Project Forestcover
Team Members

Part 1

Exponential cost function

1. Choice of the Heuristic Functions

For cost function $c_1 = e^{(h_2 - h_1)}$, we choose heuristic functions as follows:

Let K be a constant, defined as equal to 0 if $\Delta d \leq |\Delta h|$, and equal to $\Delta d - |\Delta h|$ otherwise.

- Case 1: if $\Delta h < 0$, and $\Delta d > |\Delta h|$

$$h = |\Delta h| * e^{-1} + K \tag{1}$$

- Case 2: if $\Delta h = 0$

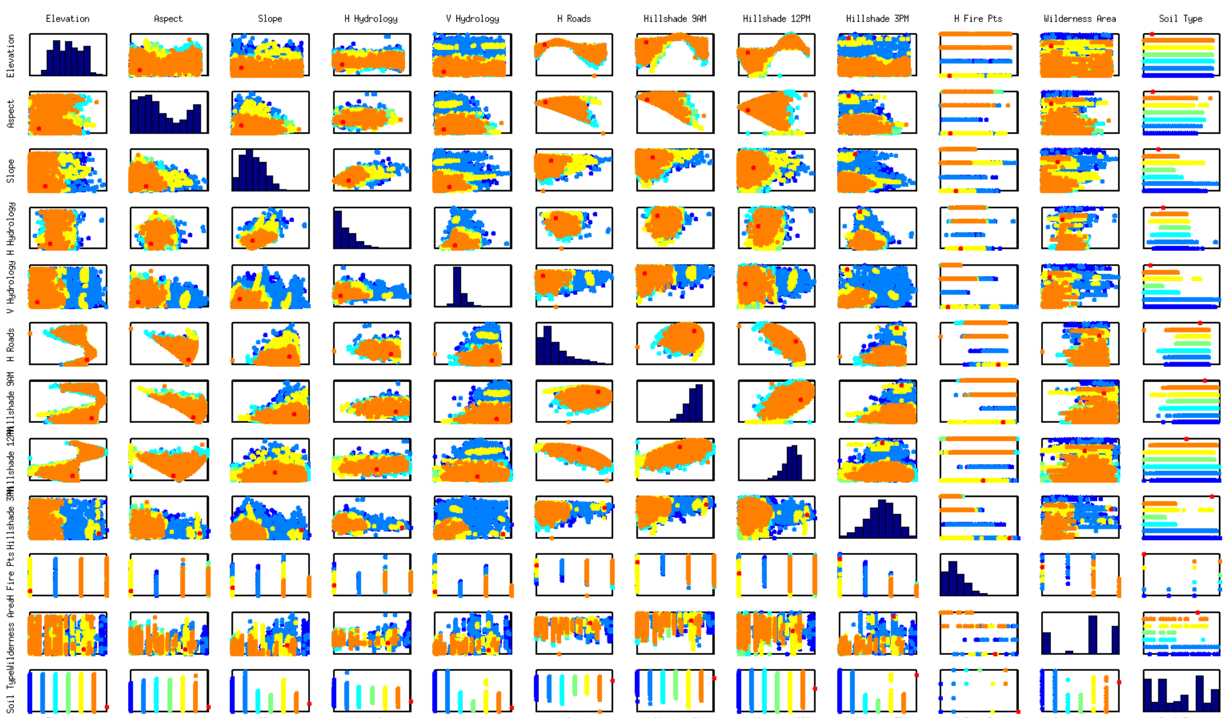
$$h = \Delta d + K \tag{2}$$

- Case 3: if $\Delta h > 0$, and $\Delta d \leq |\Delta h|$

$$h = |\Delta h| * e + K \tag{3}$$

- Case 4: This is a special case, so we discuss it seperately. If $\Delta h < 0$, and $\Delta d < |\Delta h|$

$$h = \Delta d * e^{\frac{\Delta h}{\Delta d}} + K \tag{4}$$



2. Proof of the heuristic functions

- Choice of K: Without the constant K, the heuristic function represents the minimal cost to go from p_1 to p_2 if the magnitude of Δd is less than or equal to Δh. In other words, the function underestimates the cost to change our altitude from p_1's height to that of the goal state. But that minimal change occurs in a number of steps less than the magnitude of Δh. So, without K, in all cases our function is a true underestimate, but by adding K (with a value greater than 0 when our goal is further away in horizontal distance than it is in height difference) we obtain a larger underestimate, which takes into account the need to travel an additional amount after the goal height has been reached. Case 2 of this proof provides more detail as to why the remaining distance to the goal is an acceptable underestimate.

- Case 1: As with case 3, by comparing the geometric mean and arithmetic mean of the optimal path from p_1 to p_2, we obtain a function which provides a lower bound on the cost of an optimal path to the goal. This function is minimized when the number of steps is equal to the negative of $-\Delta h = |\Delta h|$ (because the absolute value of a negative number is the negation of that number). Thus by the same equation as in case 1, we obtain $\Delta h \times e^{\frac{\Delta h}{|\Delta h|}} = \Delta h \times e^{-1}$.

- given by $\Delta d * e^0 = \Delta d$, which is the cost of going from p_1 to p_2 without any change in elevation, is an underestimate of the true cost.

- Case 3: Let the true cost from p_1 to goal be C, and it takes x moves to reach the goal; let h_i be the height of nodes along this path; let h_x be the height of the xth node, i.e. goal node. (We use h_2 for the height of goal in other cases, but in this case, h_2 is the height of 2nd node along this path, as h_i is the ith node alone this path)

$$C = \underbrace{e^{h_2-h_1} + e^{h_3-h_2} + e^{h_4-h_3} + ... + e^{h_x-h_{x-1}}}_{\text{total of } x \text{ terms}} \tag{5}$$

$$> x\sqrt[x]{e^{h_2-h_1} \times e^{h_3-h_2} \times e^{h_4-h_3} \times ... \times e^{h_x-h_{x-1}}} \tag{6}$$

$$= x\sqrt[x]{e^{h_2-h_1+h_3-h_2+h_4-h_3+...+h_x-h_{x-1}}} \tag{7}$$

$$= x\sqrt[x]{e^{h_x-h_1}} \tag{8}$$

But $h_x - h_1$ is Δh, so let F be a function of x.

$$F(x) = x\sqrt[x]{e^{\Delta h}}$$

and F(x) is a lower bound of the true cost C. We want to know the minimum value of this function F to get an lower bound of F, and this minimum value is apparently admissible, thus we can use it as our heuristic function. We simply need to take derivative of F to obtain the minimum value.

$$\frac{d}{dx}F(x) = \frac{d}{dx}x\sqrt[x]{e^{\Delta h}} = 0$$

gives when $x = \Delta h$, F has minimum value. Thus

$$F_{min} = \Delta h \sqrt[\Delta h]{e^{\Delta h}} = \Delta h \times e$$

and this is the heristic function when $\Delta h > 0$.

For case 1, we use same process, but Δh is a negative number. Thus F has minimum when $x = -\Delta h$ since x, the number of moves, must be a positive number, so we get

$$F_{min} = |\Delta h| \sqrt[|\Delta h|]{e^{\Delta h}} = \Delta h \times e^{-1}$$

and these two heuristic functions are guaranteed to be the smallest cost, thus admissible.

- Case 4: Because we are going downhill by a number of steps less than the height difference between p_1 and p_2. The minimum cost to p_2 is obtained by a direct route of Δd steps at a steady downward pace. The negative exponent is equal to the average height decrease necessary to reach p_2's height. Because this combination of factors maintains the smallest average step cost (going downhill each time) and the minimal number of steps (exactly equal to our diagonal distance from p_1 to p_2), this represents the absolute minimum cost to reach p_2 and thus is an underestimate of the true cost.

Ratio cost function

1. Choice of the Heuristic Functions

For cost function $c_2 = \frac{h_2}{h_1+1}$, we choose heuristic funtions as follows:

- Case 1: if $\Delta d = 0$
$$h = 0 \tag{9}$$

- Case 2: if $\Delta d = 1$
$$h = \frac{h_2}{h_1 + 1} \tag{10}$$

- Case 3: if $h_1 \leq 1$
$$h = \Delta d * 0.5 \tag{11}$$

- Case 4: if $\Delta d \geq h_1$
$$h = \log h_1 + \frac{h_1}{2} - \frac{1}{2} + \frac{\Delta d - h_1}{2} \tag{12}$$

- Case 5: Otherwise
$$h = \log \Delta d - \frac{1}{2} + \frac{\Delta d}{2} \tag{13}$$

2. Proof of the Heuristic Functions

General Description of Heuristic Philosophy: We relax the cost of climbing in height by allowing any increase in height to cost exactly 1. This is a lower bound on the cost of an upward step by our cost function, so this is always an underestimate of the true cost of an upward climb. Because the cost of upward movement is essentially nullified with this relaxation, in every case the optimal path is a combination of descent from our current height, coupled with, if further distance remains to be travelled, horizontal movement at a height of 1 (the lower we are, the cheaper is to move and maintain the same height, and our data is provided with the stipulation that no adjacent tiles have a height of 0).

- Case 1: If we are at the goal, the heuristic returns an estimate of 0.

$$\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + ... + \tfrac{1}{n+1} = \textstyle\sum_{i=1}^{n+1} \tfrac{1}{i}$$

- Case 2: The goal is one step away. The minimal cost to get there is to go there.

- Case 3: Our current height is already minimal, so the optimal path from p_1 to p_2 (remembing our relaxation which says that it costs us nothing to climb in the last step to the goal's height) is to continue to travel at a height of 1 until we reach the goal's tile. Since this cost would only hold on a real map if the goal's height was also 0 or 1, this is an underestimate of the true cost.

- Case 4: We are at a height greater than 1 and our height is greater than our horizontal distance to the goal. The optimal strategy is some combination of gradual (or not so gradual) descent coupled with lateral movement at a height of 1. We cannot descend all the way to our goal, because each descent decreases our elevation by 1 and our elevation is less than the distance to the goal. So we underestimate the cost of descending to ground level by assuming that we make h_1 steps, all of which have an absolute minimum cost of $\frac{1}{h_c+1}$, where h_c is our height at that stage of our descent.

  For example, if we start at height 10, and we will ultimately have to travel 50 tiles to reach our goal, we can descend for at most 10 tiles. From height 10, the minimal cost of descent is $\frac{1}{11}$. We underestimate the total cost of descent by allowing that cost, which would actually take us to an elevation of 1, to represent the cost of travelling 1 tile closer to our goal but remaining at a height of 9 (the longer we can stay elevated, the longer we can descend and incur costs less than one half, which we will be forced to accept once we have reached the bottom elevation). Staying elevated, since we are underestimating by using 1 always in the numerator, will allow each lateral step to have a cost equal to $\frac{1}{h_c+1}$, which is always less than one half. We could simply multiply h_1 by $\frac{1}{h_1+1}$, but by allowing the denominator to decrease gives us a larger underestimate and is hence preferable.

  So, the underestimated cost of descending from height 10 while moving laterally 10 tiles is equal to the sum of $\frac{1}{11} + \frac{1}{10} + \frac{1}{9} + ... + \frac{1}{2}$. This can be recognized as the first h_1 terms of the harmonic series. A lower bound on the partial sums of the harmonic series is given by: $\log n + \frac{1}{2n} - \frac{1}{2} \geq \log n - \frac{1}{2}$, which is the bound used to provide our underestimate of the cost of descent.

  This underestimate is generally far less than the true cost, but it has the advantage of being in all cases less than or equal to the true cost, and thus admissible.

- Case 5: This case is essentially the same as the previous case, except that our lateral distance to travel is less than our current height, so the minimal number of steps is equal to Δd, not h_1, and thus we consider only the first Δd terms of the harmonic series in calculating our lower bound.