Review (3.1,3.3)

$f(x) = F'(x)$

$F(x) = \int_{-\infty}^{x} f(u)\,du.$

- Continuous random variable and its c.d.f.
- Probability density function (p.d.f.), its properties.
- Mean, variance, moment generating function
- Percentiles for p.d.f.

Today's Lecture (3.4)

- Uniform distribution, its mean, variance, m.g.f.
- Exponential distribution and its mean, variance.
- Memoryless properties of exponential distribution.

Uniform distribution

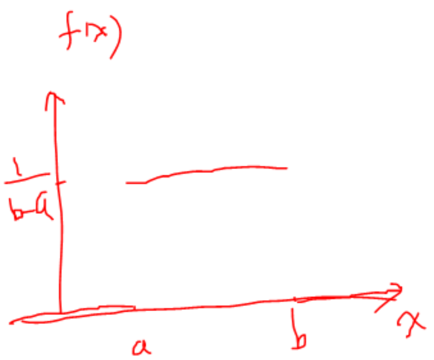
X has a uniform distribution on the interval $[a, b]$ if X is equally likely to fall anywhere in the interval $[a, b]$. We write $X \sim U(a, b)$. The p.d.f of X is

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

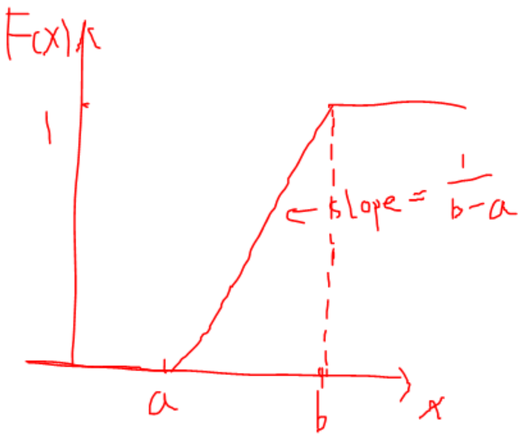
Distribution function $F_X(x)$:

$$F_X(x) = \begin{cases} 0, & \text{if } x < a, \\ \frac{x-a}{b-a}, & \text{if } a \leq x \leq b, \\ 1, & \text{if } x > b. \end{cases}$$

Plot of $f_X(x)$ and $F_X(x)$:



constant on the support



Mean, variance and m.g.f. of uniform distribution

If $X \sim U(a,b)$, then

$E(X - \frac{a+b}{2})^2 = \int_a^b (x - \frac{a+b}{2})^2 \cdot \frac{1}{b-a}\, dx$

- $E(X) = \frac{a+b}{2}$, Var$(X) = \frac{(b-a)^2}{12}$.
- $M(t) = E(e^{tX}) = \frac{e^{tb}-e^{ta}}{t(b-a)}$ if $t \neq 0$, 1 if $t = 0$.

Example: Customers arrive randomly at a bank teller's window. Given that one customer arrived during a particular 10-minute period, let X equal the time within the 10 minutes that the customer arrived. If $X \sim U(0,10)$, find

- the p.d.f. of X

$$f_X(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1/10 & \text{if } 0 < x < 10 \\ 0 & \text{if } x \geq 10 \end{cases}$$

- $P(3 \le X \le 7)$

$= \int_3^7 f_X(x)\, dx = \frac{4}{10}$

- The average arrival time.

$EX = \frac{0+10}{2} = 5$

- $\sigma^2 = \frac{(10-0)^2}{12} = \frac{100}{12}$

Poisson Process

Example: the number of volcano eruptions that would occur in the next 1000 years in the world.

Question: When will the next volcano erupt in the world? Or how long we have to wait for the next volcano?

Define X to be a continuous random variable giving the time waited before the next volcano, starting from now.

Suppose that $\{N_t : t > 0\}$ (the number of volcanoes to have occurred by time t) forms a Poisson process with rate λ. So the distribution of $N_t \sim$ Pois (λt).

N: number

- When $x \geq 0$,

$$
\begin{aligned}
F_X(x) &= P(X \leq x) \\
&= P(\text{amount of time waited for next volcano} \leq x) \\
&= P(\text{at least one volcano between now and time } x) \\
&= P(\#\text{ volcanoes between now and time } x \text{ is} \geq 1) \\
&= P(N_x \geq 1) = 1 - P(N_x = 0) \\
&= 1 - \frac{(\lambda x)^0}{0!} e^{-\lambda x} = 1 - e^{-\lambda x}
\end{aligned}
$$

- When $x < 0$,

$$
\begin{aligned}
F_X(x) &= P(X \leq x) \\
&= P(\text{less than 0 time before next volcano}) = 0
\end{aligned}
$$

waiting time is a continous random variable

Poisson process with intensity λ

Waiting time is Exponential $(\frac{1}{\lambda})$.

Exponential distribution

Definition: Exponential distribution is the distribution of the waiting time (time between events) in a Poisson process with rate λ. Let $\theta = 1/\lambda$, we write Exp(θ).

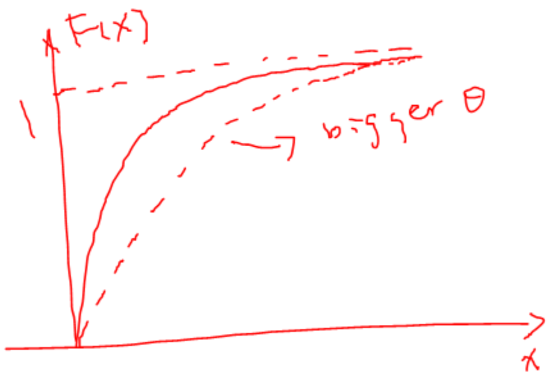
- Distribution function:

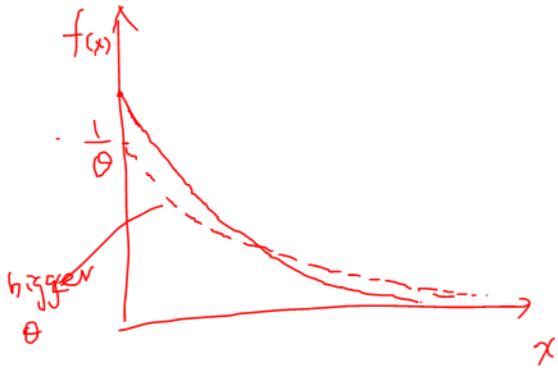
$$F_X(x) = P(X \le x) = \begin{cases} 1 - e^{-x/\theta} & \text{for } x \ge 0 \\ 0 & \text{for } x < 0 \end{cases}$$

- Probability density function:

$$f_X(x) = F'_X(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta} & \text{for } x \ge 0 \\ 0 & \text{for } x < 0 \end{cases}$$

Plot:





Note:

$$P(X > x) = 1 - F(x) = \begin{cases} e^{-\frac{x}{\theta}} & x > 0 \\ 1 & x \le 0 \end{cases}$$

Example

1. What is the probability that the next volcanic eruption in the world occurs within the next 50 years? $(\lambda = \frac{1}{10})$

X = # of eruptions in 50 years

$X \sim Pois(50\lambda) \sim Pois(5)$

Method 1: $P(X \geq 1) = 1 - P(X=0) = 1 - e^{-5} \cdot \frac{5^0}{0!} = 1 - e^{-5}$

Method 2: Let Y be waiting time until first eruption

$Y \sim Exp(\frac{1}{\lambda}) \sim Exp(10)$

$P(Y \leq 50) = F_Y(50) = 1 - e^{-50/10} = 1 - e^{-5}$

2. What is the probability that there will be at most two volcano eruptions in the world within next 50 years?

$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$= e^{-5} \times \left(\frac{5^0}{0!} + \frac{5^1}{1!} + \frac{5^2}{2!}\right)$

Method 2: Y_1: time util the first eruption

Y_2: waiting time from 1st eruption to the 2nd eruption

$Y_1 \sim Exp(10)$ and Y_1 independent of Y_2

$Y_2 \sim Exp(10)$

what's $P(Y_1 + Y_2 \leq 50)$?

We'll explain in ch 3.5 Gamma distn.

Mean, variance, m.g.f. of Exponential distribution

$$f(x) = \frac{1}{\theta}e^{-x/\theta}, \qquad 0 \le x < \infty$$

$$M(t) \ =$$

Then

$$\begin{aligned} M'(t) \ &= \ \frac{\theta}{(1-\theta t)^2} \\ M''(t) \ &= \ \frac{2\theta^2}{(1-\theta t)^3} \\ \mu \ &= \ M'(0) = \theta \\ \sigma^2 \ &= \ M''(0) - [M'(0)]^2 = \theta^2 \end{aligned}$$

So λ is the mean number of events in the unit interval, then $\theta = 1/\lambda$ is the mean waiting time for the first event. For example, if $\lambda = 10$ is the mean number of events per minute; then the mean waiting time for the first event is $1/10$ of a minute.

Memoryless property

Example: Suppose the life of my laptop X has an exponential distribution with a mean life of 1000 days. What is the probability that

- The life of my laptop is longer than 500 days.
- The life of my laptop is longer than 2000 days.
- Suppose that I have used my laptop for 500 days, what is the probability that it will last for another 2000 days?