Mathematical Framework of Quantum Gravity Based on Knot Theory

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Outline

Motivation

- Challenge: Quantum gravity requires non-perturbative approach
- Key idea: Use knot theory as mathematical framework
- Main tools:
    - Quantum groups
    - Spin networks
    - Loop quantum gravity

Quantum Groups and 6j-Symbols

- Quantum deformation $U_q(\mathfrak{g})$
- Key structure:

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}_q$$

- Properties:
  - Tetrahedral symmetry
  - Quantum Racah identity

Crossing Relations

- R-matrix:

$$R_{j_1 j_2} = q^{H_{j_1} \otimes H_{j_2}/2} P_{j_1 j_2}$$

- Yang-Baxter equation:

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

- Reidemeister moves invariance

Geometric Operators

- Area operator:

$$\hat{A}_q(S) = l_P^2 \sum_{p \in S \cap \Gamma} \sqrt{[j_p]_q [j_p + 1]_q}$$

- Volume operator:

$$\hat{V}_q(R) = l_P^3 \sum_{v \in R} \sqrt{|\hat{Q}_v|}$$

- Discrete spectrum

Holonomy-Flux Algebra

- Holonomy:

$$h_\gamma[A] = \mathcal{P} \exp \int_\gamma A$$

- Flux:

$$E(S, f) = \int_S f^i \epsilon_{abc} E_i^a dx^b \wedge dx^c$$

- Cross relations:

$$[E(S, f), h_\gamma] = i\hbar\kappa\beta(S, \gamma) X^f h_\gamma$$

Quantum Black Holes

- Horizon quantum states
- Area spectrum:

$$A = 8\pi\gamma l_P^2 \sum_i \sqrt{j_i(j_i+1)}$$

- Entropy calculation:

$$S = \frac{A}{4l_P^2} + \text{corrections}$$

Cosmological Applications

- Quantum cosmology
- Big bounce scenario
- Modified dispersion relations
- Quantum gravity corrections

Future Directions

- Semiclassical limit
- Matter coupling
- Observational tests
- Connection to other approaches

Conclusions

- Mathematical consistency
- Physical predictions
- Experimental challenges
- Future prospects