Lecture Notes

2301107 Calculus I for Engineering
Section 4

First Semester, Academic Year 2024

Chapter 4 Applications of Differentiation

Paisan Nakmahachalasint

Department of Mathematics and Computer Science

Faculty of Science, Chulalongkorn University

Chapter 4 Applications of Differentiation

4.1 Maximum and Minimum Values

Absolute and Local Extreme Values

Definition 1 (Absolute Extreme Values)

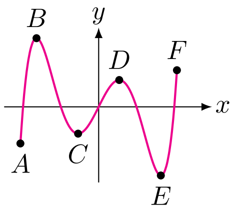
Let c be a number in the domain D of a function f. Then $f(c)$ is

- the absolute maximum value of f on D if $f(c) \geq f(x)$ for all x in D.
- the absolute minimum value of f on D if $f(c) \leq f(x)$ for all x in D.

Definition 2 (Local Extreme Values)

The number $f(c)$ is

- a local maximum value of f if $f(c) \geq f(x)$ when x is near c.
- a local minimum value of f if $f(c) \leq f(x)$ when x is near c.



The absolute maximum: B

The absolute minimum: E

Local maxima: B, D

Local minima: C, E

Some authors may define the endpoints A and F as local extrema.

Theorem 1 (The Extreme Value Theorem)

If f is continuous on a closed interval $[a, b]$,

then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$

at some numbers c and d in $[a, b]$.

Critical Numbers and the Closed Interval Method

Theorem 2 (Fermat's Theorem)

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Remark If $f'(c) = 0$, then f may not attain an extremum value at c.

For instance, $f(x) = x^3$ has $f'(0) = 0$, but f has no extremum at 0.

Definition 3 (Critical Numbers)

A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

The Closed Interval Method

To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum values; the smallest of these values is the absolute minimum values.

Example 1 Find the absolute maximum and minimum values of the function

$$f(x) = 2x^3 - 3x^2 - 12x + 6, \qquad -3 \leq x \leq 3.$$

4.3 What Derivatives Tell Us about the Shape of a Graph

What Does f' Say about f?

Definition 4 (Increasing and Decreasing Functions)

More generally
strictly increasing: $f(x_1) < f(x_2)$
non-decreasing: $f(x_1) \le f(x_2)$

A function f is called increasing on an interval I if

$$f(x_1) < f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I.$$

More generally
strictly decreasing: $f(x_1) > f(x_2)$
non-increasing: $f(x_1) \ge f(x_2)$

It is called decreasing on I if

$$f(x_1) > f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I.$$

Theorem 3 (Increasing/Decreasing Test)

1. *If* $f'(x) > 0$ *on an interval, then* f *is increasing on that interval.*

2. *If* $f'(x) < 0$ *on an interval, then* f *is decreasing on that interval.*

Remark If a function is continuous at an endpoint, then we may extend the interval I in which the function is increasing/decreasing to include that endpoint.

Example 2 Find the intervals in which $f(x) = x^3 - 27x + 18$ is increasing or decreasing.

The First Derivative Test

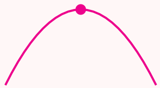
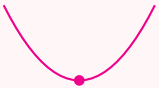
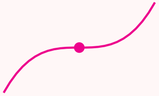
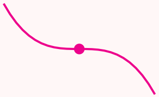
Theorem 4 (The First Derivative Test)

Suppose that c is a critical number of a continuous function f.

1. *If f' changes from positive to negative at c, then f has a local maximum at c.*
2. *If f' changes from negative to positive at c, then f has a local minimum at c.*
3. *If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.*

Remark

Consider $f'(c^-)$ when $x < c$ and $f'(c^+)$ when $x > c$.

$f'(c^-)$	$f'(c^+)$	**Conclusion**	**Graph**
$+$	$-$	A local maximum	
$-$	$+$	A local minimum	
$+$	$+$	No local extremum	
$-$	$-$	No local extremum	

Example 3 Find the local maximum and minimum values of $f(x) = x^4 - 4x^3$.

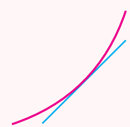
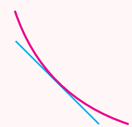
What Does f'' Say about f?

Definition 5 (Concavity)

If the graph of f lies above all of its tangents on an interval I,
then f is called concave upward on I.

If the graph of f lies below all of its tangents on I,
then f is called concave downward on I.

Remark Classification of graph according to the concavity and the increasingness.

	increasing	**decreasing**
concave upward		
concave downward		

Theorem 5 (Concavity Test)

1. *If $f''(x) > 0$ on an interval I, then the graph of f is concave upward on I.*

2. *If $f''(x) < 0$ on an interval I, then the graph of f is concave downward on I.*

Definition 6 (Inflection Point)

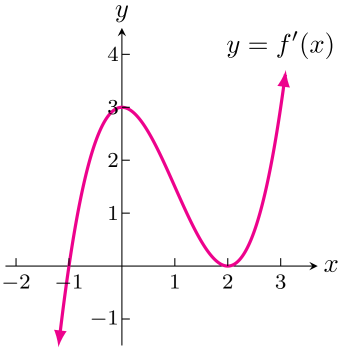
A point P on a curve $y = f(x)$ is called an inflection point
if f is continuous there and the curve changes from concave upward to concave downward
or from concave downward to concave upwared at P.

Remark If $f''(a) = 0$, the point $\big(a, f(a)\big)$ may or may not be an inflection point of $y = f(x)$.

For instance, $f(x) = x^4$ has $f''(x) = 12x^2 \geq 0$ and $f''(0) = 0$,
but $(0, 0)$ is not an inflection point of $y = f(x)$.

Example 4 Find the intervals in which the graph of $f(x) = x^4 - 4x^3$ is concave upward or downward. Indicate the inflection points on the graph.

Example 5 Use the given graph of $y = f'(x)$ to find the following.



1. The open intervals on which f is increasing/decreasing.
2. The values of x where f has a local extremum.
3. The open intervals on which f is concave upward/downward.
4. The values of a where (a, b) is a point of inflection.

The Second Derivative Test

Theorem 6 (The Second Derivative Test)

Suppose f'' is continuous near c.

1. *If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.*
2. *If $f'(c) = 0$ and $f''(c) < 0$, then $f(x)$ has a local maximum at c.*

Remark

The Second Derivative Test is inconclusive when $f''(c) = 0$.

This test also fails when $f''(c)$ does not exist.

In such cases, the First Derivative Test must be used.

In fact, when both tests apply, the First Derivative Test is often the easier one to use.

4.4 Indeterminate Forms and l'Hospital's Rule

Example 6 We know that derivatives are defined in terms of limits.

Exploit the derivative of an appropriate function to evaluate $\lim_{x\to 1} \dfrac{\ln x}{x-1}$.

Indeterminate Forms (Types $\frac{0}{0}, \frac{\infty}{\infty}$)

In general, If we have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$,

then this limit may or may not exist and is called an indeterminate form of type $\dfrac{0}{0}$.

Similary, if $f(x) \to \infty$ (or $-\infty$) and $g(x) \to \infty$ (or $-\infty$) when $x \to a$,

then $\displaystyle\lim_{x \to a} \frac{f(x)}{g(x)}$ is called an indeterminate form of type $\dfrac{\infty}{\infty}$.

L'Hospital's Rule

Theorem 7 (L'Hospital's Rule)

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

- $\lim_{x\to a} f(x) = 0$ *and* $\lim_{x\to a} g(x) = 0$, *that is,* $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ *is of type* $\dfrac{0}{0}$, *or*
- $\lim_{x\to a} f(x) = \pm\infty$ *and* $\lim_{x\to a} g(x) = \pm\infty$, *that is* $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ *is of type* $\dfrac{\infty}{\infty}$.

Then

$$\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Remark L'Hospitcal's Rule can also be applied to one-sided limits or limits at ∞ or $-\infty$.

Example 7 Find $\lim_{x \to 0} \dfrac{\sin x + \tan x}{e^x - e^{-x}}$.

Example 8 Find $\lim_{x\to\infty} \frac{x^3}{e^{2x}}$.

Remark

1. L'Hospital's Rule can be applied with the condition that $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ exists or is ∞ or $-\infty$.

   If $\frac{f'(x)}{g'(x)}$ oscillates or $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ does not exist, then l'Hospital's Rule may not be applied.

   For instance, if $f(x) = x + \sin x$ and $g(x) = x$, then $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \frac{x+\sin x}{x} = 1$.

   But $\lim\limits_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim\limits_{x\to\infty} \frac{1+\cos x}{1}$ does not exist and is not ∞ nor $-\infty$.

2. L'Hospital's Rule alone does not necessarily lead to a conclusion.

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} \overset{\left(\frac{\infty}{\infty}\right)}{=} \lim_{x\to\infty} \frac{1}{\left(\frac{x}{\sqrt{x^2+1}}\right)} = \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x}$$

$$\overset{\left(\frac{\infty}{\infty}\right)}{=} \lim_{x\to\infty} \frac{\left(\frac{x}{\sqrt{x^2+1}}\right)}{1} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$$

Indeterminate Products (Type $0 \cdot \infty$)

If $f(x) \to 0$ and $g(x) \to \infty$ (or $-\infty$) as $x \to a$,
then it is not clear what the value of $\lim\limits_{x \to a} \big[f(x) \cdot g(x)\big]$, if any, will be.

This kind of limit is called an indeterminate form of type $0 \cdot \infty$.

Example 9 Find $\lim_{x\to\infty} x \ln\left(\dfrac{x-1}{x+1}\right)$.

Indeterminate Differences (Type $\infty - \infty$)

If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, then the limit

$$\lim_{x \to a} \left[f(x) - g(x) \right]$$

is called an indeterminate form of type $\infty - \infty$.

To find the limit of this type, we try to convert the difference into a quotient, so that we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example 10 Find $\lim_{x \to 0} \left(\frac{1}{\ln(x+1)} - \frac{1}{x} \right)$.

Indeterminate Powers (Type $0^0, \infty^0, 1^\infty$)

Several indeterminate forms arise from the limit

$$\lim_{x\to a} \left[f(x)\right]^{g(x)}$$

1. $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ type 0^0
2. $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$ type ∞^0
3. $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$ type 1^∞

Each of these three cases can be treated either by taking the natural logarithm:

$$\text{let } y = \left[f(x)\right]^{g(x)}, \quad \text{then} \quad \ln y = g(x) \ln f(x)$$

or by writing the function as an exponential:

$$\left[f(x)\right]^{g(x)} = e^{g(x) \ln f(x)}$$

In either method we are led to the indeterminate product $g(x) \ln f(x)$, which is of type $0 \cdot \infty$.

Example 11 Find $\lim_{x\to 0^+} x^x$.

Example 12 Find $\lim\limits_{x\to\infty} \left(2^x + 3^x\right)^{1/x}$.

Example 13 Find $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x$ when a is a constant.

Example 14 Find $\lim\limits_{x\to 0}(1 + x - \sin x)^{1/x^3}$.

Summary of Indeterminate Forms

There are 7 types of indeterminate forms:

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0\cdot\infty,\quad \infty-\infty,\quad 1^\infty,\ \infty^0,\ 0^0$$

1. Types $\dfrac{0}{0}$, $\dfrac{\infty}{\infty}$, $0\cdot\infty$ can be evaluated by applying l'Hospital's Rule.

   Type $0\cdot\infty$ must be converted into either Types $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ before applying the rule.

2. Type $\infty-\infty$ needs an algebraic manipulation into another indeterminate form before applying the l'Hospital's Rule.

3. Types 1^∞, ∞^0, 0^0 are indeterminate forms arisen from $\lim\limits_{x\to a} f(x)^{g(x)}$.

   We can first evaluate $\lim\limits_{x\to a} g(x)\ln f(x)$, which is of Type $0\cdot\infty$.

   If $\lim\limits_{x\to a} g(x)\ln f(x) = L$, then $\lim\limits_{x\to a} f(x)^{g(x)} = e^L$.

4.5 Summary of Curve Sketching

A. **Domain**: The set of values of x for which $f(x)$ is defined.

B. **Intercepts**: y-intercept $= f(0)$. x-intercepts can be found from $f(x) = 0$.

C. **Symmetry**

(i) Even Function: $f(-x) = f(x)$: graph is symmetric about the y-axis.

(ii) Odd Function: $f(-x) = -f(x)$: graph is symmetric about the origin ($180°$ rotation).

D. **Asymptotes**

(i) Horizontal Asymptotes: $y = \lim\limits_{x\to\pm\infty} f(x)$

(ii) Vertical Asymptotes: $\lim\limits_{x\to a} f(x) = \pm\infty \implies x = a$

E. **Interval of Increase or Decrease**: increasing if $f'(x) > 0$ and decreasing if $f'(x) < 0$.

F. **Local Maximum or Minimum Values**: Critical numbers and the $1^{\text{st}}/2^{\text{nd}}$ Derivative Tests.

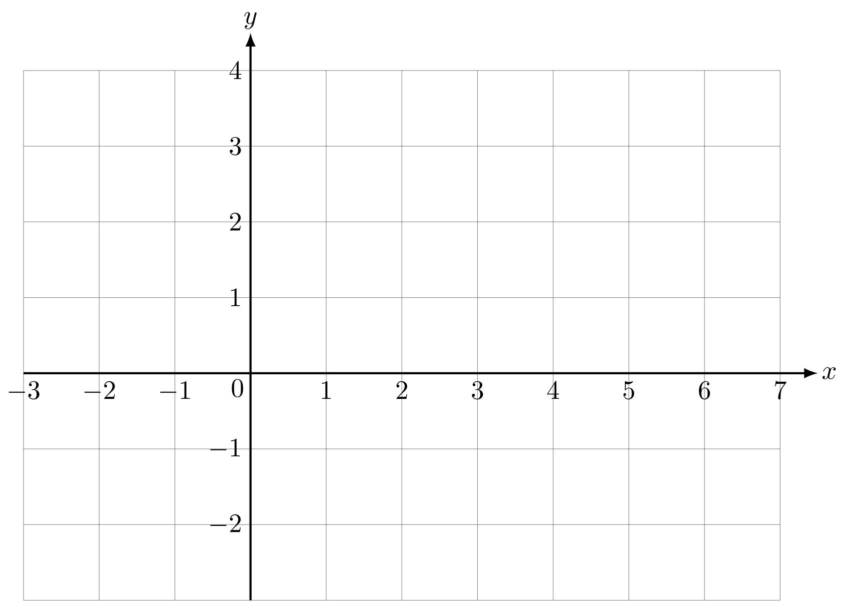
G. **Concavity and Points of Inflection**: concave upward if $f''(x) > 0$ and downward if $f''(x) < 0$.

Example 15 Sketch the graph of a continuous function $f\colon \mathbb{R} - \{2\} \to \mathbb{R}$ satisfying

$$\lim_{x\to-\infty} f(x) = 3, \quad \lim_{x\to 2^-} f(x) = -\infty, \quad \lim_{x\to 2^+} f(x) = \infty, \quad \lim_{x\to\infty} f(x) = -2$$

as well as the following conditions.

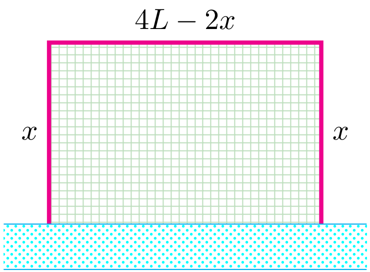
x	$f(x)$	$f'(x)$	$f''(x)$
$x < 0$		$-$	$-$
$x = 0$	2	$-$	0
$0 < x < 1$		$-$	$+$
$x = 1$	1	0	0
$1 < x < 2$		$-$	$-$
$2 < x < 3$		$-$	$+$
$x = 3$	0	0	0
$3 < x < 4$		$-$	$-$
$x = 4$	-1	$-$	0
$x > 4$		$-$	$+$



4.7 Optimization Problems

Example 16 A farmer has fencing of length $4L$ and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river.

What is the largest possible area?



Example 17 Find the smallest outer surface area of a cylindrical can with the volume V.