

Numerical Methods (ENUME 2019) – Project Assignment B: Approximation of functions

1. Make a graph of the function: $f(x) = -\cos(\pi x)e^{-x-\frac{1}{3}}$ for $x \in [-1, 1]$ and indicate a sequence of its values which will be next used for approximation:

$$\{y_n = f(x_n) | n = 1, 2, \dots, N\}, \text{ where } x_n = -1 + 2 \frac{n-1}{N-1}$$

Repeat this exercise for $N = 10, 20$ and 30 .

2. Develop a program for the least-squares approximation of the function $f(x)$ on the basis of the data $\{(x_n, y_n) | n = 1, \dots, N\}$, using the operator of pseudoinversion "\" implemented in MATLAB. Use the B-spline functions defined by the formula:

$$B_{S_k}(x) = B_S(2(x - x'_k) + 2) \quad B_S(x) = \begin{cases} x^3 & x \in [0, 1) \\ -3(x-1)^3 + 3(x-1)^2 + 3(x-1) + 1 & x \in [1, 2) \\ 3(x-2)^3 - 6(x-2)^2 + 4 & x \in [2, 3) \\ -(x-3)^3 + 3(x-3)^2 - 3(x-3) + 1 & x \in [3, 4] \end{cases}$$

$$\text{where } x'_k = -1 + 2 \frac{k-1}{K-1} \quad \text{for } k = 1, 2, \dots, K$$

as a basis of linearly independent functions. Check the correctness of the program for several pairs of the values of N and K . Add the results of approximation to the corresponding graphs made according to the instruction provided in Section 1.

3. Carry out a systematic investigation of the dependence of the accuracy of approximation on the values of N and K . Use the following accuracy indicators for this purpose:

$$\delta_2(K, N) = \frac{\|\hat{f}(x; K, N) - f(x)\|_2}{\|f(x)\|_2} \quad (\text{the root-mean-square error})$$

$$\delta_\infty(K, N) = \frac{\|\hat{f}(x; K, N) - f(x)\|_\infty}{\|f(x)\|_\infty} \quad (\text{the maximum error})$$

where $\hat{f}(x; K, N)$ is an approximating function obtained for N and K . Make the three-dimensional graphs of the functions $\delta_2(K, N)$ and $\delta_\infty(K, N)$ for $N \in \{5, \dots, 50\}$ and $K < N$.

4. Carry out a systematic investigation of the dependence of the indicators $\delta_2(K, N)$ and $\delta_\infty(K, N)$ on the standard deviation $\sigma_y \in [10^{-5}, 10^{-1}]$ of random errors the data used for approximation are corrupted with. For this purpose:

- Generate the error-corrupted data according to the formula:

$$\tilde{y}_n = y_n + \Delta \tilde{y}_n \quad \text{for } n = 1, \dots, N$$

where $\{\Delta \tilde{y}_n\}$ are pseudorandom numbers following the zero-mean normal distribution with the variance σ_y^2 , obtained by means of the MATLAB operator **randn**.

- For each value of the standard deviation σ_y , determine the values \tilde{N} and \tilde{K} minimising $\delta_2(K, N)$ and compute $\delta_{2,MIN}(\sigma_y) \equiv \delta_2(\tilde{K}, \tilde{N})$.
- Approximate the sequence of pairs $\langle \sigma_y, \delta_{2,MIN}(\sigma_y) \rangle$, determined for several dozen values of σ_y , by means of the MATLAB operator **polyfit**; present the result of approximation using the logarithmic scale on both axes.