LUPM
H. Reboul / 05/06/2024

Cosmological Theory and Cosmogravity

The cosmology summary presented below focuses on the relations that are used in the code simulations. At the beginning, however, it contains a brief presentation of general relativity and its connection with cosmology : the study of the Universe as a whole.

1 General Relativity

General relativity is a **Geometric Relativistic Theory of Gravitation (GRTG)** i.e. a theory that *«relies»* the geometry of the container (space-time) to its content (matter-energy) and which, locally, is in line with special relativity [1].

2 Cosmological Principle

The Cosmological Principle postulates that the universe is homogeneous and isotropic, i.e. the same everywhere (3D space) and in all directions ... so the only possible changes are with time, and this time is the same everywhere, since nothing "moves" [2]. This is the hypothesis of a very strong and "strange" symmetry [3]

It might have seemed even stranger, in 1917, to think that the observed sky (then at close range) was uniform on a much larger scale. But gradually, observations of stars, and later of the Cosmic Microwave Background (Radiation) (CMB or CMBR) confirmed the plausibility of this principle, at least at large distances (today at $d > 10^{24}$24 m), and the CMB shows us a plasma content which, around 380,000 years after the big bang, still was remarkably homogeneous with, in mass by unit volume, ρ has a a $\delta\rho/\rho \approx 10^{-5}$.

3 Friedmann-Lemaitre-Robertson-Walker Metric

Any GRTG associated with the cosmological principle implies that the symmetry of the content (matter-energy) is the same as that of the container (space-time) and that the most general "metric" ds^2 of the 4D space-time length element is that of Friedmann-Lemaitre-Robertson-Walker (FLRW) [4].

$$\mathbf{ds^2} = -\mathbf{R^2(t)}\left[\frac{\mathbf{dr^2}}{\mathbf{1-kr^2}} + \mathbf{r^2 d}\theta^2 + \mathbf{r^2}\sin^2\theta\mathbf{d}\varphi^2\right] + \mathbf{c^2dt^2} \quad \textbf{avec} \quad \mathbf{k = -1,\ \ 0\ \ ou\ \ +1} \tag{1}$$

1. Special relativity (Einstein, 1905) already "linked" space and time (Minkowski space-time) on the one hand, and matter and energy on the other with the equivalence $E = mc^2$.
2. does not change location
3. At our scale, we're much more accustomed to thinking about landscapes and structures that change from one point in space to another, but very little or very slowly over time, rather than the opposite.
4. established more and more generally between 1922 and 1935 by the 4 authors

where r is a radial coordinate and $R(t)$ the "scale factor", a real, defined, positive function of the variable t that multiplies the distance between fixed points in space. The 3 possibilities for k define the 3 types of monoconnex spatial topology compatible with the hypotheses : S^3 ($_{[\text{hyper}]}$-spherical space), E^3 (Euclidean space), H^3 ($_{[\text{hyper}]}$-hyperbolic space)[5]

Time t is cosmic time. It is orthogonal (independent) of spatial coordinates). The Cosmological Principle implies it is the same for all observers at rest (constant r, θ and φ) but traditionally referred to as *"comoving"*.

The $H(t)$ expansion rate is defined as the logarithmic derivative of R(t) : $H(t) \stackrel{def}{=} \dot{R}(t)/R(t)$. Its present value $H_0 = H(t_0)$ is the "Hubble-Lemaître" constant. Its dimension is the inverse of a time. For historical reasons of method of measurement it is often expressed in km s^{-1} Mpc^{-1} (1 pc $\stackrel{def}{=}$ 3.085677581491 10^{16} m.)

By following the trajectory of the photons (for which $ds = 0$) between emission and reception we deduce that if a source emits two light signals at the times t_e and $t_e + dt_e$, an observer will receive them at the times t_0 and $t_0 + dt_0$ with :

$$\mathbf{\frac{dt_0}{dt_e} = \frac{R(t_0)}{R(t_e)}} \tag{2}$$

Applied to the very small T-periods or wavelengths of light it can be written :

$$\mathbf{\frac{R(t_0)}{R(t_e)} \approx \frac{T_0}{T_e} = \frac{\lambda_0}{\lambda_e} \stackrel{def}{=} 1 + z} \tag{3}$$

z being the cosmological redshift z[6].

4 Universe and General Relativity

Einstein's general relativity, published in 1915-1916, was first applied to local gravitation (Schwarzschild, 1916). In 1917[7], Einstein was the first to apply it to the Universe and obtained a static model, for which he added an "universal constant" to his equation, stating : *"We can indeed add to the left of the field equation the fundamental tensor $g_{\mu\nu}$ multiplied by a provisionally unknown universal constant λ, without destroying the general covariance : we substitute to the field equation"* :

$$\mathbf{\frac{1}{2} g_{\mu\nu}} \mathcal{R} - \mathcal{R}_{\mu\nu} - \mathbf{\Lambda g_{\mu\nu}} = \mathbf{\frac{8\pi G}{c^4}} \mathcal{T}_{\mu\nu} \tag{4}$$

Note that with the addition of the g term, the left-hand side of the equation becomes the most general tensor whose covariant derivative is zero.

5. Cosmological Principle implies that curvature courbure (positive, negative or zero) must be the same everywhere
6. There are other causes of redshift, but they're only local since they are related to inhomogeneities or displacements.
7. Eintein, A., 1917, Sitzungsberichte der Preussischen Akad.d. Wiss., 142

Since the lower-case letter λ generally designates wavelengths, the cosmological constant has become Λ in the usual writing.

$g_{\mu\nu}$ are the coefficients of the metric. Because of its symmetry, only the cross terms $(mu = \nu)$ of FLRW (1) are non-zero :

$$g_{11} = -\frac{R(t)^2}{1-kr^2}, \quad g_{22} = -R(t)r^2, \quad g_{33} = -R(t)r^2\sin^2\theta, \quad g_{44} = c^2 \tag{5}$$

5 Friedmann-Lemaitre Universe Models

With general relativity as RGTG and by introducing the coefficients $(g_{\mu\nu})$ of dr^2, $d\theta^2$, $\mathrm{d}\varphi^2$ and dt^2 of the RW metric into Einstein's equation it becomes 2 differential equations, the Friedmann-Lemaître equations FL1 and FL2, on the scale factor R(t) in which $p(t)$ and $\rho(t)$ (pressure and density) are the only two physical parameters describing the content : p and ρ are, according to the cosmological principle, spatially homogeneous (therefore not a function of r,θ or φ) and isotropic (therefore scalar, even for pressure) [8] : .

$$-\frac{\mathbf{k}}{\mathbf{R^2}} - \frac{\mathbf{\dot{R}^2}}{\mathbf{c^2R^2}} - \frac{\mathbf{2\ddot{R}}}{\mathbf{Rc^2}} + \mathbf{\Lambda} = \frac{\mathbf{8\pi Gp}}{\mathbf{c^4}} \quad (\mathbf{FL1}) \quad \textbf{et} \quad \frac{\mathbf{k}}{\mathbf{R^2}} + \frac{\mathbf{\dot{R}^2}}{\mathbf{c^2R^2}} - \frac{\mathbf{\Lambda}}{\mathbf{3}} = \frac{\mathbf{8\pi G\rho}}{\mathbf{3c^2}} \quad (\mathbf{FL2}) \tag{6}$$

FL3 may be deducted from FL1 et FL2 :

$$\frac{\mathbf{d(\rho R^3)}}{\mathbf{dR}} + \mathbf{3p}\frac{\mathbf{R^2}}{\mathbf{c^2}} = \mathbf{0} \quad (\mathbf{FL3}) \tag{7}$$

FL2 can also be written :

$$\mathbf{\dot{R}^2 = H_\circ^2\, R_\circ^2} \left[\mathbf{\Omega_{r\circ}}\frac{\mathbf{R_\circ^2}}{\mathbf{R^2}} + \mathbf{\Omega_{m\circ}}\frac{\mathbf{R_\circ}}{\mathbf{R}} + \mathbf{\Omega_{\Lambda\circ}}\frac{\mathbf{R^2}}{\mathbf{R_\circ^2}} + \mathbf{\Omega_{k\circ}}\right] \tag{8}$$

with

$$\mathbf{\Omega_r(t)} \stackrel{\text{def}}{=} \frac{\mathbf{8\pi G\rho_r(t)}}{\mathbf{3H^2(t)}}\ , \ \mathbf{\Omega_m(t)} \stackrel{\text{def}}{=} \frac{\mathbf{8\pi G\rho_m(t)}}{\mathbf{3H^2(t)}}\ , \ \mathbf{\Omega_\Lambda(t)} \stackrel{\text{def}}{=} \frac{\mathbf{\Lambda c^2}}{\mathbf{3H^2(t)}} \quad \text{et} \quad \mathbf{\Omega_k(t)} \stackrel{\text{def}}{=} -\frac{\mathbf{kc^2}}{\mathbf{R^2(t)H^2(t)}} \tag{9}$$

G is the universal gravitational constant. $\Omega_r(t)$ is the radiation density parameter (light or ultra-relativistic particles that have the same equation of state : $p = \frac{1}{3}\rho c^2$). $\Omega_m(t)$ is the total density parameter of matter (including dark and non-baryonic) and $\Omega_\Lambda(t)$ is the density parameter of Λ. It is also called a reduced cosmological constant (but $\Omega_\Lambda(t)$ is generally not a constant).

$\Omega_k(t)$ is the curvature density parameter (or reduced curvature). It depends on k, which represents the spatial (3D) curvature of the universe.

— If k = 1, 3D space is hyper-spherical
— If k = 0, 3D-space is Euclidean (in this case $\Omega_k(t) = 0 \ \ \forall t$)
— If k = -1, 3D-space is [hyper]-hyperbolic.

8. fluids in the contents are therefore *perfect fluids*

Ω_{r0} is the better known density parameter because the light energy density ρ_{r0} is essentially that of the Cosmic Microwave Background Radiation (CMBR). One can probably add to it that of the primordial neutrinos not yet detected and which in the simplest hypothesis have an energy density equal to $\sim 68\%$ of that of the CMBR (see next section).

From FL2 :

$$\mathbf{\Omega_r(t) + \Omega_m(t) + \Omega_\Lambda(t) + \Omega_k(t) = 1} \quad \forall \mathbf{t} \tag{10}$$

So, we only need to measure three of the four Ω_i at any given time to determine the fourth. Since today Ω_{r0} is well known and, moreover, less than 10^{-4}, the knowledge of our model of the universe is essentially linked to the measurements of Ω_{m0} and $\Omega_{\Lambda 0}$.

The interactive graph allows to simulate different universes with a simple click on the diagram, choosing the values of Ω_{m0} and $\Omega_{Lambda0}$ (defaulting values are (those of the Λ-CDM model from the 2015 results of ESA's Planck mission). Note that this approximation is only valid for times when Ω_{m0} and $\Omega_{Lambda0}$ are dominant.

It's more accurate to modify the present numerical values of the cosmological parameters. For Ω_{r0}, since the CMB has a black-body thermal spectrum, the temperature T_0 should be chosen.

There's also an option to force $\Omega_k = 0 = \Omega_{k0}$. Finally, the code also allows to take account for primordial neutrinos (see next paragraph) and achieve a reasonable precision beyond the first second for the observed universe.

The volumetric mass of black body radiation ρ_r ($\rho_r = u_r/c^2$ with u_r volumetric energy) is defined only by its temperature

$$\rho_r = \frac{4\sigma T^4}{c^3} \quad \text{avec} \quad \sigma = \frac{2\pi^5 k^4}{15 h^3 c^2} \tag{11}$$

By default the fundamental constants have the values of our universe but they can be modified in the simulations (in the multiverse hypothesis they could be different) :

- Boltzmann k = 1.38064852 10^{-23} m^2.kg.s^{-2}.K^{-1}
- Planck h = 6.62607004 10^{-34} m^2.kg.s^{-1}
- gravitation G = 6.67385 10^{-11} m^3.kg^{-1}.s^{-2}
- speed of light in a vacuum c = 299792458 m.s^{-1}.

With the reduced coordinates : $\mathbf{a} \overset{\mathbf{def}}{=} \mathbf{R(t)/R(t_0)}$ and $\tau \overset{\mathbf{def}}{=} \mathbf{H_0(t - t_0)}$ we get the links [9] between a and z and the first and second derivatives of $a(\tau)$:

$$\mathbf{\frac{da}{d\tau} = \left[\frac{\Omega_{r0}}{a^2} + \frac{\Omega_{m0}}{a} + \Omega_{\Lambda 0}\ a^2 + \Omega_{k0}\right]^{\frac{1}{2}}} \tag{12}$$

$$\mathbf{\frac{d^2a}{d\tau^2} = -\frac{\Omega_{r0}}{a^3} - \frac{1}{2}\frac{\Omega_{m0}}{a^2} + \Omega_{\Lambda 0}\ a} \tag{13}$$

with the initial conditions $a(0) \overset{def}{=} \frac{da}{d\tau}(0) \overset{def}{=} 1$.

Thus the data H_0, Ω_{r0}, Ω_{m0}, and $\Omega_{\Lambda 0}$ make it possible to solve the differential equation above and thus to know (with the hypothesis of the model) $a(\tau)$ for all τ (and from there

9. Then $a = (1+z)^{-1}$ or $z = (1-a)/a$ if z is z is purely cosmological

$a(t)$ for all t) and thus to draw its graph (which is the first action of the simulation part).

We can, of course, question the measured values of our universe's parameters or simply imagine different universes. Depending on the Ω_i values chosen, we then obtain different models with or without singularity(ies). Depending on the values of the selected Ω_i, different patterns with or without singularity(ies) are obtained. If $H_0 > 0$:

— Universe with Big-Bang and no Big Crunch.
— Universe with Big-Bang and Big-Crunch
— Universe without Big-Bang but no Big Crunch
— Universe without singularity

In our current universe, the radiation density parameter is very low ($\Omega_{r0} < 10^{-4}$) while Ω_{m0} and $\Omega_{\Lambda 0}$ are close to 1/3 and 2/3. In these conditions, it's essentially Ω_{m0} and $\Omega_{\Lambda 0}$ that determine the type of universe : the separators in the Ω_{m0} and $\Omega_{\Lambda 0}$ field are shown in the interactive diagram on the right of the simulation window.

6 primordial neutrinos

Although not yet detectable due to their very low energies, and despite the question of their very low (but not zero) mass, they are probably as numerous as photons and with a Planck spectrum (black body), but at a lower temperature due to their earlier decoupling from matter. Translated into density, their grouping with RFC photons amounts [10] to multiplying the parameter Ω_r of the photons by 1.6913. Cosmogravity Universe allows to choose whether or not to take into account Ω_{r0} of the CMBR only, that of CMBR and neutrinos or none of them : by default, RFC and neutrinos are taken into account in the calculation.

7 Calculation of Durations and Ages

Still following the trajectory of the photons, we may deduce the relations (between distances, time, redshifts, apparent diameters, ...) that are used in Cosmogravity (notably in the toolbox of the "Adjunct computations" window). These expressions often use the $E(x)$ function. This function is deduced from :

$$\frac{H(z)}{H_\circ} \stackrel{def}{=} E^{\frac{1}{2}}(z) = \left[\Omega_{r\circ}(1+z)^4 + \Omega_{m\circ}(1+z)^3 + (1-\Omega_{m\circ}-\Omega_{r\circ}-\Omega_{\Lambda\circ})(1+z)^2 + \Omega_{\Lambda\circ}\right]^{\frac{1}{2}} \tag{14}$$

when defining :

$$\mathbf{E(x) \stackrel{def}{=} \Omega_{r\circ}(1+x)^4 + \Omega_{m\circ}(1+x)^3 + (1-\Omega_{m\circ}-\Omega_{r\circ}-\Omega_{\Lambda\circ})(1+x)^2 + \Omega_{\Lambda\circ}} \tag{15}$$

This gives simple expressions for the link between dt and dz along a line of sight. From $z \stackrel{def}{=} R_0/R - 1$ we deduce $dz = -(1+z)\ H(t)dt$ and thus : $dt = -H_0^{-1}(1+z)^{-1}E^{-1/2}(z)dz$.

By integrating this relation we obtain the durations according to the redshifts :

$$\mathbf{t_2 - t_1 = \frac{1}{H_\circ}\int_{z_2}^{z_1}(1+x)^{-1}E^{-\frac{1}{2}}(x)\ dx} \tag{16}$$

10. approximation with an effective number of neutrino flavours of 3.045 instead of 3 to compensate for the non-total decoupling of neutrinos during electron-positron annihilation (around t $\sim$ 1s)

We deduce the expression of the âge ($t_\circ$) of an FL Big Bang universe according to its parameters $H_\circ$ and $\Omega_{i\circ}$ and that of the age of this universe when light received today with a redshift z was emitted :

$$\mathbf{t_\circ = \frac{1}{H_\circ}\int_0^\infty (1+x)^{-1} E^{-\frac{1}{2}}(x)\ dx} \qquad \mathbf{t_e = \frac{1}{H_\circ}\int_z^\infty (1+x)^{-1} E^{-\frac{1}{2}}(x)\ dx} \tag{17}$$

Cosmogravity also allows the inverse calculation : z as a function of t.

Note on cosmological redshift and calculations ... in the future :

For a universe with $a(t)$ increasing continuously from 0 to infinity like our standard ΛCDM model, $z(t_0) = 0$ at the present time, $z \to +\infty$ the more we observe in the past ... and $z \to -1$ in the future. To sum up : $-1 < z < \infty$. Of course, negative z (from increasing $a(t)$ models) are not observable today. but they do allow us to ...calculate the future (cf. sections 7 and 8) such as, for example, the present distance (and therefore the measurable z) of the most distant objects to which we could send a receptive message in a finite time : our present cosmological event horizon [11].

8 Distances Calculation

Expression of the "metric distance" d_m is obtained by integrating the trajectory of a photon from its emission at time t_e and coordinate r until it is received at time $t_\circ$ in $r = 0$ with a cosmological redshift z.

Depending on whether the spatial curvature is negative (d_{m-}), null ($d_{m\circ}$), or positive (d_{m+}) [12] :

$$\mathbf{d_{m-} = \frac{c}{H_\circ \mid \Omega_{k\circ} \mid^{\frac{1}{2}}}} \sinh\left\{\mid \mathbf{\Omega_{k\circ}} \mid^{\frac{1}{2}} \int_0^{\mathbf{z_c}} \mathbf{E^{-\frac{1}{2}}(x)dx}\right\} \tag{18}$$

$$\mathbf{d_{m\circ} = \frac{c}{H_\circ}\int_0^{z_c} E^{-\frac{1}{2}}(x)dx} \tag{19}$$

$$\mathbf{d_{m+} = \frac{c}{H_\circ \mid \Omega_{k\circ} \mid^{\frac{1}{2}}}} \sin\left\{\mid \mathbf{\Omega_{k\circ}} \mid^{\frac{1}{2}} \int_0^{\mathbf{z_c}} \mathbf{E^{-\frac{1}{2}}(x)dx}\right\} \tag{20}$$

Cosmogravity also allows the reverse calculation : z according to d.

9 Horizon calculation

In Cosmogravity's main Universe window, the distances and spectral shifts of these horizons are calculated for present time t_0 (d_{p0} and z_{p0} for the particle horizon, and d_{e0} and z_{e0} for the event horizon). for the event horizon). In the "<Calculette cosmologique"> page, the calculation for other times is available : (d_{pt} and z_{pt} for particles and d_{et} and z_{pt} for events).

11. see section 9 on cosmological horizons

12. The three expressions can be summed up into one by defining a function $S_k(x)$ (or *sinn (x)*), $S_k(x) \overset{def}{=} \sinh x,\ x$ or $\sin x$ depending on $k = -1,\ 0,$ or$+1$: $\mathbf{d_m = \frac{c}{H_\circ|\Omega_{k\circ}|^{\frac{1}{2}}}\ \ S_k}\left\{\mid \mathbf{\Omega_{k\circ}} \mid^{\frac{1}{2}} \int_0^{\mathbf{z_c}} \mathbf{E^{-\frac{1}{2}}(x)dx}\right\}$

9.1 Cosmological particles horizon

The cosmological particle horizon is the greatest distance d_p from which we can receive a signal. It is also the greatest distance and offset of objects from which we can receive a signal in a finite time a signal emitted today (d_{p0} and z_{p0}) or at another time (d_{pt} and z_{pt}). These greater distances correspond to an infinite z and are therefore expressed (see the definition of S_k in the note in section 8) :

$$d_{p0} = \frac{c}{H_\circ \mid \Omega_{k\circ} \mid^{\frac{1}{2}}} \mid \Omega_{k\circ} \mid^{\frac{1}{2}} \int_0^\infty E^{-\frac{1}{2}}(x)dx \tag{21}$$

$$d_{pt} = \frac{c}{H_\circ \mid \Omega_{k\circ} \mid^{\frac{1}{2}}} \mid \Omega_{k\circ} \mid^{\frac{1}{2}} \int_{z(t)}^\infty E^{-\frac{1}{2}}(x)dx \tag{22}$$

Cosmogravity also calculates the inverse problem : calculation of z_{p0}) or z_{pt} ? ? ? ?

9.2 Cosmological event horizon

The cosmological event horizon is the greatest distance d_e at which we can send a signal that will be received in a finite time. It is also the longest distance to an object from which we can receive an in a finite time. Cosmogravity calculates it in the main page for the present time t_0. and for any other time in the cosmological calculator.

$$d_{e0} = \frac{c}{H_\circ \mid \Omega_{k\circ} \mid^{\frac{1}{2}}} \mid \Omega_{k\circ} \mid^{\frac{1}{2}} \int_{-1}^0 E^{-\frac{1}{2}}(x)dx \tag{23}$$

$$d_{et} = \frac{c}{H_\circ \mid \Omega_{k\circ} \mid^{\frac{1}{2}}} \mid \Omega_{k\circ} \mid^{\frac{1}{2}} \int_{-1}^{z(t)} E^{-\frac{1}{2}}(x)dx \tag{24}$$

Cosmogravity also calculates the inverse problem for this horizon : calculation of z_{e0} or z_{et}.

10 Spectral shift drift calculation

Over time, the observed cosmological spectral shift of a fixed object varies with the scale factor $a(t)$. Measuring this change seems difficult because the drift over a few years or tens of years is very small, and it is polluted by other causes of spectral shift (essentially the Doppler-Fizeau shift due to true displacement). But with the performance of the next very large surveys, the signal-to-noise ratio could be considerably improved by multiplication of the number of measurements and measurable objects.

Assuming that the shift is purely cosmological (Universe Theory, equation 3) :

$$z(t_0) = \frac{a(t_0)}{a(t_e)} - 1 \tag{25}$$

and following Balbi and Quercellini, 2007 [13]

$$z(t_0 + \Delta t_0) = \frac{a(t_0 + \Delta t_0)}{a(t_e + \Delta t_e)} - 1 \tag{26}$$

13. Balbi and Quercellini, 2007, “The time evolution of cosmological redshift as a test of dark energy” MNRAS 382, 1623, https ://articles.adsabs.harvard.edu/pdf/2007MNRAS.382.1623B

$$\Delta z = z(t_0 + \Delta t_0) - z(t_0) = \frac{a(t_0 + \Delta t_0)}{a(t_e + \Delta t_e)} - \frac{a(t_0)}{a(t_e)} \tag{27}$$

With an expansion to order 1 in $\Delta t/t$ (*"humanely"* $\Delta t/t \ll 1$)

$$\Delta z \simeq \Delta t_0 \Big[\frac{dota(t_0) - \dot{a}(t_e)}{a(t_e)}] \tag{28}$$

As $H(z) \stackrel{def}{=} \dot{a}(z)/a(z)$ the spectral shift drift can be expressed ($\Delta t_0 \ll t_0$) :

$$\frac{deltaz}{deltat_0} \simeq [H_0(1+z) - H(z)] \tag{29}$$

Spectral shift drift is therefore linked to the function $H(z)$ (see equation 14), which depends on the cosmological parameters H_0 and Ω_{i0}. Measuring this z would be an independent method for gaining a better understanding of cosmic fluids and, in particular the time evolution of their density parameters, and hence their equations of state. The dimension of $\Delta z/\Delta t_0$ is the inverse of a time. Its practical unit is the year^{-1}.

Note : There is no redshift drift for a z such that $H(z) = (1+z)H_0$

.

11 Calculation of angular diameters

Due to the radial trajectory of the photons received in an isotropic space-time the apparent diameter of an object (or the angular difference between two sources of the same z is defined at the time of emission, i.e. at the time when the distance from the source was $(1+z)$ times smaller than at the time of observation. More mathematically the metric of the surface $r = r$ and $t = t_e$ is that of an Euclidean 2-sphere of radius $R_e r = R_0 r/(1+z)$ and the Euclidean relation between linear diameter D_e (at time t_e), metric distance d_m and apparent diameter ϕ_0 is :

$$\boldsymbol{\phi_0 = \frac{D_e(1+z)}{d_m}} \tag{30}$$

$d_A \stackrel{def}{=} d_m/(1+z)$ is called "apparent diameter distance" to find the classic relationship $\phi = D/d_A$. Cosmogravity also allows the reverse calculation : z according to ϕ.

12 Calculating photometry

With a calculation similar to the previous one but on the "wave surfaces" emitted by a source and arriving on the observer, we show that the brightness E_0 (orthogonal illuminance) of a source of intensity I_e (flux emitted per unit of solid angle) in the direction of the observer and of cosmological redshift z presents (in the absence of absorption on the path) an observed brightness

$$\mathbf{E_0 = \frac{I_e}{d_m^2(1+z)^2}} \tag{31}$$

Assuming the source intensity is isotropic, its luminosity (emitted power) : $L_e = 4\ \pi\ I_e$ and :

$$\mathbf{E_0} = \frac{\mathbf{L_e}}{\mathbf{4\ \pi\ d_m^2 (1+z)^2}} \tag{32}$$

$\mathbf{d_L} \overset{\text{def}}{=} \mathbf{d_m(1+z)}$ is the "distance luminosity" (which recovers the classical relationship between brightness and distance, as we did above with the "angular diameter distance".

Magnitudes and distance moduli. Brightness and luminosity have impressive dynamics in astronomy (the brightness of the sun is $\sim 10^{25}$ times that of the galaxies that will be detected (in 2027 ?) by the ELT telescope). The concept of "magnitude" has been used since antiquity, and has then been defined as "apparent magnitude" m as a logarithmic scale of brightness E with $\mathbf{m} \overset{\text{def}}{=} \mathbf{-2.5 \log_{10} E + cte}$. And we call "absolute magnitude" M the apparent magnitude he source would have if located at a distance of 10 pc, so it's a logarithmic measure of luminosity L.

The difference $m - M$ for a star is called the "distance modulus" μ. It's easy to deduce that, in a transparent transparent space,

$$\mu \overset{\text{def}}{=} \mathbf{m - M = -5 + 5 \log_{10} d_{pc}} \tag{33}$$

Both m and M can be "bolometric" (all wavelengths and with a perfect receiver) or depend on the the spectral sensitivity curve of the instrumentation but in all cases the distance modulus retains the above expression (assuming perfect transparency of the medium between source and observer). If we observe "Luminosity standards" we know their luminosity (hence their M) and measuring their m gives us their distance. Obviously, for cosmological distances it's the "Luminosity distance" $d_L = d_m(1+z)$ to be used in the classical relationship :

$$\mu \overset{\text{def}}{=} \mathbf{m - M = -5 + 5 \log_{10} d_L(pc)} \tag{34}$$

13 Single-fluid models

Analytical solutions exist for some of Friedmann-Lemaître's universe models. This is notably (but not only) the case for those for which a single density parameter Ω_i is non-zero. These particular models are sometimes approximations at certain times of a multi-fluid FL universe.

Since $\mathbf{\Omega_m(t) + \Omega_r(t) + \Omega_\Lambda(t) + \Omega_k(t) = 1} \ \ \forall \mathbf{t}$, if only one of the Ω_i is non-zero then it's always equal to one. Four cases present themselves : $\Omega_m(t)$ =1, $\Omega_r(t)$ =1, $\Omega_\Lambda(t)$ =1 and $\Omega_k(t)$ =1.

13.1 Matter, $\Omega_m = 1 \ \ \forall t$

For this universe "of dust" (non-relativistic matter) by Einstein-de Sitter (1932) $p \approx 0$ and : $\rho_m R^3 = cte = \rho_{m0} R_0^3$

The Friedmann-Lemaître equation FL2 then leads (by taking as origin (t=0) of time that of the singularity : $R(t=0) = 0$) to :

$$\mathbf{R(t) = (6\pi G \rho_{m0} R_0^3)^{\frac{1}{3}}\ t^{\frac{2}{3}}} \tag{35}$$

As :

$$\Omega_m(t) \overset{déf}{=} \frac{8\pi G\rho_m(t)}{3H^2(t)}, \qquad \rho_{m0} = \frac{3H_0^2}{8\pi G} \tag{36}$$

$$H \overset{déf}{=} \frac{\dot{R}}{R}, \qquad H(t) = \frac{2}{3\ t} \quad \Rightarrow \quad \mathbf{H_0} = \frac{\mathbf{2}}{\mathbf{3\ t_0}} \tag{37}$$

Then

$$\mathbf{a(t)} \overset{\mathbf{def}}{=} \frac{\mathbf{R(t)}}{\mathbf{R_0}} = \left[\frac{\mathbf{3H_0}}{\mathbf{2}}\right]^{\mathbf{2/3}} \mathbf{t^{\frac{2}{3}}} \tag{38}$$

13.2 Radiation, $\Omega_r = 1 \quad \forall t$

For this entity : $p_r = \frac{1}{3}\rho_r c^2$ and (FL3) : $\rho_r R^4 = cte = \rho_{r0} R_0^4$

In this Weinberg universe, the Friedmann-Lemaître equation FL2 becomes

$$R^2\dot{R}^2 = (8\pi\ G/3)\rho_{r0}R_0^4 \quad \Rightarrow \quad \mathbf{R(t)} = \left[\frac{\mathbf{32\pi G\rho_{r0}R_0^4}}{\mathbf{3}}\right]^{\frac{1}{4}} \mathbf{t^{\frac{1}{2}}} \tag{39}$$

$$\Omega_r(t) \overset{déf}{=} \frac{8\pi G\rho_r(t)}{3H^2(t)}, \quad \Rightarrow \quad \rho_{r0} = \frac{3H_0^2}{8\pi G} \tag{40}$$

$$H \overset{déf}{=} \frac{\dot{R}}{R}, \quad \Rightarrow \quad H = \frac{1}{2\ t} \quad \Rightarrow \quad \mathbf{H_0} = \frac{\mathbf{1}}{\mathbf{2\ t_0}} \tag{41}$$

Then

$$\mathbf{a(t)} \overset{\mathbf{déf}}{=} \frac{\mathbf{R(t)}}{\mathbf{R_0}} = [\mathbf{2H_0}]^{\frac{1}{2}} \mathbf{t^{\frac{1}{2}}} \tag{42}$$

Note : Since the radiation energy density of a black body is

$$u_r = \rho_r c^2 = \frac{4\sigma T^4}{c} \quad \text{and} \quad \Omega_r \overset{déf}{=} \frac{8\pi G\rho_r}{3H^2} = 1 \tag{43}$$

the temperature T_r depends only on H :

$$T_r = \left[\frac{3H^2c^3}{32\pi G\sigma}\right]^{\frac{1}{4}} \quad \text{with} \quad \sigma = \frac{2\pi^5 k^4}{15c^2h^3} \quad \text{et} \quad \mathbf{T_r} = \left[\frac{\mathbf{45c^5h^3}}{\mathbf{64\pi^6 Gk^4}}\right]^{\frac{1}{4}} \mathbf{H^{\frac{1}{2}}} \tag{44}$$

.

13.3 Cosmological constant, $\Omega_\Lambda = 1 \quad \forall t$

For this de-Sitter universe the Friedmann-Lemaître equation FL2 leads to :

$$\frac{\dot{R}^2}{R^2} = \frac{\Lambda c^2}{3} = cte \tag{45}$$

Then $\Lambda \geq 0$ and

$$\mathbf{H} = \pm\ \mathbf{c} \left[\frac{\mathbf{\Lambda}}{\mathbf{3}}\right]^{\mathbf{1/2}} = \mathbf{cte} = \mathbf{H_0} \tag{46}$$

and

$$\mathbf{a(t)} \overset{\mathbf{déf}}{=} \frac{\mathbf{R(t)}}{\mathbf{R_0}} = \mathbf{e^{H_0\cdot(t-t_0)}} \tag{47}$$

13.4 Curvature, $\Omega_k = 1 \quad \forall t$

The sum of Friedmann-Lemaître equations FL1+FL2 results for this Milne's universe to : $\ddot{R}$ =0. Therefore :

$$R(t) = \alpha t + \beta \quad , \quad \dot{R} = \alpha = cte \quad \text{et} \quad H_0 \overset{d\acute{e}f}{=} \frac{\dot{R}_0}{R_0} = \frac{\alpha}{R_0} \tag{48}$$

As a result

$$\mathbf{a(t)} \overset{\mathbf{def}}{=} \frac{\mathbf{R(t)}}{\mathbf{R_0}} = \mathbf{H_0 t} + \frac{\beta}{\mathbf{R_0}} = \mathbf{H_0 \cdot t + cte} \tag{49}$$

Taking for the origin of time that of a=0 :

$$\mathbf{a(t) = H_0 \cdot t} \quad \text{et} \quad \mathbf{t_0 = (H_0)^{-1}} \tag{50}$$

14 Alternative cosmologies

14.1 A short history

The ΛCDM model (with Λ geometric constant) is, with different parameter values, the one published by G. Lemaître in 1931. However it is the Einstein-de-Sitter (EdS) model of 1932 (without Λ that Einstein had denied) that has subsequently been the most widely considered. In the 1980s measurements of expansion rate and density of matter, baryonic or not, with $\Omega_{m0} \approx 0.3$, gave an age of the universe smaller than that of the oldest stars (13 billion years), whereas this difficulty disappeared with an $\Omega_{\Lambda 0} \approx 0.7$.

But it wasn't until 1998 that the measurements on SNIa-type supernovae confirmed an $\Omega_\Lambda 0 > 0$ and that balloon observations («Boomerang» and «Maxima») revealed the apparent diameter of the first acoustic peak of the cosmic background radiation. Crossing observational constraints derived from the SNIa supernova brightness-redshift relations with those of the apparent diameter ($\phi \approx 1°$) of the first RFC acoustic peak, the ΛCDM model with $\Omega_{m0} \approx 0.3$ and $\Omega_{\Lambda 0} \approx 0.7$ resulting in an age of ∼13.8 billion years, has become the standard cosmological model, still in use in 2023.

14.2 Tensions

But new measurements have revealed "tensions" with the standard ΛCDM model. These include :

— "H_0 tension" : direct measurements of luminosity standards give $H_0 \approx 73\ km\ s^{-1}\ Mpc^{-1}$ while those deduced more indirectly from the RFC agree towards $H_0 \approx 67\ km\ s^{-1}\ Mpc^{-1}$ with statistical differences of several σ.

— "σ_8 tension" : The observed structuring of the universe on a scale of about $8h^{-1}$ Mpc is significantly weaker than that deduced from simulations of structures deduced from RFC anisotropies.

— The observed dipole component of the RFC interpreted as the Doppler-Fizeau effect of our displacement (369 ± 1) km/s) relative to this radiation (z ∼ 1000) is close to, but significantly different in modulus and orientation, from that observed at distances on quasars (z∼ 2).

14.3 New theories

At the same time, new cosmological theories have been proposed.
- Inflations : these theories don't necessarily call into question the ΛCDM model, but they would make it possible to explain the «initial» values (before $\sim 10^{-30}$ s) of its parameters
- Dark Energy
- and many others ...

Cosmogravity now includes simulations of universes with Dark Energy (DE).

15 Dark Energy

15.1 Relativistic equations of state

The matter, the radiation, do intervene only through their *equation of state* $p = p(\rho)$ in the energy-momentum tensor of a perfect fluid. We can thus, while remaining within the framework of Friedmann-Lemaître's equations, take into account different known or hypothetical entities i, characterized by a state equation like $p_i = w_i \rho_i c^2$. The total density and pressure of a universe at n fluids... are then :
$\rho = \sum_{i=1}^{n} \rho_i \quad et \quad p = c^2 \sum_{i=1}^{n} w_i \rho_i$.

For the *dust* (non-relativistic matter) $p \ll \rho c^2$ and $w_d = w_{nr} \approx 0$. For radiation (light or ultra-relativistic particles) $w_r = 1/3$.

The evolution of the scale factor a and that of other cosmological parameters can be generalized to a mixture of fluid n and i. Restricting ourselves, for example, to fluids of constant w_i we obtain for the evolution of the rate of expansion $H = \dot{a}/a$:

$$\frac{H}{H_\circ} = \left[\sum_{i=1}^{n} \Omega_i a^{-3(1+w_i)}\right]^{\frac{1}{2}} = \left[\sum_{i=1}^{n} \Omega_i (1+z)^{3(1+w_i)}\right]^{\frac{1}{2}} \tag{51}$$

The introduction of the terms of the other fluids in the expressions of the metric distance d_m opens the way to observational tests and to the constraint of the new parameters, by inverting the relation {parameters} $\longrightarrow$ {observable}, since the latter, such as brightness or apparent diameter, depend on d_m [14].

15.2 From Λ to Dark Energy

The cosmological constant Λ is equivalent to (and can be interpreted as) [15] a fluid of vacuum, Lorentz's invariant, of parameter $w_\Lambda = w_v = -1$ and density $\rho_{DE} = \frac{\Lambda c^2}{8\Pi G}$. This substitution leaves Einstein's equations mathematically unchanged (and consequently those of Friedmann-Lemaître if we keep the cosmological principle).

14. Multiplying the number of free parameters in the model with the same data obviously expands the uncertainties on each one when inverting

15. Lemaître, 1934, Proc.National. Acad. Sciences USA, vol 20, pp12-17

On the other hand, the geometrical cosmological constant Λ can pose (by its constancy) a severe initial adjustment problem [16] and one can try to replace it by a physical fluid. If this new entity, for the moment speculative, is described by a parameter state equation w this parameter must be spatially constant (cosmological principle) but different from -1, or even variable with time t (i.e. with z and a).

The geometric nature of Λ may also be one of the motivations for its replacement by a physical entity. It should be noted, however, that in Einstein's 1917 field equation (4), which includes Λ and has much greater mathematical generality, if we can «add» new physical fluids on the left-hand side of the equation, "replacing" the *Lambda* term with a physical fluid on the right is a modification of the mathematical equation (4) unless this DE (Dark Energy) fluid has parameter $w_{DE} = cte = -1$ and density density $rho_{DE} = fracLambdac^2 8\Pi G$.

It is sometimes distinguished between dark energy ($w_{DE}(z) \neq -1$), the quintessence ($w_Q \neq \quad cte$), the phantom energy ($w_{PE} < -1$) ...but it has now become customary to generalize the name *« dark energy»* to all possibilities and to use the CPL [17] with two parameters w_0 and w_1 (or $w_a = w_1$) :

$$w(z) = w_\circ + w_1 \frac{z}{1+z} \qquad \text{or} \qquad w(a) = w_0 + w_a(1-a) \tag{52}$$

In this representation Λ would appear as the special case of a dark energy of parameters $w_0 = -1$ and $w_1 = 0$.

Observations allow us to constrain the area of our universe in a plane (w_0, w_1) as we do for the field of $(\Omega_{m0}, \Omega_{\Lambda 0})$.

At more than 100 years [18] Λ the geometric cosmological constant or its physical equivalent ($w_0 = -1$ and $w_1 = 0$) remains compatible at less than 2 σ with the observational constraints.

15.3 Calculation

As with $E(x)$ for the model with the constant Λ, functions simplify the writing of relations, $Y(x)$ and $F(x)$:

$$\mathbf{Y(x)} \overset{\text{def}}{=} \exp\{\mathbf{-3(1+w_0+w_1)\log x - 3w_1(1-x)}\} \tag{53}$$

$$\mathbf{F(x)} \overset{\text{def}}{=} \left[\frac{\mathbf{H(x)}}{\mathbf{H_0}}\right]^{\mathbf{2}} = \mathbf{(1+x)^2\Omega_{k0} + (1+x)^3\Omega_{m0} + (1+x)^4\Omega_{r0} + Y((1+x)^{-1})\Omega_{DE0}} \tag{54}$$

Thus the first and second derivatives of $a(\tau)$ become :

$$\frac{\mathbf{da}}{\mathbf{d\tau}} = \left[-\frac{\mathbf{\Omega_{r0}}}{\mathbf{a^2}} + \frac{\mathbf{\Omega_{m0}}}{\mathbf{a}} + \mathbf{\Omega_{DE0}\ a^2Y(a) + \Omega_{k0}}\right]^{\frac{1}{2}} \tag{55}$$

16. its very small but constant value makes it insignificant in the the primordial universe, but if its value were larger, the early acceleration of the expansion would have prevented the formation of the structures
17. Chevalier & Polarski 2001,Int. J. Mod. Phys. D10,213; Linder 2003 Phys. Rev. Lett. 90.091301
18. Einstein, 1917, Sitzungsberichte der Koniglich PreuBischen Akademie der Wissenschaften (Berlin), Seite 142-152

$$\mathbf{\frac{d^2a}{d\tau^2} = -\frac{\Omega_{r0}}{a^3} - \frac{1}{2}\frac{\Omega_{m0}}{a^2} + \Omega_{DE0}\left[aY(a) + \frac{a^2}{2}\frac{dY}{da}\right]} \tag{56}$$

and that of metric distance :

$$\mathbf{d_m = \frac{c}{H_\circ \mid \Omega_{k\circ} \mid^{\frac{1}{2}}} \quad S_k\left\{\mid \Omega_{k\circ} \mid^{\frac{1}{2}} \int_0^{z_c} F^{-\frac{1}{2}}(x)dx\right\}} \tag{57}$$

The introduction of terms from other fluids into the d_m metric distance expressions paves the way for observational tests and the constraint of the new parameters, by inverting the relationship parameters $\longrightarrow$ observables, since the latter, such as brightness or apparent diameter, depend on $d_m$footnoteEvidently, multiplying the number of free model parameters with the same same data widens the uncertainties on each during inversion process.

With dark energy, if we define $\rho_{DE} \stackrel{def}{=} \rho_{DE0}Y(a)$ and $\Omega_{DE} = \dfrac{8\pi G\rho_{DE}}{3H^2} = \dfrac{8\pi G\rho_{DE0}Y(a)}{3H^2}$ we find the closing equation (2) :

$$\mathbf{\Omega_r(t) + \Omega_m(t) + \Omega_{DEN}(t) + \Omega_k(t) = 1 \quad \forall t} \tag{58}$$

The sequence of expressions used in the calculations of E, d_L, d_A, θ, d_{LT}, l ...is identical to those of the model with Λ replacing $E(x)$ by $F(x)$.

15.4 Big Rip and Big Fall

Substituting the cosmological constant Λ by a fluid whose parameter w can become ($p \stackrel{def}{=} w\rho c^2$) less than -1 opens the door to a fourth possibility of accident for the universe. In addition to the Big Bang, the Big Crunch and the Big Chill (or Big Freeze) it is possible to get a "Big Rip", i.e. an infinite expansion of $a(t)$ in a finite time. Unlike the Big Chill of the standard model where the expansion rate H tends to a constant (smaller than its current value) which preserves the not currently expanding structures, the expansion rate of a Big Rip Universe itself tends to infinity in a finite time and then disintegrate right down to the smallest material structures... It is thus an existentially extreme singularity for matter, just as extreme as the $a = 0$ of the "Big Bang". The present (2023) observational constraints of our universe in the plane (w_0, w_1) are compatible with $w_0 = -1$ and $w_1 = 0$ (i.e. the geometric cosmological constant Λ) but also with slightly different values of w_0 and w_1 that predict a Big Rip.

For the dark energy option (with the CPL parametrisation), Cosmogravity first looks for a Big Crunch. If (and only if) the answer is negative then there are 4 possibilities :

1. If $w_1 < 0$ then $w \to +\infty$ with $z \to -1$ no Big Rip.
2. If $w_1 > 0$ then $w \to -\infty$ with $z \to -1$ Big Rip
3. If $w_1 = 0$ and $w_0 < -1$ then w(constant) < -1 Big Rip
4. If $w_1 = 0$ and $w_0 > -1$ then w(constant) > -1 no Big Rip

In the above Big Rip cases (2 and 3), the time $t_{BR} - t_0$ remaining before the Big Rip is then

$$\mathbf{t_{BR} - t_0 = \frac{1}{H_\circ}\int_{-1}^{0}(1+x)^{-1}F^{-\frac{1}{2}}(x)\ dx} \tag{59}$$

The "duration of the universe" is obviously $t_0 + t_{BR}$.

The “Big Fall” (unpatented name) is the symmetrical of the Big Rip with an asymptote but in the past.

15.5 Dark Energy and cosmological horizons

The calculations are the same as for Λ, replacing E(x) by F(x) (section15.3).