Lecture Notes

2301107 Calculus I for Engineering
Section 4

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Chapter 7 Techniques of Integration

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7.1 Integration by Parts

Integration by Parts: Indefinite Integrals

The Product Rule states that if u and v are differentiable functions, then

$$d(uv) = u\,dv + v\,du$$

We can rearrange this equation as

$$uv = \int u\;dv + \int v\;du,$$

which can be rewritten in the notation of indefinite integrals as

$$\boxed{\int u\,dv = uv - \int v\,du}$$

This formula is call the formula for integration by parts.

Example 1 Find $\int \ln x \, dx$.

Example 2 Find $\displaystyle\int x\cos x\, dx.$

Remark

Our aim in using integration by parts is to obtain a simpler integral than the one we started with. Therefore we have to make a good choice on u and v.

1. Choose dv such that we can integrate to find v easily. The constant of integration is never needed and in any case will never affect the final answer.

$$\begin{aligned}\int u\ d(v+C) &= u(v+C) - \int (v+C)\ du \\ &= uv+uC - \int v\ du - \int C\ du \\ &= uv+uC - \int v\ du - uC \\ &= uv - \int v\ du\end{aligned}$$

2. Try choosing u to give a simple derivative u'.

   A polynomial may be a first try, but it does not necessarily have to be so, as a proper choice of u may as well depends on the choice of dv.

Example 3 Find $\displaystyle\int x^n \ln x \, dx$, where $n \neq -1$.

Example 4 Evaluate $\displaystyle\int x^2 e^x \, dx$ by using integration by parts twice.

Example 5 Evaluate $\displaystyle\int e^x \cos x \, dx$.

Integration by Parts: Definite Integrals

If the formula for integration by parts is combined with FTC2, we obtain

$$\boxed{\int_{x=a}^{x=b} u\ dv = uv\Big|_{x=a}^{x=b} - \int_{x=a}^{x=b} v\ du}$$

Example 6 Evaluate $\displaystyle\int_0^1 \arctan x\ dx$.

7.2 Trigonometric Integrals

Integrals of Powers of Sine and Cosine

Try a substitution $u = \sin x$ or $u = \cos x$ along with the identity $\sin^2 x + \cos^2 x = 1$.

Note that $d(\sin x) = \cos x\, dx$ and $d(\cos x) = -\sin x\, dx$.

Example 7 Find $\displaystyle\int \cos^3 x\, dx$.

Example 8 Evaluate $\displaystyle\int_0^{\pi} \sin^3 x \cos^4 x \, dx.$

Example 9 Find $\displaystyle\int \frac{\sin^3 x}{(\cos x)^{1/3}}\,dx.$

If the powers of both sine and cosine are even, we could use a half-angle trigonometric formula

$$\cos^2 x = \tfrac{1}{2}(1+\cos 2x) \quad \text{and} \quad \sin^2 x = \tfrac{1}{2}(1-\cos 2x)$$

to reduce the integrand first, then proceed as is.

Example 10 Find $\displaystyle\int \sin^2 x \cos^2 x \, dx$.

Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

1. If n is odd, i.e. $n = 2k+1$, substitute $u = \sin x$.

$$\begin{aligned}\int \sin^m x \cos^{2k+1} x \, dx &= \int \sin^m x \left(\cos^2 x\right)^k \cdot \cos x \, dx \\ &= \int \sin^m x \left(1 - \sin^2 x\right)^k \, d(\sin x)\end{aligned}$$

2. If m is odd, i.e. $m = 2k+1$, substitute $u = \cos x$.

$$\begin{aligned}\int \sin^{2k+1} x \cos^n x \, dx &= \int \left(\sin^2 x\right)^k \cos^n x \cdot \sin x \, dx \\ &= -\int \left(1 - \cos^2 x\right)^k \cos^n x \, d(\cos x)\end{aligned}$$

3. If both m and n are even, use the half-angle identities

$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x) \quad \text{and} \quad \sin^2 x = \tfrac{1}{2}(1 - \cos 2x).$$

Integrals of Powers of Secant and Tangent

Try a substitution $u = \tan x$ or $u = \sec x$ along with the identity $\sec^2 x = 1 + \tan^2 x$.

Note that $d(\tan x) = \sec^2 x\, dx$ and $d(\sec x) = \sec x \tan x\, dx$.

Example 11 Find $\displaystyle\int \sec^4 x \tan^8 x\, dx$.

Example 12 Evaluate $\int \sec^5 x \, \tan^3 x \, dx$.

Strategy for Evaluating $\displaystyle\int \sec^m x \tan^n x \, dx$

1. If m is even, i.e. $m = 2k$, substitute $u = \tan x$.

$$\int \sec^{2k} x \, \tan^n x \, dx = \int \left(\sec^2 x\right)^{k-1} \tan^n x \cdot \sec^2 x \, dx$$
$$= \int \left(1 + \tan^2 x\right)^{k-1} \tan^n x \, d(\tan x)$$

2. If n is odd, i.e. $n = 2k + 1$, substitute $u = \sec x$.

$$\int \sec^m x \, \tan^{2k+1} x \, dx = \int \sec^{m-1} x \, \left(\tan^2 x\right)^k \cdot \sec x \tan x \, dx$$
$$= \int \sec^{m-1} x \, \left(\sec^2 x - 1\right)^k \, d(\sec x)$$

3. If m is odd and n is even, use the identity $\sec^2 x = 1 + \tan^2 x$ to reduce the integrand first, then we may apply integration by parts.

Example 13 Find $\displaystyle\int \sec^4 x\,(\tan x)^{1/3}\,dx$.

Example 14 Evaluate $\displaystyle\int \sec^3 x \, dx$.

Integrals of powers of cosecant and cotangent can be done by a substitution $u = \cot x$ or $u = \csc x$ along with the identity $\csc^2 x = 1 + \cot^2 x$. If necessary, we may use integration by parts.

Note that $d(\cot x) = -\csc^2 x\, dx$ and $d(\csc x) = -\csc x \cot x\, dx$.

Example 15 Evaluate $\displaystyle\int \cot^3 x\, dx$.

Using Product Identities

To evaluate the integrals

(a) $\displaystyle\int \sin mx \cos nx\, dx$, (b) $\displaystyle\int \sin mx \sin nx\, dx$, or (c) $\displaystyle\int \cos mx \cos nx\, dx$,

use the corresponding identity:

(a) $\sin A \cos B = \frac{1}{2}\big[\sin(A-B) + \sin(A+B)\big]$

(b) $\sin A \sin B = \frac{1}{2}\big[\cos(A-B) - \cos(A+B)\big]$

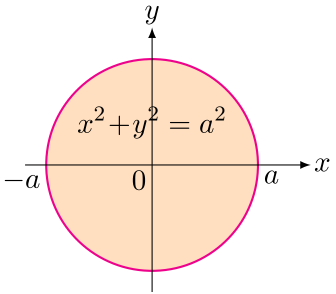
(c) $\cos A \cos B = \frac{1}{2}\big[\cos(A-B) + \cos(A+B)\big]$

Example 16 Evaluate $\int \sin 3x \cos 5x \; dx$.

Example 17 Evaluate $\displaystyle\int_0^{\pi} \sin\frac{x}{2}\cos\frac{x}{6}\,dx.$

7.3 Trigonometric Substitution

Example 18 Show that the area of a circle of radius a is πa^2.



Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta,\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta,\quad 0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Integrands with $\sqrt{a^2-x^2}$ where $a>0$

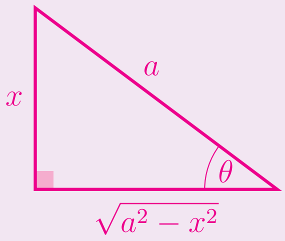
1. Substitute $x = a\sin\theta$ where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

   We have $dx = a\cos\theta\, d\theta$ and $\sqrt{a^2-x^2} = a\cos\theta$.

2. For indefinite integrals, we can return to the original variable x by the aid of the following diagram.

$$\theta = \arcsin\frac{x}{a} \qquad \sin\theta = \frac{x}{a}$$

$$\cos\theta = \frac{\sqrt{a^2-x^2}}{a} \qquad \tan\theta = \frac{x}{\sqrt{a^2-x^2}}$$



Example 19 Find $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx.$

Example 20 Find $\int_0^1 \sqrt{x - x^2}\, dx$.

Example 21 Find $\displaystyle\int_0^1 \sqrt{x^3 - x^8}\, dx$.

Integrands with $\sqrt{x^2+a^2}$ where $a>0$

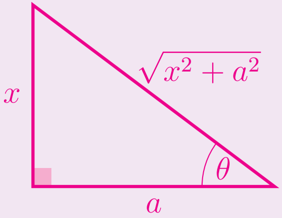
1. Substitute $x = a\tan\theta$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

   We have $dx = a\sec^2\theta\, d\theta$ and $\sqrt{x^2+a^2} = a\sec\theta$.

2. For indefinite integrals, we can return to the original variable x by the aid of the following diagram.

$$\theta = \arctan\frac{x}{a} \qquad \tan\theta = \frac{x}{a}$$

$$\sin\theta = \frac{x}{\sqrt{x^2+a^2}} \qquad \cos\theta = \frac{a}{\sqrt{x^2+a^2}}$$



Example 22 Evaluate $\displaystyle\int \frac{1}{\sqrt{4x^2+9}}\,dx.$

Integrands with $\sqrt{x^2 - a^2}$ where $a > 0$

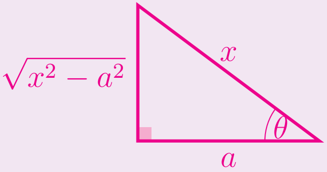
1. Substitute $x = a\sec\theta$ where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$.

   We have $dx = a\sec\theta\tan\theta\, d\theta$ and $\sqrt{x^2 - a^2} = a\tan\theta$.

2. For indefinite integrals, we can return to the original variable x by the aid of the following diagram.

$$\theta = \operatorname{arcsec}\frac{x}{a}, \qquad \cos\theta = \frac{a}{x},$$

$$\sin\theta = \frac{\sqrt{x^2 - a^2}}{x}, \qquad \tan\theta = \frac{\sqrt{x^2 - a^2}}{a}$$



Example 23 Evaluate $\displaystyle\int_{2\sqrt{2}}^{4} \frac{x}{\sqrt{x^2-4}}\,dx.$

7.4 Integration of Rational Functions by Partial Fractions

The method of partial fractions is decomposition of rational expressions into simpler fractions. One of well-known examples is the observation that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

which can be applied to the evaluation of the following summation:

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k(k+1)} &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \frac{1}{1} - \frac{1}{n+1} \\ &= \frac{n}{n+1}\end{aligned}$$

The method of partial fractions allows us to integrate a rational expression by first decomposing the integrand into simpler fractions and then integrating each fraction in the decomposition.

Example 24 Evaluate $\displaystyle\int \frac{1}{x^2 + x}\, dx.$

Example 25 Find constants A and B such that $\dfrac{x+4}{x^2+3x+2} = \dfrac{A}{x+1} + \dfrac{B}{x+2}$ for all x.

Definition 1 (Rational Functions)

Let $P(x)$ and $Q(x)$ be polynomial functions.

We will call the rational function $\dfrac{P(x)}{Q(x)}$

- proper if $\deg(P) < \deg(Q)$, or
- improper if $\deg(P) \geq \deg(Q)$.

Remark

An improper rational function $\dfrac{P(x)}{Q(x)}$ can be written as

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)},$$

where $\dfrac{R(x)}{Q(x)}$ is proper, i.e. $\deg(R) < \deg Q$.

Example 26 Find $\displaystyle\int \frac{x^3 - 2x}{x + 1}\, dx.$

The Method of Partial Fractions

Factorize the denominator, then add partial fractions corresponding to each factor.

1. Distinct Factors: Add a partial fraction for each individual factor in the following manner.
   - Linear Factor $ax+b$: add $\dfrac{A}{ax+b}$.
   - Quadratic Factor ax^2+bx+c: add $\dfrac{Ax+B}{ax^2+bx+c}$.
2. Repeated Factors: Add partial fractions up to the number of times it is repeated., e.g.
   - Linear Factor $(ax+b)^3$: add $\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\dfrac{A_3}{(ax+b)^3}$.
   - Quadratic Factor $(ax^2+bx+c)^2$: add $\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}$.

Example 27 For the denominator $(2x+3)(4x+5)^2(2x^2+3x+2)(3x^2+4x+3)^2$, we construct the partial fractions as

$$\frac{A}{2x+3}+\frac{B_1}{4x+5}+\frac{B_2}{(4x+5)^2}+\frac{Cx+D}{2x^2+3x+2}+\frac{E_1x+F_1}{3x^2+4x+3}+\frac{E_2x+F_2}{(3x^2+4x+3)^2}.$$

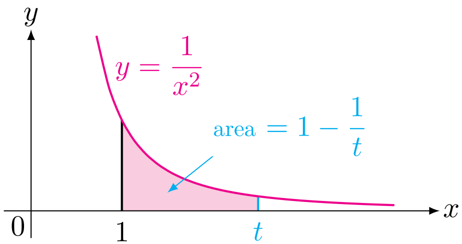
Example 28 Evaluate $\displaystyle\int_2^3 \frac{x+1}{(x-1)^2}\,dx.$

7.8 Improper Integrals

An improper integral is a definite integral $\int_a^b f(x)\,dx$ where the interval $[a,b]$ is infinite or f has an infinite discontinuity in $[a,b]$

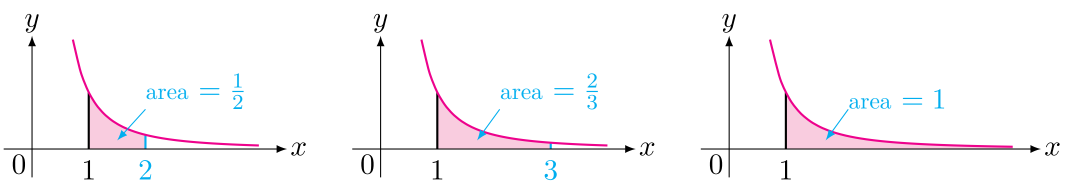
Type 1: Infinite Intervals

Consider the unbounded region under the curve $y=\frac{1}{x^2}$ on the interval $[1,t]$, where $t>1$.



The area of the region is $A(t)=\displaystyle\int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^t = 1-\frac{1}{t}$.

As $t\to\infty$, we have $\displaystyle\lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left(1-\tfrac{1}{t}\right)=1$.



We define $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = 1.$

Definition 2 (Definition of an Improper Integral of Type 1)

1. If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx \quad \text{provided this limit exists.}$$

2. If $\int_t^b f(x)\,dx$ exists for every number $t \leq b$, then

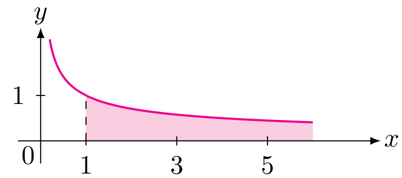
$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx \quad \text{provided this limit exists.}$$

   The improper integrals $\int_a^{\infty} f(x)\,dx$ and $\int_{-\infty}^{b} f(x)\,dx$ are called convergent if the corresponding limit exists and divergent if the limit does not exist.

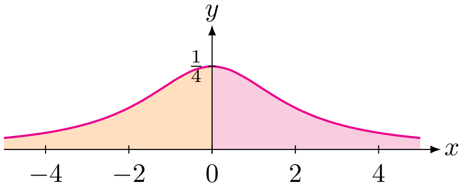
3. If both $\int_a^{\infty} f(x)\,dx$ and $\int_{-\infty}^{a} f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx \qquad \text{where } a \text{ can be any real number.}$$

Example 29 Determine whether the integral $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ is convergent or divergent.



Example 30 Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+4}\,dx$.



Type 2: Discontinuous Integrands

Definition 3 (Definition of an Improper Integral of Type 2)

1. If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx \quad \text{if this limit exists.}$$

2. If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx \quad \text{if this limit exists.}$$

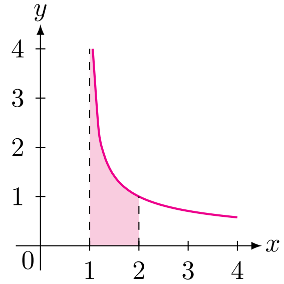
   The improper integrals $\displaystyle\int_a^b f(x)\,dx$ is called convergent if the corresponding limit exists and divergent if the limit does not exist.

3. If f has discontinuity at c, where $a < c < b$,

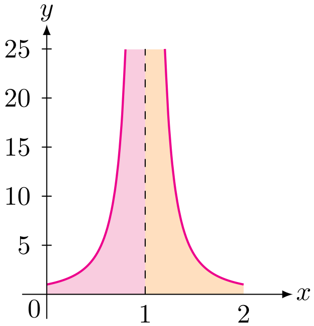
   and both $\displaystyle\int_a^c f(x)\,dx$ and $\displaystyle\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Example 31 Evaluate $\displaystyle\int_1^2 \frac{1}{\sqrt{x-1}}\,dx.$



Example 32 Determine whether $\displaystyle\int_0^2 \frac{1}{(x-1)^2}\,dx$ converges or diverges.



Example 33 **(Improper Integral of Both Type 1 and Type 2)**

Evaluate $\displaystyle\int_0^\infty \frac{1}{(x+1)\sqrt{x}}\,dx.$