STAT 400 Fall 2011

Examples for 10/10/2011
(Chapter 4.2)

Covariance and Correlation Coefficient

Covariance of X and Y

$$\sigma_{XY} = \mathrm{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = E(XY) - \mu_X \mu_Y$$

(a) $\mathrm{Cov}(X, X) = \mathrm{Var}(X)$;

(b) $\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)$;

(c) $\mathrm{Cov}(a X + b, Y) = a\,\mathrm{Cov}(X, Y)$;

(d) $\mathrm{Cov}(X + Y, W) = \mathrm{Cov}(X, W) + \mathrm{Cov}(Y, W)$.

$$\mathrm{Cov}(a X + b Y, c X + d Y)$$
$$= a c\,\mathrm{Var}(X) + (a d + b c)\,\mathrm{Cov}(X, Y) + b d\,\mathrm{Var}(Y).$$

$$\mathrm{Var}(a X + b Y) = \mathrm{Cov}(a X + b Y, a X + b Y)$$
$$= a^2\,\mathrm{Var}(X) + 2 a b\,\mathrm{Cov}(X, Y) + b^2\,\mathrm{Var}(Y).$$

0. Find in terms of σ_X^2, σ_Y^2, and σ_{XY}:

a) $\mathrm{Cov}(2X + 3Y, X - 2Y)$,

b) $\mathrm{Var}(2X + 3Y)$,

c) $\mathrm{Var}(X - 2Y)$.

Correlation coefficient of X and Y

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)} \cdot \sqrt{\mathrm{Var}(Y)}} = E\left[\left(\frac{X-\mu_X}{\sigma_X}\right), \left(\frac{Y-\mu_Y}{\sigma_Y}\right)\right]$$

(a) $-1 \le \rho_{XY} \le 1$;

(b) ρ_{XY} is either $+1$ or -1 if and only if X and Y are linear functions of one another.

If random variables X and Y are independent, then

$$E(g(X) \cdot h(Y)) = E(g(X)) \cdot E(h(Y)).$$

$\Rightarrow$ $\mathrm{Cov}(X, Y) = \sigma_{XY} = 0$, $\mathrm{Corr}(X, Y) = \rho_{XY} = 0$.

2. Consider the following joint probability distribution $p(x, y)$ of two random variables X and Y:

	y			
x	0	1	2	$p_X(x)$
1	0.15	0.10	0	0.25
2	0.25	0.30	0.20	0.75
$p_Y(y)$	0.40	0.40	0.20	1.00

Recall:

$E(X) = 1.75$,

$E(Y) = 0.8$,

$E(XY) = 1.5$.

Find $\mathrm{Cov}(X, Y) = \sigma_{XY}$ and $\mathrm{Corr}(X, Y) = \rho_{XY}$.

1. Let the joint probability density function for (X, Y) be

$$f(x,y) = \begin{cases} 60\,x^2\,y & 0 \le x \le 1,\ 0 \le y \le 1,\ x + y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $\mathrm{Cov}(X,Y) = \sigma_{XY}$ and $\mathrm{Corr}(X,Y) = \rho_{XY}$.

Recall: $f_X(x) = 30\,x^2\,(1-x)^2, \quad 0 < x < 1,$ $\qquad E(X) = \frac{1}{2},$

$f_Y(y) = 20\,y\,(1-y)^3, \quad 0 < y < 1,$ $\qquad E(Y) = \frac{1}{3},$ $\qquad E(XY) = \frac{1}{7}.$

4. Let the joint probability density function for (X, Y) be

$$f(x,y) = \begin{cases} 12\,x\,(1-x)\,e^{-2y} & 0 \le x \le 1,\ y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Find $\mathrm{Cov}(X,Y) = \sigma_{XY}$ and $\mathrm{Corr}(X,Y) = \rho_{XY}$.

Recall: $f_X(x) = 6\,x\,(1-x), \quad 0 < x < 1.$ $\qquad f_Y(y) = 2\,e^{-2y}, \quad y > 0.$

3. Let the joint probability density function for (X, Y) be

$$f(x,y)=\begin{cases} x+y & 0\le x\le 1, 0\le y\le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $\operatorname{Cov}(X,Y) = \sigma_{XY}$ and $\operatorname{Corr}(X,Y) = \rho_{XY}$.

Recall: $f_X(x) = x+\frac{1}{2}, \quad 0 < x < 1.$ $\qquad f_Y(y) = y+\frac{1}{2}, \quad 0 < y < 1.$

Review exercises on (ch4.1)

5. Suppose the probability density functions of T_1 and T_2 are

$$f_{T_1}(x) = \alpha\, e^{-\alpha x}, \quad x > 0, \qquad f_{T_2}(y) = \beta\, e^{-\beta y}, \quad y > 0,$$

respectively. Suppose T_1 and T_2 are independent. Find $P(2T_1 > T_2)$.

6. Let X and Y be two independent random variables, X has a Geometric distribution with the probability of "success" $p = 1/3$, Y has a Poisson distribution with mean 3. That is,

$$p_X(x) = \left(\frac{1}{3}\right)\cdot\left(\frac{2}{3}\right)^{x-1}, \qquad x = 1, 2, 3, \dots,$$

$$p_Y(y) = \frac{3^y e^{-3}}{y!}, \quad y = 0, 1, 2, 3, \dots.$$

a) Find $P(X = Y)$.

b) Find $P(X = 2Y)$.