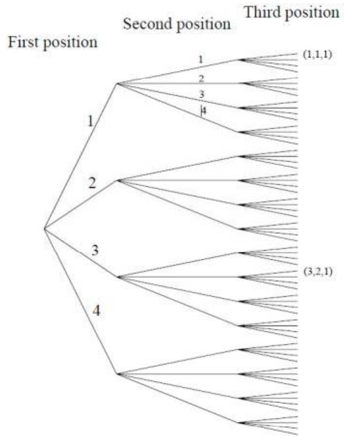
Review(1.3 Methods of Enumeration)

- Multiplication principle.
- Permutation and combination.
- Sampling with/without replacement.
- Ordered/unordered sample.

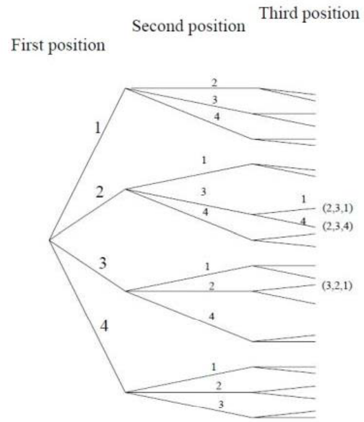
Example: Draw r numbers from the set $\{1, 2, \cdots, n\}$. Let $n = 4, r = 3$.

- Sampling with/without replacement.
- Ordered/unordered sample.

Ordered, with replacement



Ordered, without replacement



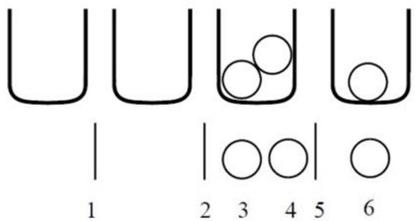
Unordered, without replacement

- n choose r

Unordered, with replacement

Taking $n = 4$, $\mathrm{r} = 3$, possible outcomes are $\{3,3,4\}$, $\{1,2,4\}$, $\{2,2,2\}$, etc. The trick to solve this counting problem is to represent the outcomes in a different way, via an ordered vector $(x_1,\ldots,x_n)$ representing how many times an element in $\{1,\ldots,4\}$ occurs. For example, $\{3,3,4\}$ corresponds to $(0,0,2,1)$.

Now place $\mathrm{r} = 3$ balls into $n = 4$ urns, numbered $1,\ldots,4$. Then $(0,0,2,1)$ means that the third urn has 2 balls and the fourth urn has 1 ball. One way to distribute the balls over the urns is to distribute $n-1 = 3$ "separators" and $\mathrm{r} = 3$ balls over $n - 1 + \mathrm{r} = 6$ positions.



Summary

The number of possibilities to sample with or without replacement in order or unordered r elements from a set of n distinct elements are summarized in the following table:

Sampling	in order	without order
without replacement	$\frac{n!}{(n-r)!}$	$\binom{n}{r}$
with replacement	n^r	$\binom{n+r-1}{r}$

Distinguishable Permutation

Suppose that a set contains n objects or two types, r of one type and $n-r$ of the other type. What is the number of distinguishable permutations of these n objects?______

Special case : 2 red balls, 2 blue balls, $n = 4, r = 2$.

Definition: Each of the ${}_nC_r$ permutations of n objects, r of one type and $n-r$ of another type, is called a *distinguishable permutation*

Binomial Coefficients

Binomial Coefficients: ${}_nC_r = \binom{n}{r}$

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} b^r a^{n-r}$$

- $\binom{n}{r} = \binom{n}{n-r}$
- $0 = \Sigma_{r=0}^{n} (-1)^r \binom{n}{r}$.
- $2^n = \Sigma_{r=0}^{n} \binom{n}{r}$.

Example: What is the coefficient of x^3y^4 in the expansion of $(x+y)^7$.

Multinomial Coefficients

In general, if we have $n = \Sigma_{i=1}^{r} n_i$ objects with r types, with n_1 objects of type 1, n_2 objects of type 2, and n_r objects of type r. The number of distinguishable permutations of the n objects is

$$\binom{n}{n_1, n_2, \cdots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}$$

Example: How many ordered arrangements are there of the letters in the word STATISTICS?______

Example: We have 5 black balls, 6 blue balls, and 7 yellow balls, we put them into 18 boxes (one balls in each box). How many possible arrangements?______

Example: In how many ways can the set of nucleotides (A,A,G,G,G,G,C,C,C,T,T,T,T,T,T)
be arranged in a sequence of 15 letters?______

Today's lecture: 2.1 Discrete Random Variables

- Random variable, space.
- Discrete random variable, probability mass function and its properties.
- Bar graph, probability histogram.

Random variable

Definition: Given a random experiment with an outcome space S, a function X that assigns to each element s in S one and only one real number $X(s) = x$ is called a *random variable*. The random variable X is discrete if the set of real values it can take is finite or countable, e.g. $\{0, 1, 2, 3, \cdots, \}$.

Example: Paul went to the car dealer and buy a car from Ferrari, Porsche, or BMW.

Random experiment: which car?

Random variable: X gives numbers to the possible outcomes.

$$\begin{aligned} \text{Ferrari} &\Rightarrow X = 1 \\ \text{Porsche} &\Rightarrow X = 2 \\ \text{BMW} &\Rightarrow X = 3 \end{aligned}$$

Definition: The space (support) of X is the set of real numbers $\{x : X(s) = x, s \in S\}$. In the book, the space of X is denoted by S, same as sample space.

- Space of $X = S =$____________(pick a car)

In many instances, one can think of the space of X as being the outcome space, but $\cdots$

Example: Wheel of fortune.
One spins a wheel and look at the angle of the pointer. If the angle is within $[0^o, 180^o)$, then one gets 1000 dollars, otherwise he/she gets 0 dollars. Let X be the dollar amount the person gets.

- What is the outcome space ?

- What is the space for X ?

- Is X a discrete random variable?

Probability mass function

The probability mass function (p.m.f.) $f(x)$, for a discrete random variable X, is given by

$$f(x) = P(X = x), \quad x \in S$$

Example: Which car?

Outcome	Ferrari	Porsche	BMW
x	1	2	3
$f(x) = P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{4}{6}$

Or $f(x) = \begin{cases} 1/6 & \text{if } x = 1 \\ 1/6 & \text{if } x = 2 \\ 4/6 & \text{if } x = 3 \\ 0 & \text{otherwise} \end{cases}$

Properties of probability mass function

- $0 \le f(x) \le 1$ for all x.
- $\Sigma_{x \in S} f(x) = 1$.
- $P(X \in A) = \Sigma_{x \in A} f(x)$.

e.g. in the car example, $P(X \in \{1, 2\}) =$ ______.

For each of the following, determine the constant c so that $f(x)$ is a p.m.f. for some random variable X.

- $f(x) = x/c$, $x = 1, 2, \cdots, 10$.

- $f(x) = c(1/3)^x$, $x = 1, 2, 3, \cdots$.

Bar graph and probability histogram

Draw a bar graph and a probability histogram for the probability mass function

x	1	2	3	4	5
$f(x)$	0.1	0.3	0.2	0.3	0.1

Bar graph:

Probability histogram

Discrete R.V. Examples

- *Discrete Uniform example* : Let X be the result of tossing a fair die. The probability mass function of X is:

x	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

- *Discrete nonuniform example:* Roll two dice
  - X_1 - number on the first die
  - X_2 - number on the second die
  - $X = X_1 + X_2$ - total number of points
    (a function of random variables is again a random variable)

(X_1, X_2)	X	(X_1, X_2)	X	(X_1, X_2)	X
(1,1)	2	(3,1)	4	(5,1)	6
(1,2)	3	(3,2)	5	(5,2)	7
(1,3)	4	(3,3)	6	(5,3)	8
(1,4)	5	(3,4)	7	(5,4)	9
(1,5)	6	(3,5)	8	(5,5)	10
(1,6)	7	(3,6)	9	(5,6)	11
(2,1)	3	(4,1)	5	(6,1)	7
(2,2)	4	(4,2)	6	(6,2)	8
(2,3)	5	(4,3)	7	(6,3)	9
(2,4)	6	(4,4)	8	(6,4)	10
(2,5)	7	(4,5)	9	(6,5)	11
(2,6)	8	(4,6)	10	(6,6)	12

x	2	3	4	5	6	7	8	9	10	11	12
$P(X=x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$