Examples for 10/05/2011 (Chapter 10.5)

Markov's Inequality:

Let $u(\mathrm{X})$ be a non-negative function of the random variable X.

If $\mathrm{E}[u(\mathrm{X})]$ exists, then, for every positive constant c,

$$\mathrm{P}(u(\mathrm{X}) \ge c) \le \frac{\mathrm{E}[u(\mathrm{X})]}{c}.$$

Chebyshev's Inequality:

Let X be any random variable with mean μ and variance σ^2. For any $\varepsilon > 0$,

$$\mathrm{P}(|\mathrm{X}-\mu| \ge \varepsilon) \le \frac{\sigma^2}{\varepsilon^2}$$

or, equivalently,

$$\mathrm{P}(|\mathrm{X}-\mu| < \varepsilon) \ge 1 - \frac{\sigma^2}{\varepsilon^2}$$

Setting $\varepsilon = k\,\sigma$, $k > 1$, we obtain

$$\mathrm{P}(|\mathrm{X}-\mu| \ge k\,\sigma) \le \frac{1}{k^2}$$

or, equivalently,

$$\mathrm{P}(|\mathrm{X}-\mu| < k\,\sigma) \ge 1 - \frac{1}{k^2}$$

That is, for any $k > 1$, the probability that the value of any random variable will be within k standard deviations of its mean is at least $1 - \frac{1}{k^2}$.

Example 1: Suppose $\mu = \mathrm{E}(\mathrm{X}) = 17$, $\sigma = \mathrm{SD}(\mathrm{X}) = 5$.

Consider interval $(9, 25) = (17-8, 17+8)$. $\Rightarrow$ $k = \frac{8}{5} = 1.6$.

$$\Rightarrow \quad \mathrm{P}(9 < \mathrm{X} < 25) = \mathrm{P}(|\mathrm{X}-\mu| < 1.6\,\sigma) \ge 1 - \frac{1}{1.6^2} = \mathbf{0.609375}.$$

Example 2: Suppose $\mu = \mathrm{E}(\mathrm{X}) = 17$, $\sigma = \mathrm{SD}(\mathrm{X}) = 5$.

Suppose also that the distribution of X is symmetric about the mean.

Consider interval $(10, 30) = (17 - 7, 17 + 13) = (\mu - 1.4\,\sigma, \mu + 2.6\,\sigma)$.

$$\mathrm{P}(10 < \mathrm{X} < 24) = \mathrm{P}(|\mathrm{X} - \mu| < 1.4\,\sigma) \geq 1 - \frac{1}{1.4^2} \approx 0.490.$$

$$\mathrm{P}(4 < \mathrm{X} < 30) = \mathrm{P}(|\mathrm{X} - \mu| < 2.6\,\sigma) \geq 1 - \frac{1}{2.6^2} \approx 0.852.$$

Since the distribution of X is symmetric about the mean,

$$\mathrm{P}(10 < \mathrm{X} < 17) \geq \frac{0.490}{2} = 0.245, \qquad \mathrm{P}(17 < \mathrm{X} < 30) \geq \frac{0.852}{2} = 0.426.$$

$$\Rightarrow \qquad \mathrm{P}(10 < \mathrm{X} < 30) \geq 0.245 + 0.426 = \mathbf{0.671}.$$

Example 3: Consider a discrete random variable X with p.m.f.

$$\mathrm{P}(\mathrm{X} = -1) = \tfrac{1}{2}, \qquad \mathrm{P}(\mathrm{X} = 1) = \tfrac{1}{2}.$$

Then $\mu = \mathrm{E}(\mathrm{X}) = 0$, $\sigma^2 = \mathrm{Var}(\mathrm{X}) = \mathrm{E}(\mathrm{X}^2) = 1$.

$$\Rightarrow \qquad \mathrm{P}(|\mathrm{X} - \mu| \geq \sigma) = \mathrm{P}(|\mathrm{X}| \geq 1) = 1. \qquad (k = 1)$$

$$\mathrm{P}(|\mathrm{X} - \mu| < \sigma) = \mathrm{P}(|\mathrm{X}| < 1) = 0.$$

Example 4: (Chebyshev's Inequality cannot be improved)

Let $a > 0$, $0 < p < \tfrac{1}{2}$. Consider a discrete random variable X with p.m.f.

$$\mathrm{P}(\mathrm{X} = -a) = p, \qquad \mathrm{P}(\mathrm{X} = 0) = 1 - 2p, \qquad \mathrm{P}(\mathrm{X} = a) = p.$$

Then $\mu = \mathrm{E}(\mathrm{X}) = 0$, $\sigma^2 = \mathrm{Var}(\mathrm{X}) = \mathrm{E}(\mathrm{X}^2) = 2pa^2$.

Let $k = \dfrac{1}{\sqrt{2p}} > 1$. Then $k\,\sigma = a$.

$$\Rightarrow \qquad \mathrm{P}(|\mathrm{X} - \mu| \geq k\,\sigma) = \mathrm{P}(|\mathrm{X}| \geq a) = 2p = \frac{1}{k^2}.$$

$$\mathrm{P}(|\mathrm{X} - \mu| < k\,\sigma) = \mathrm{P}(|\mathrm{X}| < a) = 1 - 2p = 1 - \frac{1}{k^2}.$$

Jensen’s Inequality:

If g is convex on an open interval I and X is a random variable whose support is contained in I and has finite expectation, then

$$\mathrm{E}[g(\mathrm{X})] \geq g[\mathrm{E}(\mathrm{X})].$$

If g is strictly convex then the inequality is strict, unless X is a constant random variable.

$\Rightarrow \quad \mathrm{E}(\mathrm{X}^2) \geq [\mathrm{E}(\mathrm{X})]^2 \quad \Leftrightarrow \quad \mathrm{Var}(\mathrm{X}) \geq 0$

$\Rightarrow \quad \mathrm{E}(e^{t\mathrm{X}}) \geq e^{t\mathrm{E}(\mathrm{X})} \quad \Rightarrow \quad \mathrm{M}_{\mathrm{X}}(t) \geq e^{t\mu}$

$\Rightarrow \quad \mathrm{E}\left(\frac{1}{\mathrm{X}}\right) \geq \frac{1}{\mathrm{E}(\mathrm{X})}$ for a positive random variable X

$\Rightarrow \quad \mathrm{E}[\ln \mathrm{X}] \leq \ln \mathrm{E}(\mathrm{X})$ for a positive random variable X