STAT 400 **Examples for 10/07/2011** Fall 2011

(Chapter 4.1)

Let X and Y be two discrete random variables. The **joint probability mass function** $p(x, y)$ is defined for each pair of numbers (x, y) by

$$p(x, y) = \mathrm{P}(\mathrm{X} = x \text{ and } \mathrm{Y} = y).$$

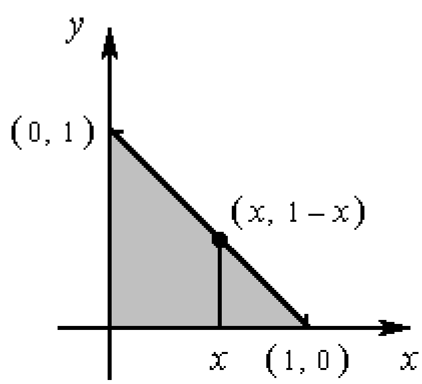
Let A be any set consisting of pairs of (x, y) values. Then

$$\mathrm{P}((\mathrm{X}, \mathrm{Y}) \in A) = \sum_{(x,y) \in A}\sum p(x, y).$$

Let X and Y be two continuous random variables. Then $f(x, y)$ is the **joint probability density function** for X and Y if for any two-dimensional set A

$$\mathrm{P}((\mathrm{X}, \mathrm{Y}) \in A) = \iint_A f(x, y)\, dx\, dy.$$

1. A Nut Company markets cans of deluxe mixed nuts containing almonds, cashews, and peanuts. Suppose the net weight of each can is exactly 1 lb, but the weight contribution of each type of nut is random. Because the three weights sum to 1, a joint probability model for any two gives all necessary information about the weight of the third type. Let X = the weight of almonds in a selected can and Y = the weight of cashews.



Then the region of positive density is D = $\{ (x, y) : 0 \le x \le 1, 0 \le y \le 1, x + y \le 1 \}$.

Let the joint probability density function for (X, Y) be

$$f(x, y) = \begin{cases} 60\, x^2\, y & 0 \le x \le 1,\ 0 \le y \le 1,\ x + y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a) Verify that $f(x, y)$ is a legitimate probability density function.

b) Find the probability that the two types of nuts together make up less than 50% of a can. That is, find the probability $P(X + Y < 0.50)$. (Find the probability that peanuts make up over 50% of a can.)

c) Find the probability that there are more almonds than cashews in a can. That is, find the probability $P(X > Y)$.

d) Find the probability that there are at least twice as many cashews as there are almonds. That is, find the probability $P(Y \geq 2X)$.

The **marginal probability mass functions** of X and of Y are given by

$$p_X(x) = \sum_{\text{all } y} p(x, y), \qquad p_Y(y) = \sum_{\text{all } x} p(x, y).$$

The **marginal probability density functions** of X and of Y are given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy, \qquad f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx.$$

e) Find the marginal probability density function for X.

f) Find the marginal probability density function for Y.

If $p(x, y)$ is the joint probability mass function of (X, Y) OR $f(x, y)$ is the joint probability density function of (X, Y), then

discrete

$$\mathrm{E}\big(g(\mathrm{X}, \mathrm{Y})\big) = \sum_{\text{all } x} \sum_{\text{all } y} g(x, y) \cdot p(x, y)$$

continuous

$$\mathrm{E}\big(g(\mathrm{X}, \mathrm{Y})\big) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \cdot f(x, y)\, dx\, dy$$

g) Find $\mathrm{E}(\mathrm{X})$, $\mathrm{E}(\mathrm{Y})$, $\mathrm{E}(\mathrm{X}+\mathrm{Y})$, $\mathrm{E}(\mathrm{X} \cdot \mathrm{Y})$.

h) If 1 lb of almonds costs the company \$1.00, 1 lb of cashews costs \$1.50, and 1 lb of peanuts costs \$0.60, what is the expected total cost of the content of a can?

2. Consider the following joint probability distribution $p(x, y)$ of two discrete random variables X and Y:

$x \ \backslash \ y$	0	1	2	
1	0.15	0.10	0	
2	0.25	0.30	0.20	

a) Find $P(X + Y = 2)$.

b) Find $P(X > Y)$.

c) Find the (marginal) probability distributions $p_X(x)$ of X and $p_Y(y)$ of Y.

x	$p_X(x)$

y	$p_Y(y)$

d) Find $E(X), E(Y), E(X + Y), E(X \cdot Y)$.

3.

Trinomial distribution: For $x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq n$,

$$\begin{aligned} f(x_1, x_2) &= P(X_1 = x_1, X_2 = x_2) \\ &= \frac{n!}{x_1! x_2! (n - x_1 - x_2)!} p_1^{x_1} p_2^{x_2} (1 - p_1 - p_2)^{n - x_1 - x_2} \end{aligned}$$

Independent Random Variables

Consider the following joint probability distribution $p(x, y)$ of two random variables X and Y:

$x \setminus y$	0	1	2	
1	0.15	0.10	0	0.25
2	0.25	0.30	0.20	0.75
	0.40	0.40	0.20	

Recall: A and B are independent if and only if $\mathrm{P}(\mathrm{A} \cap \mathrm{B}) = \mathrm{P}(\mathrm{A}) \cdot \mathrm{P}(\mathrm{B})$.

a) Are events $\{X = 1\}$ and $\{Y = 1\}$ independent?

Def Random variables X and Y are **independent** if and only if

discrete $p(x, y) = p_{\mathrm{X}}(x) \cdot p_{\mathrm{Y}}(y)$ for all x, y.

continuous $f(x, y) = f_{\mathrm{X}}(x) \cdot f_{\mathrm{Y}}(y)$ for all x, y.

$\mathrm{F}(x, y) = \mathrm{P}(\mathrm{X} \le x, \mathrm{Y} \le y)$. $f(x, y) = \partial^2 \mathrm{F}(x, y) / \partial x \, \partial y$.

Def Random variables X and Y are **independent** if and only if

$$\mathrm{F}(x, y) = \mathrm{F}_{\mathrm{X}}(x) \cdot \mathrm{F}_{\mathrm{Y}}(y) \quad \text{for all } x, y.$$

b) Are random variables X and Y independent?

1. Let the joint probability density function for (X, Y) be

$$f(x,y)=\begin{cases} 60\,x^2\,y & 0 \le x \le 1,\ 0 \le y \le 1,\ x+y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Recall: $f_X(x) = 30\,x^2\,(1-x)^2, \quad 0 < x < 1,$

$f_Y(y) = 20\,y\,(1-y)^3, \quad 0 < y < 1.$

Are random variables X and Y independent?

3. Let the joint probability density function for (X, Y) be

$$f(x,y)=\begin{cases} x+y & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Are X and Y independent?

4. Let the joint probability density function for (X, Y) be

$$f(x,y)=\begin{cases} 12\,x\,(1-x)\,e^{-2y} & 0 \le x \le 1, y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Are X and Y independent?

If random variables X and Y are independent, then

$$E(g(X) \cdot h(Y)) = E(g(X)) \cdot E(h(Y)).$$