STAT 400 Fall 2011

Homework #5

(10 points)

(due Friday, October 7th, by 3:00 p.m.)

No credit will be given without supporting work.

1. Suppose a random variable X has the following probability density function:

$$f(x) = \begin{cases} 1/x & 1 \le x \le C \\ 0 & \text{otherwise} \end{cases}$$

a) What must the value of C be so that $f(x)$ is a probability density function?

For $f(x)$ to be a probability density function, we must have:

1) $f(x) \ge 0$, 2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_{1}^{C} \frac{1}{x}\,dx = \ln C - \ln 1 = \ln C.$$

Therefore, $C = \boldsymbol{e}$.

b) Find $\mathrm{P}(\mathrm{X} < 2)$.

$$\mathrm{P}(\mathrm{X} < 2) = \int_{-\infty}^{2} f(x)\,dx = \int_{1}^{2} \frac{1}{x}\,dx = \ln 2 - \ln 1 = \mathbf{\ln 2}.$$

c) Find $\mathrm{P}(\mathrm{X} < 3)$.

$$\mathrm{P}(\mathrm{X} < 3) = \int_{-\infty}^{3} f(x)\,dx = \int_{1}^{e} \frac{1}{x}\,dx = \ln e - \ln 1 = \mathbf{1}.$$

d) Find $\mu_{\mathrm{X}} = \mathrm{E}(\mathrm{X})$.

$$\mu_{\mathrm{X}} = \mathrm{E}(\mathrm{X}) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_{1}^{e} x \cdot \frac{1}{x}\,dx = \int_{1}^{e} 1\,dx = \boldsymbol{e} - \mathbf{1}.$$

e) Find $\sigma_X^2 = \mathrm{Var}(X)$.

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = \int_1^e x^2 \cdot \frac{1}{x}\,dx = \int_1^e x\,dx = \frac{e^2-1}{2}.$$

$$\sigma_X^2 = \mathrm{Var}(X) = E(X^2) - [E(X)]^2 = \frac{-e^2+4e-3}{2} \approx \mathbf{0.242}.$$

2. Suppose a random variable X has the following probability density function:

$$\mathbf{f}(x) = \begin{cases} \mathbf{C}\,|\,\mathbf{x-2}\,| & 0 \le \mathbf{x} \le 3 \\ 0 & \textit{otherwise} \end{cases}$$

a) What must the value of C be so that $f(x)$ is a probability density function?

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_0^3 \mathbf{C}\,|\,\mathbf{x-2}\,|\,dx = \mathbf{C}\left(\int_0^2 \mathbf{(2-x)}\,dx + \int_2^3 \mathbf{(x-2)}\,dx\right) = \frac{5}{2}C.$$

$$C = \mathbf{\frac{2}{5}}.$$

b) Find the cumulative distribution function $F(x) = P(X \le x)$.

$$F(x) = \int_{-\infty}^{x} f(y)\,dy = \begin{cases} 0 & x<0 \\ \mathbf{\frac{2}{5}}\int_0^x |\,\mathbf{y-2}\,|\,dx = \mathbf{\frac{2}{5}}\int_0^x \mathbf{(2-y)}dx = \begin{cases} \dfrac{4x-x^2}{5} & 0 \le x \le 2 \\ \dfrac{x^2-4x+4}{5} & 2 < x \le 3 \end{cases} \\ 1 & x > 3 \end{cases}.$$

c) Find the median of the probability distribution of X.

Need $m = ?$ such that (Area to the left of m) $= \int_{-\infty}^{\mathbf{m}} \mathbf{f}(\mathbf{x})\,d\mathbf{x} = \mathbf{F(m)} = \frac{1}{2}$.

First we found that F(2)=4/5 > 0.5, so m should be smaller than 2, namely m is in [0,2). Then solve the equation

$$\frac{4x-x^2}{5}=0.5 \quad \Rightarrow x^2-4x+2.5=0 \quad \Rightarrow x=\frac{4\pm\sqrt{6}}{2}$$

The median is less than 2, so m = $\frac{4-\sqrt{6}}{2}$ = **0.775**.

d) Find $\mu_X = E(X)$.

$$\mu_X = \int_{-\infty}^{\infty} \mathbf{x}\cdot\mathbf{f}(\mathbf{x})\,d\mathbf{x} = \frac{2}{5}\int_0^3 \mathbf{x}\cdot|x-2|\,d\mathbf{x} = \frac{2}{5}\int_0^2 \mathbf{x}\cdot(2-x)\,d\mathbf{x}+\frac{2}{5}\int_2^3 \mathbf{x}\cdot(x-2)\,d\mathbf{x}=\frac{16}{15}$$

=**1.067**.

e) Find the moment-generating function of X, $M_X(t)$.

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx}\cdot f(x)dx = \int_0^3 e^{tx}\cdot\frac{2}{5}|\mathbf{x-2}|dx.$$

$$=\frac{2}{5}\left(\int_0^2 e^{tx}\cdot(2-x)\,dx+\int_2^3 e^{tx}\cdot(x-2)\,dx\right)$$

$$=\left(\frac{1}{t^2}e^{tx}+\frac{2}{t}e^{tx}-\frac{1}{t}e^{tx}\right)\Bigg|_0^2+\left(\frac{1}{t}xe^{tx}-\frac{1}{t^2}e^{tx}-\frac{2}{t}e^{tx}\right)\Bigg|_2^3 \qquad t\neq 0.$$

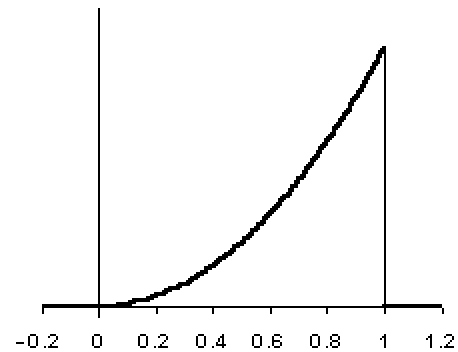
$$=-\frac{2}{t}-\frac{1}{t^2}-\frac{4}{t}e^{2t}+\frac{2}{t^2}e^{2t}+\frac{1}{t}e^{3t}-\frac{1}{t^2}e^{3t}$$

$M_X(0)$ =1.

3. Let X be a continuous random variable with the probability density function

$$f(x) = k \cdot x^2, \qquad 0 \le x \le 1,$$

$$f(x) = 0, \qquad \text{otherwise.}$$



a) What must the value of k be so that $f(x)$ is a probability density function?

1) $f(x) \ge 0,$ 2) $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1.$

$$1 = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_0^1 k \cdot x^2\,\mathrm{d}x = k \cdot \int_0^1 x^2\,\mathrm{d}x$$

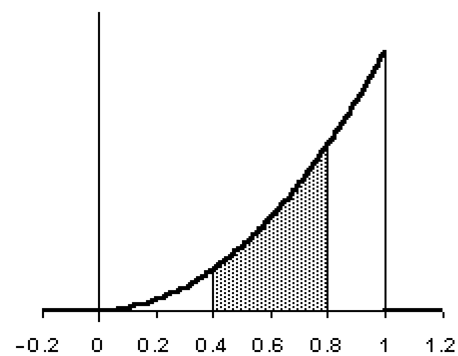
$$= k \cdot \left(\frac{x^3}{3} \right) \Bigg|_0^1 = k \cdot \left(\frac{1}{3} \right) = \frac{k}{3}. \qquad \Rightarrow \qquad k = \mathbf{3}.$$

b) Find the probability P(0.4 ≤ X ≤ 0.8).

$$P(0.4 \le X \le 0.8) = \int_{0.4}^{0.8} f(x)\,\mathrm{d}x$$

$$= \int_{0.4}^{0.8} 3 \cdot x^2\,\mathrm{d}x = x^3 \Big|_{0.4}^{0.8}$$

$$= 0.8^3 - 0.4^3 = \mathbf{0.448}.$$



c) Find the median of the distribution of X.

Need $m = ?$ such that (Area to the left of m) $= \int_{-\infty}^{m} f(x)\,\mathrm{d}x = \frac{1}{2}.$

$$\frac{1}{2} = \int_{-\infty}^{m} f(x)\,\mathrm{d}x = \int_0^m 3 \cdot x^2\,\mathrm{d}x = x^3 \Big|_0^m = m^3.$$

$$m = \sqrt[3]{1/2} = \mathbf{0.7937}.$$

d) Find $\mu_X = \mathrm{E}(X)$.

$$\mathrm{E}(X) = \mu_X = \int_{-\infty}^{\infty} x \cdot f(x)\,\mathrm{d}x = \int_0^1 x \cdot \left(3 \cdot x^2\right)\mathrm{d}x = 3 \cdot \int_0^1 x^3\,\mathrm{d}x.$$

$$= 3 \cdot \left(\frac{x^4}{4}\right)\Bigg|_0^1 = \mathbf{\frac{3}{4}} = \mathbf{0.75}.$$

e) Find $\sigma_X = \mathrm{SD}(X)$.

$$\mathrm{Var}(X) = \sigma_X^2 = \left[\int_{-\infty}^{\infty} x^2 \cdot f(x)\,\mathrm{d}x\right] - \left(\mu_X\right)^2 = \left[\int_0^1 3 \cdot x^4\,\mathrm{d}x\right] - \left(\frac{3}{4}\right)^2$$

$$= 3 \cdot \left(\frac{x^5}{5}\right)\Bigg|_0^1 - \left(\frac{3}{4}\right)^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80} = 0.0375.$$

$$\sigma_X = \mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)} = \sqrt{0.0375} = \mathbf{0.19365}.$$

f) Find the moment-generating function of X, $M_X(t)$.

$$M_X(t) = \mathrm{E}(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\,dx = \int_0^1 e^{tx} \cdot 3x^2\,dx.$$

$$u = 3x^2, \qquad dv = e^{tx}\,dx,$$

$$du = 6x\,dx, \qquad v = \frac{1}{t}e^{tx}.$$

$$M_X(t) = \int_0^1 e^{tx} \cdot 3x^2\,dx = \left(3x^2 \cdot \frac{1}{t}e^{tx}\right)\Bigg|_0^1 - \int_0^1 \left(\frac{1}{t}e^{tx} \cdot 6x\right)dx$$

$$= \frac{3}{t}e^t - \int_0^1 \left(\frac{1}{t}e^{tx} \cdot 6x\right)dx$$

$$u = 6x, \qquad dv = = \frac{1}{t} e^{tx} \, dx,$$

$$du = 6 \, dx, \qquad v = \frac{1}{t^2} e^{tx}.$$

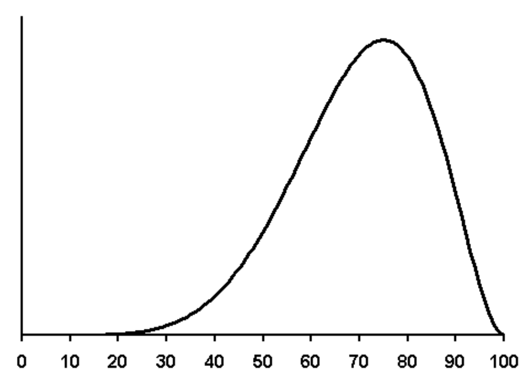
$$\mathrm{M}_X(t) = \frac{3}{t} e^{t} - \int_0^1 \left(\frac{1}{t} e^{tx} \cdot 6x \right) dx = \frac{3}{t} e^{t} - \left(6x \cdot \frac{1}{t^2} e^{tx} \right) \Bigg|_0^1 - \int_0^1 \left(\frac{1}{t^2} e^{tx} \cdot 6 \right) dx$$

$$= \frac{3}{t} e^{t} - \frac{6}{t^2} e^{t} + \left(\frac{6}{t^3} e^{tx} \right) \Bigg|_0^1 = \frac{3}{t} e^{t} - \frac{6}{t^2} e^{t} + \frac{6}{t^3} e^{t} - \frac{6}{t^3}, \qquad t \neq 0.$$

$$\mathrm{M}_X(0) = 1.$$

4. A simple model for describing mortality in the general population in a particular country is given by the probability density function

$$f(y) = \frac{252}{10^{18}} \; y^6 \, (100 - y)^2, \qquad 0 < y < 100.$$



a) Verify that $f(y)$ is a valid probability density function.

1. $f(y) \geq 0$ for each y; ✓

2. $\displaystyle\int_{-\infty}^{\infty} f(y) \, dy = 1.$

$$\int_{-\infty}^{\infty} f(y) \, dy = \int_0^{100} \frac{252}{10^{18}} \; y^6 \, (100 - y)^2 \, dy = \int_0^1 252 \; x^6 \, (1 - x)^2 \, dx$$

$$= 252 \cdot \left[\frac{1}{7} x^7 - 2 \cdot \frac{1}{8} x^8 + \frac{1}{9} x^9 \right] \Bigg|_0^1 = 252 \cdot \frac{2}{504} = 1. \qquad \checkmark$$

b) Based on this model, which event is more likely

A: a person dies between the ages of 70 and 80

or

B: a person lives past age 80?

A: $$\int_{70}^{80} \frac{252}{10^{18}} y^6 (100-y)^2 \, dy = \int_{0.7}^{0.8} 252 \, x^6 (1-x)^2 \, dx$$

$$= 252 \cdot \left[\frac{1}{7} x^7 - 2 \cdot \frac{1}{8} x^8 + \frac{1}{9} x^9 \right] \Bigg|_{0.7}^{0.8}$$

$$\approx 0.7382 - 0.4628 = 0.2754.$$

B: $$\int_{80}^{100} \frac{252}{10^{18}} y^6 (100-y)^2 \, dy = \int_{0.8}^{1.0} 252 \, x^6 (1-x)^2 \, dx$$

$$= 252 \cdot \left[\frac{1}{7} x^7 - 2 \cdot \frac{1}{8} x^8 + \frac{1}{9} x^9 \right] \Bigg|_{0.8}^{1.0}$$

$$\approx 1 - 0.7382 = 0.2618.$$

A is more likely.

c) Given that a randomly selected individual just celebrated his 60th birthday, find the probability that he will live past age 80.

$$\mathrm{P}(\text{over } 80 \mid \text{over } 60) = \frac{\mathrm{P}(\text{over } 80 \cap \text{over } 60)}{\mathrm{P}(\text{over } 60)} = \frac{\displaystyle\int_{80}^{100} \frac{252}{10^{18}} y^6 (100-y)^2 \, dy}{\displaystyle\int_{60}^{100} \frac{252}{10^{18}} y^6 (100-y)^2 \, dy}$$

$$\approx \frac{1-0.7382}{1-0.2318} = \frac{0.2618}{0.7682} \approx \mathbf{0.3408}.$$

d) Find the value of y that maximizes $f(y)$ (**mode**).

$$f'(y) = \frac{252}{10^{18}}\left[6y^5(100-y)^2 - 2y^6(100-y)\right]$$

$$= \frac{252}{10^{18}}\, y^5(100-y)\left[6(100-y)-2y\right]$$

$$= \frac{252}{10^{18}}\, y^5(100-y)\left[600-8y\right] = 0.$$

$\Rightarrow$ $y = 0$, $y = 100$ (not max), $y =$ **75** years (max).

e) Find the (average) life expectancy.

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f(y)\,dy = \int_0^{100} \frac{252}{10^{18}}\, y^7 (100-y)^2\, dy = \int_0^1 252\cdot 100\; x^6(1-x)^2\, dx$$

$$= 252\cdot 100\cdot\left[\frac{1}{8}x^8 - 2\cdot\frac{1}{9}x^9 + \frac{1}{10}x^{10}\right]\Bigg|_0^1 = 252\cdot 100\cdot\frac{2}{720} = \mathbf{70} \text{ years.}$$

5. An insurance policy reimburses a loss up to a benefit limit of 10. The policyholder's loss, Y, follows a distribution with density function:

$$f(y) = \begin{cases} \dfrac{2}{y^3} & \text{if } \; y > 1 \\[2ex] 0 & \text{otherwise} \end{cases}$$

a) What is the expected value and the variance of the policyholder's loss?

$$\mathrm{E}(\text{Loss}) = \int_1^\infty y \cdot \frac{2}{y^3}\, dy = -\frac{2}{y}\Big|_1^\infty = 2.$$

$$\mathrm{E}(\text{Loss}^2) = \int_1^\infty y^2 \cdot \frac{2}{y^3}\, dy = -2\ln y\Big|_1^\infty \text{ is not finite.} \quad \Rightarrow \mathrm{Var}(\text{Loss}) \text{ is not finite.}$$

b) What is the expected value and the variance of the benefit paid under the insurance policy?

$$\text{The benefit paid under the insurance policy} = \begin{cases} y & \text{for } \; 1 < y \le 10 \\[2ex] 10 & \text{for } \; y \ge 10 \end{cases}$$

$$\mathrm{E}(\text{Benefit Paid}) = \int_1^{10} y \cdot \frac{2}{y^3}\, dy + \int_{10}^\infty 10 \cdot \frac{2}{y^3}\, dy = -\frac{2}{y}\Big|_1^{10} - \frac{10}{y^2}\Big|_{10}^\infty = 1.9.$$

$$\mathrm{E}(\text{Benefit Paid}^2) = \int_1^{10} y^2 \cdot \frac{2}{y^3}\, dy + \int_{10}^\infty 10^2 \cdot \frac{2}{y^3}\, dy = -2\ln y\Big|_1^{10} - \frac{100}{y^2}\Big|_{10}^\infty$$

$$= 2\ln 10 + 1.$$

$$\mathrm{Var}(\text{Benefit Paid}) = 2\ln 10 + 1 - 1.9^2 = 2\ln 10 - 2.61 \approx 1.99517.$$

6. Suppose that number of accidents at the Monstropolis power plant follows the Poisson process with the average rate of 0.40 accidents per week. Assume all weeks are independent of each other.

a) Find the probability that at least 2 accidents will occur in one week.

1 week $\Rightarrow$ $\lambda = 0.40$.

Need $P(X \geq 2) = ?$ Poisson distribution: $P(X = x) = \dfrac{\lambda^{x} \cdot e^{-\lambda}}{x!}$

$$P(X \geq 2) = 1 - [\,P(X = 0) + P(X = 1)\,] = 1 - \left[\frac{0.40^{0} \cdot e^{-0.40}}{0!} + \frac{0.40^{1} \cdot e^{-0.40}}{1!}\right]$$

$$= 1 - [\,0.6703 + 0.2681\,] = \mathbf{0.0616}.$$

b) Find the probability that 4 accidents will occur in two months (8 weeks).

8 weeks $\Rightarrow$ $\lambda = 0.40 \cdot 8 = 3.2$. $P(X = 4) = \dfrac{3.2^{4} \cdot e^{-3.2}}{4!} = \mathbf{0.1781}.$

c) Find the probability that there will be 5 accident-free weeks in two months (8 weeks).

Let Y = the number of accident-free weeks in two months (8 weeks).

Then Y has Binomial distribution, $n = 8$, $p = 0.6703$ (Poisson, $\lambda = 0.40$)

$$P(Y = 5) = {}_{8}C_{5} \cdot (0.6703)^{5} \cdot (1 - 0.6703)^{3} = \mathbf{0.2716}.$$

d) Find the probability that the first accident would occur during the fourth week.

T_1 has Exponential distribution with $\lambda = 0.40$ or $\theta = {}^1\!/_{0.4} = 2.5$.

$$P(3 < T_1 < 4) = \int_3^4 0.4\,e^{-0.4t}\,dt = e^{-1.2} - e^{-1.6} \approx \mathbf{0.0993}.$$

OR

$$P(3 < T_1 < 4) = P(T_1 > 3) - P(T_1 > 4) = P(X_3 = 0) - P(X_4 = 0)$$
$$= P(\text{Poisson}(1.2) = 0) - P(\text{Poisson}(1.6) = 0) = 0.301 - 0.202 = \mathbf{0.099}.$$

OR

$$P\begin{pmatrix} \text{no accidents during} & \text{AND} & \text{at least one accident} \\ \text{the first three weeks} & & \text{during the fourth week} \end{pmatrix}$$
$$= P(X_3 = 0) \times P(X_1 \ge 1) = 0.301 \times (1 - 0.670) \approx \mathbf{0.0993}.$$

OR

Week 1		Week 2		Week 3		Week 4	
no accident		no accident		no accident		accident(s)	
0.670	$\times$	0.670	$\times$	0.670	$\times$	0.330	$\approx$ **0.0993**.

e) Find the probability that the third accident would occur during the fifth week.

T_3 has Gamma distribution with $\alpha = 3$ and $\lambda = 0.40$ or $\theta = {}^1\!/_{0.4} = 2.5$.

$$P(4 < T_3 < 5) = P(T_3 > 4) - P(T_3 > 5) = P(X_4 \le 2) - P(X_5 \le 2)$$
$$= P(\text{Poisson}(1.6) \le 2) - P(\text{Poisson}(2.0) \le 2) = 0.783 - 0.677 = \mathbf{0.106}.$$

OR

$$P(4 < T_3 < 5) = \int_4^5 \frac{0.4^3}{\Gamma(3)}\, t^{3-1}\, e^{-0.4t}\,dt = \int_4^5 \frac{0.4^3}{2}\, t^2\, e^{-0.4t}\,dt \approx \mathbf{0.1067}.$$

OR

$$P\begin{pmatrix}\text{two accidents during} \\ \text{the first four weeks}\end{pmatrix} \text{ AND } \begin{pmatrix}\text{at least one accident} \\ \text{during the fifth week}\end{pmatrix}$$

$$+ \; P\begin{pmatrix}\text{one accident during} \\ \text{the first four weeks}\end{pmatrix} \text{ AND } \begin{pmatrix}\text{at least two accident} \\ \text{during the fifth week}\end{pmatrix}$$

$$+ \; P\begin{pmatrix}\text{no accidents during} \\ \text{the first four weeks}\end{pmatrix} \text{ AND } \begin{pmatrix}\text{at least three accident} \\ \text{during the fifth week}\end{pmatrix}$$

$$= P(X_4 = 2) \times P(X_1 \geq 1) + P(X_4 = 1) \times P(X_1 \geq 2) + P(X_4 = 0) \times P(X_1 \geq 3)$$

$$= (0.783 - 0.525) \times (1 - 0.670) + (0.525 - 0.202) \times (1 - 0.938) + 0.202 \times (1 - 0.992) \approx \mathbf{0.107}.$$

From the textbook:

3.3-24 (a),(b) (3.2-24 (a),(b))

(a) $P(X > 2000) = \int_{2000}^{\infty} (2x/1000^2) e^{-(x/1000)^2} dx = \left[-e^{-(x/1000)^2}\right]_{2000}^{\infty} = e^{-4}$;

(b)

$$\begin{aligned}
\left[-e^{-(x/1000)^2}\right]_{\pi_{0.75}}^{\infty} &= 0.25 \\
e^{-(\pi_{0.75}/1000)^2} &= 0.25 \\
-(\pi_{0.75}/1000)^2 &= \ln(0.25) \\
\pi_{0.75} &= 1177.41;
\end{aligned}$$

3.4-4 (3.3-4)

X is $U(4, 5)$;

(a) $\mu = 9/2$; (b) $\sigma^2 = 1/12$; (c) 0.5.

3.4-8 (3.3-8)

(a) $f(x) = \left(\frac{2}{3}\right) e^{-2x/3}, \qquad 0 \le x < \infty;$

(b) $P(X > 2) = \int_2^\infty \frac{2}{3} e^{-2x/3}\,dx = \left[-e^{-2x/3}\right]_2^\infty = e^{-4/3}.$

3.5-2 (3.4-2)

Either use integration by parts or

$$\begin{aligned} F(x) &= P(X \le x) \\ &= 1 - \sum_{k=0}^{\alpha-1} \frac{(\lambda x)^k e^{-\lambda x}}{k!}. \end{aligned}$$

Thus, with $\lambda = 1/\theta = 1/4$ and $\alpha = 2$,

$$\begin{aligned} P(X < 5) &= 1 - e^{-5/4} - \left(\frac{5}{4}\right) e^{-5/4} \\ &= 0.35536. \end{aligned}$$

3.5-8 (3.4-8)

(a) W has a gamma distribution with $\alpha = 7,\ \theta = 1/16$.

(b) Using Table III in the Appendix,

$$\begin{aligned} P(W \leq 0.5) &= 1 - \sum_{k=0}^{6} \frac{8^k e^{-8}}{k!} \\ &= 1 - 0.313 = 0.687, \end{aligned}$$

because here $\lambda w = (16)(0.5) = 8$.

3.6-6 (5.2-6)

$M(t) = e^{166t + 400t^2/2}$ so

(a) $\mu = 166$; **(b)** $\sigma^2 = 400$;

(c) $P(170 < X < 200) = P(0.2 < Z < 1.7) = 0.3761$;

(d) $P(148 \leq X \leq 172) = P(-0.9 \leq Z \leq 0.3) = 0.4338$.

3.6-14 (5.2-14)

(a) $P(X > 22.07) = P(Z > 1.75) = 0.0401$;

(b) $P(X < 20.857) = P(Z < -1.2825) = 0.10$.

Thus the distribution of Y is $b(15,\ 0.10)$
and from Table II in the Appendix, $P(Y \leq 2) = 0.8159$.