1. Gain of z^*: Why the gain of z^* is at least $m \cdot \frac{0.2}{1-g(0)}$?
2. Gain of u: After the compensation step, why does the gain of u have a value of $m(y*)$?
3. Ratio Computation: Could you delve deeper into the methodology of computing the ratio? Was this derived analytically or did you use a specific computational tool to arrive at this?

Lemma 5.2. *Suppose u is active and matched earlier than v in $M(y_u, y_v)$. If v is matched with some x such that $w_{vx} \geq \frac{1}{2}$ in $M_u(y_u, y_v)$, then we have $\alpha_u + \alpha_v \geq g(y_u)$ in $M(y_u, y_v)$.*

Proof. By Fact 5.1, we have $\alpha_u \geq g(y_u) - h(y_u)$. If v remains matched with x in $M(y_u, y_v)$, then the lemma holds since $\alpha_u + \alpha_v \geq g(y_u) - h(y_u) + \frac{1}{2}m(y_v) \geq g(y_u)$ (recall that $\min_y\{m(y)\} \geq 5 \cdot \max_x\{h(x)\}$.). If u does not have a victim or only needs to send 0 compensation to its victim, then we are also done.

Otherwise, by Lemma 5.1, the symmetric difference between $M(y_u, y_v)$ and $M_u(y_u, y_v)$ is an alternating path starting from u that contains (v, x). Let z be matched by u in $M(y_u, y_v)$ and z^* be the victim of u. Since the edge z^* matches in $M(y_u, y_v)$ appears no later than (v, x) in the alternating path, by Lemma 5.1, the perturbed weight of this edge is at least $(1 - g(\min\{y_v, y_x\}))\frac{1}{2} \geq 0.2$ (recall that $g(\cdot) \in [0.4, 0.6]$). Hence the gain of z^* in $M(y_u, y_v)$ before the compensation step is at least $m(y_{z^*}) \cdot \frac{0.2}{1-g(0)} = m(y_{z^*}) \cdot \frac{1}{3}$. Consequently the gain of u after the compensation step is $\alpha_u \geq (g(y_u) - h(y_u))w_{uz} + m(y_{z^*}) \cdot \frac{1}{3} \geq g(y_u)$. □

Additionally, while reading, I noticed a couple of potential typos:

- In the section referring to "Fact 5.1", it seems like it should be labeled as "Lemma 5.1".

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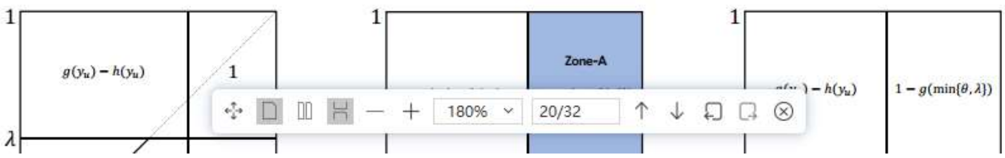
the definition of θ (recall that $y_u > \theta$). The same argument implies that v is not matched strictly earlier. Hence $\alpha_u + \alpha_v = 1$ when $y_u > \theta$ and $y_v > \lambda$ (recall that u, v are perfect partners, and hence they do not need to send any compensation).

Next, we consider the case when $y_u < \theta$ and $y_v > \lambda$. We have (1) v is not matched strictly earlier than u; and (2) u is not passively matched strictly earlier than v. Because in either case, increasing y_u does not change the matching status of v, which means that v is not matched with u in $M(1, y_v)$, contradicting the definition of λ. Hence when $y_u < \theta$ and $y_v > \lambda$, either u is passively matched by v, or u is active and matched earlier than v. By Lemma 5.1 we have $\alpha_u + \alpha_v \geq g(y_u) - h(y_u)$[13]. Symmetrically, we have $\alpha_u + \alpha_v \geq g(y_v) - h(y_v)$ when $y_u > \theta$ and $y_v < \lambda$.

Finally, we consider the case when $y_u < \theta$ and $y_v < \lambda$. We show that in this case we must have $\alpha_u + \alpha_v \geq \min\{g(y_u), g(y_v)\}$. Suppose u is matched earlier than v. Then u must be active, as otherwise u is also passive in $M(y_u, 1)$. By definition of λ, we know that in $M_u(y_u, y_v)$, the edge v matches has weight at least $w_{uv} = 1$. Applying Lemma 5.2, we have $\alpha_u + \alpha_v \geq g(y_u)$. Similarly, when v is matched earlier than u, we have $\alpha_u + \alpha_v \geq g(y_v)$. Therefore, $\alpha_u + \alpha_v \geq \min\{g(y_u), g(y_v)\}$. Given that g is non-decreasing, we have $\alpha_u + \alpha_v \geq g(y_u)$ when $y_u < y_v$ and $\alpha_u + \alpha_v \geq g(y_v)$ when $y_v < y_u$. In summary, we have (refer to Figure 6a)

$$\mathop{\mathbf{E}}_{y_u, y_v}[\alpha_u + \alpha_v] \geq (1-\lambda)\int_0^\theta \Big(g(y_u) - h(y_u)\Big)dy_u + (1-\theta)\int_0^\lambda \Big(g(y_v) - h(y_v)\Big)dy_v + \int_0^\lambda (\lambda - y_u)g(y_u)dy_u + \int_0^\lambda (\theta - y_v)g(y_v)dy_v + (1-\theta)(1-\lambda). \tag{8}$$



In another section, I believe the word "expect" might be intended to be "except".

LEMMA 6.1 (MONOTONICITY). *Consider any matching $M(\vec{y})$ and any vertex $u \in L$, if we fix the ranks of all vertices expect u, the weight of the edge u matches is non-increasing w.r.t. $y_u \in [0, 1]$.*