Lecture Notes

2301107 Calculus I for Engineering
Section 4

First Semester, Academic Year 2024

Chapter 5 Integrals

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4.9 Antiderivatives

The Antiderivative of a Function

Definition 1

A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

$$\begin{tikzcd} F(x) \arrow[r, bend left, "\text{Differentiate}"] & f(x) \arrow[l, bend left, "\text{Antidifferentiate}"] \end{tikzcd}$$

Remark

We know that $\dfrac{dC}{dx} = 0$ for any constant C, If $F(x)$ is an antiderivative of $f(x)$, then $F(x) + C$ will also be an antiderivative of $f(x)$, where C is an arbitrary constant.

Theorem 1

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Example 1 Find the most general antiderivatives of each of the following functions.

1. $f(x) = \sin x$
2. $f(x) = x^n,\ n \neq -1$
3. $f(x) = \dfrac{1}{x}$

Antidifferentiation Formulas

Let F and G be antiderivatives of f and g, respectively.

Function	**Particular antiderivative**	**Function**	**Particular antiderivative**
$cf(x)$	$cF(x)$	$\cos x$	$\sin x$
$f(x)+g(x)$	$F(x)+G(x)$	$\sin x$	$-\cos x$
x^n $(n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\sec^2 x$	$\tan x$
$\dfrac{1}{x}$	$\ln\lvert x\rvert$	$\sec x \tan x$	$\sec x$
e^x	e^x	$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x$
b^x	$\dfrac{b^x}{\ln b}$	$\dfrac{1}{x^2+1}$	$\arctan x$

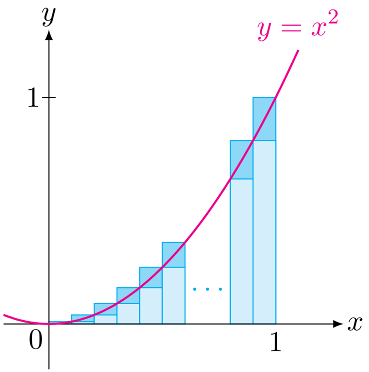
Example 2 Let $F(x)$ and $G(x)$ be antiderivatives of $3f(x)$ and $2g(x)+2$, respectively. Let $h(x)=2f(x)+3g(x)$.

1. Find the most general antiderivative of $h(x)$.

2. If $F(1)=G(1)=1$, find the antiderivative $H(x)$ of $h(x)$ for which $H(1)=1$.

5.1 The Area Problem

Example 3 Show that the area of the region that lies under the curve $y = x^2$, where $0 \le x \le 1$ is $\frac{1}{3}$.



Example 4 Determine a region whose area is equal to $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\sqrt{1+\frac{3i}{n}}$.

5.2 The Definite Integral

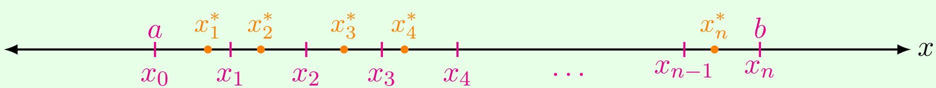
The Definite Integral

Definition 2 (Definition of a Definite Integral)

If f is a function defined for $a \le x \le b$,

we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{1}{n}(b - a)$.

Let $x_0(= a), x_1, x_2, \dots, x_n(= b)$ be the endpoints of these subintervals and let $x_1^*, x_2^*, \dots, x_n^*$ be any sample points in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$.



Then the definite integral of f from a to b is

upper limit integrand

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \overbrace{\boxed{\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x}}^{\text{Riemann sum}}$$

lower limit

provided that this limit exists and gives the same value for all possible choices of sample points.

If it does exist, we say that f is integrable on $[a, b]$.

Theorem 2

If f is continuous on $[a,b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a,b]$; that is, the definite integral $\displaystyle\int_a^b f(x)\,dx$ exists.

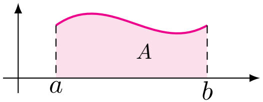
Area under a Curve

From the Riemann sum $\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$, we can see that

$f\left(x_i^*\right)\Delta x$ is the area of the rectangle of height $f\left(x_i^*\right)$ and width Δx. $\left(f\left(x_i^*\right)\Delta x < 0 \text{ if } f\left(x_i^*\right) < 0\right)$

Therefore, $\int_a^b f(x)\,dx$ may be geometrically interpreted as follows.

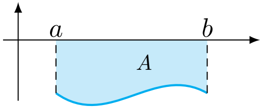
1. If $f(x) \geq 0$ on the interval $[a, b]$:



$$\int_a^b f(x)\,dx = A$$

$\int_a^b f(x)\,dx =$ the area of the region between the curve $y = f(x)$ and the x axis from a to b.

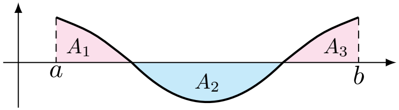
2. If $f(x) \leq 0$ on the interval $[a, b]$:



$$\int_a^b f(x)\,dx = -A$$

$\int_a^b f(x)\,dx =$ negative of the area of the region between the curve $f(x)$ and the x-axis from a to b.

3. If $f(x)$ has positive and negative values on the interval $[a, b]$:



Areas are calculated separately according to the signs of $f(x)$. $\int_a^b f(x)\,dx = A_1 - A_2 + A_3$

Example 5 Find the definite integrals by graphing and finding the area under a curve.

1. $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$

2. $\displaystyle\int_{-1}^1 \sqrt{2-x^2}\,dx$

Example 6 Find the definite integrals by graphing and finding the area under a curve.

1. $\displaystyle\int_0^3 (x-1)\ dx$

2. $\displaystyle\int_{-1}^2 |2x-1|\ dx$

Properties of the Definite Integral

From the Riemann sum $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$,

there are a special case and an extension as follows.

1. Special case when $b = a$: Since $\Delta x = \frac{1}{n}(b-a) = \frac{1}{n}(a-a) = 0$, therefore

$$\boxed{\int_a^a f(x)\,dx = 0}$$

2. When the upper limit b and the lower limit a in interchanged: Δx will change sign.

   The definition should be extended to

$$\boxed{\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx}$$

Theorem 3 (Properties of the Integral)

1. $\displaystyle\int_a^b c\ dx = c(b-a),$ *where c is any constant*

2. $\displaystyle\int_a^b \big[f(x) \pm g(x)\big]\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$

3. $\displaystyle\int_a^b c\,f(x)\ dx = c\int_a^b f(x)\ dx,$ *where c is any constant*

4. $\displaystyle\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$ *(If f is integrable only on $[a, b]$, then c must be in $[a, b]$.)*

5.3 The Fundamental Theorem of Calculus

The **Fundamental Theorem of Calculus (FTC)** established a connection between two branches of calculus:

- **Differential Calculus** which arises from the tangent problem and a rate of change of a function
- **Integral Calculus** which arises from the area problem and accumulation of small values

The Fundamental Theorem of Calculus gives the precise inverse relationship between the derivative and the integral.

This relationship is exploited and used to develop calculus into a systematic mathematical method. In particular, we may compute areas and integrals very easily without having to compute the Riemann sum.

The First Fundamental Theorem of Calculus

Theorem 4 (FTC1)

If f is continuous on $[a,b]$, the the function g defined by

$$g(x) = \int_a^x f(t)\,dt \quad \text{when } a \leq x \leq b$$

is continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) = f(x)$.

Remark

1. Intuitively, $g(x+h) - g(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt$.

   When $h \to 0$, the interval $[x, x+h]$ will be small, and thus $\int_x^{x+h} f(t)\,dt \approx f(x) \cdot h$.

   Therefore $g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f(x)$.

2. A function f continuous on $[a,b]$ always has an antiderivative, $F(x) = \int_a^x f(t)\,dt$, but FTC1 provides no specific method to find an explicit form of the antiderivative.

Example 7 Find the derivatives of the following functions.

1. $f(x) = \displaystyle\int_1^x \sin(t^2)\,dt$
2. $g(x) = \displaystyle\int_1^{x^3} \sin(t^2)\,dt$
3. $h(x) = \displaystyle\int_{x^2}^{x^3} \sin(t^2)\,dt$

The Second Fundamental Theorem of Calculus

Theorem 5 (FTC2)

if f is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function F such that $F' = f$.

Remark

We may write $F(x)\Big|_a^b$ or $F(x)\Big]_a^b$ to indicate the difference $F(b) - F(a)$.

Example 8 Evaluate the integral $\int_a^b x^n \, dx$ when $n \geq 0$ is a constant.

Remark We should be careful when $n < 0$.

For instance, when $n = -2$, the integration $\cancel{\int_{-1}^{2} \frac{1}{x^2} \, dx = -\frac{1}{x}\Big|_{-1}^{2} = -\frac{1}{2} - 1 = -\frac{3}{2}}$ is invalid

because $\lim_{x \to 0} \frac{1}{x^2} = \infty$ and thus $f(x) = \frac{1}{x^2}$ is discontinuous on $[-1, 2]$.

This integral is an example of an improper integral which will be studied later.

Example 9 Evaluate $\displaystyle\int_{-2}^{-1} \frac{1}{x}\, dx.$

Differentiation and Integration as Inverse Processes

Theorem 6 (The Fundamental Theorem of Calculus)

Suppose f is continuous on $[a, b]$.

1. *If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.*
2. *$\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is $F' = f$.*

Remark

1. FTC1 can be rewritten as $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$ which says that

   if we integrate a continuous function f and then differentiate the result,

   we arrive back at the original funciton f.

2. FTC2 can be rewritten as $\displaystyle\int_a^x F'(t)\,dt = F(x) - F(a)$ which says that

   if we differentiate a function F and then integrate the result,

   we arrive back at the original function F, except for the constant $F(a)$.

5.4 Indefinite Integral and the Net Change Theorem

Indefinite Integral

$$\boxed{\int f(x)\,dx \quad \text{means} \quad F'(x) = f(x)}$$

Example 10 Find the general indefinite integral $\displaystyle\int (x^2 + 3\sin x)\,dx$.

Table of Indefinite Integrals

$$\int cf(x)\ dx = c\int f(x)\ dx$$

$$\int \left[f(x)+g(x)\right]\ dx = \int f(x)\ dx + \int g(x)\ dx$$

$$\int k\ dx = kx + C$$

$$\int x^n\ dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int \frac{1}{x}\ dx = \ln|x| + C$$

$$\int e^x\ dx = e^x + C$$

$$\int b^x\ dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\ dx = -\cos x + C$$

$$\int \cos x\ dx = \sin x + C$$

$$\int \sec^2 x\ dx = \tan x + C$$

$$\int \csc^2 x\ dx = -\cot x + C$$

$$\int \sec x \tan x\ dx = \sec x + C$$

$$\int \csc x \cot x\ dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\ dx = \arctan x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\ dx = \arcsin x + C$$

Example 11 Evaluate $\displaystyle\int \frac{\sin x}{\cos^2 x}\,dx$.

5.5 The Substitution Rule

An integral like $\int 2x\sqrt{x^2+1}\;dx$ cannot be evaluated using fundamental properties of integrals that we have previously studied.

Suppose we introduce a new quantity $u = x^2 + 1$.

Then the differential of u is $du = 2x\;dx$.

$$\int 2x\sqrt{x^2+1}\;dx = \int \sqrt{x^2+1}\cdot 2x\;dx = \int \sqrt{u}\;du$$

$$= \tfrac{2}{3}u^{3/2} + C = \tfrac{2}{3}\big(x^2+1\big)^{3/2} + C$$

This method of integration is known as the Substitution Rule.

In general, the Substitution Rule can be applied to an integral in the form $\int f\big(g(x)\big)\,g'(x)\;dx$.
If we make the change of variable or substitution $u = g(x)$,
and calculate the differential $du = g'(x)\;dx$, then we have

$$\int f\big(g(x)\big)\,g'(x)\;dx = \int f(u)\;du.$$

Substitution: Indefinite Integrals

Theorem 7 (The Substitution Rule)

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f\big(g(x)\big)\, g'(x)\, dx = \int f(u)\, du.$$

Remark

The Substitution Rule for integration is equivalent to the Chain Rule for differentiation.

Example 12 Evaluate $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\,dx.$

Example 13 Evaluate $\displaystyle\int \frac{1}{x^2}\cos\left(\frac{\pi}{x}\right)\,dx.$

Example 14 Evaluate $\displaystyle\int \frac{\arctan x}{x^2+1}\,dx.$

Example 15 Evaluate $\displaystyle\int \tan x \; dx$.

Example 16 Evaluate $\displaystyle\int \sec x \; dx$.

Integration of Trigonometric Functions

$$\int \sin x \; dx = -\cos x + C$$

$$\int \cos x \; dx = \sin x + C$$

$$\begin{aligned}\int \tan x \; dx &= \ln|\sec x| + C \\ &= -\ln|\cos x| + C\end{aligned}$$

$$\begin{aligned}\int \cot x \; dx &= \ln|\sin x| + C \\ &= -\ln|\csc x| + C\end{aligned}$$

$$\begin{aligned}\int \sec x \; dx &= \ln|\sec x + \tan x| + C \\ &= -\ln|\sec x - \tan x| + C\end{aligned}$$

$$\begin{aligned}\int \csc x \; dx &= \ln|\csc x - \cot x| + C \\ &= -\ln|\csc x + \cot x| + C\end{aligned}$$

Substitution: Definite Integrals

Theorem 8 (The Substitution Rule for Definite Integrals)

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f\big(g(x)\big)\, g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

Example 17 Evaluate $\displaystyle\int_1^2 \frac{\log_2 x}{x}\,dx.$

Example 18 Evaluate $\displaystyle\int_0^1 \frac{x+1}{\sqrt{x^2+2x+2}}\,dx.$