Some Useful Math for the Analysis of Algorithms

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This note covers some of the mathematical preliminaries needed to enjoy [SW], the textbook *Algorithms, 4th Edition* by Sedgewick and Wayne, and is targeted for the Algorithms and Data Structures course at ITU Copenhagen.

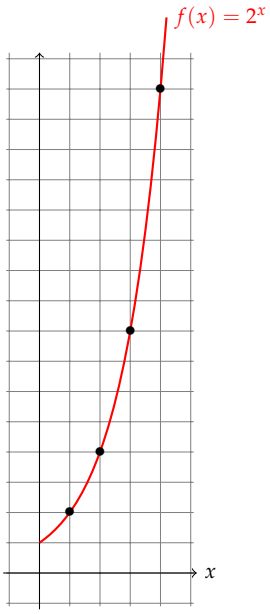
Some functions encountered in the analysis of algorithms

Exponentiation

For a positive integer n (*the exponent*) and a positive real number b (*the base*) we define a shorthand for iterated multiplication, called exponentiation:

$$b^n = \underbrace{b \times b \times \cdots \times b}_{n \text{ times}} .$$

We read b^n as "b to the power of n" or simply "b to the n". (In algorithms, b is often 2, sometimes it's 10. In maths and physics, it's often the number $\mathrm{e} = 2.7182\ldots$ and one then uses the notation $\exp(n)$ for e^n. The function exp is called *the exponential function*.)



Exercise1. For $b > 1$ and $n, m \geq 1$, which of the following equations is true: $b^{nm} = b^n + b^m$, or $b^{nm} = (b^n)^m$?
Exercise2. For $n, m \geq 1$, true or false: $b^{n+m} = b^n b^m$? $b^{n-m} = b^n / b^m$?
Exercise3. For $a, b > 1$ and $n, m \geq 1$, which is true: $(ab)^n = a^n + b^n$? $(ab)^n = a^n b^n$? $(a+b)^n = a^n + b^n$?
Exercise4. What would be a useful convention for what b^0 should mean?
Exercise5. Express as a power of 2: $2 \cdot 2^2$, $(2^2)^3$, $2^{(2^2)}$, $2^n + 2^n$.

There is a way to define what b^x should mean even when x is negative, or not even an integer, so that $x \mapsto b^x$ becomes a smooth function defined for all reals that agrees with b^n when $x = n$ is an integer. This definition requires calculus and lies outside of our scope. The notation makes good sense all the way, for example b^{-1} equals $1/b$ so that $b^{N+1} b^{-1} = b^N$, and $b^{\frac{1}{2}}$ equals $\sqrt{b}$ so that $(b^{\frac{1}{2}})^2 = b^{\frac{1}{2} \cdot 2} = b$.

Positional notation

One of Civilisation's big breakthroughs is to use exponentiation to denote numbers in a very compact way. When we write 7143, we mean

$$7 \cdot 10^3 + 1 \cdot 10^2 + 4 \cdot 10^1 + 3 \cdot 10^0 .$$

To see how clever this is, compare it to the notation used in ancient Rome (MMMMMMMCXLIII), or worse, to unary notation. Our notation is called "digital" or "base 10" (because $b = 10$). Positional notation is particularly clever when performing arithmetic, and

digit, noun, from Latin *digitus* 'finger, toe'. 1. Any of the numerals 0 to 9. 2. A finger or toe.

many people are comfortable using this, having been trained to find even complicated operations like $100 - 1 = 99$ natural. (Compare to the more straightforward $\mathrm{C} - \mathrm{I} = \mathrm{IC}$.)

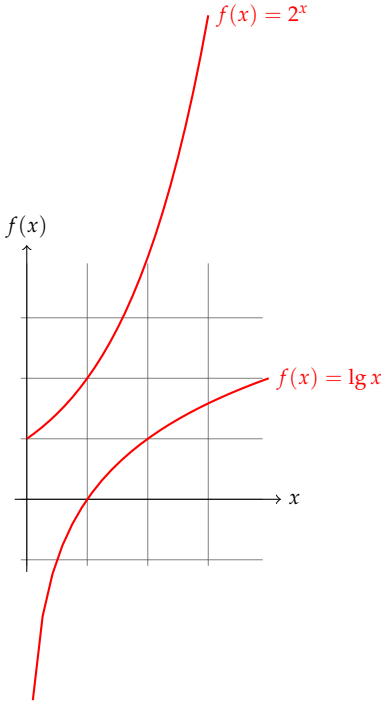
Binary notation is the positional number system using the base $b = 2$. It has only two numerals (called "binary digits" or "bits"), 0 and 1. The binary number 1101111100111 has value

$$2^{12} + 2^{11} + 2^9 + 2^8 + 2^7 + 2^6 + 2^5 + 2^2 + 2^1 + 2^0$$

which happens to be 7143 in base 10.

```
0 000
1 001
2 010
3 011
4 100
5 101
6 110
7 111
```

Exercise6. Express 8 in binary. What is the binary number 10010 in base 10?
Exercise7. Add the binary numbers 100101 and 1011 using the school algorithm for addition.
Exercise8. Multiply the binary numbers 100101 and 1011 using the school algorithm for multiplication.



Logarithms

The *binary logarithm* $\lg N$ is defined for every positive real number N and satisfies $2^{\lg N} = N$. Roughly, "how often you need to double 1 to get N."

More generally, the *logarithm base b* for any real $b > 1$ is a function defined for all positive reals N by $\log_b N = x$ where $b^x = N$. (To see that this does indeed define $\log_b N$ uniquely, we would need to observe that the function $x \mapsto b^x$ is a bijection from the reals to the positive reals for $b > 1$. But at this point we have stopped pretending to give a rigorous presentation anyway.)

Note that $\log_1 5$ or $\lg 0$ and $\lg(-1)$ are just meaningless collections of ink, like $8 + \times$ or $\frac{\surd}{4}$.

Exercise9. What is $\lg 2$? What is $\log_b b$?
Exercise10. What is $\lg 2^x$? What is $\log_b b^x$?
Exercise11. For $b > 1$ and $N, M > 0$, which is true: $\log_b(NM) = (\log_b N) + (\log_b M)$? $\log_b(NM) = (\log_b N)(\log_b M)$? $\log_b(N + M) = (\log_b N)(\log_b M)$? $\log_b(N + M) = (\log_b N) + (\log_b M)$?
Exercise12. Which is true: $\log_b(N^x) = x \log_b N$? $\log_b(N^x) = (\log_b N)^x$?
Exercise13. True or false: $\log_a N = \log_b N / \log_b a$?
Exercise14. Find $\lg 1024$, $\lg \frac{1}{4}$, $\lg 8$.

Floors and ceilings

The *floor function* $\lfloor x \rfloor$ maps any real number x to the largest integer less than or equal to x. Likewise, the *ceiling function* $\lceil x \rceil$ maps any real number x to the smallest integer greater than or equal to x.

Exercise15. Find $\lfloor \frac{1}{2} \rfloor$, $\lfloor -\frac{1}{2} \rfloor$, $\lfloor \pi \rfloor$, $\lfloor 7 \rfloor$.
Exercise16. Find $\lceil \frac{1}{2} \rceil$, $\lceil -\frac{1}{2} \rceil$, $\lceil \pi \rceil$, $\lceil 7 \rceil$.
Exercise17. Find $\lfloor \frac{1}{2} + \lceil \frac{3}{2} \rceil \rfloor$, $\lfloor \frac{1}{2} \cdot \lceil \frac{5}{2} \rceil \rfloor$.
Exercise18. For real x, true or false: $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$.

Exercise19. For all real x, true or false: $\lceil -x \rceil = -\lfloor x \rfloor$, $\lfloor -x \rfloor = -\lceil x \rceil$.
Exercise20. For reals x and y, true of false: $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$. $\lceil xy \rceil = \lceil x \rceil \lceil y \rceil$.
Exercise21. For real x and integer n, true or false: $\lfloor x + n \rfloor = \lfloor x \rfloor + n$, $\lceil x + n \rceil = \lceil x \rceil + n$,

A particularly interesting function is $\lfloor \log_b N \rfloor$, the largest integer not greater than $\log_b N$. This is 1 less than the number of bits needed to express N in base b. For example, in base $b = 10$ we need $\lfloor \log_{10} 99 \rfloor + 1 = 2$ digits to express 99. (In other words, $\lfloor \log_{10} N \rfloor$ function is *really easy* to compute in your head when N is written in base 10: Just count the number of digits and subtract 1. It's one of the most natural properties of a number. Similarly, if the number is written in binary, $\lfloor \lg N \rfloor$ is just the number of bits minus 1.)

Exercise22. How often can you divide a number N by two until you reach 1 or less?

Factorials

The factorial function is another iterated multiplication, defined for an integer $N \geq 1$ as $N! = N \times (N-1) \times \cdots \times 1$. By convention, we also define $0! = 1$.

Exercise23. How big is $N!$ compared to 2^N and N^N?
Exercise24. Find N such that $N!$ is roughly the number of elementary particles in the universe.
Exercise25. Find N such that $N!$ is roughly the number of nanoseconds since the beginning to Time.

Except for "fricking huge" it's difficult to get a feeling for the value of $N!$. A sometimes useful fact, the proof of which is beyond the scope of this note, is Stirling's approximation,

$$N! \sim \sqrt{2\pi N}\left(\frac{N}{\mathrm{e}}\right)^N.$$

Binomial coefficients

The number of k element subsets of $\{1, \ldots, N\}$ is denoted $C(N, k)$. Note that we are talking about sets, and remember that the sets $\{1, 2, 4\}$ and $\{1, 4, 2\}$ are the same. For $N = 3$ and $k = 2$ there are 3 such sets:

$$\{1, 2\}, \{2, 3\}, \{1, 3\}.$$

A confusing way to define $C(N, k)$ is to say "the number of ways to pick k elements from N". I find this confusing because we're *not* actually interested in the fact that $\{1, 2\}$ can be picked in two ways ("first pick 1, then 2", or "first pick 2, then 1".)

Nevertheless, many textbooks use this.

By contrast, consider the number of *ordered* sequences of k elements from $\{1, \ldots, N\}$. For $N = 3$ and $k = 2$, the possible sequences

are:

$$(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\,.$$

In general, there are N^k such sequences, so this number does not deserve special notation.

Exercise26. Which values do (r,s,t) take? What is the final value of C in terms of N?

```
C = 0
for (int r = 0; r < N; r++)
    for (int s = 0; s < N; s++)
        for (int t = 0; t < N; t++)
            C++;
```

Exercise27. Which values do (r,s,t) take? What is the final value of C in terms of N?

```
C = 0
for (int r = 0; r < N-2; r++)
    for (int s = r+1; s < N-1; s++)
        for (int t = s+1; t < N; t++)
            C++;
```

A popular notation is $C(N,k) = \binom{N}{k}$. We'll stick to that. By convention, we agree that the number of ways to pick 0 elements is 1 (namely "by not doing anything"), so $\binom{N}{0} = 1$. You can't pick a nonempty set from an empty set, so $\binom{0}{k} = 0$ for $k \geq 1$. And, perversely but consequently, $\binom{0}{0} = 1$. These conventions turn out to lead to pleasant notation and are consistent with our other choices, such as $0! = 1$.

The numbers also appear as coefficients in the evaluation of binomials, which are expressions of the form $(x+y)^N$. For example in $(x+y)^2 = x^2 + 2xy = y^2$, the coefficients are $1 = \binom{2}{0}$, $2 = \binom{2}{1}$, and $1 = \binom{2}{0}$. In general, we have the Binomial Theorem,

$$(x+y)^N = \binom{N}{0}x^N y^0 + \binom{N}{1}x^{N-1}y^1 + \cdots + \binom{N}{k}x^{N-k}y^k + \cdots + \binom{N}{N-1}x^1 y^{N-1} + \binom{N}{0}x^0 y^N\,.$$

This is why $\binom{N}{k}$ is often called a *binomial coefficient*.

Exercise28. Find the values $\binom{3}{0}$, $\binom{3}{1}$, $\binom{3}{2}$, and $\binom{3}{3}$ by factoring out $(x+y)^3$.

Exercise29. Find $\binom{N}{0} + \binom{N}{1} + \cdots + \binom{N}{N}$.

Binomial coefficients can be expressed in terms of factorials using the following identity:

$$\binom{N}{k} = \frac{N!}{k!(N-k)!}\,.$$

This formula is surprisingly useless both for getting an intuition about the size of $\binom{N}{k}$ and for actually computing it (because the numbers are so huge). Still, it's good enough for some small numbers:

Exercise30. Express $\binom{N}{1}$, $\binom{N}{2}$, and $\binom{N}{3}$ as polynomials in N.

Harmonic numbers

Sums

[To be written]

Growth Rates

To express the running time of algorithms, we say things like "Bottom-up Mergesort requires between $\frac{1}{2}N \lg N$ and $N \lg N$ compares and $\sim 2N \lg N$ data moves" using plain English ("between"), tilde notation, and a well-defined cost model (comparisons, data moves).

Sometimes, this level of detail is exactly right. But in many instances, we are satisfied with more coarse-grained expressions and would like to say things like "Mergesort runs in linearithmic time," without specifying what is counted (comparisons? moves? arithmetic? array accesses?) or what the leading coefficient is.

Let f and g be functions. Then f is $O(g)$ if there exist positive constants C and N_0 such that $|f(N)| \leq |Cg(N)|$ holds for all $N \geq N_0$.

We pronounce this as "f is big oh of g." It is often written $f = O(g)$ or $f \in O(g)$ or $f \leq O(g)$, and called *asymptotic*, *Landau*, or just *big-O notation*.

Exercise31. State the running times of the following code snippet as a function of N in asymptotic notation.

```
C = 0
for (int i = 0; i < N-2; i++)
    for (int j = i+1; j < N-1; j++)
        for (int k = j+1; k < N; k++)
            C++;
```

Exercise32. As above:

```
C = 0
if (N > 10)
    for (int i = 0; i < N; i++)
        for (int j = 0; j < N; j++)
            C++;
else
    for (int i = 0; i < N; i++)
            C++;
```

Exercise33. As above:

```
C = 0
if (N < 10)
    for (int i = 0; i < N; i++)
        for (int j = 0; j < N; j++)
            C++;
else
    for (int i = 0; i < N; i++)
            C++;
```

Exercise34. True or false: if $f \sim g$ then f is $O(g)$.
Exercise35. True or false: if f is $O(g)$ then $f \sim g$.
Exercise36. True or false: if $a_k N^k + a_{k-1} N^{k-1} + \cdots a_1 N + a_0$ is $O(N^k)$.
Exercise37. True or false: N is $O(N^2)$, N is $O(N)$, $N + \log N$ is $O(N)$, $N \log N$ is $O(N)$.
Exercise38. Order the following functions by their growth rate, that is, write them as a list $f_1(N), f_2(N), \ldots$, such that $f_i(N)$ is $O(f_{i+1}(N))$:

$$10^{10} \qquad N^2 \qquad \log N \qquad \log\log N \qquad (\log N)^2 \qquad \sqrt{\log N} \qquad N! \qquad N^N$$

$$N^2 \log N \qquad N \log N \log\log N \qquad N^{3/2} \qquad 2^N \qquad 2^{N^2} \qquad 3^N \qquad \sqrt{N} \log N \qquad \log(N^2+1)\,.$$