CS 6316 Machine Learning

Neural Networks

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Overview

1. From Logistic Regression to Neural Networks
2. Expressive Power of Neural Networks
3. Learning Neural Networks
4. Computation Graph

From Logistic Regression to Neural Networks

Logistic Regression

- An unified form for $y \in \{-1, +1\}$

$$p(Y = +1 \mid \boldsymbol{x}) = \frac{1}{1 + \exp(-\langle \boldsymbol{w}, \boldsymbol{x} \rangle)} \tag{1}$$

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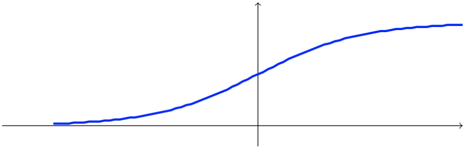
- The sigmoid function $\sigma(a)$ with $a \in \mathbb{R}$

$$\sigma(a) = \frac{1}{1 + \exp(-a)} \tag{2}$$

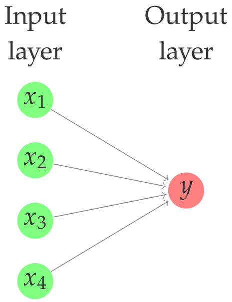


Graphical Representation

- A specific example of LR

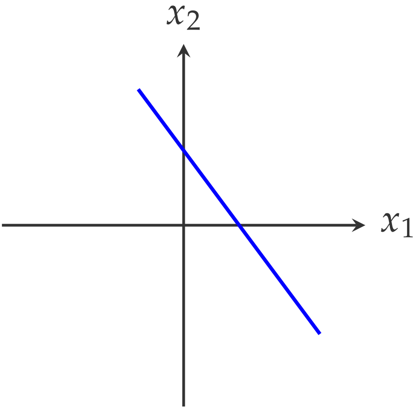
$$p(Y = 1 \mid \boldsymbol{x}) = \sigma(\sum_{j=1}^{4} w_j \boldsymbol{x}_{\cdot,j}) \tag{3}$$

- The graphical representation of this LR model is



Capacity of a LR

Logistic regression gives a linear decision boundary



From LR to Neural Networks

Build upon logistic regression, a simple neural network can be constructed as

$$
\begin{aligned}
z_k &= \sigma(\sum_{i=1}^{d} w_{k,j}^{(1)} x_{\cdot,i}) \quad k \in [K] && (4) \\
P(y = 1 \mid \boldsymbol{x}) &= \sigma(\sum_{k=1}^{K} w_k^{(o)} z_k) && (5)
\end{aligned}
$$

- $\boldsymbol{x} \in \mathbb{R}^d$: d-dimensional input
- $y \in \{-1, +1\}$ (binary classification problem)

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- $\boldsymbol{x} \in \mathbb{R}^d$: d-dimensional input
- $y \in \{-1, +1\}$ (binary classification problem)
- $\{w_{k,i}^{(1)}\}$ and $\{w_k^{(o)}\}$ are two sets of the parameters, and
- K is the number of hidden units, each of them has the same form as a LR.

Mathematical Formulation

- Element-wise formulation

$$z_k = \sigma(\sum_{j=1}^{d} w_{k,j}^{(1)} x_{\cdot,j}) \quad k \in [K] \tag{6}$$

$$P(y = +1 \mid \boldsymbol{x}) = \sigma(\sum_{k=1}^{K} w_k^{(o)} z_k) \tag{7}$$

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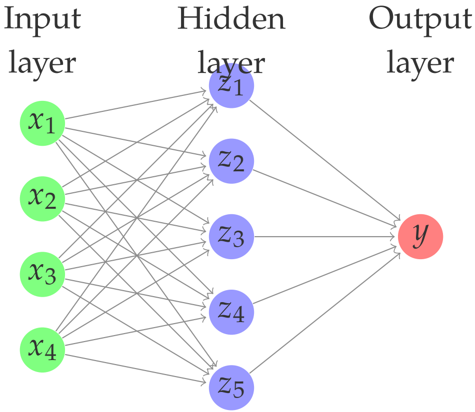
- Matrix-vector formulation

$$\boldsymbol{z} = \sigma(\mathbf{W}^{(1)} \boldsymbol{x}) \tag{8}$$

$$P(y = +1 \mid \boldsymbol{x}) = \sigma((\boldsymbol{w}^{(o)})^{\top} \boldsymbol{z}) \tag{9}$$

where $\mathbf{W}^{(1)} \in \mathbb{R}^{K \times d}$ and $\mathbf{w}^{(o)} \in \mathbb{R}^{K}$

Graphical Representation

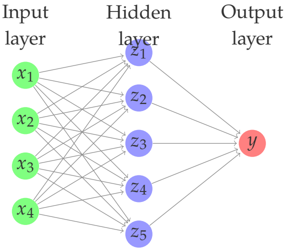


- Depth: 2 (two-layer neural network)
- Width: 5 (the maximal number of units in each layer)

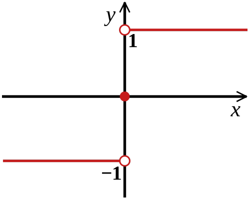
Hypothesis Space

The hypothesis space of neural networks is usually defined by the architecture of the network, which includes

- the nodes in the network,
- the connections in the network, and
- the activation function (e.g., σ)

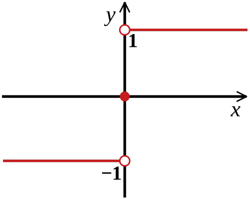


Other Activation Functions



(a) Sign function

Other Activation Functions



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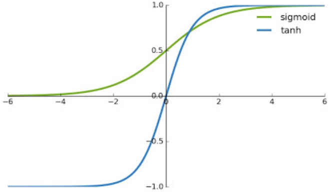
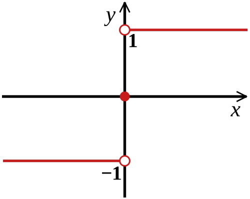


(b) Tanh function

Other Activation Functions



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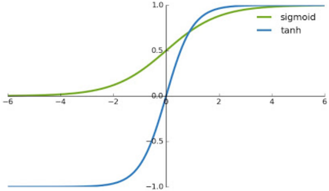
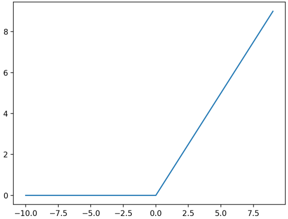


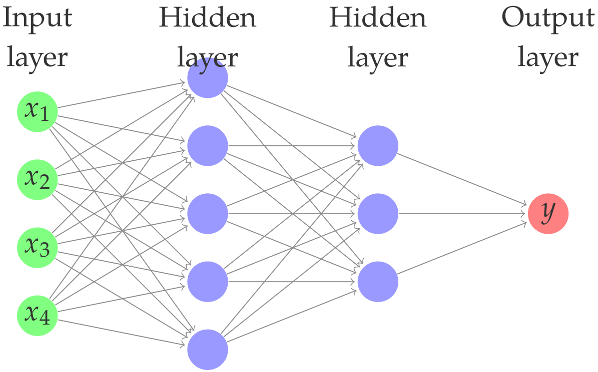
(b) Tanh function



(c) ReLU function [Jarrett et al., 2009]

Another Network/Hypothesis Space

Simply increasing the number of layers or increase the number of hidden units, we can create another hypothesis space



Expressive Power of Neural Networks

Two-layer NNs with Sign Function

Consider a neural network defined by the following functions

$$z_k = \mathrm{sign}\left(\sum_{j=1}^{d} w_{k,j}^{(1)} x_{\cdot,j}\right) \quad k \in [K] \tag{10}$$

$$h(\boldsymbol{x}) = \mathrm{sign}\left(\sum_{k=1}^{K} w_{k}^{(o)} z_k\right) \tag{11}$$

where $\mathrm{sign}(a)$ is the sign function.

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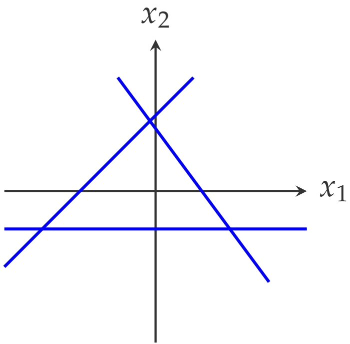
where $\mathrm{sign}(a)$ is the sign function.

$h(\boldsymbol{x})$ can be rewritten as

$$h(\boldsymbol{x}) = \mathrm{sign}\left(\sum_{k=1}^{K} w_k^{(o)} \cdot \mathrm{sign}(\sum_{j=1}^{d} w_{k,i}^{(1)} x_{\cdot,j})\right) \tag{12}$$

Decision Boundary

$h(\boldsymbol{x})$ is defined by a combination of K linear predictors



Similar conclusion applies to other activation functions.

[Shalev-Shwartz and Ben-David, 2014, Page 274]

Universal Approximation Theorem

Restrict the inputs $x_{\cdot,j} \in \{-1, +1\} \forall j \in [d]$ as binary

Universal Approximation Theorem

For every d, there exists a two-layer neural network (Equations 10 – 11), such that this hypothesis space contains all functions from $\{-1, +1\}^d$ to $\{-1, +1\}$

[Shalev-Shwartz and Ben-David, 2014, Section 20.3]

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- The minimal size of network that satisfies the theorem is exponential in d
- Similar results hold for σ as the activation function

[Shalev-Shwartz and Ben-David, 2014, Section 20.3]

Learning Neural Networks

Neural Network Predictions

Consider a binary classification problem with $\mathcal{Y} = \{-1, +1\}$,

- A two-layer neural network gives the following prediction as

$$P(Y = +1 \mid \boldsymbol{x}) = \sigma\left((\boldsymbol{w}^{(o)})^{\mathsf{T}}\sigma(\mathbf{W}^{(1)}\boldsymbol{x})\right) \tag{13}$$

  where $\{\boldsymbol{w}^{(o)}, \mathbf{W}^{(1)}\}$ are the parameters

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  where $\{\boldsymbol{w}^{(o)}, \mathbf{W}^{(1)}\}$ are the parameters
- Assume the ground-truth label is y, it can be defined as an empirical function

$$q(Y = y' \mid \boldsymbol{x}) = \delta(y', y) = \begin{cases} 1 & y' = y \\ 0 & y' \neq y \end{cases} \tag{14}$$

Cross Entropy

Given one data point, The loss function of a neural network is usually defined as the cross entropy of the prediction distribution p and the empirical distribution p

$$\begin{aligned} H(q,p) &= -q(Y=+1 \mid \boldsymbol{x}) \log p(Y=+1 \mid \boldsymbol{x}) \\ &\quad -q(Y=-1 \mid \boldsymbol{x}) \log p(Y=-1 \mid \boldsymbol{x}) \end{aligned} \tag{15}$$

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Since q is defined with a Delta function, Depending on y, we have

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H(q, p) = \begin{cases} -\log p(Y = +1 \mid \boldsymbol{x}) & Y = +1 \\ -\log p(Y = -1 \mid \boldsymbol{x}) & Y = -1 \end{cases} \tag{16}
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It is equivalent to the negative log-likelihood (NLL) function used in learning LR.

ERM

- Given a set of training example $S = \{(\boldsymbol{x}_i, y_i)\}_{i=1}^{m}$, the loss function is defined as

$$L(\boldsymbol{\theta}) = -\sum_{i=1}^{m} p(y_i \mid \boldsymbol{x}_i) \tag{17}$$

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- For example, $\boldsymbol{\theta} = \{\boldsymbol{w}^{(o)}, \mathbf{W}^{(1)}\}$, for the previously defined two-layer neural network
- Just like learning a LR, we can use gradient-based learning algorithm

Gradient-based Learning

A simple scratch of gradient-based learning

1. Compute the gradient of $\boldsymbol{\theta}$, $\frac{\partial L(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$

[1] More detail will be discussed in the next lecture

Gradient-based Learning

A simple scratch of gradient-based learning

1. Compute the gradient of $\boldsymbol{\theta}$, $\frac{\partial L(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$
2. Update the parameter with the gradient

$$\boldsymbol{\theta}^{(\text{new})} \leftarrow \boldsymbol{\theta}^{(\text{old})} - \eta \cdot \left.\frac{\partial L(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}^{(\text{old})}} \tag{18}$$

   where η is the learning rate

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   where η is the learning rate

3. Go back step 1 until it converges[1]

[1]More detail will be discussed in the next lecture

Gradient Computation

Consider the two-layer neural network with one training example $(\boldsymbol{x}, y)$, to further simplify the computation, we assume $y = +1$

$$\log p(y \mid \boldsymbol{x}) = \log \sigma \left((\boldsymbol{w}^{(o)})^{\mathsf{T}} \sigma(\mathbf{W}^{(1)} \boldsymbol{x}) \right) \tag{19}$$

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$$\log p(y \mid \boldsymbol{x}) = \log \sigma\left((\boldsymbol{w}^{(o)})^\top \sigma(\mathbf{W}^{(1)}\boldsymbol{x})\right) \tag{19}$$

The gradient with respect to $\boldsymbol{w}^{(o)}$ is

$$\frac{\partial L(\boldsymbol{\theta})}{\partial \boldsymbol{w}^{(o)}} = -\frac{\partial \log \sigma\left(\cdot\right)}{\partial \sigma\left(\cdot\right)} \cdot \frac{\partial \sigma\left((\boldsymbol{w}^{(o)})^\top \sigma(\mathbf{W}^{(1)}\boldsymbol{x})\right)}{\partial (\boldsymbol{w}^{(o)})^\top \sigma(\mathbf{W}^{(1)}\boldsymbol{x})} \cdot \frac{\partial (\boldsymbol{w}^{(o)})^\top \sigma(\mathbf{W}^{(1)}\boldsymbol{x})}{\partial \boldsymbol{w}^{(o)}} \tag{20}$$

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$$\begin{aligned}
\frac{\partial L(\boldsymbol{\theta})}{\partial \boldsymbol{w}^{(o)}} &= -\frac{\partial \log \sigma\left(\cdot\right)}{\partial \sigma\left(\cdot\right)} \cdot \frac{\partial \sigma\left((\boldsymbol{w}^{(o)})^{\mathsf{T}} \sigma(\mathbf{W}^{(1)} \boldsymbol{x})\right)}{\partial (\boldsymbol{w}^{(o)})^{\mathsf{T}} \sigma(\mathbf{W}^{(1)} \boldsymbol{x})} \cdot \frac{\partial (\boldsymbol{w}^{(o)})^{\mathsf{T}} \sigma(\mathbf{W}^{(1)} \boldsymbol{x})}{\partial \boldsymbol{w}^{(o)}} \\
&= -\left\{1 - \sigma\left((\boldsymbol{w}^{(o)})^{\mathsf{T}} \sigma(\mathbf{W}^{(1)} \boldsymbol{x})\right)\right\} \cdot \sigma(\mathbf{W}^{(1)} \boldsymbol{x})
\end{aligned} \tag{20}$$

which is in the similar form as the LR updating equation.

Gradient Computation (II)

The gradient with respect to $W^{(1)}$ is

$$\begin{aligned}\frac{\partial L(\boldsymbol{\theta})}{\partial \boldsymbol{w}^{(o)}} \quad = \quad & -\frac{\partial \log \sigma\Big(\cdot\Big)}{\partial \sigma\Big(\cdot\Big)} \cdot \frac{\partial \sigma\Big((\boldsymbol{w}^{(o)})^\top \sigma(\mathbf{W}^{(1)}\boldsymbol{x})\Big)}{\partial (\boldsymbol{w}^{(o)})^\top \sigma(\mathbf{W}^{(1)}\boldsymbol{x})} \\ & \cdot \frac{\partial (\boldsymbol{w}^{(o)})^\top \sigma(\mathbf{W}^{(1)}\boldsymbol{x})}{\partial \sigma(\mathbf{W}^{(1)}\boldsymbol{x})} \cdot \frac{\partial \mathbf{W}^{(1)}\boldsymbol{x}}{\partial \mathbf{W}^{(1)}} \end{aligned} \tag{21}$$

Gradient Computation (II)

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- Both of them are the applications of the chain rule in calculus plus some derivatives of basic functions
- In the literature of neural networks, it is called the back-propagation algorithm [Rumelhart et al., 1986]

Computation Graph

Forward Operations

Consider the example of a two-layer neural network

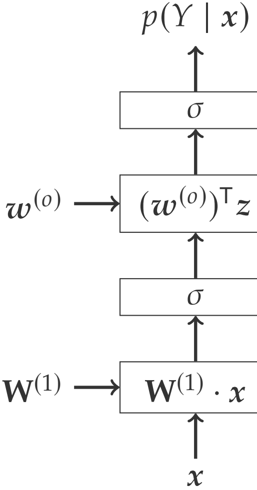
$$P(Y = +1 \mid \boldsymbol{x}) = \sigma\left((\boldsymbol{w}^{(o)})^\top \sigma(\mathbf{W}^{(1)}\boldsymbol{x})\right) \tag{22}$$

A neural network is a composition of some basic functions and operations. For example

- $\sigma(\cdot)$
- matrix transpose $(\boldsymbol{w}^{(o)})^\top$
- matrix-vector multiplication $\mathbf{W}^{(1)}\boldsymbol{x}$

Forward Graph

The computation graph of the two-layer neural network[2]



[2]For simplicity, the transpose operation is ignored from the graph

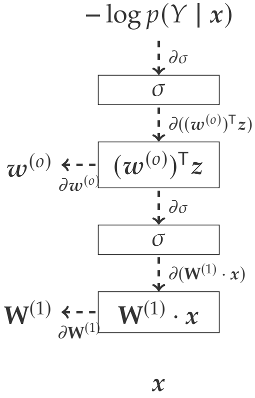
Backward Operations

Similarly, the gradient of neural network parameters are computed with a series of backward operations associated with the derivative of some basic function. For example

- $\frac{\partial \sigma(x)}{\partial x} = \sigma(x)(1 - \sigma(x))$
- $\frac{\partial \boldsymbol{a}^\top \boldsymbol{x}}{\partial \boldsymbol{x}} = \boldsymbol{a}$
- $\frac{\partial \log(x)}{\partial x} = \frac{1}{x}$
- $\frac{\partial \mathbf{W}\boldsymbol{x}}{\partial \boldsymbol{x}} = \begin{bmatrix} \boldsymbol{x}^\top \\ \vdots \\ \boldsymbol{x}^\top \end{bmatrix}$

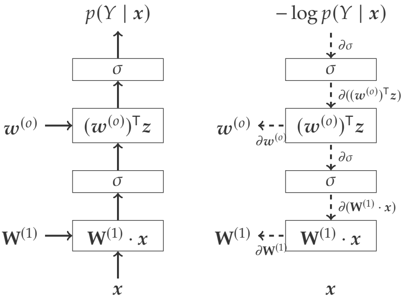
Backward Graph

With the chain rule, gradient of the loss function with respect to any parameter can be computed backward step-by-step along the path



Computation Graph

Perform the forward/backward step with a graph of basic operations (e.g., PyTorch, Tensorflow)



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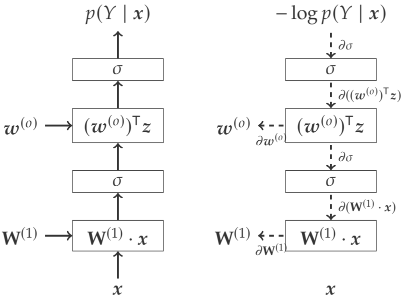

- Modular implementation: implement each module with its forward/backward operations together
- Automatic differentiation: automatically run with the backward step

What is Deep Learning?

Definition

Deep Learning is building a system by assembling parameterized modules into a (possibly dynamic) computation graph, and training it to perform a task by optimizing the parameters using a gradient-based method.

[LeCun, 2020, AAAI 2020 Keynote]

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