Chapter 1.3 Methods of Enumeration

- Multiplication principle.
- Permutation and combination.
- Sampling with/without replacement.
- Ordered/unordered sample.

Multiplication Principle

In general, if there are p experiments and the first has N_1 possible outcomes, the second N_2, ..., and the pth N_p possible outcomes, then there are a total of $N_1 \times N_2 \times \ldots N_p$ possible outcomes for the p experiments.

Example:
A DNA molecule is a sequence of four types of nucleotides, denoted by *A; G;C and T. For a molecule* 1 million (10^6) units long, how many different possible sequences?

Permutation

Example: Suppose that 3 positions are to be filled with 3 different objects (O1,O2,O3). How many possible arrangements are there?

In general, there are *n!* ways to fill in n positions with *n* different objects, where

$$n! = n \times (n - 1) \times \ldots \times 2 \times 1; \; 0! = 1.$$

Definition: Each of the *n!* arrangements of *n* different objects is called a **permutation** of the *n* objects.

${}_nP_r$

If only r positions are to be filled with objects selected from n different objects, $r \leq n$, the number of possible ordered arrangements is

$$ {}_nP_r = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!} $$

- Each of the ${}_nP_r$ arrangements is called **a permutation of the *n* objects taken *r* at a time.**

Example

There are 5 seats in a classroom and 3 students registered for the class. How many seating arrangements do we have?

- Using Multiplication rule:
- E1: S1 has 5 choices ,
- E2: S2 has 4 choices,
- E3: S3 has 3 choices
- # of arrangements for composite events E1E2E3 is:
- $5 \times 4 \times 3 = 5!/2! = 5!/(5-3)! = 60$.

- There are only 3 seats in a classroom and 5 students are registered for the class. How many seating arrangements do we have?

With or without replacement

- **Sampling with replacement:** an object is selected and then replaced (put it back) before the next object is selected.

Example: In Illinois, license plates have seven numbers. How many distinct such plates are possible?

- **Sampling without replacement**: an object is NOT replaced (put it back) after it has been selected.

Example: How many four-letter code words are possible using the letters in HOPE if the letters may not be repeated.

Ordered or unordered sample

- If r objects are selected from a set of n objects, and *the order of selection is noted*, the selected set of r objects is called an **ordered sample of size r.**

Example: the number of ways of selecting a president, a vice president in a club which has 5 members.
20

- If the *order of the selection is not noted,* the selected set is called **unordered sample of size r.**

Example: the number of ways of selecting two presidents in a club which has 5 members.
20/2=10

	President	VP
[1,]	"member1"	"member2"
[2,]	"member1"	"member3"
[3,]	"member1"	"member4"
[4,]	"member1"	"member5"
[5,]	"member2"	"member1"
[6,]	"member2"	"member3"
[7,]	"member2"	"member4"
[8,]	"member2"	"member5"
[9,]	"member3"	"member1"
[10,]	"member3"	"member2"
[11,]	"member3"	"member4"
[12,]	"member3"	"member5"
[13,]	"member4"	"member1"
[14,]	"member4"	"member2"
[15,]	"member4"	"member3"
[16,]	"member4"	"member5"
[17,]	"member5"	"member1"
[18,]	"member5"	"member2"
[19,]	"member5"	"member3"
[20,]	"member5"	"member4"

President President

Combination (Unordered Sample)

Example: If you select 5 cards from a card deck of 54, you are typically only interested in the cards you have, not in the order in which you received them. How many different

combinations of 5 cards out of 54 are there?

In general, the number of subsets of size r that can be selected from n different objects is

$$ {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!} $$

Each of the ${}_nC_r$ unordered subsets in called a *combination of n objects taken r at a time.*

Binomial coefficients

- The number ${}_nC_r$ are also called **binomial coefficients**

$$(a+b)^n = (a+b)(a+b)\cdots(a+b)$$

Convert to math model:
We have n ball labeled by "a", *n* ball labeled by "b" and *n* boxes labeled by 1,…*n*. We randomly draw one ball each time and put it in one box so that each box only has exactly one ball. How many possible combinations for each?

		# combinations
We have all the black balls	b^n	1
We have one white ball and n-1 black balls	ab^{n-1}	${}_nC_1 = n$
….		
We have r white ball and n-r black ball	$a^r b^{n-r}$	${}_nC_r = \dfrac{n!}{r!(n-r)!}$

Binomial coefficients (Cont'd)

- Properties of binomial coefficients

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

$$\binom{n}{r} = \binom{n}{n\text{-}r} \quad \binom{n}{0} = \binom{n}{n} = 1 \quad \binom{n}{1} = \binom{n}{n\text{-}1} = n$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

- Pascal triangle (Homework 2)

Distinguishable and indistinguishable

Example 1:
How many ordered arrangements are there of the letters in the word STATISTICS?
10 letters in total, 1 A, 1 C, 2 I, 3 S and 3 T.

$$\binom{10}{1}\binom{9}{1}\binom{8}{2}\binom{6}{3}\binom{3}{3} = 50400$$

(unordered, distinguishable, without replacement)

Example 2:
We have 5 black balls, 6 blue balls, and 7 yellow balls, we put them into 18 boxes (one balls in each box). How many possible arrangements?

$$\binom{18}{5}\binom{13}{6}\binom{7}{7} = \frac{18!}{5!6!7!}$$

(unordered, distinguishable, without replacement)

Distinguishable and indistinguishable (Cont'd)

Definition. Given n objects with r of one type, and $n - r$ of another type, there are $_nC_r$ **distinguishable permutations** of the n objects.

- In general, given n objects, of which n_1 are of one type, n_2 are of second type … and n_k are of the last type, with $n = n_1 + n_2 + \ldots + n_k$.

  The **number of distinguishable permutations** of is given by:

$$\binom{n}{n_1\, n_2\, n_3 \ldots n_k} = \frac{n!}{n_1!\; n_2!\; n_3! \ldots n_k!}$$

Example

Suppose that of 100 applicants for a job 50 were women and 50 were men, all equally qualified. Further suppose that the company hired 2 women and 8 men. *How likely is this outcome under the assumption that the company does not discriminate?*

Solution: How many ways are there to choose

- *10 out of 100 applicants?*
- *2 out of 50 female applicants and 8 out of 50 male* applicants?
- Thus the chance of this event is ____________.

Summary

- The number of possibilities to sample with or without replacement in order or unordered r elements from a set of n distinct elements are summarized in the following table:

Sampling	in order	without order
without replacement	$\frac{n!}{(n-r)!}$	$\binom{n}{r}$
with replacement	n^r	$\binom{n+r-1}{r}$ ※

※ Sampling with replacement, without order is optional. Derivation NOT required!

Birthday Problem

What is the probability that at least two students have the same birthday in a classroom that has *n* students?

Solution: Let X be the number of students share the same birthday.

$P(X \geq 2) = 1 - P(X < 2)$

$P(X < 2) = P($ *nobody share the same birthday*$) = \dfrac{{}_{365}P_r}{365^r}$

$$P(X \geq 2) = 1 - \frac{365 \times 364 \times \cdots \times (365 - r + 1)}{365^r} = 1 - (1 - \frac{0}{365})(1 - \frac{1}{365}) \times \cdots \times (1 - \frac{r-1}{365})$$

$$= 1 - \exp(\sum_{i=1}^{r} \log(1 - \frac{i-1}{365})) \approx 1 - \exp(-\sum_{i=1}^{r} \frac{i-1}{365}) = 1 - \exp(-\frac{r(r-1)}{2 \times 365})$$

For r=57, $P(X \geq 2) \approx 0.99$

Here, we applied $\log(1 - x) = x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \cdots$