Lecture Notes

2301107 Calculus I for Engineering
Section 4

First Semester, Academic Year 2024

Chapter 3 Differentiation Rules

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3.1 Derivatives of Polynomials and Exponential Functions

Polynomial Functions

Theorem 1 (Derivative of Constant Functions and Power Functions)

1. $\dfrac{dc}{dx} = 0$ *(Constant Functions)*
2. $\dfrac{dx}{dx} = 1$ *(Identity Functions)*
3. *If n is any real number, then* $\dfrac{dx^n}{dx} = nx^{n-1}$. *(The Power Rule)*

Theorem 2 (New Derivatives from Old)

1. $\dfrac{d}{dx}\big[c\,f(x)\big] = c\,\dfrac{d}{dx}f(x)$ *(The Constant Multiple Rule)*
2. $\dfrac{d}{dx}\big[f(x) + g(x)\big] = \dfrac{d}{dx}f(x) + \dfrac{d}{dx}g(x)$ *(The Sum Rule)*
3. $\dfrac{d}{dx}\big[f(x) - g(x)\big] = \dfrac{d}{dx}f(x) - \dfrac{d}{dx}g(x)$ *(The Difference Rule)*

Example 1

$f(x)$	$f'(x)$
$\dfrac{1}{x} + \dfrac{1}{x^2}$	
$\sqrt{x}$	
$\dfrac{1}{\sqrt{x}}$	
$\dfrac{2x^3 + 3x^2 + 5x + 7}{\sqrt{x}}$	

Exponential Functions

Example 2 Try to compute the derivative of the exponential function $f(x) = b^x$ using the definition of a derivative.

Definition 1 (Definition of the Number e)

$$e \text{ is the number such that } \lim_{h\to 0}\frac{e^h-1}{h}=1.$$

Theorem 3 (Derivative of the Natural Exponential Function)

$$\frac{de^x}{dx}=e^x$$

3.2 The Product and Quotient Rules

The Product Rule

Theorem 4

If f and g are both differentiable, then

$$\frac{d}{dx}\big[f(x)g(x)\big] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)].$$

Example 3 Let $f(x) = xe^x$. Find 1. $f'(x)$ 2. $f''(x)$.

The Quotient Rule

Theorem 5

If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

Example 4 If $f(x) = \dfrac{g(x)}{\sqrt{x}}$, where $g(4) = 2$ and $g'(4) = 3$, find $f'(4)$.

Theorem 6 (Table of Differentiation Formula)

$$\frac{dc}{dx} = 0 \qquad \frac{dx^n}{dx} = nx^{n-1} \qquad \frac{de^x}{dx} = e^x$$

$$(cf)' = cf' \qquad (f+g)' = f' + g' \qquad (f-g)' = f' - g'$$

$$(uv)' = u'v + uv' \qquad \left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$$

3.3 Derivatives of Trigonometric Functions

Derivatives of Trigonometric Functions

Example 5 Find the derivative of $f(x) = \sin x$.

Example 6 Find the derivative of $f(x) = \cos x$.

Example 7 Find the derivative of $f(x) = \tan x$.

Theorem 7

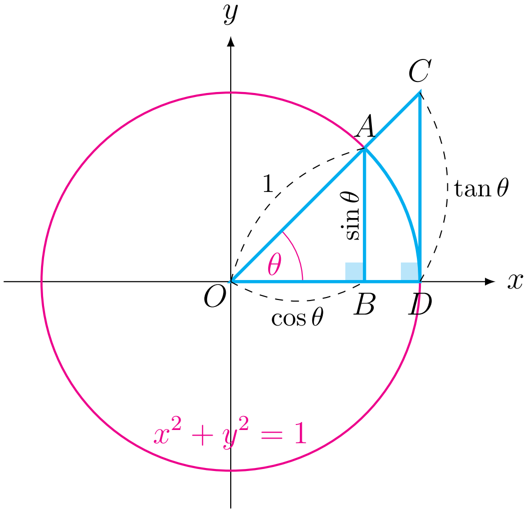
$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Two Special Trigonometric Limits

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$



$$\boxed{\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}=\frac{1}{2}}$$

Remark

$$\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}=0$$

Example 8 Find $\lim\limits_{x\to 0} \dfrac{\tan 2x}{\sin 3x}$.

Example 9 Find $\lim\limits_{x \to 2} \dfrac{\tan(x^2 + x - 6)}{x^2 - 4}$.

3.4 The Chain Rule

Theorem 8 (The Chain Rule)

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f\big(g(x)\big)$ is differentiable and F' is given by the product

$$F'(x) = f'\big(g(x)\big) \cdot g'(x)$$

$$\begin{tikzcd} f\big(g(x)\big) \\ g(x) \arrow[u, dash, "f"] \\ x \arrow[u, dash, "g"] \arrow[uu, dash, bend right, "f \circ g"'] \end{tikzcd}$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Theorem 9 (The Power Rule Combined with the Chain Rule)

If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{du^n}{dx} = nu^{n-1}\,\frac{du}{dx}.$$

Alternatively,

$$\frac{d}{dx}\big[g(x)\big]^n = n\big[g(x)\big]^{n-1}g'(x).$$

Example 10 Find the derivative of $f(x) = \exp\left[\sin^2(x^3)\right]$.

Example 11 Suppose that $f(1) = 1$ and $f'(1) = 5$.

1. Find $g'(1)$ if $g(x) = f(x^3)$.
2. Find $h'(1)$ if $h(x) = f\big(f(x^2)\big)$.

Derivatives of General Exponential Functions

$$b^x = e^{x \ln b}$$

Theorem 10

$$\frac{db^x}{dx} = b^x \ln b$$

Example 12 Find the derivative of $f(x) = 2^{\tan x}$.

3.5 Implicit Differentiation

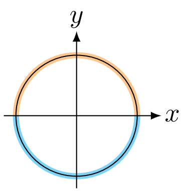
Implicitly Defined Functions

We write $y = f(x)$ to express y explicitly in terms of x.

However, y may be defined implicitly in terms of x by a relation between x and y such as $x^2 + y^2 = 25$ or $x^3 + y^3 = 2xy$.

In some cases it is possible to solve such an equation for y as an explict function of x.

For instantce, if we solve $x^2 + y^2 = 25$ for y, we get $y = \sqrt{25 - x^2}$ and $y = -\sqrt{25 - x^2}$.



It is not easy to solve $x^3 + y^3 = 2xy$ for y explicitly as a function of x.

Implicit Differentiation

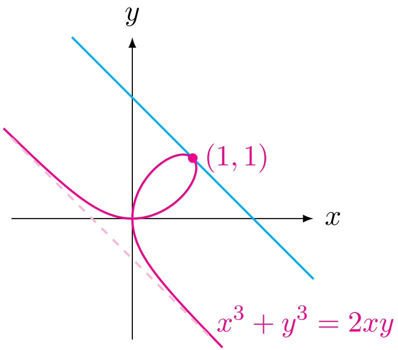
Fortunately, we don't need to solve an equation for y explicitly in terms of x in order to find the derivative of y. Instead, we can use the method of implicit differentiation.

Example 13 If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$.

Then find an equation of the tangent line to the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

Example 14

Find the tangent to the curve $x^3 + y^3 = 2xy$ at the point $(1, 1)$.



3.6 Derivatives of Logarithmic and Inverse Trigonometric Functions

Derivatives of Inverse Functions

Theorem 11

Let f be a one-to-one function and let $g = f^{-1}$. If $y = f(x)$ and $x = g(y)$, then

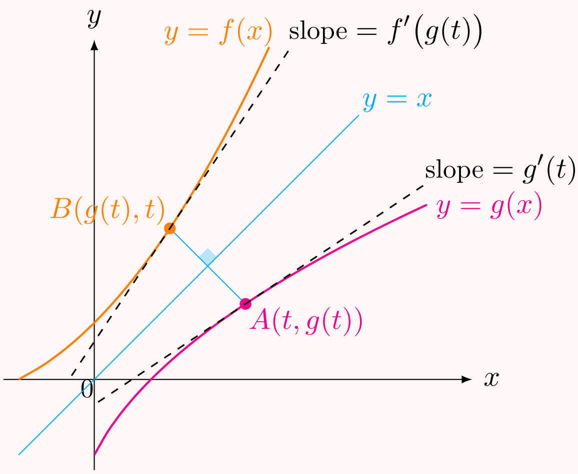
$$\frac{dy}{dx} \cdot \frac{dx}{dy} = 1.$$

Equivalently,

$$g'(x) = \frac{1}{f'\big(g(x)\big)}.$$

Remark [Geometric Interpretation of a Derivative of an Inverse Function]

The graphs of $y = g(x)$ and $x = f(y)$ are symmetric with respect to the line $y = x$.



The tangent to the curve $y = g(x)$ at the point $\big(t, g(t)\big)$ has the slope of $g'(t)$.

The tangent to the curve $y = f(x)$ at the point $\big(g(t), t\big)$ has the slope of $f'\big(g(t)\big)$.

These two slopes are reciprocal; that is, $g'(t) = \dfrac{1}{f'\big(g(t)\big)}$.

Example 15 Let $f(x) = x^3 + x^5$ and let $g(x) = f^{-1}(x)$. Find $g'(2)$.

Derivatives of Logarithmic Functions

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

Example 16 Find 1. $\dfrac{d}{dx}\big[\ln(-x)\big]$ when $x < 0$ 2. $\dfrac{d}{dx}\big[\ln|x|\big]$ when $x \neq 0$

Example 17 Let $k \neq 0$ be a constant. Find $\dfrac{d}{dx}\big[\ln(kx)\big]$.

Example 18 Find $\dfrac{d}{dx}\left[\log_b |x|\right]$.

Example 19 Find $\dfrac{d}{dx}\left[\,\log_2(\sin x^2)\right]$.

Logarithmic Differentiation

Example 20 Let n be any real number. Show that $\dfrac{dx^n}{dx} = nx^{n-1}$.

Example 21 Find $f'(1)$ if $f(x) = x^{3x^2}$.

Example 22 Find $f'(1)$ if $f(x) = \dfrac{x^{1/3}\sqrt{2x^2-1}}{(3x-2)^2}$.

The Number e as a Limit

$$\boxed{e = \lim_{x \to 0} (1+x)^{1/x}}$$

$$\boxed{e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n}$$

Derivatives of Inverse Trigonometric Functions

Example 23 Differentiate $y = \arcsin x$.

Example 24 Differentiate $y = \arctan x$.

Example 25 Differentiate $y = \operatorname{arcsec} x$.

Theorem 12

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\operatorname{arccsc} x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\operatorname{arccot} x) = -\frac{1}{1+x^2}$$

Function	**Domain**	**Range**	
arcsin	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	
arccos	$[-1, 1]$	$[0, \pi]$	
arctan	$(-\infty, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	
arccsc	$(-\infty, -1] \cup [1, \infty)$	$(0, \frac{\pi}{2}] \cup (\pi, \frac{3\pi}{2}]$	nonstandard
arcsec	$(-\infty, -1] \cup [1, \infty)$	$[0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$	definition
arccot	$(-\infty, \infty)$	$(0, \pi)$	

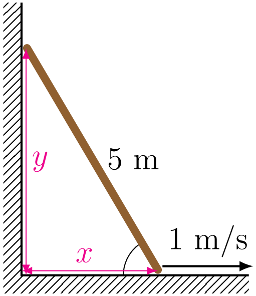
Example 26 Find $f'(x)$ if $f(x) = \arctan(x^2)$.

Example 27 Find $f'(x)$ if $f(x) = \operatorname{arcsec}(1+\sqrt{x})$.

3.9 Related Rates

Related rates problems involves the rate of change of one quantity in terms of the rate of change of another quantity. The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time.

Example 28 A ladder of length 5 m rests against a vertical wall. If the bottom of the ladder slides away from the wall at a speed of 1 m/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 3 m from the wall?

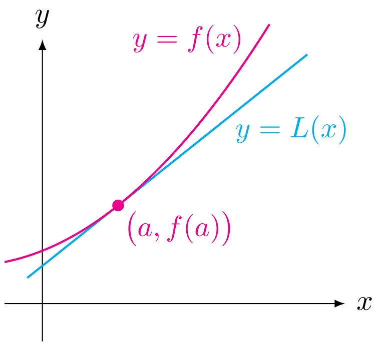


Example 29 Car A is traveling east away from an intersection at 60 km/h and Car B is traveling south towards the intersection at 40 km/h.

Find the rate of change of the distance between these two cars when Car A and Car B are 400 m and 300 m from the intersection,

3.10 Linear Approximation and Differentials

Linearization and Approximation



The tangent to the curve $y = f(x)$ at the point $\big(a, f(a)\big)$ has an equation $y = f(a) + f'(a)(x - a)$.

The linear function $L(x) = f(a) + f'(a)(x - a)$ is called the linearization of f at a.

The approximation $f(x) \approx L(x)$ or

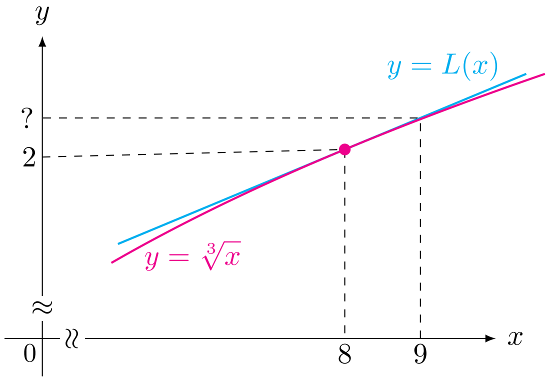
$$\boxed{f(x) \approx f(a) + f'(a)(x-a) \qquad \text{when } x \approx a}$$

is called linear approximation of f at a.

We may also write the linear approximation as

$$\boxed{f(x+\Delta x) \approx f(x) + f'(x)\,\Delta x \qquad \text{when } \Delta x \approx 0}$$

Example 30 Approximate $\sqrt[3]{9}$ using a linear approximation of $f(x) = \sqrt[3]{x}$ at 8.



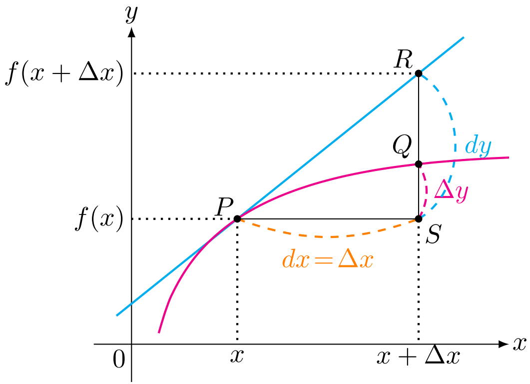
Example 31 Approximate $\sqrt{3.1^2 + 3.95^2}$ using a linear approximation.

Differentials

If $y = f(x)$, where f is differentiable function, then the differential dx is an independent variable (dx can be given the value of any real number).

The differential dy is then defined in terms of dx b

$$\boxed{dy = f'(x)\,dx}$$



Let $P\big(x, f(x)\big)$ and $Q\big(x + \Delta x, f(x + \Delta x)\big)$ be points on the graph of f and let $dx = \Delta x$. The corresponding change in y is

$$\boxed{\Delta y = f(x + \Delta x) - f(x)}$$

The slope of the tangent line is $f'(x)$.

Thus $dy = f'(x)\,dx$ represents the amount that the tangent line changes, whereas Δy represents the amount that the curve $y = f(x)$ changes.

Notice that $\Delta y \approx dy$ when $dx \approx 0$.

Example 32 Calculate Δy and dy for $y = f(x) = x^2 + x$.

1. When x changes from 2 to 2.1.
2. When x changes from 2 to 2.5.

Example 33 The radius of a sphere was measured and found to be 10 cm with a possible error in measurement of at most 0.05 cm. What are the maximum error and the maximum relative error in using this value of the radius to compute the volume of the sphere?