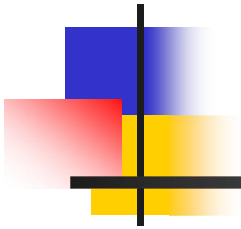


Programming Languages and Compilers (CS 421)



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<http://www.cs.illinois.edu/class/cs421/>

Based in part on slides by Mattox Beckman, as updated by Vikram Adve, Gul Agha and Elsa Gunter



Type Declarations

- n *Type declarations*: explicit assignment of types to variables (signatures to functions) in the code of a program
 - n Must be checked in a strongly typed language
 - n Often not necessary for strong typing or even static typing (depends on the type system)



Type Inference

- n *Type inference*: A program analysis to assign a type to an expression from the program context of the expression
 - n Fully static type inference first introduced by Robin Miller in ML
 - n Haskell, OCAML, SML all use type inference
 - n Records are a problem for type inference



Format of Type Judgments

n A *type judgement* has the form

$$\Gamma \vdash \text{exp} : \tau$$

n Γ is a typing environment

n Supplies the types of variables and functions

n Γ is a list of the form $[x : \sigma , \dots]$

n exp is a program expression

n τ is a type to be assigned to exp

n \vdash pronounced “turnstile”, or “entails” (or “satisfies”)



Example Valid Type Judgments

- $n \quad [] \vdash \text{true or false} : \text{bool}$
- $n \quad [x : \text{int}] \vdash x + 3 : \text{int}$
- $n \quad [p : \text{int} \rightarrow \text{string}] \vdash p(5) : \text{string}$



Format of Typing Rules

Assumptions

$$\frac{\Gamma_1 \vdash \text{exp}_1 : \tau_1 \quad \dots \quad \Gamma_n \vdash \text{exp}_n : \tau_n}{\Gamma \vdash \text{exp} : \tau}$$

Conclusion

- n Idea: Type of expression determined by type of components
- n Rule without assumptions is called an *axiom*
- n Γ may be omitted when not needed



Format of Typing Rules

Assumptions

$$\frac{\Gamma_1 \vdash \text{exp}_1 : \tau_1 \quad \dots \quad \Gamma_n \vdash \text{exp}_n : \tau_n}{\Gamma \vdash \text{exp} : \tau}$$

Conclusion

- Γ , exp , τ are *parameterized* environments, expressions and types - *i.e.* may contain *meta-variables*



Axioms - Constants

$\vdash n : \text{int}$ (assuming n is an integer constant)

$\vdash \text{true} : \text{bool}$

$\vdash \text{false} : \text{bool}$

- n These rules are true with any typing environment
- n n is a meta-variable



Axioms - Variables

Notation: Let $\Gamma(x) = \sigma$ if $x : \sigma \in \Gamma$ and there is no $x : \tau$ to the left of $x : \sigma$ in Γ

Variable axiom:

$$\frac{}{\Gamma \vdash x : \sigma} \quad \text{if } \Gamma(x) = \sigma$$



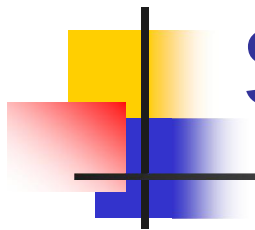
Simple Rules - Arithmetic

Primitive operators ($\oplus \in \{ +, -, *, \dots \}$):

$$\frac{\Gamma \vdash e_1 : \text{int} \quad \Gamma \vdash e_2 : \text{int}}{\Gamma \vdash e_1 \oplus e_2 : \text{int}}$$

Relations ($\sim \in \{ <, >, =, \leq, \geq \}$):

$$\frac{\Gamma \vdash e_1 : \text{int} \quad \Gamma \vdash e_2 : \text{int}}{\Gamma \vdash e_1 \sim e_2 : \text{bool}}$$

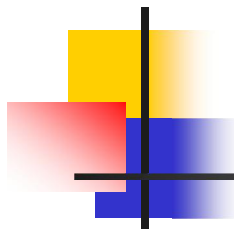


Simple Rules - Booleans

Connectives

$$\frac{\Gamma \vdash e_1 : \text{bool} \quad \Gamma \vdash e_2 : \text{bool}}{\Gamma \vdash e_1 \ \&\& \ e_2 : \text{bool}}$$

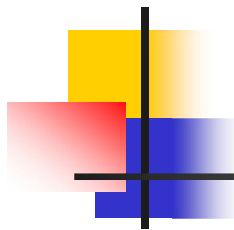
$$\frac{\Gamma \vdash e_1 : \text{bool} \quad \Gamma \vdash e_2 : \text{bool}}{\Gamma \vdash e_1 \ || \ e_2 : \text{bool}}$$



Simple Example

- n Let $\Gamma = [x:\text{int} ; y:\text{bool}]$
- n Show $\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}$
- n Start building the proof tree from the bottom up

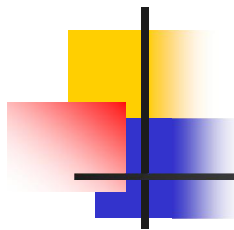
$$\frac{\quad}{\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}} ?$$



Simple Example

- n Let $\Gamma = [x:\text{int} ; y:\text{bool}]$
- n Show $\Gamma \vdash y \mid\mid (x + 3 > 6) : \text{bool}$
- n Which rule has this as a conclusion?

$$\frac{?}{\Gamma \vdash y \mid\mid (x + 3 > 6) : \text{bool}}$$



Simple Example

- n Let $\Gamma = [x:\text{int} ; y:\text{bool}]$
- n Show $\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}$
- n Booleans: \parallel

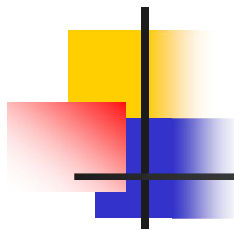
$$\frac{\Gamma \vdash y : \text{bool} \quad \Gamma \vdash x + 3 > 6 : \text{bool}}{\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}}$$



Simple Example

- n Let $\Gamma = [x:\text{int} ; y:\text{bool}]$
- n Show $\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}$
- n Pick an assumption to prove

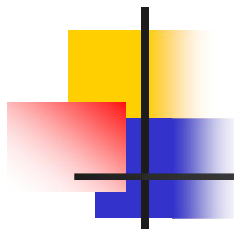
$$\frac{\frac{?}{\Gamma \vdash y : \text{bool}} \quad \Gamma \vdash x + 3 > 6 : \text{bool}}{\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}}$$



Simple Example

- n Let $\Gamma = [x:\text{int} ; y:\text{bool}]$
- n Show $\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}$
- n Which rule has this as a conclusion?

$$\frac{\frac{?}{\Gamma \vdash y : \text{bool}} \quad \Gamma \vdash x + 3 > 6 : \text{bool}}{\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}}$$



Simple Example

- n Let $\Gamma = [x:\text{int} ; y:\text{bool}]$
- n Show $\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}$
- n Axiom for variables

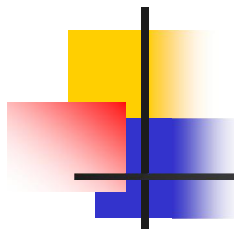
$$\frac{\Gamma \vdash y : \text{bool} \quad \Gamma \vdash x + 3 > 6 : \text{bool}}{\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}}$$



Simple Example

- n Let $\Gamma = [x:\text{int} ; y:\text{bool}]$
- n Show $\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}$
- n Pick an assumption to prove

$$\frac{\frac{}{\Gamma \vdash y : \text{bool}} \quad \frac{\frac{}{\Gamma \vdash x + 3 > 6 : \text{bool}}}{?}}{\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}}$$



Simple Example

- n Let $\Gamma = [x:\text{int} ; y:\text{bool}]$
- n Show $\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}$
- n Which rule has this as a conclusion?

$$\frac{\frac{\Gamma \vdash y : \text{bool}}{\Gamma \vdash y : \text{bool}} \quad \frac{\Gamma \vdash x + 3 > 6 : \text{bool}}{\Gamma \vdash x + 3 > 6 : \text{bool}}}{\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}} \quad ?$$



Simple Example

- n Let $\Gamma = [x:\text{int} ; y:\text{bool}]$
- n Show $\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}$
- n Arithmetic relations

$$\frac{\frac{}{\Gamma \vdash y : \text{bool}} \quad \frac{\Gamma \vdash x + 3 : \text{int} \quad \Gamma \vdash 6 : \text{int}}{\Gamma \vdash x + 3 > 6 : \text{bool}}}{\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}}$$



Simple Example

- n Let $\Gamma = [x:\text{int} ; y:\text{bool}]$
- n Show $\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}$
- n Pick an assumption to prove

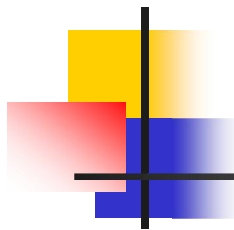
$$\frac{\frac{\Gamma \vdash y : \text{bool}}{\Gamma \vdash y : \text{bool}} \quad \frac{\Gamma \vdash x + 3 : \text{int} \quad \frac{\Gamma \vdash 6 : \text{int}}{?}}{\Gamma \vdash x + 3 > 6 : \text{bool}}}{\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}}$$



Simple Example

- n Let $\Gamma = [x:\text{int} ; y:\text{bool}]$
- n Show $\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}$
- n Which rule has this as a conclusion?

$$\frac{\frac{\Gamma \vdash y : \text{bool}}{\Gamma \vdash y : \text{bool}} \quad \frac{\Gamma \vdash x + 3 : \text{int} \quad \frac{\Gamma \vdash 6 : \text{int}}{?}}{\Gamma \vdash x + 3 > 6 : \text{bool}}}{\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}}$$



Simple Example

- n Let $\Gamma = [x:\text{int} ; y:\text{bool}]$
- n Show $\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}$
- n Axiom for constants

$$\frac{\frac{}{\Gamma \vdash y : \text{bool}} \quad \frac{\Gamma \vdash x + 3 : \text{int} \quad \overline{\Gamma \vdash 6 : \text{int}}}{\Gamma \vdash x + 3 > 6 : \text{bool}}}{\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}}$$



Simple Example

- n Let $\Gamma = [x:\text{int} ; y:\text{bool}]$
- n Show $\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}$
- n Pick an assumption to prove

$$\frac{\frac{\Gamma \vdash y : \text{bool}}{\Gamma \vdash y : \text{bool}} \quad \frac{\frac{\Gamma \vdash x + 3 : \text{int}}{\Gamma \vdash x + 3 : \text{int}} \quad \frac{\Gamma \vdash 6 : \text{int}}{\Gamma \vdash 6 : \text{int}}}{\Gamma \vdash x + 3 > 6 : \text{bool}}}{\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}}$$



Simple Example

- n Let $\Gamma = [x:\text{int} ; y:\text{bool}]$
- n Show $\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}$
- n Which rule has this as a conclusion?

$$\frac{\frac{\Gamma \vdash y : \text{bool}}{\Gamma \vdash y : \text{bool}} \quad \frac{\frac{\Gamma \vdash x + 3 : \text{int}}{\Gamma \vdash x + 3 : \text{int}} \quad \frac{\Gamma \vdash 6 : \text{int}}{\Gamma \vdash x + 3 > 6 : \text{bool}}}{\Gamma \vdash x + 3 > 6 : \text{bool}}}{\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}} \quad ?$$



Simple Example

- n Let $\Gamma = [x:\text{int} ; y:\text{bool}]$
- n Show $\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}$
- n Arithmetic operations

$$\frac{\frac{\frac{\Gamma \vdash x : \text{int} \quad \Gamma \vdash 3 : \text{int}}{\Gamma \vdash x + 3 : \text{int}} \quad \Gamma \vdash 6 : \text{int}}{\Gamma \vdash x + 3 > 6 : \text{bool}} \quad \Gamma \vdash y : \text{bool}}{\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}}$$



Simple Example

- n Let $\Gamma = [x:\text{int} ; y:\text{bool}]$
- n Show $\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}$
- n Pick an assumption to prove

$$\frac{\frac{\frac{\Gamma \vdash x : \text{int}}{\Gamma \vdash x + 3 : \text{int}} \quad \frac{\Gamma \vdash 3 : \text{int}}{\Gamma \vdash 6 : \text{int}}}{\Gamma \vdash x + 3 > 6 : \text{bool}} \quad \Gamma \vdash y : \text{bool}}{\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}}$$

Simple Example

- n Let $\Gamma = [x:\text{int} ; y:\text{bool}]$
- n Show $\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}$
- n Which rule has this as a conclusion?

$$\begin{array}{c}
 \frac{\Gamma \vdash x : \text{int} \quad \frac{\Gamma \vdash 3 : \text{int}}{?}}{\Gamma \vdash x + 3 : \text{int}} \quad \frac{}{\Gamma \vdash 6 : \text{int}} \\
 \frac{\Gamma \vdash y : \text{bool} \quad \Gamma \vdash x + 3 > 6 : \text{bool}}{\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}}
 \end{array}$$



Simple Example

- n Let $\Gamma = [x:\text{int} ; y:\text{bool}]$
- n Show $\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}$
- n Axiom for constants

$$\frac{\frac{\Gamma \vdash x : \text{int} \quad \overline{\Gamma \vdash 3 : \text{int}}}{\Gamma \vdash x + 3 : \text{int}} \quad \overline{\Gamma \vdash 6 : \text{int}}}{\frac{\Gamma \vdash y : \text{bool} \quad \Gamma \vdash x + 3 > 6 : \text{bool}}{\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}}}$$



Simple Example

- n Let $\Gamma = [x:\text{int} ; y:\text{bool}]$
- n Show $\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}$
- n Pick an assumption to prove

$$\frac{\frac{\frac{\Gamma \vdash x : \text{int}}{\Gamma \vdash x + 3 : \text{int}} \quad \frac{\Gamma \vdash 3 : \text{int}}{\Gamma \vdash 6 : \text{int}}}{\Gamma \vdash x + 3 > 6 : \text{bool}} \quad \Gamma \vdash y : \text{bool}}{\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}}$$



Simple Example

- n Let $\Gamma = [x:\text{int} ; y:\text{bool}]$
- n Show $\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}$
- n Which rule has this as a conclusion?

$$\frac{\frac{\frac{\Gamma \vdash x : \text{int}}{\Gamma \vdash x + 3 : \text{int}} \quad \frac{\Gamma \vdash 3 : \text{int}}{\Gamma \vdash 6 : \text{int}}}{\Gamma \vdash x + 3 > 6 : \text{bool}} \quad \Gamma \vdash y : \text{bool}}{\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}}$$



Simple Example

- n Let $\Gamma = [x:\text{int} ; y:\text{bool}]$
- n Show $\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}$
- n Axiom for variables

$$\frac{\frac{\frac{\Gamma \vdash x : \text{int}}{} \quad \frac{\Gamma \vdash 3 : \text{int}}{}}{\Gamma \vdash x + 3 : \text{int}}} \quad \frac{\Gamma \vdash 6 : \text{int}}{} \quad \frac{\Gamma \vdash y : \text{bool} \quad \Gamma \vdash x + 3 > 6 : \text{bool}}{\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}}$$



Simple Example

- n Let $\Gamma = [x:\text{int} ; y:\text{bool}]$
- n Show $\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}$
- n No more assumptions! DONE!

$$\frac{\frac{\frac{\Gamma \vdash x : \text{int}}{} \quad \frac{\Gamma \vdash 3 : \text{int}}{}}{\Gamma \vdash x + 3 : \text{int}} \quad \frac{\Gamma \vdash 6 : \text{int}}{} \quad \frac{\Gamma \vdash y : \text{bool}}{} \quad \frac{\Gamma \vdash x + 3 > 6 : \text{bool}}{}}{\Gamma \vdash y \parallel (x + 3 > 6) : \text{bool}}$$



Type Variables in Rules

n If_then_else rule:

$$\frac{\Gamma \vdash e_1 : \text{bool} \quad \Gamma \vdash e_2 : \tau \quad \Gamma \vdash e_3 : \tau}{\Gamma \vdash (\text{if } e_1 \text{ then } e_2 \text{ else } e_3) : \tau}$$

n τ is a type variable (meta-variable)

n Can take any type at all

n All instances in a rule application must get same type

n Then branch, else branch and if_then_else must all have same type

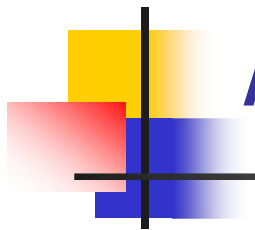


Function Application

n Application rule:

$$\frac{\Gamma \vdash e_1 : \tau_1 \rightarrow \tau_2 \quad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash (e_1 e_2) : \tau_2}$$

n If you have a function expression e_1 of type $\tau_1 \rightarrow \tau_2$ applied to an argument of type τ_1 , the resulting expression has type τ_2



Application Examples

$$\frac{\Gamma \vdash \text{print_int} : \text{int} \rightarrow \text{unit} \quad \Gamma \vdash 5 : \text{int}}{\Gamma \vdash (\text{print_int } 5) : \text{unit}}$$

$n \quad e_1 = \text{print_int}, \quad e_2 = 5,$

$n \quad \tau_1 = \text{int}, \quad \tau_2 = \text{unit}$

$$\frac{\Gamma \vdash \text{map print_int} : \text{int list} \rightarrow \text{unit list} \quad \Gamma \vdash [3;7] : \text{int list}}{\Gamma \vdash (\text{map print_int } [3; 7]) : \text{unit list}}$$

$n \quad e_1 = \text{map print_int}, \quad e_2 = [3; 7],$

$n \quad \tau_1 = \text{int list}, \quad \tau_2 = \text{unit list}$



Fun Rule

- n Rules describe types, but also how the environment Γ may change
- n Can only do what rule allows!
- n fun rule:

$$\frac{[x : \tau_1] + \Gamma \vdash e : \tau_2}{\Gamma \vdash \text{fun } x \rightarrow e : \tau_1 \rightarrow \tau_2}$$



Fun Examples

$$\frac{[y : \text{int}] + \Gamma \vdash y + 3 : \text{int}}{\Gamma \vdash \text{fun } y \rightarrow y + 3 : \text{int} \rightarrow \text{int}}$$

$$\frac{[f : \text{int} \rightarrow \text{bool}] + \Gamma \vdash f \ 2 :: [\text{true}] : \text{bool list}}{\Gamma \vdash (\text{fun } f \rightarrow f \ 2 :: [\text{true}]) : (\text{int} \rightarrow \text{bool}) \rightarrow \text{bool list}}$$



Let and Let Rec

n let rule:

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad [x : \tau_1] + \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (\text{let } x = e_1 \text{ in } e_2) : \tau_2}$$

n let rec rule:

$$\frac{[x : \tau_1] + \Gamma \vdash e_1 : \tau_1 \quad [x : \tau_1] + \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (\text{let rec } x = e_1 \text{ in } e_2) : \tau_2}$$



- n The above system can't handle polymorphism as in OCAML
- n No type variables in type language (only meta-variable in the logic)
- n Would need:
 - n Object level type variables and some kind of type quantification
 - n **let** and **let rec** rules to introduce polymorphism
 - n Explicit rule to eliminate (instantiate) polymorphism



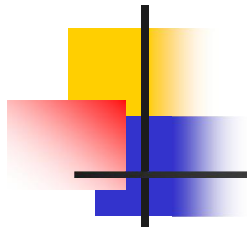
Example

n Which rule do we apply?

?

| - (let rec one = 1 :: one in
 let x = 2 in
 fun y -> (x :: y :: one)) : int → int
list

9/22/09



Proof of 1

n Which rule?

$$[\text{one} : \text{int list}] \vdash (1 :: \text{one}) : \text{int list}$$



Proof of 1

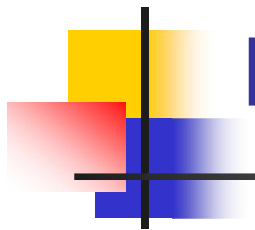
n Application

③

$$\frac{[one : \text{int list}] \vdash ((::) 1) : \text{int list} \rightarrow \text{int list}}{[one : \text{int list}] \vdash (1 :: one) : \text{int list}}$$

④

$$\frac{[one : \text{int list}] \vdash one : \text{int list}}{[one : \text{int list}] \vdash (1 :: one) : \text{int list}}$$



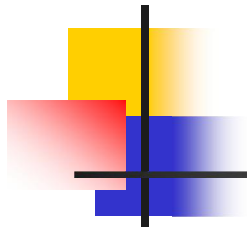
Proof of 3

Constants Rule

$$\frac{[one : int\ list] \vdash \quad (>::) : int \rightarrow int\ list \rightarrow int\ list}{[one : int\ list] \vdash ((>::) 1) : int\ list \rightarrow int\ list}$$

Constants Rule

$$\frac{[one : int\ list] \vdash \quad 1 : int}{[one : int\ list] \vdash ((>::) 1) : int\ list \rightarrow int\ list}$$



Proof of 4

n Rule for variables

$$\frac{}{[one : int\ list] \ |- \ one:int\ list}$$



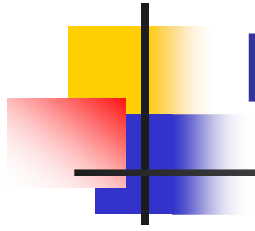
Proof of 2

n Constant

⑤ $[x:\text{int}; \text{one} : \text{int list}] \vdash$
 $\text{fun } y \rightarrow$
 $(x :: y :: \text{one}))$

$[\text{one} : \text{int list}] \vdash 2:\text{int} \quad : \text{int} \rightarrow \text{int list}$

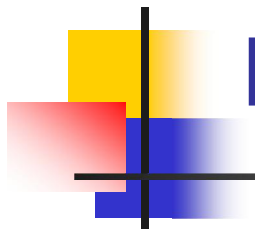
$[\text{one} : \text{int list}] \vdash (\text{let } x = 2 \text{ in}$
 $\text{fun } y \rightarrow (x :: y :: \text{one})) : \text{int} \rightarrow \text{int list}$



Proof of 5

?

$[x:\text{int}; \text{one} : \text{int list}] \vdash \text{fun } y \rightarrow (x :: y :: \text{one}))$
 $: \text{int} \rightarrow \text{int list}$

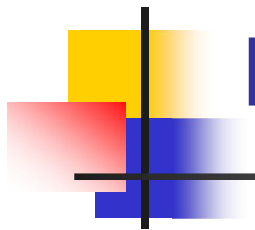


Proof of 5

?

$$[y:\text{int}; x:\text{int}; \text{one} : \text{int list}] \vdash (x :: y :: \text{one}) : \text{int list}$$

$$[x:\text{int}; \text{one} : \text{int list}] \vdash \text{fun } y \rightarrow (x :: y :: \text{one})) \\ : \text{int} \rightarrow \text{int list}$$



Proof of 5

⑥

$[y:\text{int}; x:\text{int}; \text{one} : \text{int list}] \vdash$
 $((::) x):\text{int list} \rightarrow \text{int list}$

$[y:\text{int}; x:\text{int}; \text{one} : \text{int list}] \vdash (x :: y :: \text{one}) : \text{int list}$

$[x:\text{int}; \text{one} : \text{int list}] \vdash \text{fun } y \rightarrow (x :: y :: \text{one})$

$: \text{int} \rightarrow \text{int list}$

⑦

$[y:\text{int}; x:\text{int}; \text{one} : \text{int list}] \vdash$

$(y :: \text{one}) : \text{int list}$



Proof of 6

Constant

Variable

[...] |- (::)

: int → int list → int list

[...; x:int;...] |- x:int

[y:int; x:int; one : int list] |- ((::) x)

:int list → int list



Proof of 7

Pf of 6 [y/x]

Variable

•
•
•

$[y:\text{int}; \dots] \vdash ((::) y)$	$[\dots; \text{one: int list}] \vdash$
$\text{:int list} \rightarrow \text{int list}$	one: int list
$[y:\text{int}; x:\text{int}; \text{one} : \text{int list}] \vdash (y :: \text{one}) :$	
int list	



Curry - Howard Isomorphism

- n Type Systems are logics; logics are type systems
- n Types are propositions; propositions are types
- n Terms are proofs; proofs are terms
- n Functions space arrow corresponds to implication; application corresponds to modus ponens



Curry - Howard Isomorphism

n Modus Ponens

$$\frac{A \Rightarrow B \quad A}{B}$$

- Application

$$\frac{\Gamma \vdash e_1 : \alpha \rightarrow \beta \quad \Gamma \vdash e_2 : \alpha}{\Gamma \vdash (e_1 \ e_2) : \beta}$$