Part 4 Weighted Vertex Cover

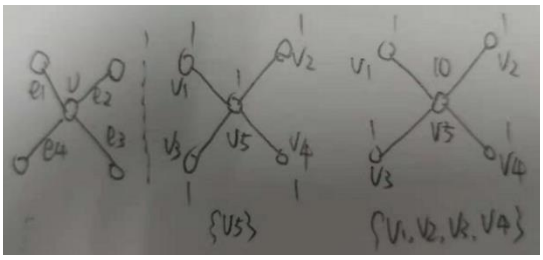
(**Definition 1**) (**Minimum Weight Vertex Cover**, VC) Given a graph (V, E), in which there is a weight w_v associated with each vertex $v \in V$, find **a subset of vertexes** that

(1) covers all the edges in E;

(2) the **total (vertex) weight** is minimized.

Notes: The minimum weight vertex cover is an **NP-hard problem**.

Notes: In the following example, vertex v covers edges $\{e_1, e_2, e_3, e_4\}$. For the first weighted graph, the minimum weight vertex cover is $\{v_5\}$ and the total weight is 1. For the second weighted graph, the minimum weighted vertex cover is $\{v_1, v_2, v_3, v_4\}$ and the total weight is 4.



(**Definition 2**) (**Integer Linear Program**, ILP) The **minimum weight vertex cover** problem can be formulated as an equivalent **integer liner program** problem. For each vertex v, one can associate it with a variable $x_v \in \{0,1\}$, where $x_v = 1$ when vertex v is picked and $x_v = 0$, otherwise. The objective function of **ILP** is as follow:

$$
\begin{aligned}
&\min \textstyle\sum_{v \in V} w_v x_v \\
&\text{s.t. } x_v + x_u \geq 1 \text{ for each edge } (u,v) \in E \\
&\quad x_v \in \{0,1\} \text{ for each vertex } v \in V
\end{aligned}
$$

(**Definition 3**) (**LP Relaxation**) One can replace the 'integrity constraint' (i.e., $x_v \in \{0,1\}$) by allowing $0 \leq x_v \leq 1$ (for each vertex $v \in V$), which relaxes the **ILP** w.r.t the minimum weight vertex problem as an **LP**. In this case, <u>the solution to **ILP** is also a **valid** (feasible) **solution** to the **LP relaxation**</u>. In other words, <u>**LP relaxation** provides a **good lower bound** of **ILP**</u>.

Especially, the relaxed constraint $0 \leq x_v \leq 1$ (for each vertex $v \in V$) can be simply replaced by $x_v \geq 0$, because if we set $x_v > 1$, it satisfies the constraint $x_u + x_v \geq 1$ (regardless of the setting of x_u) but it will <u>**increase the value of the objective function**</u> that we want to minimize. In other words, if $x_v > 1$ is a feasible value for LP relaxation (i.e., satisfying the corresponding constraint $x_u + x_v \geq 1$), there exist **a smaller setting** for x_v that can further improve the objective function value while **satisfying the constraint** $x_u + x_v \geq 1$. In summary, LP Relaxation can be formulated as follow:

$$\min \sum_{v\in V} w_v x_v$$
$$\text{s.t. } x_v + x_u \geq 1 \text{ for each edge } (u,v) \in E,$$
$$x_v \geq 0 \text{ for each vertex } v \in V.$$

Notes: **ILP** is an **NP-hard problem**, while **LP** is in the **Class** *P*, i.e., there exist polynomial-time algorithm for LP. Typical polynomial-time algorithms for LP include:

[1] Khochiyan's Ellipsoid Method (1979)

[2] Karmaker's Interior Point Method (1984)

(**Algorithm 1**) (**Deterministic Rounding**)

Step 1: Solve the LP Relaxation to obtain an optimal (fractional(分数的)) solution **x***;

Step 2: Let $S = \{v : x_v^* \geq 1/2\}$.

Step 3: Output *S*.

(**Theorem 1**) The output of **Algorithm 1** *S* is a **vertex cover**.

Proof of **Theorem 1**. For an arbitrary edge (*u*, *v*) in the original graph, consider the optimal solution **x*** of **Algorithm 1**, where we have

$$x_u^* + x_v^* \geq 1.$$

When $x_u^* < 1/2$ and $x_v^* < 1/2$, i.e., both *u* and *v* aren't picked, we have

$$x_u^* + x_v^* < 1,$$

which means **x*** is not a feasible solution of LP Relaxation. Only when at least one vertex *u* or *v* has the value x_u (or x_v)≥1/2, **x*** can satisfy the aforementioned constraint, where at least one vertex *u* or *v* is picked. Hence, *S* (i.e., the output of **Algorithm 1**) is a valid vertex cover.

(**Theorem 2**) $w(S) \leq 2Opt(I)$, where *w*(*S*) is the total weight of vertexes in *S* (i.e., the output of **Algorithm 1**) and *Opt*(*I*) is the weight of the **optimal vertex cover**, with *I* referred to the instance (input) of the vertex cover problem.

Proof of **Theorem 2**. Let **x*** be the optimal solution of **LP Relaxation**. Let $x_v' = 1$ if $x_v^* \geq 1/2$ and $x_v' = 0$ if $x_v^* < 1/2$. We have

$$w(S) = \sum_{v\in S} w_v = \sum_v w_v x_v' \leq \sum_v w_v 2 \cdot x_v^* = 2\sum_v w_v x_v^* = 2Opt_{LP}(I),$$

where $Opt_{LP}(I)$ represents the **optimal value** of **LP Relaxation** w.r.t. instance *I*.

Moreover, since **LP Relaxation** provides **a lower bound** for **ILP**, we also have

$$Opt_{LP}(I) \leq Opt(I).$$

Hence, we further have

$$w(S) \leq 2Opt_{LP}(I) \leq 2Opt(I).$$

The **approximation ratio** of **Algorithm 1** is

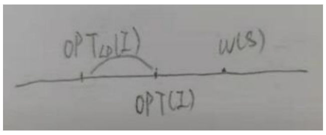
$$\frac{w(S)}{Opt(I)} \le \frac{2Opt(I)}{Opt(I)} = 2 \cdot$$

Notes: Our analysis will never give the **approximation ratio** that is better than the gap between $Opt_{LP}(I)$ and $Opt(I)$. Concretely, let *r* be the <u>bound of the approximation ratio</u> we get. In our analysis, we use $w(S)/Opt_{LP}(I)$ to determine *r* and <u>let *r* be the estimated approximation ratio</u>, where we have

$$r = \frac{w(S)}{Opt(I)} \le \frac{w(S)}{Opt_{LP}(I)} \le 2 = r \cdot$$

Moreover, we also have

$$\frac{Opt(I)}{Opt_{LP}(I)} \le \frac{w(S)}{Opt_{LP}(I)} \le r \cdot$$



(**Definition 4**) Let *Π* be a (minimization or maximization) optimization problem. Let *I* be an instance (input) of *Π*. Let $Opt_{\Pi}(I)$ be the **objective function value** the **optimal solution** to instance *I*. We may write it as $Opt(I)$ if the problem is implicit. Typically, we can model *Π* by an **integer linear program** (ILP) problem. Then, $Opt(I)$ is the objective value of **ILP**.

Let $Opt_{LP}(I)$ be the objective function of the optimal solution to the **LP Relaxation** (for instance *I*). If our analysis is <u>based on the lower bound provided by the LP Relaxation</u> (i.e., compares the cost of the solution *w*(*S*) with $Opt_{LP}(I)$), then the **best approximation guarantee** we can show is

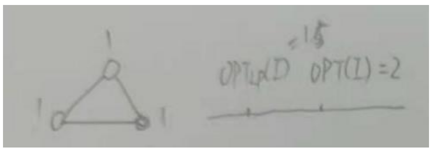
$$\sup_{I} \frac{Opt_{\Pi}(I)}{Opt_{LP}(I)} \cdot$$

This quantity is called as the '**Integrality Gap**'. Assume that we're dealing with the <u>**minimization problem**</u>.

Notes: Different **LP Relaxation** of the same problem *Π* can have different **integrality gap**, which depends on how you relax the ILP. We say that <u>a LP Relaxation is **EXACT** if the **integrality gap** is 1</u> w.r.t. $Opt_{\Pi}(I)/Opt_{LP}(I)$. Our analysis shows that for the **Vertex Cover** problem, the **integrality gap** is <u>less than or equal to 2</u>.

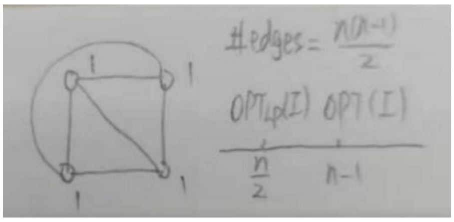
Notes: For minimization problem, **integrality gap** is the **worst case ratio** over all instances of *Π* of the optimal value of the **ILP** and **LP relaxation**.

Notes: For the following the example regarding **Vertex Cover**,



we have $Opt(I) = 2$ and $Opt_{LP}(I) = 1.5$. Since $Opt(I)/Opt_{LP}(I) = 4/3$, we know that the integrality gap is **at least** 4/3.

Notes: For the **Vertex Cover** on a complete graph with *n* vertexes (denoted as K_n),



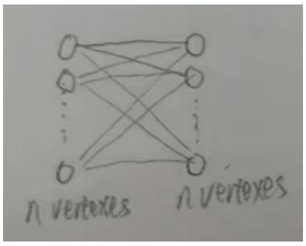
we have $Opt(I) = (n-1)$ and $Opt_{LP}(I) = n/2$. Since $Opt(I)/Opt_{LP}(I) = 2(n-1)/n$, we known that the integrality gap is **at least** $2(n-1)/n$. When n=3, we have the same result with the previous example (i.e., the triangle). With the increase of *n*, the value of $2(n-1)/n$ goes closer to 2.

(**Remark 1**) (1) The best approximation ratio known for Vertex Cover is $2-\Theta(1/\sqrt{\log n})$.

(2) It's an open problem that to obtain an (2-ε)-approximation for any fixed ε>0. Alternatively, to prove that is NP-hard to achieve this approximation ratio.

(3) Unless *P*=*NP*, there's no 1.36-appromxation for Vertex Cover (i.e., the 'hardest result').

(**Example 1**) A **tight example** of **Vertex Cover** is as follow. For an $n \times n$ bipartite graph (a **complete bipartite graph**),



we have $Opt_{VC}(I) = n$, if we chose the vertexes in one side.

Furthermore, if we set $x_v = 1/2$ for each vertex $v \in V$, the **LP Relaxation** can satisfy all the constraints (i.e., $x_u + x_v \geq 1$ w.r.t. to each edge (*u*, *v*)), in which we have $Opt_{LP}(I) = 0.5 \times 2n = n$.

Based on the optimal solution of **LP Relaxation**, **Algorithm 1** select all the 2*n* nodes in the graph, with $w(S) = 2n$. Hence, the **approximation ratio** of **Algorithm 1** is $w(S)/Opt_{LP}(I) = 2$.

(**Definition 5**) According to **Definition 3**, the **Vertex Cover** problem can be approximated via the **LP Relaxation**:

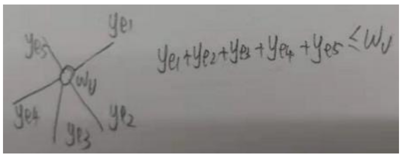
$$\begin{aligned} &\min \textstyle\sum_{v \in V} w_v x_v \\ &\text{s.t. } x_v + x_u \geq 1 \text{ for each } (u,v) \in E \\ &\quad x_v \geq 0 \text{ for each vertex } v \in V \end{aligned}$$

Let the **LP Relaxation** be the **primal LP**, we have the corresponding **dual LP**:

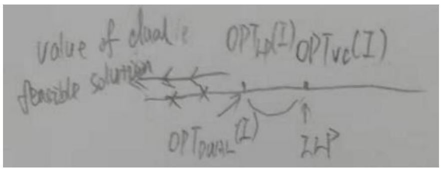
$$\max \sum_{e\in E} y_e$$
$$\text{s.t.} \sum_{\substack{e \text{ induced} \\ \text{to } v}} y_e \le w_v \text{ for each vertex } v \in V .$$
$$y_e \ge 0 \text{ for each edge } e \in E$$

Especially, y_e can be considered as the '**non-negative charge**' assigned to an edge *e*.

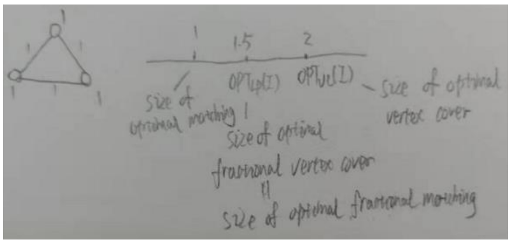
We want to assign the 'charge' to each edge such that the **total charge of all the edges** is **as large as possible**. The constraint is that for each vertex *v*, the **total charge** of edges induced on *v* is **at most** w_v. For example,



It follows **weak duality** that the total charge $\sum_{e\in E} y_e$ for any **dual feasible solution** is a **lower bound** of the **Minimum Vertex Cover problem**.



Notes: Consider the following (triangle) example:



The optimal size of the **Minimum Vertex Cover problem** is $Opt_{VC}(I) = 2$, where 2 vertexes are picked to cover all the edges. The **optimal fractional size** of the **LP Relaxation** (i.e., the **Primal LP**) is $Opt_{LP}(I) = 1.5$, where the 'charge' of each vertex is 0.5.

Moreover, the **optimal fractional size** of the **Dual LP** is also 1.5, where the 'charge' of each edge is 0.5. When we let $y_e \in \{0,1\}$ (rather than $y_e \ge 0$) in the **Dual LP**, it becomes the **Maximum Matching problem**. The **optimal size** of the **Maximum Matching problem** is 1, where only 1 edge can be picked.

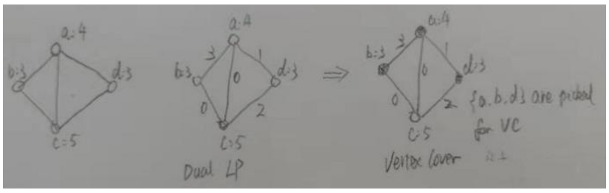
(**Theorem 3**) For any graph *G*, by the **weak duality**, the size of **Maximum Matching** is at most the size of **Minimum Vertex Cover**.

Proof of **Theorem 3** (**Direct Combinatorial Argument**). For every edge of a **matching**, one of its end points must be included in the **vertex cover**. And there may still be some vertexes not covered by the **matching**. Hence, the size of a **vertex cover** should be **greater than or equal to** the size of a **matching**.

Notes: **Theorem 3** can also be proved via the **Weak Duality**. Let the **LP Relaxation** of **Minimum Vertex Set** be the **Primal LP**. Then, the **ILP** of the **Dual LP** is the equivalent description for the **Maximum Matching problem**.

The optimal solution to Maximum Matching is a **feasible solution** to the **Dual LP**, while the optimal solution to Minimum Vertex Cover is a **feasible solution** to the **Primal LP**. By **Weak Duality**, <u>the size of **Maximum Matching** provides a **lower bound** for the size of **Minimum Vertex Cover**</u>, which finish the proof.

(**Example 2**) Consider the following graph and a **feasible solution** to the **Dual LP** (i.e., charging the edges with constraints related to the induced vertexes).



We define a vertex v is '**tight**' if the total charge of edges incident to v is equal to w_v. We pick the 'tight' vertices, which can be a **feasible solution** to **Vertex Cover**. In the example, the 'tight' vertexes $\{a, b, c\}$ are picked as a feasible solution, with the **objective value** (4+3+3)=10.

Note that the **sum of the 'charges'** assigned to all the **edges** is (3+1+2+0+0)=6, which is a lower bound of the **Vertex Cover**.

(**Algorithm 2**) (**Primal-Dual Algorithm**)

Step 1: Initialize $\mathbf{x}=[0,0,\cdots,0]\in\Re^{|V|}$ and $\mathbf{y}=[0,0,\cdots,0]\in\Re^{|E|}$. (Note that now **x** is infeasible, but **y** is feasible.)

Step 2: For each edge $e=(u,v)\in E$

(1) If edge e is **not covered** (i.e., $x_u<1$ and $x_v<1$):

- 'Raise' the dual variable y_e until one of its end points u or v becomes 'tight'.

(2) If u is 'tight': $x_u\leftarrow 1$.

(3) If v is 'tight': $x_v\leftarrow 1$.

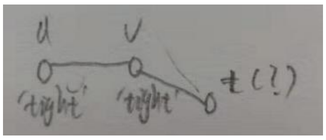
Step 3: output the Vertex Cover **x**.

Notes: **Algorithm 2** construct (i) <u>a **valid integral solution x** to the **ILP formulation** of **Vertex Cover**</u> and (ii) <u>a **feasible solution y** to the **Dual LP**</u>.

Notes: An edge (u, v) is uncovered, if $x_u<1$ and $x_v<1$.

Notes: A vertex v is said to be '**tight**', if $\sum_{e \text{ incident to } v} y_e = w_v$.

Notes: For an edge $e=(u, v)$, <u>if both u and v are 'tight'</u>, we <u>select both u and v</u>. Consider the following example. When <u>u and v are 'tight'</u> and we <u>only pick u</u>, <u>edge (v, t) may not be included in the output</u> if $w_t>w_v$. Concretely, since v has already become 'tight', we cannot further raise the charge of (v, t) to make t 'tight'.



Notes: The correctness of **Algorithm 2** is obvious. Because for **each edge**, the algorithm makes sure that at least one of the end points is selected in the **Vertex Cover**.

Notes: The vector **y** at the end of the algorithm is **a feasible solution** for the **Dual LP**, because we raise the its 'charge' until some of the vertexes are 'tight', i.e., no vertex ever goes '**over tight**'.

(**Theorem 4**) The **approximation ratio** of **Algorithm 2** is 2.

Proof 1 of **Theorem 4**. By **Weak Duality**, we have

$$\sum_{e\in E} y_e \le Opt_{LP}(I) \le Opt_{VC}(I) \,. \ (1)$$

Let S be the vertex cover output of **Algorithm 2** and the total weight of S be $w(S)=\sum_{v\in V} w_v x_v$. We further have

$$w(S) \le 2\sum_{e\in E} y_e \,, \ (2)$$

because (i) the charge of each edge contributes to the weights of its **both end points** but (ii) not all the vertices are selected in **Algorithm 2** (i.e., only the 'tight' vertices are picked). Concretely, we have the following derivation:

$$w(S)=\sum_{v \text{ is tight}} w_v = \sum_{v \text{ is tight}} \sum_{\substack{e \text{ incident} \\ \text{to } v}} y_e = \sum_{e\in E} y_e \cdot i_e \le \sum_{e\in E} 2y_e \,,$$

where $i_e \in \{1,2\}$ is defined as the number of 'tight' vertices that edge e is incident to.

By (1) and (2), we have

$$w(S) \le 2\sum_{e\in E} y_e \le 2Opt_{VC}(I) \,.$$

Hence, the **approximation ratio** of **Algorithm 2** is

$$\frac{w(S)}{Opt_{VC}(I)} \le \frac{2Opt_{VC}(I)}{Opt_{VC}(I)} = 2 \,.$$

(**Remark 2**) Now we have two algorithms for the **Minimum Vertex Cover** problem, which are the deterministic rounding algorithm (i.e., **Algorithm 1**) and primal-dual algorithm (i.e., **Algorithm 2**). In **Algorithm 1**, we solve the solution to the LP relaxation, while in **Algorithm 2**, it doesn't request solving an LP (i.e., **Algorithm 2** is purely 'combinatorial'), but we use the **LP-Duality** concept to **design** and **analyze Algorithm 2**.

In fact, we can also analyze **Algorithm 2 without** using the **LP-Duality** concept. Namely, we have two different proofs for **Theorem 4**, where we can avoid 'weak duality' altogether by showing the lower bound as follow (i.e., replace the first step in **Proof 1** of **Theorem 4** with another analysis).

Proof 2 of **Theorem 4**. Let C* be an **optimal vertex cover**. Then, we have

$$Opt_{VC}(I)=\sum_{v\in C^*} w_v \ge \sum_{v\in C^*} \sum_{\substack{e \text{ indicent} \\ \text{to } v}} y_e \,.$$

Since C^* is an optimal vertex cover, it should cover all the edges in the graph and each edge should be counted at **least once**. Hence, we further have

$$Opt_{VC}(I) \geq \sum_{v \in C^*} \sum_{\substack{e \text{ incident} \\ \text{to } v}} y_e \geq \sum_{e \in E} y_e,$$

which has the same result as in the first step of the **Proof 1** of **Theorem 4**. The rest of the proof is the same as that in the **Proof 1** of **Theorem 4**.

In summary, we use the **LP-Duality** concept to design **Algorithm 2**. Then, we cloud throw this concept out of the window.