Review(2.4)

- Bernoulli distribution and its mean, variance.
- Binomial distribution and its p.m.f., properties.
- Cumulative distribution function and its properties.
- Geometric distribution and Negative binomial distribution

A Property of Binomial distribution

If $X \sim b(n,p)$, then $(n-X) \sim b(n, 1-p)$.

Example: An urn contains 9 blue balls and 6 red balls. The balls are well mixed. Draw one ball each time, and record the color, then put it back (with replacement). Let X be the number of times we get blue balls among 10 draws. What is the distribution of X, and what is the probability that there are at least 9 blue balls out of 10 draws? (0.0464)

$X \sim b(n,p) = b(10, \frac{9}{15})$

$P(X \geq 9) = \ldots$

$Y = \#$ of red balls $\quad Y = n - X$

$Y \sim b(10, \frac{6}{15})$

$P(Y \leq 1) = P(Y=0) + P(Y=1)$

$\boxed{\text{use excel to calculate}}$

Binomial v.s. Hypergeometric distribution

Example: A jar has N marbles, S of them are orange and N - S are blue. Suppose 3 marbles are selected. Find the probability that there are 2 orange marbles in the sample, if the selection is done ...

X = # orange marbles

with replacement

(a) N =10, S=4

$p = S/N = 0.4$ $n=3$

$X \sim b(3, 0.4)$

$P(X=2) = \binom{3}{2} 0.4^2 \times 0.6 = 0.288$

(b) N= 100, S =40

the same

0.288

(c) N=1000, S=400

same

0.288

without replacement

Hyper geometric

$P(X=2) = \frac{\binom{4}{2}\binom{6}{1}}{\binom{10}{3}} = 0.3$

$P(X=2) = \frac{\binom{40}{2}\binom{60}{1}}{\binom{100}{3}} = 0.2894$

$P(X=2) = \frac{\binom{400}{2}\binom{600}{1}}{\binom{1000}{3}} = 0.2881$

	Binomial	Hypergeometric
	with replacement	without replacement
Probability	$P(X=x) = \binom{n}{x} \cdot p^x \cdot (1-p)^{n-x}$	$P(X=x) = \frac{\binom{S}{x} \cdot \binom{N-S}{n-x}}{\binom{N}{n}}$
Expected Value	$E(X) = n \cdot p$	$E(X) = n \cdot \frac{S}{N}$
Variance	$Var(X) = n \cdot p \cdot (1-p)$	$Var(X) = n \cdot \frac{S}{N} \cdot \left(1 - \frac{S}{N}\right) \cdot \frac{N-n}{N-1}$

when $N \gg n$

If the population size is large (compared to the sample size) Binomial Distribution can be used regardless of whether sampling is with or without replacement.

to approximate hyper geometric

Geometric and Negative binomial distribution

Example: Suppose that during practice, a basket ball player can make a free throw 80% of the time. Furthermore, assume that a sequence of free throw shooting can be thought of as independent Bernoulli trials.

- Let X_1 equal the number of free throws that this player must attempt to make a total of 1 shot. What is the distribution of X_1.

$X_1 \sim Geo(0.8)$

- Let X_2 equal the minimum number of free throws that this player must attempt to make a total of 10 shots. What is the distribution of X_2.

$X_2 \sim$ negative binomial $(10, 0.8)$

$x = 10, 11, 12, \ldots$

$P(X_2 = x) = \binom{x-1}{9} 0.2^{x-10} \times 0.8^{10}$.

recall: p.m.f two ways to characterize
c.d.f the distribution of random variable

Today's Lecture (2.5)

- Moment-generating function (m.g.f.) and its properties
- Calculating mean and variance through m.g.f.

Moment generating function

Definition: Let X be a random variable of the discrete type with p.m.f. $f(x)$ and space S. If there is a positive number h such that

$$E[e^{tX}] = \sum_{x \in S} e^{tx} f(x)$$

exists and is finite for $-h < t < h$, then the function of t defined by $M(t) = E[e^{tX}]$ is called the moment-generating function (m.g.f) of X.

- If the space of X is $\{b_1, b_2, b_3, \cdots\}$, the moment generating function is given by

$$Ee^{tX} = M(t) = e^{tb_1} f(b_1) + e^{tb_2} f(b_2) + e^{tb_3} f(b_3) + \cdots.$$

  where $f(b_i) = P(X = b_i)$.

- If the moment generating function exists for t in an open interval containing zero, it uniquely determines the distribution of the random variable.

Example: What is m.g.f. of the following distribution?

x	1	2	3	4	5
$f(x)$	0.1	0.3	0.2	0.3	0.1

$M(t) = Ee^{tX}$

$= e^{t} \cdot 0.1 + e^{2t} \cdot 0.3 + e^{3t} \cdot 0.2 + e^{4t} \cdot 0.3 + e^{5t} \cdot 0.1$

Property of m.g.f

If the moment-generating function exists in an open interval containing zero, then $E(X^r) = M^{(r)}(0)$. In particular,

- $\mu = M'(0)$
- $\sigma^2 = E[X^2] - [E(X)]^2 = M''(0) - [M'(0)]^2$

Binomial distribution $b(n, p)$: see Example 2.5-3

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \cdots, n$$

- Step 1: The m.g.f $M(t)$ is

$$M(t) = Ee^{tX} = \sum_{x=0}^{n} e^{tx} \cdot f(x) = \sum_{x=0}^{n} e^{tx} \cdot \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (p \cdot e^t)^x \times (1-p)^{n-x}$$

$$\overset{*}{=} [p \cdot e^t + (1-p)]^n$$

- Step 2: Calculate the derivatives of $M(t)$,
  - $M'(t) = n[(1-p) + pe^t]^{n-1}(pe^t)$
  - $M''(t) = n(n-1)[(1-p) + pe^t]^{n-2}(pe^t)^2 + n[(1-p) + pe^t]^{n-1}(pe^t)$
- Step 3: Calculate μ, σ^2 based on $M'(t)$ and $M''(t)$.
  - $\mu = E(X) = M'(0) = np$
  - $\sigma^2 = E(X^2) - \mu^2 = M''(0) - [M'(0)]^2 = np(1-p)$

$*$: recall $(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} a^x b^{n-x}$

The m.g.f. of Geometric distribution

Recall: We say that X has a *geometric distribution* if

$$f(x) = p(1-p)^{x-1}, \; x = 1, 2, 3, \cdots$$

$$\begin{aligned} M(t) &= \sum_{x=1}^{\infty} e^{tx} f(x) \\ &= \sum_{x=1}^{\infty} e^{tx} p(1-p)^{x-1} \\ &= p(1-p)^{-1} \sum_{x=1}^{\infty} [e^t(1-p)]^x \\ * &= p(1-p)^{-1} \frac{e^t(1-p)}{1-e^t(1-p)} = \frac{pe^t}{1-e^t(1-p)} \end{aligned}$$

Note that $e^t(1-p) < 1$, i.e. $t < -\log(1-p)$. So

$$\begin{aligned} M'(t) &= \frac{pe^t}{[1-e^t(1-p)]^2} \\ M''(t) &= \frac{pe^t[1+(1-p)e^t]}{[1-e^t(1-p)]^3} \end{aligned}$$

So

$$\begin{aligned} \mu &= M'(0) = \frac{1}{p} \\ \sigma^2 &= M''(0) - (M'(0))^2 \\ &= \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2} \end{aligned}$$

$*$: recall: $\sum_{x=1}^{\infty} a^x = \frac{a}{1-a}$ if $|a| < 1$.

Negative Binomial distribution

Definition: We say that X has a negative binomial distribution if

$$f(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \ \ x = r, r+1, r+2, \cdots$$

In this case, $\mu = r/p$ and $\sigma^2 = r(1-p)/p^2$ and

$$M(t) = \frac{(pe^t)^r}{[1-(1-p)e^t]^r}, \text{ where } t < -\log(1-p)$$

See textbook Pg 4.

*: recall $(1-w)^{-r} = \sum_{x=r}^{\infty} \binom{x-1}{r-1} w^{x-r}$ if $|w| < 1$.

A note about distribution names

Discrete distributions often get their names from mathematical power series.

- Binomial probabilities sum to 1 because of the Binomial Theorem:

  $(p+(1-p))^n =<$ sum of Binomial probabilities $>= 1$

- Negative Binomial probabilities sum to 1 by the Negative Binomial expansion: i.e. the Binomial expansion with a negative power, $-r$:

  $p^r(1-(1-p))^{-r} =<$ sum of NegBin probabilities $>= 1$

- Geometric probabilities sum to 1 because they form a Geometric series:

  $p\sum\limits_{x=0}^{\infty}(1-p)^x =<$ sum of Geometric probabilities $>= 1$

A note about discrete distribution in EXCEL

- Hypergeometric with N = population size, S = number of "successes" in the population, n = sample size.
  X = number of "successes" in the sample without replacement.
  =HYPGEOMDIST(x, n, S, N) gives $P(X = x)$;

- Binomial, X = number of "successes" in n independent trials.
  =BINOMDIST$(x, n, p, 0)$ gives $P(X = x) = \binom{n}{x}p^x(1-p)^{n-x}$;
  =BINOMDIST$(x, n, p, 1)$ gives $P(X \leq x)$;

- Negative Binomial, X = number of independent trials until the r-th "success"
  = NEGBINOMDIST$(x-r, r, p)$ gives $P(X = x) = \binom{x-1}{r-1}p^r(1-p)^{x-r}$.

Q: How to get geometric distribution from Excel?