Statistics 400 / Mathematics 463
Midterm Exam 1 (AL1) Version A
Sept, 28th 2011, Wednesday, 11:00-11:50am

Name: Solution ______________ Student I.D. ______________

Discussion Section (circle one): AD1 AD2 AD3 AD4

1. Please **print** your name and student ID number in the above space and circle the discussion section number.
2. This is a closed book, closed-notes examination. You should have a calculator and a single two-sided page of notes that you may refer to.
3. Please provide the answers in the space provided. If you do not have enough space, please use the back of a nearby page or ask for additional blank paper. Make sure you sign any loose pages.
4. In order to receive full credit for a problem, you should show all of your work and explain your reasoning. Good work can receive substantial partial credit even if the final answer is incorrect.
5. In most cases the later parts of a question do not require the answers to earlier parts. You should try all parts of a problem even if you get stuck on an early part.

Question	Points
Question 1 (12 points)	
Question 2 (12 points)	
Question 3 (24 points)	
Question 4 (20 points)	
Question 5 (20 points)	
Question 6 (12+5 (Bonus) points)	
Total (105 points)	

1. **[12 points]** Suppose that E and F are two events such that $P(E) = 0.7$ and $P(F) = 0.5$.

(a)**[6 points]** Prove that $P(E \cap F) \geq 0.2$.

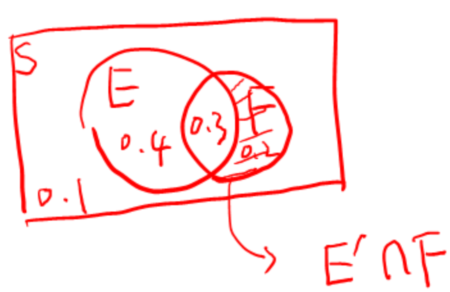
$$1 \geq P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

$$\Rightarrow P(E \cap F) \geq P(E) + P(F) - 1$$

$$= 0.2$$

(b)**[6 points]** Suppose we know $P(E \cap F) = 0.3$, find out $P(E'|F)$.

$$P(E'|F) = \frac{P(E' \cap F)}{P(F)} = \frac{P(F) - P(E \cap F)}{P(F)} = \frac{0.2}{0.5} = 0.4$$



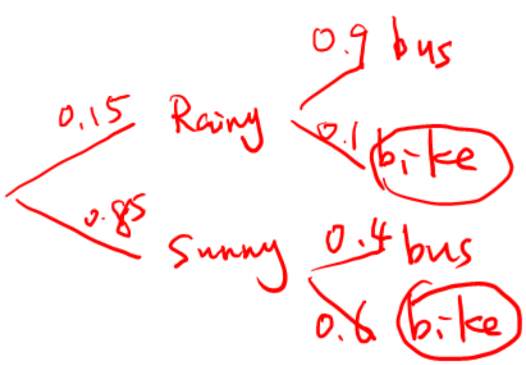
2. **[12 points]** Every day, Tom makes a return trip to campus either by bus, or by bike. If it is raining, he takes the bus with probability 0.9 and bikes with probability 0.1. If it is sunny, he takes the bus with probability 0.4, and bikes with probability 0.6. Suppose it rains with probability 0.15 and is sunny with probability 0.85.

   Define the following events for Tom's travel to campus:

   $A = \{\text{by bus}\}$, $B = \{\text{by bike}\}$, $R = \{\text{rainy}\}$, $S = \{\text{sunny}\}$.

   (a) **[6 points]** Find $P(B)$.

$P(B) = 0.1 \times 0.15 + 0.6 \times 0.85 = 0.525$



   (b) **[6 points]** If you are told that Tom came to campus by bike on a particular day, what is the probability that it was raining on that day?

$$P(R|B) = \frac{P(R \cap B)}{P(B)} = \frac{0.15 \times 0.1}{0.525} = 0.029$$

3. **[24 points]** Suppose that 5% of all copies of a particular textbook fail a certain binding strength test. Let X denote the number among the 15 randomly selected copies that fail the test. $X \sim b(15, 0.05)$

(a) **[6 points]** What is the probability that at least 14 **pass** the test?

$$P(X \le 1) = P(X=0) + P(X=1)$$
$$= \binom{15}{0} 0.05^0\, 0.95^{15} + \binom{15}{1} 0.05^1\, 0.95^{14}$$
$$= 0.829$$

(b) **[6 points]** As the textbooks are finished with the binding process, they are randomly selected and their bindings are checked. What is the probability that someone has to check 4 books to get the first failure of the strength test?

Let Z be # of books checked to get the 1st failure

Then Z is geometric $(\frac{1}{20})$

$$P(Z=4) = 0.95^3 \times 0.05 = 0.0429$$

(c) **[6 points]** Let Y be the minimum number of books checked to get the fourth failure of the strength test. What is the distribution of Y? (Please either give the name of the distribution with parameter values or write down the p.m.f. function)

Y is negative binomial with $r = 4$, $p = 0.05$

the p.m.f. is $P(Y=y) = \binom{y-1}{3} 0.05^4\, 0.95^{y-4}$

(d) **[6 points]** Suppose someone checks 100 books,can you approximate the probability that she sees less than 3 failures? Show your work. Let X be # of failures then $X \sim b(100, 0.05)$

use poisson approximation

$\lambda = np = 100 \times 0.05 = 5$

$$P(X<3) \approx P(\text{pois}(5) < 3) = P(\text{Pois}(5) \le 2)$$
$$= e^{-5} \cdot \left[\frac{5^0}{0!} + \frac{5^1}{1!} + \frac{5^2}{2!}\right] = 0.125$$

4. **[22 points]** Max, Martin and Mike are three deer hunters who shoot simultaneously at a nearby sheepdog. It can be assumed that Max has a 80% chance of hitting the sheepdog, Martin has a 70% chance, Mike has a 40% chance and they shoot independently.

   (a) **[6 points]** What is the chance that the sheepdog is hit by all of the three hunters?

   $0.8 \times 0.7 \times 0.4 = 0.224$

   (b) **[6 points]** What is the chance that the sheepdog is hit by exactly one bullet?

   {only Max hit} + {only Martin hit} + {only Mike hit}

   $0.8 \times 0.3 \times 0.6 + 0.2 \times 0.7 \times 0.6 + 0.2 \times 0.3 \times 0.4 = 0.252$

   (c) **[10 points]** What is the probability that Max hit the dog if the sheepdog is hit? (Hint: first find the probability that the sheepdog is hit by at least one bullet)

   P(sheepdog is hit) = 1 − P(none of the boys hit it)

   $= 1 - 0.2 \times 0.3 \times 0.6 = 0.964$

   $$P(\text{Max hit it} \mid \text{sheep dog is hit}) = \frac{P(\text{Max hit})}{P(\text{sheep dog is hit})}$$

   $$= \frac{0.8}{0.964} = 0.830$$

+5 bonus

5. [**20 points**] The probability density function of a random variable X is given by

$$f(x) = c|x+2| \text{ if } 0 < x < 3.$$

(a) [~~5~~ 7 **points**] Calculate the value of c.

$$1 = \int_0^3 f(x)\,dx = \int_0^3 c|x+2|\,dx = c\int_0^3 (x+2)\,dx$$

$$= c\left(\frac{x^2}{2}+2x\right)\Big|_0^3 = c\times\left(\frac{9}{2}+6\right) = \frac{21}{2}c$$

$$\Rightarrow \quad c = \frac{2}{21}$$

(b) [~~5~~ 7 **points**] Find the cumulative distribution function of X.

$$F(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \int_0^x f(u)\,du = \frac{2}{21}\int_0^x (u+2)\,du = \frac{x^2}{21}+\frac{4}{21}x & x \in (0,3) \\ 1 & \text{if } x \ge 3 \end{cases}$$

(c) [~~5~~ 6 **points**] What is the mean of X?

$$EX = \int_0^3 f(x)\,dx = \frac{2}{21}\int_0^3 x(x+2)\,dx$$

$$= \frac{2}{21}\left(\frac{x^3}{3}+x^2\right)\Big|_0^3 = 12/7 = 1.714$$

bonus (d) [**5 points**] What is the median of X?

median m is where $F(m) = \frac{1}{2}$ and $m \in (0,3)$

from b) $\frac{1}{21}(m^2+4m) = \frac{1}{2}$

$$\Rightarrow \quad m^2+4m-\frac{21}{2} = 0$$

$$\Rightarrow \quad m = \frac{-4 \pm \sqrt{58}}{2}$$

since $m > 0$, so $m = \frac{-4+\sqrt{58}}{2} = 1.808$

6. **[12+5 (bonus) points]** Three fair dice are tossed.

(a) **[3 points]** What is the probability that the three dice turn up all sixes?

$$\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{216}$$

(b) **[4 points]** What is the probability that the three dice turn up all the same number? the dice can be all ones, all twos, ..., all sixes

$$\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times 6 = \frac{1}{36}$$

(c) **[5 points]** What is the probability that the three dice turn up three different numbers?

Note, $P(\text{all different}) \neq 1 - P(\text{all the same})$

$$P(\text{all different}) = \frac{{}_6P_3}{6^3} = \frac{6 \times 5 \times 4}{6 \times 6 \times 6} = \frac{5}{9}$$

(d) **[bonus 5 points]** What is the probability that two of the dice turn up one number, while the rest one of the dice turn up some other number? (For example, (4, 5, 4).) There are 3 ways to have two fours and one five,

i.e. (4, 5, 4) (5, 4, 4) (4, 4, 5)

And there're ${}_6P_2$ ways to pick up 2 ordered number

So, the probability is $\frac{3 \times {}_6P_2}{6^3} = \frac{3 \times 6 \times 5}{6 \times 6 \times 6} = \frac{5}{12}$.