having $D > 0$ does not clearly counteract the "common-pool" distortion. In addition, it introduces departures from tax-smoothing and expenditure-smoothing, and thus it appears that representatives would not choose to issue any debt.

Problem 11.17

The probability density function of T is given by

$$(1)\quad f(T) = \begin{cases} \dfrac{1}{2X} & \text{if} \quad \mu - x \le T \le \mu + x \\ 0 & \text{otherwise.} \end{cases}$$

The associated cumulative distribution function is given by

$$(2)\quad F(T) = \begin{cases} 0 & \text{if} \quad T < \mu - X \\ \dfrac{T - (\mu - X)}{2X} & \text{if} \quad \mu - x \le T \le \mu + x \\ 1 & \text{if} \quad T > \mu - X. \end{cases}$$

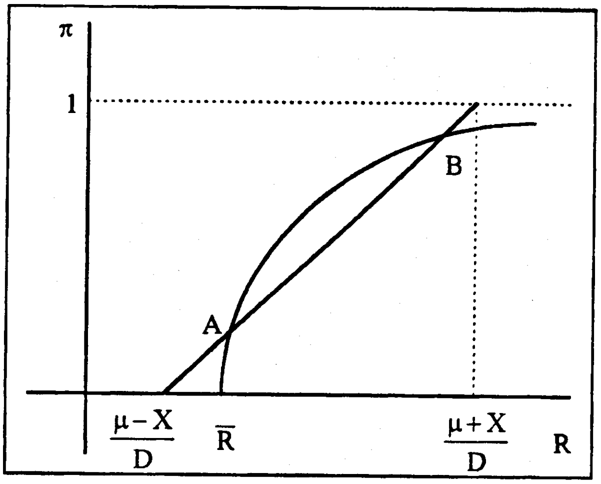
The probability of a default equals the probability that tax revenue, T, is less than the amount due on the debt, RD, and thus equals F(RD). So from equation (2), we can see that the probability of default, π, is given by

$$(3)\quad \pi = F(RD) = \begin{cases} 0 & \text{if} \quad RD < \mu - X \quad \text{or} \quad R < (\mu - X)/D \\ \dfrac{RD - (\mu - X)}{2X} & \text{if} \quad \mu - x \le RD \le \mu + x \quad \text{or} \quad (\mu - x)/D \le R \le (\mu + x)/D \\ 1 & \text{if} \quad RD > \mu + X \quad \text{or} \quad R > (\mu + X)/D. \end{cases}$$

The other equilibrium condition describing combinations of R and π for which investors are willing to hold the economy's debt is still given by

$$(4)\quad \pi = \frac{R - \bar{R}}{R}.$$

Equations (3) and (4) are depicted in the figure at right. This shows the possible situation of multiple equilibria. Under the plausible dynamics described in the text, the equilibrium at A is stable whereas the equilibrium at B is not. Another stable equilibrium occurs when investors are unwilling to hold the economy's debt at any interest rate.



(a) A rise in μ represents an upward shift in the distribution of possible tax revenue without a change in its dispersion. The probability of default line shifts to the right by the change in μ. The locus given by equation (4) is unaffected. The stable equilibrium would now involve a lower interest factor and a lower probability of default.

(b) A fall in X represents a decrease in the dispersion of possible tax revenue without a change in its expected value. The locus given by equation (4) is unaffected. The slope of the probability of default line is given by $\partial\pi/\partial R = D/2X$. Thus this line essentially rotates; it becomes steeper over a smaller range and still goes through the point ($\pi = 1/2$, $R = \mu/D$). If the original intersection between the two equilibrium conditions was at $RD < \mu$ or $R < \mu/D$ (as in the case depicted in the figure above), the new stable equilibrium would involve a lower interest factor and a lower probability of default.