Lecture Notes

2301107 Calculus I for Engineering
Section 4

First Semester, Academic Year 2024

Chapter 6 Applications of Integration

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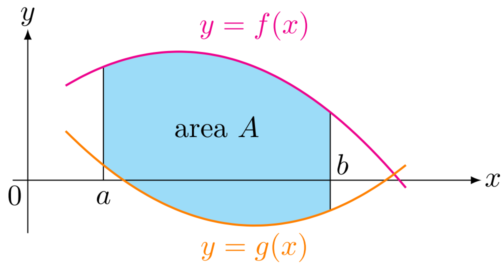
Chapter 6 Applications of Integration

6.1 Areas Between Curves

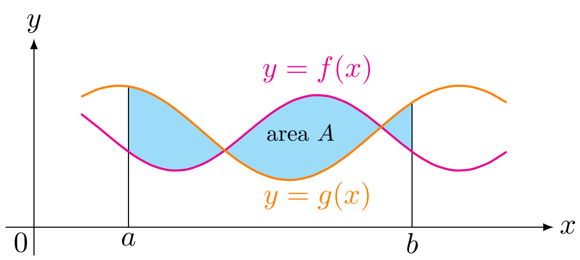
Area Between Curves: Integrating With Respect to x

The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continuous and a$f(x) \geq g(x)$ for all $x \in [a, b]$, is given by



$$A = \int_a^b \big(f(x) - g(x)\big)\, dx$$

The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is given by



$$A = \int_a^b \big|f(x) - g(x)\big|\, dx$$

Example 1 Find the area of the region bounded by the curves $y = x^2$ and $y = 2x - x^2$.

Example 2 Find the area of the region bounded by the curve $y = 2\sqrt{x}$ where $0 \leq x \leq 1$, and the lines $x = 2y$ and $x + y = 3$.

Area Between Curves: Integrating With Respect to y

Example 3 Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = x + 1$.

6.2 Volumes

Find Volume Using Cross-Sectional Area

Let S be a solid that lies between $x = a$ and $x = b$.

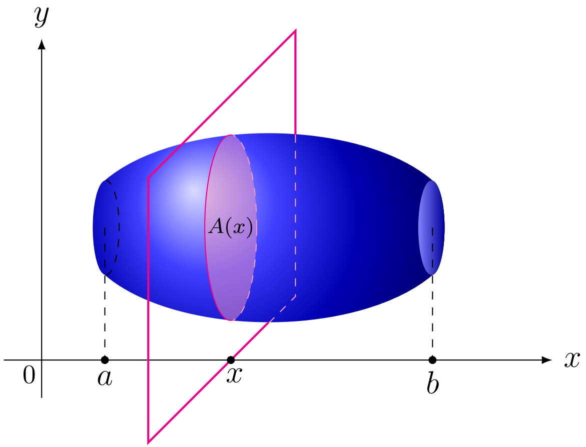
If the cross-sectional area of S in the plane P_x,

through x and perpendicular to the x-axis, is $A(x)$,

where A is a continuous function,

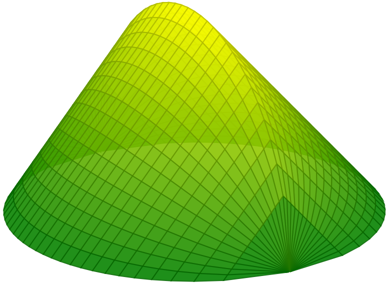
then the volume of S is given by

$$\boxed{V = \int_a^b A(x)\,dx}$$

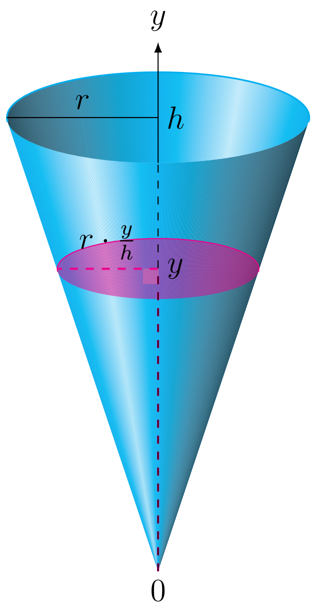


Example 4 Show that a sphere of radius r has the volume of $\frac{4}{3}\pi r^3$.

Example 5 Find the volume of a solid with a circular base of radius r and the parallel cross sections perpendicular to the base are equilateral triangles.

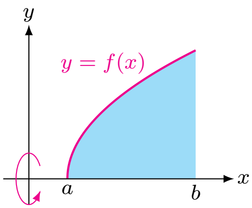


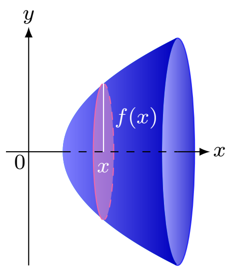
Example 6 Show that a circular cone whose base is a circle of radius r and height is h has the volume of $\frac{1}{3}\pi r^2 h$.

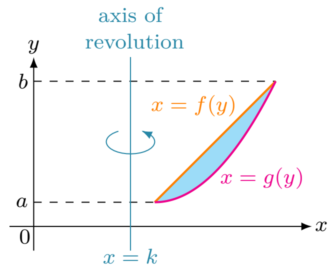


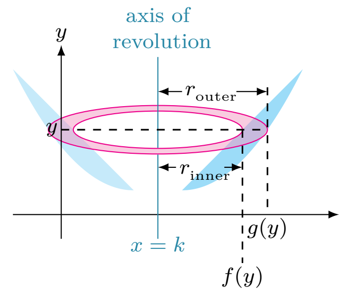
Volumes of Solids of Revolution

If we revolve a region about a line, we obtain a solid of revolution. The cross sections perpendicular to the axis of rotation are circles (disks) or circular rings (washers).









Area of Disks:

$$A(x) = \pi(\text{radius})^2 = \pi\big(f(x)\big)^2$$

$$\therefore\ V = \int_a^b A(x)\,dx$$

Area of Washers:

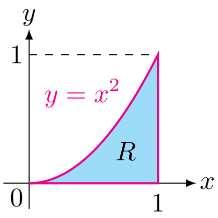
$$A(y) = \pi\left((\text{outer radius})^2 - (\text{inner radius})^2\right)$$

$$= \pi\left(\big(g(y) - k\big)^2 - \big(f(y) - k\big)^2\right)$$

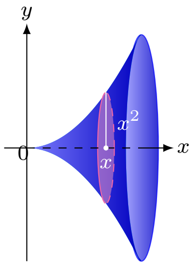
$$\therefore\ V = \int_a^b A(y)\,dy$$

This method of find the volumes of solids of revolution is called the method of disks and washers.

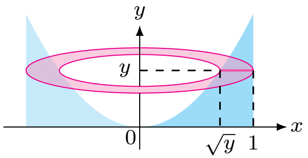
Example 7 Find the volume of the solid obtained by rotating the region R enclosed by the curve $y = x^2$, where $0 \le x \le 1$.

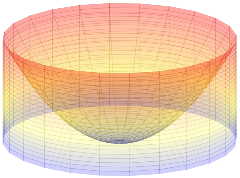


1. About the x-axis.

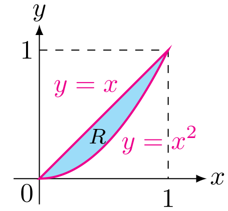


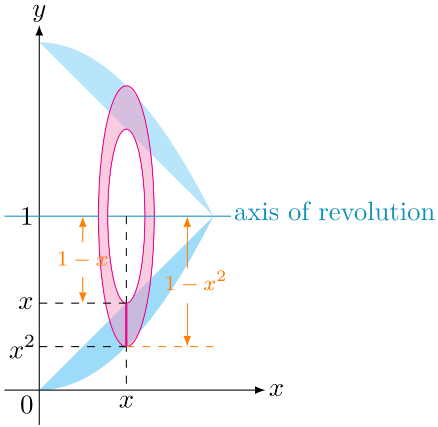
2. About the y-axis.





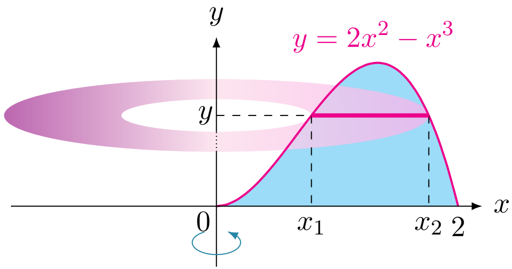
Example 8 The region R enclosed by the curves $y = x$ and $y = x^2$ is rotated about the line $y = 1$. Find the volume of the resulting solid.





6.3 Volumes by Cylindrical Shells

The region R is enclosed by the curves $y = 2x^2 - x^3$ and the x-axis.

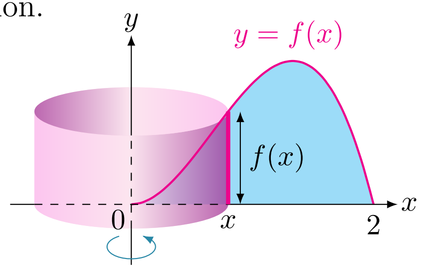


Consider the problem of finding the volume of the solid obtained by rotating the region R about the y-axis.
If we slide perpendicular to the y-axis, we get a washer of which the area is $A(y) = \pi(x_2^2 - x_1^2)$, where $0 \le x_1 \le x_2 \le 2$ and x_1, x_2 are the roots of $y = 2x^2 - x^3$ for each y, which are complicated (but still possible) to solve.

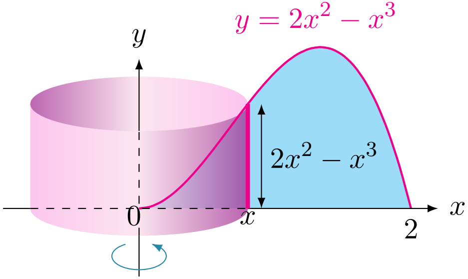
Another method called the method of cylindrical shells allows us to find the volume easily by considering a cylindrical shell which encircle the axis of revolution.
The area of the cylindrical shell can be found from

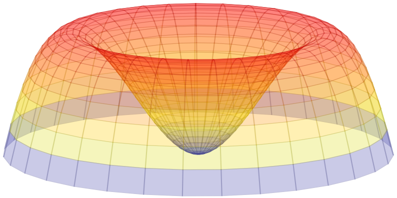
$$\boxed{\text{area of cylindrical shell} = (\text{circumference}) \times (\text{height})}$$

which can then be integrated to yield the volume of the solid.

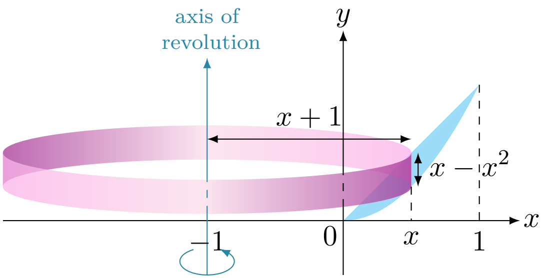


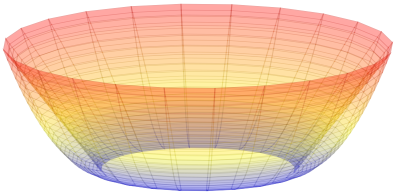
Example 9 Find the volume of the solid obtained by rotating about the y-axis the region R bounded by the curves $y = 2x^2 - x^3$ and $y = 0$.





Example 10 Use cylindrical shells to find the volume of the solid obtained by rotating about the line $x = -1$ the region R enclosed by the curves $y = x$ and $y = x^2$.





Disks and Washers versus Cylindrical Shells

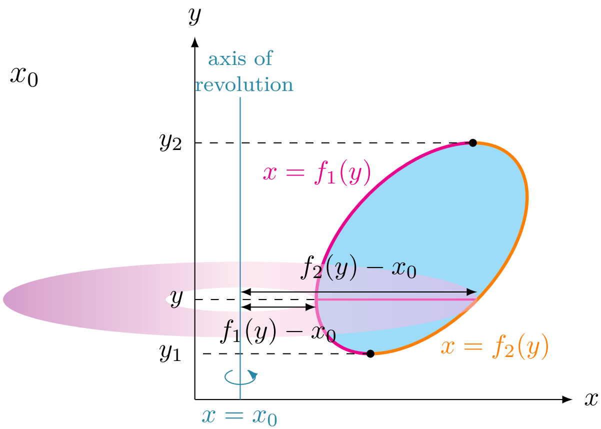
Disks and Washers when rotated about the line $x = x_0$

$$A(y) = \underbrace{\pi\left[(\text{outer radius})^2 - (\text{inner radius})^2\right]}^{\text{area of washer}}$$

$$= \pi\left[\big(f_2(y) - x_0\big)^2 - \big(f_1(y) - x_0\big)^2\right]$$

$$V = \int_{y_1}^{y_2} A(y)\,dy$$

integrate along the axis of revolution



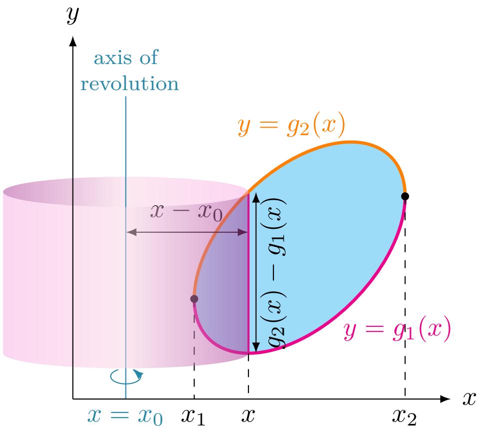
Cylindrical Shells when rotated about the line $x = x_0$

$$A(x) = \overbrace{(\text{circumference}) \times (\text{height})}^{\text{area of cylindrical shell}}$$

$$= 2\pi(x - x_0)\big(g_2(x) - g_1(x)\big)$$

$$V = \int_{x_1}^{x_2} A(x)\,dx$$

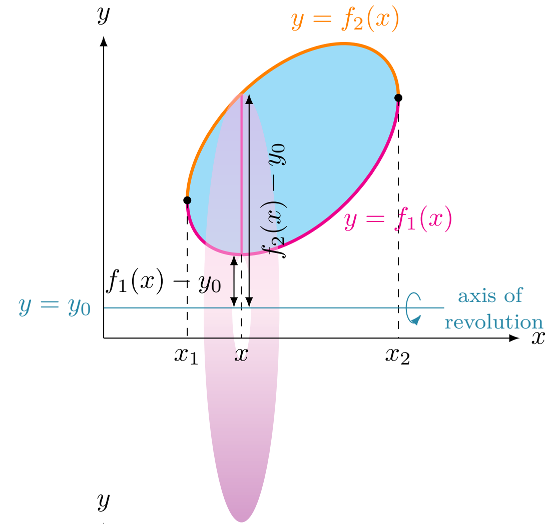
integrate perpendicular to the axis of revolution



Disks and Washers when rotated about the line $y = y_0$

$$A(x) = \pi \overbrace{\left[(\text{outer radius})^2 - (\text{inner radius})^2\right]}^{\text{area of washer}}$$
$$= \pi\left[\big(f_2(x) - y_0\big)^2 - \big(f_1(x) - y_0\big)^2\right]$$

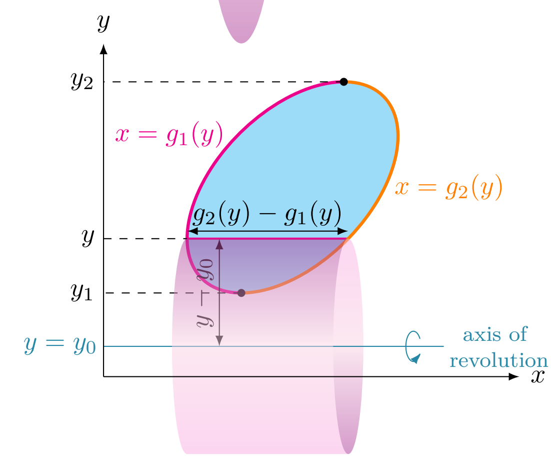
$$V = \int_{x_1}^{x_2} A(x)\,dx$$ integrate along the axis of revolution



Cylindrical Shells when rotated about the line $y = y_0$

$$A(y) = \overbrace{(\text{circumference}) \times (\text{height})}^{\text{area of cylindrical shells}}$$
$$= 2\pi(y - y_0)\big(g_2(y) - g_1(y)\big)$$

$$V = \int_{y_1}^{y_2} A(y)\,dy$$ integrate perpendicular to the axis of revolution

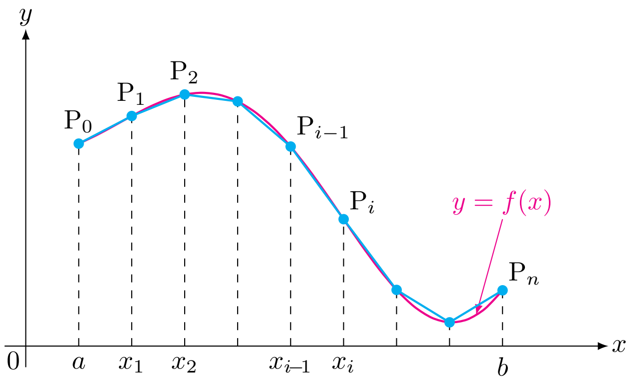


8.1 Arc Length

Consider the problem of finding the arc length of a curve $y = f(x)$ when $a \le x \le b$.

Divide the interval $[a, b]$ into n subintervals with endpoints $\mathrm{P}_0, \mathrm{P}_1, \dots, \mathrm{P}_n$

and equal width $\Delta x = \frac{1}{n}(b - a)$.



The length of the segment $\mathrm{P}_{i-1}\mathrm{P}_i$ is

$$\mathrm{P}_{i-1}\mathrm{P}_i = \sqrt{(\Delta x)^2 + (y_i - y_{i-1})^2}$$
$$= \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}\, \Delta x.$$

Define the length L of the curve as

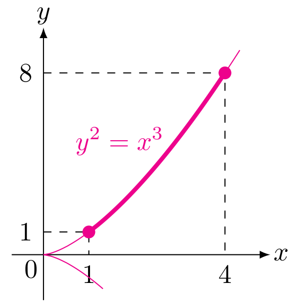
$$L = \lim_{n \to \infty} \sum_{i=1}^{n} (\mathrm{P}_{i-1}\mathrm{P}_i).$$

Theorem 1 (The Arc Length Formula)

If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \le x \le b$, is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Example 11 Find the length of the arc of the curve $y^2 = x^3$ between the points $(1, 1)$ and $(4, 8)$.



Example 12 Find the length of the astroid $x^{2/3} + y^{2/3} = 1$.