Be sure to show all your work; your partial credit might depend on it.
No credit will be given without supporting work.
The exam is closed book and closed notes.
You are allowed to use a calculator and one 8.5" x 11" sheet with notes on it.

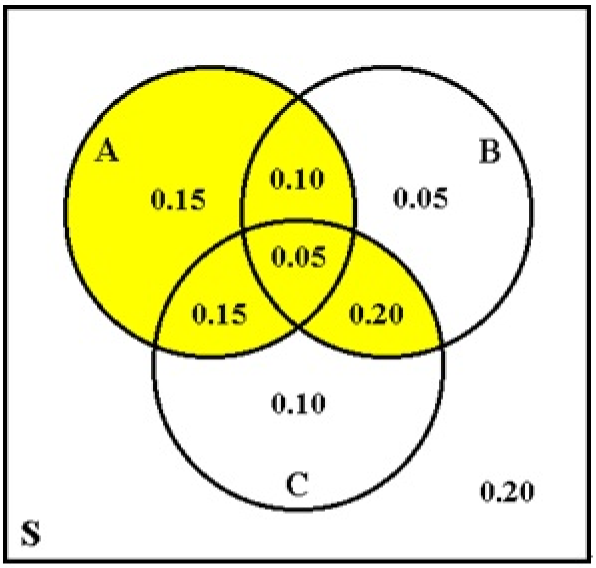
1. Suppose $P(A) = 0.45, P(B) = 0.40, P(C) = 0.50, P(A \cap B) = 0.15, P(A \cap C) = 0.20, P(B \cap C) = 0.25, P(A \cap B \cap C) = 0.05$.

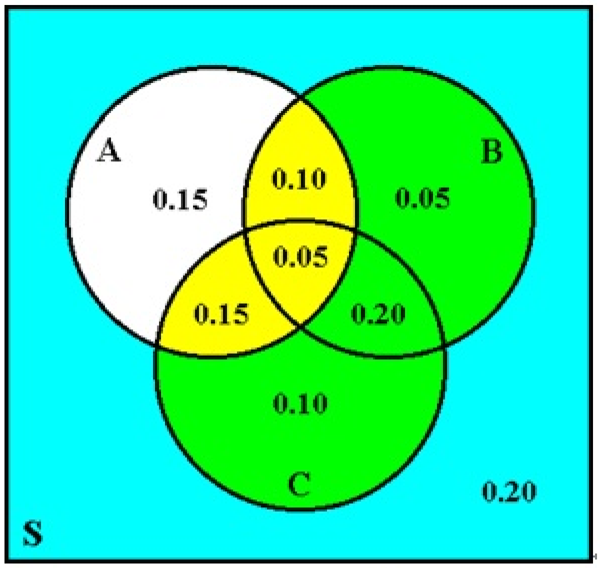
a) Find $P(A \cup C)$.

b) Find $P(A \cup (B \cap C))$.

c) Find $P(B \cup C | A')$.

a) $\quad P(A \cup C) = P(A) + P(C) - P(A \cap C) = 0.45 + 0.50 - 0.20 = \mathbf{0.75}$.





b) $\quad P(A \cup (B \cap C)) = \mathbf{0.65}$.

c) $\quad P(B \cup C \mid A') = \dfrac{P(A' \cap (B \cup C))}{P(A')}$

$$= \frac{0.35}{0.55} = \mathbf{\frac{7}{11}} = \mathbf{0.63\overline{63}}.$$

2. Tom's lecture at UIUC often finishes late. Suppose that the lecture finishes W minutes late, where W is a discrete random variable with the following probability mass function:

w	0	1	2	3	4	5
$f_W(w) = P(W = w)$	0.4	a	b	0.10	0.05	0.05

Here a and b are missing values to be calculated. Tom calculates that $E(W) = 1.3$.

a) Find the values of a and b.

Since $f_W(w)$ is a p.m.f., we have

$$0.4 + a + b + 0.1 + 0.05 + 0.05 = 1$$

Further, $E(W) = 1.3$ is equivalent to the following

$$0 \times 0.4 + a + 2b + 3 \times 0.10 + 4 \times 0.05 + 5 \times 0.05 = 1.3$$

So we have two equations and two unknown quantities a and b. Thus $a = 0.25$ and $b = 0.15$.

b) Find the standard deviation of W, σ_W.

$$\begin{aligned} \mathsf{Var}(W) &= E(W^2) - [E(W)]^2 \\ &= 0^2 \times 0.4 + 1^2 \times 0.25 + 2^2 \times 0.15 + 3^2 \times 0.10 + 4^2 \times 0.05 + 5^2 \times 0.05 - 1.3^2 \\ &= 2.11 \end{aligned}$$

so $\sigma_W = \sqrt{\mathsf{Var}(W)} = 1.453$.

3. The probability that a circuit board coming off an assembly line needs rework is 0.15. Suppose that 12 boards are tested and all boards are independent of each other.

a) What is the probability that exactly 4 will need rework?

Let X be the number of circuit boards that need rework. Then $X \sim b(12, 0.15)$.

$$P(X = 4) = \binom{12}{4} 0.15^4 \times 0.85^8 = 0.06828$$

b) What is the probability that at least one needs rework?

$$P(X \geq 1) = 1 - P(X = 0) = 1 - 0.85^{12} = 0.8578$$

c) What is the probability that at most two needs rework?

$$\begin{aligned} P(X \le 2) &= P(X=0)+P(X=1)+P(X=2) \\ &= \binom{12}{0}0.15^0 0.85^{12} + \binom{12}{1}0.15^1 0.85^{11} + \binom{12}{2}0.15^2 0.85^{10} \\ &= 0.1422 + 0.3012 + 0.2924 = 0.7358 \end{aligned}$$

4. Sixty-five percent (65%) of all women who submit to pregnancy tests are actually pregnant. A certain pregnancy test gives a *false positive* result with probability 0.02 and a *valid positive* result with probability 0.99. We say that this test has sensitivity 0.99 and specificity $1-0.02 = 0.98$. In other words, a pregnant woman gives a positive test value with probability 0.99. A non-pregnant woman gives a negative test result with probability 0.98.

a) Among women who submit to a pregnancy test, what fraction of the tests are positive?

Let $T =$"test positive" and $A =$"actually pregnant." Then $P(A) = 0.65$, $P(T|A) = 0.99$, $P(T|A') = 0.02$.

$$\begin{aligned} P(T) &= P(T \cap A) + P(T \cap A') \\ &= P(A)P(T|A) + P(A')P(T|A') \\ &= 0.65 \times 0.99 + 0.35 \times 0.02 \\ &= 0.6505(\text{or} = 65.05\%) \end{aligned}$$

b) If a particular woman's test is indeed positive, what is the probability that she is actually pregnant?

$$P(A|T) = \frac{P(T \cap A)}{P(T)} = \frac{P(T|A)P(A)}{P(T)} = \frac{0.65 \times 0.99}{0.6505} = 0.9892(\text{or} = 98.92\%)$$

Alternatively, using Bayes's theorem,

$$\begin{aligned} P(A|T) &= \frac{P(T|A)P(A)}{P(A)P(T|A) + P(A')P(T|A')} \\ &= \frac{0.65 \times 0.99}{0.65 \times 0.99 + 0.35 \times 0.02} = 0.9892(\text{or} = 98.92\%) \end{aligned}$$

c) If a particular woman's test is negative, what is the probability that she is actually pregnant?

$$\begin{aligned} P(A|T') &= \frac{P(A \cap T')}{P(T')} = \frac{P(T'|A)P(A)}{1-P(T)} \\ &= \frac{(1-P(T|A))P(A)}{1-P(T)} = \frac{(1-0.99)0.65}{1-0.6505} = 0.0186(\text{or} = 1.86\%) \end{aligned}$$

5. The probability density function of a random variable X is given by

$$f(x) = c(x-1)(2-x) \text{ if } 1 < x < 2.$$

a) Calculate the value of c.

The density should integrate to one, i.e.

$$\int_1^2 c(x-1)(2-x)dx = c/6 = 1$$

Hence $c = 6$.

b) Find the cumulative distribution function of X.

$$F(x) = P(X \leq x) = \int_1^x f(y)dy = 3(x-1)^2 - 2(x-1)^3 = -2x^3 + 9x^2 - 12x + 5.$$

6. According to an airline industry report, roughly 1 piece of luggage out of every 200 that are checked is lost. Suppose that a frequent-flying businesswoman will be checking 120 bags over the course of the next year. Approximate the probability that she will lose 2 or more pieces of luggage.

Let X be the number of luggage she will lose. Then $X \sim b(n = 120, p = 1/200)$. Then X is approximately Poisson($n \times p = 0.6$). Therefore

$$P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - e^{-0.6} - e^{-0.6} \times 0.6 = 0.122$$