Review(2.4)

- Bernoulli distribution and its mean, variance.
- Binomial distribution and its p.m.f., properties.
- Cumulative distribution function and its properties.
- Geometric distribution and Negative binomial distribution

A Property of Binomial distribution

If $X \sim b(n, p)$, then $(n - X) \sim b(n, 1 - p)$.

Example: An urn contains 9 blue balls and 6 red balls. The balls are well mixed. Draw one ball each time, and record the color, then put it back (with replacement). Let X be the number of times we get blue balls among 10 draws. What is the distribution of X, and what is the probability that there are at least 9 blue balls out of 10 draws? (0.0464)

Binomial v.s. Hypergeometric distribution

Example: A jar has N marbles, S of them are orange and N - S are blue. Suppose 3 marbles are selected. Find the probability that there are 2 orange marbles in the sample, if the selection is done ...

with replacement | without replacement

(a) N =10, S=4

(b) N= 100, S =40

(c) N=1000, S=400

	Binomial	**Hypergeometric**
	with replacement	without replacement
Probability	$P(X=x)=\binom{n}{x}\cdot p^x\cdot(1-p)^{n-x}$	$P(X=x)=\dfrac{\binom{S}{x}\cdot\binom{N-S}{n-x}}{\binom{N}{n}}$
Expected Value	$E(X)=n\cdot p$	$E(X)=n\cdot\dfrac{S}{N}$
Variance	$Var(X)=n\cdot p\cdot(1-p)$	$Var(X)=n\cdot\dfrac{S}{N}\cdot\left(1-\dfrac{S}{N}\right)\cdot\dfrac{N-n}{N-1}$

If the population size is large (compared to the sample size) Binomial Distribution can be used regardless of whether sampling is with or without replacement.

Geometric and Negative binomial distribution

Example: Suppose that during practice, a basket ball player can make a free throw 80% of the time. Furthermore, assume that a sequence of free throw shooting can be thought of as independent Bernoulli trials.

- Let X_1 equal the number of free throws that this player must attempt to make a total of 1 shot. What is the distribution of X_1.

- Let X_2 equal the minimum number of free throws that this player must attempt to make a total of 10 shots. What is the distribution of X_2.

Today's Lecture (2.5)

- Moment-generating function (m.g.f.) and its properties
- Calculating mean and variance through m.g.f.

Moment generating function

Definition: Let X be a random variable of the discrete type with p.m.f. $f(x)$ and space S. If there is a positive number h such that

$$E[e^{tX}] = \sum_{x \in S} e^{tx} f(x)$$

exists and is finite for $-h < t < h$, then the function of t defined by $M(t) = E[e^{tX}]$ is called the moment-generating function (m.g.f) of X.

- If the space of X is $\{b_1, b_2, b_3, \cdots\}$, the moment generating function is given by

  $$M(t) = e^{tb_1} f(b_1) + e^{tb_2} f(b_2) + e^{tb_3} f(b_3) + \cdots.$$

  where $f(b_i) = P(X = b_i)$.

- If the moment generating function exists for t in an open interval containing zero, it uniquely determines the distribution of the random variable.

Example: What is m.g.f. of the following distribution?

x	1	2	3	4	5
$f(x)$	0.1	0.3	0.2	0.3	0.1

Property of m.g.f

If the moment-generating function exists in an open interval containing zero, then $E(X^r) = M^{(r)}(0)$. In particular,

- $\mu = M'(0)$
- $\sigma^2 = E[X^2] - [E(X)]^2 = M''(0) - [M'(0)]^2$

Binomial distribution $b(n, p)$:

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \cdots, n$$

- Step 1: The m.g.f $M(t)$ is

- Step 2: Calculate the derivatives of $M(t)$,
  - $M'(t) = n[(1-p) + pe^t]^{n-1}(pe^t)$
  - $M''(t) = n(n-1)[(1-p) + pe^t]^{n-2}(pe^t)^2 + n[(1-p) + pe^t]^{n-1}(pe^t)$
- Step 3: Calculate μ, σ^2 based on $M'(t)$ and $M''(t)$.
  - $\mu = E(X) = M'(0) = np$
  - $\sigma^2 = E(X^2) - \mu^2 = M''(0) - [M'(0)]^2 = np(1-p)$

The m.g.f. of Geometric distribution

Recall: We say that X has a *geometric distribution* if

$$f(x) = p(1-p)^{x-1}, \ x = 1, 2, 3, \cdots$$

$$\begin{aligned} M(t) &= \sum_{x=1}^{\infty} e^{tx} f(x) \\ &= \sum_{x=1}^{\infty} e^{tx} p(1-p)^{x-1} \\ &= p(1-p)^{-1} \sum_{x=1}^{\infty} [e^t(1-p)]^x \\ &= p(1-p)^{-1} \frac{e^t(1-p)}{1-e^t(1-p)} = \frac{pe^t}{1-e^t(1-p)} \end{aligned}$$

Note that $e^t(1-p) < 1$, i.e. $t < -\log(1-p)$. So

$$\begin{aligned} M'(t) &= \frac{pe^t}{[1-e^t(1-p)]^2} \\ M''(t) &= \frac{pe^t[1+(1-p)e^t]}{[1-e^t(1-p)]^3} \end{aligned}$$

So

$$\begin{aligned} \mu &= M'(0) = \frac{1}{p} \\ \sigma^2 &= M''(0) - (M'(0))^2 \\ &= \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2} \end{aligned}$$

Negative Binomial distribution

Definition: We say that X has a negative binomial distribution if

$$f(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \ x = r, r+1, r+2, \cdots$$

In this case, $\mu = r/p$ and $\sigma^2 = r(1-p)/p^2$ and

$$M(t) = \frac{(pe^t)^r}{[1-(1-p)e^t]^r}, \text{ where } t < -\log(1-p)$$

A note about distribution names

Discrete distributions often get their names from mathematical power series.

- Binomial probabilities sum to 1 because of the Binomial Theorem:

  $(p+(1-p))^n =<$ sum of Binomial probabilities $>= 1$

- Negative Binomial probabilities sum to 1 by the Negative Binomial expansion: i.e. the Binomial expansion with a negative power, $-r$:

  $p^r(1-(1-p))^{-r} =<$ sum of NegBin probabilities $>= 1$

- Geometric probabilities sum to 1 because they form a Geometric series:

  $p\sum_{x=0}^{\infty}(1-p)^x =<$ sum of Geometric probabilities $>= 1$

A note about discrete distribution in EXCEL

- Hypergeometric with N = population size, S = number of "successes" in the population, n = sample size.
  X = number of "successes" in the sample without replacement.
  =HYPGEOMDIST(x, n, S, N) gives $P(X = x)$;

- Binomial, X = number of "successes" in n independent trials.
  =BINOMDIST$(x, n, p, 0)$ gives $P(X = x) = \binom{n}{x}p^x(1-p)^{n-x}$;
  =BINOMDIST$(x, n, p, 1)$ gives $P(X \leq x)$;

- Negative Binomial, X = number of independent trials until the r-th "success"
  = NEGBINOMDIST$(x-r, r, p)$ gives $P(X = x) = \binom{x-1}{r-1}p^r(1-p)^{x-r}$.