Examples for 10/03/2011

1. At *Initech*, the salaries of the employees are normally distributed with mean μ = $36,000 and standard deviation σ = $5,000.

a) Mr. Smith is paid $42,000. What proportion of the employees of *Initech* are paid less that Mr. Smith?

$$\mathrm{P}(\mathrm{X} < 42{,}000) = \mathrm{P}\left(\mathrm{Z} < \frac{42{,}000 - 36{,}000}{5{,}000}\right) = \mathrm{P}(\mathrm{Z} < 1.20) = \mathbf{0.8849}.$$

b) What proportion of the employees have their salaries over $40,000?

$$\mathrm{P}(\mathrm{X} > 40{,}000) = \mathrm{P}\left(\mathrm{Z} > \frac{40{,}000 - 36{,}000}{5{,}000}\right) = \mathrm{P}(\mathrm{Z} > 0.80) = 1 - 0.7881 = \mathbf{0.2119}.$$

c) Suppose 10 *Initech* employees are randomly and independently selected. What is the probability that 3 of them have their salaries over $40,000?

Let Y = number of employees (out of 10) who have salaries over $40,000.
Then Y has Binomial distribution, $n = 10$, $p = \mathbf{0.2119}$ (see (b)).

$$\mathrm{P}(\mathrm{Y} = 3) = {}_{10}\mathrm{C}_3 \cdot (0.2119)^3 \cdot (0.7881)^7 = \mathbf{0.2156}.$$

d) What proportion of the employees have their salaries between $30,000 and $40,000?

$$\mathrm{P}(30{,}000 < \mathrm{X} < 40{,}000) = \mathrm{P}\left(\frac{30{,}000 - 36{,}000}{5{,}000} < \mathrm{Z} < \frac{40{,}000 - 36{,}000}{5{,}000}\right)$$

$$= \mathrm{P}(-1.2 < \mathrm{Z} < 0.80) = 0.7881 - 0.1151 = \mathbf{0.6730}.$$

e) Mrs. Jones claims that her salary is high enough to just put her among the highest paid 15% of all employees working at *Initech*. Find her salary.

Need $x = ?$ such that $P(X > x) = 0.15$. $\quad$ (area to the right is 0.15)

First, need $z = ?$ such that $P(Z > z) = 0.15$.

$z = 1.04$.

$X = \mu + \sigma Z$. $\quad x = 36{,}000 + 5{,}000 \times 1.04 = \mathbf{\$41{,}200}$.

f) Ms. Green claims that her salary is so low that 90% of the employees make more than she does. Find her salary.

Need $x = ?$ such that $P(X > x) = 0.90$. $\quad$ (area to the right is 0.90)

First, need $z = ?$ such that $P(Z > z) = 0.90$.

$z = -1.28$.

$X = \mu + \sigma Z$. $\quad x = 36{,}000 + 5{,}000 \times (-1.28) = \mathbf{\$29{,}600}$.

2. Suppose that the lifetime of ***Outlast*** batteries is normally distributed with mean $\mu = 240$ hours and unknown standard deviation. Suppose also that 20% of the batteries last less than 219 hours. Find the standard deviation of the distribution of the lifetimes.

Need $\sigma = ?$ $\quad$ Know $P(X < 219) = 0.20$.

First, need $z = ?$ such that $P(Z < z) = 0.20$.

$z = -0.84$.

$X = \mu + \sigma Z$. $\quad 219 = 240 + \sigma \times (-0.84)$.

$$-21 = \sigma \times (-0.84).$$

$$\sigma = \mathbf{25} \text{ hours}.$$

Let X be normally distributed with mean μ and standard deviation σ.
Then the moment-generating function of X is

$$M_X(t) = e^{\mu t + \sigma^2 t^2/2}.$$

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2} dx$$

$$= \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} \cdot \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = e^{\mu t + \sigma^2 t^2/2} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(z-\sigma t)^2/2} dz$$

$$= e^{\mu t + \sigma^2 t^2/2},$$

since $\frac{1}{\sqrt{2\pi}} e^{-(z-\sigma t)^2/2}$ is the probability density function of a $N(\sigma t, 1)$ random variable.

Let $Y = aX + b$. Then $M_Y(t) = e^{bt} M_X(at)$.
Therefore, Y is normally distributed with mean $a\mu + b$ and variance $a^2\sigma^2$ (standard deviation $|a|\sigma$).

1. (continued)

g) All *Initech* employees receive a memo instructing them to put away 4% of their salaries plus \$100 per month (\$1,200 per year) in a special savings account to supplement Social Security. What proportion of the employees would put away more than \$3,000 per year?

$Y = 0.04\,X + 1{,}200$. $\quad\quad P(Y > 3{,}000) = ?$

$Y > 3{,}000 \quad \Leftrightarrow \quad X > 45{,}000.$

$$P(X > 45{,}000) = P\left(Z > \frac{45{,}000 - 36{,}000}{5{,}000}\right) = P(Z > 1.80) = 1 - 0.9641 = \mathbf{0.0359}.$$

OR

$\mu_Y = 0.04 \times 36{,}000 + 1{,}200 = \$2{,}640$, $\sigma_Y = 0.04 \times 5{,}000 = \$200$.

$$P(Y > 3{,}000) = P\left(Z > \frac{3{,}000 - 2{,}640}{200}\right) = P(Z > 1.80) = 1 - 0.9641 = \mathbf{0.0359}.$$