# Constructing an Iterated Fibre Bundle: A Simple Example

## Introduction

This document demonstrates the explicit construction of an iterated fibre bundle through a simple example. The goal is to construct a two-layer fibre bundle where the base space of the first fibre bundle serves as the total space of the second.

## Construction of the Iterated Fibre Bundle

We construct the iterated fibre bundle in the following steps:

### Step 1: Define the First Fibre Bundle

Choose the base space $B = S^1$ (a circle). Let the fibre $F_1 = S^1$. Define the total space $E_1 = S^1 \times S^1$, which is a torus $T^2$. Define the projection map $\pi_1 : T^2 \to S^1$ as:

$$\pi_1(\theta_1, \theta_2) = \theta_1,$$

where $(\theta_1, \theta_2)$ are points on $T^2$. Check the fibre structure:

- For any $\theta_1 \in S^1$, the fibre is:

  $$\pi_1^{-1}(\theta_1) = \{\theta_1\} \times S^1,$$

  which is homeomorphic to $S^1$. Thus, $F_1 \to E_1 \to B$ forms a fibre bundle.

### Step 2: Define the Second Fibre Bundle

Let the total space of the first fibre bundle $E_1 = T^2$ serve as the base space for the second fibre bundle. Choose the fibre $F_2 = \mathbb{R}$. Define the total space $E_2 = T^2 \times \mathbb{R}$. Define the projection map $\pi_2 : T^2 \times \mathbb{R} \to T^2$ as:

$$\pi_2((\theta_1, \theta_2), r) = (\theta_1, \theta_2),$$

where $((\theta_1, \theta_2), r)$ are points in $E_2$. Check the fibre structure:

- For any $(\theta_1, \theta_2) \in T^2$, the fibre is:

  $$\pi_2^{-1}((\theta_1, \theta_2)) = \{(\theta_1, \theta_2)\} \times \mathbb{R},$$

  which is homeomorphic to $\mathbb{R}$. Thus, $F_2 \to E_2 \to E_1$ forms a fibre bundle.

### Step 3: The Iterated Fibre Bundle

Combining the two layers of fibre bundles, we obtain the iterated fibre bundle:

$$F_2 \to E_2 \to B,$$

where $E_2 = T^2 \times \mathbb{R}$ is the total space, $B = S^1$ is the base space, and the final fibre is $F_2 = \mathbb{R}$.

## Verification

1. The first fibre bundle $F_1 \to E_1 \to B$ is valid because for each $\theta_1 \in S^1$, the fibre $\pi_1^{-1}(\theta_1) \cong S^1$ is a connected space. 2. The second fibre bundle $F_2 \to E_2 \to E_1$ is valid because for each $(\theta_1, \theta_2) \in T^2$, the fibre $\pi_2^{-1}((\theta_1, \theta_2)) \cong \mathbb{R}$ is a trivial fibre. 3. Since fibre bundles can be composed, the resulting iterated fibre bundle $F_2 \to E_2 \to B$ satisfies the definition.

## Conclusion

This construction explicitly demonstrates a simple example of an iterated fibre bundle, where $F_2 = \mathbb{R}$, $E_2 = T^2 \times \mathbb{R}$, and $B = S^1$.