# 1 Examples

The examples below illustrate the use of the Simulink implementation above.

**Example 1.6** (interconnection of hybrid systems $\mathcal{H}_1$ (bouncing ball) and $\mathcal{H}_2$ (moving platform)) Consider a bouncing ball ($\mathcal{H}_1$) bouncing on a platform ($\mathcal{H}_2$) at some initial height and converging to the ground at zero height. This is an interconnection problem because the current states of each system affect the behavior of the other system. In this interconnection, the bouncing ball will contact the platform, bounce back up, and cause a jump in height of the platform so that it gets closer to the ground. After some time, both the ball and the platform will converge to the ground. In order to model this system, the output of the bouncing ball becomes the input of the moving platform, and vice versa. For the simulation of the described system with regular data where $\mathcal{H}_1$ is given by

$$f_1(\xi, u_1, v_1) := \begin{bmatrix} \xi_2 \\ -\gamma - b\xi_2 + v_{11} \end{bmatrix}, C_1 := \{(\xi, u_1) \mid \xi_1 \geq u_1, u_1 \geq 0\} \tag{1}$$

$$g_1(\xi, u_1, v_1) := \begin{bmatrix} \xi_1 + \alpha_1 \xi_2^2 \\ e_1|\xi_2| + v_{12} \end{bmatrix}, D_1 := \{(\xi, u_1) \mid \xi_1 = u_1, u_1 \geq 0\}, y_1 = h_1(\xi) := \xi_1 \tag{2}$$

where $\gamma, b, \alpha_1 > 0, e_1 \in [0, 1)$, $\xi = [\xi_1, \xi_2]^\top$ is the state, $y_1 \in \mathbb{R}$ is the output, $u_1 \in \mathbb{R}$ and $v_1 = [v_{11}, v_{12}]^\top \in \mathbb{R}^2$ are the inputs, and the hybrid system $\mathcal{H}_2$ is given by

$$f_2(\eta, u_2, v_2) := \begin{bmatrix} \eta_2 \\ -\eta_1 - 2\eta_2 + v_{12} \end{bmatrix}, C_2 := \{(\eta, u_2) \mid \eta_1 \leq u_2, \eta_1 \geq 0\} \tag{3}$$

$$g_2(\eta, u_2, v_2) := \begin{bmatrix} \eta_1 - \alpha_2|\eta_2| \\ -e_2|\eta_2| + v_{22} \end{bmatrix}, D_2 := \{(\eta, u_2) \mid \eta_1 = u_2, \eta_1 \geq 0\}, y_2 = h_2(\eta) := \eta_1 \tag{4}$$

where $\alpha_2 > 0, e_2 \in [0, 1)$, $\eta = [\eta_1, \eta_2]^\top \in \mathbb{R}^2$ is the state, $y_2 \in \mathbb{R}$ is the output, and $u_2 \in \mathbb{R}$ and $v_2 = [v_{21}, v_{22}]^\top \in \mathbb{R}^2$ are the inputs.

Therefore, the interconnection may be defined by the input assignment

$$u_1 = y_2, \qquad u_2 = y_1. \tag{5}$$

The signals $v_1$ and $v_2$ are included as external inputs in the model in order to simulate the effects of environmental perturbations, such as a wind gust, on the system.

The MATLAB scripts in each of the function blocks of the implementation above are given as follows. The constants for the interconnected system are $\gamma = 0.8$, $b = 0.1$, and $\alpha_1, \alpha_2 = 0.1$.

- For hybrid system $\mathcal{H}_1$:

  Flow map

```
function xdot = f(x, u)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Matlab Function  Author: Ricardo Sanfelice
%
% Project: Simulation of a hybrid system (interconnection)
%
% Name: f.m
%
% Description: Flow map
%
% Version: 1.0
% Required files: -
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

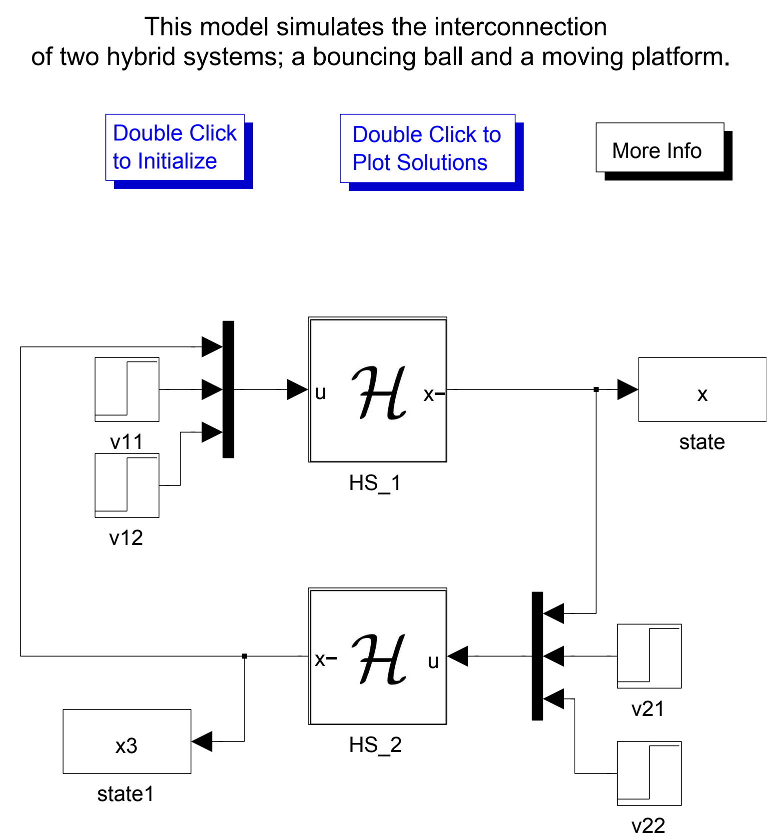



Figure 1: MATLAB/Simulink implementation of interconnected hybrid systems $\mathcal{H}_1$ and $\mathcal{H}_2$

```

% state
xi1 = x(1);
xi2 = x(2);

%input
y2 = u(1);
v11 = u(2);
v12 = u(3);

% flow map
%xdot=f(x,u);
xi1dot = xi2;
xi2dot = -0.8-0.1*xi2+v11;

xdot = [xi1dot;xi2dot];
```

Flow set

```
function v  = C(x, u)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Matlab Function  Author: Ricardo Sanfelice
%
```

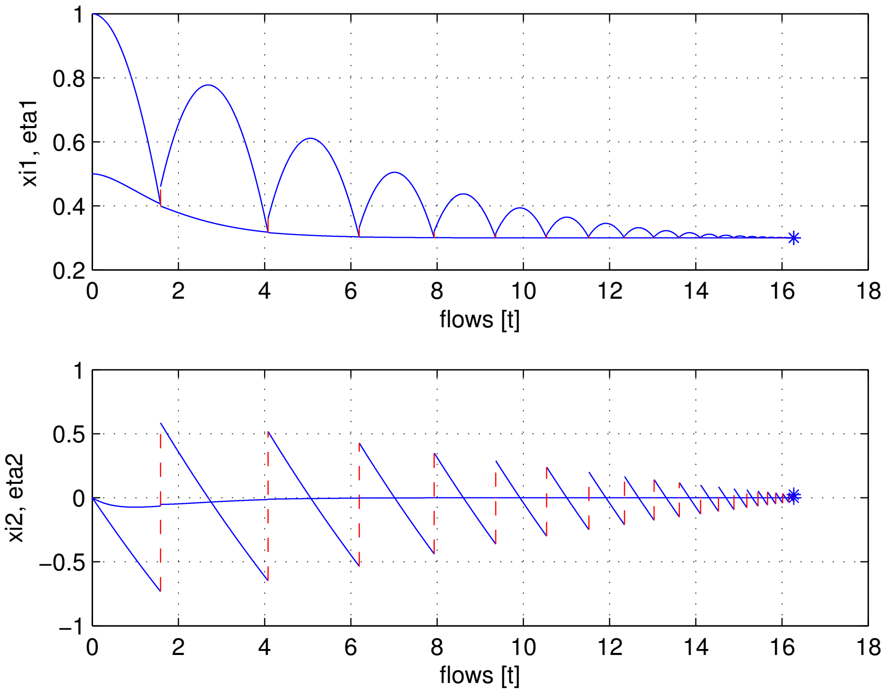
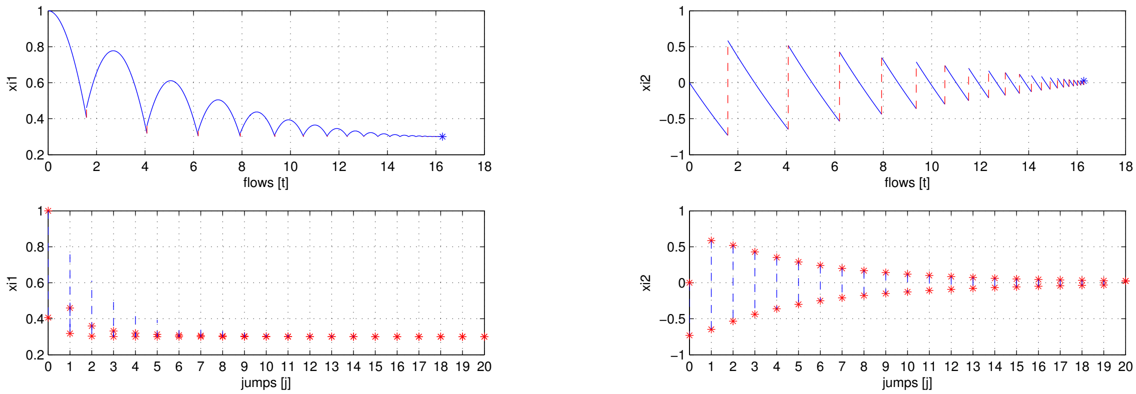



Figure 2: Solution of Example 1.6: height and velocity

```
% Project: Simulation of a hybrid system (interconnection)
%
% Name: C.m
%
% Description: Flow set
%
% Version: 1.0
% Required files: -
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% state
xi1 = x(1);
xi2 = x(2);

%input
y2 = u(1);
v11 = u(2);
v12 = u(3);

if (xi1 >= y2)   % flow condition
    v = 1;  % report flow
else
    v = 0;    % do not report flow
end
```

Jump map




(a) Height

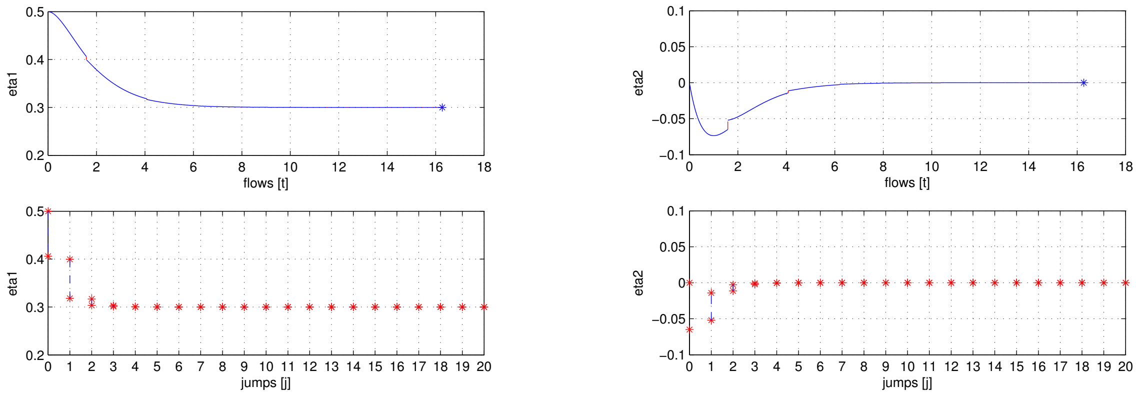
(b) Velocity

Figure 3: Solution of Example 1.6 for system $\mathcal{H}_1$

```
function xplus = g(x, u)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Matlab Function  Author: Ricardo Sanfelice
%
% Project: Simulation of a hybrid system (interconnection)
%
% Name: g.m
%
% Description: Jump map
%
% Version: 1.0
% Required files: -
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% state
xi1 = x(1);
xi2 = x(2);

%input
y2 = u(1);
v11 = u(2);
v12 = u(3);

xi1plus=y2+0.1*xi2^2;
xi2plus=0.8*abs(xi2)+v12;

xplus = [xi1plus;xi2plus];
```

Jump set

```
function v  = D(x, u)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```




(a) Height    (b) Velocity

Figure 4: Solution of Example 1.6 for system $\mathcal{H}_2$

```
% Matlab Function  Author: Ricardo Sanfelice
%
% Project: Simulation of a hybrid system (interconnection)
%
% Name: D.m
%
% Description: Jump set
%
% Version: 1.0
% Required files: -
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% state
xi1 = x(1);
xi2 = x(2);

%input
y2 = u(1);
v11 = u(2);
v12 = u(3);


if (xi1 <= y2) % jump condition
    v = 1;  % report jump
else
    v = 0;   % do not report jump
end
```

- For hybrid system $\mathcal{H}_2$:

  Flow map

```
function xdot = f(x, u)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
% Matlab Function  Author: Ricardo Sanfelice
%
% Project: Simulation of a hybrid system (interconnection)
%
% Name: f.m
%
% Description: Flow map
%
% Version: 1.0
% Required files: -
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


% state
eta1 = x(1);
eta2 = x(2);

%input
y1 = u(1);
v21 = u(2);
v22 = u(3);

% flow map
eta1dot = eta2;
eta2dot = -eta1-2*eta2+v21;

xdot = [eta1dot;eta2dot];
```

Flow set

```
function v  = C(x, u)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Matlab Function  Author: Ricardo Sanfelice
%
% Project: Simulation of a hybrid system (interconnection)
%
% Name: C.m
%
% Description: Flow set
%
% Version: 1.0
% Required files: -
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% state
eta1 = x(1);
eta2 = x(2);

%input
y1 = u(1);
v21 = u(2);
v22 = u(3);

```

```
if (eta1 <= y1)  % flow condition
    v = 1;  % report flow
else
    v = 0;   % do not report flow
end
```

Jump map

```
function xplus = g(x, u)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Matlab Function  Author: Ricardo Sanfelice
%
% Project: Simulation of a hybrid system (interconnection)
%
% Name: g.m
%
% Description: Jump map
%
% Version: 1.0
% Required files: -
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% state
eta1 = x(1);
eta2 = x(2);

%input
y1 = u(1);
v21 = u(2);
v22 = u(3);

% jump map
eta1plus = y1-0.1*abs(eta2);
eta2plus = -0.8*abs(eta2)+v22;

xplus = [eta1plus;eta2plus];
```

Jump set

```
function v  = D(x, u)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Matlab Function  Author: Ricardo Sanfelice
%
% Project: Simulation of a hybrid system
%
% Name: D.m
%
% Description: Jump set
%
% Version: 1.0
% Required files: -
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

```

```
% state
eta1 = x(1);
eta2 = x(2);

%input
y1 = u(1);
v21 = u(2);
v22 = u(3);


if (eta1 >= y1) % jump condition
    v = 1;  % report jump
else
    v = 0;   % do not report jump
end
```

A solution to the interconnection of hybrid systems $\mathcal{H}_1$ and $\mathcal{H}_2$ with $T = 18$, $J = 20$, $rule = 1$, is depicted in Figure 2. Both the projection onto $t$ and $j$ are shown. A solution to the hybrid system $\mathcal{H}_1$ is depicted in Figure 3(a) (height) and Figure 3(b) (velocity). A solution to the hybrid system $\mathcal{H}_2$ is depicted in Figure 4(a) (height) and Figure 4(b) (velocity).

These simulations reflect the expected behavior of the interconnected hybrid systems.

For MATLAB/Simulink files of this example, see Examples/Example_1.6.

$\square$