Appendix 1: Continuous Time: First Order Differential Equations

1. Introduction to First-order differential equations

An **ordinary differential equation** is an equation of the form

$G(t, y(t), y'(t), y''(t), \ldots) = 0$ for all t,

where G is a known function and y is an unknown function, $y'(t)$ is the derivative of y with respect to t, $y''(t)$ is the second derivative of y with respect to t, and so on.

To solve this equation we need to find a *function* y that satisfies the equation for all values of t.

The name "differential equation" is qualified by "ordinary" to reflect the fact that only one variable, y, is involved.

In general a differential equation that has a solution has many solutions, each corresponding to a different set of "initial conditions".

Ex 1: If our differential equation is $y'(t) - 1 = 0$, then $y(t) = t + C$ is a solution for any value of C.

If we know that $y(0) = 0$, for example, then we have $C = 0$; or if $y(1) = 2$, then $C = 1$.

A differential equation together with an initial condition is called an **initial value problem**.

If ***only*** the *first* derivative $y'(t)$ of y is involved then the equation is a **first-order ordinary differential equation**.

The independent variable is often denoted t to reflect the fact that it represents time. However, in some cases the independent variable has a different interpretation.

First-order ordinary differential equations

A first-order ordinary differential equation takes the form: $G(t, y'(t), y(t)) = 0$ for all t,

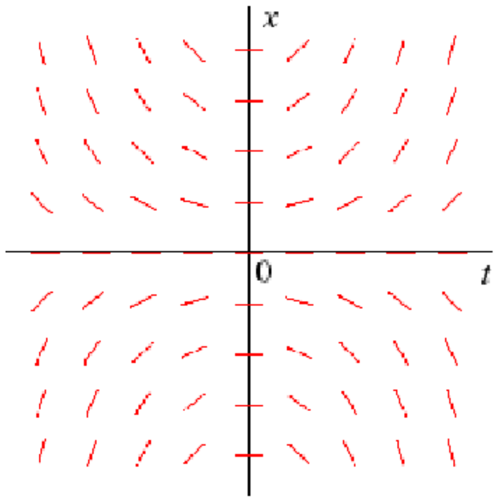
which we can alternatively write as $y'(t) = F(t, y(t))$ for all t.

The first derivative $y' \equiv dy/dt$ is the only one that can appear in a first-order differential equation, but it may enter in various powers: dy/dt, $(dy/dt)^2$, or $(dy/dt)^3$. The highest power attained by the derivative in the equation is referred to as the ***degree*** of the differential equation.

Such an equation may be difficult, or even impossible, to solve explicitly. One way to get a feel for the qualitative character of the solution, without calculating a solution explicitly, is graphical.

To illustrate this method, consider the equation $y'(t) = y(t)t$.

For any pair (t, y) we can find the value of $y'(t)$, and plot this slope in a **direction diagram** in which the axes are t (horizontal) and y (vertical). For example, at $(0, 0)$ we have $y'(t) = 0$; at $(2, 2)$ we have $y'(t) = 4$. The following figure plots the slopes (indicated by short line segments) at several points. (In this figure the grid size is 1/2 unit.)



2. First Order Linear Differential Equations with Constant Coefficient and Constant Term

In case the derivative $y' \equiv dy/dt$ appears only in a first degree, and so does the dependent variable y, and no product of the form $y(dy/dt)$ occurs, then the equation is said to be ***linear***.

Consider a general first-order linear differential equation:

$$\frac{dy}{dt} + u(t) \cdot y = w(t),$$

where u and w are two functions of t, as is y.

Special Case:

$$u = \text{constant} \quad \text{and} \quad w = \text{constant}$$

A: Homogeneous Case $(u,\ w = \text{constant} \ \text{ and } \ w = 0)$

$$\frac{dy}{dt} + ay = 0\,,\ \ a = \text{constant}$$

This differential equation is said to be ***homogeneous***, on account of the zero constant term.

Solution: $\dfrac{dy}{dt} + ay = 0 \ \Leftrightarrow\ \dfrac{1}{y}\dfrac{dy}{dt} = -a \ \Leftrightarrow\ \dfrac{1}{y}dy = -adt$

$$\Leftrightarrow\ \int \frac{1}{y}dy = -\int adt \ \Rightarrow\ \ln y = -at + C$$

$$\Rightarrow\ y(t) = e^{-at+C} = e^{C}e^{-at} \equiv Ae^{-at}, \text{ where } \ e^{C} \equiv A.$$

… general solution (A is arbitrary constant)

$\Rightarrow\ y(t) = y(0)e^{-at}$… Definite solution.

注意：1. $y(0)$的特殊意義為使解滿足初始條件的唯一值

2. 解$y(t) = y(0)e^{-at}$不是一個數值，而是時間的函數

3. 解不為任何導數或微分式，只要代入 t 值就可以得到該時間點的 y 值。

Ex 2: $\dfrac{dy}{dt} - 2y = 0;\ \ y(0) = 9$

$$\Leftrightarrow\ \frac{dy}{y} = 2dt \ \Rightarrow\ \ln y = -2t + C \ \Rightarrow\ y(t) = e^{-2t+C} = Ae^{-2t}$$

Initial condition: $y(0) = 9$

$$t = 0,\ y(t=0) = Ae^{-2\cdot 0} = A \ \Rightarrow\ 9 = A$$

Definite solution: $y(t) = 9e^{-2t}$

B. Non-homogeneous Case (u=常數 and w=常數但非零)

$$\frac{dy}{dt} + ay = b\,,\ \ a, b = \text{constant}$$

其完整解 y 包含兩部分：

(1) complementary function y_c

y_c是 reduced equation: $\dfrac{dy}{dt} + ay = 0$的一般解

(2) particular integral y_p

y_p是 complete equation: $\dfrac{dy}{dt} + ay = b$的 any particular solution

(a) 求y_c

解$\dfrac{dy}{dt} + ay = 0$微分方程式: $\Rightarrow$ $y(t) = Ae^{-at}$ (但它並非完整解，它只是解的一部分)

說明：若$y(t) = Ae^{-at}$是完整微分方程式$\dfrac{dy}{dt} + ay = b$的解，則他將會滿足$\dfrac{dy}{dt} + ay = b$。

檢驗：$y(t) = Ae^{-at}$ $\Rightarrow$ $\dfrac{dy}{dt} = -aAe^{-at}$

將 $y(t) = Ae^{-at}$ 和 $\dfrac{dy}{dt} = -aAe^{-at}$ 代入完整微分方程式 $\dfrac{dy}{dt} + ay = b$中，結果發現

$$-aAe^{-at} + aAe^{-at} \neq b$$

因此，$y(t) = Ae^{-at}$ 並非完整微分方程 $\dfrac{dy}{dt} + ay = b$ 的全解。

(b) 求 y_p

因為 y_p 是 complete equation: $\dfrac{dy}{dt} + ay = b$ 的 any particular solution，所以先試解最簡單形式

假設解 $y_p = k$ (常數)，若此解為真，則該解將滿足完整微分方程式 $\dfrac{dy}{dt} + ay = b$

檢驗：$y_p = k \;\Rightarrow\; \dfrac{dy_p}{dt} = 0$

將 $y_p = k$ 和 $\dfrac{dy_p}{dt} = 0$ 代入完整微分方程式 $\dfrac{dy}{dt} + ay = b$ 中，

結果發現 $0 + ak = b$

所以，k 必須滿足 $k = \dfrac{b}{a},\ a \neq 0$。因此，特解(particular solution)為 $y_p = k = \dfrac{b}{a},\ a \neq 0$。

(c) 完整解(complete solution)的一般式 $y = y_c + y_p$

$y(t) = y_c + y_p = Ae^{-at} + \dfrac{b}{a},\ a \neq 0\ldots$ General solution

(d) 利用初始條件求解定解(definite solution)

若 initial condition 為 $t = 0$ 時，y 值為 $y(0)$

則 $t = 0$ 代入一般解 $y(t) = Ae^{-at} + \dfrac{b}{a}$ 中，

得 $y(0) = Ae^{-a\cdot 0} + \dfrac{b}{a} = A + \dfrac{b}{a} \;\Rightarrow\; A = y(0) - \dfrac{b}{a}$

所以，definite solution 為 $y(t) = \left[y(0) - \dfrac{b}{a}\right]e^{-at} + \dfrac{b}{a},\ a \neq 0$。

Verification: 若$y(t)=\left[y(0)-\frac{b}{a}\right]e^{-at}+\frac{b}{a}$為真解，則代回原微分方程式$\frac{dy}{dt}+ay=b$，應該可以成立。

$y(t)=\left[y(0)-\frac{b}{a}\right]e^{-at}+\frac{b}{a}$ → 對 time 微分得$\frac{dy}{dt}=-a\left[y(0)-\frac{b}{a}\right]e^{-at}$

將上兩式代回原微分方程 $\frac{dy}{dt}+ay=b$

$\rightarrow\ -a\left[y(0)-\frac{b}{a}\right]e^{-at}+a\left\{\left[y(0)-\frac{b}{a}\right]e^{-at}+\frac{b}{a}\right\}=b\ \rightarrow\ b=b$；

所以其為真解

$t=0$代入驗算，得$y(t=0)=\left[y(0)-\frac{b}{a}\right]e^{-a\cdot 0}+\frac{b}{a}=y(0)$，恒成立。

例題 1：初始條件$y(0)=10$，求$\frac{dy}{dt}+2y=6$的解。

(a) y_c解：求$\frac{dy}{dt}+2y=0$的齊次解，得$y_c(t)=Ae^{-2t}$

(b) y_p 解：求$\frac{dy}{dt}+2y=6$的特解，令$y_p=k$代入後，得

$0+2k=6\ \Rightarrow\ k=3$

(c)完整解 y：$y(t)=y_c+y_p=Ae^{-2t}+3$

(d)定解：因為$t=0$時，$y(0)=10$，將之代入完整解，得

$y(0)=10=Ae^{-0}+3\ \Rightarrow\ A=10-3=7$

因此，定解為$y(t)=7e^{-2t}+3$

例題 2：初始條件$y(1)=1$，求$\frac{dy}{dt}+4y=0$的解。

(a) y_c解：求$\frac{dy}{dt}+4y=0$的齊次解，得$y_c(t)=Ae^{-4t}$

(b) y_p 解：求 $\dfrac{dy}{dt}+4y=0$ 的特解，令 $y_p=k$ 代入後，得

$$0+4k=0 \;\Rightarrow\; k=0$$

(c)完整解 y：$y(t)=y_c+y_p=Ae^{-4t}+0$

(d)定解：因為 $t=1$時，$y(1)=1$，將之代入完整解，得

$$y(1)=1=Ae^{-4}+0 \;\Rightarrow\; A=1/e^{-4}=e^{4}$$

因此，定解為$y(t)=e^{4}e^{-4t}$

注意：若$a=0$，則上述微分方程式$\dfrac{dy}{dt}+ay=b$

$\rightarrow$ 退化成 $\dfrac{dy}{dt}=b$

此時，解$y(t)=\left[y(0)-\dfrac{b}{a}\right]e^{-at}+\dfrac{b}{a}$已經不再有意義。

亦即，$y(t)=\left[y(0)-\dfrac{b}{a}\right]e^{-at}+\dfrac{b}{a}$不是$\dfrac{dy}{dt}=b$的解。

那微分方程式$\dfrac{dy}{dt}=b$的解為何呢？

想法一：$\dfrac{dy}{dt}=b \;\rightarrow\; dy=b\cdot dt \;\rightarrow\; \int dy=\int b\cdot dt$

$\rightarrow\; y(t)=bt+c$, c 為任意常數

若初始條件為：$t=0$時，y 值為$y(0)$，則$y(0)=b\cdot 0+c=c$

因此定解為：$y(t)=bt+y(0)$

想法二：(a) 求齊次解y_c: 齊次微分方程式$\dfrac{dy}{dt}+0y=0$，其解為

$$y_c=Ae^{-at}=Ae^{-0t}=A,\; A \text{ 為任意常數}$$

(b) 求特定解y_p: 完整微分方程式$\dfrac{dy}{dt}=b$，令最簡單的特定解為$y_p=k\cdot t$，

將之代入 $\frac{dy}{dt} = b$ 中，(因為 $\frac{dy_p}{dt} = k$) 得 $k = b$，因此 $y_p = b \cdot t$

(c) 完整解 $y(t) = y_c + y_p = A + bt$

(d) 定解：將 $t = 0$ 代入完整解 $y(t) = A + bt$，得

$y(0) = A + b \cdot 0 = A$

因此，定解為 $y(t) = y(0) + bt$

例題 3：期初條件為 $y(0) = 5$，解微分方程式 $\frac{dy}{dt} = 2$。

自行練習

Application 1: ***Dynamics of market price***

Framework: $Q_d = \alpha - \beta P, \quad \alpha, \beta > 0$

$Q_s = -\gamma + \delta P, \quad \gamma, \delta > 0$

$\Rightarrow$ 均衡價格 $\bar{P} = \frac{\alpha + \gamma}{\beta + \delta}$，均衡數量 $\bar{Q} = \alpha - \beta \bar{P} = \frac{\alpha\delta - \beta\gamma}{\beta + \delta}$

若市場不均衡時，則價格 P 與數量 Q 會隨著時間調整；

問題是：

這個調整過程，價格 P 會隨著時間的經過而收斂(到新均衡)嗎？

The Time Path: 當市場發生超額需求(超額供給)時，價格在下一瞬間會上升(下跌)

$$\frac{dP}{dt} = j(Q_d - Q_s), \quad j > 0$$

其中，j 為市場的調整係數(adjustment coefficient)；

$Q_d - Q_s$ 為超額需求(excess demand)

因此，均衡的定義可以改寫成：

$$\boxed{Q_d = Q_s \quad \Leftrightarrow \quad \frac{dP}{dt} = 0}$$

$$\frac{dP}{dt} = j\cdot[\alpha - \beta P + \gamma - \delta P] = j(\alpha+\gamma) - j(\beta+\delta)P$$

$$\Leftrightarrow \quad \frac{dP}{dt} + j(\beta+\delta)P = j(\alpha+\gamma)$$

$$(\text{hint: } \frac{dy}{dt} \quad + \quad a\cdot y \quad = \quad b)$$

Solution: 求均衡解：

$$\text{均衡條件：} Q_d = Q_s \quad \Leftrightarrow \quad \frac{dP}{dt} = 0$$

$$\Rightarrow \quad \frac{dP}{dt} + j(\beta+\delta)P = j(\alpha+\gamma)$$

$$\Rightarrow \quad \overline{P} = \frac{\alpha+\gamma}{\beta+\delta}$$

(a)求齊次解 P_c:

齊次微分方程式 $\frac{dP}{dt} + j(\beta+\delta)P = 0$，其解為

$P_c = Ae^{-j(\beta+\delta)t}$, A 為任意常數

(b)求特定解 P_p:

完整微分方程式為 $\frac{dP}{dt} + j(\beta+\delta)P = j(\alpha+\gamma)$，故令特定解為 $P_p = k$，

所以 $\frac{dP_p}{dt} = 0$，將之代入完整微分方程式中，

得 $0 + j(\beta+\delta)k = j(\alpha+\gamma)$，因此 $P_p = k = \frac{\alpha+\gamma}{\beta+\delta} = \overline{P}$

(c)求完整解：

$$P(t) = P_c + P_p = Ae^{-j(\beta+\delta)t} + \overline{P}$$

(d)定解：假設在$t = 0$時，初始價格為$P(0)$

則$P(0) = Ae^{-j(\beta+\delta)\cdot 0} + \overline{P} = A + \overline{P}$，

因此$A = P(0) - \overline{P}$

所以，定解為$P(t) = [P(0) - \overline{P}]\cdot e^{-j(\beta+\delta)t} + \overline{P}$

Dynamic Stability of Equation

價格 ***P*** 會隨著時間的經過而收斂(到新均衡)嗎？

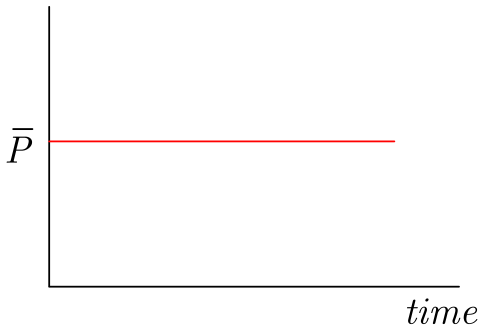
即檢驗$t \to \infty$，$P(t) \to ?$

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \{[P(0) - \overline{P}]\cdot e^{-j(\beta+\delta)t} + \overline{P}\}$$

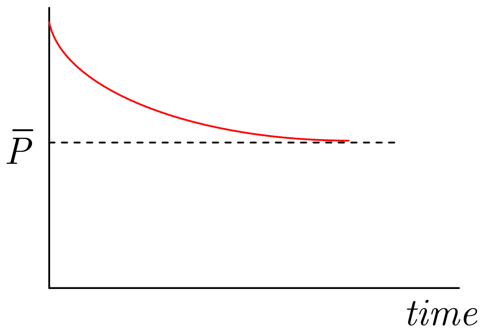
$$= \lim_{t\to\infty} [P(0) - \overline{P}]\cdot e^{-j(\beta+\delta)t} + \lim_{t\to\infty} \overline{P} = \overline{P}$$

as $\beta+\delta<0$

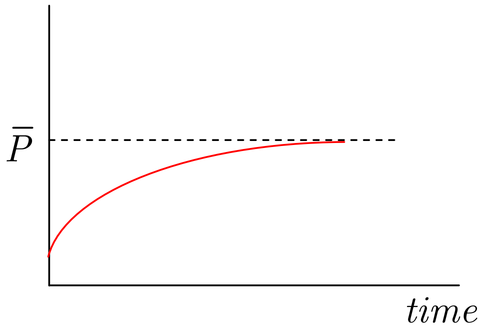
If $P(0) = \overline{P} \;\Rightarrow\; P(t) = \overline{P}$



If $P(0) > \overline{P} \Rightarrow\; P(t) = [P(0) - \overline{P}]e^{-j(\beta+\delta)t} + \overline{P} > \overline{P}$



If $P(0) < \overline{P} \Rightarrow \ P(t) = [P(0) - \overline{P}]e^{-j(\beta+\delta)t} + \overline{P} < \overline{P}$



動態安定的經濟詮釋：

當 $P(0) \neq \overline{P}$ 時，則 $P(t) = [P(0) - \overline{P}]e^{-j(\beta+\delta)t} + \overline{P}$

在 $j(\beta+\delta) > 0$ 條件下，當 $t \to \infty$ 時，$P(t) \to \overline{P}$ $\ (\because e^{-j(\beta+\delta)t} \to 0)$

因此，$j(\beta+\delta) > 0$ 是價格(此商品市場)具有動態安定的條件。

若 $j > 0$ (意義：市場發生超額需求 ED，價格 P 會上升而非下跌)

則動態安定的條件：$j(\beta+\delta) > 0 \longrightarrow$ 變成 $\beta+\delta > 0$

$\beta+\delta > 0 \quad \Leftrightarrow \quad \delta \ > \ -\beta$

供給線斜率 > 需求線斜率

例：若一商品市場其 demand curve 為負斜率、supply curve 為正斜率，則該市場是否具動態安定性？

因為 demand curve 為負斜率、supply curve 為正斜率滿足了動態安定性條件 $\delta > -\beta$。

例：若一商品市場其 demand curve 與 supply curve 均為正斜率，則該市場是否具動態安定性？

Ans: 不一定

3. First Order Linear Differential Equations with Variable Coefficient and Variable Term

一般化的線性一階微分方程式：$\dfrac{dy}{dt} + u(t)\cdot y = w(t)$，其 time path?

$u(t)$：變係數

$w(t)$：變數項

A. Homogeneous Case: ($w(t)=0$)

$$\frac{dy}{dt} + u(t)\cdot y = 0$$

$$\begin{aligned}
\frac{dy}{dt} + u(t)\cdot y = 0 \;\Leftrightarrow\; & \frac{1}{y}dy = -u(t)\cdot dt \\
\Rightarrow\; & \int \frac{1}{y}dy = -\int u(t)\cdot dt \\
\Rightarrow\; & \ln y + c = -\int u(t)\cdot dt \\
\Rightarrow\; & e^{\ln y} = e^{-\int u(t)\cdot dt - c} = e^{-\int u(t)\cdot dt}e^{-c} = Ae^{-\int u(t)\cdot dt} \\
\Rightarrow\; & y_c(t) = Ae^{-\int u(t)\cdot dt}\text{, } A \text{ 為任意常數}
\end{aligned}$$

比較：$\dfrac{dy}{dt} + a\cdot y = 0$的解 $\Rightarrow$ $y_c(t) = Ae^{-a\cdot t}$

例題：求解微方:$\dfrac{dy}{dt} + 3t^2 y = 0$

$$\begin{aligned}
\frac{dy}{dt} + 3t^2 y = 0 \;\Leftrightarrow\; & \frac{1}{y}dy = -3t^2 dt \\
\Rightarrow\; & y(t) = e^{-c}e^{-\int 3t^2 dt} = Ae^{-t^3}
\end{aligned}$$

B. Nonhomogeneous Case: $(w(t) \neq 0)$

$$\frac{dy}{dt} + u(t) \cdot y = w(t)$$

非齊次型的一般化線性一階微分方程式的解是不太容易求得的
$\rightarrow$ 可以使用 *Exact* differential equation 來說明其求解的過程

微分方程式$\dfrac{dy}{dt} + u(t) \cdot y = w(t)$的一般解為：

$$y(t) = e^{-\int u(t)dt} \cdot \left[A + \int w(t) \cdot e^{\int u(t)dt} dt \right],$$

A 為任意常數(由初始條件決定之)

補充：(公式的推導)

If $\Phi(t)$ is a fundamental (matrix) solution of the homogeneous linear system

$$\frac{dy}{dt} = -u(t) \cdot y \qquad \text{(i.e. } \Phi(t) = A e^{-\int u(t)dt} \text{)}$$

then every solution of the ***non-homogeneous*** system

$$\frac{dy}{dt} = -u(t) \cdot y + w(t)$$

is given by

$$\phi(t) = \Phi(t) \cdot \left[\Phi^{-1}(t_0) \cdot \phi(t_0) + \int_{t_0}^{t} \Phi^{-1}(s) \cdot w(s) \cdot ds \right]$$

for any real to $t_0 \in (-\infty, \infty)$.

Proof:

Since the solution of the homogeneous system:

$$\frac{dy}{dt} = -u(t) \cdot y$$

is

$$\Phi(t) = A\,e^{-\int u(t)dt},$$

$$\boxed{\frac{d\Phi(t)}{dt} = -u(t) \cdot \Phi(t)}$$

then the *general* solution of the homogeneous system can be written as

$$\phi(t) = c \cdot \Phi(t)\,, \tag{1}$$

where c is an arbitrary constant.

We wish to satisfy the non-homogeneous equation

$$\frac{dy}{dt} = -u(t) \cdot y + w(t)$$

by the same expression as Eq.(1) but allow c to be a function of t; this explains the same of the theorem as variation of constants.

Rewrite Eq.(1) with c now being a variable, as

$$\phi(t) = c(t) \cdot \Phi(t), \tag{2}$$

and differentiate, to get

$$\begin{aligned}\frac{d\phi(t)}{dt} &= \Phi(t) \cdot \frac{dc(t)}{dt} + c(t) \cdot \frac{d\Phi(t)}{dt} \\ &= \Phi(t) \cdot \frac{dc(t)}{dt} + c(t) \cdot [-u(t) \cdot \Phi(t)] \\ &= \Phi(t) \cdot \frac{dc(t)}{dt} - u(t) \cdot c(t) \cdot \Phi(t) \\ &= \underline{\Phi(t) \cdot \frac{dc(t)}{dt}} - u(t) \cdot \phi(t) \end{aligned} \tag{3}$$

(Recall the differential equation $\frac{dy}{dt} = -u(t) \cdot y + \underline{w(t)}$)

In order for Eq.(3) to satisfy the non-homogeneous equation, it must be the case that

$$\Phi(t)\cdot\frac{dc(t)}{dt}=w(t)$$

This follows from inspection of Eq.(2). Thus, we have

$$\frac{dc(t)}{dt}=w(t)\cdot\Phi^{-1}(t),$$

or equivalently,

$$c(t)=c_0+\int w(s)\cdot\Phi^{-1}(s)\cdot ds.$$

As a result, Eq.(2) can be rewritten as

$$\phi(t)=\Phi(t)\cdot\left[c_0+\int w(\tau)\cdot\Phi^{-1}(\tau)\cdot d\tau\right], \qquad (4)$$

where $\Phi(\tau)=Ae^{-\int u(\tau)d\tau}$.

In Eq.(4), if we integrate from t_0 to t, then the solution $\phi(t)$ will satisfy the initial condition $\phi(t_0)$. That is,

$$\phi(t_0)=c(t_0)\cdot\Phi(t_0)$$

or, $c(t_0)=\phi(t_0)\cdot\Phi^{-1}(t_0)$.

If, instead, the initial condition is $\phi(t_0)=y_0$, then the solution of the non-homogeneous equation becomes

$$\phi(t)=\Phi(t)\cdot\left[\phi(t_0)\cdot\Phi^{-1}(t_0)+\int_{t_0}^{t} w(\tau)\cdot\Phi^{-1}(\tau)\cdot d\tau\right],$$

where $\Phi(\tau)=Ae^{-\int u(\tau)d\tau}$ and $\Phi^{-1}(\varsigma)=A^{-1}e^{-\int u(\tau)d\tau}$, which establishes the theorem.

Ex: 求解$\dfrac{dy}{dt}+2ty=t$

Sol:
$$
\begin{aligned}
y(t) &= e^{-\int 2t\cdot dt}\left[A+\int te^{\int 2t\cdot dt}dt\right] \\
&= e^{-(t^2+k)}\left[A+\int te^{t^2+k}dt\right] \\
&= e^{-k}e^{-t^2}\left[A+e^{k}\int te^{t^2}dt\right] = e^{-k}e^{-t^2}A+e^{-k}e^{-t^2}e^{k}\int te^{t^2}dt \\
&= e^{-k}e^{-t^2}A+e^{-t^2}\int t\,e^{t^2}dt = e^{-k}e^{-t^2}A+e^{-t^2}\left[\frac{1}{2}\int e^{t^2}dt^2\right] \\
&= e^{-k}e^{-t^2}A+e^{-t^2}\left[\frac{1}{2}e^{t^2}+c\right] = e^{-k}e^{-t^2}A+\frac{1}{2}+ce^{-t^2} \\
&= \frac{1}{2}+(e^{-k}A+c)e^{-t^2} = \frac{1}{2}+Be^{-t^2}, \qquad \text{where } B\equiv e^{-k}A+c
\end{aligned}
$$

Ex: 求解$\dfrac{dy}{dt}+4ty=4t$

Ans: $y(t)=1+Be^{-2t^2}$，B 為任意常數

Homework: 求解$\dfrac{dy}{dt}+4ty=4e^{t}$

Recall:

對比常數係數、常數項的微分方程式：$\dfrac{dy}{dt}+ay=b$

$\Rightarrow\quad u(t)=a \ \text{ and } \ w(t)=b$

$$
\Rightarrow\quad y(t)=e^{-\int a\ dt}\cdot\left[A+\int b\cdot e^{\int a\ dt}dt\right]=e^{-at}\cdot\left[A+\int b\cdot e^{at}dt\right]
$$

$$
=e^{-at}\cdot\left[A'+\frac{b}{a}e^{at}\right]=A'e^{-at}+\frac{b}{a}
$$

4. Exact Differential Equations

正合微分方程式可以用來說明$\frac{dy}{dt}+u(t)\cdot y=w(t)$的解：

$$y(t)=e^{-\int u(t)dt}\cdot\left[A+\int w(t)\cdot e^{\int u(t)dt}dt\right]$$

基本上，正合微分方程式可以是**線性**微分方程式，也可以是**非線性**微分方程式

<u>定義：</u>

兩變數函數$F(y,t)$的全微分為

$$dF(y,t)=\frac{\partial F}{\partial y}dy+\frac{\partial F}{\partial t}dt$$

當此微分等於 0，則$dF(y,t)=\frac{\partial F}{\partial y}dy+\frac{\partial F}{\partial t}dt=0$為正合微分方程

Ex: 令函數$F(y,t)=y^2t+k,\qquad k=\text{constant}$

其全微分：

$$dF(y,t)=2yt\cdot dy+y^2\cdot dt$$

因此，正合微分方程式為：

$$2yt\cdot dy+y^2\cdot dt=0$$

$$\text{或}\ \frac{dy}{dt}+\frac{y^2}{2yt}=0$$

檢驗：

給定一微分方程式，我們要如何判斷此微分方程式是否為 Exact D.E.?

一般微分方程式：$M \cdot dy + N \cdot dt = 0$，若存在一個函數$F(y,t)$，其中$M = \dfrac{\partial F}{\partial y}$、$N = \dfrac{\partial F}{\partial t}$，且滿足$\dfrac{\partial M}{\partial t} = \dfrac{\partial N}{\partial y}$ (亦即 Young's Theorem: $\dfrac{\partial^2 F}{\partial t \partial y} = \dfrac{\partial^2 N}{\partial y \partial t}$)，則此$dF = 0$即為 Exact Differential Equation。

Ex: $2yt \cdot dy + y^2 \cdot dt = 0$是否為正合微分方程式?

檢驗條件：$\dfrac{\partial M}{\partial t} = \dfrac{\partial N}{\partial y}$

$$\frac{\partial M}{\partial t} = 2y\,,\quad \frac{\partial N}{\partial y} = 2y$$

P.s. 對於 y 出現在 M, N 中的形式，並沒有任何限制，所以 Exact D.E. 可以是 y 的非線性函數

Ex: $(y^2 t + t^2) \cdot dy + (yt^2 + 2y) \cdot dt = 0$是否為正合微分方程式?

求解：

那如何求解正合微分方程式呢

當你已經確認$dF(y,t) = \dfrac{\partial F}{\partial y} dy + \dfrac{\partial F}{\partial t} dt = 0$為正合微分方程式，則

解$\int dF(y,t) = 0 \;\Leftrightarrow\;$ 解$F(y,t) = c$

Step 1: $\because \; M = \dfrac{\partial F}{\partial y} \quad \therefore \; F$ 函數必包含了 y 變數

$\therefore$ 假設$F(y,t) = \int M \cdot dy + \psi(t)$

假設的 F 函數中出現 $\psi(t)$，是因為 $\dfrac{\partial F}{\partial y}$ 的偏微分將 t 視為常數

Step 2: 若假設的 F 函數正確，則它須滿足 Exact D.E.的定義。

因此，$F(y,t)=\int M\cdot dy+\psi(t)$ 需滿足

$$\frac{\partial F(y,t)}{\partial t}=\frac{\partial\left[\int M\cdot dy+\psi(t)\right]}{\partial t}=N$$

Step 3: 由上關係式求得 $\psi(t)$ 函數，並將求得的 $\psi(t)$ 函數代回假設的 $F(y,t)=\int M\cdot dy+\psi(t)$

Step 4: 最後將求得的 $F(y,t)$ 函數代入解 $F(y,t)=c$ 之中，即得該正合微方的解。

Ex: 求解 $2yt\cdot dy+y^2\cdot dt=0$

先檢驗是否為正合微分方程式

$$\because\ \frac{\partial M}{\partial t}=2y=\frac{\partial N}{\partial y},\ \therefore\ \text{它為正合微分方程式}$$

1. 假設 $F(y,t)=\int 2yt\cdot dy+\psi(t)=y^2t+\psi(t)$

2. $\dfrac{\partial F(y,t)}{\partial t}=N$ 須成立

$$\Leftrightarrow\ \frac{\partial[y^2t+\psi(t)]}{\partial t}=y^2+\psi'(t)=N=y^2$$

$$\Leftrightarrow\ \psi'(t)=0$$

3. $\psi'(t)=0\ \Leftrightarrow\ \psi(t)=k=\text{constant}$

因此，$F(y,t)=y^2t+k$

4. 所以，正合微分方程式的解為：$F(y,t)=y^2t+k=c$

或 $y^2t=c'\ \ (c'\equiv c-k)$、或$y=c't^{-1/2}$

Ex: 求解$(t+2y)\cdot dy+(y+3t^2)\cdot dt=0$

先檢驗是否為正合微分方程式

$$\because\ \frac{\partial M}{\partial t}=1=\frac{\partial N}{\partial y},\ \therefore\ \text{它為正合微分方程式}$$

1. 假設$F(y,t)=\int (t+2y)\cdot dy+\psi(t)=ty+y^2+\psi(t)$

2. $\dfrac{\partial F(y,t)}{\partial t}=N$須成立

$$\Leftrightarrow\ \frac{\partial[ty+y^2+\psi(t)]}{\partial t}=y+\psi'(t)=N=y+3t^2$$

$$\Leftrightarrow\ \psi'(t)=3t^2$$

3. $\psi'(t)=3t^2\ \Leftrightarrow\ \psi(t)=t^3+k$

因此，$F(y,t)=ty+y^2+t^3+k$

4. 正合微分方程式的解為：$F(y,t)=ty+y^2+t^3+k=c$

或$ty+y^2+t^3=c'\ \ (c'\equiv c-k)$

以上四步驟亦可以用在 non-exact differential equation

只要對非正合微分方程式乘上一積分因子(共同因子)，可以使得**非正合**微分方程式轉為**正合**微分方程式

Ex: 求解$2t\cdot dy+y\cdot dt=0$

先檢驗是否為正合微分方程式

$$\because\ \frac{\partial M}{\partial t}=2\neq 1=\frac{\partial N}{\partial y},\ \therefore\ \text{它為非正合微分方程式}$$

兩邊同時乘上一積分因子 $\boldsymbol{y}$

$$\Rightarrow \; 2ty \cdot dy + y^2 \cdot dt = 0 \ldots \text{ 已為正合微分方程式}$$

1. 假設 $F(y,t) = \int 2ty \cdot dy + \psi(t) = ty^2 + \psi(t)$
2. $\dfrac{\partial F(y,t)}{\partial t} = N$ 須成立

$$\Leftrightarrow \; \frac{\partial[ty^2 + \psi(t)]}{\partial t} = y^2 + \psi'(t) = N = y^2$$

$$\Leftrightarrow \; \psi'(t) = 0$$

3. $\psi'(t) = 0 \;\Leftrightarrow\; \psi(t) = k$

   因此，$F(y,t) = ty^2 + k$
4. 正合微分方程式的解為：$F(y,t) = ty^2 + k = c$

   或 $ty^2 = c' \;\; (c' \equiv c - k)$

Alternative Way:

$$2t \cdot dy + y \cdot dt = 0 \;\Leftrightarrow\; \frac{1}{y} \cdot dy = -\frac{1}{2t} \cdot dt$$

$$\Leftrightarrow \; \int \frac{1}{y} \cdot dy = -\frac{1}{2} \int \frac{1}{t} \cdot dt \ldots\ldots$$

正合微分方程式可以用來說明 $\dfrac{dy}{dt} + u(t) \cdot y = w(t)$ 的解：

$$\frac{dy}{dt} + u(t) \cdot y = w(t) \;\rightarrow\; dy - [w(t) - u(t) \cdot y] dt = 0$$

積分因子：$I = e^{\int u \cdot dt}$

$$\rightarrow \; \underbrace{I}_{M} \cdot dy + \underbrace{I[u \cdot y - w]}_{N} \cdot dt = 0$$

檢驗是否為正合微分方程式：

$$\frac{\partial M}{\partial t} = \frac{\partial I}{\partial t} = \frac{\partial}{\partial t}[e^{\int u \cdot dt}] = u \cdot e^{\int u \cdot dt}$$

$$\frac{\partial N}{\partial y}=\frac{\partial\{I\,[u\cdot y-w]\}}{\partial y}=\frac{\partial}{\partial y}\{e^{\int u\cdot dt}\,[u\cdot y-w]\}=e^{\int u\cdot dt}\cdot u$$

$$\therefore\ \frac{\partial M}{\partial t}=\frac{\partial N}{\partial y}$$

求解：

Step 1: Let $F(y,t)=\int I\cdot dy+\psi(t)=\int(e^{\int u\cdot dt})\cdot dy+\psi(t)$

$$\rightarrow\quad F(y,t)=y\cdot e^{\int u\cdot dt}+\psi(t)$$

Step 2: $\dfrac{\partial F(y,t)}{\partial t}=N$

$$\rightarrow\quad \frac{\partial[y\cdot e^{\int u\cdot dt}+\psi(t)]}{\partial t}=yu\cdot e^{\int u\cdot dt}+\psi'(t)=N=I\,[u\cdot y-w]$$

$$\rightarrow\quad yu\cdot e^{\int u\cdot dt}+\psi'(t)=e^{\int u\cdot dt}\,[u\cdot y-w]$$

$$\rightarrow\quad \psi'(t)=-we^{\int u\cdot dt}$$

Step 3: $\psi'(t)=-we^{\int u\cdot dt}\quad\rightarrow\quad \psi(t)=-\int(we^{\int u\cdot dt})\cdot dt$

$$F(y,t)=y\cdot e^{\int u\cdot dt}-\int(we^{\int u\cdot dt})\cdot dt$$

Step 4: Thus, $F(y,t)=y\cdot e^{\int u\cdot dt}-\int(we^{\int u\cdot dt})\cdot dt=c$

$$\rightarrow\quad y=e^{-\int u\cdot dt}\cdot\left[c+\int(we^{\int u\cdot dt})\cdot dt\right]$$

Ex: 求解$dy+(2ty-t)dt=0$

積分因子：$I=\exp(\int 2t\cdot dt)=\exp(t^2+c)=Ae^{t^2},\ (\because\ u=2t)$

$\rightarrow\quad Ae^{t^2}dy+Ae^{t^2}(2ty-t)dt=0$

$\ldots\ldots$

5. first-order and first-degree 的非線性微分方程式

一般式：

$$f(y,t)\cdot dy+g(y,t)\cdot dt=0\text{，或}$$
$$\frac{dy}{dt}=h(y,t)$$

其中，y, t 的乘冪(power)沒有限制

在 Exact D.E.中，y 的 power 可以是高次的

例如：$2yt\cdot dy+y^2\cdot dt=0$ 即是

若兩邊同除 y，雖然使其為線性微分方程式，但卻非正合微分方程式，所以正合微分方程式經常視為非線性(nonlinear)

僅討論兩種類型

A. Separable Variables

在適當條件下，$f(y,t)\cdot dy+g(y,t)\cdot dt=0$

可以改寫成：$h(y)\cdot dy+k(t)\cdot dt=0$

Ex: 求解 $3y^2\cdot dy-t\cdot dt=0$

$$\begin{aligned}3y^2\cdot dy-t\cdot dt=0 \;&\Rightarrow\; 3y^2\cdot dy=t\cdot dt\\ &\Rightarrow\; \int 3y^2\cdot dy=\int t\cdot dt\\ &\Rightarrow\; y^3+c_1=\frac{1}{2}t^2+c_2\\ &\Rightarrow\; y^3=\frac{1}{2}t^2+c\text{，}(c=c_2-c_1)\end{aligned}$$

Ex: 求解 $2t\cdot dy+y\cdot dt=0$

乍看下非 separable，但適當轉換即可成 separable

兩邊同除 $2ty$，

$$\frac{1}{y}dy + \frac{1}{2t}dt = 0$$

$$\Leftrightarrow \int \frac{1}{y} \cdot dy = -\int \frac{1}{2t} \cdot dt$$

$$\Leftrightarrow \ln y + \frac{1}{2}\ln t = c \quad \Leftrightarrow \quad \ln(yt^{1/2}) = c$$

$$\Leftrightarrow yt^{1/2} = e^c \equiv A \quad \Leftrightarrow \quad y = At^{-1/2}$$

B. Equations Reducible to the Linear Form (Bernoulli Equation)

型式：$\dfrac{dy}{dt} + R(t) \cdot y = T(t) \cdot y^m, \quad m \neq 0,1$ (nonlinear form)

簡化步驟：

1. 同除y^m： $y^{-m}\dfrac{dy}{dt} + R(t) \cdot y^{1-m} = T(t)$

2. 變數轉換：令$z \equiv y^{1-m}$，則$\dfrac{dz}{dt} = (1-m)y^{-m}\dfrac{dy}{dt}$

$$\rightarrow \quad \frac{1}{1-m}\frac{dz}{dt} = y^{-m}\frac{dy}{dt}$$

轉換後的微分方程式：

$$\frac{1}{1-m}\frac{dz}{dt} + R(t) \cdot z(t) = T(t)$$

3. $\dfrac{1}{1-m}\dfrac{dz}{dt} + R(t) \cdot z(t) = T(t)$

$\rightarrow \quad dz + (1-m)[R(t) \cdot z(t) - T(t)] \cdot dt = 0 \ldots$一階線性微方(for z)

Ex: 求解$\dfrac{dy}{dt} + ty = 3ty^2$

$$\rightarrow \quad y^{-2}\frac{dy}{dt} + ty^{-1} = 3t \quad (\text{同除} y^2)$$

令 $z = y^{-1} \;\rightarrow\; \dfrac{dz}{dt} = -y^{-2}\dfrac{dy}{dt}$

改寫微分方程式成：

$$-\frac{dz}{dt} + tz = 3t$$

(Recall: $y(t) = e^{-\int u(t)dt} \cdot \left[A + \int w(t) \cdot e^{\int u(t)dt} dt\right]$)

…………

$\Rightarrow \;\; z(t) = Ae^{\frac{1}{2}t^2} + 3$

因為令 $z = y^{-1}$，所以 $y(t) = 1/(Ae^{\frac{1}{2}t^2} + 3)$

Ex: 求解 $\dfrac{dy}{dt} + \dfrac{1}{t}y = y^3$

$\rightarrow \;\; y^{-3}\dfrac{dy}{dt} + ty^{-2} = 1$ (同除 y^3)

令 $z = y^{-2} \;\rightarrow\; \dfrac{dz}{dt} = -2y^{-3}\dfrac{dy}{dt}$

改寫微分方程式成：

$$-\frac{1}{2}\frac{dz}{dt} + tz = 1$$

(Recall: $y(t) = e^{-\int u(t)dt} \cdot \left[A + \int w(t) \cdot e^{\int u(t)dt} dt\right]$)

…………

$\Rightarrow \;\; z(t) = At^2 - 2t$

因為令 $z = y^{-2}$，所以 $y(t) = (At^2 - 2t)^{-1/2}$

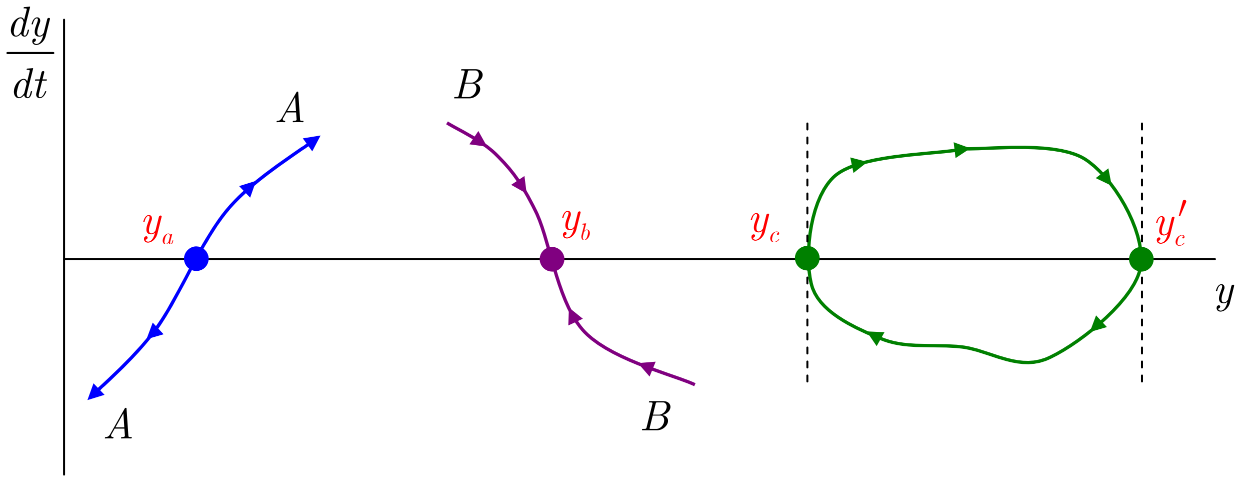
6. The Qualitative-Graphic Approach (定性圖形分析)

有時候 differential equation 無法求出數量解，但從圖形分析可以得到它的時間路徑(time path)。(時間路徑用以判斷$y(t)$能否收斂)

The Phase Diagram (以一階微分方程式：$\frac{dy}{dt} = f(y)$ 為例)

$f(y)$可以是線性，或非線性函數

三類相位圖(phase diagram)



(1)通過橫軸時，切線斜率為正

(2)通過橫軸時，切線斜率為負

(3)通過橫軸時，切線斜率為無窮大

注意事項：

(i) $\frac{dy}{dt} > 0$, y隨著時間增加而增加；
$\frac{dy}{dt} < 0$, y隨著時間增加而減少

(ii) 均衡水準$\overline{y}$，只發生在橫軸且$\frac{dy}{dt} = 0$

⇒找均衡點，需考慮相線與 y(橫)軸交點，並注意其動態安定性

例題 1：畫出$\dfrac{dy}{dt} = 3y + 3$的相位圖 $f(y)$

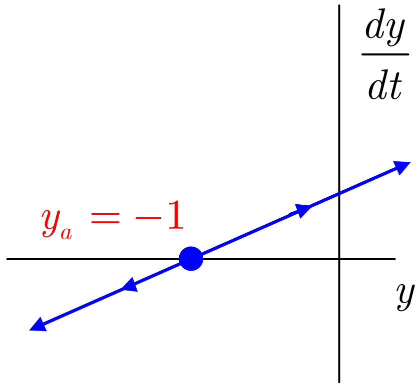
均衡點：

發生在$\dfrac{dy}{dt} = 0$的地方，亦即$3\overline{y} + 3 = 0$，$\therefore \overline{y} = -1$

相位線(在均衡點附近)之斜率與曲度：

$$\frac{\partial(dy\,/\,dt)}{\partial y} = 3 > 0 \quad \text{..... 正斜率}$$

$$\frac{\partial^2(dy\,/\,dt)}{\partial y^2} = 0 \quad \text{..... 直線}$$



Excise 1：畫出$\dfrac{dy}{dt} = \sqrt{y} - 2$ 的相位圖

Excise 2：畫出$\dfrac{dy}{dt} = -2\ln(y) + 2$ 的相位圖

例題 2：畫出$\dfrac{dy}{dt} = -y^2 + 4$ 的相位圖

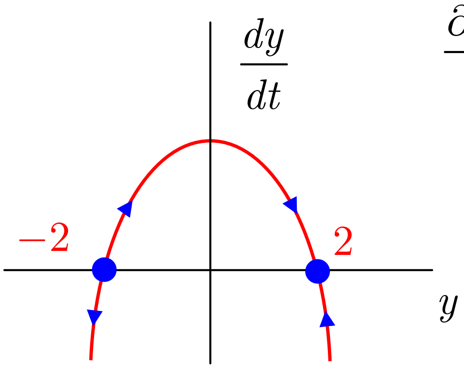
均衡點：

發生在$\dfrac{dy}{dt} = 0$的地方，亦即$-\overline{y}^2 + 4 = 0$，$\therefore \overline{y} = \pm 2$

相位線(在均衡點附近)之斜率與曲度：

$$\frac{\partial(dy\,/\,dt)}{\partial y} = -2\overline{y} \begin{matrix} < 0 \\ > 0 \end{matrix}, \quad \text{if } \overline{y} = \begin{cases} 2 \\ -2 \end{cases}$$

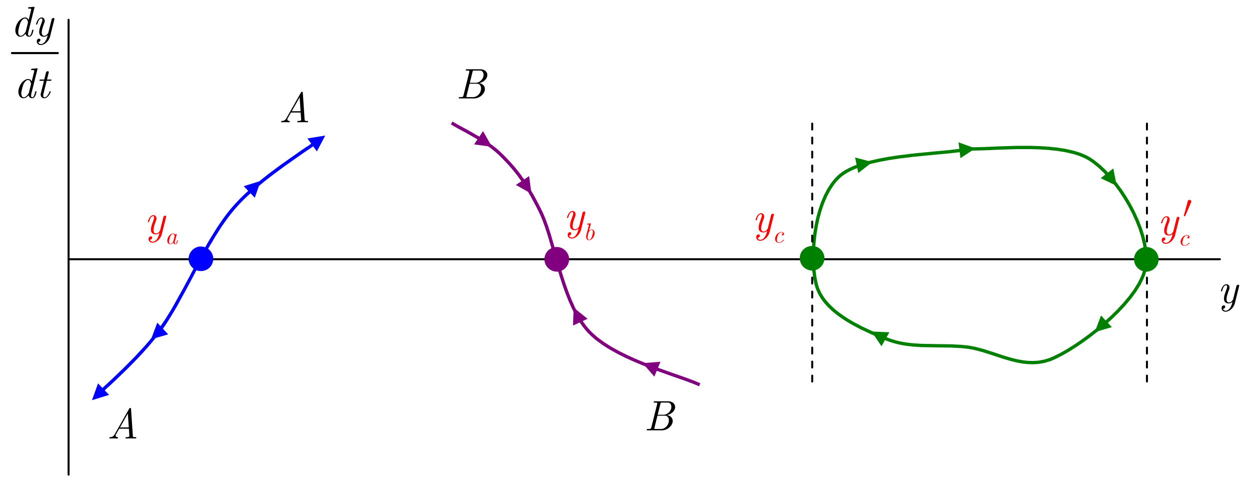
$$\frac{\partial^2(dy\,/\,dt)}{\partial y^2} = -2 < 0 \quad \text{..... concave function}$$



Homework 1：畫出$\dfrac{dy}{dt} = -y^3 + 3y^2 + 9y - 2$ 的相位圖

Homework 2：畫出$\dfrac{dy}{dt} = y^2 + 1$ 的相位圖

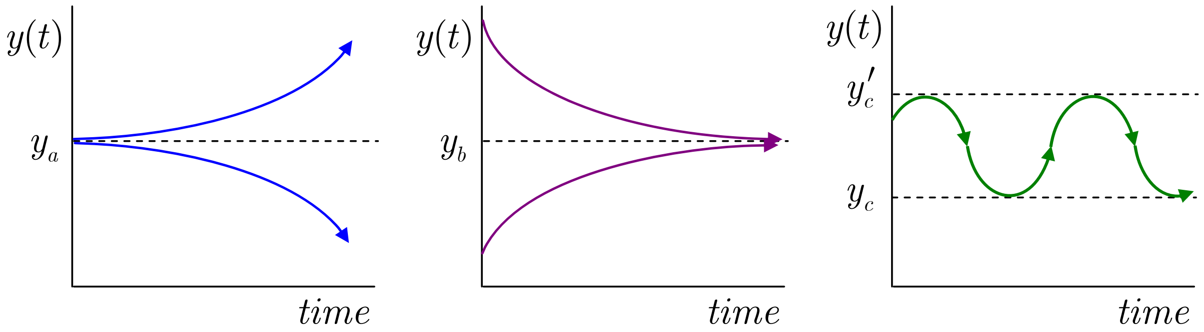
上述三類相位圖所對應的時間路徑(time path)為：



相位線與橫軸交點之斜率為正值

相位線與橫軸交點之斜率為負值

相位線與橫軸交點之斜率為(正負)無窮大



y_a: dynamic **instability** (發散)

y_b: dynamic **stability** (收斂)

y_c，y_c': limit **cycle** (循環波動)

以之前所學過的微分方程式$\dfrac{dy}{dt}+ay=b$ (或$\dfrac{dy}{dt}=-ay+b$)為例

其解：$y(t)=\left[y(0)-\dfrac{b}{a}\right]e^{-at}+\dfrac{b}{a}$ $(a\neq 0)$

(1) $a<0$, $y(t)$ diverges from equilibrium y_a

$$t\to\infty,\ \Rightarrow\ e^{-at}\to\infty,\ \Rightarrow\ y(t)\to\infty$$

(2) $a>0$, $y(t)$ converges to equilibrium y_b

$$t\to\infty,\ \Rightarrow\ e^{-at}\to 0,\ \Rightarrow\ y(t)\to\frac{b}{a}$$

若$a=0$，其解為$y(t)=y(0)+bt$

$$t\to\infty,\ \Rightarrow\ bt\to\infty,\ \Rightarrow\ y(t)\to\infty$$

Summary:

A first-order differential equation is **autonomous** if it takes the form $y'(t)=F(y(t))$ (i.e. the value of $y'(t)$ does not depend independently on the variable t).

An **equilibrium state** of such an equation is a values of x for which $F(\overline{y})=0$. (If $F(\overline{y})=0$ then $y'(t)=0$, so that the value of y does not change.)

A phase diagram indicates the direction in which y is changing for a "representative" collections of values of y. To construct such a diagram, plot the function F, which gives the value of $y'(t)$.

For values of y at which the graph of F is above the y-axis we have $y'(t) > 0$, so that y is increasing; for values of y at which the graph is below the y-axis we have $y'(t) < 0$, so that y is decreasing. A value of y for which $F(\overline{y}) = 0$ is an equilibrium state.

We say that an equilibrium $\overline{y}$ is (locally) **stable** if, after a small departure from the equilibrium, the value of y approaches $\overline{y}$. From the phase diagram, you can see that

- if $F(\overline{y}) = 0$ and $F'(\overline{y}) < 0$ then $\overline{y}$ is a **stable** equilibrium
- if $F(\overline{y}) = 0$ and $F'(\overline{y}) > 0$ then $\overline{y}$ is an **unstable** equilibrium.

7. Application: Solow Growth Model

A. 模型架構：

(1) 使用兩種生產要素：資本(K)、勞動(L)

生產函數表示成：$Q = f(K, L),\ \ K, L > 0,$

$$f_K > 0,\ f_L > 0,\ f_{KK} < 0,\ f_{LL} < 0$$

(2) f 為線性齊次函數

$$\therefore\ \ Q = L \cdot f(\frac{K}{L}, 1) \equiv L \cdot \phi(k),\ \text{where}\ \ k \equiv K / L$$

$$f_K \equiv MPP_K = \phi'(k) > 0,$$

$$f_{KK} = \frac{\partial \phi'(k)}{\partial K} = \frac{\partial \phi'(k)}{\partial k}\frac{\partial k}{\partial K} = \phi''(k) \cdot \frac{1}{L} < 0, \quad \Leftrightarrow \quad \phi''(k) < 0$$

(3) 產出 Q 的固定儲蓄比例 s $(\equiv MPS)$被用於投資，即

$$\dot{K}\ (\equiv \frac{dK}{dt}) = sQ$$

勞動力成長率為(正值)固定數λ，即

$$\frac{\dot{L}}{L}\ (\equiv \frac{dL\,/\,dt}{L}) = \lambda > 0$$

(4) 所有要素（資本與勞工）均充分就業

B. 推論過程：

均衡條件：$I = S$

$$\Rightarrow\ \dot{K} = sQ\ \Rightarrow\ \dot{K} = sL\cdot\phi(k)$$

又因為 $k \equiv \dfrac{K}{L}\ \Rightarrow\ \dot{k} = \dfrac{L\dot{K} - K\dot{L}}{L^2} = \dfrac{L[sL\cdot\phi(k)]}{L^2} - \dfrac{K}{L}\dfrac{\dot{L}}{L}$

$$\Rightarrow\ \boxed{\dot{k} = s\cdot\phi(k) - k\cdot\lambda}$$ ...k 的一階微分方程式

因為 $\phi(k)$ 為一般函數，∴只能對 $\dot{k} = s\phi(k) - k\lambda$ 進行定性分析

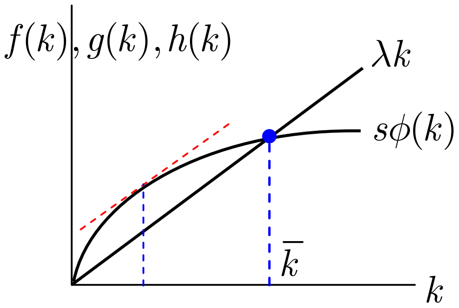
C. 定性分析：

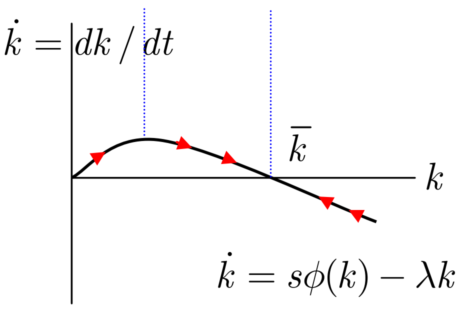
令 $g(k) \equiv s\phi(k),\ \ h(k) \equiv \lambda k$，

則 $\dot{k} = s\phi(k) - k\lambda,$ 其中 $\phi' > 0,\ \phi'' < 0$

$$\equiv g(k) - h(k)$$

將 g 和 h 函數畫在圖形上，其交點滿足 $g(k) = h(k)$，亦即 $\dot{k} = 0$





說明：(1) 均衡條件：$\dot{k}=0 \;\Leftrightarrow\; s\phi(\bar{k})-\bar{k}\lambda=0$

求得均衡的每人資本$\bar{k}$

(2) 相位圖的斜率與曲度：

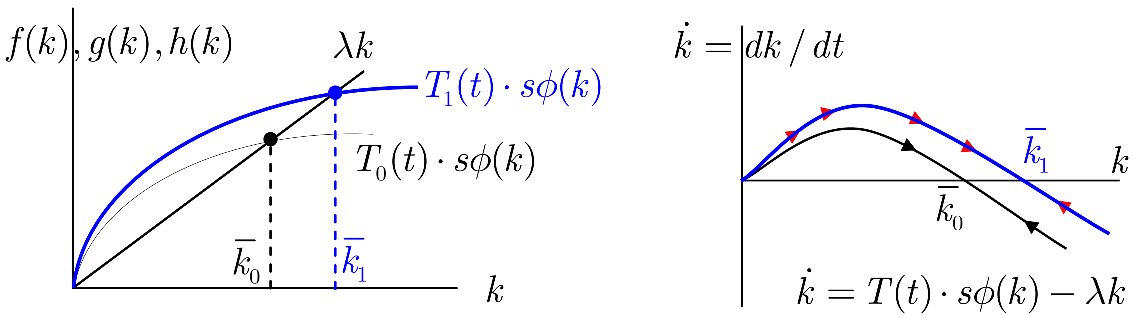
$$\text{斜率：}\frac{\partial \dot{k}}{\partial k}=s\phi'(k)-\lambda \gtreqless 0$$

$$\text{曲度：}\frac{\partial^2 \dot{k}}{\partial k^2}=s\phi''(k)<0 \quad \text{..... concave function}$$

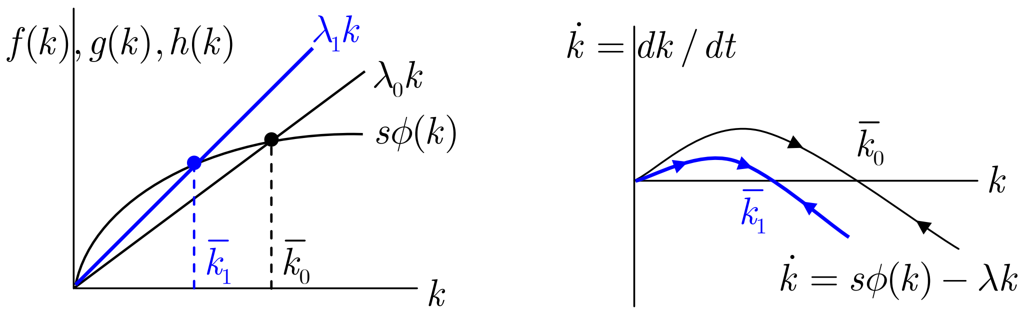
D. 比較靜態：

(1) 技術進步

令生產函數為$Q=T(t)\cdot f(K,L)$，$\dfrac{dT}{dt}>0$



(2) 人口成長率上升 ($\lambda\uparrow$；$\lambda_0<\lambda_1$)



(3) 儲蓄率上升 ($s\uparrow$) ..... 練習題

E. 明確生產函數的實例

假設生產函數為$Q = K^{\alpha} L^{1-\alpha}$，則$\phi(k) = \dfrac{Q}{L} = k^{\alpha}$

每人資本的動態方程式：$\dot{k} = sk^{\alpha} - \lambda k$

….. Bernoulli equation

$\Rightarrow \ \dot{k} + \lambda k = sk^{\alpha}$

均衡條件：$\dot{k} = 0 \ \Leftrightarrow \ \lambda \bar{k} = s\bar{k}^{\alpha} \ \Leftrightarrow \ \bar{k}(\lambda - s\bar{k}^{\alpha-1}) = 0$

$\therefore$ 均衡每人資本$\bar{k} = 0$ or $\bar{k} = \left(\dfrac{s}{\lambda}\right)^{\frac{1}{1-\alpha}}$

(1) 圖形分析：

類似前述 (自行練習)

(2) 數學解

$$\dot{k} + \lambda k = sk^{\alpha} \overset{\text{同除}k^{a}}{\Rightarrow} k^{-\alpha} \cdot \dot{k} + \lambda k^{1-\alpha} = s$$

令$z \equiv k^{1-\alpha}$

$$\begin{aligned}
&\Rightarrow \ \dot{z} \equiv (1-\alpha)k^{-\alpha} \cdot \dot{k} \\
&\Rightarrow \ \frac{1}{1-\alpha}\dot{z} + \lambda \cdot z = s \\
&\Rightarrow \ \dot{z} + (1-\alpha)\lambda \cdot z = (1-\alpha)s \\
&\Rightarrow \ z(t) = \frac{s}{\lambda} + Ae^{-(1-\alpha)\lambda \cdot t} \\
&\Rightarrow \ k^{1-\alpha}(t) = \frac{s}{\lambda} + Ae^{-(1-\alpha)\lambda \cdot t} \\
&\qquad\qquad\quad = \bar{k}^{1-\alpha} + Ae^{-(1-\alpha)\lambda \cdot t}
\end{aligned}$$

$$\text{or} \ \Rightarrow \ k(t) = \left[\bar{k}^{1-\alpha} + Ae^{-(1-\alpha)\lambda \cdot t}\right]^{\frac{1}{1-\alpha}}$$

初始條件：$t=0$時，每人資本為$k(0)$

$$\Rightarrow\ k^{1-\alpha}(0)=\overline{k}^{1-\alpha}+Ae^{-(1-\alpha)\lambda\cdot 0}=\overline{k}^{1-\alpha}+A$$

$$\Rightarrow\ A=k^{1-\alpha}(0)-\overline{k}^{1-\alpha}\text{，其中 }\overline{k}^{1-\alpha}=\frac{s}{\lambda}$$

$$\begin{aligned}\therefore\ k^{1-\alpha}(t)&=\overline{k}^{1-\alpha}+\left(k^{1-\alpha}(0)-\overline{k}^{1-\alpha}\right)e^{-(1-\alpha)\lambda\cdot t}\\&=\frac{s}{\lambda}+\left(k^{1-\alpha}(0)-\frac{s}{\lambda}\right)e^{-(1-\alpha)\lambda\cdot t}\end{aligned}$$

As $t\to\infty$, $k^{1-\alpha}\to\dfrac{s}{\lambda}$ or $k\to\left(\dfrac{s}{\lambda}\right)^{1/(1-\alpha)}$