END-FAITHFUL SPANNING TREES IN GRAPHS WITHOUT NORMAL SPANNING TREES

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Abstract. Schmidt characterised the class of rayless graphs by an ordinal rank function, which makes it possible to prove statements about rayless graphs by transfinite induction. Halin asked whether Schmidt's rank function can be generalised to characterise other important classes of graphs. In this paper we answer Halin's question in the affirmative: we characterise two important classes of graphs by an ordinal rank function.

Seymour and Thomas have characterised for every uncountable cardinal κ the class of graphs without a T_κ minor. We extend their characterisations by an ordinal rank function, one for every uncountable cardinal κ.

Another largely open problem raised by Halin asks for a characterisation of the class of graphs with an end-faithful spanning tree. A well-studied subclass is formed by the graphs with a normal spanning tree. We determine a larger subclass, the class of normally traceable graphs, which consists of the connected graphs with a rayless tree-decomposition into normally spanned parts. Investigating the class of normally traceable graphs further we prove that, for every normally traceable graph, having a rayless spanning tree is equivalent to all its ends being dominated. Our proofs rely on a characterisation of the class of normally traceable graphs by an ordinal rank function that we provide.

1. Introduction

Schmidt [6, 14] characterised the class of rayless graphs by an ordinal rank function, which makes it possible to prove statements about rayless graphs by transfinite induction. For example, Bruhn, Diestel, Georgakopoulos and Sprüssel [1, 6] proved the unfriendly partition conjecture for the class of rayless graphs in this way.

At the turn of the millennium, Halin [10] asked in his legacy collection of problems whether Schmidt's rank can be generalised to characterise other important classes of graphs besides the class of rayless graphs. In this paper we answer Halin's question in the affirmative: we characterise two important classes of graphs by an ordinal rank function.

As our first main result, we characterise for every uncountable cardinal κ the class of graphs without a T_κ minor by an ordinal rank function that we call the κ-rank (recall that T_κ denotes the κ-branching tree):

Theorem 1. *For every graph G and every uncountable cardinal κ the following assertions are equivalent:*

(i) *G contains no T_κ minor;*
(ii) *G has a κ-rank.*

This extends Seymour and Thomas' characterisations [16]. We remark that, for regular uncountable cardinals κ, they also showed that a graph contains a T_κ minor if and only if it contains a subdivision of T_κ.

Our second main result addresses another largely open problem raised by Halin. Call a spanning tree T of a graph G *end-faithful* if the natural map $\varphi\colon \Omega(T) \to \Omega(G)$ satisfying $\omega \subseteq \varphi(\omega)$ is bijective. Here, $\Omega(T)$ and $\Omega(G)$ denote the set of ends of

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T and of G, respectively. Halin [9] conjectured that every connected graph has an end-faithful spanning tree. However, Seymour and Thomas [15] and Thomassen [17] constructed uncountable counterexamples; for instance, there exists a connected graph that has precisely one end but all whose spanning trees must contain a subdivision of $T_{\aleph_1}$. Ever since, it has been an open problem to characterise the class of graphs that admit an end-faithful spanning tree.

Normal spanning trees are important examples of end-faithful spanning trees. Given a graph G, a rooted tree $T \subseteq G$ is *normal* in G if the endvertices of every T-path in G are comparable in the tree-order of T, cf. [6]. Call a set U of vertices of a graph G *normally spanned* in G if U is contained in a tree $T \subseteq G$ that is normal in G. The graph G is *normally spanned* if $V(G)$ is normally spanned in G, i.e., if G has a normal spanning tree. Thus, every normally spanned graph has an end-faithful spanning tree.

A second existence result for end-faithful spanning trees is due to Polat [13] and directly addresses the counterexamples by Seymour and Thomas and by Thomassen: every connected graph that does not contain a subdivision of $T_{\aleph_1}$ has an end-faithful spanning tree.

As our second main result, we determine a new subclass of the class of graphs with an end-faithful spanning tree. Call a connected graph G *normally traceable* if it has a rayless tree-decomposition into parts that are normally spanned in G. For the definition of tree-decompositions see [6].

Theorem 2. *Every normally traceable graph has an end-faithful spanning tree.*

Our theorem easily extends the two known existence results for end-faithful spanning trees: On the one hand, every normally spanned graph has a trivial tree-decomposition into one normally spanned part. On the other hand, every connected graph without a subdivision of $T_{\aleph_1}$ has a rayless tree-decomposition into countable parts by the characterisation of Seymour and Thomas [16], and countable vertex sets are normally spanned.

In both cases, the extension is proper: The $\aleph_1$-*branching trees with tops* are the graphs obtained from the rooted $T_{\aleph_1}$ by selecting uncountably many rooted rays and adding for every selected ray R a new vertex, its *top*, and joining it to infinitely many vertices of R [8]. Every $T_{\aleph_1}$ with tops has a star-decomposition into normally spanned parts where $T_{\aleph_1}$ forms the central part and each top plus its neighbours forms a leaf's part. However, not every $T_{\aleph_1}$ with tops has a normal spanning tree [8, 12], and every $T_{\aleph_1}$ with tops contains $T_{\aleph_1}$ as a subgraph.

As our third main result, we extend two existence results on rayless spanning trees. Recall that a vertex v of a graph G *dominates* a ray $R \subseteq G$ if there is an infinite v–R fan in G. An end of G is *dominated* if one (equivalently: each) of its rays is dominated, see [6]. For a connected graph G, having a rayless spanning tree is equivalent to all the ends of G being dominated if G is normally spanned [4] or if G does not contain a subdivision of $T_{\aleph_1}$ [13]. Our third main result extends these results, and any $T_{\aleph_1}$ with all tops witnesses that this extension is proper.

Theorem 3. *For every normally traceable graph G, having a rayless spanning tree is equivalent to all the ends of G being dominated.*

Finally, as our fourth main result we characterise the class of normally traceable graphs by an ordinal rank function that we call the normal rank:

Theorem 4. *For every graph G the following assertions are equivalent:*

(i) *G is normally traceable;*
(ii) *G has a normal rank.*

We use this in the proofs of all our results on normally traceable graphs.

This paper is organised as follows. Section 2 provides the tools and terminology that we use throughout this paper. In Section 3 we introduce the κ-rank and prove Theorem 1. Then, in Section 4 we introduce the normal rank and prove Theorem 4. We prove Theorem 2 in Section 5 and we prove Theorem 3 in Section 6.

2. Tools and terminology

Any graph-theoretic notation not explained here can be found in Diestel's textbook [6]. A non-trivial path P is an *A-path* for a set A of vertices if P has its endvertices but no inner vertex in A.

Recall that a *comb* is the union of a ray R (the comb's *spine*) with infinitely many disjoint finite paths, possibly trivial, that have precisely their first vertex on R. The last vertices of those paths are the *teeth* of this comb. Given a vertex set U, a *comb attached to U* is a comb with all its teeth in U, and a *star attached to U* is a subdivided infinite star with all its leaves in U. The following lemma is [6, Lemma 8.2.2], also see the series [2, 3, 4, 5].

Lemma 2.1 (Star-Comb Lemma)**.** *Let G be any connected graph and let $U \subseteq V(G)$ be infinite. Then G contains either a comb attached to U or a star attached to U.*

An *end* of a graph G, as defined by Halin [9], is an equivalence class of rays of G, where a ray is a one-way infinite path. Here, two rays are said to be *equivalent* if for every finite vertex set $X \subseteq V(G)$ both have a subray (also called *tail*) in the same component of $G - X$. So in particular every end ω of G chooses, for every finite vertex set $X \subseteq V(G)$, a unique component $C(X, \omega)$ of $G - X$ in which every ray of ω has a tail. In this situation, the end ω is said to *live* in $C(X, \omega)$. The set of ends of a graph G is denoted by $\Omega(G)$. We use the convention that Ω always denotes the set of ends $\Omega(G)$ of the graph named G.

Let us say that an end ω of a graph G is contained *in the closure* of M, where M is either a subgraph of G or a set of vertices of G, if for every finite vertex set $X \subseteq V(G)$ the component $C(X, \omega)$ meets M. Equivalently, ω lies in the closure of M if and only if G contains a comb attached to M with its spine in ω. We write $\partial_\Omega M$ for the subset of Ω that consists of the ends of G lying in the closure of M.

A subset X of a poset $P = (P, \leq)$ is *cofinal* in P, and $\leq$, if for every $x \in X$ there is a $p \in P$ with $p \geq x$. We say that a rooted tree $T \subseteq G$ contains a set U *cofinally* if $U \subseteq V(T)$ and U is cofinal in the tree-order of T. We remark that the original statement of the following lemma also takes critical vertex sets in the closure of T or U into account.

Lemma 2.2 ([2, Lemma 2.13])**.** *Let G be any graph. If $T \subseteq G$ is a rooted tree that contains a vertex set U cofinally, then $\partial_\Omega T = \partial_\Omega U$.*

Suppose that H is any subgraph of G and $\varphi\colon \Omega(H) \to \Omega(G)$ is the natural map satisfying $\eta \subseteq \varphi(\eta)$ for every end η of H. Furthermore, suppose that a set $\Psi \subseteq \Omega(G)$ of ends of G is given. We say that H is *end-faithful* for Ψ if $\varphi \restriction \varphi^{-1}(\Psi)$ is injective and $\mathrm{im}(\varphi) \supseteq \Psi$. And H *reflects* Ψ if φ is injective with $\mathrm{im}(\varphi) = \Psi$. A spanning tree of G that is end-faithful for all the ends of G is *end-faithful*.

Lemma 2.3 ([2, Lemma 2.11])**.** *If G is any graph and $T \subseteq G$ is any normal tree, then T reflects the ends of G in the closure of T.*

Given any graph G, a set $U \subseteq V(G)$ of vertices is *dispersed* in G if there is no end in the closure of U in G. Equivalently, U is dispersed if and only if G contains no comb attached to U. In [11], Jung proved that normally spanned sets of vertices can be characterised in terms of dispersed vertex sets:

Theorem 2.4 (Jung [11, Satz 6]; [2, Theorem 3.5])**.** *Let G be any graph. A vertex set $U \subseteq V(G)$ is normally spanned in G if and only if it is a countable union of dispersed sets. In particular, G is normally spanned if and only if $V(G)$ is a countable union of dispersed sets.*

3. Ranking T_κ-free graphs

In this section we characterise for every uncountable cardinal κ the class of graphs without a T_κ minor by an ordinal rank function that we call the κ-rank.

Suppose that κ is any infinite cardinal. Let us assign *κ-rank* 0 to all the graphs of order less than κ. Given an ordinal $\alpha > 0$, we assign *κ-rank* α to every graph G that does not already have a κ-rank $< \alpha$ and which has a set X of less than κ many vertices such that every component of $G - X$ has some κ-rank $< \alpha$. Note that the $\aleph_0$-rank is Schmidt's rank [6, 14].

The κ-rank behaves quite similarly to Schmidt's rank [6, p. 243]: When disjoint graphs G_i have κ-ranks $\xi_i < \alpha$, their union clearly has a κ-rank of at most α; if the union is finite, it has κ-rank $\max_i \xi_i$. Induction on α shows that subgraphs of graphs of κ-rank α also have a κ-rank of at most α. Conversely, joining less than κ many new vertices to a graph, no matter how, will not change its κ-rank.

Not every graph has a κ-rank. Indeed, an inflated κ-branching tree cannot have a κ-rank, since deleting less than κ many of its vertices always leaves a component that contains another inflated κ-branching tree. As subgraphs of graphs with a κ-rank also have a κ-rank, this means that only graphs without a T_κ minor can have a κ-rank. But all these do:

Theorem 1. *For every graph G and every uncountable cardinal κ the following assertions are equivalent:*

(i) *G contains no T_κ minor;*
(ii) *G has a κ-rank.*

Hence the κ-rank characterises the class of graphs without a T_κ minor.

Our proof relies upon a theorem by Seymour and Thomas [16] that we recall here. For every set M we denote by $[M]^{<\kappa}$ the set of all subsets of M of cardinality $< \kappa$. Now, given a graph G, we write $\mathscr{C}_X$ for the set of components of $G - X$ for every set $X \subseteq V(G)$ of vertices. An *escape* of order κ in G is a function σ which assigns to each $X \in [V(G)]^{<\kappa}$ the vertex set $V[\mathscr{C}] := \bigcup \{ V(C) \mid C \in \mathscr{C} \}$ of a subset $\mathscr{C} \subseteq \mathscr{C}_X$ in such a way that:

(i) if $X \subseteq Y$, then $\sigma(Y) \subseteq \sigma(X)$,
(ii) if $X \subseteq Y$, then for $\sigma(X) = V[\mathscr{C}]$ every component $C \in \mathscr{C}$ intersects $\sigma(Y)$, and
(iii) $\sigma(\emptyset) \neq \emptyset$.

We speak of (i), (ii) and (iii) as the first, second and third *escape axioms*. We remark that Seymour and Thomas' escapes can in fact be seen as more general predecessors of directions which describe the ends of a graph by a theorem of Diestel and Kühn [7].

Theorem 3.1 ([16, Theorem 1.3])**.** *For every graph G and every uncountable cardinal κ the following assertions are equivalent:*

(i) *G contains a T_κ minor;*
(ii) *G has an escape of order κ.*

We are now ready to prove Theorem 1:

Proof of Theorem 1. We show the equivalence $\neg(\mathrm{i}) \leftrightarrow \neg(\mathrm{ii})$. The forward implication has already been pointed out above. For the backward implication suppose that G has no κ-rank; we show that G must contain a T_κ minor. By Theorem 3.1 it suffices to find an escape of order κ in G. We define a candidate σ for such an escape as follows. Given any vertex set $X \in [V(G)]^{<\kappa}$ we call a component C of $G - X$ *bad* if it has no κ-rank, and we let $\sigma(X) := V[\mathscr{C}]$ for the collection $\mathscr{C}$ of all the bad components of $G - X$. It remains to show that σ satisfies all three escape axioms.

Having no κ-rank is closed under taking supergraphs, so the first axiom holds. For the second axiom, let any two vertex sets $X \subseteq Y \in [V(G)]^{<\kappa}$ be given, and consider any component $C \in \mathscr{C}$ for $\sigma(X) = V[\mathscr{C}]$. Then $C - Y$ must have a component that has no κ-rank, and this component then is bad as desired. Finally, the third axiom holds because the graph G must have a bad component. □

4. Normally traceable graphs

In this section we characterise the class of normally traceable graphs by an ordinal rank function that we call the normal rank.

Let G be any connected graph. A connected subgraph $H \subseteq G$ has *normal rank* 0 in G if the vertex set of H is normally spanned in G. Given an ordinal $\alpha > 0$, a connected subgraph $H \subseteq G$ has *normal rank* α in G if it does not already have a normal rank $< \alpha$ in G and if there is a vertex set $X \subseteq V(H)$ that is normally spanned in G such that every component of $H - X$ has some normal rank $< \alpha$ in G.

The graph G has *normal rank* α for an ordinal α if G has normal rank α in G.

Theorem 4. *For every connected graph G the following assertions are equivalent:*

(i) *G is normally traceable;*
(ii) *G has a normal rank.*

Moreover, if G has a tree-decomposition witnessing that G is normally traceable, then G has normal rank at most the rank of the decomposition tree. Conversely, if G has a normal rank, then G is normally traceable and this is witnessed by a tree-decomposition whose decomposition tree has as rank the normal rank of G.

Before we prove this theorem, we point out a few properties of the normal rank.

Lemma 4.1. *Let G be any connected graph.*

(i) *If G has $\aleph_1$-rank α, then G has some normal rank $\leq \alpha$.*
(ii) *There are graphs that have a normal rank but that have neither an $\aleph_1$-rank nor a normal spanning tree.*

Proof. (i) We show that every connected subgraph $H \subseteq G$ of $\aleph_1$-rank α has normal rank $\leq \alpha$ in G, by induction on α; for $H = G$ this establishes (i). Any connected countable subgraph of G is normally spanned in G by Jung's Theorem 2.4, so the base case holds. For the induction step suppose that $\alpha > 0$. We find a countable vertex set $X \subseteq V(H)$ so that every component of $H - X$ has some $\aleph_1$-rank $< \alpha$. As X is countable it is also normally spanned in G. By the induction hypothesis every component of $H - X$ has normal rank $< \alpha$ in G. Hence X witnesses that H has normal rank $\leq \alpha$ in G.

(ii) Let G be any $T_{\aleph_1}$ with all tops and all edges between each top and its corresponding ray. Then G has normal rank 1 because $G - T_{\aleph_1}$ consists only of isolated vertices. However, G has no $\aleph_1$-rank by Theorem 1, and G has no normal spanning tree as pointed out in [8, 12]. □

Lemma 4.2. *Let $H \subseteq H' \subseteq G$ be any three connected graphs.*

(i) *If H' has normal rank α in G, then H has normal rank $\leq \alpha$ in G.*
(ii) *If H has normal rank α in G, then H has normal rank $\leq \alpha$ in H'.*
In particular, if H has normal rank α in G, then H has normal rank $\leq \alpha$.

Proof. (i) Induction on α. If $\alpha = 0$, then the vertex set of H' is normally spanned in G; in particular, the vertex set of $H \subseteq H'$ is normally spanned in G.

Otherwise $\alpha > 0$. Then there exists a vertex set $X \subseteq V(H')$ that is normally spanned in G such that every component of $H' - X$ has normal rank $< \alpha$ in G. Every component of $H - X$ is contained in a component of $H' - X$ and hence has normal rank $< \alpha$ in G by the induction hypothesis. Thus, H has normal rank $\leq \alpha$ in G.

(ii) Induction on α. If $\alpha = 0$, then the vertex set of H is normally spanned in G. In particular, by Jung's Theorem 2.4, the vertex set of H is normally spanned in $H' \subseteq G$, so H has normal rank 0 in H' as desired.

Otherwise $\alpha > 0$. Then there exists a vertex set $X \subseteq V(H)$ that is normally spanned in G such that every component of $H - X$ has normal rank $< \alpha$ in G. Note that X is also normally spanned in $H' \subseteq G$ by Jung's Theorem 2.4. By the induction hypothesis, every component of $H - X$ has normal rank $< \alpha$ in H'. Thus, H has normal rank $\leq \alpha$ in H'. □

Proof of Theorem 4. Let G be any connected graph. To show the equivalence (i)$\leftrightarrow$(ii) together with the 'moreover' part of the theorem, it suffices to show the following two assertions:

(1) If G has a tree-decomposition witnessing that G is normally traceable, then G has a normal rank which is at most the rank of the decomposition tree.
(2) If G has a normal rank, then G is normally traceable and this is witnessed by a tree-decomposition whose decomposition tree has rank at most the normal rank of G.

(1) We show that every connected subgraph $H \subseteq G$ that has a rayless tree-decomposition $(T, \mathcal{V})$ into parts that are normally spanned in G does have normal rank $\leq \alpha$ in G for α the rank of T. We prove this by induction on α; for $H = G$ and α equal to the rank of the decomposition tree of some tree-decomposition of G witnessing that G is normally traceable we obtain (1). If H and $(T, \mathcal{V})$ are such that $\alpha = 0$, then T is finite, and hence the union of all the parts in $\mathcal{V}$ is normally spanned in G by Jung's Theorem 2.4; in particular, $V(H)$ is normally spanned in G and hence has normal rank 0 in G.

Otherwise H and $(T, \mathcal{V})$ are such that $\alpha > 0$. Let $W \subseteq V(T)$ be any finite vertex set such that every component of $T - W$ has rank $< \alpha$. Then the vertex set

$X := \bigcup_{t\in W} V_t \subseteq V(H)$ is normally spanned in G by Jung's Theorem 2.4. Every component of $H - X$ is contained in $\bigcup_{t\in T'} G[V_t]$ for some component T' of $T - W$, so by the induction hypothesis every component of $H - X$ has normal rank $< \alpha$ in G. Thus, H has normal rank $\le \alpha$ in G.

(2) Suppose that G is any connected graph that has a normal rank. We show that every connected subgraph $H \subseteq G$ of normal rank α in G has a rayless tree-decomposition $(T, \mathcal{V})$ into parts that are normally spanned in G such that T has rank $\le \alpha$, by induction on the normal rank α of H in G; for $H = G$ this establishes (2). If $\alpha = 0$, then $V(H)$ is normally spanned in G and the trivial tree-decomposition of H into the single part $V(H)$ is as desired.

Otherwise $\alpha > 0$. Then there exists a vertex set $X \subseteq V(H)$ that is normally spanned in G such that every component of $H - X$ has normal rank $< \alpha$ in G. By the induction hypothesis, every component C of $H - X$ has a rayless tree-decomposition $(T_C, \mathcal{V}_C)$ with $\mathcal{V}_C = (\, V_C^t \mid t \in T_C \,)$ such that every part is normally spanned in G and the rank of T_C is $< \alpha$. Without loss of generality the trees T_C are pairwise disjoint. We choose from every tree T_C an arbitrary node $t_C \in T_C$. Then we let the tree T be obtained from the disjoint union $\bigcup_C T_C$ by adding a new vertex t_* that we join to all the chosen nodes t_C. We define the family $\mathcal{V} = (\, V^t \mid t \in T \,)$ by letting $V^t := V_C^t \cup X$ for all $t \in T_C \subseteq T$ and $V^{t_*} := X$. Then $(T, \mathcal{V})$ is a rayless tree-decomposition of H into parts that are normally spanned in G by Jung's Theorem 2.4, and the rank of T is $\le \alpha$ because every component of $T - t_*$ has rank $< \alpha$. $\square$

5. End-faithful spanning trees

In this section we prove that every normally traceable graph has an end-faithful spanning tree. Our proof requires some preparation.

Lemma 5.1. *Let G be any graph and let $\Psi \subseteq \Omega(G)$ be any set of ends of G. Furthermore, let $H \subseteq G$ be any spanning forest that reflects Ψ and let C be any component of H. If a spanning tree T of G arises from H by adding one D–C edge for every component $D \neq C$ of H, then T reflects Ψ.* $\square$

Lemma 5.2. *Let G be any graph with a spanning tree $T \subseteq G$ that reflects a set $\Psi \subseteq \Omega(G)$ and let $R \subseteq G$ be a ray from some end in Ψ. Then there exists a spanning tree $T' \subseteq G$ that reflects Ψ and contains R.*

Moreover, T' can be chosen such that no end other than the end of R lies in the closure of the symmetric difference $E(T)\triangle E(T')$ (viewed as a subgraph of G).

The 'moreover' part of the lemma says that T and T' differ only locally. Note that there may also be no end in the closure of $E(T)\triangle E(T')$.

Proof. Given $T \subseteq G$, Ψ and R, we root T arbitrarily and write ω for the end of R in G. Furthermore, we write R_T for the unique rooted ray in T that is equivalent to R, and we pick a sequence $P_0, P_1, \ldots$ of pairwise disjoint R–R_T paths in G. We write C for the comb $C := R \cup \bigcup_n P_n$ consisting of R and all the paths P_n, and we write U for the vertex set of the subtree $\lceil C \rceil_T$ of T. Note that $R_T \subseteq \lceil C \rceil_T$ because the paths $P_0, P_1, \ldots$ meet R_T infinitely often. By standard arguments we have $\partial_\Omega C = \{\omega\}$, and so $\partial_\Omega U = \{\omega\}$ follows by Lemma 2.2. Since T reflects Ψ and $\lceil C \rceil_T$ contains only rays from ω, we deduce that $\lceil C \rceil_T$ is either rayless or one-ended. As $\lceil C \rceil_T$ contains the ray R_T, it is one-ended.

Next, we define an edge set $F \subseteq E(\lceil C \rceil_T)$, as follows. If R has a tail in R_T, then we set $F = \emptyset$. Otherwise R has no tail in R_T. Then we select infinitely many pairwise edge-disjoint C-paths $Q_0, Q_1, \ldots$ in the ray R_T (these exist because R has no tail in R_T). We choose one edge of every path Q_n and we let F consist of all the chosen edges, completing the definition of F.

The graph $(\lceil C \rceil_T \cup C) - F$ is a connected subgraph of G and inside it, we extend C arbitrarily to a spanning tree T_R. Then T_R has vertex set U, and T_R reflects $\{\omega\}$: Every ray R' in T_R that is disjoint from R meets at most one component of $C - R$ because C and R' are contained in the tree T_R, and hence R' must have a tail in $\lceil C \rceil_T - C$. But $\lceil C \rceil_T$ contains just one rooted ray, namely the ray R_T, and either R_T contains a tail of R or F consists of infinitely many edges of R_T, contradicting the existence of R' in $T_R \subseteq (\lceil C \rceil_T \cup C) - F$. It remains to extend T_R to a spanning tree of G reflecting Ψ. For this, we consider the collection $\{\, T_i \mid i \in I \,\}$ of all the components of $T - U$. By the choice of U, every end ω' of G other than ω is still represented by an end of one of the trees T_i: Indeed, if ω' is an end of G other than ω, then it does not lie in the closure of U, and hence every ray in ω' has a tail that avoids U. In particular, every ray in T that lies in ω' has some tail that avoids U. Therefore, the union of T_R and all the trees T_i is a spanning forest of G reflecting Ψ.

We extend this spanning forest to a spanning tree T' by adding all the T_i–T_R edges of T for every $i \in I$ (note that T contains precisely one T_i–T_R edge for every $i \in I$ as $T \cap G[U] = \lceil C \rceil_T$ is connected). Then T' reflects Ψ again by Lemma 5.1. To see $\partial_\Omega(E(T) \triangle E(T')) \subseteq \{\omega\}$ recall $\partial_\Omega G[U] = \{\omega\}$ and note that the symmetric difference is contained in $G[U]$ entirely. $\square$

Lemma 5.3. *Let G be any graph and let $X \subseteq V(G)$ be any vertex set.*

(i) *Every end of G is contained in the closure of X in G or in the closure of some component of $G - X$ in G.*
(ii) *Every end of G that is contained in the closure of two distinct components of $G - X$ in G is also contained in the closure of X in G.*

Proof. (i) Let ω be any end of G and let $R \in \omega$ be any ray. Then either the vertex set of R intersects X infinitely, or R has a tail that is contained in some component C of $G - X$. In the first case, ω is contained in the closure of X, and in the second case it is contained in the closure of C in G.

(ii) Let C and C' be two distinct components of $G - X$ and suppose that ω is any end of G that is contained in the closure of both C and C' in G. If $S \subseteq V(G)$ is any finite vertex set, then the component $C(S, \omega)$ meets both C and C'. As X separates C and C' in G it follows that $C(S, \omega)$ meets X as well. We conclude that ω is contained in the closure of X in G. $\square$

Lemma 5.4. *Let G be any connected graph, let $X \subseteq V(G)$ be normally spanned in G and let C be any component of $G - X$ so that $G[C \cup X]$ is connected. If C has normal rank ξ in G, then $G[C \cup X]$ has normal rank $\leq \xi$.*

Proof. Suppose that C is a component of $G - X$ that has normal rank ξ in G. If $\xi = 0$, then $V(C)$ is normally spanned in G and $G[C \cup X]$ has a normal spanning tree by Jung's Theorem 2.4, so $G[C \cup X]$ has normal rank 0 as desired. Otherwise there is a vertex set $Y \subseteq V(C)$ that is normally spanned in G and satisfies that every component of $C - Y$ has normal rank $< \xi$ in G. Note that $X \cup Y$ is normally spanned in G by Jung's Theorem 2.4. Therefore $X \cup Y$ witnesses that $G[C \cup X]$ has normal rank $\leq \xi$ in G. Finally, Lemma 4.2 (ii) implies that $G[C \cup X]$ has normal rank $\leq \xi$. □

Theorem 2. *Every normally traceable graph has an end-faithful spanning tree.*

Proof. By Theorem 4 we may prove the statement via induction on the normal rank of G. If G has normal rank 0, then it has a normal spanning tree, and normal spanning trees are end-faithful. For the induction step suppose that G has normal rank $\alpha > 0$, and let $X \subseteq V(G)$ be any vertex set that is normally spanned in G and satisfies that every component of $G - X$ has normal rank $< \alpha$ in G. By replacing X with the vertex set of any normal tree in G that contains X, we may assume that X is the vertex set of a normal tree $T_{\text{NT}} \subseteq G$; indeed, every component of $G - X$ still has normal rank $< \alpha$ in G by Lemma 4.2 (i). Note that, by Lemma 2.3, the tree T_{NT} reflects the ends of G in the closure of X.

By Lemma 5.3 (i), every end of G is contained in the closure of X in G or in the closure of some component of $G - X$. And by Lemma 5.3 (ii), every end of G that is contained in the closure of two distinct components of $G - X$ in G is also contained in the closure of X in G. Thus, by Lemma 5.1 it suffices to find in each component C of $G - X$ a spanning forest H_C so that every component of H_C sends an edge in G to T_{NT} and so that H_C reflects $\partial_\Omega C \setminus \partial_\Omega X$.

For this, consider any component C of $G - X$. Let P be the (possibly one-way infinite) path in T_{NT} that is formed by the down-closure of $N(C)$ in T_{NT}. Then by Lemma 5.4 the graph $G[C \cup P]$ has normal rank $< \alpha$, and therefore satisfies the induction hypothesis. Hence we find an end-faithful spanning tree T_C of $G[C \cup P]$. By Lemma 5.2 we may assume that the path P is a subgraph of T_C if this path is a ray. It is straightforward to check that $H_C := T_C - X$ is as desired. □

6. Rayless spanning trees

In this section we prove that for every normally traceable graph G, having a rayless spanning tree is equivalent to all the ends of G being dominated. Our proof requires the following theorem:

Theorem 6.1 ([4, Theorem 1])**.** *Let G be any graph and let $U \subseteq V(G)$ be normally spanned. Then there is a rayless tree $T \subseteq G$ that includes U if and only if all the ends of G in the closure of U are dominated in G.*

Theorem 3. *For every normally traceable graph G, having a rayless spanning tree is equivalent to all the ends of G being dominated.*

Proof. Let G be any normally traceable graph. The forward implication is clear. By Theorem 4 we may prove the backward implication via induction on the normal rank of G. For this, we suppose that every end of G is dominated. If G has normal rank 0, then it is normally spanned. Thus, by Theorem 6.1, the graph G has a rayless spanning tree. For the induction step suppose that G has normal rank $\alpha > 0$, and let $X \subseteq V(G)$ be any vertex set that is normally spanned in G and satisfies that every component of $G - X$ has normal rank $< \alpha$ in G. By replacing X with any normal tree in G that contains X, we may assume that X is the vertex set of a normal tree $T_{\text{NT}} \subseteq G$; indeed, every component of $G - X$ still has normal rank $< \alpha$ in G by Lemma 4.2 (i).

We claim that it suffices to find in every component C of $G-X$ a rayless spanning forest H_C such that every component of H_C sends an edge in G to X. This can be seen as follows. Suppose that we find such a rayless spanning forest H_C in every component C of $G-X$. By Theorem 6.1 we find a rayless tree $T_{\text{RL}} \subseteq G$ that contains $X = V(T_{\text{NT}})$. Then we set $H'_D := H_C \cap D$ for every component D of $G - T_{\text{RL}}$ and the component C of $G-X$ containing it. Now consider the spanning forest H of G that is the union of all forests H'_D with the tree T_{RL}. Then a rayless spanning tree of G arises from H by Lemma 5.1.

To complete the proof, we show that every component C of $G-X$ has a rayless spanning forest H_C. So let C be any component of $G-X$. If the neighbourhood $N(C) \subseteq T_{\text{NT}}$ is finite, then we let P be the path in T_{NT} formed by the down-closure of $N(C)$ in T_{NT}. Otherwise, we let P be the union of the ray R formed by the down-closure of $N(C)$ in T_{NT} with a star in G attached to R. Then every end of the graph $G[C \cup P]$ is dominated, and by Lemma 5.4 it satisfies the induction hypothesis. Hence we find a rayless spanning tree T_C of $G[C \cup P]$, and $H_C := T_C - X$ is as desired. □

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