CS 566 A2 HW1 | Jiankun Dong

1) $S_{n+1} = \frac{(n+1)(n+2)}{2} = \frac{n(n+1)}{2} + (n+1)$

$1^3+2^3+\cdots+(n+1)^3 = 1+\cdots+n^3+(n+1)^3$

$\because (n+1) \neq (n+1)^3$ for $n \in N+$

$\therefore S = \frac{n(n+1)}{2}$ is not correct formula for $1^3+\cdots+n^3$.

Correct formula:

$S_1 = 1^3 \quad S_2 = 1^3+2^3$

$$n^4 - (n-1)^4 = (n^2+(n-1)^2)(n^2-(n-1)^2) = (2n^2-2n+1)(2n-1)$$
$$= 4n^3-6n^2+4n-1$$

$$(n-1)^4-(n-2)^4 = 4(n-1)^3-6(n-1)^2+4(n-1)-1$$
$$\vdots$$
$$2^4-1^4 = 4\cdot 2^3-6\cdot 2^2+4\cdot 2-1.$$

$\therefore$ Adding left and right together:

$$n^4-1^4 = \left[4\left(\sum_{i=1}^{n} i^3\right) - 4\cdot 1^3\right] - 6\left(\sum_{i=1}^{n} i^2 - 1^2\right) + 4\left(\frac{(1+n)n}{2}-1\right) - (n-1)$$

$$\therefore \cancel{n^4+1} \quad n^4 - 1 = 4\left(\sum_{i=1}^{n} i^3\right) - 6\sum_{i=1}^{n} i^2 + 2n^2+n-1$$

For $\sum_{i=1}^{n} i^2$ we know $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ (proof appended on next page).

$$\therefore n^4-1 = 4\sum_{i=1}^{n} i^3 - n(n+1)(2n+1) + 2n^2+n-1 = 4\sum_{i=1}^{n} i^3 - (2n^3+3n^2+n) + 2n^2+n-1$$

$$n^4-1 = 4\sum_{i=1}^{n} i^3 - 2n^3-n^2-1$$

$$\therefore \sum_{i=1}^{n} i^3 = \frac{n^4+2n^3+n^2}{4} = \boxed{\frac{n^2(n+1)^2}{4}}$$

For $\sum_{i=1}^{n} i^2$: $n=1$ leftside $=1$ $\quad \frac{n(n+1)(2n+1)}{6}\Big|_{n=1} = \frac{2\times 3}{6} = 1$ left = right.

For $n+1$ case: leftside $= \sum_{i=1}^{n} i^2 + (n+1)^2$

$$\text{rightside} = \frac{(n+1)(n+2)(2n+3)}{6} = \frac{(n+1)(n+2)(2n+1)}{6} + \frac{2(n+1)(n+2)}{6}$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{2(n+1)(2n+1) + 2(n+1)(n+2)}{6}$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{2n^2+3n+1+n^2+3n+2}{3}$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{3(n^2+2n+1)}{3} = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$= \sum_{i=1}^{n} i^2 + (n+1)^2$$

$\therefore$ leftside $=$ rightside

$$\therefore \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

2) When $f(n) < g(n)$:

$n^2 + 4n < 14n + 39$.

$n^2 - 10n - 39 < 0$

$(n-13)(n+3) < 0$

$\because n \geq 0$

$\therefore$ when $n \in [0, 13)$, $n \in N^+$, ~~f(n)~~ algorithm 1 faster than algorithm 2.

3). Refer to ~~proof~~ problem 1, I've proved that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3+3n^2+n}{6} = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

$$\Theta\left(\sum_{i=1}^{n} i^2\right) = \Theta(n^3) + \Theta(n^2) + \Theta(n) = \Theta(n^3) \neq \Theta(n^2).$$

4).

```
Findk (k, array[]) {
    if (len(array) > 4) {        // for Array size at least 5:
        for (i=0; i< mod(len(array), 5); i++) {   // Check every 5 entry)
            int curr = i·5;
            if (k == array[curr]) return curr;
            if (k < array[curr]) {         // check precedly 4
                ~~for (j~~
                for (j = curr-1; j > curr-5; j--) {
                    ~~if (j<0) return NULL; (Array < 5 entry~~
                    if (k == array[j]) return j;
                }
                return null;  // k not in array.
            }
        }
    } else { for (i=0; i< (len(array)); i++) { if(array[i]==k) return i;}
        return Null;  // No k in array
    }}
```

Find k_When array size ≤ 4

Worst Case:

Assume n entries in Array:

~~Total~~ Total checks $= 4 + (n \bmod 5)$

5). Algorithm 1:

$X^N = X \cdot X^{(N-1)}$; $X^0 = 1$:

```
func (X, N){
    return   helperfunc (X, N, 1);
}
helperfunc (X, N, result){
    if ( N==0 ) return (result);
    return  helper(X, (N-1), result * X );
}
```

This function has $O(N)$

Algorithm 2:

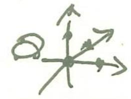
$n = 2^m$ $X^n = ((X^2)^2)^2 \ldots$

```
func2 (X, n) {
    int m = log (n, 2) ;
    for ( i=0, i<m, i++ ) { x = X^2 ; }
    return X;
}
```

this function has $O\{\log n\}$

$\therefore$ Algorithm 2 is more efficient.

6). $i = 1$ 1 times 

a) $i = 2$ $1 + 3 = 4$ times 

$i = 3$: $4 + 6 = 10$ times.

$i = n$: $S_n = S_{(n-1)} + \frac{n(n+1)}{2}$

$S_{(n-1)} - S_{(n-2)} = \frac{(n-1)n}{2}$

$\vdots$

$S_{(2)} - S_{(1)} = \frac{2 * (2+1)}{2} = 3.$

$S_{(1)} = 1.$

Adding left and right side:

$$(S_n - S_{(n-1)}) + (S_{(n-1)} - S_{(n-2)}) + \dots + (S_{(2)} - S_{(1)}) + S_{(1)} = \frac{n(n+1)}{2} + \frac{(n-1)n}{2} + \dots + 1$$

$$S_n = \sum_{i=1}^{n} \frac{i(i+1)}{2} = \frac{1}{2}\sum_{i=1}^{n} i^2 + \frac{1}{2}\sum_{i=1}^{n} i$$

(refer to problem 1 for prof $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$)

$$= \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right)$$

$$= \frac{n(n+1)(2n+4)}{12}$$

$$\therefore S_n = \frac{n(n+1)(n+2)}{6}$$

Checking for $n = N+1$

Right Side: $S_{(N+1)} = \frac{(N+2)(N+1)(N+3)}{6} = \frac{N(N+1)(N+2)}{6} + \frac{3}{6}(N+1)(N+2) = \frac{N(N+1)(N+2)}{6} + \frac{1}{2}(N+1)(N$

Left Side: $S_{(N+1)} = S_N + \sum_{i=1}^{N+1} i = S_N + \frac{1}{2}(N+1)(N+1+1) = S_N + \frac{1}{2}(N+1)(N+2)$

$\therefore$ Right side = left side

$\therefore$ $S_n = \frac{n(n+1)(n+2)}{6}$ is correct.

6) b) . $\Theta(S_N) = \Theta\left\{\frac{n(n+1)(n+2)}{6}\right\} = \Theta\left\{\frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n\right\}$

$= \Theta\{n^3 + n^2 + n\}$

$\therefore \quad \Theta\{S_N\} = \Theta\{n^3\}$