STAT 400 Fall 2011

Homework #2

(10 points)

(due Friday, September 9, by 3:00 p.m.)

No credit will be given without supporting work.

1. Bob is applying for a job with five companies. At the first company, he is in the final group of four applicants, one of which will be chosen for the position. At two of the five companies, Bob is one of ten candidates; and at the last two companies, he is in an early stage of application in a pool of 25 candidates. Assuming that all companies make their decisions independently of each other, and that Bob is as likely to be chosen as any other applicant, what is the probability of getting at least one job offer?

"at least one" = "either 1st **or** 2nd **or** 3rd **or** 4th **or** 5th" = union.

P(at least one) = 1 − P(none).

"none" = "not 1st **and** not 2nd **and** not 3rd **and** not 4th **and** not 5th".

$$P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5) = 1 - P(A_1' \cap A_2' \cap A_3' \cap A_4' \cap A_5')$$

since the companies are independent

$$= 1 - P(A_1') \cdot P(A_2') \cdot P(A_3') \cdot P(A_4') \cdot P(A_5')$$

$$= 1 - (0.75) \cdot (0.90) \cdot (0.90) \cdot (0.96) \cdot (0.96) = 1 - 0.56 = \mathbf{0.44}.$$

OR

$$P(\text{at least one job offer}) = 0.25 + (0.75 \cdot 0.10) + (0.75 \cdot 0.90 \cdot 0.10)$$

$$+ (0.75 \cdot 0.90 \cdot 0.90 \cdot 0.04) + (0.75 \cdot 0.90 \cdot 0.90 \cdot 0.96 \cdot 0.04) = \mathbf{0.44}.$$

OR

$$P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5) = P(A_1) + P(A_2) + P(A_3) + P(A_4) + P(A_5)$$

$$- P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_1 \cap A_4) - P(A_1 \cap A_5) - P(A_2 \cap A_3)$$

$$
\begin{aligned}
&- P(A_2 \cap A_4) - P(A_2 \cap A_5) - P(A_3 \cap A_4) - P(A_3 \cap A_5) - P(A_4 \cap A_5) \\
&+ P(A_1 \cap A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_4) + P(A_1 \cap A_2 \cap A_5) \\
&+ P(A_1 \cap A_3 \cap A_4) + P(A_1 \cap A_3 \cap A_5) + P(A_1 \cap A_4 \cap A_5) \\
&+ P(A_2 \cap A_3 \cap A_4) + P(A_2 \cap A_3 \cap A_5) + P(A_2 \cap A_4 \cap A_5) \\
&+ P(A_3 \cap A_4 \cap A_5) \qquad - P(A_1 \cap A_2 \cap A_3 \cap A_4) - P(A_1 \cap A_2 \cap A_3 \cap A_5) \\
&- P(A_1 \cap A_2 \cap A_4 \cap A_5) - P(A_1 \cap A_3 \cap A_4 \cap A_5) - P(A_2 \cap A_3 \cap A_4 \cap A_5 \\
&+ P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) \;=\; \mathbf{\ldots}
\end{aligned}
$$

2. At *Initech*, 50% of all employees surf the Internet during work hours. 20% of the employees surf the Internet <u>and</u> play *Solitaire* during work hours. It is also known that 60% of the employees either surf the Internet <u>or</u> play *Solitaire* (or both) during work hours.

$$P(\text{ Internet }) = 0.50, \qquad P(\text{ Internet} \cap \textit{Solitaire}\text{ }) = 0.20,$$
$$P(\text{ Internet} \cup \textit{Solitaire}\text{ }) = 0.60.$$

a) What proportion of the employees play *Solitaire* during work hours?

$P(\text{ Internet} \cup \textit{Solitaire}\text{ }) = P(\text{ Internet }) + P(\textit{ Solitaire }) - P(\text{ Internet} \cap \textit{Solitaire}\text{ })$

$0.60 = 0.50 + P(\textit{ Solitaire }) - 0.20$

$P(\textit{ Solitaire }) = \mathbf{0.30}$.

	Solitaire	*Solitaire′*	
Internet	**0.20**	0.30	**0.50**
Internet′	0.10	0.40	0.50
	0.30	0.70	1.00

b) <u>If it is known that an employee surfs the Internet during work hours</u>, what is the probability that he/she also plays *Solitaire*?

$$P(\,Solitaire \mid \text{Internet}\,) = \frac{P(\,Solitaire \cap \text{Internet}\,)}{P(\,\text{Internet}\,)} = \frac{0.20}{0.50} = \mathbf{0.40}.$$

c) Suppose an employee does not play *Solitaire* during work hours. What is the probability that he/she surfs the Internet?

$$P(\,\text{Internet} \mid Solitaire'\,) = \frac{P(\,\text{Internet} \cap Solitaire'\,)}{P(\,Solitaire'\,)} = \frac{0.30}{0.70} = \mathbf{\frac{3}{7}} \approx \mathbf{0.4286}.$$

d) Are events {an employee surfs the Internet during work hours} and {an employee plays *Solitaire* during work hours} independent? ***Justify your answer***.

$P(\,\text{Internet} \cap Solitaire\,) = 0.20$, $\quad P(\,\text{Internet}\,) \cdot P(\,Solitaire\,) = 0.50 \cdot 0.30 = 0.15$.

Since $P(\,\text{Internet} \cap Solitaire\,) \neq P(\,\text{Internet}\,) \cdot P(\,Solitaire\,)$, {Internet} and {*Solitaire*} are **NOT independent**.

OR

$P(\,Solitaire\,) = 0.30$, $\quad P(\,Solitaire \mid \text{Internet}\,) = 0.40$.

Since $P(\,Solitaire \mid \text{Internet} \neq P(\,Solitaire\,)$, {Internet} and {*Solitaire*} are **NOT independent**.

3. During two-and-a-half years of research, bio-psychologist Onur Güntürkün discovered that when people kiss, they turn their heads to the right roughly twice as often as to the left. (Güntürkün, O. Human behaviour: Adult persistence of head-turning asymmetry. *Nature*, **421**, 711, (2003).)

Suppose the probability that a person would turn his/her head to the right is $^2/_3$, and the probability that a person would turn his/her head to the left is $^1/_3$. A couple is planning a kiss on Valentine's day. Assume that their choice of which way to turn their heads is independent of each other.

a) What is the probability that they would both turn their heads to the right (and kiss)?

Person 1 / Person 2	Right	Left	
Right	$^2/_3 \times {^2/_3}$ **$^4/_9$** kiss	$^1/_3 \times {^2/_3}$ **$^2/_9$** bump noses	$^2/_3$
Left	$^2/_3 \times {^1/_3}$ **$^2/_9$** bump noses	$^1/_3 \times {^1/_3}$ **$^1/_9$** kiss	$^1/_3$
	$^2/_3$	$^1/_3$	1.00

P(both turn their heads to the right) = **$^4/_9$**.

b) What is the probability that they would bump noses (i.e., choose the opposite direction to turn their heads)?

P(bump noses) = $^2/_9 + {^2/_9} =$ **$^4/_9$**.

Similar arguments with $^3/_4$ and $^1/_4$ predict phenotypic ratios of 9:3:3:1 in F_2 offspring in Mendel's dihybrid crosses.

4. From a group of 16 male and 9 female armadillos, Noah must choose two to travel on his ark. Unable to distinguish between male and female armadillos, Noah must choose at random.

a) Noah chooses the **two** armadillos at random. Compute the probability that Noah gets two armadillos of the opposite sex (i.e., one male and one female armadillo).

Do not choose the same armadillo twice $\Rightarrow$ without replacement.

Does not matter whether a female is chosen first and a male is chosen second, or a male is chosen first and a female is chosen second.

$$P(\text{F M}) + P(\text{M F}) = {}^{9}\!/_{25} \cdot {}^{16}\!/_{24} + {}^{16}\!/_{25} \cdot {}^{9}\!/_{24}$$

$$= {}^{288}\!/_{600} = {}^{12}\!/_{25} = \mathbf{0.48}.$$

OR

$$P(\text{ 1 F and 1 M }) = \frac{{}_{9}C_{1} \cdot {}_{16}C_{1}}{{}_{25}C_{2}} = \frac{9 \cdot 16}{300} = \frac{144}{300} = \frac{12}{25} = \mathbf{0.48}.$$

b) In order to improve his chances of selecting at least one male and one female armadillo, Noah decides to "cheat" and select **three** armadillos to travel on his ark. Compute the probability that Noah gets at least one male and one female armadillo.

8 possible outcomes:

M M M	F F F	"bad" outcomes
M M F	F F M	"good" outcomes
M F M	F M F	
M F F	F M M	

$$1 - [\, P(\text{M M M}) + P(\text{F F F}) \,] = 1 - [\, {}^{16}\!/_{25} \cdot {}^{15}\!/_{24} \cdot {}^{14}\!/_{23} + {}^{9}\!/_{25} \cdot {}^{8}\!/_{24} \cdot {}^{7}\!/_{23} \,]$$

$$= 1 - [\, {}^{3360}\!/_{13800} + {}^{504}\!/_{13800} \,]$$

$$= 1 - {}^{3864}\!/_{13800} = 1 - 0.28 = \mathbf{0.72}.$$

OR

P(M M F) + P(M F M) + P(M F F) + P(F M M) + P(F M F) + P(F F M)

$$= {}^{16}/_{25} \cdot {}^{15}/_{24} \cdot {}^{9}/_{23} + {}^{16}/_{25} \cdot {}^{9}/_{24} \cdot {}^{15}/_{23} + {}^{16}/_{25} \cdot {}^{9}/_{24} \cdot {}^{8}/_{23}$$

$$+ {}^{9}/_{25} \cdot {}^{16}/_{24} \cdot {}^{15}/_{23} + {}^{9}/_{25} \cdot {}^{16}/_{24} \cdot {}^{8}/_{23} + {}^{9}/_{25} \cdot {}^{8}/_{24} \cdot {}^{16}/_{23}$$

$$= {}^{2160}/_{13800} + {}^{2160}/_{13800} + {}^{1152}/_{13800}$$

$$+ {}^{2160}/_{13800} + {}^{1152}/_{13800} + {}^{1152}/_{13800}$$

$$= {}^{9936}/_{13800} = \mathbf{0.72}.$$

OR

{ 2 F and 1 M } = { M F F , F M F , F F M }
{ 1 F and 2 M } = { M M F , M F M , F M M }

$$P(\text{ 2 F and 1 M }) + P(\text{ 1 F and 2 M }) = \frac{{}_{9}C_{2} \cdot {}_{16}C_{1}}{{}_{25}C_{3}} + \frac{{}_{9}C_{1} \cdot {}_{16}C_{2}}{{}_{25}C_{3}}$$

$$= \frac{36 \cdot 16}{2300} + \frac{9 \cdot 120}{2300} = \frac{1656}{2300} = \mathbf{0.72}.$$

5. Does a monkey have a better chance of rearranging

I I I L L N O S to spell I L L I N O I S

or

E E E E N N S S T to spell T E N N E S S E E ?

I I I L L N O S $\rightarrow$ I L L I N O I S $\dfrac{8!}{3! \cdot 2! \cdot 1! \cdot 1! \cdot 1!} = 3360$ ways.

E E E E N N S S T $\rightarrow$ T E N N E S S E E $\dfrac{9!}{4! \cdot 2! \cdot 2! \cdot 1!} = 3780$ ways.

Spelling I L L I N O I S is a bit more likely than spelling T E N N E S S E E.

6. An electronic device has four independent components. Two of those four are new, and have a reliability of 0.80 each, one is old, with 0.75 reliability, and one is very old, and its reliability is 0.50.

Let $A_i = \{\, i^{\text{th}}$ component is functional $\}$.

Then $P(A_1) = P(A_2) = \mathbf{0.80}$, $P(A_3) = \mathbf{0.75}$, $P(A_4) = \mathbf{0.50}$.

a) Suppose that the device works if all four components are functional. What is the probability that the device will work when needed?

"all four" = "1st **and** 2nd **and** 3rd **and** 4th" = intersection.

$P(A_1 \cap A_2 \cap A_3 \cap A_4)$ since the components are independent

$$= P(A_1) \cdot P(A_2) \cdot P(A_3) \cdot P(A_4) = (0.80) \cdot (0.80) \cdot (0.75) \cdot (0.50) = \mathbf{0.24}.$$

b) Suppose that the device works if at least one of the four components is functional. What is the probability that the device will work when needed?

"at least one" = "either 1st **or** 2nd **or** 3rd **or** 4th **or** 5th" = union.

P(at least one) = 1 – P(none).

"none" = "not 1st **and** not 2nd **and** not 3rd **and** not 4th **and** not 5th".

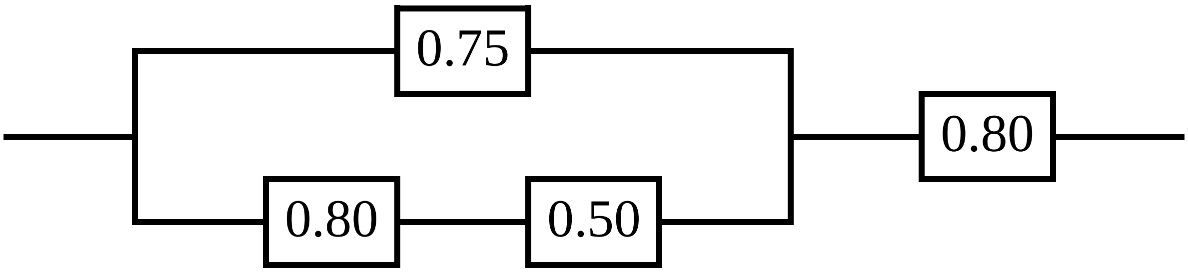
$$P(A_1 \cup A_2 \cup A_3 \cup A_4) = 1 - P(A_1' \cap A_2' \cap A_3' \cap A_4')$$

since the components are independent

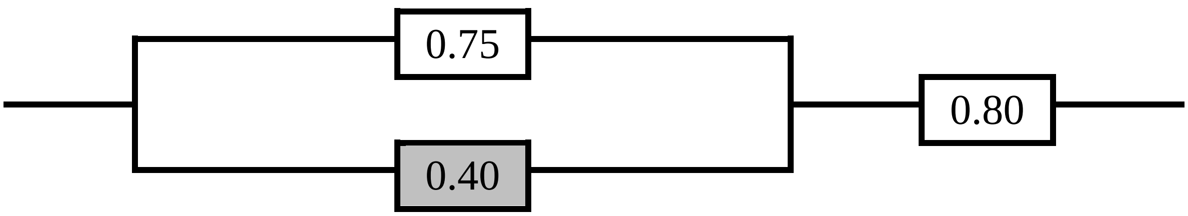
$$= 1 - P(A_1') \cdot P(A_2') \cdot P(A_3') \cdot P(A_4')$$

$$= 1 - (0.20) \cdot (0.20) \cdot (0.25) \cdot (0.50) = 1 - 0.005 = \mathbf{0.995}.$$

c) Suppose that the four components are connected as shown on the diagram below. Find the reliability of the system.



$0.80 \times 0.50 = 0.40$.



$1 - 0.25 \times 0.60 = 0.85$. OR $0.75 + 0.40 - 0.75 \times 0.40 = 0.85$.

$0.85 \times 0.80 = \mathbf{0.68}$.

OR

$$\begin{aligned} \mathrm{P}(\text{top} \cup \text{bottom}) &= \mathrm{P}(\text{top}) + \mathrm{P}(\text{bottom}) - \mathrm{P}(\text{top} \cap \text{bottom}) \\ &= 0.75 \cdot 0.80 + (0.80 \cdot 0.50) \cdot 0.80 - 0.75 \cdot (0.80 \cdot 0.50) \cdot 0.80 \\ &= 0.6 + 0.32 - 0.24 = \mathbf{0.68}. \end{aligned}$$

7. Three prisoners, A, B and C, are in separate cells and sentenced to death. The governor has selected one of them at random to be pardoned. The warden knows which one is pardoned, but is not allowed to tell. Prisoner A begs the warden to let him know the identity of one of the others who is going to be executed. "If B is to be pardoned, give me C's name. If C is to be pardoned, give me B's name. And if I'm to be pardoned, flip a coin to decide whether to name B or C."

The warden tells A that B is to be executed. Prisoner A is pleased because he believes that his probability of surviving has gone up from 1/3 to 1/2, as it is now between him and C. Prisoner A secretly tells C the news, who is also pleased, because he reasons that A still has a chance of 1/3 to be the pardoned one, but his chance has gone up to 2/3. What is the correct answer?

C is correct.

Let E_A, E_B, E_C be the event that the corresponding prisoner will be pardoned, and D the event that the warden mentions prisoner B as the one not being pardoned, then, from Bayes' Theorem, the posterior probability of A being pardoned, is:

$$P(E_A \mid D) = \frac{P(D \mid E_A)P(E_A)}{P(D \mid E_A)P(E_A) + P(D \mid E_B)P(E_B) + P(D \mid E_C)P(E_C)} = \frac{\frac{1}{2} \times \frac{1}{3}}{\frac{1}{2} \times \frac{1}{3} + 0 \times \frac{1}{3} + 1 \times \frac{1}{3}} = \frac{1}{3}.$$

The posterior probability of C being pardoned, is

$$P(E_C \mid D) = 1 - P(E_A \mid D) - P(E_B \mid D) = 1 - \frac{1}{3} - 0 = \frac{2}{3}.$$

Or, from Bayes' Theorem,

$$P(E_C \mid D) = \frac{P(D \mid E_C)P(E_C)}{P(D \mid E_A)P(E_A) + P(D \mid E_B)P(E_B) + P(D \mid E_C)P(E_C)} = \frac{1 \times \frac{1}{3}}{\frac{1}{2} \times \frac{1}{3} + 0 \times \frac{1}{3} + 1 \times \frac{1}{3}} = \frac{2}{3}.$$

An intuitive explanation

Initially Prisoner A has a 1/3 chance of pardon. And knowing whether “B” or “C” will be executed does not change his chance. After he hears B will be executed, the chance of he gets a pardon is still 1/3. And if A will not get the pardon himself it must only be going to C. That means there is a 2/3 chance for C to get a pardon.

From the textbook:

1.3-4

(a) $4\times 5 \times 2 =$ **40**.

(b) $2\times 2 \times 2 =$ **8**.

1.3-6

(a) $4\times\binom{6}{3} =$ **80**.

(b) $4\times 2^6 =$ **256**.

(c) The number of unordered samples of size 3 that can be selected out of 4 objects is $\binom{4-1+3}{3} = \binom{6}{3} =$ **20**.

1.5-8

$$P(O\,O\,B) + P(O\,B\,O) + P(B\,O\,O) = \frac{1}{6}\cdot\frac{2}{6}\cdot\frac{3}{6} + \frac{1}{6}\cdot\frac{4}{6}\cdot\frac{3}{6} + \frac{5}{6}\cdot\frac{2}{6}\cdot\frac{3}{6} = \mathbf{\frac{2}{9}}.$$

1.5-16

(a) With replacement.

The chance of select the WIN ball is $\frac{1}{5}$ each time. The probability that I win during my k-th draw is $(\frac{4}{5})^k \frac{1}{5}$, hence, the total probability of I win the game is $\sum_{k=0}^{\infty} (\frac{4}{5})^{2k} \frac{1}{5} = \frac{1}{1-(4/5)^2} \times \frac{1}{5} = \mathbf{\frac{5}{9}}$.

(b) Without replacement.
I can only win the game during the first draw, or the third draw, or the fifth draw.
The probability of winning is $\frac{1}{5} + \frac{4}{5}\cdot\frac{3}{4}\cdot\frac{1}{3} + \frac{4}{5}\cdot\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{1} = \mathbf{\frac{3}{5}}$.

1.6-8

$$P(\text{Repair}) = 0.40 \times 0.10 + 0.30 \times 0.05 + 0.20 \times 0.03 + 0.10 \times 0.02$$
$$= 0.040 + 0.015 + 0.006 + 0.002 = 0.063.$$

$$P(B_i \mid \text{Repair}) = \frac{P(B_i \cap \text{Repair})}{P(\text{Repair})}$$

i	1	2	3	4
$P(B_i \mid \text{Repair})$	$\mathbf{\frac{40}{63}}$	$\mathbf{\frac{15}{63}}$	$\mathbf{\frac{6}{63}}$	$\mathbf{\frac{2}{63}}$

1.6-10

(a) $P(AD) = 0.02 \times 0.92 + 0.98 \times 0.05 = 0.0184 + 0.0490 = \mathbf{0.0674}$.

(b) $P(N \mid AD) = \frac{0.0490}{0.0674} = \mathbf{0.727}$; $P(A \mid AD) = \frac{0.0184}{0.0674} = \mathbf{0.273}$.

(c) $P(N \mid ND) = \frac{0.98 \times 0.95}{0.02 \times 0.08 + 0.98 \times 0.95} = \frac{0.9310}{0.9326} = \mathbf{0.9983}$;

$P(A \mid ND) = \frac{0.0016}{0.9326} = \mathbf{0.0017}$.

(d) Yes, particularly those in part (b).

2.1-10

(a) $$\frac{\binom{3}{1}\cdot\binom{47}{9}}{\binom{50}{10}} = \frac{\frac{3!}{1!\cdot 2!}\cdot\frac{47!}{9!\cdot 38!}}{\frac{50!}{10!\cdot 40!}} = \frac{3\cdot 10\cdot 40\cdot 39}{50\cdot 49\cdot 48} = \frac{39}{98} = 0.39796.$$

(b) $$\frac{\binom{3}{0}\cdot\binom{47}{10}}{\binom{50}{10}}+\frac{\binom{3}{1}\cdot\binom{47}{9}}{\binom{50}{10}} = \frac{1\cdot\frac{47!}{10!\cdot 37!}}{\frac{50!}{10!\cdot 40!}}+\frac{39}{98} = \frac{40\cdot 39\cdot 38}{50\cdot 49\cdot 48}+\frac{39}{98}$$

$$= \frac{247}{490}+\frac{195}{490} = \frac{221}{245} = 0.90204.$$

2.1-14

P(at least one underweight) = 1 – P(none underweight)

$$= 1-\frac{17}{20}\cdot\frac{16}{19}\cdot\frac{15}{18}\cdot\frac{14}{17}\cdot\frac{13}{16} = 1-\frac{91}{228} = \frac{137}{228} \approx \mathbf{0.60}.$$