Chapter 1.3 Methods of Enumeration

- Multiplication principle.
- Permutation and combination.
- Sampling with/without replacement.
- Ordered/unordered sample.

Multiplication Principle

In general, if there are p experiments and the first has N_1 possible outcomes, the second N_2, ..., and the pth N_p possible outcomes, then there are a total of $N_1 \times N_2 \times \ldots N_p$ possible outcomes for the p experiments.

Example:
A DNA molecule is a sequence of four types of nucleotides, denoted by *A; G;C and T. For a molecule* 1 million (10^6) units long, how many different possible sequences?

Permutation

Example: Suppose that 3 positions are to be filled with 3 different objects (O1,O2,O3). How many possible arrangements are there?

In general, there are $n!$ ways to fill in n positions with n different objects, where

$$n! = n \times (n - 1) \times \ldots \times 2 \times 1; \; 0! = 1.$$

Definition: Each of the $n!$ arrangements of n different objects is called a **permutation** of the n objects.

${}_nP_r$

If only r positions are to be filled with objects selected from n different objects, $r \le n$, the number of possible ordered arrangements is

$${}_nP_r = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

- Each of the ${}_nP_r$ arrangements is called **a permutation of the n objects taken r at a time.**

Example

There are 5 seats in a classroom and 3 students registered for the class. How many seating arrangements do we have?

- Using Multiplication rule:
- E1: S1 has 5 choices ,
- E2: S2 has 4 choices,
- E3: S3 has 3 choices
- # of arrangements for composite events E1E2E3 is:
- $5 \times 4 \times 3 = 5!/2! = 5!/(5-3)! = 60$.

- There are only 3 seats in a classroom and 5 students are registered for the class. How many seating arrangements do we have?

With or without replacement

- **Sampling with replacement:** an object is selected and then replaced (put it back) before the next object is selected.

Example: In Illinois, license plates have seven numbers. How many distinct such plates are possible?

- **Sampling without replacement**: an object is NOT replaced (put it back) after it has been selected.

Example: How many four-letter code words are possible using the letters in HOPE if the letters may not be repeated.

Ordered or unordered sample

- If r objects are selected from a set of n objects, and *the order of selection is noted*, the selected set of r objects is called an **ordered sample of size r.**

Example: the number of ways of selecting a president, a vice president in a club which has 5 members.
20

- If the *order of the selection is not noted,* the selected set is called **unordered sample of size r.**

Example: the number of ways of selecting two presidents in a club which has 5 members.
20/2=10

President VP

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[1,] "member1" "member2"
[2,] "member1" "member3"
[3,] "member1" "member4"
[4,] "member1" "member5"
[5,] "member2" "member1"
[6,] "member2" "member3"
[7,] "member2" "member4"
[8,] "member2" "member5"
[9,] "member3" "member1"
[10,] "member3" "member2"
[11,] "member3" "member4"
[12,] "member3" "member5"
[13,] "member4" "member1"
[14,] "member4" "member2"
[15,] "member4" "member3"
[16,] "member4" "member5"
[17,] "member5" "member1"
[18,] "member5" "member2"
[19,] "member5" "member3"
[20,] "member5" "member4"
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President President

Combination (Unordered Sample)

Example: If you select 5 cards from a card deck of 54, you are typically only interested in the cards you have, not in the order in which you received them. How many different *combinations of 5 cards out of* 54 are there?

In general, the number of subsets of size r that can be selected from n different objects is

$$_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Each of the $_nC_r$ unordered subsets in called a *combination of n objects taken r at a time.*

Binomial coefficients

- The number ${}_nC_r$ are also called **binomial coefficients**

$$(a+b)^n = (a+b)(a+b)\cdots(a+b)$$

Convert to math model:
We have n ball labeled by "a", n ball labeled by "b" and n boxes labeled by 1,...n. We randomly draw one ball each time and put it in one box so that each box only has exactly one ball. How many possible combinations for each?

		# combinations
We have all the black balls	b^n	1
We have one white ball and n-1 black balls	ab^{n-1}	${}_nC_1 = n$
....		
We have r white ball and n-r black ball	$a^r b^{n-r}$	${}_nC_r = \frac{n!}{r!(n-r)!}$

Binomial coefficients (Cont'd)

- Properties of binomial coefficients

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

$$\binom{n}{r} = \binom{n}{n\text{-}r} \quad \binom{n}{0} = \binom{n}{n} = 1 \quad \binom{n}{1} = \binom{n}{n\text{-}1} = n$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

- Pascal triangle (Homework 2)

Distinguishable and indistinguishable

Example 1:
How many ordered arrangements are there of the letters in the word STATISTICS?
10 letters in total, 1 A, 1 C, 2 I, 3 S and 3 T.

$$\binom{10}{1}\binom{9}{1}\binom{8}{2}\binom{6}{3}\binom{3}{3} = 50400$$

(unordered, distinguishable, without replacement)

Example 2:
We have 5 black balls, 6 blue balls, and 7 yellow balls, we put them into 18 boxes (one balls in each box). How many possible arrangements?

$$\binom{18}{5}\binom{13}{6}\binom{7}{7} = \frac{18!}{5!6!7!}$$

(unordered, distinguishable, without replacement)

Distinguishable and indistinguishable (Cont'd)

Definition. Given n objects with r of one type, and $n - r$ of another type, there are ${}_nC_r$ **distinguishable permutations** of the n objects.

- In general, given n objects, of which n_1 are of one type, n_2 are of second type ... and n_k are of the last type, with $n = n_1 + n_2 + \ldots + n_k$.
The **number of distinguishable permutations** of is given by:

$$\binom{n}{n_1\, n_2\, n_3 \ldots n_k} = \frac{n!}{n_1!\; n_2!\; n_3! \ldots n_k!}$$

Example

Suppose that of 100 applicants for a job 50 were women and 50 were men, all equally qualified. Further suppose that the company hired 2 women and 8 men. *How likely is this outcome under the assumption that the company does not discriminate?*

Solution: How many ways are there to choose

- *10 out of 100 applicants?*
- *2 out of 50 female applicants and 8 out of 50 male* applicants?
- Thus the chance of this event is ____________.

Summary

- The number of possibilities to sample with or without replacement in order or unordered r elements from a set of n distinct elements are summarized in the following table:

Sampling	in order	without order
without replacement	$\frac{n!}{(n-r)!}$	$\binom{n}{r}$
with replacement	n^r	$\binom{n+r-1}{r}$※

※ Sampling with replacement, without order is optional. Derivation NOT required!

Birthday Problem

What is the probability that at least two students have the same birthday in a classroom that has *n* students?

Solution: Let X be the number of students share the same birthday.

$P(X \geq 2) = 1 - P(X < 2)$

$P(X < 2) =$ P(*nobody share the same birthday*) $= \dfrac{{}_{365}P_r}{365^r}$

$$P(X \geq 2) = 1 - \frac{365 \times 364 \times \cdots \times (365 - r + 1)}{365^r} = 1 - (1 - \frac{0}{365})(1 - \frac{1}{365}) \times \cdots \times (1 - \frac{r-1}{365})$$

$$= 1 - \exp(\sum_{i=1}^{r} \log(1 - \frac{i-1}{365})) \approx 1 - \exp(-\sum_{i=1}^{r} \frac{i-1}{365}) = 1 - \exp(-\frac{r(r-1)}{2 \times 365})$$

For r=57, $P(X \geq 2) \approx 0.99$

Here, we applied $\log(1 - x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$