University of Toronto
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MAT137Y Test 2

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Duration: 110 minutes

No Aids Permitted

Sample Solutions

1. For each question below, only your final answer will be graded. **No justification is necessary.**

(1a) (2 points) Let $f(x) = e^{\sin(x\sin(x))}$ and calculate $f'(x)$.

Final Answer

$f'(x) = e^{\sin(x\sin(x))}\cos(x\sin(x))(\sin(x) + x\cos(x))$

(1b) (2 points) The function $f(x) = x^3 + x + 3$ is one-to-one. Calculate $f^{-1}(5)$ and $(f^{-1})'(5)$.

Final Answer

$f^{-1}(5) = 1$

Final Answer

$(f^{-1})'(5) = \frac{1}{4}$

(1c) (2 points) Let $h(x) = \arcsin(\sqrt{\frac{1+x}{2}})$ and calculate $h'(0)$.

Final Answer

$h'(0) = \frac{1}{2}$

(1d) (2 points) Let $F(x) = x^{(\ln x)^3}$ and calculate $F'(e)$.

Final Answer

$F'(e) = 4$

For each question below, only your final answer will be graded. **No justification is necessary.**

2. Let $f(x) = \sqrt{x}$ with domain $D = [0, \infty)$. For every point $a \in D$, let $L_a : D \to \mathbb{R}$ be the linear approximation to f at the point a, if such an approximation is defined.

(2a) (2 points) Find an expression for $L_1(x)$ and $L_4(x)$.

Final Answer

$L_1(x) = \frac{1}{2}(x-1)+1 \ or \ \frac{1}{2}x + \frac{1}{2}$

Final Answer

$L_4(x) = \frac{1}{4}(x-4)+2 \ or \ \frac{1}{4}x + 1$

(2b) (3 points) For $x \in D$, we would like to approximate $f(x)$. Let C be the set of $x \in D$ such that $L_1(x)$ is a *strictly better* approximation than $L_4(x)$ (i.e., has less error).

1. We can claim that 1.1 is in C.

   ✓ True ◯ False ◯ Not enough information to determine.

2. Express C in interval notation.

   Final Answer

   $C = [0, 2)$

For each question below, only your final answer will be graded. **No justification is necessary.**

3. Determine if the following statements are True or False for all differentiable functions f with domain $\mathbb{R}$.

(3a) (1 point) If f has a local maximum at 2, then $f'(2) = 0$.

✓ True	◯ False

(3b) (1 point) If $f(2) = 2$ and $f(4) = 2$, then there is a $c \in (2, 4)$ such that f has a local extrema at c.

✓ True	◯ False

(3c) (1 point) If $f(1) = 4$ and $f'(x) \geq 4$ for all $x \in \mathbb{R}$, then $f(3) \geq 10$.

✓ True	◯ False

4. (4a) (2 points) Let f be a function defined on an interval I.
Write down the definition of "f is (strictly) decreasing on I".

Final Answer

We say that f is (strictly) decreasing on I if $\forall x_1, x_2 \in I,\ x_1 < x_2 \implies f(x_1) > f(x_2)$.
OR
We say that f is (strictly) decreasing on I if $\forall x_1, x_2 \in I,\ f(x_1) < f(x_2) \implies x_1 > x_2$.

(4b) (2 points) State the Extreme Value Theorem. Make sure to specify all assumptions.

Final Answer

Let $a, b \in \mathbb{R}$ with $a < b$. Let f be a function defined on the interval $[a, b]$.
IF f is continuous on $[a, b]$ THEN f has a maximum and a minimum on $[a, b]$.

5. Let f and g be functions with domain $\mathbb{R}$ and define the function h by

$$h(x) = \begin{cases} f(x) & \text{if } x > 0 \\ g(x) & \text{if } x \leq 0 \end{cases}.$$

For each of the following, give an example of f and g such that h has the desired property or explain why no such example exists. You should give f and g explicitly in math expressions.

(5a) (2 points) h is continuous at 0 but not differentiable at 0.
There are a lot of such f and g. Here is one pair:

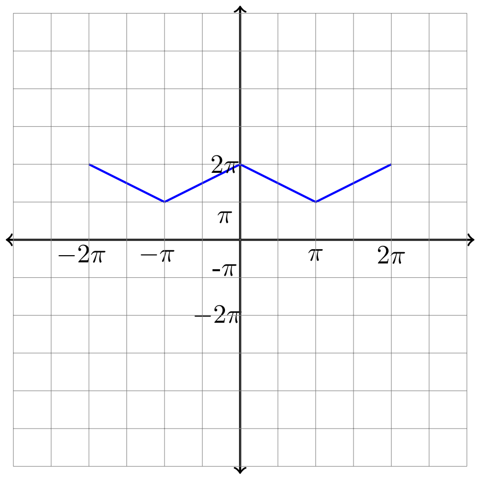
$$f(x) = x, \quad g(x) = -x$$

(5b) (2 points) $h'(0)$ exists and $h''(0)$ does not exist.
There are a lot of such f and g. Here is one pair:

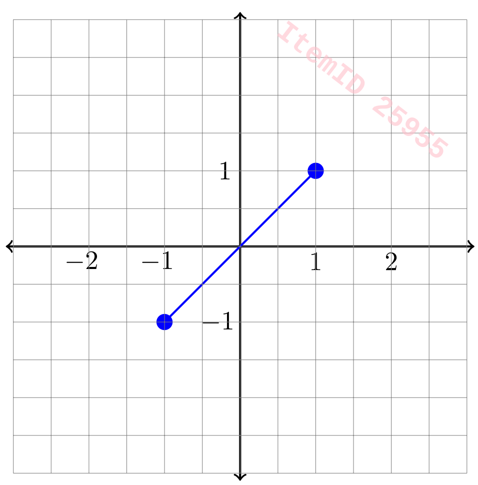
$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) + 1, & x \neq 0 \\ 1, & x = 0 \end{cases}, \quad g(x) = 1$$

(5c) (2 points) $h''(0)$ exists and $h'(0)$ does not exist.
We cannot find such f and g. This is because if $h''(0)$ exists, we can prove $h'(0)$ exists.
Since $h''(0)$ exists, we have h' is differentiable at 0.
Since h' is differentiable at 0, we have h' is continuous at 0.
Therefore, $h'(0)$ exists according to the definition of continuity.

6. (4 points) Let $g : [\pi, 2\pi] \to [-1, 1]$ be the restriction of $\cos x$ to the interval $[\pi, 2\pi]$.
Define g^{-1} to be the inverse function of g. Sketch the following graphs on the given intervals. Make sure to label your axes.
$y = g^{-1}(\cos(x))$ on $[-2\pi, 2\pi]$



$y = \cos(g^{-1}(x))$ on $[-1, 1]$



7. Let $a, b \in \mathbb{R}$. Let g be a function with domain $\mathbb{R}$.

(7a) (2 points) Write the definition "g is differentiable everywhere".

g is differentiable everywhere if

$$\forall x \in \mathbb{R}, \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \textit{ exists} .$$

g is differentiable everywhere if

$$\forall x \in \mathbb{R}, \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = L \textit{ for some } L \in \mathbb{R}.$$

(7b) (4 points) We know the following:

- $g(0) = a$
- g is differentiable at 0 and $g'(0) = b$
- for every $x, y \in \mathbb{R}$, $g(x+y) = g(x) + g(y) + 2xy - a$

Prove that g is differentiable everywhere and find a formula for g'.

We want to show $\forall x \in \mathbb{R}, \lim\limits_{h \to 0} \frac{g(x+h)-g(x)}{h} = L \textit{ for some } L \in \mathbb{R}.$

Proof: Let $x \in \mathbb{R}$ be fixed.

By the definition of the derivative, we have

$$\begin{aligned} g'(x) &= \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h \to 0} \frac{g(x) + g(h) + 2xh - a - g(x)}{h} \qquad \textit{(by the third hypothesis)} \\ &= \lim_{h \to 0} \frac{g(h) + 2xh - a}{h} \\ &= \lim_{h \to 0} \left[\frac{g(h) - a}{h} + 2x \right] \end{aligned}$$

Note that $g'(0) = \lim\limits_{h \to 0} \frac{g(h)-g(0)}{h} = \lim\limits_{h \to 0} \frac{g(h)-a}{h}$ by using the definition of the derivative as a limit and the first hypothesis.

Also, we know $g'(0) = b$ from the second hypothesis.

Thus, we have $\lim\limits_{h \to 0} \frac{g(h)-a}{h}$ exists and equals to $f'(0) = b$.

Therefore, we can use the limit laws of sums to get

$$g'(x) = \lim_{h \to 0} \left[\frac{g(h) - a}{h} + 2x \right] = \lim_{h \to 0} \left[\frac{g(h) - a}{h} \right] + \lim_{h \to 0} 2x = f'(0) + 2x = b + 2x$$

Final Answer

$\forall x \in \mathbb{R}, g'(x) = 2x + b$

8. Let f and g be two differentiable functions with domain $\mathbb{R}$. Suppose that for all $x \in \mathbb{R}$

$$f'(x) = -2g(x), g'(x) = 2f(x)$$

(8a) (3 points) Prove that there is a constant C such that $f^2(x) + g^2(x) = C$ for all $x \in \mathbb{R}$.
Proof: Define a function $F(x) = f^2(x) + g^2(x)$.
Since f and g are differentiable with domain $\mathbb{R}$, we have $F(x)$ is differentiable with domain $\mathbb{R}$.
And $F'(x) = 2f(x)f'(x) + 2g(x)g'(x)$.
Plugging in $f'(x) = -2g(x), g'(x) = 2f(x)$ into $F'(x)$, we have $\forall x \in \mathbb{R}$

$$F'(x) = 2f(x)f'(x) + 2g(x)g'(x) = 2f(x) \cdot (-2g(x)) + 2g(x) \cdot 2f(x) = 0$$

Apply the theorem in 5.9, we can conclude there is a constant C such that $F(x) = C$ for all $x \in \mathbb{R}$.
This completes the proof.

(8b) (1 point) Suppose that $f(0) = 0$ and $g(0) = -3$.What is C?

Final Answer

$C = 9$

(8c) (1 point) (**Bonus problem**) Give an example of f and g which satisfy the following properties:

- f and g are differentiable with domain $\mathbb{R}$
- $f(0) = 0$, $g(0) = -3$
- $\forall x \in \mathbb{R}$, $f'(x) = -2g(x)$, $g'(x) = 2f(x)$

Final Answer

$f(x) = 3\sin(2x)$

Final Answer

$g(x) = -3\cos(2x)$