Examples for 10/07/2011

Let X and Y be two discrete random variables. The **joint probability mass function** $p(x, y)$ is defined for each pair of numbers (x, y) by

$$p(x, y) = \mathrm{P}(\mathrm{X} = x \text{ and } \mathrm{Y} = y).$$

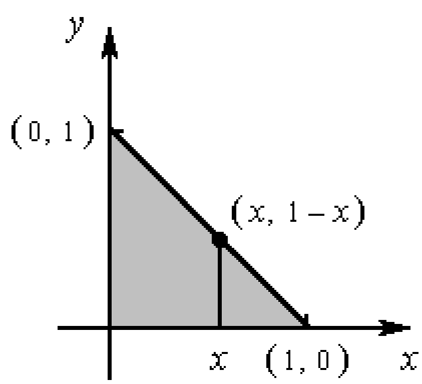
Let A be any set consisting of pairs of (x, y) values. Then

$$\mathrm{P}\big((\mathrm{X}, \mathrm{Y}) \in A\big) = \sum_{(x,y) \in A} \sum p(x, y).$$

Let X and Y be two continuous random variables. Then $f(x, y)$ is the **joint probability density function** for X and Y if for any two-dimensional set A

$$\mathrm{P}\big((\mathrm{X}, \mathrm{Y}) \in A\big) = \iint_A f(x, y)\, dx\, dy.$$

1. A Nut Company markets cans of deluxe mixed nuts containing almonds, cashews, and peanuts. Suppose the net weight of each can is exactly 1 lb, but the weight contribution of each type of nut is random. Because the three weights sum to 1, a joint probability model for any two gives all necessary information about the weight of the third type. Let X = the weight of almonds in a selected can and Y = the weight of cashews.



Then the region of positive density is $\mathrm{D} = \{\, (x, y) : 0 \le x \le 1,\ 0 \le y \le 1,\ x + y \le 1 \,\}$.

Let the joint probability density function for (X, Y) be

$$f(x, y) = \begin{cases} 60\, x^2\, y & 0 \le x \le 1,\ 0 \le y \le 1,\ x + y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a) Verify that $f(x, y)$ is a legitimate probability density function.

1. $f(x, y) \ge 0$ for all (x, y). ✓

2. $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dx\, dy = \int_0^1 \left(\int_0^{1-x} 60\, x^2\, y\, dy \right) dx = \int_0^1 \left(30\, x^2\, (1 - x)^2 \right) dx$$

$$= \int_0^1 \left(30\, x^2 - 60\, x^3 + 30\, x^4 \right) dx = \left(10\, x^3 - 15\, x^4 + 6\, x^5 \right) \Big|_0^1 = 1. \ ✓$$

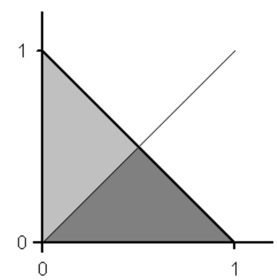
b) Find the probability that the two types of nuts together make up less than 50% of a can. That is, find the probability $P(X + Y < 0.50)$. (Find the probability that peanuts make up over 50% of a can.)

$$P(X + Y < 0.50) = \int_0^{0.5} \left(\int_0^{0.5-x} 60\,x^2\,y\,dy \right) dx = \int_0^{0.5} 30\,x^2 (0.5 - x)^2\,dx$$

$$= \int_0^{0.5} \left(7.5\,x^2 - 30\,x^3 + 30\,x^4 \right) dx = \left(2.5\,x^3 - 7.5\,x^4 + 6\,x^5 \right) \Big|_0^{0.5} = \mathbf{\frac{1}{32}} = \mathbf{0.03125}.$$

c) Find the probability that there are more almonds than cashews in a can. That is, find the probability $P(X > Y)$.

$$P(X > Y) = \int_0^{1/2} \left(\int_y^{1-y} 60\,x^2\,y\,dx \right) dy$$



$$= \int_0^{1/2} 20\,y \left(\int_y^{1-y} 3\,x^2\,dx \right) dy$$

$$= \int_0^{1/2} 20\,y \left((1-y)^3 - y^3 \right) dy$$

$$= \int_0^{1/2} 20\,y \left(1 - 3\,y + 3\,y^2 - 2\,y^3 \right) dx = \int_0^{1/2} \left(20\,y - 60\,y^2 + 60\,y^3 - 40\,y^4 \right) dx$$

$$= \left(10\,y^2 - 20\,y^3 + 15\,y^4 - 8\,y^5 \right) \Big|_0^{1/2} = \mathbf{\frac{11}{16}} = \mathbf{0.6875}.$$

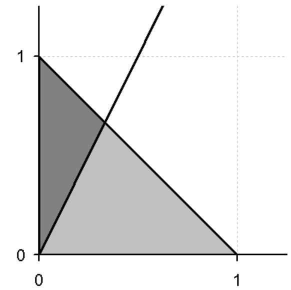
OR

$$P(X > Y) = 1 - \int_0^{1/2} \left(\int_x^{1-x} 60\,x^2\,y\,dy \right) dx = 1 - \int_0^{1/2} 30\,x^2 \left(\int_x^{1-x} 2\,y\,dy \right) dx$$

$$= 1 - \int_0^{1/2} 30\,x^2 \left((1-x)^2 - x^2 \right) dx = 1 - \int_0^{1/2} 30\,x^2 (1 - 2\,x)\,dx$$

$$= 1 - \int_0^{1/2} \left(30\,x^2 - 60\,x^3 \right) dx = 1 - \left(10\,x^3 - 15\,x^4 \right) \Big|_0^{1/2} = \mathbf{\frac{11}{16}} = \mathbf{0.6875}.$$

d) Find the probability that there are at least twice as many cashews as there are almonds. That is, find the probability $P(Y \geq 2X)$.



$$P(Y \geq 2X) = \int_0^{1/3} \left(\int_{2x}^{1-x} 60\,x^2\,y\,dy \right) dx$$

$$= \int_0^{1/3} \left(30\,x^2 \left[(1-x)^2 - (2x)^2 \right] \right) dx$$

$$= \int_0^{1/3} \left(30\,x^2 - 60\,x^3 - 90\,x^4 \right) dx$$

$$= \int_0^{1/3} \left(30\,x^2 - 60\,x^3 - 90\,x^4 \right) dx = \left(10\,x^3 - 15\,x^4 - 18\,x^5 \right) \Big|_0^{1/3}$$

$$= \frac{10}{27} - \frac{15}{81} - \frac{18}{243} = \mathbf{\frac{1}{9}}.$$

The **marginal probability mass functions** of X and of Y are given by

$$p_X(x) = \sum_{\text{all } y} p(x,y), \qquad p_Y(y) = \sum_{\text{all } x} p(x,y).$$

The **marginal probability density functions** of X and of Y are given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy, \qquad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx.$$

e) Find the marginal probability density function for X.

$$f_X(x) = \int_0^{1-x} 60\,x^2\,y\,dy = 30\,x^2 \int_0^{1-x} 2\,y\,dy = 30\,x^2\,(1-x)^2, \qquad 0 < x < 1.$$

f) Find the marginal probability density function for Y.

$$f_Y(y) = \int_0^{1-y} 60\,x^2\,y\,dx = 20\,y \int_0^{1-y} 3\,x^2\,dx = 20\,y\,(1-y)^3, \qquad 0 < y < 1.$$

If $p(x,y)$ is the joint probability mass function of (X,Y) OR $f(x,y)$ is the joint probability density function of (X,Y), then

discrete

$$\mathrm{E}(g(\mathrm{X},\mathrm{Y})) = \sum_{\text{all } x}\sum_{\text{all } y} g(x,y)\cdot p(x,y)$$

continuous

$$\mathrm{E}(g(\mathrm{X},\mathrm{Y})) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\cdot f(x,y)\,dx\,dy$$

g) Find $\mathrm{E}(\mathrm{X})$, $\mathrm{E}(\mathrm{Y})$, $\mathrm{E}(\mathrm{X}+\mathrm{Y})$, $\mathrm{E}(\mathrm{X}\cdot\mathrm{Y})$.

$$\mathrm{E}(\mathrm{X}) = \int_0^1 x\cdot 30\,x^2\,(1-x)^2\,dx = \int_0^1 \left(30\,x^3 - 60\,x^4 + 30\,x^5\right)dx$$

$$= \left(7.5\,x^4 - 12\,x^5 + 5\,x^6\right)\Big|_0^1 = \mathbf{0.5} = \mathbf{\frac{1}{2}}.$$

$$\mathrm{E}(\mathrm{Y}) = \int_0^1 y\cdot 20\,y\,(1-y)^3\,dx = \int_0^1 \left(20\,y^2 - 60\,y^3 + 60\,y^4 - 20\,y^5\right)dx$$

$$= \left(\frac{20}{3}\,y^3 - 15\,y^4 + 12\,y^5 - \frac{20}{6}\,y^6\right)\Big|_0^1 = \mathbf{\frac{1}{3}}.$$

$$\mathrm{E}(\mathrm{X}+\mathrm{Y}) = \mathrm{E}(\mathrm{X}) + \mathrm{E}(\mathrm{Y}) = \mathbf{\frac{5}{6}}.$$

$$\mathrm{E}(\mathrm{X}\cdot\mathrm{Y}) = \int_0^1\left(\int_0^{1-x} x\,y\cdot 60\,x^2\,y\,dy\right)dx = \int_0^1\left(20\,x^3\,(1-x)^3\right)dx$$

$$= \int_0^1\left(20\,x^3 - 60\,x^4 + 60\,x^5 - 20\,x^6\right)dx$$

$$= \left(5\,x^4 - 12\,x^5 + 10\,x^6 - \frac{20}{7}\,x^7\right)\Big|_0^1 = \mathbf{\frac{1}{7}}.$$

h) If 1 lb of almonds costs the company \$1.00, 1 lb of cashews costs \$1.50, and 1 lb of peanuts costs \$0.60, what is the expected total cost of the content of a can?

$$\text{Total cost} = (1.00)\,\mathrm{X} + (1.50)\,\mathrm{Y} + (0.60)\,(1-\mathrm{X}-\mathrm{Y}) = 0.6 + 0.4\,\mathrm{X} + 0.9\,\mathrm{Y}.$$

$$\mathrm{E}(\text{Total cost}) = 0.6 + 0.4\,\mathrm{E}(\mathrm{X}) + 0.9\,\mathrm{E}(\mathrm{Y}) = \mathbf{\$1.10}.$$

2. Consider the following joint probability distribution $p(x, y)$ of two random variables X and Y:

$x \setminus y$	0	1	2	
1	0.15	0.10	0	
2	0.25	0.30	0.20	

a) Find $P(X + Y = 2)$.

$P(X + Y = 2) = p(1, 1) + p(2, 0) = 0.10 + 0.25 = \mathbf{0.35}$.

b) Find $P(X > Y)$.

$P(X > Y) = p(1, 0) + p(2, 0) + p(2, 1) = 0.15 + 0.25 + 0.30 = \mathbf{0.70}$.

c) Find the (marginal) probability distributions $p_X(x)$ of X and $p_Y(y)$ of Y.

x	$p_X(x)$
1	0.25
2	0.75

y	$p_Y(y)$
0	0.40
1	0.40
2	0.20

d) Find $E(X)$, $E(Y)$, $E(X + Y)$, $E(X \cdot Y)$.

$E(X) = 1 \times 0.25 + 2 \times 0.75 = \mathbf{1.75}$.

$E(Y) = 0 \times 0.40 + 1 \times 0.40 + 2 \times 0.20 = \mathbf{0.8}$.

$E(X + Y) = 1 \times 0.15 + 2 \times 0.25 + 2 \times 0.10 + 3 \times 0.30 + 3 \times 0 + 4 \times 0.20 = \mathbf{2.55}$.

OR

$E(X + Y) = E(X) + E(Y) = 1.75 + 0.8 = \mathbf{2.55}$.

$E(X \cdot Y) = 0 \times 0.15 + 0 \times 0.25 + 1 \times 0.10 + 2 \times 0.30 + 2 \times 0 + 4 \times 0.20 = \mathbf{1.5}$.

Independent Random Variables

2. Consider the following joint probability distribution $p(x, y)$ of two random variables X and Y:

$x \setminus y$	0	1	2	
1	0.15	0.10	0	0.25
2	0.25	0.30	0.20	0.75
	0.40	0.40	0.20	

Recall: A and B are independent if and only if $P(A \cap B) = P(A) \cdot P(B)$.

a) Are events $\{X = 1\}$ and $\{Y = 1\}$ independent?

$$P(X = 1 \cap Y = 1) = p(1, 1) = 0.10 = 0.25 \times 0.40 = P(X = 1) \times P(Y = 1).$$

$\{X = 1\}$ and $\{Y = 1\}$ are **independent**.

Def Random variables X and Y are **independent** if and only if

discrete $\quad p(x, y) = p_X(x) \cdot p_Y(y) \quad$ for all x, y.

continuous $\quad f(x, y) = f_X(x) \cdot f_Y(y) \quad$ for all x, y.

$F(x, y) = P(X \leq x, Y \leq y)$. $\qquad f(x, y) = \partial^2 F(x, y) \big/ \partial x \, \partial y$.

Def Random variables X and Y are **independent** if and only if

$$F(x, y) = F_X(x) \cdot F_Y(y) \quad \text{for all } x, y.$$

b) Are random variables X and Y independent?

$$p(1, 0) = 0.15 \neq 0.25 \times 0.40 = p_X(1) \times p_Y(0).$$

X and Y are **NOT independent**.

1. Let the joint probability density function for (X, Y) be

$$f(x,y) = \begin{cases} 60\,x^2\,y & 0 \le x \le 1,\ 0 \le y \le 1,\ x + y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Recall: $f_X(x) = 30\,x^2\,(1-x)^2, \quad 0 < x < 1,$

$f_Y(y) = 20\,y\,(1-y)^3, \quad 0 < y < 1.$

Are random variables X and Y independent?

The support of (X, Y) is not a rectangle.

X and Y are **NOT independent**.

3. Let the joint probability density function for (X, Y) be

$$f(x,y) = \begin{cases} x + y & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Are X and Y independent?

$$\begin{aligned} f_1(x) &= \int_0^1 (x+y)\,dy \\ &= \left[xy + \frac{1}{2}y^2\right]_0^1 = x + \frac{1}{2}, \qquad 0 \le x \le 1\ ; \\ f_2(y) &= \int_0^1 (x+y)\,dx = y + \frac{1}{2}, \qquad 0 \le y \le 1; \\ f(x,y) &= x + y \ne \left(x + \frac{1}{2}\right)\left(y + \frac{1}{2}\right) = f_1(x) f_2(y). \end{aligned}$$

X and Y are **NOT independent**.

4. Let the joint probability density function for (X, Y) be

$$f(x,y)=\begin{cases} 12\,x(1-x)\,e^{-2y} & 0 \le x \le 1,\ y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Are X and Y independent?

$$f_{\mathrm{X}}(x) = \int_0^{\infty} 12\,x(1-x)e^{-2y}\,dy = 6\,x(1-x), \quad 0 < x < 1.$$

$$f_{\mathrm{Y}}(y) = \int_0^{1} 12\,x(1-x)e^{-2y}\,dx = 2\,e^{-2y}, \quad y > 0.$$

Since $f(x,y) = f_{\mathrm{X}}(x) \cdot f_{\mathrm{Y}}(y)$ for all x, y, X and Y are **independent**.

If random variables X and Y are independent, then

$$\mathrm{E}(g(\mathrm{X}) \cdot h(\mathrm{Y})) = \mathrm{E}(g(\mathrm{X})) \cdot \mathrm{E}(h(\mathrm{Y})).$$