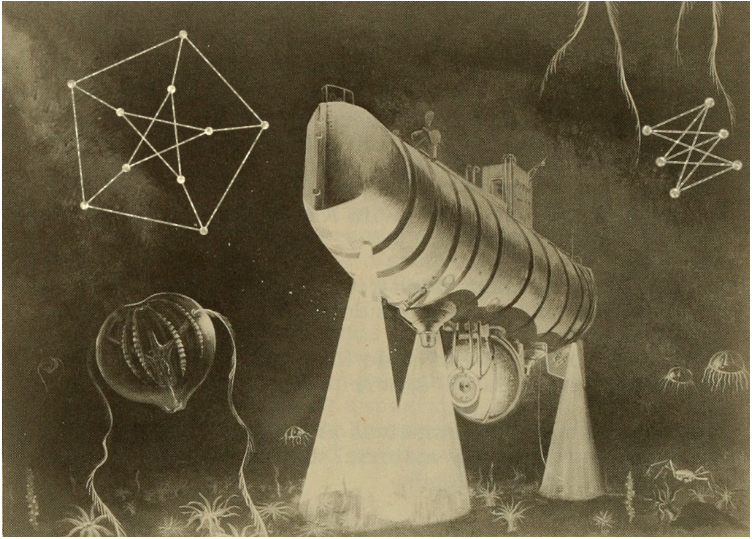
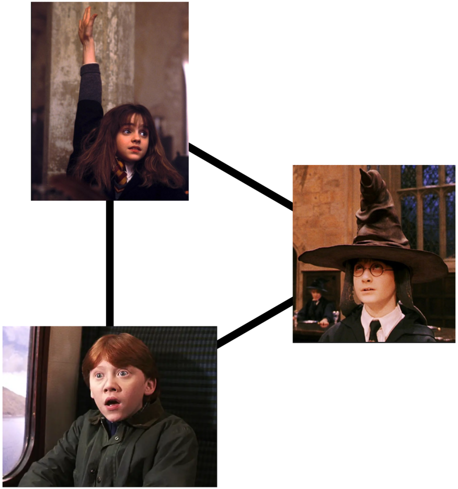
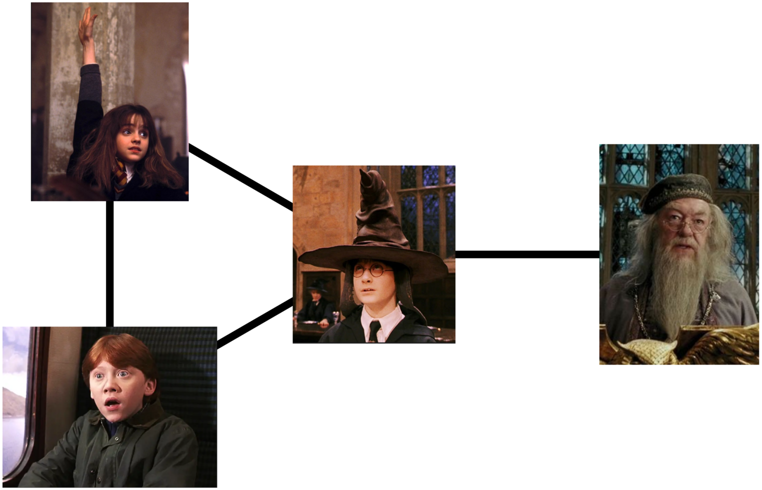
A Combinatorial Journey to the Challenger Deep of Mathematics

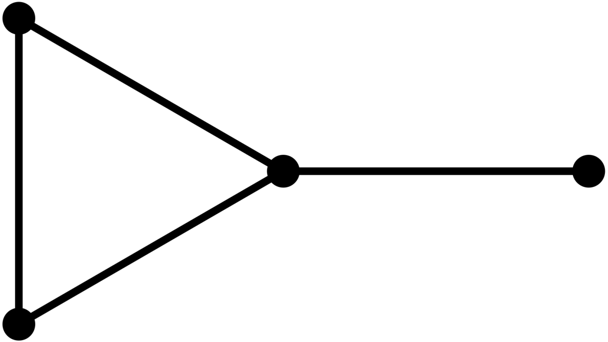
Jan Kurkofka



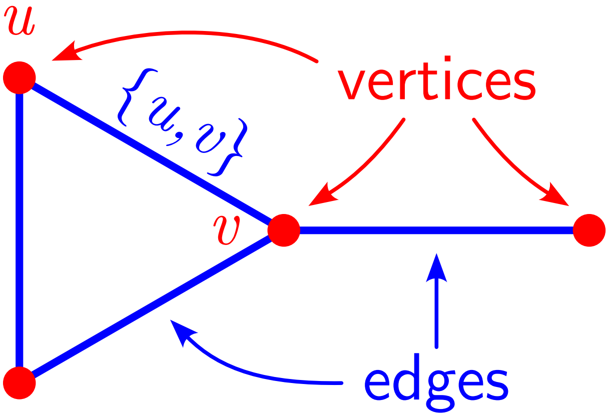


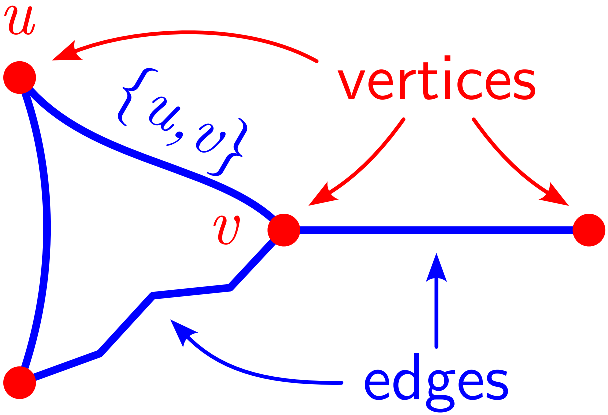




graph

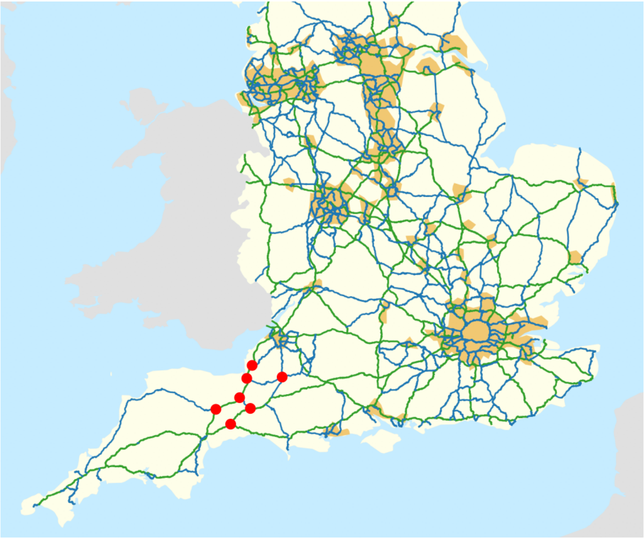


graph

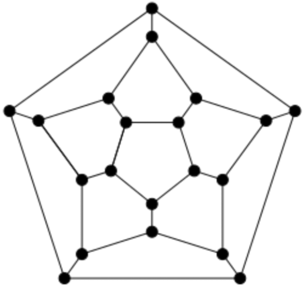


Social networks

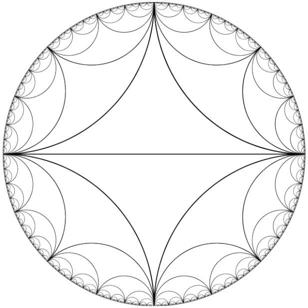
Infrastructure



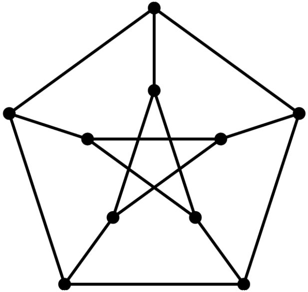
Maths!



Dodecahedron

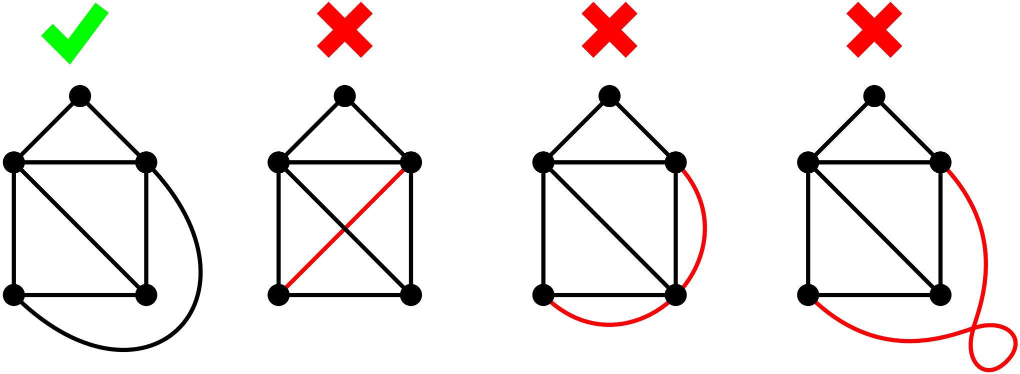


Farey graph



Petersen graph

Which graphs $\underbrace{\text{can be drawn in the plane so that no two edges cross}}_{\textit{are planar}}$?

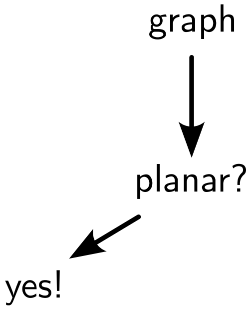


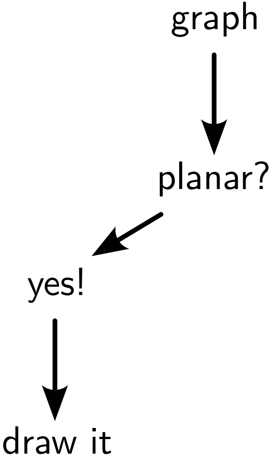
graph

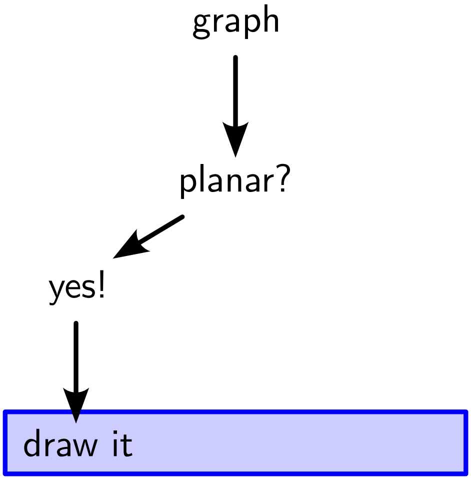
graph

$\downarrow$

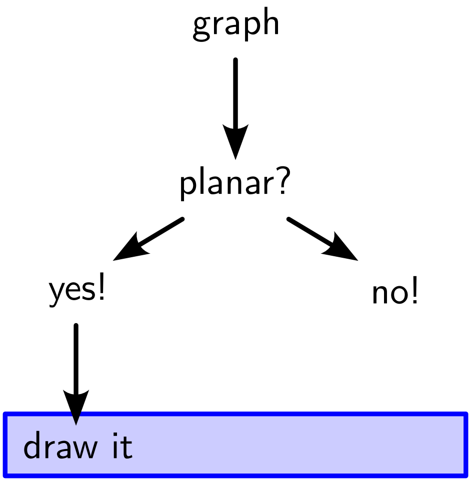
planar?


graph
planar?
yes!

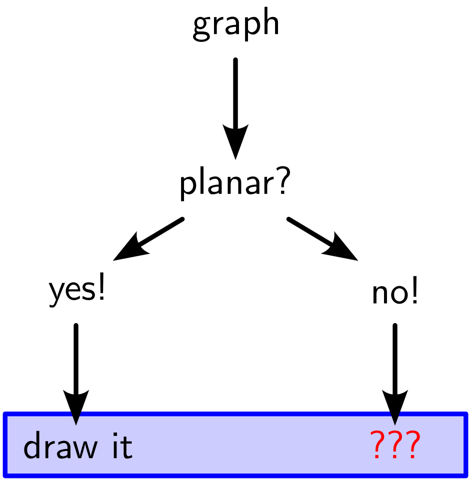

graph
planar?
yes!
draw it



Easy to verify



Easy to verify



Easy to verify

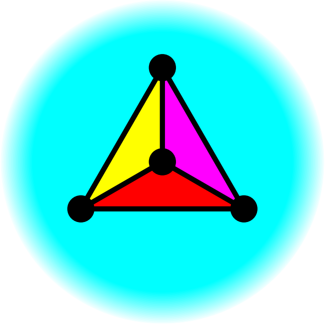
Euler's formula (1752)

For every planar drawing of a connected graph with n vertices and m edges:

$$n - m + \ell = 2$$

where ℓ is the number of faces of the drawing.

faces: connected regions of the plane minus the drawing



$$\begin{aligned} n &= 4 \\ m &= 6 \\ \ell &= 4 \end{aligned}$$

Euler's formula (1752)

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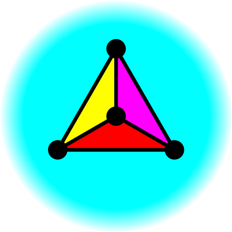

where ℓ is the number of faces of the drawing.

faces: connected regions of the plane minus the drawing

Corollary A. Every triangulation of the plane with n vertices has $3n - 6$ edges.

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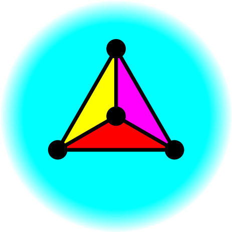


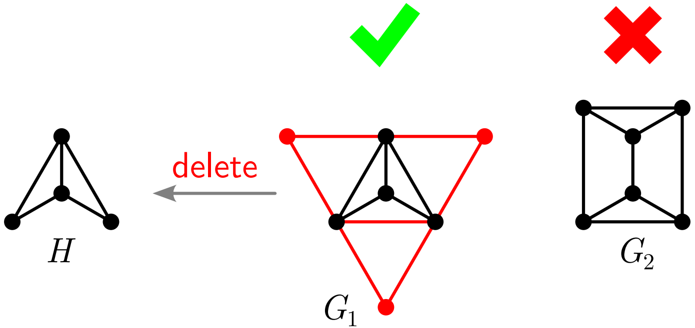
where ℓ is the number of faces of the drawing.

faces: connected regions of the plane minus the drawing

Corollary A. Every triangulation of the plane with n vertices has $3n - 6$ edges.

Corollary B. K_5 is not planar.

A graph H is a *subgraph* of a graph G if H can be obtained from G by successively deleting edges or isolated vertices.



Fact. Subgraphs of planar graphs are planar.

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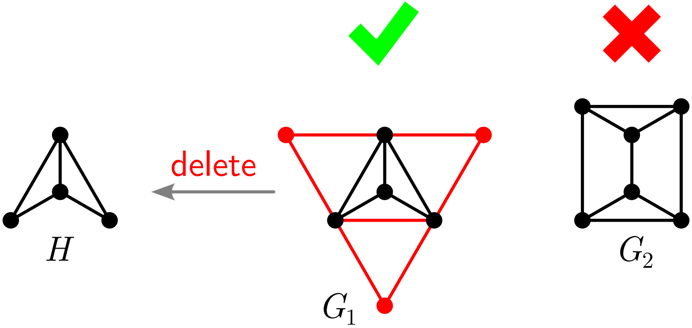
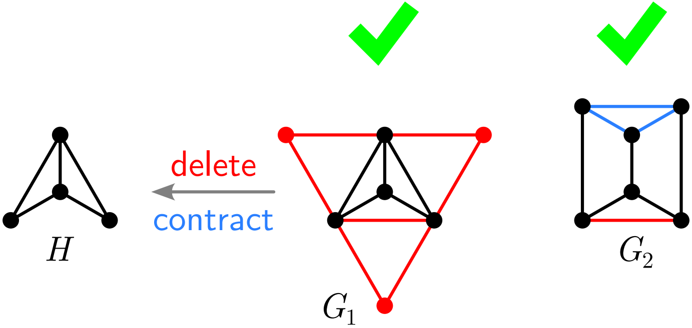


Fact. Subgraphs of planar graphs are planar.

Conjecture. Every nonplanar graph contains K_5 as a subgraph.

A graph H is a *minor* of a graph G if H can be obtained from G by successively deleting edges or isolated vertices or contracting edges.

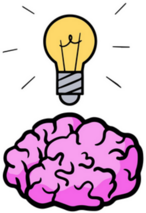


Fact. Minors of planar graphs are planar.

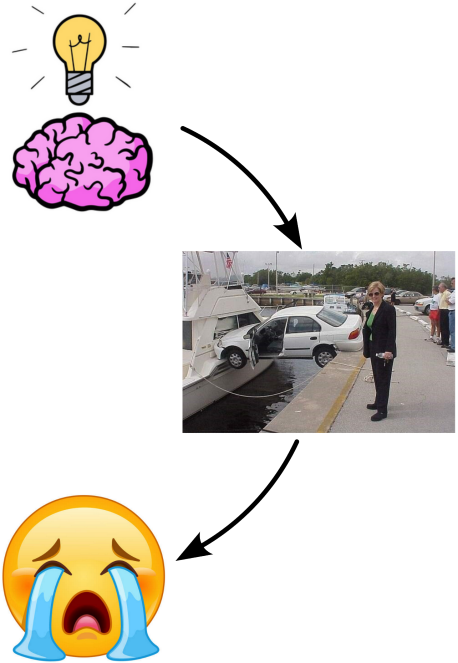
Conjecture. Every nonplanar graph contains K_5 as a minor.

Life as a Mathematician

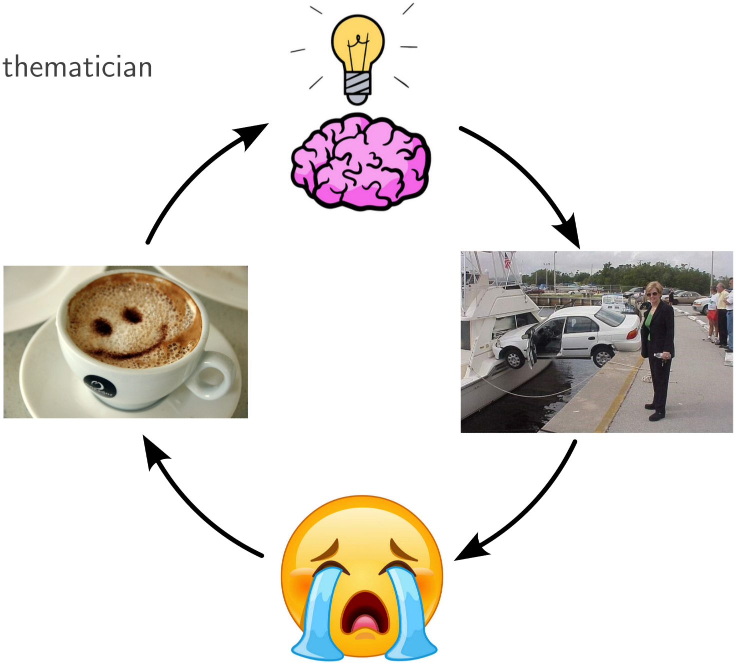
Life as a Mathematician



Life as a Mathematician



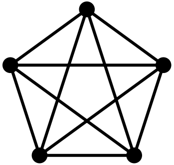
Life as a Mathematician



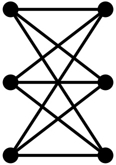
Kuratowski's theorem (1930)

For every graph G, the following assertions are equivalent:

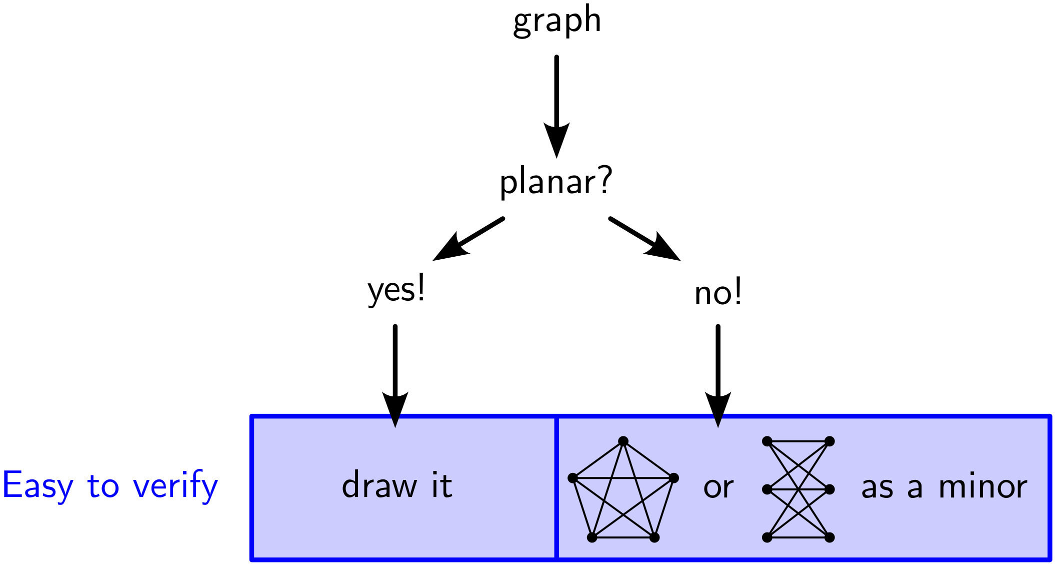
- G is planar;
- G contains neither K_5 nor $K_{3,3}$ as a minor.



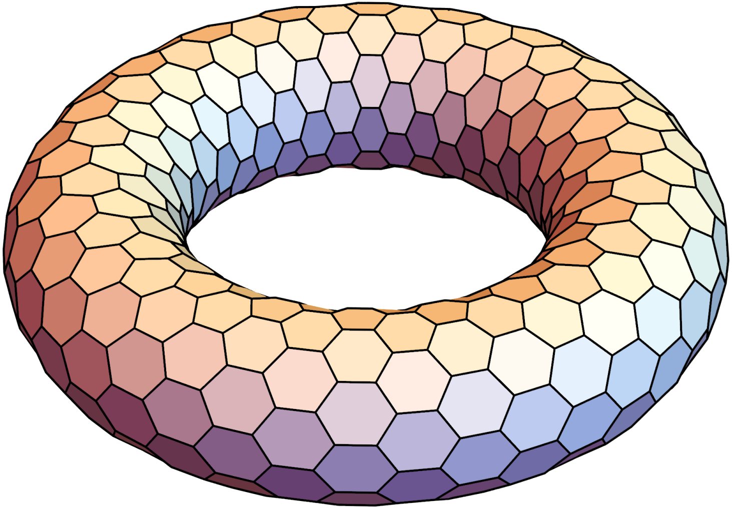
K_5



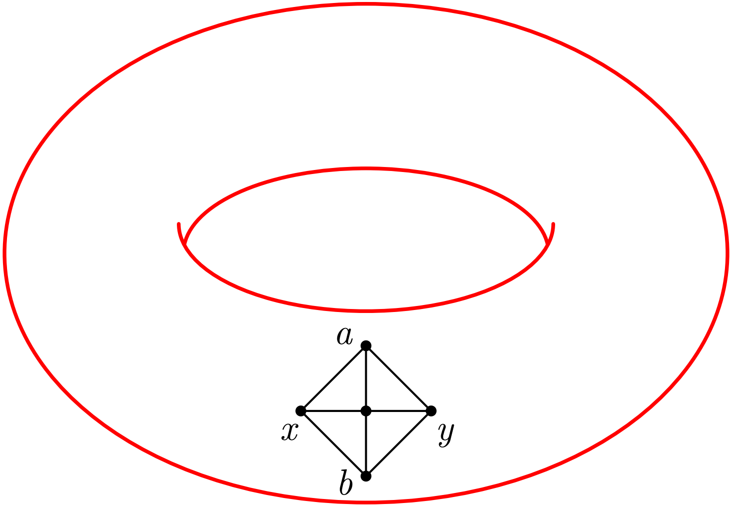
$K_{3,3}$


graph
planar?
yes!
no!
Easy to verify
draw it
or
as a minor

Is there a Kuratowski-type theorem for the torus?



Is there a planar drawing of K_5 on the torus?



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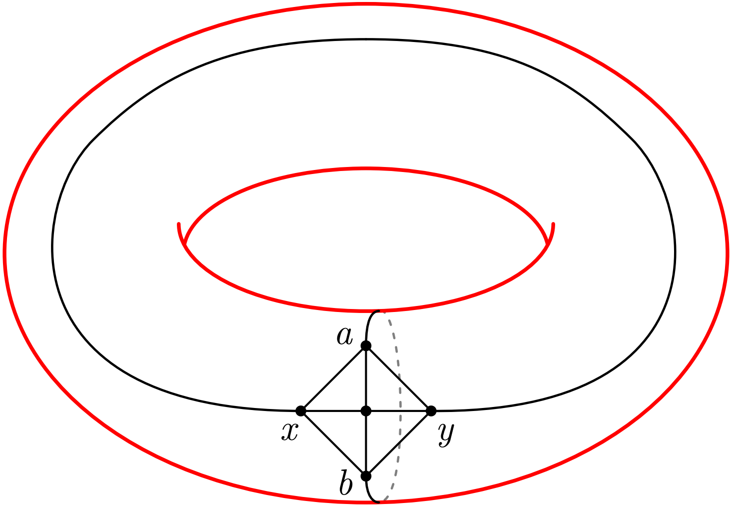
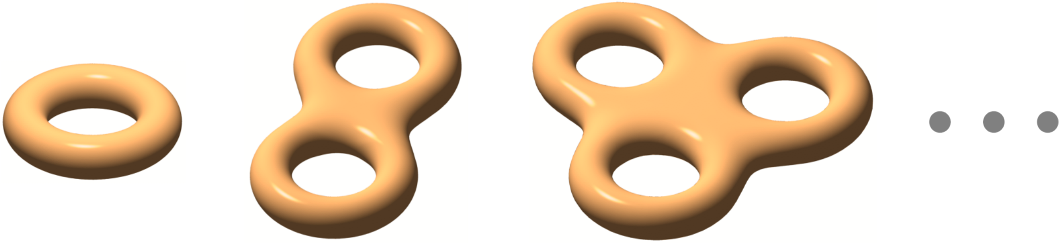
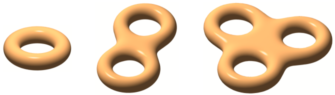


Are there Kuratowski-type theorems for other surfaces?



Conjecture (allegedly Wagner, 1960s)

For **every** graph-property $\mathcal{P}$ that is closed under taking minors

(e.g. being planar or admitting a drawing on  $\cdots$)

Conjecture (allegedly Wagner, 1960s)

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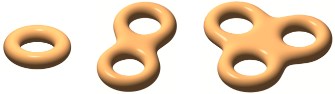
(e.g. being planar or admitting a drawing on  $\cdots$)

there exist **finitely many** graphs $X_1, \ldots, X_k$ such that the following assertions are equivalent:

- G exhibits the property $\mathcal{P}$;
- G contains none of the graphs $X_1, \ldots, X_k$ as a minor.

minor-closed graph-property $\mathcal{P}$	excluded minors $X_1, \ldots, X_k$
planar	

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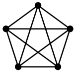
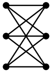
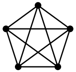
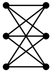
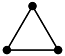
planar	 
forest	

minor-closed graph-property $\mathcal{P}$	excluded minors $X_1, \ldots, X_k$
planar	 
forest	

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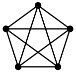
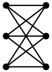
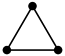
planar	 
forest	
linkless	

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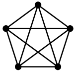
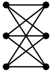
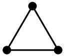
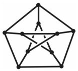
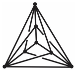
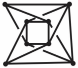
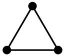
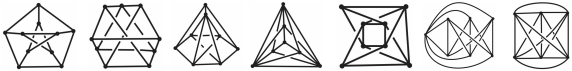
planar	 
forest	
linkless	      

minor-closed graph-property $\mathcal{P}$	excluded minors $X_1, \ldots, X_k$
planar	
forest	
linkless	
planar after deleting $\leqslant 1$ vertex	

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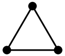
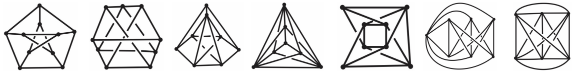
forest	
linkless	
planar after deleting $\leqslant 1$ vertex	???

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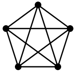
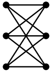
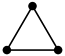
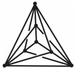
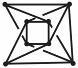
planar	 
forest	
linkless	      
planar after deleting $\leqslant 1$ vertex	$??? \geqslant 157$

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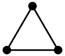
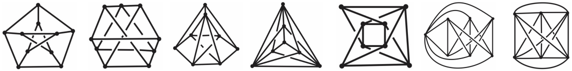
forest	
linkless	
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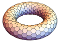
torus 	

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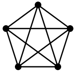
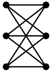
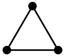
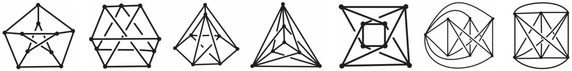
planar	 
forest	
linkless	
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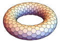
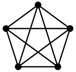
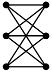
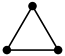
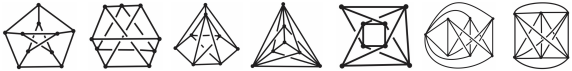
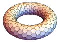
torus 	$???$

minor-closed graph-property $\mathcal{P}$	excluded minors $X_1, \ldots, X_k$
planar	 
forest	
linkless	
planar after deleting $\leqslant 1$ vertex	$??? \geqslant 157$
torus 	$??? \geqslant 17,523$

Conjecture (allegedly Wagner, 1960s)

For **every** minor-closed graph-property $\mathcal{P}$ there exist **finitely many** graphs $X_1, \ldots, X_k$ such that the following assertions are equivalent:

- G exhibits the property $\mathcal{P}$;
- G contains none of the graphs $X_1, \ldots, X_k$ as a minor.

Neil Robertson

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Corollary. For every minor-closed graph-property there exists an efficient (cubic time) algorithm for testing whether a given graph exhibits the property.

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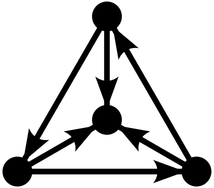

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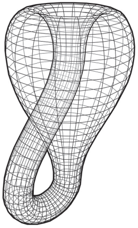
Active research



- Graph-Minor Theorem for matroids (write-up phase)
- Graph-Minor Theorem for directed graphs
- Given an explicit $\mathcal{P}$, find $X_1, \dots, X_k$ explicitly
- Algorithms to compute $X_1, \dots, X_k$ given $\mathcal{P}$

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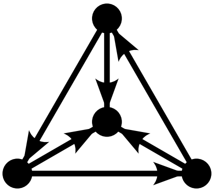

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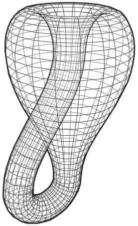

Thank you!

All feedback welcome :)