18.2 Multilevel Indexes

The indexing schemes we have described thus far involve an ordered index file. A binary search is applied to the index to locate pointers to a disk block or to a record (or records) in the file having a specific index field value. A binary search requires approximately ($\log_2 b_i$) block accesses for an index with b_i blocks because each step of the algorithm reduces the part of the index file that we continue to search by a factor of 2. This is why we take the log function to the base 2. The idea behind a **multilevel index** is to reduce the part of the index that we continue to search by bfr_i, the blocking factor for the index, which is larger than 2. Hence, the search space is reduced much faster. The value bfr_i is called the **fan-out** of the multilevel index, and we will refer to it by the symbol ***fo***. Whereas we divide the *record search space* into two halves at each step during a binary search, we divide it *n*-ways (where n = the fan-out) at each search step using the multilevel index. Searching a multilevel index requires approximately ($\log_{fo} b_i$) block accesses, which is a substantially smaller number than for a binary search if the fan-out is larger than 2. In most cases, the fan-out is much larger than 2.

A multilevel index considers the index file, which we will now refer to as the **first** (or **base**) **level** of a multilevel index, as an *ordered file* with a *distinct value* for each $K(i)$. Therefore, by considering the first-level index file as a sorted data file, we can create a primary index for the first level; this index to the first level is called the **second level** of the multilevel index. Because the second level is a primary index, we can use block anchors so that the second level has one entry for *each block* of the first level. The blocking factor bfr_i for the second level—and for all subsequent levels—is the same as that for the first-level index because all index entries are the same size; each has one field value and one block address. If the first level has r_1 entries, and the blocking factor—which is also the fan-out—for the index is $bfr_i = fo$, then the first level needs $\lceil (r_1/fo) \rceil$ blocks, which is therefore the number of entries r_2 needed at the second level of the index.

We can repeat this process for the second level. The **third level**, which is a primary index for the second level, has an entry for each second-level block, so the number of third-level entries is $r_3 = \lceil (r_2/fo) \rceil$. Notice that we require a second level only if the first level needs more than one block of disk storage, and, similarly, we require a third level only if the second level needs more than one block. We can repeat the preceding process until all the entries of some index level t fit in a single block. This block at the tth level is called the **top** index level.[4] Each level reduces the number of entries at the previous level by a factor of fo—the index fan-out—so we can use the formula $1 \le (r_1/((fo)^t))$ to calculate t. Hence, a multilevel index with r_1 first-level entries will have approximately t levels, where $t = \lceil (\log_{fo}(r_1)) \rceil$. When searching the

[4]The numbering scheme for index levels used here is the reverse of the way levels are commonly defined for tree data structures. In tree data structures, t is referred to as level 0 (zero), $t - 1$ is level 1, and so on.

index, a single disk block is retrieved at each level. Hence, t disk blocks are accessed for an index search, where t is the *number of index levels*.

The multilevel scheme described here can be used on any type of index—whether it is primary, clustering, or secondary—as long as the first-level index has *distinct values for $K(i)$ and fixed-length entries*. Figure 18.6 shows a multilevel index built over a primary index. Example 3 illustrates the improvement in number of blocks accessed when a multilevel index is used to search for a record.

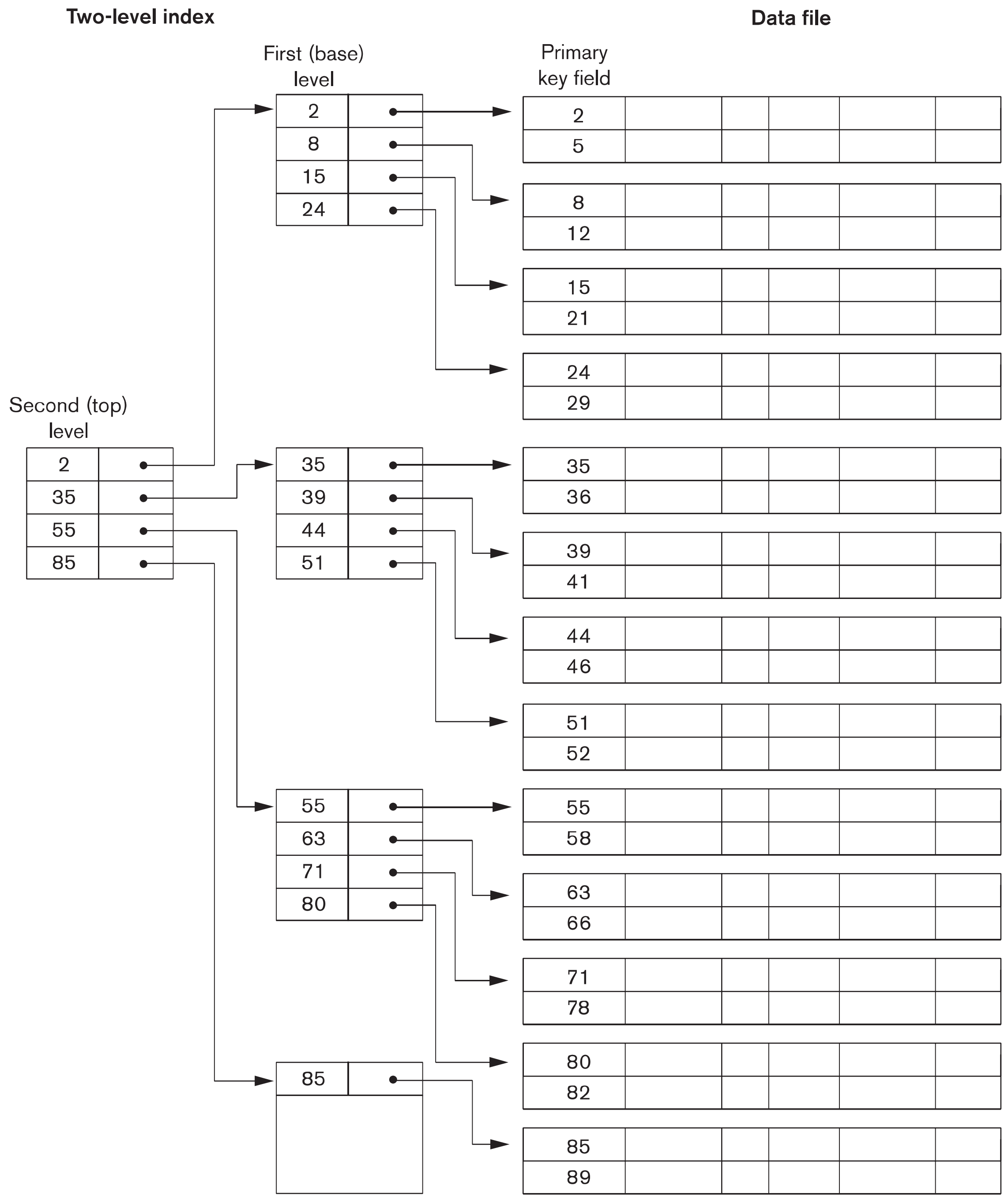
Example 3. Suppose that the dense secondary index of Example 2 is converted into a multilevel index. We calculated the index blocking factor $bfr_i = 68$ index entries per block, which is also the fan-out fo for the multilevel index; the number of first-level blocks $b_1 = 442$ blocks was also calculated. The number of second-level blocks will be $b_2 = \lceil(b_1/fo)\rceil = \lceil(442/68)\rceil = 7$ blocks, and the number of third-level blocks will be $b_3 = \lceil(b_2/fo)\rceil = \lceil(7/68)\rceil = 1$ block. Hence, the third level is the top level of the index, and $t = 3$. To access a record by searching the multilevel index, we must access one block at each level plus one block from the data file, so we need $t + 1 = 3 + 1 = 4$ block accesses. Compare this to Example 2, where 10 block accesses were needed when a single-level index and binary search were used.

Notice that we could also have a multilevel primary index, which would be nondense. Exercise 18.18(c) illustrates this case, where we *must* access the data block from the file before we can determine whether the record being searched for is in the file. For a dense index, this can be determined by accessing the first index level (without having to access a data block), since there is an index entry for *every* record in the file.

A common file organization used in business data processing is an ordered file with a multilevel primary index on its ordering key field. Such an organization is called an **indexed sequential file** and was used in a large number of early IBM systems. IBM's **ISAM** organization incorporates a two-level index that is closely related to the organization of the disk in terms of cylinders and tracks (see Section 17.2.1). The first level is a cylinder index, which has the key value of an anchor record for each cylinder of a disk pack occupied by the file and a pointer to the track index for the cylinder. The track index has the key value of an anchor record for each track in the cylinder and a pointer to the track. The track can then be searched sequentially for the desired record or block. Insertion is handled by some form of overflow file that is merged periodically with the data file. The index is recreated during file reorganization.

Algorithm 18.1 outlines the search procedure for a record in a data file that uses a nondense multilevel primary index with t levels. We refer to entry i at level j of the index as $<K_j(i), P_j(i)>$, and we search for a record whose primary key value is K. We assume that any overflow records are ignored. If the record is in the file, there must be some entry at level 1 with $K_1(i) \leq K < K_1(i + 1)$ and the record will be in the block of the data file whose address is $P_1(i)$. Exercise 18.23 discusses modifying the search algorithm for other types of indexes.

Figure 18.6
A two-level primary index resembling ISAM (Indexed Sequential Access Method) organization.



Algorithm 18.1. Searching a Nondense Multilevel Primary Index with t Levels

```
(* We assume the index entry to be a block anchor that is the first key per block. *)
$p \leftarrow$ address of top-level block of index;
for $j \leftarrow t$ step – 1 to 1 do
    begin
        read the index block (at $j$th index level) whose address is $p$;
        search block $p$ for entry $i$ such that $K_j(i) \leq K < K_j(i+1)$
     (* if $K_j(i)$
        is the last entry in the block, it is sufficient to satisfy $K_j(i) \leq K$ *);
        $p \leftarrow P_j(i)$ (* picks appropriate pointer at $j$th index level *)
    end;
    read the data file block whose address is $p$;
    search block $p$ for record with key = $K$;
```

As we have seen, a multilevel index reduces the number of blocks accessed when searching for a record, given its indexing field value. We are still faced with the problems of dealing with index insertions and deletions, because all index levels are *physically ordered files.* To retain the benefits of using multilevel indexing while reducing index insertion and deletion problems, designers adopted a multilevel index called a **dynamic multilevel index** that leaves some space in each of its blocks for inserting new entries and uses appropriate insertion/deletion algorithms for creating and deleting new index blocks when the data file grows and shrinks. It is often implemented by using data structures called B-trees and B$^+$-trees, which we describe in the next section.

18.3 Dynamic Multilevel Indexes Using B-Trees and B$^+$-Trees

B-trees and B$^+$-trees are special cases of the well-known search data structure known as a **tree.** We briefly introduce the terminology used in discussing tree data structures. A **tree** is formed of **nodes**. Each node in the tree, except for a special node called the **root**, has one **parent** node and zero or more **child** nodes. The root node has no parent. A node that does not have any child nodes is called a **leaf** node; a nonleaf node is called an **internal** node. The **level** of a node is always one more than the level of its parent, with the level of the root node being *zero*.[5] A **subtree** of a node consists of that node and all its **descendant** nodes—its child nodes, the child nodes of its child nodes, and so on. A precise recursive definition of a subtree is that it consists of a node n and the subtrees of all the child nodes of n. Figure 18.7 illustrates a tree data structure. In this figure the root node is A, and its child nodes are B, C, and D. Nodes E, J, C, G, H, and K are leaf nodes. Since the leaf nodes are at different levels of the tree, this tree is called **unbalanced**.

[5]This standard definition of the level of a tree node, which we use throughout Section 18.3, is different from the one we gave for multilevel indexes in Section 18.2.

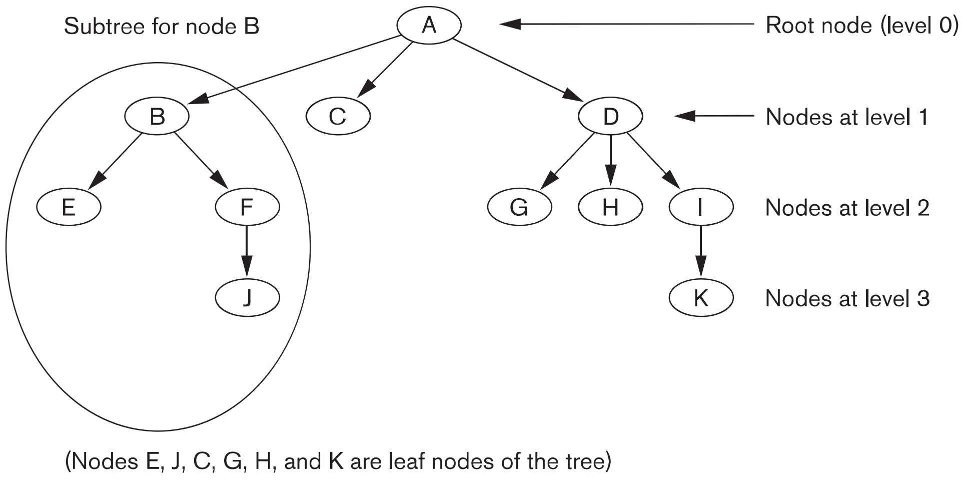


Figure 18.7
A tree data structure that shows an unbalanced tree.

In Section 18.3.1, we introduce search trees and then discuss B-trees, which can be used as dynamic multilevel indexes to guide the search for records in a data file. B-tree nodes are kept between 50 and 100 percent full, and pointers to the data blocks are stored in both internal nodes and leaf nodes of the B-tree structure. In Section 18.3.2 we discuss B^+-trees, a variation of B-trees in which pointers to the data blocks of a file are stored only in leaf nodes, which can lead to fewer levels and higher-capacity indexes. In the DBMSs prevalent in the market today, the common structure used for indexing is B^+-trees.

18.3.1 Search Trees and B-Trees

A **search tree** is a special type of tree that is used to guide the search for a record, given the value of one of the record's fields. The multilevel indexes discussed in Section 18.2 can be thought of as a variation of a search tree; each node in the multilevel index can have as many as *fo* pointers and *fo* key values, where *fo* is the index fan-out. The index field values in each node guide us to the next node, until we reach the data file block that contains the required records. By following a pointer, we restrict our search at each level to a subtree of the search tree and ignore all nodes not in this subtree.

Search Trees. A search tree is slightly different from a multilevel index. A **search tree of order** p is a tree such that each node contains *at most* $p - 1$ search values and p pointers in the order $<P_1, K_1, P_2, K_2, \ldots, P_{q-1}, K_{q-1}, P_q>$, where $q \leq p$. Each P_i is a pointer to a child node (or a NULL pointer), and each K_i is a search value from some ordered set of values. All search values are assumed to be unique.[6] Figure 18.8 illustrates a node in a search tree. Two constraints must hold at all times on the search tree:

1. Within each node, $K_1 < K_2 < \ldots < K_{q-1}$.
2. For all values X in the subtree pointed at by P_i, we have $K_{i-1} < X < K_i$ for $1 < i < q$; $X < K_i$ for $i = 1$; and $K_{i-1} < X$ for $i = q$ (see Figure 18.8).

Whenever we search for a value X, we follow the appropriate pointer P_i according to the formulas in condition 2 above. Figure 18.9 illustrates a search tree of order $p = 3$ and integer search values. Notice that some of the pointers P_i in a node may be NULL pointers.

We can use a search tree as a mechanism to search for records stored in a disk file. The values in the tree can be the values of one of the fields of the file, called the **search field** (which is the same as the index field if a multilevel index guides the search). Each key value in the tree is associated with a pointer to the record in the data file having that value. Alternatively, the pointer could be to the disk block containing that record. The search tree itself can be stored on disk by assigning each tree node to a disk block. When a new record is inserted in the file, we must update the search tree by inserting an entry in the tree containing the search field value of the new record and a pointer to the new record.

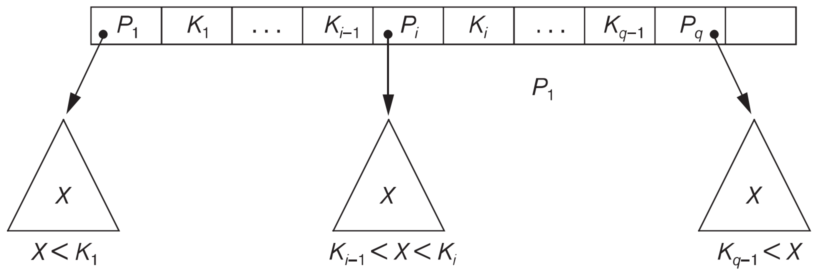


Figure 18.8
A node in a search tree with pointers to subtrees below it.

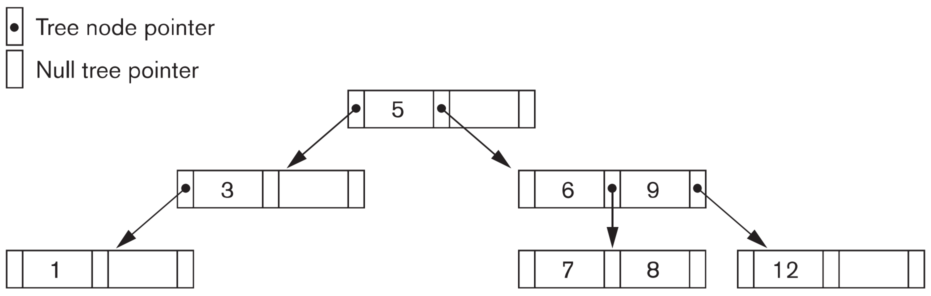


Figure 18.9
A search tree of order $p = 3$.

[6]This restriction can be relaxed. If the index is on a nonkey field, duplicate search values may exist and the node structure and the navigation rules for the tree may be modified.

Algorithms are necessary for inserting and deleting search values into and from the search tree while maintaining the preceding two constraints. In general, these algorithms do not guarantee that a search tree is **balanced**, meaning that all of its leaf nodes are at the same level.[7] The tree in Figure 18.7 is not balanced because it has leaf nodes at levels 1, 2, and 3. The goals for balancing a search tree are as follows:

- To guarantee that nodes are evenly distributed, so that the depth of the tree is minimized for the given set of keys and that the tree does not get skewed with some nodes being at very deep levels
- To make the search speed uniform, so that the average time to find any random key is roughly the same

While minimizing the number of levels in the tree is one goal, another implicit goal is to make sure that the index tree does not need too much restructuring as records are inserted into and deleted from the main file. Thus we want the nodes to be as full as possible and do not want any nodes to be empty if there are too many deletions. Record deletion may leave some nodes in the tree nearly empty, thus wasting storage space and increasing the number of levels. The B-tree addresses both of these problems by specifying additional constraints on the search tree.

B-Trees. The B-tree has additional constraints that ensure that the tree is always balanced and that the space wasted by deletion, if any, never becomes excessive. The algorithms for insertion and deletion, though, become more complex in order to maintain these constraints. Nonetheless, most insertions and deletions are simple processes; they become complicated only under special circumstances—namely, whenever we attempt an insertion into a node that is already full or a deletion from a node that makes it less than half full. More formally, a **B-tree of order p**, when used as an access structure on a *key field* to search for records in a data file, can be defined as follows:

1. Each internal node in the B-tree (Figure 18.10(a)) is of the form

   $<P_1, <K_1, Pr_1>, P_2, <K_2, Pr_2>, ..., <K_{q-1}, Pr_{q-1}>, P_q>$

   where $q \le p$. Each P_i is a **tree pointer**—a pointer to another node in the B-tree. Each Pr_i is a **data pointer**[8]—a pointer to the record whose search key field value is equal to K_i (or to the data file block containing that record).
2. Within each node, $K_1 < K_2 < ... < K_{q-1}$.
3. For all search key field values X in the subtree pointed at by P_i (the ith subtree, see Figure 18.10(a)), we have:

   $K_{i-1} < X < K_i$ for $1 < i < q$; $X < K_i$ for $i = 1$; and $K_{i-1} < X$ for $i = q$.
4. Each node has at most p tree pointers.

[7]The definition of *balanced* is different for binary trees. Balanced binary trees are known as *AVL trees*.

[8]A data pointer is either a block address or a record address; the latter is essentially a block address and a record offset within the block.

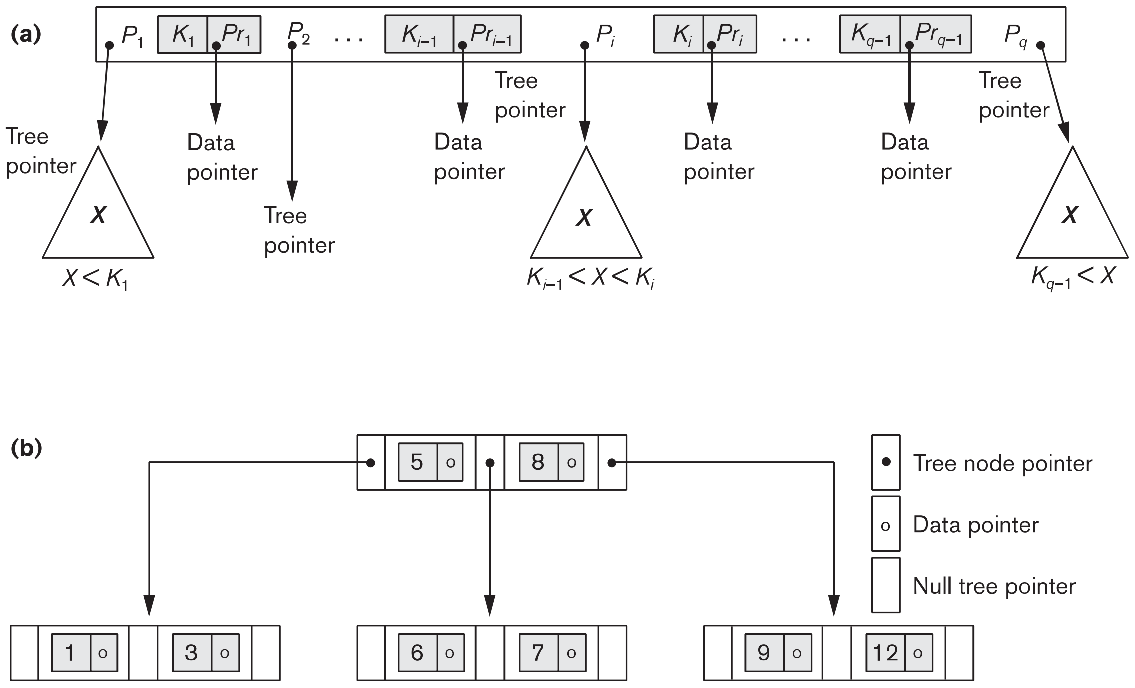


Figure 18.10
B-tree structures. (a) A node in a B-tree with $q - 1$ search values. (b) A B-tree of order $p = 3$.The values were inserted in the order 8, 5, 1, 7, 3, 12, 9, 6.

5. Each node, except the root and leaf nodes, has at least $\lceil (p/2) \rceil$ tree pointers. The root node has at least two tree pointers unless it is the only node in the tree.
6. A node with q tree pointers, $q \leq p$, has $q - 1$ search key field values (and hence has $q - 1$ data pointers).
7. All leaf nodes are at the same level. Leaf nodes have the same structure as internal nodes except that all of their *tree pointers* P_i are NULL.

Figure 18.10(b) illustrates a B-tree of order $p = 3$. Notice that all search values K in the B-tree are unique because we assumed that the tree is used as an access structure on a key field. If we use a B-tree *on a nonkey field*, we must change the definition of the file pointers Pr_i to point to a block—or a cluster of blocks—that contain the pointers to the file records. This extra level of indirection is similar to option 3, discussed in Section 18.1.3, for secondary indexes.

A B-tree starts with a single root node (which is also a leaf node) at level 0 (zero). Once the root node is full with $p - 1$ search key values and we attempt to insert another entry in the tree, the root node splits into two nodes at level 1. Only the middle value is kept in the root node, and the rest of the values are split evenly