STAT 400 Fall 2011

Homework #6
(10 points)
(due Friday, Oct 14th, by 3:00 p.m.)

No credit will be given without supporting work.

1. The "fill" problem is important in many industries, such as those making cereal, toothpaste, beer, and so on. If an industry claims that it is selling 12 ounces of its product in a container, it must have a mean greater than 12 ounces, or else FDA will crack down, although the FDA will allow a very small percentage of the containers to have less than 12 ounces.

a) If the content X of a container has a $N(12.1, \sigma^2)$ distribution, find σ such that
$P(X < 12) = 0.01$.

b) If the content X of a container has a $N(\mu, 0.05^2)$ distribution, find μ such that
$P(X < 12) = 0.01$.

2. Models of the pricing of stock options often make the assumption of a normal distribution. An analyst believes that the price of an *Initech* stock option varies from day to day according to normal distribution with mean \$9.22 and unknown standard deviation.

a) The analyst also believes that 77% of the time the price of the option is greater than \$7.00. Find the standard deviation of the price of the option.

b) Find the proportion of days when the price of the option is greater than \$10.00?

c) Following the famous "buy low, sell high" principle, the analyst recommends buying *Initech* stock option if the price falls into the lowest 14% of the price distribution, and selling if the price rises into the highest 9% of the distribution. Mr. Statman doesn't know much about history, doesn't know much about biology, doesn't know much about statistics, but he does want to be rich someday. Help Mr. Statman find the price below which he should buy *Initech* stock option and the price above which he should sell.

3. a) If e^{3t+8t^2} is the m.g.f. of the random variable X, find $P(-1 < X < 9)$.

b) Suppose X is a normally distributed random variable. Suppose also that

$$P(X > 44.4) = 0.33 \quad \text{and} \quad P(X < 45.6) = 0.7123.$$

What is the m.g.f. of X?

4. Let the random variable X have the p.d.f.

$$f(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2}, \qquad 0 < x < \infty, \qquad \text{zero elsewhere.}$$

Find the mean and the variance of X.

Hint: Compute $E(X)$ directly and $E(X^2)$ by comparing the integral with the integral representing the variance of a random variable that is $N(0, 1)$.

5. Let X be $N(\mu, \sigma^2)$ and consider the transformation $X = \ln(Y)$ or, equivalently, $Y = e^X$. (The random variable Y is said to have a the *lognormal distribution.*)

a) Find the mean and the variance of Y by first determining $E(e^X)$ and $E[(e^X)^2]$, by using the mgf of X.

b) Let X be $N(10, 4)$ and $Y = e^X$. Find $P(6{,}000 < Y < 18{,}000)$.

6. Let X and Y have the joint probability density function

$$f_{X,Y}(x, y) = \begin{cases} x + 4y & 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the marginal p.d.f.s $f_X(x)$ and $f_Y(y)$.

b) Are X and Y independent? *Justify your answer.*

c) Find $\text{Cov}(X, Y)$.

7. In Neverland, annual income is normally distributed with mean \$47,500 and standard deviation \$12,500. Neverland's IRS divides everyone into three groups: "low income" (below \$40,000), "middle income" (between \$40,000 and \$60,000), and "high income" (over \$60,000). Every year, the IRS audits 1% of the individuals in the "low income" group, 2.5% of the individuals in the "middle income" group, and 7% of the individuals in the "high income" group. Suppose that the individuals to be audited at selected at random.

a) What proportion of Neverland's population falls into each of the three income groups? ["Hint": The answers should add up to 100%.]

b) What proportion of Neverland's population was audited last year?

c) Suppose Mr. Smith was audited last year. What is the probability that Mr. Smith is in "low income" group? "Middle income" group? "High income" group? ["Hint": The answers should add up to 100%.]

d) Suppose Mr. Jones is not in the "low income" group. What is the probability that Mr. Jones will be audited this year?

e) Are events {a person was audited last year} and {a person is in "middle income" group} independent? ***Justify your answer***.

f) Are events {a person was audited last year} and {a person is in "middle income" group} mutually exclusive? ***Justify your answer***.

8. Dick and Jane have agreed to meet for lunch between noon (0:00 p.m.) and 1:00 p.m. Denote Jane's arrival time by X, Dick's by Y, and suppose X and Y are independent with probability density functions

$$f_X(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad\qquad f_Y(y) = \begin{cases} 2y & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability that Jane arrives before Dick. That is, find $P(X < Y)$.

9. Suppose that the random variables X and Y have joint p.d.f. $f(x, y)$ given by

$$f(x, y) = C\,x^2 y, \qquad 0 < x < y, \quad x + y < 2.$$

a) Sketch the support of (X, Y). That is, sketch $\{\, 0 < x < y, \quad x + y < 2 \,\}$.

b) What must the value of C be so that $f(x, y)$ is a valid joint p.d.f.?

c) Find $P(X + Y < 1)$.

d) Find the marginal probability density function for X.

e) Find the marginal probability density function for Y.

From the textbook:

10.5-2 (4.7-2)

4.1-1

4.1-8 (4.1-6)

4.2-8 (a), (b), (c), (d)

4.2-12 (a), (b)