# Lecture Notes

## 2301107 Calculus I for Engineering
## Section 4

First Semester, Academic Year 2024

# Chapter 2 Limits and Derivatives

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# Chapter 2 Limits and Derivatives

# 2.2 The Limit of a Function

## Finding Limits Numerically and Graphically

**Example 1** Let's investigate the behavior of $f(x) = \dfrac{2^x - 1}{x}$ for values of $x$ near 0.

**Remark** We write

1. $x \to a$ for "$x$ approaches $a$"
2. $\lim\limits_{x \to a} f(x) = L$ for "$f(x) \to L$ as $x \to a$"

**Definition 1 (Intuitive Definition of a Limit)**

Suppose $f(x)$ is defined when $x$ is near the number $a$.

(This means that $f$ is defined on some open interval that contains $a$, except possibly at $a$ itself.)

Then we write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as $x$ approaches $a$, equals $L$"

if we can make the values of $f(x)$ arbitrarily close to $L$ (as close to $L$ as we like)

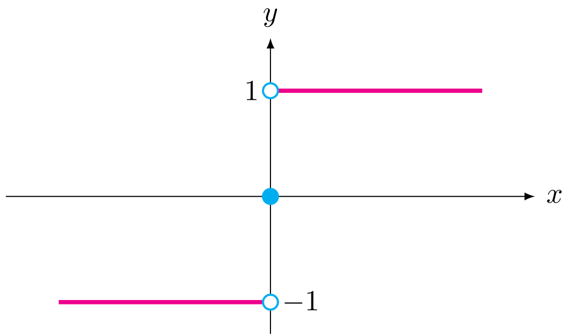
by restricting $x$ to be sufficiently close to $a$ (on either side of $a$) but not equal to $a$.

**Remark** If a limit of a function exists, the limit must have a unique value.

# One-Sided Limits

**Example 2** **(The Sign Function)**

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$




**Remark** If $x \neq 0$, then $\dfrac{|x|}{x} = \operatorname{sgn}(x)$.

**Definition 2 (Intuitive Definition of One-Sided Limits)**

We write $\lim\limits_{x\to a^-} f(x) = L$ and say that the left-hand limit of $f(x)$ as $x$ approaches $a$

[or the limit of $f(x)$ as $x$ approaches $a$ from the left]

is equal to $L$ if we can make the values of $f(x)$ arbitrarily close to $L$

by restricting $x$ to be sufficiently close to $a$ with $x < a$.

We write $\lim\limits_{x\to a^+} f(x) = L$ and say that the right-hand limit of $f(x)$ as $x$ approaches $a$

[or the limit of $f(x)$ as $x$ approaches $a$ from the right]

is equal to $L$ if we can make the values of $f(x)$ arbitrarily close to $L$

by restricting $x$ to be sufficiently close to $a$ with $x > a$.

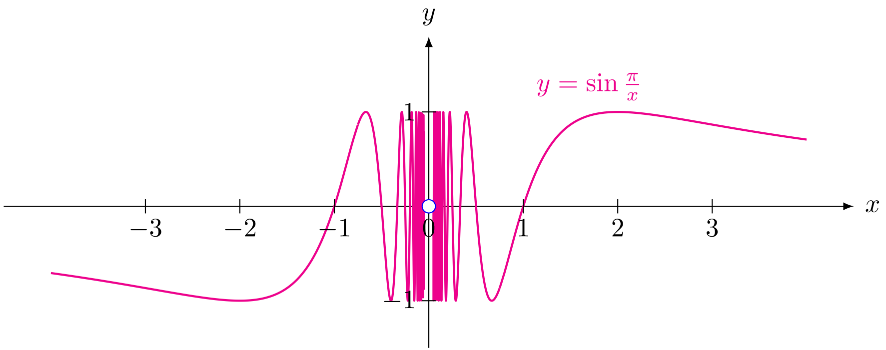
**Remark** $\lim\limits_{x\to a} f(x) = L$ if and only if $\lim\limits_{x\to a^-} f(x) = L$ and $\lim\limits_{x\to a^+} f(x) = L$

**Example 3** Let $f(x) = \dfrac{x}{\sqrt[4]{x^2}}$. Find 1. $\lim\limits_{x\to 0^+} f(x)$ 2. $\lim\limits_{x\to 0^-} f(x)$ 3. $\lim\limits_{x\to 0} f(x)$.

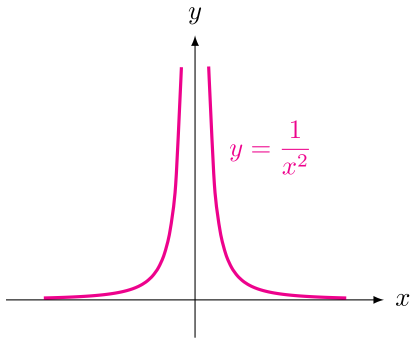
**Remark** $\sqrt[4]{x^2} = \sqrt{|x|}$

## How Can a Limit Fail to Exist?

**Example 4** Investigate $\lim_{x\to 0} \sin \dfrac{\pi}{x}$.



**Example 5** Investigate $\lim\limits_{x\to 0} \dfrac{1}{x^2}$.



## Infinite Limits; Vertical Asymptotes

**Definition 3 (Intuitive Definition of an Infinite Limit)**

Let $f$ be a function defined on both sides of $a$, except possibly at $a$ itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking $x$ sufficiently close to $a$, but not equal to $a$.

Let $f$ be a function defined on both sides of $a$, except possibly at $a$ itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

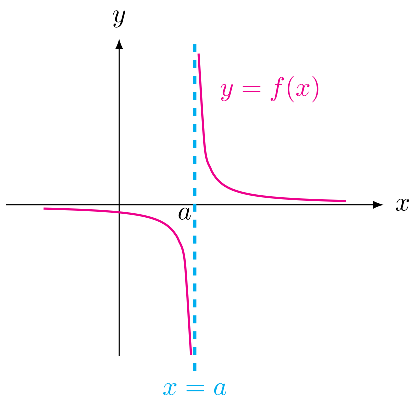
means that the values of $f(x)$ can be made arbitrarily large negative by taking $x$ sufficiently close to $a$, but not equal to $a$.

**Definition 4 (Vertical Asymptotes)**

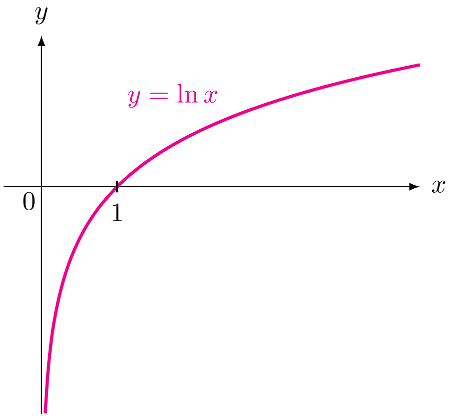
The vertical line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$

$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty$$



**Example 6** The vertical asymptote of $f(x) = \ln x$.




$$\lim_{x \to 0^+} \ln x = -\infty$$

## 2.3 Calculating Limits Using the Limit Laws

### Properties of Limits

**Theorem 1 (Limit Laws)**

*Suppose that c ia a constant and the limits* $\lim\limits_{x\to a} f(x)$ *and* $\lim\limits_{x\to a} g(x)$ *exist. Then*

1. $\lim\limits_{x\to a}\big[f(x)+g(x)\big] = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$

2. $\lim\limits_{x\to a}\big[f(x)-g(x)\big] = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x)$

3. $\lim\limits_{x\to a}\big[c\,f(x)\big] = c\lim\limits_{x\to a} f(x)$

4. $\lim\limits_{x\to a}\big[f(x)g(x)\big] = \lim\limits_{x\to a} f(x)\cdot\lim\limits_{x\to a} g(x)$

5. $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ $\quad$ *if* $\lim\limits_{x\to a} g(x) \neq 0$

**Theorem 2 (Limits Laws (Cont'd))**

*Let $n$ be a positive integer. Suppose that the limit $\lim_{x\to a} f(x)$ exist. Then*

6. $\lim_{x\to a} \big[f(x)\big]^n = \big[\lim_{x\to a} f(x)\big]^n$

7. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ *(If $n$ is even, we assume that $\lim_{x\to a} f(x) > 0$.)*

**Theorem 3 (Special Limits)**

*Let $n$ be a positive integer. Then*

1. $\lim_{x\to a} c = c$ *when $c$ is a constant.*

2. $\lim_{x\to a} x = a$

3. $\lim_{x\to a} x^n = a^n$

4. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ *(If $n$ is even, we assume that $a > 0$.)*

## Evaluating Limits by Direct Substitution

**Definition 5 (Polynomial Functions and Rational Functions)**

1. If $f(x)$ is a polynomial of $x$, we say that $f$ is a polynomial function.

2. If there exist polynomials $P(x)$ and $Q(x)$ such that $f(x) = \dfrac{P(x)}{Q(x)}$, we say that $f$ is a rational function.

**Remark** A polynomial function is always a rational function.

**Example 7** Which of the followings are polynomial or rational functions?

$$\frac{x+1}{x^2+1} \qquad \frac{1}{x} \qquad x^2 \qquad 1$$

$$|x| \qquad \frac{1}{\sqrt{x}} \qquad \frac{|x|}{|x|^3}$$

**Theorem 4 (Direct Substitution Property)**

*If $f$ is a polynomial or rational function and $a$ is in the domain of $f$, then*

$$\lim_{x \to a} f(x) = f(a).$$

**Example 8** Find $\displaystyle\lim_{x \to 1} \frac{x^2 + 2x + 3}{x^3 + 2x^2 + 1}$.

**Remark** Functions that have the Direct Substitution Property are called continuous at $a$.

**Example 9** Evaluate $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{x^2 + x - 2}$.

**Remark** if $f(x) = g(x)$ when $x \neq a$, then $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x)$, provided the limits exist.

**Example 10** Find $\lim_{x\to 1} \dfrac{x^3-1}{x^4-1}$.

**Example 11** Find $\lim_{x \to 0} \dfrac{x^2}{\sqrt{x^2+9}-3}$.

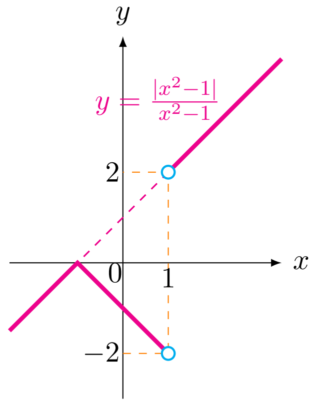
## Using One-Sided Limits

**Theorem 5**

$$\lim_{x \to a} f(x) = L \qquad \textit{if and only if} \qquad \lim_{x \to a^-} f(x) = L \quad \textit{and} \quad \lim_{x \to a^+} f(x) = L$$

**Example 12** Show that $\lim\limits_{x \to 0} \dfrac{|x|}{x}$ does not exist.

**Example 13** Investigate $\lim\limits_{x \to 1} \dfrac{|x^2 - 1|}{x - 1}$.



## The Squeeze Theorem

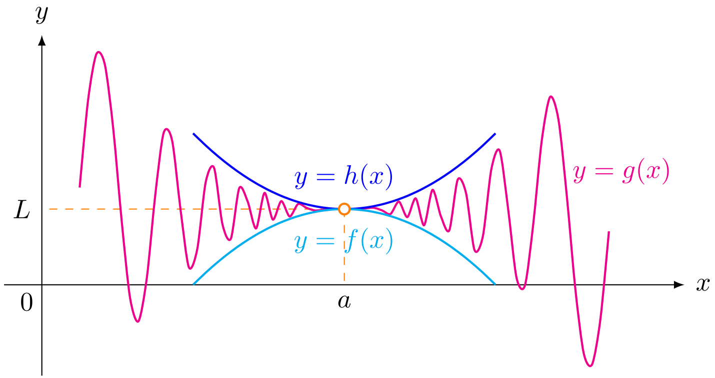
**Theorem 6 (The Squeeze Theorem)**

*If* $f(x) \leq g(x) \leq h(x)$ *when* $x$ *is near* $a$ *(except possibly at* $a$*) and*

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x),$$

*then*

$$\lim_{x\to a} g(x) = L$$



**Example 14** Find $\lim_{x \to 0} x^2 \sin \frac{1}{x}$.

# 2.5 Continuity

## Continuity of a Function

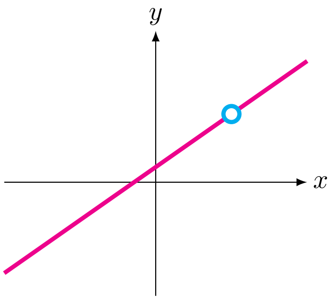
**Definition 6**

A function $f$ is *continuous at a number* $a$ if

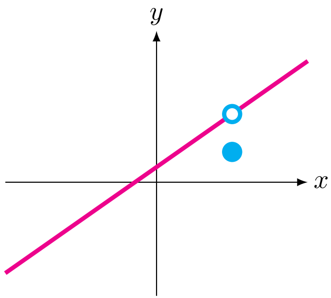
$$\lim_{x \to a} f(x) = f(a).$$

**Remark** Continuity of $f$ at $a$ implicitly requires three things.

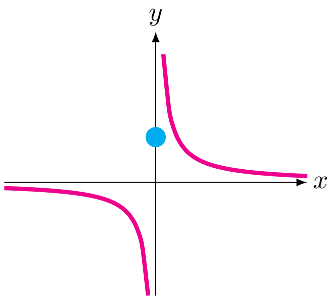
1. $f(a)$ is defined (that is, $a$ is in the domain of $f$)
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$



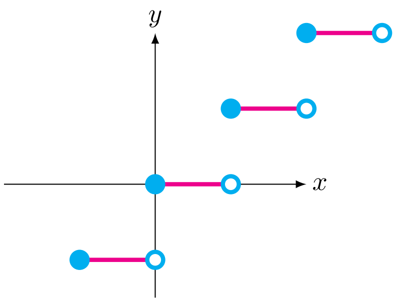

A removable discontinuity




A removable discontinuity




An infinite discontinuity




Jump discontinuities

**Example 15** Classify all discontinuities of $f(x) = \dfrac{(|x| + x)(x^2 - 4x + 3)}{(x^2 - 2x)(x^2 - 3x + 2)}$.

**Definition 7**

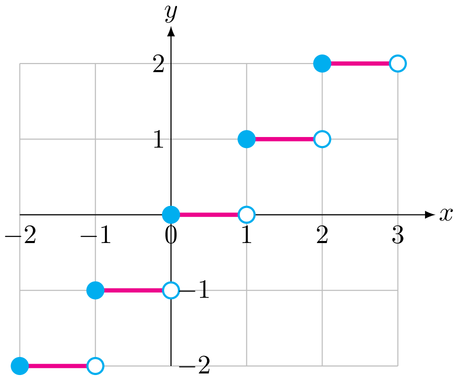
A function $f$ is continuous from the right at a number $a$ if

$$\lim_{x \to a^+} f(x) = f(a)$$

and $f$ is continuous from the left at $a$ if

$$\lim_{x \to a^-} f(x) = f(a).$$

**Example 16** Let $f(x) = \lfloor x \rfloor$ be the floor function.



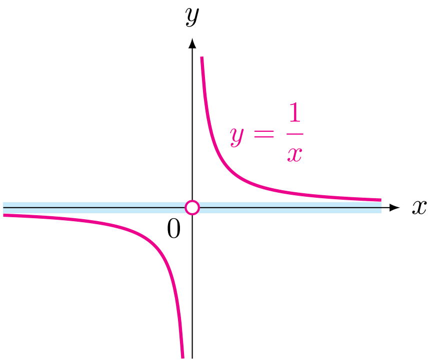

1. At which number is $f$ continuous from the right?
2. At which number is $f$ continuous from the left?
3. At which number is $f$ continuous?

**Definition 8 (Continuity on an Interval)**

A function $f$ is continuous on an interval if it is continuous at every number in the interval.

(If $f$ is defined only on one side of an endpoint of the interval,
we understand continuous at the endpoint to mean
continuous from the right or continuous from the left.)

**Example 17** Is $f(x) = \dfrac{1}{x}$ continuous on its domain?



## Properties of Continuous Functions

**Theorem 7**

*If $f$ and $g$ are continuous at $a$ and $c$ is a constant,*

*the following functions are also continuous at $a$:*

1. $f + g$
2. $f - g$
3. $cf$
4. $fg$
5. $\dfrac{f}{g}$ *if* $g(a) \neq 0$

**Theorem 8**

1. *Any polynomial is continuous everywhere;*
   *that is, it is continuous on* $\mathbb{R} = (-\infty, \infty)$.

2. *Any rational function is continuous wherever it is defined;*
   *that is, it is continuous on its domain.*

**Theorem 9**

*The following types of functions are continuous at every number in their domains:*

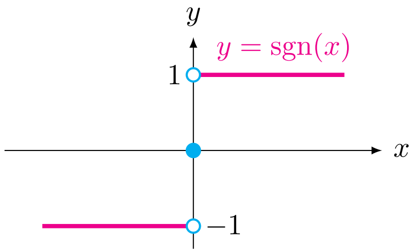
- *polynomials*
- *rational functions*
- *root functions*
- *trigonometric functions*
- *inverse trigonometric functions*
- *exponential functions*
- *logarithmic functions*

**Theorem 10**

*if $f$ is continuous at $b$ and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f\big(g(x)\big) = f(b)$.*
*In other words,*

$$\lim_{x\to a} f\big(g(x)\big) = f\left(\lim_{x\to a} g(x)\right).$$

**Example 18** Find 1. $\operatorname{sgn}\left(\lim_{x\to 0} x\right)$ 2. $\lim_{x\to 0} \operatorname{sgn}(x)$

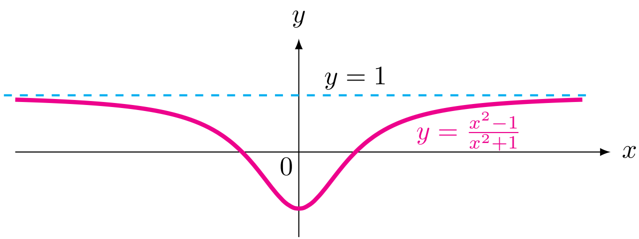



**Remark** The sgn function is not continuous at 0 and $\operatorname{sgn}\left(\lim_{x\to 0} x\right) \neq \lim_{x\to 0} \operatorname{sgn}(x)$.

# 2.6 Limits at Infinity; Horizontal Asymptotes

## Limits at Infinity and Horizontal Asymptotes

**Example 19** Investigate the behavior of $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$ as $x$ becomes large.




**Remark** We write

$x \to \infty$ to mean "$x$ is infinitely large", and

$x \to -\infty$ to mean "$x$ is infinitely large negative"

**Definition 9 (Intuitive Definition of a Limit at Infinity)**

Let $f$ be a function defined on some interval $(a, \infty)$. Then

$$\lim_{x\to\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to $L$ by requiring $x$ to be sufficiently large.

Let $f$ be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x\to-\infty} f(x) = L$$

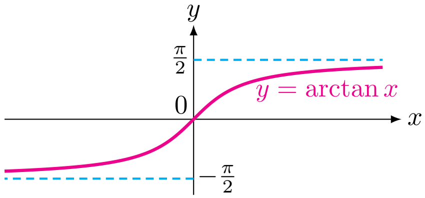
means that the values of $f(x)$ can be made arbitrarily close to $L$ by requiring $x$ to be sufficiently large negative.

**Definition 10**

The line $y = L$ is called a horizontal symptote of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L.$$

**Example 20** Find the horizontal asymptotes of $y = \arctan x$.



# Evaluating Limits at Infinity

**Theorem 11**

*If* $r > 0$ *is a rational number, then*

$$\lim_{x \to \infty} \frac{1}{x^r} = 0.$$

*If* $r > 0$ *is a rational number such that* $x^r$ *is defined for all* $x$*, then*

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0.$$

**Example 21** Find $\lim_{x\to\infty} \dfrac{2x^2+3x+1}{\sqrt{x^4+x^2+1}}$.

**Example 22** Find $\lim_{x\to\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)$.

**Theorem 12**

1. $\lim_{x\to\infty} f(x) = L$ *if and only if* $\lim_{t\to 0^+} f\left(\frac{1}{t}\right) = L$

2. $\lim_{x\to-\infty} f(x) = L$ *if and only if* $\lim_{t\to 0^-} f\left(\frac{1}{t}\right) = L$

**Example 23** Show that $\lim_{x\to 0^-} e^{1/x} = 0$.

**Remark** $\lim_{x\to-\infty} e^{x} = 0$

## Infinite Limits at Infinity

The notation

$$\lim_{x\to\infty} f(x) = \infty$$

is used to indicate that the values of $f(x)$ become large as $x$ becomes large.

Similar meanings are attached to the following symbols:

$$\lim_{x\to-\infty} f(x) = \infty \qquad \lim_{x\to\infty} f(x) = -\infty \qquad \lim_{x\to-\infty} f(x) = -\infty$$

**Remark** $\lim_{x\to\infty} e^x = \infty$

# 2.7 Derivatives and Rates of Change

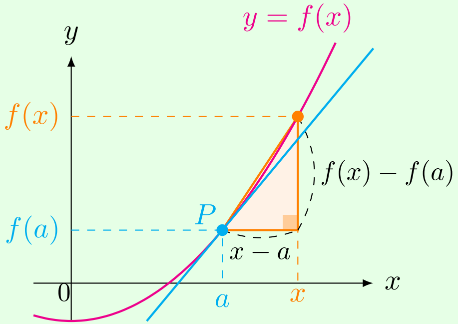
## Tangents

**Definition 11**

The tangent line to the curve $y = f(x)$ at the point $P\big(a, f(a)\big)$ is the line through $P$ with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

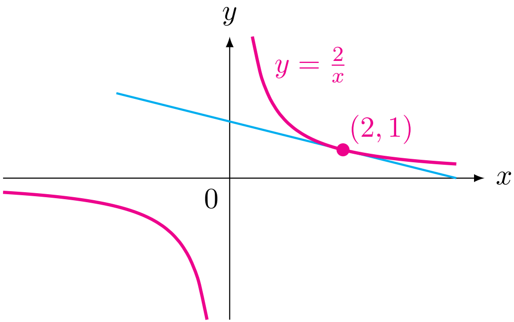
provided that this limit exists.




**Remark** Equivalently,

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

**Example 24** Find an equation of the tangent line to the parabola $y = \dfrac{2}{x}$ at the point $(2, 1)$.



## Derivatives

**Definition 12**

The derivative of a function $f$ at a number $a$, denoted by $f'(a)$, is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

**Remark** Equivalently,

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

**Example 25** Find the derivative of the function $f(x) = x^2 + 2x + 3$ at the number $a$.

**Remark** The tangent line to $y = f(x)$ at $\big(a, f(a)\big)$ is the line through $\big(a, f(a)\big)$ whose slope is equal to $f'(a)$, the derivative of $f$ at $a$.

**Example 26** Find the derivative of the function $f(x) = \dfrac{1}{\sqrt{x}}$ at the number $a > 0$.

**Example 27** If $f'(1) = 5$, find the following limits:

1. $\displaystyle\lim_{h\to 0} \frac{f(1+2h) - f(1)}{h}$
2. $\displaystyle\lim_{h\to 0} \frac{f(1-h) - f(1)}{h}$
3. $\displaystyle\lim_{h\to 0} \frac{f(1+h) - f(1-h)}{h}$.

## Rates of Change

Suppose $y$ is a quantity that depends on another quantity $x$.

Thus $y$ is a function of $x$ and we write $y = f(x)$.

If $x$ changes from $x_1$ to $x_2$, then the change in $x$ (also called the increment of $x$) is $\Delta x = x_2 - x_1$ and the corresponding change in $y$ is $\Delta y = f(x_2) - f(x_1)$.

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the average rate of change of $y$ with respect to $x$ over the interval $[x_1, x_2]$.

**Remark** instantaneous rate of change $= \displaystyle\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to $x$ when $x = a$.

# 2.8 The Derivative as a Function

## The Derivative Function

The derivative of a function $f$ at a fixed number $a$ is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

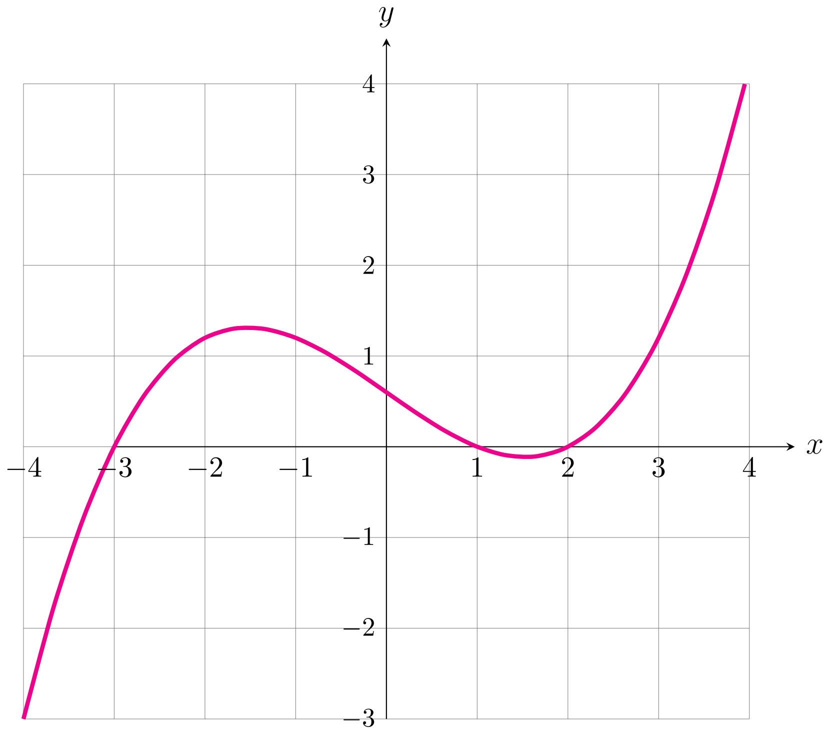
Here we change our point of view and let the number $a$ vary.

If we replace $a$ by a variable $x$, we obtain

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

We can regard $f'$ as a new function, called the derivative of $f$.

**Example 28** Use the graph of a function $f$ to sketch the graph of the derivative $f'$.



**Example 29** If $f(x) = \sqrt{x}$, find the derivative of $f$. State the domain of $f'$.

**Example 30** Find equations of lines through the point $(1, -6)$ that are tangent to the parabola $y = x^2 + 2x$.

**Example 31** Find a parabola whose tangent line at $(1, 2)$ has equation $y = 5x - 3$ and the parabola passes through the origin.

## Other Notations

If we use the traditional notation $y = f(x)$ to indicate that the independent variable is $x$ and the dependent variable is $y$, then some common alternative notations for the derivative are as follows:

$$\boxed{f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)}$$

The symbols D and $\dfrac{d}{dx}$ are called differential operators because they indicate the operation of differentiation, which is the process of calculating a derivative.

The symbol $\dfrac{dy}{dx}$ should not be regarded as a ration (for the time being); it is simply a synonym for $f'(x)$. We can rewrite the definition of derivative in the form

$$\boxed{\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}}$$

If we want to indicate the value of a derivative $\dfrac{dy}{dx}$ at a specific number $a$, we use the notation

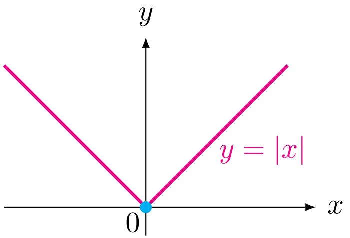
$$\left.\frac{dy}{dx}\right|_{x=a} \qquad \text{or} \qquad \left.\frac{dy}{dx}\right]_{x=a}$$

**Definition 13**

A function $f$ is differentiable at $a$ if $f'(a)$ exists.

It is differentiable on an open interval $(a, b)$ [or $(a, \infty)$ or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

**Example 32** Where is the function $f(x) = |x|$ differentiable?



**Theorem 13**

*If $f$ is differentiable at $a$, then $f$ is continuous at $a$.*

***Proof***

$$\lim_{x\to a}(f(x)-f(a)) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot(x-a) = f'(a)\cdot 0 = 0$$

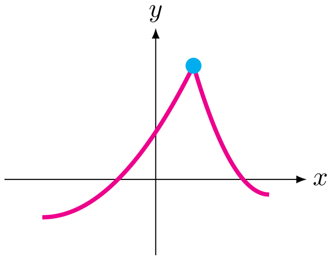
Therefore $\lim_{x\to a} f(x) = f(a)$. $\square$

**Remark**

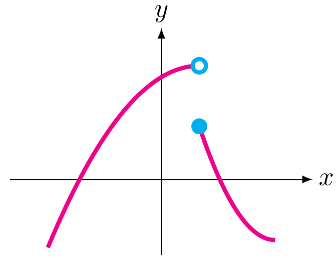
1. If a function $f$ is discontinuous at $a$, then $f$ is not differentiable at $a$.
2. The converse of the above theorem is false; that is, there are functions that are continuous but not differentiable.

   For instance, the function $f(x) = |x|$ is continuous at 0 but not differentiable at 0.

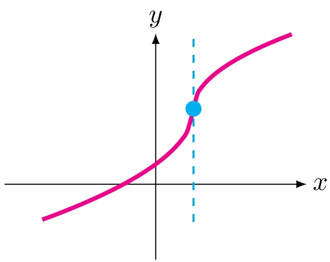
# How Can a Function Fail to be Differentiable?




A corner




A discontinuity




A vertical tangent

## Higher Derivatives

Let $y = f(x)$. We define higher derivatives of $f$ as follows.

- $f'$ is the first derivative of $f$.
- $(f')'$ is the second derivative of $f$ and is denoted by $y''$ or $f''$ or $\dfrac{d^2y}{dx^2}$.
- $(f'')'$ is the third derivative of $f$ and is denoted by $y'''$ or $f'''$ or $\dfrac{d^3y}{dx^3}$.
- The $n^{\text{th}}$ derivative of $f$ is denoted by $y^{(n)}$ or $f^{(n)}$ or $\dfrac{d^ny}{dx^n}$ and is defined by

$$f^{(n)} = (f^{(n-1)})' \qquad \text{or} \qquad \frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)$$

For convenience, we may also write $f^{(0)}$ in the meaning of $f$; that is $f^{(0)} = f$.