Hybrid Equations (HyEQ) Toolbox v2.04

A Toolbox for Simulating Hybrid Systems in MATLAB/Simulink®

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Abstract

This note describes the Hybrid Equations (HyEQ) Toolbox implemented in MATLAB/Simulink for the simulation of hybrid dynamical systems. This toolbox is capable of simulating individual and interconnected hybrid systems where multiple hybrid systems are connected and interact such as a bouncing ball on a moving platform, fireflies synchronizing their flashing, and more. The Simulink implementation includes four basic blocks that define the dynamics of a hybrid system. These include a flow map, flow set, jump map, and jump set. The flows and jumps of the system are computed by the integrator system which is comprised of blocks that compute the continuous dynamics of the hybrid system, trigger jumps, update the state of the system and simulation time at jumps, and stop the simulation. We also describe a "lite simulator" which allows for faster simulation.

Contents

1 Introduction

To get started, a webinar introducing the HyEQ Toolbox is available at
`http://www.mathworks.com/videos/hyeq-a-toolbox-for-simulation-of-hybrid-dynamical-systems-81992.html`
A free two-step registration is required by Mathworks.

A hybrid system is a dynamical system with continuous and discrete dynamics. Several mathematical models for hybrid systems have appeared in literature. In this paper, we consider the framework for hybrid systems used in [1, 2, 3, 4], where a hybrid system $\mathcal{H}$ on a state space $\mathbb{R}^n$ with input space $\mathbb{R}^m$ is defined by the following objects:

- A set $C \subset \mathbb{R}^n \times \mathbb{R}^m$ called the *flow set*.
- A function $f\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ called the *flow map*.
- A set $D \subset \mathbb{R}^n \times \mathbb{R}^m$ called the *jump set*.
- A function $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ called the *jump map*.

We consider the simulation in MATLAB/Simulink of hybrid systems $\mathcal{H} = (C, f, D, g)$ written as

$$\mathcal{H}: \qquad x, \quad u \in \mathbb{R}^m, \quad y \in \mathbb{R}^n \qquad \begin{cases} x & = f(x,u) & (x,u) \in C \\ x^+ & = g(x,u) & (x,u) \in D \\ y & = h(x) \end{cases} \tag{1}$$

The flow map f defines the continuous dynamics on the flow set C, while the jump map g defines the discrete dynamics on the jump set D. These objects are referred to as the *data* of the hybrid system $\mathcal{H}$, which at times is explicitly denoted as $\mathcal{H} = (C, f, D, g)$. We illustrate this framework in a simple, yet rich in behavior, hybrid system.

Example 1.1 (bouncing ball system) Consider a model for a bouncing ball written as

$$f(x) := \begin{bmatrix} x_2 \\ -\gamma \end{bmatrix}, \; C := \left\{ x \in \mathbb{R}^2 \mid x_1 \geq 0 \right\} \tag{2}$$

$$g(x) := \begin{bmatrix} 0 \\ -\lambda x_2 \end{bmatrix}, \; D := \left\{ x \in \mathbb{R}^2 \mid x_1 \leq 0, \; x_2 \leq 0 \right\} \tag{3}$$

where $\gamma > 0$ is the gravity constant and $\lambda \in [0, 1)$ is the restitution coefficient. In this model, we consider the ball to be bouncing on a floor at a height of 0. This model is re-visited as an example in Section 3 and Section 5. □

The remainder of this note is organized as follows. In Section 2, we describe how to install the HyEQ Toolbox in MATLAB. In Section 3, we introduce the Lite HyEQ Simulator for solving hybrid systems without inputs. In Section 4, we introduce the HyEQ Simulator implemented in Simulink for solving single and interconnected hybrid systems with inputs. In Section 5, we work through several examples for the simulation of single and interconnected hybrid systems. In Section 6, we give directions to where the simulator files can be downloaded.

2 Installation

The following procedure describes how to install the Hybrid Equations (HyEQ) Toolbox in MATLAB. This installation adds useful `.m` files to the MATLAB library and several blocks to the Simulink block library.

Steps for installation:

1. Download the HyEQ Toolbox from MATLAB Central or the author's website at `https://hybrid.soe.ucsc.edu/softw`
2. Extract all files and save in any place (except the root folder).
3. Open MATLAB and change the current folder to the folder where the `install.m` is located.
4. Type `install` in the command window and hit enter to run the file `install.m`.
5. Follow the on-screen prompts. Must answer yes to the question:

```
Add toolbox permanently into your startup path (highly recommended)? Y/E/N [Y]: y
```

6. Once installation has finished, close and then reopen MATLAB.

Now the HyEQ Toolbox is ready for use.

If you wish to uninstall the HyEQ Toolbox from MATLAB, simply run the `tbclean.m` file inside the `HyEQ_Toolbox_V2_04` folder, and follow the on-screen prompts.

3 Lite HyEQ Simulator: A stand-alone MATLAB code for simulation of hybrid systems without inputs

One way to simulate hybrid systems is to use ODE function calls with events in MATLAB. Such an implementation gives fast simulation of a hybrid system.

In the lite HyEQ solver, four basic functions are used to define the *data* of the hybrid system $\mathcal{H}$ as in (1) (without inputs):

- The flow map is defined in the MATLAB function `f.m`. The input to this function is a vector with components defining the state of the system x. Its output is the value of the flow map f.
- The flow set is defined in the MATLAB function `C.m`. The input to this function is a vector with components defining the state of the system x. Its output is equal to 1 if the state belongs to the set C or equal to 0 otherwise.
- The jump map is defined in the MATLAB function `g.m`. Its input is a vector with components defining the state of the system x. Its output is the value of the jump map g.
- The jump set is defined in the MATLAB function `D.m`. Its input is a vector with components defining the state of the system x. Its output is equal to 1 if the state belongs to D or equal to 0 otherwise.

Our Lite HyEQ Simulator uses a main function `run.m` to initialize, run, and plot solutions for the simulation, functions `f.m, C.m, g.m,` and `D.m` to implement the data of the hybrid system, and `HyEQsolver.m` which will solve the differential equations by integrating the continuous dynamics, $\dot{x} = f(x)$, and jumping by the update law $x^+ = g(x)$. The ODE solver called in `HyEQsolver.m` initially uses the initial or most recent step size, and after each integration, the algorithms in `HyEQsolver.m` check to see if the solution is in the set C, D, or neither. Depending on which set the solution is in, the simulation is accordingly reset following the dynamics given in f or g, or the simulation is stopped. This implementation is fast because it also does not store variables to the workspace and only uses built-in ODE function calls.

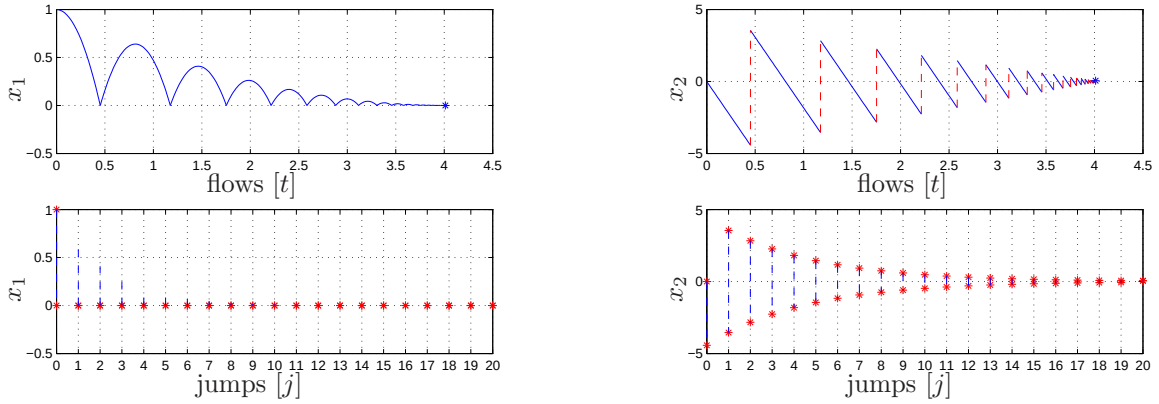
Time and jump horizons are set for the simulation using `TSPAN = [TSTART TFINAL]` as the time interval of the simulation and `JSPAN = [JSTART JSTOP]` as the interval for the number of discrete jumps allowed. The simulation stops when either the time or jump horizon, i.e. the final value of either interval, is reached.

The example below shows how to use the HyEQ solver to simulate a bouncing ball.

Example 3.1 (bouncing ball with Lite HyEQ Solver) Consider the hybrid system model for the bouncing ball with data given in Example 1.1.

For this example, we consider the ball to be bouncing on a floor at zero height. The constants for the bouncing ball system are $\gamma = 9.81$ and $\lambda = 0.8$. The following procedure is used to simulate this example in the Lite HyEQ Solver:

- Inside the MATLAB script `run.m`, initial conditions, simulation horizons, a rule for jumps, ode solver options, and a step size coefficient are defined. The function `HyEQsolver.m` is called in order to run the simulation, and a script for plotting solutions is included.
- Then the MATLAB functions `f.m, C.m, g.m, D.m` are edited according to the data given above.
- Finally, the simulation is run by clicking the run button in `run.m` or by calling `run.m` in the MATLAB command window.



(a) Height

(b) Velocity

Figure 1: Solution of Example 3.1

Example code for each of the MATLAB files `run.m, f.m, C.m, g.m,` and `D.m` is given below.

```
1
2 %-------------------------------------------------------------------------
3 % Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
4 % https://hybrid.soe.ucsc.edu/software
5 % http://hybridsimulator.wordpress.com/
6 % Filename: run_ex1_2a.m
7 %-------------------------------------------------------------------------
8 % Project: Simulation of a hybrid system (bouncing ball)
9 %-------------------------------------------------------------------------
10 %-------------------------------------------------------------------------
11 %   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
12 %   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
13 %   Copyright @ Hybrid Systems Laboratory (HSL),
14 %   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00
15
16 function run_ex1_2a
```

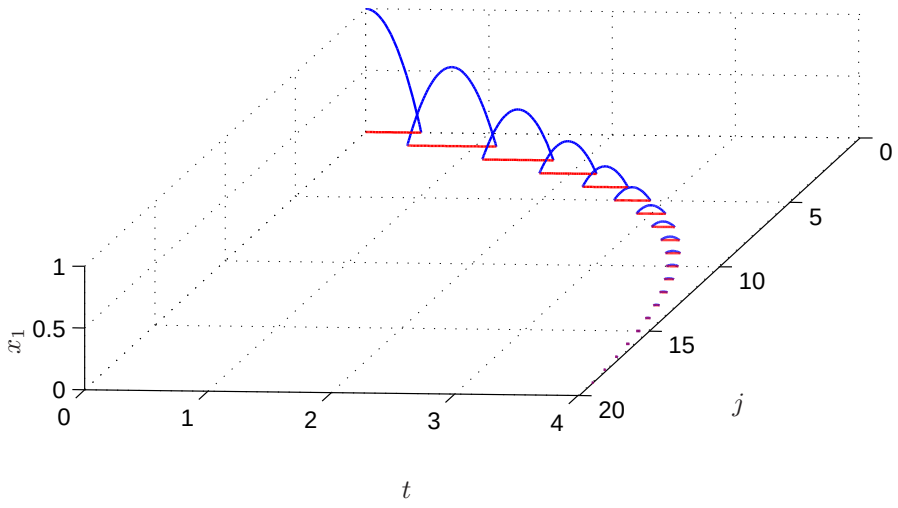


Figure 2: Hybrid arc corresponding to a solution of Example 3.1: height

```
% initial conditions
x1_0 = 1;
x2_0 = 0;
x0 = [x1_0;x2_0];

% physical variables
global gamma lambda
gamma = -9.81;  % gravity constant
lambda = 0.8;   % restitution coefficent

% simulation horizon
TSPAN=[0 10];
JSPAN = [0 20];

% rule for jumps
% rule = 1 -> priority for jumps
% rule = 2 -> priority for flows
rule = 1;

options = odeset('RelTol',1e-6,'MaxStep',.1);

% simulate
[t j x] = HyEQsolver( @f_ex1_2a,@g_ex1_2a,@C_ex1_2a,@D_ex1_2a,...
    x0,TSPAN,JSPAN,rule,options);

% plot solution
```

```
figure(1) % position
clf
subplot(2,1,1), plotHarc(t,j,x(:,1));
grid on
ylabel('x_1 position')
subplot(2,1,2), plotHarc(t,j,x(:,2));
grid on
ylabel('x_2 velocity')

% plot phase plane
figure(2) % position
clf
plotHarcColor(x(:,1),j,x(:,2),t);
xlabel('x_1')
ylabel('x_2')
grid on

% plot hybrid arc
figure(3)
plotHybridArc(t,j,x)
xlabel('j')
ylabel('t')
zlabel('x1')

```

Flow map

```

function xdot = f_ex1_2a(x)
%--------------------------------------------------------------------------
% Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
% https://hybrid.soe.ucsc.edu/software
% http://hybridsimulator.wordpress.com/
% Filename: f_ex1_2a.m
%--------------------------------------------------------------------------
% Project: Simulation of a hybrid system (bouncing ball)
% Description: Flow map
%--------------------------------------------------------------------------
%--------------------------------------------------------------------------
%   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
%   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
%   Copyright @ Hybrid Systems Laboratory (HSL),
%   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00

% state
x1 = x(1);
x2 = x(2);

global gamma

% differential equations
xdot = [x2 ; gamma];
end
```

Flow set

```
1
2 function [value] = C_ex1_2a(x)
3 %-------------------------------------------------------------------------
4 % Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
5 % https://hybrid.soe.ucsc.edu/software
6 % http://hybridsimulator.wordpress.com/
7 % Filename: C_ex1_2a.m
8 %-------------------------------------------------------------------------
9 % Description: Flow set
10 % Return 0 if outside of C, and 1 if inside C
11 %-------------------------------------------------------------------------
12 %-------------------------------------------------------------------------
13 %   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
14 %   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
15 %   Copyright @ Hybrid Systems Laboratory (HSL),
16 %   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00
17
18
19 x1 = x(1);
20
21 if x1 >= 0
22     value = 1;
23 else
24     value = 0;
25 end
26 end
```

Jump map

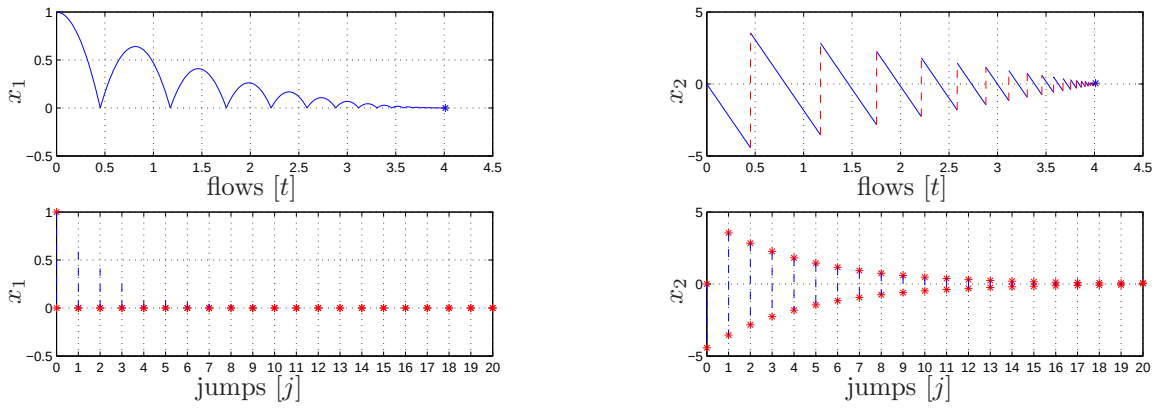
```
1
2 function xplus = g_ex1_2a(x)
3 %-------------------------------------------------------------------------
4 % Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
5 % https://hybrid.soe.ucsc.edu/software
6 % http://hybridsimulator.wordpress.com/
7 % Filename: g_ex1_2a.m
8 %-------------------------------------------------------------------------
9 % Project: Simulation of a hybrid system (bouncing ball)
10 % Description: Jump map
11 %-------------------------------------------------------------------------
12 %-------------------------------------------------------------------------
13 %   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
14 %   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
15 %   Copyright @ Hybrid Systems Laboratory (HSL),
16 %   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00
17
18 % state
19 x1 = x(1);
20 x2 = x(2);
21
22 global lambda
23
24 xplus = [-x1 ; -lambda*x2];
25 end
```

Jump set

```
1
2 function inside = D_ex1_2a(x)
3 %-------------------------------------------------------------------------
4 % Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
5 % https://hybrid.soe.ucsc.edu/software
6 % http://hybridsimulator.wordpress.com/
7 % Filename: D_ex1_2a.m
8 %-------------------------------------------------------------------------
9 % Description: Jump set
10 % Return 0 if outside of D, and 1 if inside D
11 %-------------------------------------------------------------------------
12 %-------------------------------------------------------------------------
13 %   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
14 %   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
15 %   Copyright @ Hybrid Systems Laboratory (HSL),
16 %   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00
17
18 x1 = x(1);
19 x2 = x(2);
20 if (x1 <= 0 && x2 <= 0)
21     inside = 1;
22 else
23     inside = 0;
24 end
25 end
```

A solution to the bouncing ball system from $x(0,0) = [1,0]^\top$ and with $TSPAN = [0\ \ 10], JSPAN = [0\ \ 20]$, $rule = 1$, is depicted in Figure 3(a) (height) and Figure 3(b) (velocity). Both the projection onto t and j are shown. Figure 2 depicts the corresponding hybrid arc for the position state.

For MATLAB files of this example, see Examples/Example_1.2.



(a) Height

(b) Velocity

Figure 3: Solution of Example 3.1

3.1 Solver Function

The solver function `HyEQsolver` solves the hybrid system using three different functions as shown below. First, the flows are calculated using the built-in ODE solver function ODE45 in MATLAB. If the solution leaves the flow set `C`, the discrete event is detected using the function `zeroevents` as shown in Section 3.1.1. When the state jumps, the next value of the state is calculated via the jump map `g` using the function `jump` as shown in Section 3.1.2.

```
1
2 function [t j x] = HyEQsolver(f,g,C,D,x0,TSPAN,JSPAN,rule,options,solver,E)
3 %HYEQSOLVER solves hybrid equations.
4 %   Syntax: [t j x] = HYEQSOLVER(f,g,C,D,x0,TSPAN,JSPAN,rule,options,solver,E)
5 %   computes solutions to the hybrid equations
6 %
7 %   \dotx = f(x,t,j)  x \in C x^+ = g(x,t,j)  x \in D
8 %
9 %   where x is the state, f is the flow map, g is the jump map, C is the
10 %   flow set, and D is the jump set. It outputs the state trajectory (t,j)
11 %   -> x(t,j), where t is the flow time parameter and j is the jump
12 %   parameter.
13 %
14 %   x0 defines the initial condition for the state.
15 %
16 %   TSPAN = [TSTART TFINAL] is the time interval. JSPAN = [JSTART JSTOP] is
17 %       the interval for discrete jumps. The algorithm stop when the first
18 %       stop condition is reached.
19 %
20 %   rule (optional parameter) - rule for jumps
21 %       rule = 1 (default) -> priority for jumps rule = 2 -> priority for
22 %       flows
23 %
24 %   options (optional parameter) - options for the solver see odeset f.ex.
25 %       options = odeset('RelTol',1e-6);
26 %       options = odeset('InitialStep',eps);
27 %
28 %   solver (optional parameter. String) - selection of the desired ode
29 %       solver. All ode solvers are suported, exept for ode15i.  See help
30 %       odeset for detailed information.
31 %
32 %   E (optional parameter) - Mass matrix [constant matrix | function_handle]
33 %       For problems:
34 %       E*\dotx = f(x) x \in C
35 %       x^+ = g(x)  x \in D
36 %       set this property to the value of the constant mass matrix. For
37 %       problems with time- or state-dependent mass matrices, set this
38 %       property to a function that evaluates the mass matrix. See help
39 %       odeset for detailed information.
40 %
41 %         Example: Bouncing ball with Lite HyEQ Solver
42 %
43 %         % Consider the hybrid system model for the bouncing ball with data given in
44 %         % Example 1.2. For this example, we consider the ball to be bouncing on a
45 %         % floor at zero height. The constants for the bouncing ball system are
46 %         % \gamma=9.81 and \lambda=0.8. The following procedure is used to
```

```
%          % simulate this example in the Lite HyEQ Solver:
%
%          % * Inside the MATLAB script run_ex1_2.m, initial conditions, simulation
%          % horizons, a rule for jumps, ode solver options, and a step size
%          % coefficient are defined. The function HYEQSOLVER.m is called in order to
%          % run the simulation, and a script for plotting solutions is included.
%          % * Then the MATLAB functions f_ex1_2.m, C_ex1_2.m, g_ex1_2.m, D_ex1_2.m
%          % are edited according to the data given below.
%          % * Finally, the simulation is run by clicking the run button in
%          % run_ex1_2.m or by calling run_ex1_2.m in the MATLAB command window.
%
%          % For further information, type in the command window:
%          web(['Example_1_2.html']);
%
%          % Define initial conditions
%          x1_0 = 1;
%          x2_0 = 0;
%          x0   = [x1_0; x2_0];
%
%          % Set simulation horizon
%          TSPAN = [0 10];
%          JSPAN = [0 20];
%
%          % Set rule for jumps and ODE solver options
%          %
%          % rule = 1 -> priority for jumps
%          %
%          % rule = 2 -> priority for flows
%          %
%          % set the maximum step length. At each run of the
%          % integrator the option 'MaxStep' is set to
%          % (time length of last integration)*maxStepCoefficient.
%          %  Default value = 0.1
%
%          rule               = 1;
%
%          options            = odeset('RelTol',1e-6,'MaxStep',.1);
%
%          % Simulate using the HYEQSOLVER script
%          % Given the matlab functions that models the flow map, jump map,
%          % flow set and jump set (f_ex1_2, g_ex1_2, C_ex1_2, and D_ex1_2
%          % respectively)
%
%          [t j x] = HYEQSOLVER( @f_ex1_2,@g_ex1_2,@C_ex1_2,@D_ex1_2,...
%              x0,TSPAN,JSPAN,rule,options,'ode45');
%
%          % plot solution
%
%          figure(1) % position
%          clf
%          subplot(2,1,1),plotflows(t,j,x(:,1))
%          grid on
%          ylabel('x1')
%
```

```
101 %         subplot(2,1,2),plotjumps(t,j,x(:,1))
102 %         grid on
103 %         ylabel('x1')
104 %
105 %         figure(2) % velocity
106 %         clf
107 %         subplot(2,1,1),plotflows(t,j,x(:,2))
108 %         grid on
109 %         ylabel('x2')
110 %
111 %         subplot(2,1,2),plotjumps(t,j,x(:,2))
112 %         grid on
113 %         ylabel('x2')
114 %
115 %         % plot hybrid arc
116 %
117 %         figure(3)
118 %         plotHybridArc(t,j,x)
119 %         xlabel('j')
120 %         ylabel('t')
121 %         zlabel('x1')
122 %
123 %         % plot solution using plotHarc and plotHarcColor
124 %
125 %         figure(4) % position
126 %         clf
127 %         subplot(2,1,1), plotHarc(t,j,x(:,1));
128 %         grid on
129 %         ylabel('x_1 position')
130 %         subplot(2,1,2), plotHarc(t,j,x(:,2));
131 %         grid on
132 %         ylabel('x_2 velocity')
133 %
134 %
135 %         % plot a phase plane
136 %         figure(5) % position
137 %         clf
138 %         plotHarcColor(x(:,1),j,x(:,2),t);
139 %         xlabel('x_1')
140 %         ylabel('x_2')
141 %         grid on
142 %
143 %-------------------------------------------------------------------------
144 % Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
145 % https://hybrid.soe.ucsc.edu/software
146 % http://hybridsimulator.wordpress.com/
147 % Filename: HYEQSOLVER.m
148 %-------------------------------------------------------------------------
149 %   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
150 %   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
151 %   Copyright @ Hybrid Systems Laboratory (HSL),
152 %   Revision: 0.0.0.4 Date: 04/6/2017 16:26:00
153
154
```

```
155 if ~exist('rule','var')
156     rule = 1;
157 end
158
159 if ~exist('options','var')
160     options = odeset();
161 end
162 if exist('E','var') && ~exist('solver','var')
163     solver = 'ode15s';
164 end
165 if ~exist('solver','var')
166     solver = 'ode45';
167 end
168 if ~exist('E','var')
169     E = [];
170 end
171 % mass matrix (if existent)
172 isDAE = false;
173 if ~isempty(E)
174     isDAE = true;
175     switch isa(E,'function_handle')
176         case true % Function E(x)
177             M = E;
178             options = odeset(options,'Mass',M,'Stats','off',...
179                 'MassSingular','maybe','MStateDependence','strong',...
180                 'InitialSlope',f_hdae(x0,TSPAN(1)));
181         case false % Constant double matrix
182             M = double(E);
183             options = odeset(options,'Mass',M,'Stats','off',...
184                 'MassSingular','maybe','MStateDependence','none');
185     end
186 end
187
188 odeX = str2func(solver);
189 nargf = nargin(f);
190 nargg = nargin(g);
191 nargC = nargin(C);
192 nargD = nargin(D);
193
194
195
196 % simulation horizon
197 tstart = TSPAN(1);
198 tfinal = TSPAN(end);
199 jout = JSPAN(1);
200 j = jout(end);
201
202 % simulate
203 tout = tstart;
204 [rx,cx] = size(x0);
205 if rx == 1
206     xout = x0;
207 elseif cx == 1
208     xout = x0.';
```

```
else
    error('Error, x0 does not have the proper size')
end

% Jump if jump is prioritized:
if rule == 1
    while (j<JSPAN(end))
        % Check if value it is possible to jump current position
        insideD = fun_wrap(xout(end,:).',tout(end),j,D,nargD);
        if insideD == 1
            [j tout jout xout] = jump(g,j,tout,jout,xout,nargg);
        else
            break;
        end
    end
end
fprintf('Completed: %3.0f%%',0);
while (j < JSPAN(end) && tout(end) < TSPAN(end))
    options = odeset(options,'Events',@(t,x) zeroevents(x,t,j,C,D,...
        rule,nargC,nargD));
    % Check if it is possible to flow from current position
    insideC = fun_wrap(xout(end,:).',tout(end),j,C,nargC);
    if insideC == 1
        if isDAE
            options = odeset(options,'InitialSlope',f(xout(end,:).',tout(end)));
        end
        [t,x] = odeX(@(t,x) fun_wrap(x,t,j,f,nargf),[tout(end) tfinal],...
            xout(end,:).', options);
        nt = length(t);
        tout = [tout; t];
        xout = [xout; x];
        jout = [jout; j*ones(1,nt)'];
    end

    %Check if it is possible to jump
    insideD = fun_wrap(xout(end,:).',tout(end),j,D,nargD);
    if insideD == 0
        break;
    else
        if rule == 1
            while (j<JSPAN(end))
                % Check if it is possible to jump from current position
                insideD = fun_wrap(xout(end,:).',tout(end),j,D,nargD);
                if insideD == 1
                    [j tout jout xout] = jump(g,j,tout,jout,xout,nargg);
                else
                    break;
                end
            end
        else
            [j tout jout xout] = jump(g,j,tout,jout,xout,nargg);
        end
    end
    fprintf('\b\b\b\b%3.0f%%',max(100*j/JSPAN(end),100*tout(end)/TSPAN(end)));
```

```
end
t = tout;
x = xout;
j = jout;
fprintf('\nDone\n');
end

```

3.1.1 Events Detection

```

function [value,isterminal,direction] = zeroevents(x,t,j,C,D,rule,nargC,nargD)
switch rule
    case 1 % -> priority for jumps
        isterminal(1) = 1; % InsideC
        isterminal(2) = 1; % Inside(C \cap D)
        isterminal(3) = 1; % OutsideC
        direction(1) = -1; % InsideC
        direction(2) = -1; % Inside(C \cap D)
        direction(3) =  1; % OutsideC
    case 2 %(default) -> priority for flows
        isterminal(1) = 1; % InsideC
        isterminal(2) = 0; % Inside(C \cap D)
        isterminal(3) = 1; % OutsideC
        direction(1) = -1; % InsideC
        direction(2) = -1; % Inside(C \cap D)
        direction(3) =  1; % OutsideC
end

insideC = fun_wrap(x,t,j,C,nargC);
insideD = fun_wrap(x,t,j,D,nargD);
outsideC = -fun_wrap(x,t,j,C,nargC);


value(1) = 2*insideC;
value(2) = 2-insideC - insideD;
value(3) = 2*outsideC;

end

```

3.1.2 Jump Map

```

function [j tout jout xout] = jump(g,j,tout,jout,xout,nargfun)
% Jump
j = j+1;
y = fun_wrap(xout(end,:).',tout(end),jout(end),g,nargfun);
% Save results
tout = [tout; tout(end)];
xout = [xout; y.'];
jout = [jout; j];
end

```

3.1.3 Function Wrapper

```
1
2 function xdelta = fun_wrap(x,t,j,h,nargfun)
3 %fun_wrap   Variable input arguments function (easy use for users).
4 %   fun_wrap(x,t,j,h,nargfun) depending on the function h written by the
5 %   user, this script selects how the HyEQ solver should call that
6 %   function.
7 %    x: state
8 %    t: time
9 %    j: discrete time
10 %    h: function handle
11 %    nargfun: number of input arguments of function h
12 %---------------------------------------------------------------------------
13 % Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
14 % https://hybrid.soe.ucsc.edu/software
15 % http://hybridsimulator.wordpress.com/
16 % Filename: fun_wrap.m
17 %---------------------------------------------------------------------------
18 %   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
19 %   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
20 %   Copyright @ Hybrid Systems Laboratory (HSL),
21 %   Revision: 0.0.0.3 Date: 01/28/2016 5:12:00
22
23
24 switch nargfun
25     case 1
26         xdelta = h(x);
27     case 2
28         xdelta = h(x,t);
29     case 3
30         xdelta = h(x,t,j);
31 end
32 end
```

3.2 Software Requirements

In order to run simulations using the Lite HyEQ Simulator, MATLAB R13 or newer is required.

3.3 Configuration of Solver

Before a simulation is started, it is important to determine the needed integrator scheme, zero-cross detection settings, precision, and other tolerances. Using the default settings does not always give the most efficient or most accurate simulations. In the Lite HyEQ Simulator, these parameters are edited in the `run.m` file using

```
options = odeset(RelTol,1e-6,MaxStep ,.1);.
```

3.4 Initialization

The Lite HyEQ Simulator is initialized and run by calling the function `run.m`. Inside `run.m`, the initial conditions, simulation horizons `TSPAN` and `JSPAN`, a rule for jumps, and simulation tolerances are defined. After all of the parameters are defined, the function `HyEQsolver` is called, and the simulation runs. See below for sample code to initialize and run the bouncing ball example, Example 3.1.

```
1
2 % initial conditions
3 x1_0 = 1;
4 x2_0 = 0;
5 x0 = [x1_0;x2_0];
6 % simulation horizon
7 TSPAN=[0,10];
8 JSPAN = [0,20];
9 % rule for jumps
10 % rule = 1 -> priority for jumps
11 % rule = 2 -> priority for flows
12 rule = 1;
13 options = odeset('RelTol',1e-6,'MaxStep',.1);
14 % simulate
15 [t,j,x] = HyEQsolver(@f,@g,@C,@D,x0,TSPAN,JSPAN,rule,options);
```

3.5 Postprocessing and Plotting solutions

The function `run.m` is also used to plot solutions after the simulations is complete. See below for sample code to plot solutions to the bouncing ball example, Example 3.1.

```
1
2 % plot solution
3 figure(1) % position
4 clf
5 subplot(2,1,1),plotflows(t,j,x(:,1))
6 grid on
7 ylabel('x1')
8 subplot(2,1,2),plotjumps(t,j,x(:,1))
9 grid on
10 ylabel('x1')
11 figure(2) % velocity
12 clf
13 subplot(2,1,1),plotflows(t,j,x(:,2))
14 grid on
15 ylabel('x2')
16 subplot(2,1,2),plotjumps(t,j,x(:,2))
17 grid on
18 ylabel('x2')
19 % plot hybrid arc
20 figure(2)
21 plotHybridArc(t,j,x)
22 xlabel('j')
23 ylabel('t')
24 zlabel('x1')
25 grid on
26 view(37.5,30)
```

The following functions are used to generate the plots:

- plotarc(t,j,x,L,jstar,modificatorF,modificatorJ,resolution,DDD,true3D): plots the hybrid time domain (matrix) (t, j) versus the sate x (matrix) taking into account jumps j. If x is a matrix (n states), then the hybrid time is plotted versus the rows or columns of the matrix, whichever line up. If t and j are a matrices, then each column of x will be plotted according to the hybrid time domain composed

for each column of t and j. Depending on the input data, this function is capable of plotting several types of figures, e.g., 2D, 3D, hybrid arcs with color, etc. Next, we list several functions that specialize different types of plotting styles from plotarc. For more information, please type >> help plotarc or >> helpwin plotarc in the command window.

- plotarc3(t,j,x,L,jstar,modificatorF,modificatorJ,resolution,true3D) is a version of plotarc that specializes in figures in 3D. For more information, please type >> help plotarc3 or >> helpwin plotarc3 in the command window.

- plotflows(t,j,x,$jstar$,resolution): plots (in blue) the projection of the trajectory x onto the flow time axis t. The value of the trajectory for intervals $[t_j, t_{j+1}]$ with empty interior is marked with $*$ (in blue). Dashed lines (in red) connect the value of the trajectory before and after the jump. Figure 12(a) shows a plot created with this function.
  - plotflows(t,j,x,$jstar$): The plot is cut regarding the $jstar$ interval ($jstar = [j - initialj - final]$).
  - plotflows(t,j,x,$jstar$,resolution): Also, a maximum resolution in between jumps is given by the input variable resolution

- plotjumps(t,j,x,$jstar$,resolution): plots (in red) the projection of the trajectory x onto the jump time j. The initial and final value of the trajectory on each interval $[t_j, t_{j+1}]$ is denoted by $*$ (in red) and the continuous evolution of the trajectory on each interval is depicted with a dashed line (in blue). Figure 12(a) shows a plot created with this function.
  - plotjumps(t,j,x,$jstar$): The plot is cut regarding the $jstar$ interval ($jstar = [j - initialj - final]$).
  - plotjumps(t,j,x,$jstar$,resolution): Also, a maximum resolution in between jumps is given by the input variable resolution

- plotHybridArc(t,j,x,$jstar$,resolution): plots (in blue and red) the trajectory x on hybrid time domains. The intervals $[t_j, t_{j+1}]$ indexed by the corresponding j are depicted in the $t-j$ plane (in red). Figure **??** shows a plot created with this function.
  - plotHybridArc(t,j,x,$jstar$): The plot is cut regarding the $jstar$ interval ($jstar = [j - initialj - final]$).
  - plotHybridArc(t,j,x,$jstar$,resolution): Also, a maximum resolution in between jumps is given by the input variable resolution

- plotHarc(t,j,x,$jstar$,modificatorF,modificatorJ,resolution) is a function for plotting hybrid arcs (n states).
  - plotHarc(t,j,x): plots the trajectory x versus the hybrid time domain (t, j). If x is a matrix, then the time vector is plotted versus the rows or columns of the matrix, whichever line up.
  - plotHarc(t,j,x,$jstar$): plots the trajectory x versus the hybrid time domain (t, j), and the plot is cut regarding the $jstar$ interval ($jstar = [j_{initial}, j_{final}]$).
  - plotHarc(t,j,x,$jstar$,modificatorF,modificatorJ): ModificatorF and ModificatorJ are cell arrays that contains the standard matlab ploting modificators (type >> help plotHarc or >> helpwin plotHarc in the command window for more information).

- plotHarcColor(t,j,x,L,$jstar$,resolution) plots the trajectory x on hybrid time domain with color.
  - plotHarcColor(t,j,x,L): plots the trajectory x (vector) versus the hybrid time domain (t, j). The hybrid arc is plotted with L data as color. The input vectors t, j, x, L must have the same length.
  - plotHarcColor(t,j,x,L,$jstar$): If a specific interval in j is required, $jstar = [j_{initial}, j_{final}]$ must be provided. (type >> help plotHarcColor or >> helpwin plotHarcColor in the command window for more information)

- plotHarcColor3D(t,j,x,L,$jstar$,modificator,resolution) plots an $3D$ hybrid arc with color.

– plotHarcColor3D(t,j,x,L) plots the trajectory x (3 states) taking into account the hybrid time domain (t, j). The hybrid arc is plotted with L data as color. The input vectors t, j, x, L must have the same length and x must have three columns.

– plotHarcColor3D(t,j,x,L,$jstar$) If a specific interval in j is required, $jstar = [j_{initial}, j_{final}]$ must be provided.

– plotHarcColor3D(t,j,x,L,$jstar$,modificator) Modificator is a cell array that contains the standard matlab ploting modificators (type >> help plotHarcColor3D or >> helpwin plotHarcColor3D in the command window for more information).

4 HyEQ Simulator: A Simulink implementation for simulation of single and interconnected hybrid systems with or without inputs

The HyEQ Toolbox includes three main Simulink library blocks that allow for simulation of a hybrid system $\mathcal{H} = (C, f, D, g)$ using either externally defined functions or embedded MATLAB functions, and a single hybrid system or interconnected hybrid systems with inputs using embedded MATLAB functions. Figure 4 shows these blocks in the Simulink Library Browser.

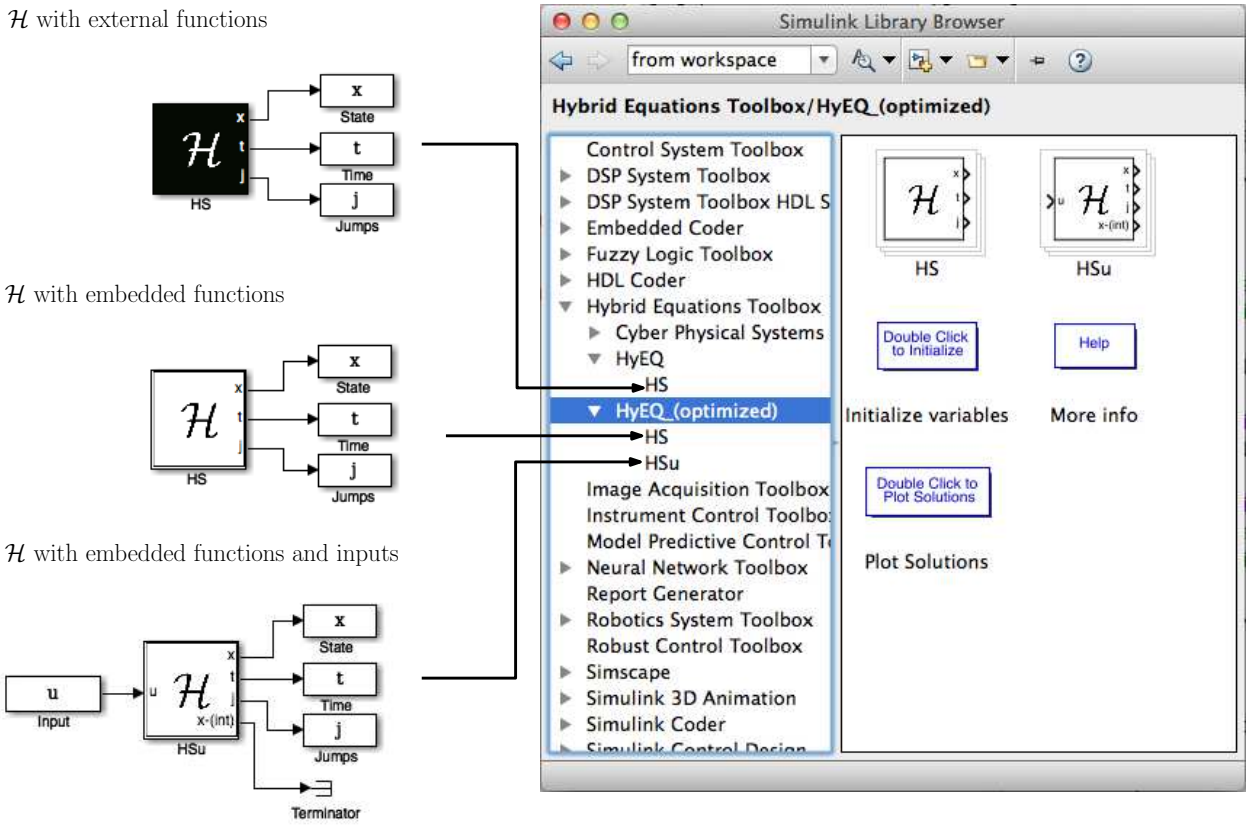


Figure 4: MATLAB/Simulink library blocks for Simulink implementation.

Figure 5 shows a Simulink implementation for simulating a hybrid system with inputs using embedded MATLAB functions. In this implementation, four basic blocks are used to define the *data* of the hybrid system $\mathcal{H}$:

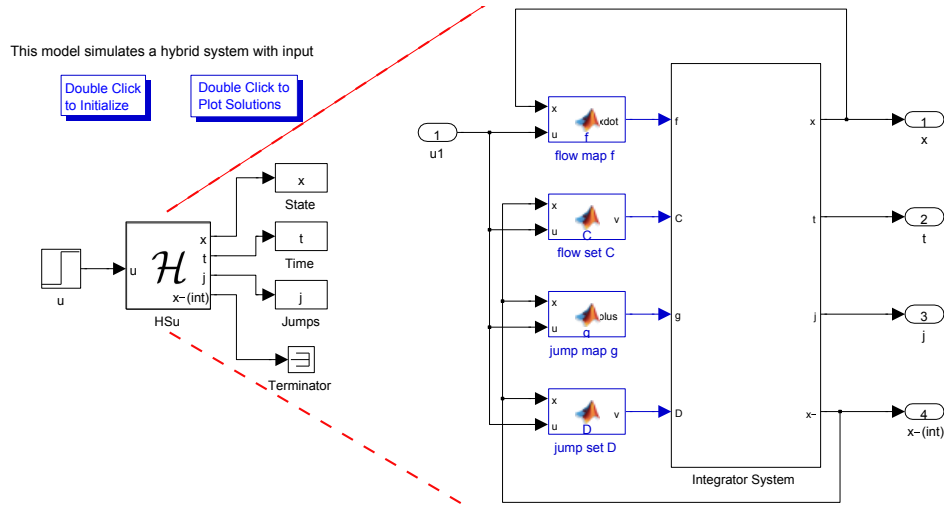


Figure 5: MATLAB/Simulink implementation of a hybrid system $\mathcal{H} = (C, f, D, g)$ with inputs.

- The flow map is implemented in an *Embedded MATLAB function block* executing the function `f.m`. Its input is a vector with components defining the state of the system x, and the input u. Its output is the value of the flow map f which is connected to the input of an integrator.
- The flow set is implemented in an *Embedded MATLAB function block* executing the function `C.m`. Its input is a vector with components x^- and input u of the *Integrator system*. Its output is equal to 1 if the state belongs to the set C or equal to 0 otherwise. The minus notation denotes the previous value of the variables (before integration). The value x^- is obtained from the state port of the integrator.
- The jump map is implemented in an *Embedded MATLAB function block* executing the function `g.m`. Its input is a vector with components x^- and input u of the *Integrator system*. Its output is the value of the jump map g.
- The jump set is implemented in an *Embedded MATLAB function block* executing the function `D.m`. Its input is a vector with components x^- and input u of the *Integrator system*. Its output is equal to 1 if the state belongs to D or equal to 0 otherwise.

In our implementation, MATLAB `.m` files are used. The file `initialization.m` is used to define initial variables before simulation. The file `postprocessing.m` is used to plot the solutions after a simulation is complete. These two `.m` files are called by double-clicking the *Double Click to...* blocks at the top of the Simulink Model (see Section 4.6 for more information on these `.m` files and their use).

4.1 Specialized library for simulation of Cyber-Physical Systems

The HyEQ Toolbox includes a series of blocks that model elements of a cyber-physical system (CPS). Those models are special instances of the hybrid systems blocks described above, particularly the blocks that use embedded MATLAB functions. A cyber-physical system is given by the interconnection between a physical process, *the plant*, a computer algorithm used for control, *the controller*; and the subsystems needed to interconnect the plant and the controller, i.e., *the interfaces, converters, and signal conditioners*. Most of the elements described in this section are presented in an extended form in [5].

In these notes, the temporal evolution of the variables of a cyber-physical system are captured using dynamical models. In this document, we advocate that hybrid dynamical system models can be employed to capture the behavior of cyber-physical systems. More precisely, the evolution of the continuous variables is captured by *differential equations* while the evolution of the discrete variables is captured by *difference*

equations. These equations are typically nonlinear due to the complexity of the dynamics of those variables. Furthermore, conditions determining the change of the continuous and discrete variables according to the said equations/inclusions can be conveniently captured by functions of the variables, inputs, and outputs. In Figure 6, cyber and physical blocks are implemented within the HyEQ simulator.

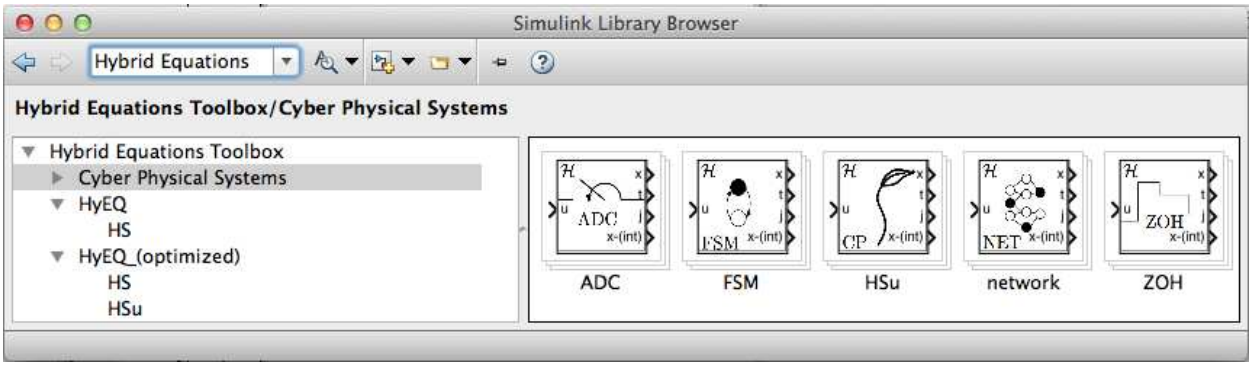


Figure 6: MATLAB/Simulink library blocks for Cyber-Physical Systems implementation.

4.1.1 Models of physical components

The physical components of a cyber-physical system include the analog elements, physical systems, and the environment. Consider a model of a physical system given by a differential equation $\dot{x} = f(x, u)$ and a output map $y = h(x, u)$. Among the many possible models available, we capture the dynamics of the physical components by using the hybrid systems framework by

$$\dot{x} = f(x, u), \qquad C := \{x \mid x \in \mathbb{R}^n\}, \tag{4a}$$

$$x^+ = g(x) = \emptyset, \qquad D := \emptyset, \tag{4b}$$

$$y = h(x, u) \tag{4c}$$

Particularly, the model in (4) can be implemented in the simulink block *HSu* in Figures 5) and 6).

4.1.2 Models of cyber components

The cyber components of a cyber-physical system include those in charge of performing computations, implementing algorithms, and transmitting digital data over networks. The tasks performed by the code (at the software level) and the logic-based mechanisms (at the circuit level) involve variables that only change at discrete events, not necessarily periodically.

We denote the state variable of the cyber components by $x_c \in \Upsilon$, where $\Upsilon \subset \mathbb{R}^{n_C}$ is the state space. The dynamics of η are defined by a difference equation with right-hand side defined by the map G_c. We let $v \in \mathcal{V} \subset \mathbb{R}^{m_C}$ denote the input signals affecting the cyber components and $\zeta \in \mathbb{R}^{r_C}$ to be the output defined by the output function κ, which is a function of the state x_c and of the input v. With these definitions, the general mathematical description of the cyber component is

$$x_c^+ = G_c(x_c, v), \quad \zeta \quad = \quad \kappa(x_c, v) \tag{5}$$

In certain cases, it would be needed to impose restrictions on the state and inputs to the cyber component. Such conditions can be modeled imposing that x_c and v belong to a subset of their state space, namely,

$$(x_c, v) \in D_c \subset \Upsilon \times \mathcal{V} \tag{6}$$

The model of the cyber components is given by (5)-(6). Next, we provide specific constructions of models of cyber components.

1) *Pure Finite State Machines.* A finite state machine (FSM) or deterministic finite automaton (DFA) is a system with inputs, states, and outputs taking values from discrete sets that are updated at discrete transitions (or jumps) triggered by its inputs. Then, given a FSM and an initial state $q_0 \in Q$, a transition to a state $q_1 = \delta(q_0, v)$ is performed when an input $v \in \Sigma$ is applied to it. After the transition, the output of the FSM is updated to $\kappa(q_1)$. This mechanism can be captured by the difference equation

$$q^+ \;=\; \delta(q,v) \qquad \zeta \;=\; \kappa(q) \qquad (q,v) \in Q \times \Sigma \tag{7}$$

This model captures the dynamics the model of the cyber components in (5)-(6) with

$$x_c = q, \qquad \Upsilon = Q, \qquad \mathcal{V} = \Sigma, \qquad G_c = \delta, \qquad D_c = \Upsilon \times \mathcal{V}$$

Note that there is no notion of time associated with the FSM model above. An example of this model is presented in Example 5.8. In addition, the block FSM in Figure 6 can be used to model these type of systems.

2) *Analog-to-Digital Converters.* Analog-to-digital converters (ADCs), or simply sampling devices, are commonly used to provide measurements of the physical systems to the cyber components. Their main function is to sample their input, which is usually the output of the sensors measuring the output y, at a given periodic rate T_s^* and to make these samples available to the embedded computer. A basic model for a sampling device consists of a timer state and a sample state. When the timer reaches the value of the sampling time T_s^*, the timer is reset to zero and the sample state is updated with the inputs to the sampling device.

The model for the sampling device we propose has both continuous and discrete dynamics. If the timer state has not reached T_s^*, then the dynamics are such that the timer state increases continuously with a constant, unitary rate. When T_s^* is reached, the timer state is reset to zero and the sample state is mapped to the inputs of the sampling device. To implement this mechanism, we employ a timer state $\tau_s \in \mathbb{R}_{\geq 0}$ and a sample state $m_s \in \mathbb{R}^{r_P}$. The input to the sampling device is denoted by $v_s \in \mathbb{R}^{r_p}$. The model of the sampling devices is

$$\dot{\tau}_s = 1, \quad \dot{m}_s = 0 \qquad \text{when} \quad \tau_s \in [0, T_s^*] \tag{8}$$

$$\tau_s^+ = 0, \quad m_s^+ = v_s \qquad \text{when} \quad \tau_s \geq T_s^* \tag{9}$$

In practice, there exists a time, usually called *the ADC acquisition time*, between the triggering of the ADC with the sampling device and the update of its output. Such a delay limits the number of samples per second that the ADC can provide. Additionally, an ADC can store and process finite-length digital words, which causes quantization. The model above omits effects such as acquisition delays and quantization effects, but those can be incorporated if needed. In particular, quantization effects can be added to the model in (8)-(9) by replacing the update law for m_s to $m_s^+ = \text{round}(v_s)$, where the function round is such that $\text{round}(v_s)$ is the closest number to v_s that the machine precision can represent.

3 *Digital-to-Analog Converters.* The digital signals in the cyber components need to be converted to analog signals for their use in the physical world. Digital-to-analog converters (DACs) perform such a task by converging digital signals into analog equivalents. One of the most common models for a DAC is the zero-order hold model (ZOH). In simple terms, a ZOH converts a digital signal at its input into an analog signal at its output. Its output is updated at discrete time instants, typically periodically, and held constant in between updates, until new information is available at the next sampling time. We will model DACs as ZOH devices with dynamics similar to (8)-(9). Let $\tau_h \in \mathbb{R}_{\geq 0}$ be the timer state, $m_h \in \mathbb{R}^{r_C}$ be the sample state (note that the value of h indicates the number of DACs in the interface), and $v_h \in \mathbb{R}^{r_C}$ be the inputs of the DAC. Its operation is as follows. When $\tau_h \leq 0$, the timer state is reset to τ_r and the sample state is updated with v_h (usually the output of the

embedded computer), where $\tau_r \in [T^{\min}, T^{\max}]$ is a random variable that models the time in-between communication instants and $T^{\min} \leq T^{\max}$. A model that captures this mechanism is given by

$$\dot{\tau}_h = -1, \quad \dot{m}_h = 0 \qquad \text{when} \quad \tau_h \in [T^{\min}, T^{\max}] \tag{10}$$

$$\tau_h^+ = \tau_r, \quad m_h^+ = v_h \qquad \text{when} \quad \tau_h \leq T^{\min} \tag{11}$$

4 *Digital Networks.* The information transfer between the physical and cyber components, or between subsystems within the cyber components, might occur over a digital communication network. The communication links bridging each of these components are not capable of continuously transmitting information, but rather, they can only transmit sampled (and quantized) information at discrete time instants. Combining the ideas in the models of the converters in the previous items, we propose a model of a digital network link that has a variable that triggers the transfer of information provided at its input, and that stores that information until new information arrives. We assume that the transmission of information occurs at instants $\{t_i\}_{i=1}^{i^*}$, $i^* \in \mathbb{N} \cup \{\infty\}$, satisfying

$$T_N^{*\min} \leq t_{i+1} - t_i \leq T_N^{*\max} \qquad \forall i \in \{1, 2, \ldots, i^* - 1\}$$

where $T_N^{*\min}$ and $T_N^{*\max}$ are constants satisfying

$$T_N^{*\min}, T_N^{*\max} \in [0, \infty]$$

and

$$T_N^{*\min} \leq T_N^{*\max}$$

and i^* is the number of transmission events, which might be finite or infinite. The constant $T_N^{*\min}$ determines the minimum possible time in between transmissions while the constant $T_N^{*\max}$ defines the maximum amount of time elapsed between transmissions. In this way, a communication channel that allows transmission events at a high rate would have $T_N^{*\min}$ small (zero for infinitely fast transmissions), while one with slow data rate would have $T_N^{*\min}$ large. The constant $T_N^{*\max}$ determines how often transmissions may take place. Note that the constants $T_N^{*\min}$ and $T_N^{*\max}$ can be generalized to functions so as to change according to other states or inputs.

At every t_i, the information at the input v_N of the communication link is used to update the internal variable m_N, which is accessible at the output end of the network and remains constant between communication events. This internal variable acts as an information buffer, which can contain not only the latest piece of information transmitted but also previously transmitted information. A record of the number of communication events is logged in the internal variable j_N. At communication events, the value of j_N gets updated and remains constant in-between events. A mathematical model capturing the said mechanism is given by

$$\dot{\tau}_N = -1, \quad \dot{m}_N = 0, \quad \dot{j}_N = 0 \qquad \text{when} \quad \tau_N \in [0, T_N^{*\max}] \tag{12}$$

$$\tau_N^+ \in [T_N^{*\min}, T_N^{*\max}], \quad m_N^+ = v_N, \quad j_N^+ = j_N + 1 \qquad \text{when} \quad \tau_N \leq 0 \tag{13}$$

Note that the update law for τ_N at jumps is given in terms of a difference inclusion, which implies that the new value of τ_N is taken from the set $[T_N^{*\min}, T_N^{*\max}]$. The dimension of the states and the input would depend on the type of components that connect to and from it, and also the size of data transmitted and buffered. Similar to the models proposed for conversion, the model of the digital link (12)-(13) does not include delays nor quantization, but such effects can be incorporated if needed.

4.2 The Integrator System

In this section we discuss the internals of the *Integrator System* shown in Figure 7.

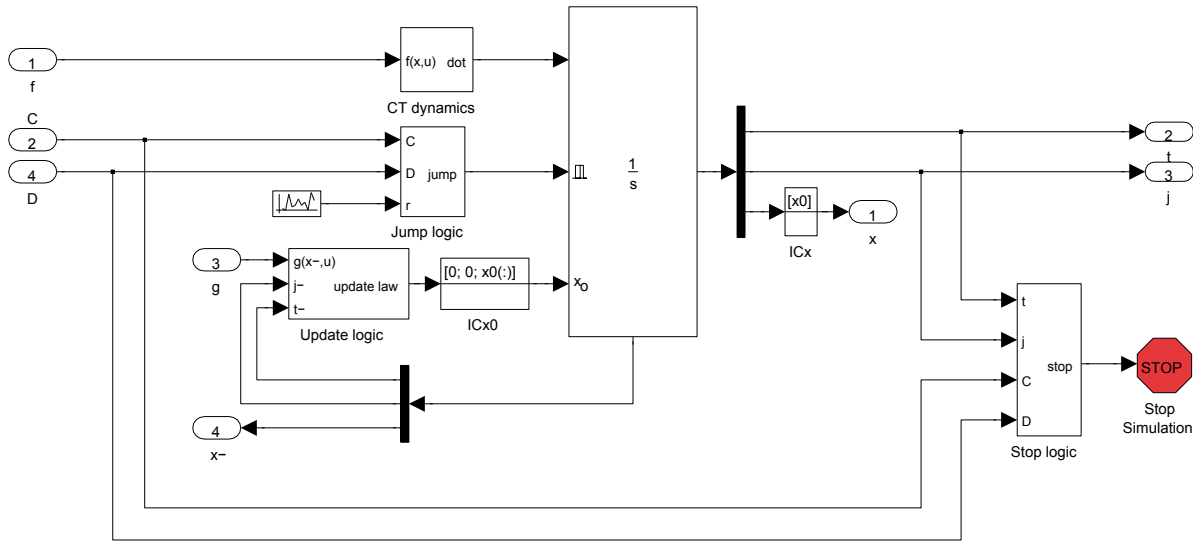


Figure 7: Integrator System

4.2.1 CT Dynamics

This block is shown in Figure 8. It defines the continuous-time (CT) dynamics by assembling the time derivative of the state $[t\ j\ x^\top]^\top$. States t and j are considered states of the system because they need to be updated throughout the simulation in order to keep track of the time and number of jumps. Without t and j, solutions could not be plotted accurately. This is given by

$$\dot{t} = 1, \qquad \dot{j} = 0, \qquad \dot{x} = f(x, u)\ .$$

Note that input port 1 takes the value of $f(x, u)$ through the output of the *Embedded MATLAB function block f* in Figure 5.

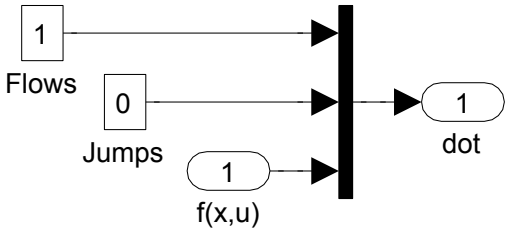


Figure 8: CT dynamics

4.2.2 Jump Logic

This block is shown in Figure 9. The inputs to the jump logic block are the output of the blocks C and D indicating whether the state is in those sets or not, and a random signal with uniform distribution in $[0, 1]$. Figure 9 shows the Simulink blocks used to implement the Jump Logic. The variable *rule* defines whether the simulator gives priority to jumps, priority to flows, or no priority. It is initialized in `initialization.m`.

The output of the Jump Logic is equal to one when:

- the output of the *D block* is equal to one and $rule = 1$,
- the output of the *C block* is equal to zero, the output of the *D block* is equal to one, and $rule = 2$,

- the output of the *C block* is equal to zero, the output of the *D block* is equal to one, and $rule = 3$,
- or the output of the *C block* is equal to one, the output of the *D block* is equal to one, $rule = 3$, and the random signal r is larger or equal than 0.5.

Under these events, the output of this block, which is connected to the integrator external reset input, triggers a reset of the integrator, that is, a jump of $\mathcal{H}$. The reset or jump is activated since the configuration of the reset input is set to "level hold", which executes resets when this external input is equal to one (if the next input remains set to one, multiple resets would be triggered). Otherwise, the output is equal to zero.

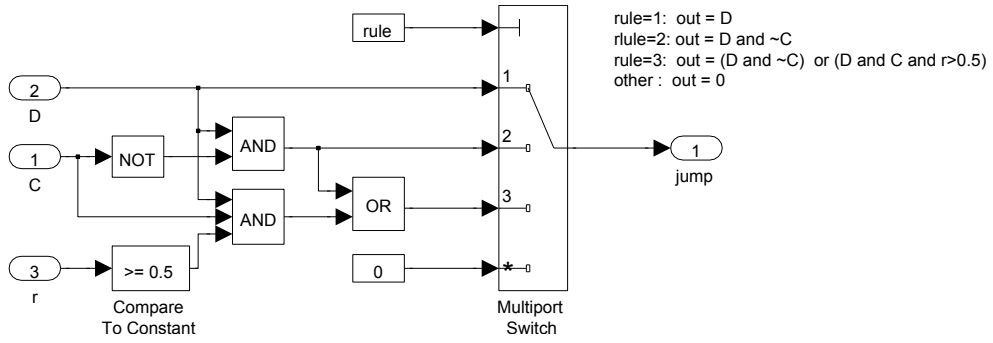


Figure 9: Jump Logic

4.2.3 Update Logic

This block is shown in Figure 10. The update logic uses the *state port* information of the integrator. This port reports the value of the state of the integrator, $[t\ j\ x^\top]^\top$, at the exact instant that the reset condition becomes true. Notice that x^- differs from x since at a jump, x^- indicates the value of the state that triggers the jump, but it is never assigned as the output of the integrator. In other words, "$x \in D$" is checked using x^- and if true, x is reset to $g(x^-, u)$. Notice, however, that u is the same because at a jump, u indicates the next evaluated value of the input, and it is assigned as the output of the integrator. The flow time t is kept constant at jumps and j is incremented by one. More precisely

$$t^+ = t^-, \qquad j^+ = j^- + 1, \qquad x^+ = g(x^-, u)$$

where $[t^-\ j^-\ x^{-\top}]^\top$ is the state that triggers the jump.

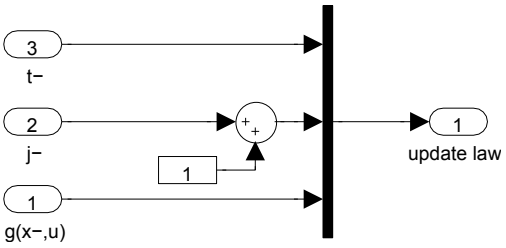


Figure 10: Update Logic

4.2.4 Stop Logic

This block is shown in Figure 11. It stops the simulation under any of the following events:

- The flow time is larger than or equal to the maximum flow time specified by T.

- The jump time is larger than or equal to the maximum number of jumps specified by J.
- The state of the hybrid system x is neither in C nor in D.

Under any of these events, the output of the logic operator connected to the *Stop block* becomes one, stopping the simulation. Note that the inputs C and D are routed from the output of the blocks computing whether the state is in C or D and use the value of x^-.

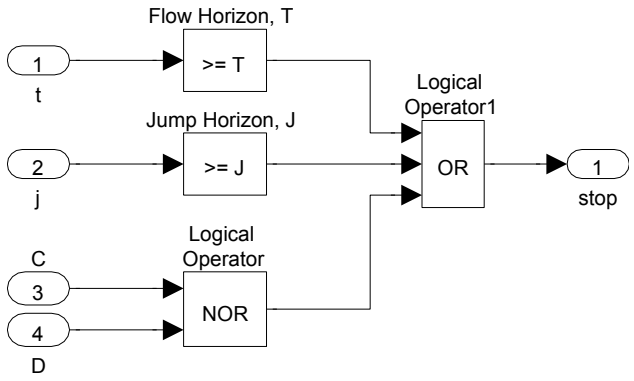


Figure 11: Stop Logic

4.3 Software Requirements

In order to run simulations of single hybrid systems using externally defined functions, MATLAB with Simulink is required.

In order to run simulations using the HyEQ Simulator with embedded MATLAB functions, MATLAB/Simulink and a supported ANSI, C, or C++ 32-bit compiler must be installed. We now briefly describe how to install necessary compilers for Windows and Mac/Linux. For more information on supported compilers, please visit `http://www.mathworks.com/support/compilers/R2012b/win64.html`.

4.3.1 Configuration of HyEQ Simulator with embedded functions for Windows

For 32-bit Windows, the LCC compiler is included with MATLAB. First, open MATLAB and then locate and choose a compiler for building MEX-files by typing

```
>> mex -setup
```

into the MATLAB command window. Then, follow the prompts as shown below.

```
>> mex -setup

Welcome to mex -setup.  This utility will help you set up
a default compiler.  For a list of supported compilers, see
http://www.mathworks.com/support/compilers/R2012a/win32.html

Please choose your compiler for building MEX-files:

Would you like mex to locate installed compilers [y]/n? y

Select a compiler:
[1] Lcc-win32 C 2.4.1

[0] None
```

```
Compiler: 1

Please verify your choices:

Compiler: Lcc-win32 C 2.4.1

Are these correct [y]/n? y

Done . . .
```

For 64-bit Windows, a C-compiler is not supplied with MATLAB. Before running the HyEQ Toolbox in MATLAB/Simulink, please follow the following steps:

1. If you don't have *Microsoft .NET Framework 4* on your computer, download and install it from `http://www.microsoft.com/en-us/download/details.aspx?id=17851`.
2. Then download and install *Microsoft Windows SDK* from `http://www.microsoft.com/en-us/download/details.asp`
3. Then perform the steps outlined above for 32-bit Windows to setup and install the compiler.

As of October 10, 2013, when installing the toolbox in Windows 8, please follow the next steps.

1. If you don't have *Microsoft .NET Framework 4* on your computer, download and install it from `http://www.microsoft.com/en-us/download/details.aspx?id=8279`.
2. Then download and install *Microsoft Windows SDK*
   - If you don't have *Visual C++ 2010 SP1* installed on your computer:
     - Download and install *Microsoft Windows SDK 7.1* from `http://www.microsoft.com/en-us/download/detai`
     - Apply the following patch from *Microsoft* onto the *SDK 7.1* installation: `http://www.microsoft.com/en-us/d`
   - If you have Visual *Visual C++ 2010 SP1* or its redistributable packages installed on your computer:
     - Uninstall the *Visual C++ 2010* redistributable packages, both x64 and x86 versions. This can be done from *Control Panel / Uninstall Programs Menu*.
     - Download and install *Microsoft Windows SDK 7.1* from `http://www.microsoft.com/en-us/download/detai`
     - Apply the following patch from *Microsoft* onto the *SDK 7.1* installation: `http://www.microsoft.com/en-us/d`
     - Reinstall the *Visual C++ 2010* redistributable packages:
       x86 version: `http://www.microsoft.com/en-us/download/details.aspx?id=5555`
       x64 version: `http://www.microsoft.com/en-us/download/details.aspx?id=14632`
3. Then perform the steps outlined above for 32-bit Windows to setup and install the compiler.

4.3.2 Configuration of HyEQ Simulator with embedded functions for Mac/Linux

From a terminal window, check that the file `gcc` is in the folder `/usr/bin`. If it is not there, make a symbolic link. You might require to install the latest version of `Xcode` first. In order to generate a symbolic link for gcc, that MATLAB can find to compile the simulation files (see `http://www.mathworks.com/support/sysreq/previous_release`
change folder to `/usr/bin` and then

```
sudo ln -s gcc gcc-4.2
```

Then, it should be possible to setup the gcc compiler in matlab as follows:

```
>> mex -setup
    Options files control which compiler to use, the compiler and link command
    options, and the runtime libraries to link against.

    Using the 'mexsh -setup' command selects an options file that is
    placed in ~/.matlab/R2013b and used by default for 'mexsh'. An options
    file in the current working directory or specified on the command line
    overrides the default options file in ~/.matlab/R2013b.

    To override the default options file, use the 'mexsh -f' command
    (see 'mexsh -help' for more information).
```

The options files available for MEX are:

```
The options files available for mexsh are:

  1: /Applications/MATLAB_R2013b.app/bin/mexopts.sh :
      Template Options file for building MEX-files


  0: Exit with no changes

Enter the number of the compiler (0-1): 1

Overwrite ~/.matlab/R2013b/mexopts.sh ([y]/n)?: Y

/Applications/MATLAB_R2013b.app/bin/mexopts.sh is being copied to
/SOME_FOLDER/mexopts.sh
```

At this point, it is possible to check if the *gcc* is properly setup by testing any of the Simulink examples with embedded functions (see Figure 4) (e.g., Examples 5.1, 5.2, 5.3, 5.4, 5.5 or 5.6).

If an error regarding "*gmake*" similar to "No supported compiler or SDK was found" and/or a warning "no such sysrooot directory: ′Developer/SDKs/MacOSX10.X.sdk′" is shown when compiling, please consider the following procedure.

A. For Matlab 2013b and previous:

   The compiler can not find the appropriate path. It is necesary to change some lines in the file "*mexopts.sh*" (copied previously in the folder "SOME_FOLDER") .

   First, locate the Xcode-SDK in your hard drive. Open a terminal window and execute the following command

   ```
   xcodebuild -version -sdk macosx10.9 Path
   ```

   which returns the location of MacOSX10.9.sdk, denoted here as SDK_FOLDER. Now, in the MATLAB command window locate the file "*mexopts.sh*" by typing

   ```
   cd /SOME_FOLDER/
   ```

   Then, open the file

   ```
   edit mexopts.sh
   ```

   and edit the lines

- `SDKROOT='/Developer/SDKs/MacOSX10.X.sdk'`
  to
  `SDKROOT='SDK_FOLDER'`
  and
- `MACOSX_DEPLOYMENT_TARGET='10.X'`
  to
  `MACOSX_DEPLOYMENT_TARGET='10.9'`

Now, it is possible to test if the compiler works. If the following error appears "`unknown type name 'char16_t'`," some flags must be changed to avoid this problem. It is required to add `-Dchar16_t=UINT16_T` and `-std=c++11` to the flags CFLAGS and CFLAGS respectively, e.g., change

- `CFLAGS="-fno-common -no-cpp-precomp -arch $ARCHS -isysroot $SDKROOT -mmacosx-version-min=$MACOSX_DEPLOYMENT_TARGET"`
  to
  `CFLAGS="-fno-common -no-cpp-precomp -arch $ARCHS -isysroot $SDKROOT -mmacosx-version-min=$MACOSX_DEPLOYMENT_TARGET -Dchar16_t=UINT16_T"`
  and
- `CXXFLAGS="-fno-common -no-cpp-precomp -fexceptions -arch $ARCHS -isysroot $SDKROOT -mmacosx-version-min=$MACOSX_DEPLOYMENT_TARGET"`
  to
  `CXXFLAGS="-fno-common -no-cpp-precomp -fexceptions -arch $ARCHS -isysroot $SDKROOT -mmacosx-version-min=$MACOSX_DEPLOYMENT_TARGET -std=c++11"`

Finally, restart matlab and test any of the aforementioned Simulink examples. for more information visit `http://www.mathworks.com/matlabcentral/answers/121315-how-to-set-the-c-compiler-of-matlab2013a-in-osx-10-9`

B. For Matlab 2014b and newer.

First, locate the Xcode-SDK in your hard drive. It may be SDK 10.9, 10.10, 10.11, here we are going to denote it as **10.X**. Open a terminal window and execute the following command

```
xcodebuild -version -sdk macosx10.X Path
```

which returns the location of MacOSX10.X.sdk. We are interested in the last portion of the path, specifically after /Applications/Xcode.app/.Contents/Developer/, denoted here as **SDK_FOLDER**. Now, locate and edit the following files

```
/Applications/MATLAB_R201??.app/bin/maci64/mexopts/clang_maci64.xml
```

```
/Applications/MATLAB_R201??.app/bin/maci64/mexopts/clang++_maci64.xml
```

Inside those files there are the lines

```
<dirExists name="$$/Platforms/MacOSX.platform/Developer/SDKs/MacOSX10.10.sdk" />
<cmdReturns name="find $$ -name MacOSX10.10.sdk" />
```

edit (or add below those lines) the lines

- `<dirExists name="$$SDK\_FOLDER" />`
  and
- `<cmdReturns name="find $$ -name MacOSX10.X.sdk" />`

Finally, restart matlab and test any of the aforementioned Simulink examples. for more information visit `https://bitbucket.org/d2d-development/d2d-software/issues/46/xcode-7-on-osx-with-matlab-r2015a-b`

4.4 Configuration of Integration Scheme

Before a simulation is started, it is important to determine the needed integrator scheme, zero-cross detection settings, precision, and other tolerances. Using the default settings does not always give the most efficient or most accurate simulations. One way to edit these settings is to open the Simulink Model, select `Simulation>Configuration Parameters>Solver`, and change the settings there. We have made this simple by defining variables for configuration parameters in the `initialization.m` file. The last few lines of the `initialization.m` file look like that given below.

```
%configuration of solver
RelTol = 1e-8;
MaxStep = .001;
```

In these lines, "RelTol = 1e-8" and "MaxStep = .001" define the relative tolerance and maximum step size of the ODE solver, respectively. These parameters greatly affect the speed and accuracy of solutions.

4.5 Initialization

When the block labeled *Double Click to Initialize* at the top of the Simulink Model is double-clicked, the simulation variables are initialized by calling the script `initialization.m`. The script `initialization.m` defines the initial conditions by defining the initial values of the state components, any necessary parameters, the maximum flow time specified by T, the maximum number of jumps specified by J, and tolerances used when simulating. These can be changed by editing the script file `initialization.m`. See below for sample code to initialize the bouncing ball example, Example 5.1.

```
% initialization for bouncing ball example
clear all
% initial conditions
x0 = [1;0];
% simulation horizon
T = 10;
J = 20;
% rule for jumps
% rule = 1 -> priority for jumps
% rule = 2 -> priority for flows
% rule = 3 -> no priority, random selection when simultaneous conditions
rule = 1;
%configuration of solver
RelTol = 1e-8;
```

It is important to note that variables called in the *Embedded MATLAB function blocks* must be added as inputs and labeled as "parameters". This can be done by opening the *Embedded MATLAB function block* selecting `Tools>Edit Data/Ports` and setting the scope to `Parameter`.

After the block labeled *Double Click to Initialize* is double-clicked and the variables initialized, the simulation is run by clicking the run button or selecting `Simulation>Start`.

4.6 Postprocessing and Plotting solutions

A similar procedure is used to define the plots of solutions after the simulation is run. The solutions can be plotted by double-clicking on the block at the top of the Simulink Model labeled *Double Click to Plot Solutions* which calls the script `postprocessing.m`. The script `postprocessing.m` may be edited to include the desired postprocessing and solution plots. See below for sample code to plot solutions to the bouncing

ball example, Example 5.1. The functions used to generate the plots are described in Section 3.5.

```
1
2 %postprocessing for the bouncing ball example
3 % plot solution
4 figure(1)
5 clf
6 subplot(2,1,1),plotflows(t,j,x)
7 grid on
8 ylabel('x')
9 subplot(2,1,2),plotjumps(t,j,x)
10 grid on
11 ylabel('x')
12 % plot hybrid arc
13 figure(2)
14 plotHybridArc(t,j,x)
15 xlabel('j')
16 ylabel('t')
17 zlabel('x')
18 grid on
19 view(37.5,30)
```

5 Examples

The examples below illustrate the use of the Simulink implementation above.

Example 5.1 (bouncing ball with input) For the simulation of the bouncing ball system with a constant input and regular data given by

$$f(x,u) := \begin{bmatrix} x_2 \\ -\gamma \end{bmatrix}, C := \left\{(x,u) \in \mathbb{R}^2 \times \mathbb{R} \mid x_1 \geq u\right\} \tag{14}$$

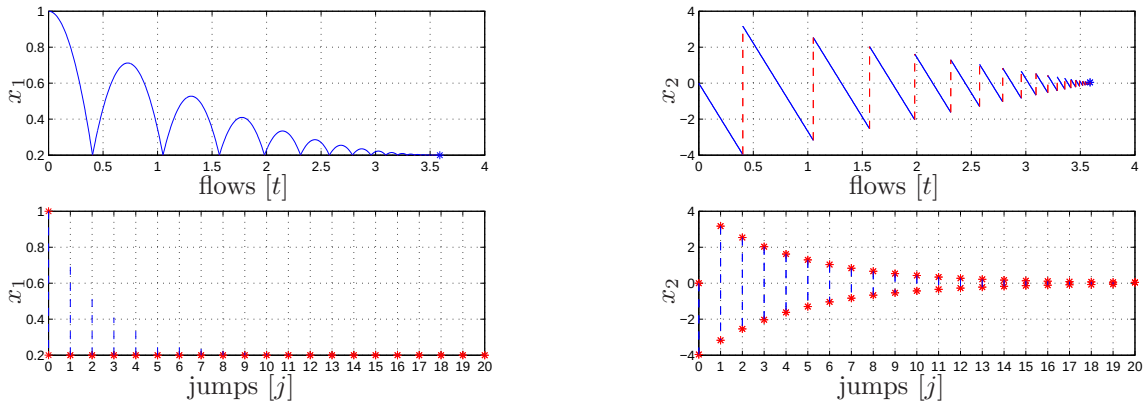
$$g(x,u) := \begin{bmatrix} u \\ -\lambda x_2 \end{bmatrix}, D := \left\{(x,u) \in \mathbb{R}^2 \times \mathbb{R} \mid x_1 \leq u \ , \ x_2 \leq 0\right\} \tag{15}$$

where $\gamma > 0$ is the gravity constant, u is the input constant, and $\lambda \in [0,1)$ is the restitution coefficient. The MATLAB scripts in each of the function blocks of the implementation above are given as follows. An input was chosen to be $u(t,j) = 0.2$ for all (t,j). The constants for the bouncing ball system are $\gamma = 9.81$ and $\lambda = 0.8$.

The following procedure is used to simulate this example using the model in the file `Example_1_2a.slx`:

- `Example_1_2a.slx` is opened in MATLAB/Simulink.
- The *Embedded MATLAB function blocks f, C, g, D* are edited by double-clicking on the block and editing the script. In each embedded function block, parameters must be added as inputs and defined as parameters by selecting `Tools>Edit Data/Ports`, and setting the scope to `Parameter`. For this example, *gamma* and *lambda* are defined in this way.
- The initialization script `initialization.m` is edited by opening the file and editing the script. The flow time and jump horizons, T and J are defined as well as the initial conditions for the state vector, x_0, and input vector, u_0, and a rule for jumps, *rule*.

- The postprocessing script `postprocessing.m` is edited by opening the file and editing the script. Flows and jumps may be plotted by calling the functions *plotflows* and *plotjumps*, respectively. The hybrid arc may be plotted by calling the function *plotHybridArc*.
- The simulation stop time and other simulation parameters are set to the values defined in `initialization.m` by selecting `Simulation>Configuration Parameters>Solver` and inputting T, $RelTol$, $MaxStep$, etc..
- The masked integrator system is double-clicked and the simulation horizons and initial conditions are set as desired.
- The block labeled *Double Click to Initialize* is double-clicked to initialize variables.
- The simulation is run by clicking the run button or selecting `Simulation>Start`.
- The block labeled *Double Click to Plot Solutions* is double-clicked to plot the desired solutions.



(a) Height

(b) Velocity

Figure 12: Solution of Example 5.1

Flow map

```
1
2 function xdot = f_ex1_2a(x)
3 %-------------------------------------------------------------------------
4 % Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
5 % https://hybrid.soe.ucsc.edu/software
6 % http://hybridsimulator.wordpress.com/
7 % Filename: f_ex1_2a.m
8 %-------------------------------------------------------------------------
9 % Project: Simulation of a hybrid system (bouncing ball)
10 % Description: Flow map
11 %-------------------------------------------------------------------------
12 %-------------------------------------------------------------------------
13 %   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
14 %   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
15 %   Copyright @ Hybrid Systems Laboratory (HSL),
16 %   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00
```

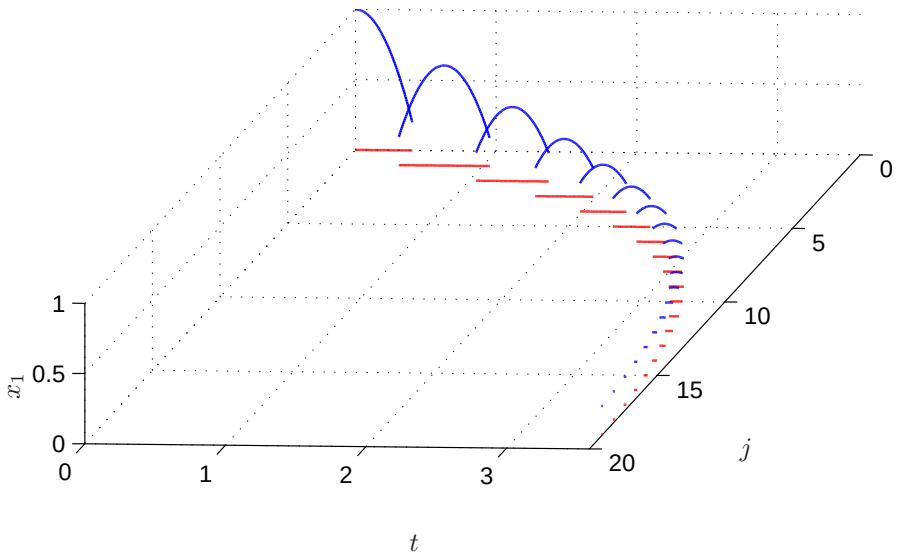


Figure 13: Hybrid arc corresponding to a solution of Example 5.1: height

```
17
18 % state
19 x1 = x(1);
20 x2 = x(2);
21
22 global gamma
23
24 % differential equations
25 xdot = [x2 ; gamma];
26 end
```

Flow set

```
1
2 function [value] = C_ex1_2a(x)
3 %-------------------------------------------------------------------------
4 % Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
5 % https://hybrid.soe.ucsc.edu/software
6 % http://hybridsimulator.wordpress.com/
7 % Filename: C_ex1_2a.m
8 %-------------------------------------------------------------------------
9 % Description: Flow set
10 % Return 0 if outside of C, and 1 if inside C
11 %-------------------------------------------------------------------------
12 %-------------------------------------------------------------------------
13 %   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
14 %   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
```

```
%   Copyright @ Hybrid Systems Laboratory (HSL),
%   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00


x1 = x(1);

if x1 >= 0
    value = 1;
else
    value = 0;
end
end
```

Jump map

```

function xplus = g_ex1_2a(x)
%-------------------------------------------------------------------------
% Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
% https://hybrid.soe.ucsc.edu/software
% http://hybridsimulator.wordpress.com/
% Filename: g_ex1_2a.m
%-------------------------------------------------------------------------
% Project: Simulation of a hybrid system (bouncing ball)
% Description: Jump map
%-------------------------------------------------------------------------
%-------------------------------------------------------------------------
%   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
%   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
%   Copyright @ Hybrid Systems Laboratory (HSL),
%   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00

% state
x1 = x(1);
x2 = x(2);

global lambda

xplus = [-x1 ; -lambda*x2];
end
```

Jump set

```

function inside = D_ex1_2a(x)
%-------------------------------------------------------------------------
% Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
% https://hybrid.soe.ucsc.edu/software
% http://hybridsimulator.wordpress.com/
% Filename: D_ex1_2a.m
%-------------------------------------------------------------------------
% Description: Jump set
% Return 0 if outside of D, and 1 if inside D
%-------------------------------------------------------------------------
%-------------------------------------------------------------------------
```

```
%   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
%   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
%   Copyright @ Hybrid Systems Laboratory (HSL),
%   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00

x1 = x(1);
x2 = x(2);
if (x1 <= 0 && x2 <= 0)
    inside = 1;
else
    inside = 0;
end
end
```

A solution to the bouncing ball system from $x(0,0) = [1,0]^\top$ and with $T = 10, J = 20$, $rule = 1$, is depicted in Figure 12(a) (height) and Figure 12(b) (velocity). Both the projection onto t and j are shown. Figure **??** depicts the corresponding hybrid arc for the position state.

These simulations reflect the expected behavior of the bouncing ball model. Note the only difference between this example and the example of a bouncing ball without a constant input is that, in this example, the ball bounces on a platform at a height of the chosen input value 0.2 rather than the ground at a value of 0.

For MATLAB/Simulink files of this example, see Examples/Example_1.2a.

□

Example 5.2 (alternate way to simulate the bouncing ball)

Consider the bouncing ball system with a constant input and regular data as given in Example 5.1. This example shows that a MATLAB function block, such as the jump set D, can be replaced with operational blocks in Simulink. Figure 14 shows this implementation. The other functions and solutions are the same as in Example 5.1.

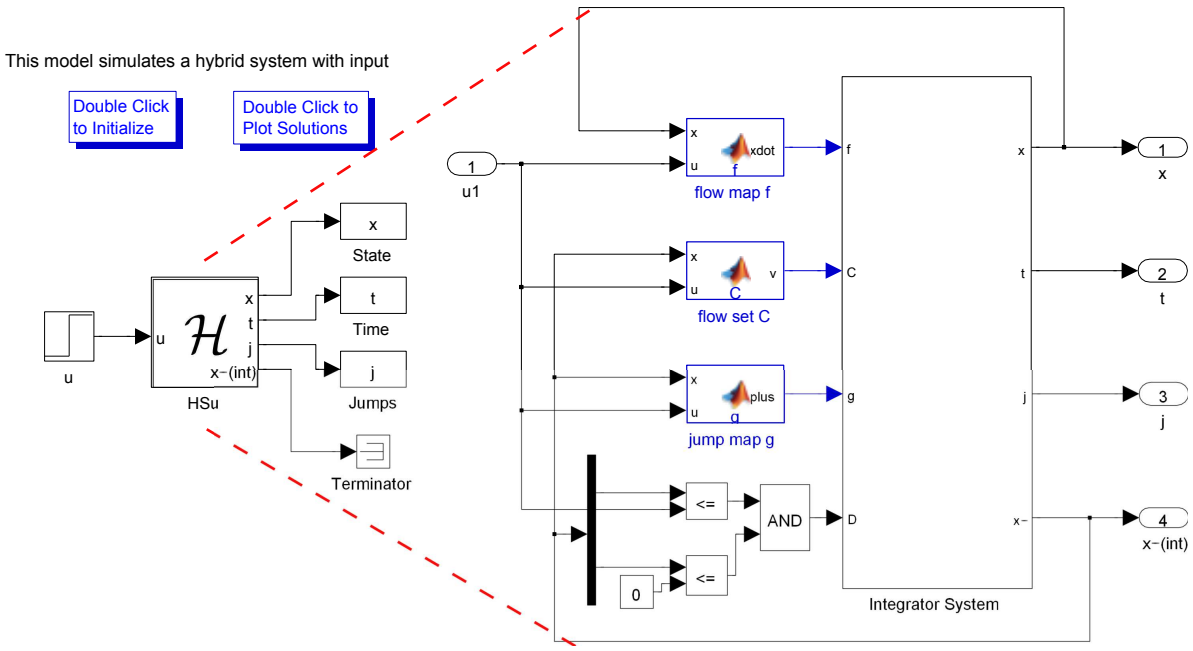


Figure 14: Simulink implementation of bouncing ball example with operator blocks

For MATLAB/Simulink files of this example, see Examples/Example_1.3 and Examples/Example_1.4.

Example 5.3 (vehicle following a track with boundaries) Consider a vehicle modeled by a Dubins vehicle model traveling along a given track with state vector $x = [\xi_1, \xi_2, \xi_3]^\top$ with dynamics given by $\dot{\xi}_1 = u\cos\xi_3$, $\dot{\xi}_2 = u\sin\xi_3$, and $\dot{\xi}_3 = -\xi_3 + r(q)$. The input u is the tangential velocity of the vehicle, ξ_1 and ξ_2 describe the vehicle's position on the plane, and ξ_3 is the vehicle's orientation angle. Also consider a switching controller attempting to keep the vehicle inside the boundaries of a track given by $\{(\xi_1, \xi_2) : -1 \leq \xi_1 \leq 1\}$. A state $q \in \{1, 2\}$ is used to define the modes of operation of the controller. When $q = 1$, the vehicle is traveling to the left, and when $q = 2$, the vehicle is traveling to the right. A logic variable r is defined in order to steer the vehicle back inside the boundary. The state of the closed-loop system is given by $x := [\xi^\top \; q]^\top$. A model of such a closed-loop system is given by

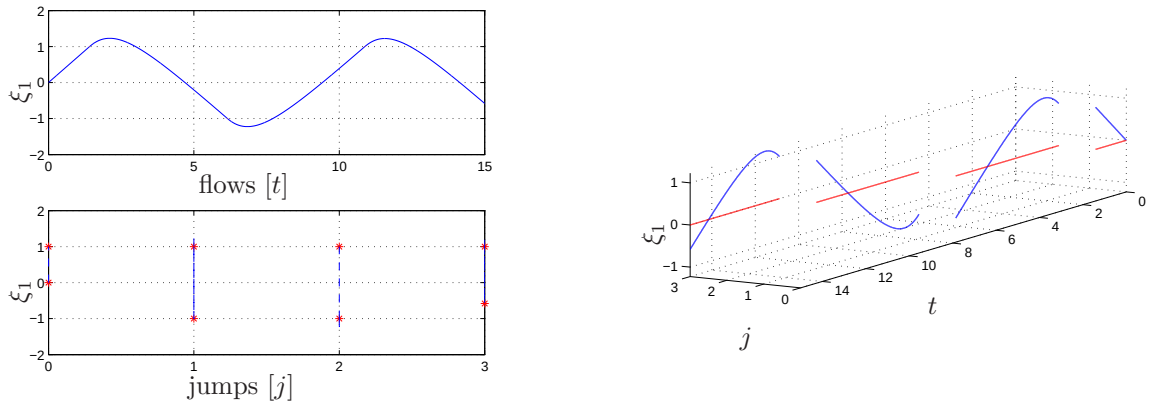
$$f(x,u) := \begin{bmatrix} \begin{bmatrix} u\cos(\xi_3) \\ u\sin(\xi_3) \\ -\xi_3 + r(q) \end{bmatrix} \\ u \end{bmatrix}, \quad r(q) := \begin{cases} \frac{3\pi}{4} & \text{if} \quad q = 1 \\ \frac{\pi}{4} & \text{if} \quad q = 2 \end{cases} \tag{16}$$

$$C := \left\{(\xi, u) \in \mathbb{R}^3 \times \{1,2\} \times \mathbb{R} \mid (\xi_1 \leq 1, q = 2) \text{ or } (\xi_1 \geq -1, q = 1)\right\}, \tag{17}$$

$$g(\xi,u) := \begin{cases} \begin{bmatrix} \xi \\ 2 \end{bmatrix} & \text{if} \quad \xi_1 \leq -1, \quad q = 1 \\ \begin{bmatrix} \xi \\ 1 \end{bmatrix} & \text{if} \quad \xi_1 \geq 1, \quad q = 2 \end{cases}, \tag{18}$$

$$D := \left\{(\xi, u) \in \mathbb{R}^3 \times \{1,2\} \times \mathbb{R} \mid (\xi_1 \geq 1, q = 2) \text{ or } (\xi_1 \leq -1, q = 1)\right\} \tag{19}$$

The MATLAB scripts in each of the function blocks of the implementation above are given as follows. The tangential velocity of the vehicle is chosen to be $u = 1$, the initial position on the plane is chosen to be $(\xi_1, \xi_2) = (0, 0)$, and the initial orientation angle is chosen to be $\xi_3 = \frac{\pi}{4}$ radians.



(a) Trajectory

(b) Hybrid arc

Figure 15: Solution of Example 5.3

Flow map

```
1
2 function xdot = f(x, u)
3
4 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
5 % Matlab Function  Author: Ricardo Sanfelice
6 %
```

```
% Project: Simulation of a hybrid system (Vehicle Traveling on a Track)
%
% Name: f.m
%
% Description: Flow map
%
% Version: 1.0
% Required files: -
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

xtemp = zeros(4,1);
xtemp = x;
x = xtemp;

% state
xi1 = x(1);      %x-position
xi2 = x(2);      %y-position
xi3 = x(3);      %orientation angle
q = x(4);

% q = 1 --> going left
% q = 2 --> going right

if q == 1
    r = 3*pi/4;
elseif q == 2
    r = pi/4;
else
    r = 0;
end

% flow map: xidot=f(xi,u);
xi1dot = u*cos(xi3);  %tangential velocity in x-direction
xi2dot = u*sin(xi3);  %tangential velocity in y-direction
xi3dot = -xi3 + r;      %angular velocity
qdot = 0;

xdot = [xi1dot;xi2dot;xi3dot;qdot];
```

Flow set

```

function v  = C(x, u)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Matlab Function  Author: Ricardo Sanfelice
%
% Project: Simulation of a hybrid system (Vehicle Traveling on a Track)
%
% Name: C.m
%
% Description: Flow set
%
% Version: 1.0
% Required files: -
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

xtemp = zeros(4,1);
xtemp = x;
x = xtemp;

% state
xi1 = x(1);      %x-position
xi2 = x(2);      %y-position
xi3 = x(3);      %orientation angle
q = x(4);

% q = 1 --> going left
% q = 2 --> going right

if ((xi1 < 1) && (q == 2)) || ((xi1 > -1) && (q == 1))  % flow condition
    v = 1;  % report flow
else
    v = 0;   % do not report flow
end
```

Jump map

```

function xplus = g(x, u)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Matlab Function  Author: Ricardo Sanfelice
%
% Project: Simulation of a hybrid system (Vehicle Traveling on a Track)
%
% Name: g.m
%
% Description: Jump map
%
% Version: 1.0
% Required files: -
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

xtemp = zeros(4,1);
xtemp = x;
x = xtemp;
% state
xi1 = x(1);      %x-position
xi2 = x(2);      %y-position
xi3 = x(3);      %orientation angle
q = x(4);

% q = 1 --> going left
% q = 2 --> going right
xi1plus=xi1;
xi2plus=xi2;
xi3plus=xi3;
qplus=q;

```

```
% jump map
if ((xi1 >= 1) && (q == 2)) || ((xi1 <= -1) && (q == 1))
   qplus = 3-q;
else
    qplus = q;
end

xplus = [xi1plus;xi2plus;xi3plus;qplus];
```

Jump set

```

function v  = D(x, u)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Matlab Function  Author: Ricardo Sanfelice
%
% Project: Simulation of a hybrid system (Vehicle Traveling on a Track)
%
% Name: D.m
%
% Description: Jump set
%
% Version: 1.0
% Required files: -
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

xtemp = zeros(4,1);
xtemp = x;
x = xtemp;

% state
xi1 = x(1);      %x-position
xi2 = x(2);      %y-position
xi3 = x(3);      %orientation angle
q = x(4);

% q = 1 --> going left
% q = 2 --> going right

 if ((xi1 >= 1) && (q == 2)) || ((xi1 <= -1) && (q == 1))  % jump condition
     v = 1;  % report jump
 else
     v = 0; % do not report jump
 end
```

A solution to the system of a vehicle following a track in $\{(\xi_1, \xi_2) : -1 \leq \xi_1 \leq 1\}$, and with $T = 15, J = 10$, $rule = 1$, is depicted in Figure 15(a) (trajectory). Both the projection onto t and j are shown. Figure 15(b) depicts the corresponding hybrid arc.

For MATLAB/Simulink files of this example, see Examples/Example_1.5.

□

Example 5.4 (interconnection of hybrid systems $\mathcal{H}_1$ (bouncing ball) and $\mathcal{H}_2$ (moving platform)) Consider a bouncing ball ($\mathcal{H}_1$) bouncing on a platform ($\mathcal{H}_2$) at some initial height and converging to the ground at zero height. This is an interconnection problem because the current states of each system affect the behavior

of the other system. In this interconnection, the bouncing ball will contact the platform, bounce back up, and cause a jump in height of the platform so that it gets closer to the ground. After some time, both the ball and the platform will converge to the ground. In order to model this system, the output of the bouncing ball becomes the input of the moving platform, and vice versa. For the simulation of the described system with regular data where $\mathcal{H}_1$ is given by

$$f_1(\xi, u_1, v_1) := \begin{bmatrix} \xi_2 \\ -\gamma - b\xi_2 + v_{11} \end{bmatrix}, C_1 := \{(\xi, u_1) \mid \xi_1 \geq u_1, u_1 \geq 0\} \tag{20}$$

$$g_1(\xi, u_1, v_1) := \begin{bmatrix} \xi_1 + \alpha_1 \xi_2^2 \\ e_1|\xi_2| + v_{12} \end{bmatrix}, D_1 := \{(\xi, u_1) \mid \xi_1 = u_1, u_1 \geq 0\}, y_1 = h_1(\xi) := \xi_1 \tag{21}$$

where $\gamma, b, \alpha_1 > 0, e_1 \in [0, 1)$, $\xi = [\xi_1, \xi_2]^\top$ is the state, $y_1 \in \mathbb{R}$ is the output, $u_1 \in \mathbb{R}$ and $v_1 = [v_{11}, v_{12}]^\top \in \mathbb{R}^2$ are the inputs, and the hybrid system $\mathcal{H}_2$ is given by

$$f_2(\eta, u_2, v_2) := \begin{bmatrix} \eta_2 \\ -\eta_1 - 2\eta_2 + v_{12} \end{bmatrix}, C_2 := \{(\eta, u_2) \mid \eta_1 \leq u_2, \eta_1 \geq 0\} \tag{22}$$

$$g_2(\eta, u_2, v_2) := \begin{bmatrix} \eta_1 - \alpha_2|\eta_2| \\ -e_2|\eta_2| + v_{22} \end{bmatrix}, D_2 := \{(\eta, u_2) \mid \eta_1 = u_2, \eta_1 \geq 0\}, y_2 = h_2(\eta) := \eta_1 \tag{23}$$

where $\alpha_2 > 0, e_2 \in [0, 1)$, $\eta = [\eta_1, \eta_2]^\top \in \mathbb{R}^2$ is the state, $y_2 \in \mathbb{R}$ is the output, and $u_2 \in \mathbb{R}$ and $v_2 = [v_{21}, v_{22}]^\top \in \mathbb{R}^2$ are the inputs.

Therefore, the interconnection may be defined by the input assignment

$$u_1 = y_2, \qquad u_2 = y_1. \tag{24}$$

The signals v_1 and v_2 are included as external inputs in the model in order to simulate the effects of environmental perturbations, such as a wind gust, on the system.

The MATLAB scripts in each of the function blocks of the implementation above are given as follows. The constants for the interconnected system are $\gamma = 0.8$, $b = 0.1$, and $\alpha_1, \alpha_2 = 0.1$.

- For hybrid system $\mathcal{H}_1$:

  Flow map

```
1
2 function xdot = f(x, u)
3
4 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
5 % Matlab Function  Author: Ricardo Sanfelice
6 %
7 % Project: Simulation of a hybrid system (interconnection)
8 %
9 % Name: f.m
10 %
11 % Description: Flow map
12 %
13 % Version: 1.0
14 % Required files: -
15 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
16
17 % state
18 xi1 = x(1);
19 xi2 = x(2);
20
```

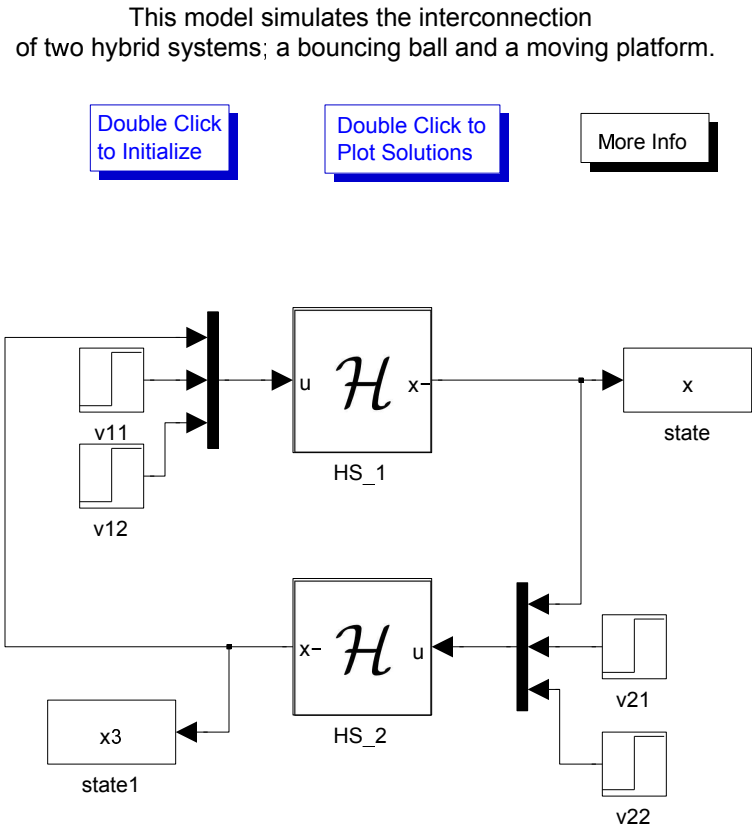


Figure 16: MATLAB/Simulink implementation of interconnected hybrid systems $\mathcal{H}_1$ and $\mathcal{H}_2$

```
%input
y2 = u(1);
v11 = u(2);
v12 = u(3);

% flow map
%xdot=f(x,u);
xi1dot = xi2;
xi2dot = -0.8-0.1*xi2+v11;

xdot = [xi1dot;xi2dot];
```

Flow set

```

function v  = C(x, u)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Matlab Function  Author: Ricardo Sanfelice
%
% Project: Simulation of a hybrid system (interconnection)
%
% Name: C.m
%
```

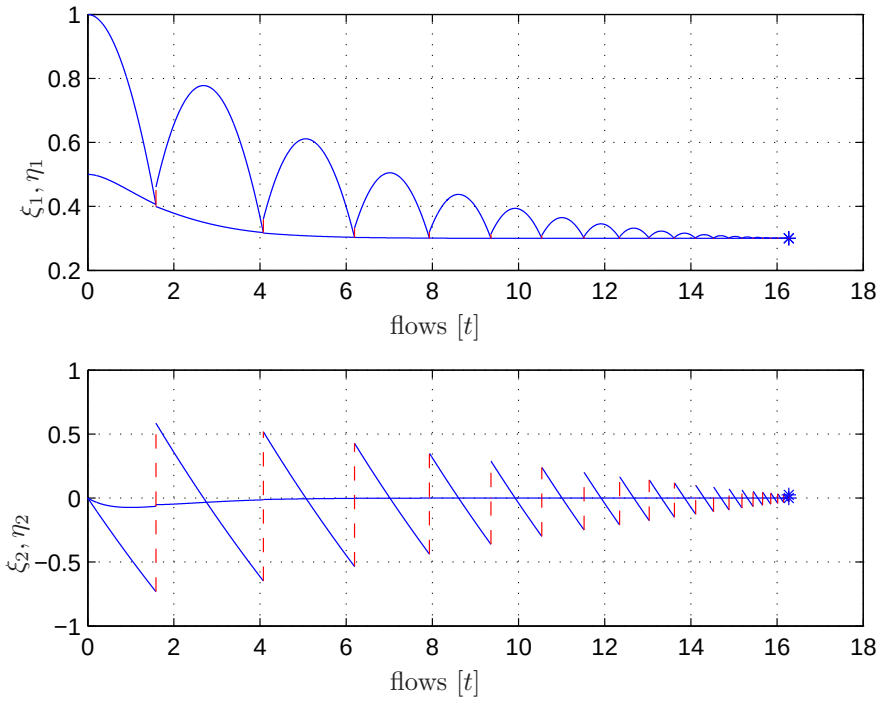
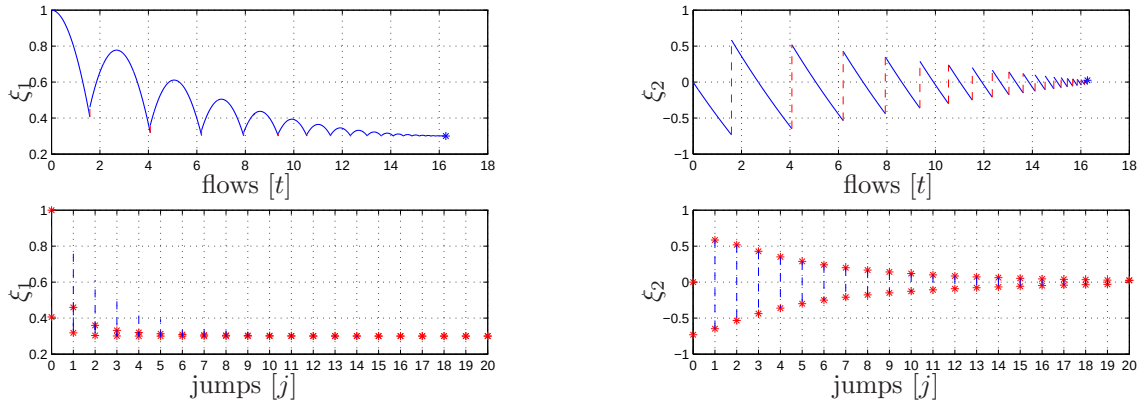


Figure 17: Solution of Example 5.4: height and velocity

```
% Description: Flow set
%
% Version: 1.0
% Required files: -
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% state
xi1 = x(1);
xi2 = x(2);

%input
y2 = u(1);
v11 = u(2);
v12 = u(3);

if (xi1 >= y2)  % flow condition
    v = 1;  % report flow
else
    v = 0;   % do not report flow
end
```

Jump map

```

function xplus = g(x, u)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```



(a) Height

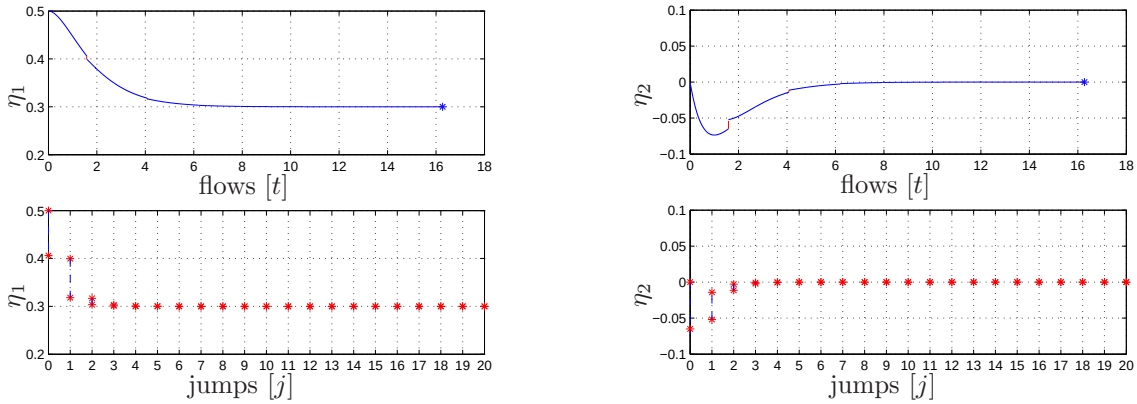
(b) Velocity

Figure 18: Solution of Example 5.4 for system $\mathcal{H}_1$

```
% Matlab Function  Author: Ricardo Sanfelice
%
% Project: Simulation of a hybrid system (interconnection)
%
% Name: g.m
%
% Description: Jump map
%
% Version: 1.0
% Required files: -
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% state
xi1 = x(1);
xi2 = x(2);

%input
y2 = u(1);
v11 = u(2);
v12 = u(3);

xi1plus=y2+0.1*xi2^2;
xi2plus=0.8*abs(xi2)+v12;

xplus = [xi1plus;xi2plus];
```

Jump set

```

function v  = D(x, u)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Matlab Function  Author: Ricardo Sanfelice
%
```



(a) Height

(b) Velocity

Figure 19: Solution of Example 5.4 for system $\mathcal{H}_2$

```
% Project: Simulation of a hybrid system (interconnection)
%
% Name: D.m
%
% Description: Jump set
%
% Version: 1.0
% Required files: -
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% state
xi1 = x(1);
xi2 = x(2);

%input
y2 = u(1);
v11 = u(2);
v12 = u(3);


if (xi1 <= y2) % jump condition
    v = 1;  % report jump
else
    v = 0;   % do not report jump
end
```

- For hybrid system $\mathcal{H}_2$:

  Flow map

```

function xdot = f(x, u)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Matlab Function  Author: Ricardo Sanfelice
```

```
%
% Project: Simulation of a hybrid system (interconnection)
%
% Name: f.m
%
% Description: Flow map
%
% Version: 1.0
% Required files: -
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% state
eta1 = x(1);
eta2 = x(2);

%input
y1 = u(1);
v21 = u(2);
v22 = u(3);

% flow map
eta1dot = eta2;
eta2dot = -eta1-2*eta2+v21;

xdot = [eta1dot;eta2dot];
```

Flow set

```
function v  = C(x, u)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Matlab Function  Author: Ricardo Sanfelice
%
% Project: Simulation of a hybrid system (interconnection)
%
% Name: C.m
%
% Description: Flow set
%
% Version: 1.0
% Required files: -
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% state
eta1 = x(1);
eta2 = x(2);

%input
y1 = u(1);
v21 = u(2);
v22 = u(3);

if (eta1 <= y1)  % flow condition
```

```
    v = 1;  % report flow
else
    v = 0;   % do not report flow
end
```

Jump map

```

function xplus = g(x, u)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Matlab Function  Author: Ricardo Sanfelice
%
% Project: Simulation of a hybrid system (interconnection)
%
% Name: g.m
%
% Description: Jump map
%
% Version: 1.0
% Required files: -
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% state
eta1 = x(1);
eta2 = x(2);

%input
y1 = u(1);
v21 = u(2);
v22 = u(3);

% jump map
eta1plus = y1-0.1*abs(eta2);
eta2plus = -0.8*abs(eta2)+v22;

xplus = [eta1plus;eta2plus];
```

Jump set

```

function v  = D(x, u)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Matlab Function  Author: Ricardo Sanfelice
%
% Project: Simulation of a hybrid system
%
% Name: D.m
%
% Description: Jump set
%
% Version: 1.0
% Required files: -
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
16
17 % state
18 eta1 = x(1);
19 eta2 = x(2);
20
21 %input
22 y1 = u(1);
23 v21 = u(2);
24 v22 = u(3);
25
26
27 if (eta1 >= y1) % jump condition
28     v = 1;  % report jump
29 else
30     v = 0;   % do not report jump
31 end
```

A solution to the interconnection of hybrid systems $\mathcal{H}_1$ and $\mathcal{H}_2$ with $T = 18, J = 20, rule = 1$, is depicted in Figure 17. Both the projection onto t and j are shown. A solution to the hybrid system $\mathcal{H}_1$ is depicted in Figure 18(a) (height) and Figure 18(b) (velocity). A solution to the hybrid system $\mathcal{H}_2$ is depicted in Figure 19(a) (height) and Figure 19(b) (velocity).

These simulations reflect the expected behavior of the interconnected hybrid systems.

For MATLAB/Simulink files of this example, see Examples/Example_1.6.

□

Example 5.5 (biological example: synchronization of two fireflies) Consider a biological example of the synchronization of two fireflies flashing. The fireflies can be modeled mathematically as periodic oscillators which tend to synchronize their flashing until they are flashing in phase with each other. A state value of $\tau_i = 1$ corresponds to a flash, and after each flash, the firefly automatically resets its internal timer (periodic cycle) to $\tau_i = 0$. The synchronization of the fireflies can be modeled as an interconnection of two hybrid systems because every time one firefly flashes, the other firefly notices and jumps ahead in its internal timer τ by $(1 + \varepsilon)\tau$, where ε is a biologically determined coefficient. This happens until eventually both fireflies synchronize their internal timers and are flashing simultaneously. Each firefly can be modeled as a hybrid

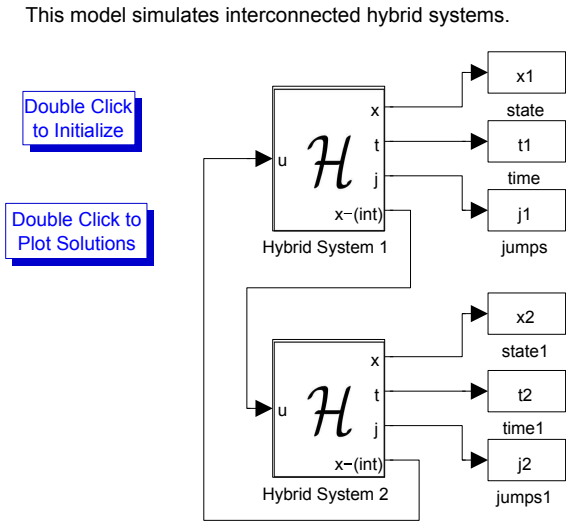
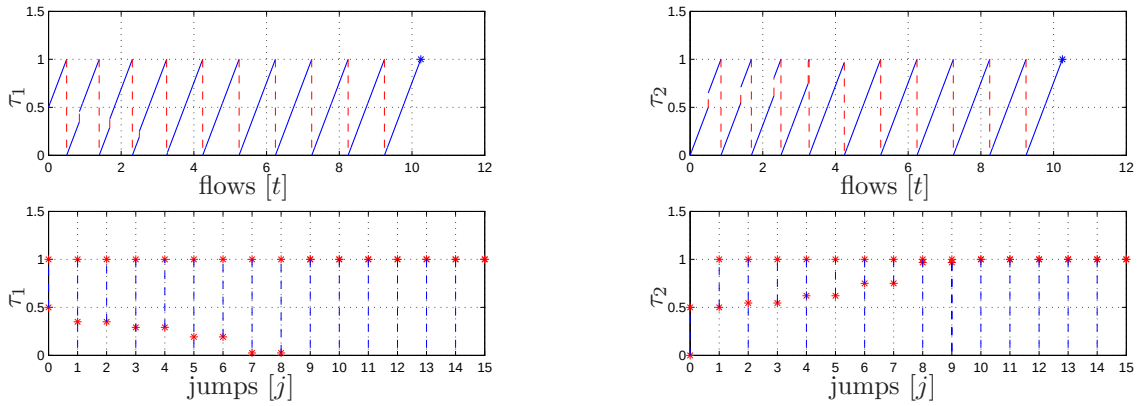


Figure 20: Interconnection Diagram for Example 5.5

system given by

$$
\begin{aligned}
f_i(\tau_i, u_i) &:= 1, && (25)\\
C_i &:= \left\{(\tau_i, u_i) \in \mathbb{R}^2 \mid 0 \leq \tau_i \leq 1\right\} \cap \left\{(\tau_i, u_i) \in \mathbb{R}^2 \mid 0 \leq u_i \leq 1\right\} && (26)\\
g_i(\tau_i, u_i) &:= \begin{cases} (1+\varepsilon)\tau_i & (1+\varepsilon)\tau_i < 1 \\ 0 & (1+\varepsilon)\tau_i \geq 1 \end{cases} && (27)\\
D_i &:= \left\{(\tau_i, u_i) \in \mathbb{R}^2 \mid \tau_i = 1\right\} \cup \left\{(\tau_i, u_i) \in \mathbb{R}^2 \mid u_i = 1\right\}. && (28)
\end{aligned}
$$

The interconnection diagram for this example is simpler than in the previous example because now no external inputs are being considered. The only event that affects the flashing of a firefly is the flashing of the other firefly. The interconnection diagram can be seen in Figure 20.



(a) Solution for system $\mathcal{H}_1$

(b) Solution for system $\mathcal{H}_2$

Figure 21: Solution of Example 5.5

For hybrid system $\mathcal{H}_i$, $i = 1, 2$:
Flow map

```
1
2 function taudot = f(tau, u)
3
4 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
5 % Matlab Function  Author: Ricardo Sanfelice
6 %
7 % Project: Simulation of a hybrid system (interconnection)
8 %
9 % Name: f.m
10 %
11 % Description: Flow map
12 %
13 % Version: 1.0
14 % Required files: -
15 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
16
17 % flow map
18 taudot = 1;
```

Flow set

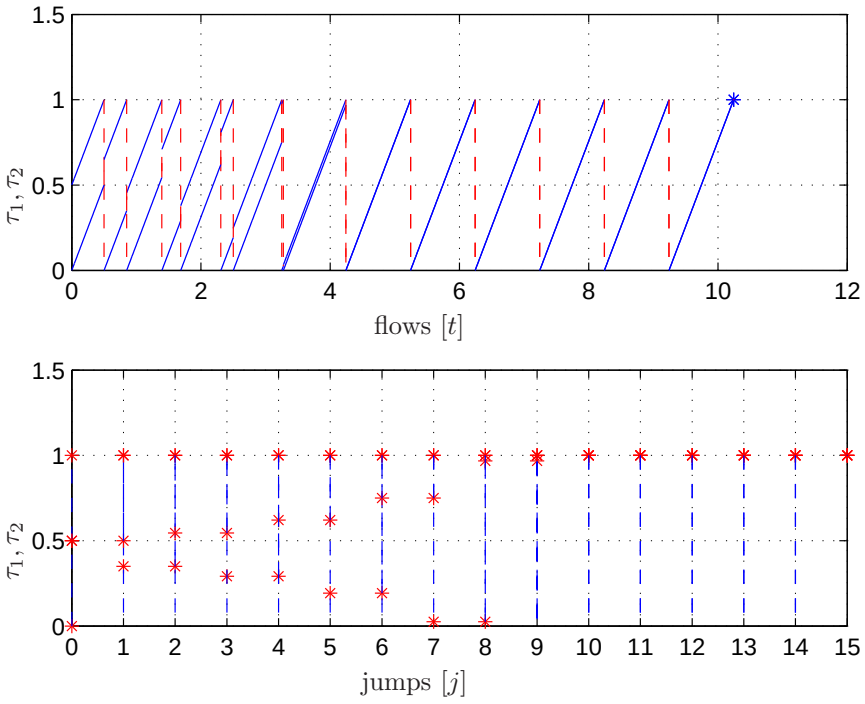


Figure 22: Solution of Example 5.5 for interconnection of $\mathcal{H}_1$ and $\mathcal{H}_2$

```
1
2 function v  = C(tau, u)
3
4 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
5 % Matlab Function  Author: Ricardo Sanfelice
6 %
7 % Project: Simulation of a hybrid system (interconnection)
8 %
9 % Name: C.m
10 %
11 % Description: Flow set
12 %
13 % Version: 1.0
14 % Required files: -
15 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
16
17 %flow set
18 if ((tau > 0) && (tau < 1)) || ((u > 0) && (u <= 1))  % flow condition
19     v = 1;  % report flow
20 else
21     v = 0;  % do not report flow
22 end
```

Jump map

```
1
2 function tauplus = g(tau, u, e)
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Matlab Function  Author: Ricardo Sanfelice
%
% Project: Simulation of a hybrid system (interconnection)
%
% Name: g.m
%
% Description: Jump map
%
% Version: 1.0
% Required files: -
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% jump map
if (1+e)*tau < 1
    tauplus = (1+e)*tau;
elseif (1+e)*tau >= 1
    tauplus = 0;
else
    tauplus = tau;
end
```

Jump set

```
function v  = D(tau, u)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Matlab Function  Author: Ricardo Sanfelice
%
% Project: Simulation of a hybrid system (interconnection)
%
% Name: D.m
%
% Description: Jump set
%
% Version: 1.0
% Required files: -
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

if (u >= 1) || (tau >= 1) % jump condition
    v = 1;  % report jump
else
    v = 0;  % do not report jump
end
```

A solution to the interconnection of hybrid systems $\mathcal{H}_1$ and $\mathcal{H}_2$ with $T = 15, J = 15$, $rule = 1$, $\varepsilon = 0.3$ is depicted in Figure 22. Both the projection onto t and j are shown. A solution to the hybrid system $\mathcal{H}_1$ is depicted in Figure 21(a). A solution to the hybrid system $\mathcal{H}_2$ is depicted in Figure 21(b).

These simulations reflect the expected behavior of the interconnected hybrid systems. The fireflies initially flash out of phase with one another and then synchronize to flash in the same phase.

For MATLAB/Simulink files of this example, see Examples/Example_1.7.

□

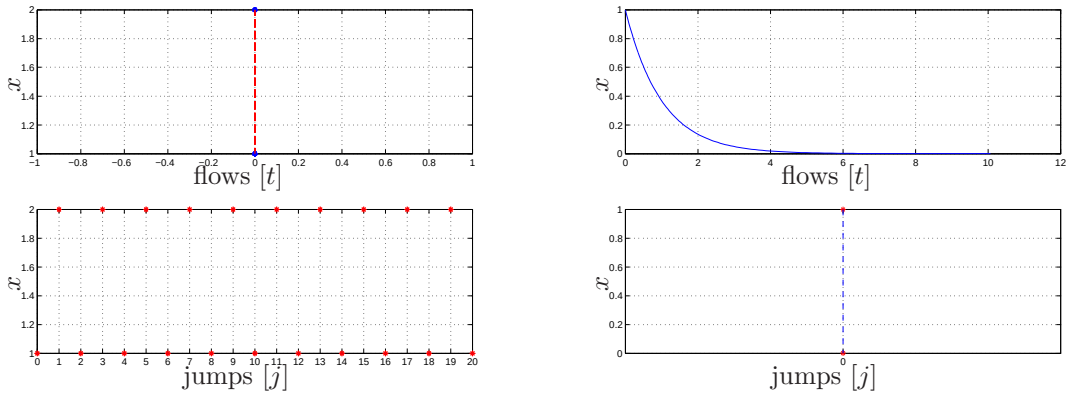
Example 5.6 (a simple mathematical example to show different type of simulation results) Consider the hybrid system with data

$$f(x) := -x,\ C := [0,1],\ g(x) := 1 + \text{mod}(x,2),\ D := \{1\} \cup \{2\}.$$

Note that solutions from $\xi = 1$ and $\xi = 2$ are nonunique. The following simulations show the use of the variable *rule* in the *Jump Logic block*.

Jumps enforced: A solution from $x0 = 1$ with $T = 10, J = 20$, $rule = 1$ is depicted in Figure 23(a). The solution jumps from 1 to 2, and from 2 to 1 repetitively.

Flows enforced: A solution from $x0 = 1$ with $T = 10, J = 20$, $rule = 2$ is depicted in Figure 23(b). The solution flows for all time and converges exponentially to zero.



(a) Forced jumps logic.

(b) Forced flows logic.

Figure 23: Solution of Example 5.6

Random rule: A solution from $x0 = 1$ with $T = 10, J = 20$, $rule = 3$ is depicted in Figure 24(a). The solution jumps to 2, then jumps to 1 and flows for the rest of the time converging to zero exponentially. Enlarging D to

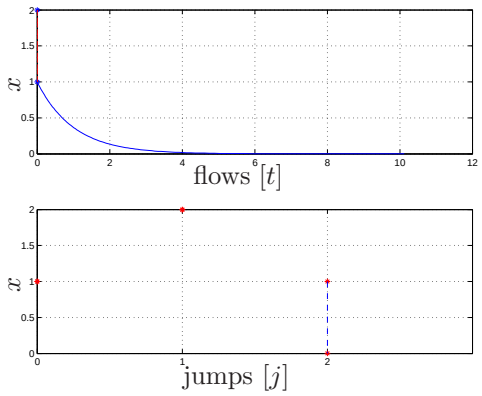
$$D := [1/50, 1] \cup \{2\}$$

causes the overlap between C and D to be "thicker". The simulation result is depicted in Figure 24(b) with the same parameters used in the simulation in Figure 24(a). The plot suggests that the solution jumps several times until $x < 1/50$ from where it flows to zero. However, Figure 24(c), a zoomed version of Figure 24(b), shows that initially the solution flows and that at $(t, j) = (0.2e-3, 0)$ it jumps. After the jump, it continues flowing, then it jumps a few times, then it flows, etc. The combination of flowing and jumping occurs while the solution is in the intersection of C and D, where the selection of whether flowing or jumping is done randomly due to using $rule = 3$.

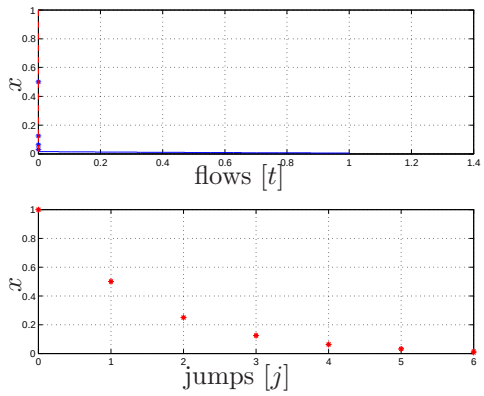
This simulation also reveals that this implementation does not precisely generate hybrid arcs. The maximum step size was set to $0.1e-3$. The solution flows during the first two steps of the integration of the flows with maximum step size. The value at $t = 0.1e-3$ is very close to 1. At $t = 0.2e-3$, instead of assuming a value given by the flow map, the value of the solution is about 0.5, which is the result of the jump occurring at $(0.2e-3, 0)$. This is the value stored in x at such time by the integrator. Note that the value of x' at $(0.2e-3, 0)$ is the one given by the flow map that triggers the jump, and if available for recording, it should be stored in $(0.2e-3, 0)$. This is a limitation of the current implementation.

The following simulations show the *Stop Logic block* stopping the simulation at different events.

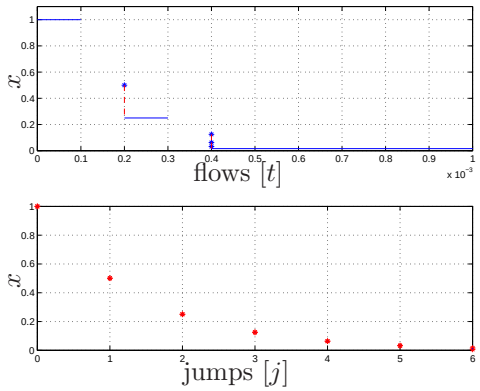
Solution outside $C \cup D$: Taking $D = \{1\}$, a simulation starting from $x0 = 1$ with $T = 10, J = 20$, $rule = 1$ stops since the solution leaves $C \cup D$. Figure 25(a) shows this.



(a) Random logic for flowing/jumping.



(b) Random logic for flowing/jumping.



(c) Random logic for flowing/jumping. Zoomed version.

Figure 24: Solution of Example 5.6

Solution reaches the boundary of C from where jumps are not possible: Replacing the flow set by $[1/2, 1]$ a solution starting from $x0 = 1$ with $T = 10, J = 20$ and $rule = 2$ flows for all time until it reaches the boundary of C where jumps are not possible. Figure 25(b) shows this.

Note that in this implementation, the Stop Logic is such that when the state of the hybrid system is not in $(C \cup D)$, then the simulation is stopped. In particular, if this condition becomes true while flowing, then the last value of the computed solution will not belong to C. It could be desired to be able to recompute the solution so that its last point belongs to the corresponding set. From that point, it should be the case that solutions cannot be continued.

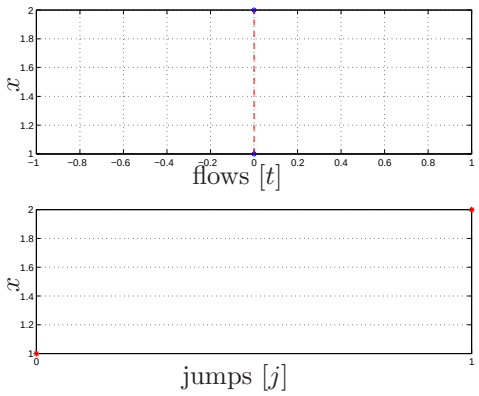
For MATLAB/Simulink files of this example, see Examples/Example_1.8.

□

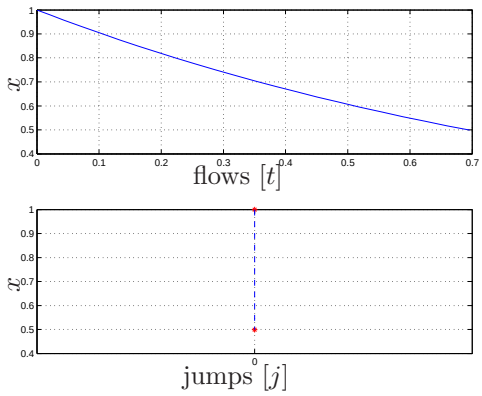
Example 5.7 (Linear time-invariant plant)

Following the model of a physical component in (4), a linear time-invariant model of the physical component is defined by

$$f(x,u) = f_P(x,u) = A_P x + B_P u, \qquad h(x,u) = h(x,u) = M_P x + N_P u$$



(a) Forced jump logic and different D.



(b) Forced flow logic.

Figure 25: Solution of Example 5.6 with premature stopping.

where A_P, B_P, M_P, and N_P are matrices of appropriate dimensions. State and input constraints can directly be embedded into the set C_P. For example, the constraint that x has all of its components nonnegative and that u has its components with norm less or equal than one is captured by

$$\begin{aligned} C_P &= \{(x,u) \in \mathbb{R}^{n_P} \times \mathbb{R}^{m_P} \mid x_i \geq 0 \ \forall i \in \{1,2,\ldots,n_P\}\} \\ &\quad \cap \{(x,u) \in \mathbb{R}^{n_P} \times \mathbb{R}^{m_P} \mid |u_i| \leq 1 \ \forall i \in \{1,2,\ldots,m_P\}\} \end{aligned}$$

For example, the evolution of the temperature of a room with a heater can be modeled by a linear-time invariant system with state x denoting the temperature of the room and with input $u = (u_1, u_2)$, where u_1 denotes whether the heater is turned on ($u_1 = 1$) or turned off ($u_1 = 0$) while u_2 denotes the temperature outside the room. The evolution of the temperature is given by

$$\dot{x} = -x + \begin{bmatrix} z_\Delta & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \text{when } (x,u) \in C_P = \{(x,u) \in \mathbb{R} \times \mathbb{R}^2 \mid u_1 \in \{0,1\}\} \tag{29}$$

where z_Δ is a constant representing the heater capacity. ◘

Example 5.8 (Implementation of a FSM)

Transitions of the state of a FSM are triggered by changes of its input v. The system can be modeled as a cascade of two systems, in which an external signal drives the FSM. The FSM is modeled as the difference equation in (7). Assuming that the input feeds the input u of the FSM, the state of the FSM is updated according to the transition function evaluated at the current input, that is, q is updated according to

$$q^+ = \delta(q,u)$$

The above model can be summarized as follows:

$$f(q,u) := \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{30}$$

$$C := \{(q,u) \in \{1,2\} \times \{1,2\} \times \{1,2\} \times \{1,2\} \mid \delta(q,u) = q\}, \tag{31}$$

$$g(q,u) := \delta(q,u) = \begin{bmatrix} 3-u_1 \\ 3-u_2 \end{bmatrix}, \tag{32}$$

$$D := \{(q,u) \in \{1,2\} \times \{1,2\} \times \{1,2\} \times \{1,2\} \mid \delta(q,u) \in \{1,2\} \times \{1,2\} \setminus q\}, \tag{33}$$

$$y := h(q) = q \tag{34}$$

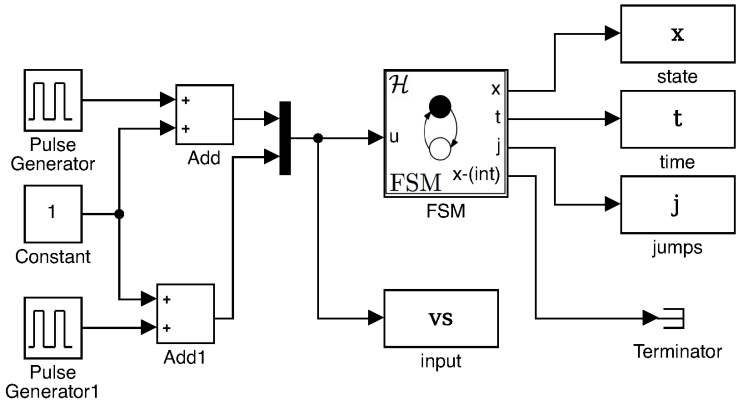


Figure 26: Simulation of a Finite State Machine (FSM) in Example 5.8.

where the input and the state are given by $u = (u_1, u_2) \in \{1, 2\} \times \{1, 2\}$, and $q = (q_1, q_2) \in \{1, 2\} \times \{1, 2\}$, respectively.

For the hybrid system FSM in Figure 26 we have the following Matlab embedded functions that describe the sets C and D and functions f and g. Figure 27 depicts the corresponding inputs and state of the FSM.

Flow map

```

function xdot = f(x, u)
%-------------------------------------------------------------------------
% Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
% https://hybrid.soe.ucsc.edu/software
% http://hybridsimulator.wordpress.com/
%-------------------------------------------------------------------------
% Project: Simulation of a hybrid system Finite state machine (FSM)
% Description: Flow map
%-------------------------------------------------------------------------
%-------------------------------------------------------------------------
%   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
%   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
%   Copyright @ Hybrid Systems Laboratory (HSL),
%   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00
%-------------------------------------------------------------------------
% flow map: xdot=f(x,u);

n = length(x); % size of the q state
qdot = zeros(n,1); % dynamics of the q state
xdot = [qdot];
```

Flow set

```

function v  = C(x, u)
%-------------------------------------------------------------------------
% Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
% https://hybrid.soe.ucsc.edu/software
% http://hybridsimulator.wordpress.com/
%-------------------------------------------------------------------------
```

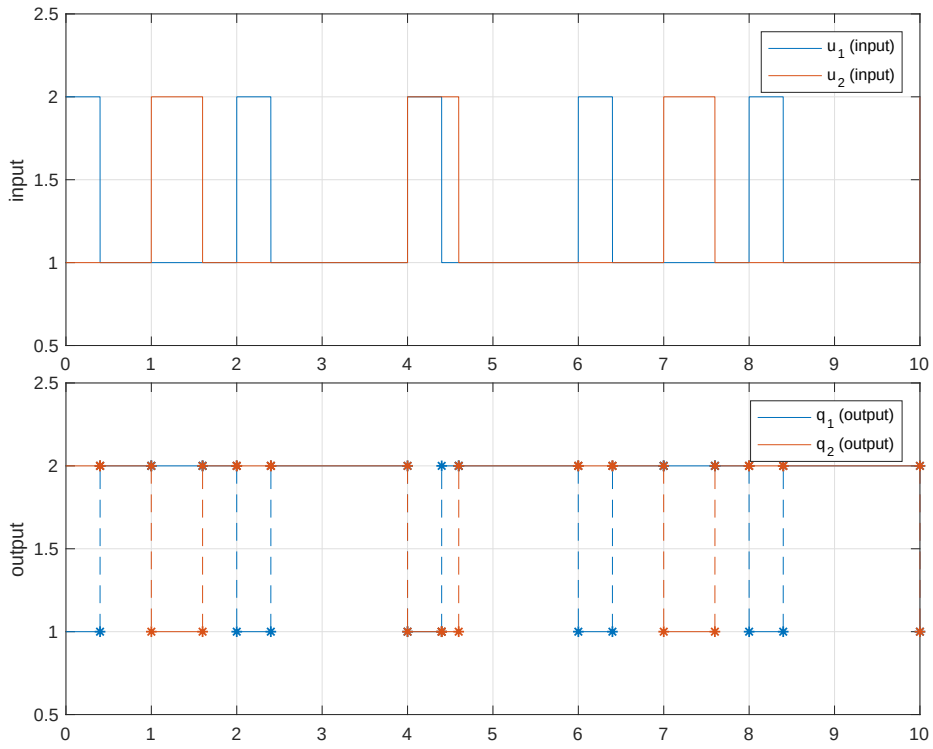


Figure 27: Finite state machine states and inputs.

```
% Project: Simulation of a hybrid system Finite state machine (FSM)
% Description: Flow set
%-------------------------------------------------------------------------
%-------------------------------------------------------------------------
%   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
%   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
%   Copyright @ Hybrid Systems Laboratory (HSL),
%   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00
%
% Check on flow conditions
% E.g.,
% if (x(1) >= u(1))  % flow condition
%     v = 1;  % report flow
% else
%     v = 0;   % do not report flow
% end

xtemp = zeros(2,1);
xtemp = x;
x = xtemp;

% n = length(x); % size of the q state
qplus = zeros(2,1); % dynamics of the q state
% e.g.,
qplus(1) = 3-u(1);
qplus(2) = 3-u(2);
```

```
34 if x(1)==qplus(1) && x(2)==qplus(2)
35     v = 1; % report flow
36 else
37     v = 0; % do not report flow
38 end
39
```

Jump map

```
1
2 function xplus = g(x, u)
3 %-------------------------------------------------------------------------
4 % Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
5 % https://hybrid.soe.ucsc.edu/software
6 % http://hybridsimulator.wordpress.com/
7 %-------------------------------------------------------------------------
8 % Project: Simulation of a hybrid system Finite state machine (FSM)
9 % Description: Jump map
10 %-------------------------------------------------------------------------
11 %-------------------------------------------------------------------------
12 %   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
13 %   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
14 %   Copyright @ Hybrid Systems Laboratory (HSL),
15 %   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00
16 %-------------------------------------------------------------------------
17 % jump map: xplus = g(x,u);
18
19 % n = length(x); % size of the q state
20 qplus = zeros(2,1); % dynamics of the q state
21 % e.g.,
22 qplus(1) = 3-u(1);
23 qplus(2) = 3-u(2);
24
25 xplus = [qplus];
26
```

Jump set

```
1
2 function v  = D(x, u)
3 %-------------------------------------------------------------------------
4 % Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
5 % https://hybrid.soe.ucsc.edu/software
6 % http://hybridsimulator.wordpress.com/
7 %-------------------------------------------------------------------------
8 % Project: Simulation of a hybrid system Finite state machine (FSM)
9 % Description: Jump set
10 %-------------------------------------------------------------------------
11 %-------------------------------------------------------------------------
12 %   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
13 %   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
14 %   Copyright @ Hybrid Systems Laboratory (HSL),
15 %   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00
16 %
17 % Check on jump conditions
```

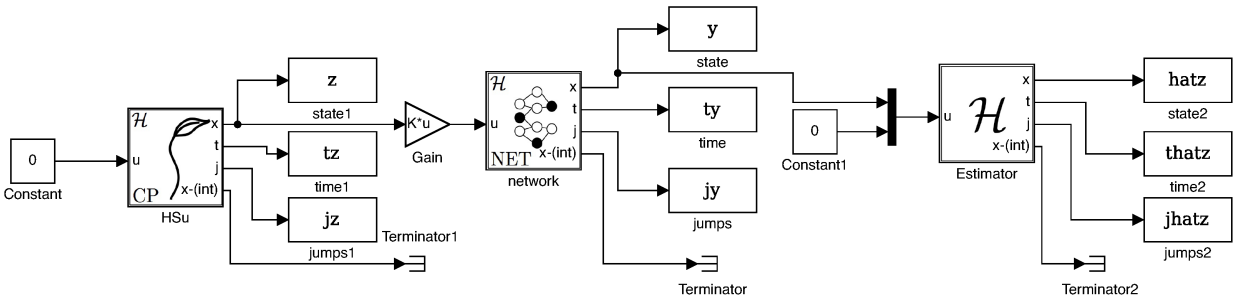


Figure 28: Estimation over a network scheme in Example 5.9.

```
% % E.g.,
% if (x(1) <= u(1)) && (x(2) <= 0)  % jump condition
%     v = 1;  % report jump
% else
%     v = 0;   % do not report jump
% end
xtemp = zeros(2,1);
xtemp = x;
x = xtemp;

% n = length(x); % size of the q state
qplus = zeros(2,1); % dynamics of the q state
% e.g.,
qplus(1) = 3-u(1);
qplus(2) = 3-u(2);
if x(1)==qplus(1) && x(2)==qplus(2)
    v = 0; % do not report jump
else
    v = 1; % report jump
end
```

□

Example 5.9 (Estimation Over a Network) Consider a physical process given in terms of the state-space model

$$\dot{x} = Ax, \qquad y = Mx, \qquad x \in \mathbb{R}^{n_P}, y \in \mathbb{R}^{r_P} \tag{35}$$

where x is the state and y is the measured output. The output is digitally transmitted through a network. At the other end of the network, a computer receives the information and runs an algorithm that takes the measurements of y to estimate the state x of the physical process. In Figure 28 the scheme of the simulation is presented. We consider an estimation algorithm with a state variable $\hat{x} \in \mathbb{R}^{n_P}$, which denotes the estimate of x, that is appropriately reset to a new value involving the information received. More precisely, denoting the transmission times by t_i and L a constant matrix to be designed, the estimation algorithm updates its state as follows

$$\hat{x}^+ = \hat{x} + L(y - M\hat{x}) \tag{36}$$

at every instant information is received. In between such events, the algorithm updates its state continuously so as to match the evolution of the state of the physical process, that is, via

$$\dot{\hat{x}} = A\hat{x} \tag{37}$$

Modeling the network as a hybrid system, which, in particular, assumes zero transmission delay, the state variables of the entire system are j_N, $\tau_N \in \mathbb{R}$, $m_N \in \mathbb{R}^{r_P}$, and $x,\ \hat{x} \in \mathbb{R}^{n_P}$. Then, transmissions occur when $\tau_N \leq 0$, at which events the state of the network is updated via

$$\tau_N^+ \in [T_N^{*\min}, T_N^{*\max}], \quad j_N^+ = j_N + 1, \qquad m_N^+ = y$$

and the state of the algorithm is updated via (36). Note that since the state of the physical process does not change at such events, we can use the following trivial difference equation to update it at the network events:

$$x^+ = x$$

In between events, the state of the network is updated as

$$\dot{\tau}_N = -1, \qquad \dot{j}_N, \qquad \dot{m}_N = 0$$

the state of the algorithm changes continuously according to (37), and the state of the physical process changes according to (35). Considering the equations above, we arbitrarily pick the following data for each of the subsystems in Figure 28:

- Physical process:

$$f_P(x,u) := Ax + Bu,\ C_P := \mathbb{R}^{n_p} \times \mathbb{R} \tag{38}$$

$$G_P(x,u) := x,\ D_P := \emptyset,\ y = h(x) := Mx \tag{39}$$

  where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -2 & 1 & -1 & 0 \\ 2 & -2 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \tag{40}$$

  $n_p = 4$, $r_p = 1$, $x \in \mathbb{R}^{n_p}$, $y \in \mathbb{R}^{r_p}$, and $u \in \mathbb{R}$.

- Network:

$$f(x_N, u_N) := \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \qquad C := \{(x_N, u_N) \mid \tau_N \geq 0\}, \tag{41}$$

$$g(x_N, u_N) := \begin{bmatrix} u_N \\ j_N + 1 \\ \tau_r \end{bmatrix}, \qquad D := \{(x_N, u_N) \mid \tau_N \leq 0\}, \quad y_N = h(x_N) := x_N \tag{42}$$

  where $\tau_r \in [T_N^{*\min}, T_N^{*\max}]$ is a random variable that models the time in-between communication instances. Also, the sate and the input are given by $x_N = (m_N, j_N, \tau_N) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}$, and $u_N = y \in \mathbb{R}^{r_p}$, respectively.

- Estimator:

$$f(x_E, u_E) := \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} x_E + \begin{bmatrix} B \\ 0 \end{bmatrix} v, \qquad C := \{(x_E, u_E) \mid j_E = j_N\}, \tag{43}$$

$$g((\hat{x}, j_E), u_E) := \begin{bmatrix} \hat{x} + L(m_N - M\hat{x}) \\ j_N \end{bmatrix}, \qquad D := \{(x,u) \mid j_E \in \mathbb{N} \setminus j_N\}, \tag{44}$$

  where L, which is designed as in [6], is given by

$$L := \begin{bmatrix} 1 \\ -0.9433 \\ -0.6773 \\ 1.6274 \end{bmatrix}, \tag{45}$$

the input and the state are given by $u_E = (x_N, v) = ((m_N, j_N, \tau_N), v) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, and $x_E = (\hat{x}, j_E) \in \mathbb{R}^4 \times \mathbb{R}$, respectively. Notice that the estimator block (Estimator) in Figure 28 is implemented using a regular hybrid system block with embedded functions.

For each hybrid system in Figure 28 (HSu, network, and Estimator) we have the following Matlab embedded functions that describe the sets C and D and functions f and g. Figure 29 depicts the corresponding hybrid arc for the position state.

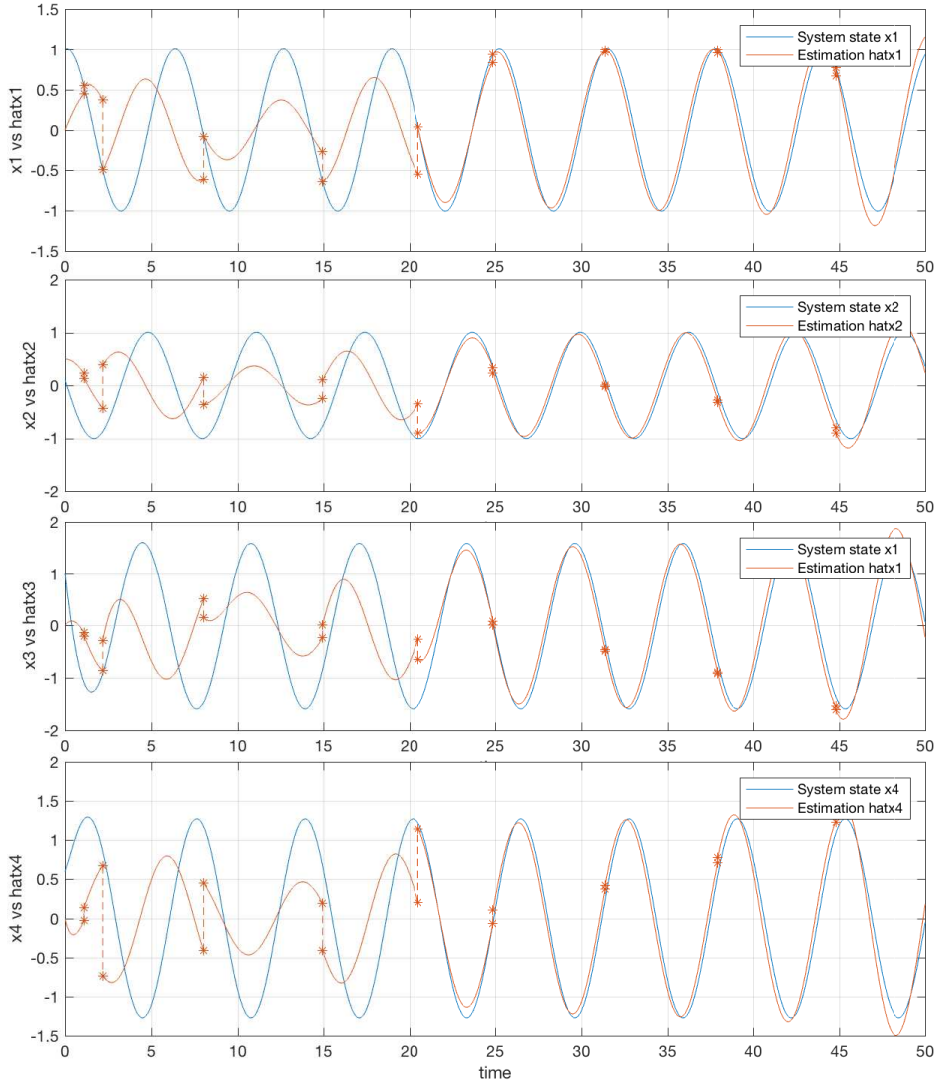


Figure 29: Estimated states Vs. continuous plant states

- Continuous process:

  Flow map

```
1
2 function zdot = f(z, u, ctes)
3 %------------------------------------------------------------------------
4 % Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
```

```
% https://hybrid.soe.ucsc.edu/software
% http://hybridsimulator.wordpress.com/
%-------------------------------------------------------------------------
% Project: Simulation of a hybrid system (plant with constraints in the
% state and the input)
% Description: Flow map
%-------------------------------------------------------------------------
%-------------------------------------------------------------------------
%   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
%   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
%   Copyright @ Hybrid Systems Laboratory (HSL),
%   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00
%-------------------------------------------------------------------------
% flow map: xdot=f(x,u);
% ctes = [A,B,M',L,P];

A = ctes(:,1:4);
B = ctes(:,5);
% M = ctes(:,6)';
% L = ctes(:,7);

zdot = A*z + B*u;
```

Flow set

```

function v  = C(x, u)
%-------------------------------------------------------------------------
% Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
% https://hybrid.soe.ucsc.edu/software
% http://hybridsimulator.wordpress.com/
%-------------------------------------------------------------------------
% Project: Simulation of a hybrid system
% Description: Flow set
%-------------------------------------------------------------------------
%-------------------------------------------------------------------------
%   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
%   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
%   Copyright @ Hybrid Systems Laboratory (HSL),
%   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00
%
% Check on flow conditions
% E.g.,
% if (x(1) >= u(1))  % flow condition
%     v = 1;  % report flow
% else
%     v = 0;   % do not report flow
% end


v = 1; % report flow

```

Jump map

```
1
2 function xplus = g(x, u)
3 %-------------------------------------------------------------------------
4 % Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
5 % https://hybrid.soe.ucsc.edu/software
6 % http://hybridsimulator.wordpress.com/
7 %-------------------------------------------------------------------------
8 % Project: Simulation of a hybrid system
9 % Description: Jump map
10 %-------------------------------------------------------------------------
11 %-------------------------------------------------------------------------
12 %   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
13 %   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
14 %   Copyright @ Hybrid Systems Laboratory (HSL),
15 %   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00
16 %-------------------------------------------------------------------------
17 % jump map: xplus = g(x,u);
18
19 xplus = zeros(4,1);
20 xplus = x;
21
22
```

Jump set

```
1
2 function v  = D(x, u)
3 %-------------------------------------------------------------------------
4 % Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
5 % https://hybrid.soe.ucsc.edu/software
6 % http://hybridsimulator.wordpress.com/
7 %-------------------------------------------------------------------------
8 % Project: Simulation of a hybrid system
9 % Description: Jump set
10 %-------------------------------------------------------------------------
11 %-------------------------------------------------------------------------
12 %   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
13 %   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
14 %   Copyright @ Hybrid Systems Laboratory (HSL),
15 %   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00
16 %
17 % Check on jump conditions
18 % % E.g.,
19 % if (x(1) <= u(1)) && (x(2) <= 0)  % jump condition
20 %     v = 1;  % report jump
21 % else
22 %     v = 0;   % do not report jump
23 % end
24
25 v = 0; % do not report jump
```

- Network:

  Flow map

```
1
2 function xdot = f(x,vs)
3 %-------------------------------------------------------------------------
4 % Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
5 % https://hybrid.soe.ucsc.edu/software
6 % http://hybridsimulator.wordpress.com/
7 %-------------------------------------------------------------------------
8 % Project: Simulation of a hybrid system Digital network (net)
9 % Description: Flow map
10 %-------------------------------------------------------------------------
11 %-------------------------------------------------------------------------
12 %   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
13 %   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
14 %   Copyright @ Hybrid Systems Laboratory (HSL),
15 %   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00
16
17 n = length(vs); % measured input size
18
19 msdot = zeros(n,1); % measured continuous dynamics
20 jdot = 0;
21 tau_sdot = -1; % Timer tau_s
22 xdot = [msdot;jdot;tau_sdot];
23
```

Flow set

```
1
2 function v  = C(x, vs, Tnmax)
3 %-------------------------------------------------------------------------
4 % Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
5 % https://hybrid.soe.ucsc.edu/software
6 % http://hybridsimulator.wordpress.com/
7 %-------------------------------------------------------------------------
8 % Project: Simulation of a hybrid system Digital network (net)
9 % Description: Flow set
10 %-------------------------------------------------------------------------
11 %-------------------------------------------------------------------------
12 %   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
13 %   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
14 %   Copyright @ Hybrid Systems Laboratory (HSL),
15 %   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00
16 %
17 % Check on flow conditions
18 % E.g.,
19 % if (x(1) >= u(1))  % flow condition
20 %     v = 1;  % report flow
21 % else
22 %     v = 0;   % do not report flow
23 % end
24
25 tau_s = x(end); % timer state
26
27 if tau_s>=0 && tau_s<= Tnmax
28     v = 1; % report flow
29 elseif tau_s> Tnmax
```

```
    v = 0; % do not report flow
else
    v = 0;
end
```

Jump map

```

function xplus = g(x, vs, Tnmax, Tnmin, tk)
%-------------------------------------------------------------------------
% Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
% https://hybrid.soe.ucsc.edu/software
% http://hybridsimulator.wordpress.com/
%-------------------------------------------------------------------------
% Project: Simulation of a hybrid system Analog-to-Digital converter (ADC)
% Description: Jump map
%-------------------------------------------------------------------------
%-------------------------------------------------------------------------
%   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
%   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
%   Copyright @ Hybrid Systems Laboratory (HSL),
%   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00

n = length(vs); % measured input size
xtemp = zeros(n+2,1);
xtemp = x;
x = xtemp;


j = x(n+1);
msplus = vs; % output = measured input
% The value tau_s is updated as a function of vs, e.g.,
tau_splus = tk(j+1); % Timer tau_s
j_plus = j+1;
xplus = [msplus;j_plus;tau_splus];



```

Jump set

```

function v  = D(x, vs)
%-------------------------------------------------------------------------
% Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
% https://hybrid.soe.ucsc.edu/software
% http://hybridsimulator.wordpress.com/
%-------------------------------------------------------------------------
% Project: Simulation of a hybrid system Analog-to-Digital converter (ADC)
% Description: Jump set
%-------------------------------------------------------------------------
%-------------------------------------------------------------------------
%   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
%   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
%   Copyright @ Hybrid Systems Laboratory (HSL),
```

```
%   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00
%
% Check on jump conditions
% % E.g.,
% if (x(1) <= u(1)) && (x(2) <= 0)  % jump condition
%     v = 1;  % report jump
% else
%     v = 0;   % do not report jump
% end


tau_s = x(end); % timer state

if tau_s>=0
    v = 0; % do not report jump
elseif tau_s<= 0
    v = 1; % report jump
else
    v = 0;
end
```

- Estimator:

  Flow map

```

function xdot = f(x, v,ctes)
%-------------------------------------------------------------------------
% Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
% https://hybrid.soe.ucsc.edu/software
% http://hybridsimulator.wordpress.com/
%-------------------------------------------------------------------------
% Project: Simulation of a hybrid system
% Description: Flow map
%-------------------------------------------------------------------------
%-------------------------------------------------------------------------
%   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
%   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
%   Copyright @ Hybrid Systems Laboratory (HSL),
%   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00
%-------------------------------------------------------------------------
% flow map: xdot=f(x,u);
% ctes = [A,B,M',L,P];

A = ctes(:,1:4);
B = ctes(:,5);
% M = ctes(:,6)';
% L = ctes(:,7);
n = length(A);
u = v(end);
z = x(1:n);
jdot = 0;
zdot = A*z + B*u;
xdot = [zdot;jdot];
```

Flow set

```
1
2 function out  = C(x, v)
3 %-------------------------------------------------------------------------
4 % Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
5 % https://hybrid.soe.ucsc.edu/software
6 % http://hybridsimulator.wordpress.com/
7 %-------------------------------------------------------------------------
8 % Project: Simulation of a hybrid system
9 % Description: Flow set
10 %-------------------------------------------------------------------------
11 %-------------------------------------------------------------------------
12 %   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
13 %   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
14 %   Copyright @ Hybrid Systems Laboratory (HSL),
15 %   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00
16 %
17 % Check on flow conditions
18 % E.g.,
19 % if (x(1) >= u(1))  % flow condition
20 %     v = 1;  % report flow
21 % else
22 %     v = 0;   % do not report flow
23 % end
24
25 j = x(end); % internal communication memory event
26 jnet = v(end-2); % communication event
27
28 if j == jnet
29     out = 1; % report flow
30 else
31     out = 0;
32 end
```

Jump map

```
1
2 function xplus = g(x, v, ctes)
3 %-------------------------------------------------------------------------
4 % Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
5 % https://hybrid.soe.ucsc.edu/software
6 % http://hybridsimulator.wordpress.com/
7 %-------------------------------------------------------------------------
8 % Project: Simulation of a hybrid system
9 % Description: Jump map
10 %-------------------------------------------------------------------------
11 %-------------------------------------------------------------------------
12 %   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
13 %   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
14 %   Copyright @ Hybrid Systems Laboratory (HSL),
15 %   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00
16 %-------------------------------------------------------------------------
17 % jump map: xplus = g(x,u);
18
```

```
A = ctes(:,1:4);
M = ctes(:,6)';
L = ctes(:,7);

n = 4;
z = zeros(4,1);
zplus = zeros(4,1);
xplus = zeros(5,1);
z = x(1:4);
jnet = v(2); % communication event
y = v(1);

jplus = jnet;
zplus = z + L*(y-M*z);
xplus = [zplus;jplus];
```

Jump set

```

function out  = D(x, v)
%-------------------------------------------------------------------------
% Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
% https://hybrid.soe.ucsc.edu/software
% http://hybridsimulator.wordpress.com/
%-------------------------------------------------------------------------
% Project: Simulation of a hybrid system
% Description: Jump set
%-------------------------------------------------------------------------
%-------------------------------------------------------------------------
%   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
%   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
%   Copyright @ Hybrid Systems Laboratory (HSL),
%   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00
%
% Check on jump conditions
% % E.g.,
% if (x(1) <= u(1)) && (x(2) <= 0)  % jump condition
%     v = 1;  % report jump
% else
%     v = 0;   % do not report jump
% end


n = length(x)-1; % measured input size
j = x(end); % internal communication memory event
jnet = v(end-2); % communication event

if j ~= jnet
    out = 1; % report jump
else
    out = 0;
end

```

□

Example 5.10 (Sample-and-hold Feedback Control) Consider a physical process modeled as in Example 5.7 and a state feedback interconnection. The algorithm (static gain) uses measurements of its output and controls the input of the physical process with the goal of steering its state to zero. Suppose the sampling device is ideal and that the signals are connected to the plant via a DAC modeled as in (10)-(11).

The interconnection between the models of the physical process, the sampling device, the finite state machine, and the DAC has the feedback topology shown in Figure 30. In particular, the output of the DAC is connected to the input u of the physical process by a matrix gain K, while the input v of the finite state machine is equal to the output y of the physical process at every sampling instant.

- Physical process:

$$f_P(x,u) := Ax + Bu,\ C_P := \mathbb{R}^2 \times \mathbb{R}^2, \tag{46}$$

$$G_P(x,u) := x,\ D_P := \emptyset,\ y = h(x) := x \tag{47}$$

  where $A = \begin{bmatrix} 0 & 1 \\ 0 & -b/m \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1/m \end{bmatrix}$, and $x = (x_1, x_2) \in \mathbb{R}^2$.

- Analog-to-Digital Converter (ADC):

$$f(x,u) := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ C := \{(x,u)\ |\ \tau_s \in [0, T_s^*]\}, \tag{48}$$

$$g(x,u) := \begin{bmatrix} u \\ 0 \end{bmatrix},\ D := \{(x,u)\ |\ \tau_s \geq T_s^*\},\ y = h(x) := x \tag{49}$$

  where $x = (m_s, \tau_s) \in \mathbb{R} \times \mathbb{R}^2$, and $u \in \mathbb{R}^2$.

- Zero-Order Hold (ZOH):

$$f(x,u) := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ C := \{(x,u)\ |\ \tau_s \in [0, T_s^*]\}, \tag{50}$$

$$g(x,u) := \begin{bmatrix} u \\ 0 \end{bmatrix},\ D := \{(x,u)\ |\ \tau_s \geq T_s^*\},\ y = h(x) := x \tag{51}$$

  where $x = (m_s, \tau_s) \in \mathbb{R} \times \mathbb{R}^2$, and $u \in \mathbb{R}^2$.

The feedback gain is given by $K = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$.

For each hybrid system in Figure 30 (HSu, ADC, and ZOH) we have the following Matlab embedded functions that describe the sets C and D and functions f and g. Figure 31 presents a numerical simulation of the interconnected system.

- Continuous process:

  Flow map

```
1
2 function xdot = f(x, u, ctes)
3 %-------------------------------------------------------------------------
4 % Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
5 % https://hybrid.soe.ucsc.edu/software
6 % http://hybridsimulator.wordpress.com/
7 %-------------------------------------------------------------------------
8 % Project: Simulation of a hybrid system (plant with constraints in the
9 % state and the input)
10 % Description: Flow map
```

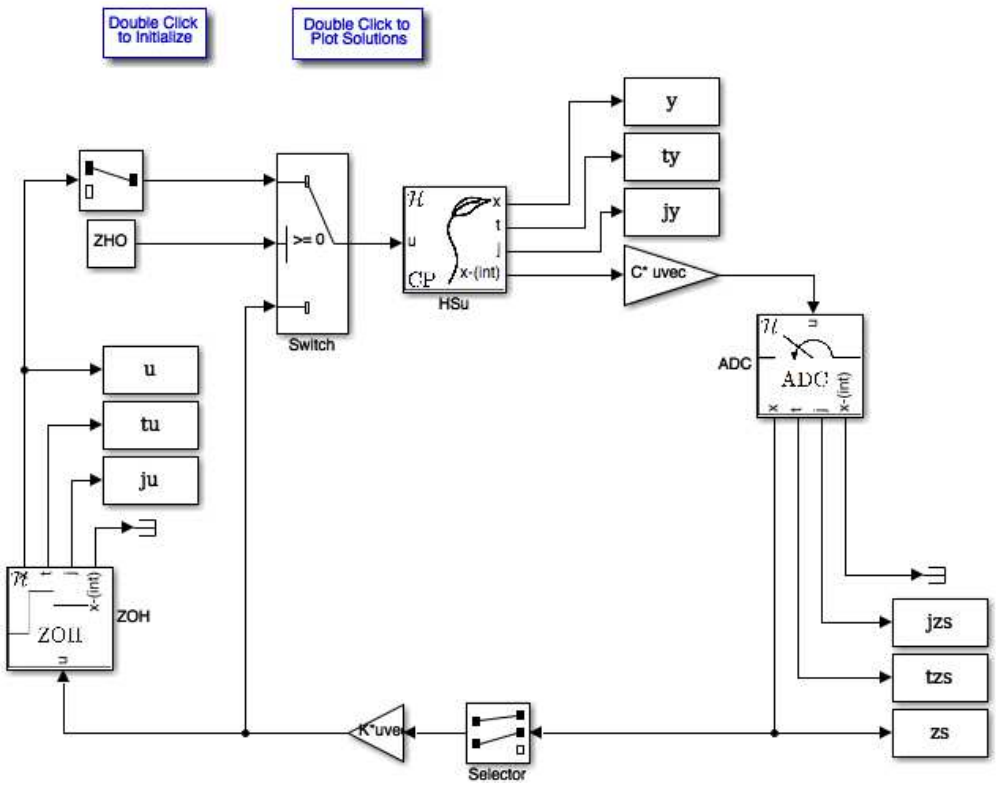


Figure 30: Sample and hold feedback interconnection for Example 5.10

```
%-------------------------------------------------------------------------
%-------------------------------------------------------------------------
%   See also PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC, PLOTHARCCOLOR,
%   PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
%   Copyright @ Hybrid Systems Laboratory (HSL),
%   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00
%-------------------------------------------------------------------------
% flow map: xdot=f(x,u);

% ctes = [A,B,C,K'];

A = ctes(:,1:2);
B = ctes(:,3);
C = ctes(:,4:5);
K = ctes(:,6)';

xdot = A*x + B*u;
```

Flow set

```

function v  = C(x, u)
%-------------------------------------------------------------------------
% Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
% https://hybrid.soe.ucsc.edu/software
```

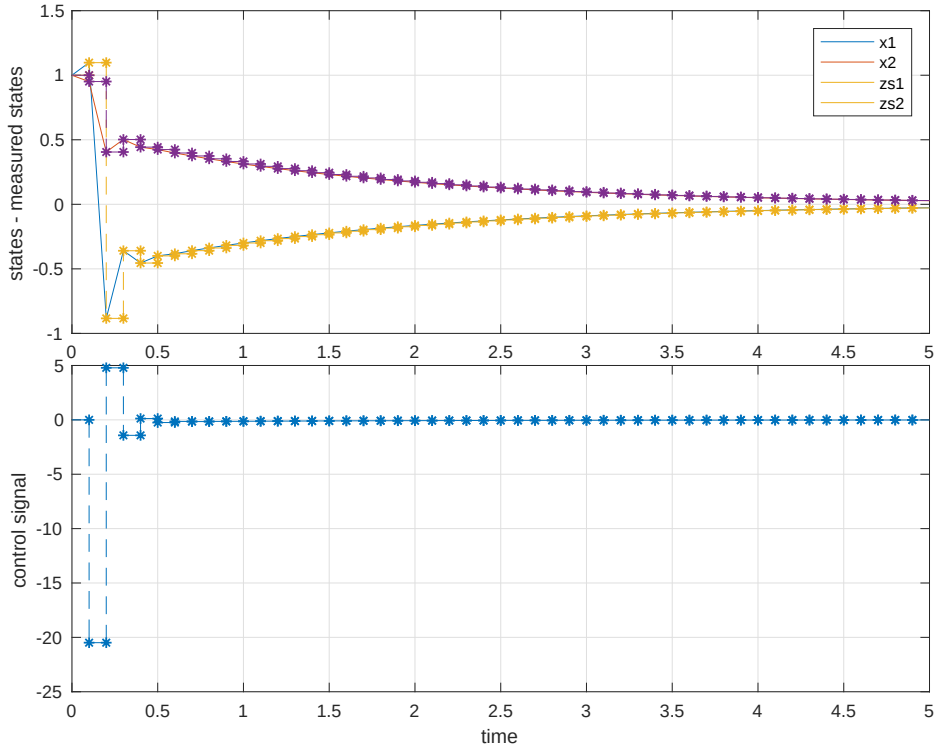


Figure 31: Measuread states and inputs.

```
% http://hybridsimulator.wordpress.com/
%-------------------------------------------------------------------------
% Project: Simulation of a hybrid system
% Description: Flow set
%-------------------------------------------------------------------------
%-------------------------------------------------------------------------
%   See also PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC, PLOTHARCCOLOR,
%   PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
%   Copyright @ Hybrid Systems Laboratory (HSL),
%   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00
%
% Check on flow conditions
% E.g.,
% if (x(1) >= u(1))  % flow condition
%     v = 1;  % report flow
% else
%     v = 0;   % do not report flow
% end


v = 1; % report flow

```

Jump map

```

function xplus = g(x, u)
```

```
%-------------------------------------------------------------------------
% Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
% https://hybrid.soe.ucsc.edu/software
% http://hybridsimulator.wordpress.com/
%-------------------------------------------------------------------------
% Project: Simulation of a hybrid system
% Description: Jump map
%-------------------------------------------------------------------------
%-------------------------------------------------------------------------
%   See also PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC, PLOTHARCCOLOR,
%   PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
%   Copyright @ Hybrid Systems Laboratory (HSL),
%   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00
%-------------------------------------------------------------------------
% jump map: xplus = g(x,u);

xplus = x;


```

Jump set

```

function v  = D(x, u)
%-------------------------------------------------------------------------
% Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
% https://hybrid.soe.ucsc.edu/software
% http://hybridsimulator.wordpress.com/
%-------------------------------------------------------------------------
% Project: Simulation of a hybrid system
% Description: Jump set
%-------------------------------------------------------------------------
%-------------------------------------------------------------------------
%   See also PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC, PLOTHARCCOLOR,
%   PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
%   Copyright @ Hybrid Systems Laboratory (HSL),
%   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00
%
% Check on jump conditions
% % E.g.,
% if (x(1) <= u(1)) && (x(2) <= 0)  % jump condition
%     v = 1;  % report jump
% else
%     v = 0;   % do not report jump
% end

v = false; % do not report jump
```

- ADC and ZOH:

  Flow map

```

function xdot = f(x,vs)
%-------------------------------------------------------------------------
```

```
% Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
% https://hybrid.soe.ucsc.edu/software
% http://hybridsimulator.wordpress.com/
%-------------------------------------------------------------------------
% Project: Simulation of a hybrid system Analog-to-Digital converter (ADC)
% Description: Flow map
%-------------------------------------------------------------------------
%-------------------------------------------------------------------------
%   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
%   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
%   Copyright @ Hybrid Systems Laboratory (HSL),
%   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00

n = length(vs); % measured input size

msdot = zeros(n,1); % measured continuous dynamics
tau_sdot = 1; % Timer tau_s
xdot = [msdot;tau_sdot];

```

Flow set

```

function v  = C(x, vs, Ts)
%-------------------------------------------------------------------------
% Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
% https://hybrid.soe.ucsc.edu/software
% http://hybridsimulator.wordpress.com/
%-------------------------------------------------------------------------
% Project: Simulation of a hybrid system Analog-to-Digital converter (ADC)
% Description: Flow set
%-------------------------------------------------------------------------
%-------------------------------------------------------------------------
%   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
%   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
%   Copyright @ Hybrid Systems Laboratory (HSL),
%   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00
%
% Check on flow conditions
% E.g.,
% if (x(1) >= u(1))  % flow condition
%     v = 1;  % report flow
% else
%     v = 0;   % do not report flow
% end

n = length(vs); % measured input size

tau_s = x(end); % timer state

if tau_s>=0 && tau_s<= Ts
    v = 1; % report flow
elseif tau_s> Ts
    v = 0; % do not report flow
else
```

```
    v = 0;
end
```

Jump map

```

function xplus = g(x, vs)
%-------------------------------------------------------------------------
% Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
% https://hybrid.soe.ucsc.edu/software
% http://hybridsimulator.wordpress.com/
%-------------------------------------------------------------------------
% Project: Simulation of a hybrid system Analog-to-Digital converter (ADC)
% Description: Jump map
%-------------------------------------------------------------------------
%-------------------------------------------------------------------------
%   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
%   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
%   Copyright @ Hybrid Systems Laboratory (HSL),
%   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00


n = length(vs) % measured input size

msplus = vs; % output = measured input
tau_splus = 0; % Timer tau_s
xplus = [msplus;tau_splus];



```

Jump set

```

function v  = D(x, vs, Ts)
%-------------------------------------------------------------------------
% Matlab M-file Project: HyEQ Toolbox @  Hybrid Systems Laboratory (HSL),
% https://hybrid.soe.ucsc.edu/software
% http://hybridsimulator.wordpress.com/
%-------------------------------------------------------------------------
% Project: Simulation of a hybrid system Analog-to-Digital converter (ADC)
% Description: Jump set
%-------------------------------------------------------------------------
%-------------------------------------------------------------------------
%   See also HYEQSOLVER, PLOTARC, PLOTARC3, PLOTFLOWS, PLOTHARC,
%   PLOTHARCCOLOR, PLOTHARCCOLOR3D, PLOTHYBRIDARC, PLOTJUMPS.
%   Copyright @ Hybrid Systems Laboratory (HSL),
%   Revision: 0.0.0.3 Date: 05/20/2015 3:42:00
%
% Check on jump conditions
% % E.g.,
% if (x(1) <= u(1)) && (x(2) <= 0)  % jump condition
%     v = 1;  % report jump
% else
%     v = 0;   % do not report jump
```

```
% end

n = length(vs); % measured input size

tau_s = x(end); % timer state

if tau_s>=0 && tau_s<= Ts
    v = 0; % do not report jump
elseif tau_s> Ts
    v = 1; % report jump
else
    v = 0;
end
```

There are numerous practical examples of systems that can be modeled within the general model for sample-and-hold feedback control defined above. For example, one "classical" example is the control of the temperature of a room by turning on and off a heater so as to keep the temperature within a desired range; see the model in (29). Another widely known example is the control of the level of a water tank.

□

6 Further Reading

Installation files for the HyEQ Toolbox described in this paper can be found at MATLAB Central and at the author's website

`https://hybrid.soe.ucsc.edu/software`.

Also, resources and examples are shared by the HyEQ Toolbox users in the blog

`http://hybridsimulator.wordpress.com`.

7 Acknowledgments

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- Bharani Malladi - The University of Arizona

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