STAT 400 / MATH 463 Fall 2011

Homework #1

(due Friday, September 2, by 3:00 p.m.)

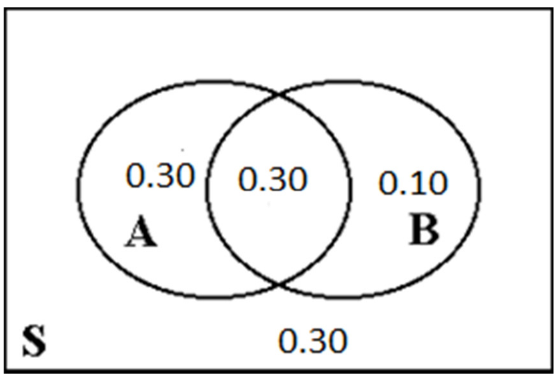
1. Suppose that $P(A) = 0.60$, $P(B) = 0.40$, $P(A \cap B) = 0.30$.
What is the probability that …

a) either A occurs or B occurs (or both);

b) B does not occur;

c) B occurs and A does not occur;

d) neither A nor B occurs;



a) $P(A \cup B) = \mathbf{0.70}$.

b) $P(B') = \mathbf{0.60}$.

c) $P(B \cap A') = \mathbf{0.10}$.

d) $P(A' \cap B') = \mathbf{0.30}$.

2. Suppose $P(A) = 0.6$, $P(B) = 0.5$, $P(C) = 0.4$,
$P(A \cap B) = 0.3$, $P(A \cap C) = 0.2$, $P(B \cap C) = 0.2$,
$P(A \cap B \cap C) = 0.1$. Find …

a) $P(A \cup B)$;

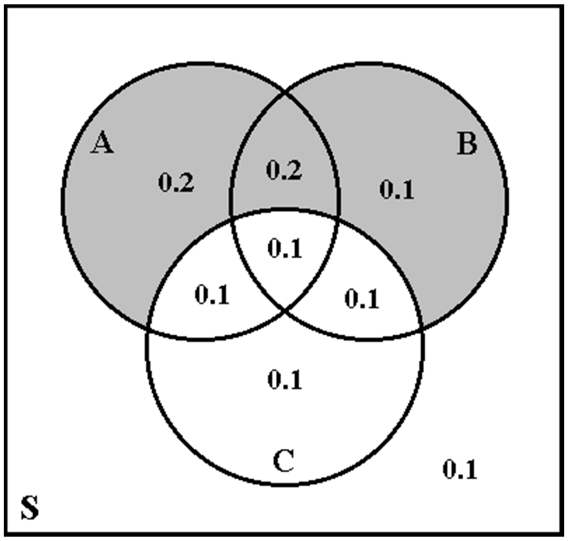
b) $P(B \cup C)$;

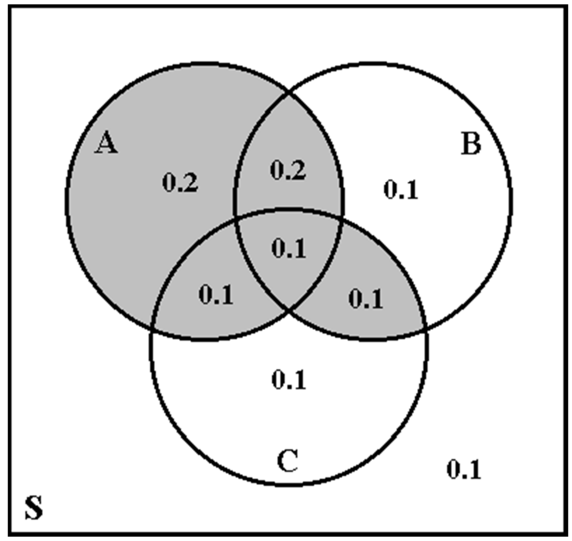
c) $P((A \cup B) \cap C')$;

d) $P(A \cup (B \cap C))$.

a) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6 + 0.5 - 0.3 = \mathbf{0.8}$.

b) $P(B \cup C) = P(B) + P(C) - P(B \cap C) = 0.5 + 0.4 - 0.2 = \mathbf{0.7}$.





c) $P((A \cup B) \cap C') = \mathbf{0.5}$.

d) $P(A \cup (B \cap C)) = \mathbf{0.7}$.

3. Suppose a baseball player steps to the plate with the intention of trying to "coax" a base on balls by never swinging at a pitch. The umpire, of course, will necessarily call each pitch either a ball (B) or a strike (S). What outcomes make up the event A, that a batter walks on the sixth pitch? Note: A batter "walks" if the fourth ball is called before the third strike.

S S B B B B	B S S B B B	B B S S B B
S B S B B B	B S B S B B	B B S B S B
S B B S B B	B S B B S B	B B B S S B
S B B B S B		

4. Consider a "thick" coin with three possible outcomes of a toss (Heads, Tails, and Edge) for which Heads and Tails are equally likely, but Edge is seven times less likely than Heads. What is the probability of Heads ?

$P(\text{Heads}) = P(\text{Tails}) = p$ for some p. $P(\text{Edge}) = p/7$.

$\mathrm{P}(\text{ Heads }) + \mathrm{P}(\text{ Tails }) + \mathrm{P}(\text{ Edge }) = 1.$

$p + p + p/7 = 1.$ $\qquad \dfrac{15}{7} p = 1.$ $\qquad \mathrm{P}(\text{ Heads }) = p = \mathbf{\dfrac{7}{15}}.$

5. Suppose S = { 0, 1, 2, 3, … } and

$$\mathrm{P}(0) = p, \qquad \mathrm{P}(k) = \frac{1}{5^k}, \quad k = 1, 2, 3, \ldots .$$

a) Find the value of p that would make this a valid probability model.

Must have $\mathrm{P}(0) + \mathrm{P}(1) + \mathrm{P}(2) + \mathrm{P}(3) + \mathrm{P}(4) + \ldots = 1.$

$$1 = p + \sum_{k=1}^{\infty} \frac{1}{5^k} = p + \sum_{k=1}^{\infty} \left(\frac{1}{5}\right)^k = p + \frac{1/5}{1 - 1/5} = p + \frac{1}{4}. \qquad p = \mathbf{\frac{3}{4}}.$$

b) Find $\mathrm{P}(\text{odd})$.

$$\mathrm{P}(1) + \mathrm{P}(3) + \mathrm{P}(5) + \mathrm{P}(7) + \ldots = \frac{1}{5^1} + \frac{1}{5^3} + \frac{1}{5^5} + \frac{1}{5^7} + \ldots = \sum_{k=0}^{\infty} \left(\frac{1}{5}\right)^{2k+1}$$

$$= \frac{1}{5} \cdot \sum_{k=0}^{\infty} \left(\frac{1}{25}\right)^k = \frac{1}{5} \cdot \frac{1}{1 - 1/25} = \mathbf{\frac{5}{24}}.$$

6. Suppose S = { 1, 2, 3, … } and $\mathrm{P}(k) = \dfrac{(\ln 2)^k}{k!}, \quad k = 1, 2, 3, \ldots .$

Is this is a valid probability model? *Justify your answer.*

- $p(x) \geq 0 \quad \forall x$ ✓

- $\sum_{\text{all } x} p(x) = 1$

$$\sum_{k=1}^{\infty} \frac{(\ln 2)^k}{k!} = \sum_{k=0}^{\infty} \frac{(\ln 2)^k}{k!} - 1 = e^{\ln 2} - 1 = 2 - 1 = 1. \qquad ✓$$

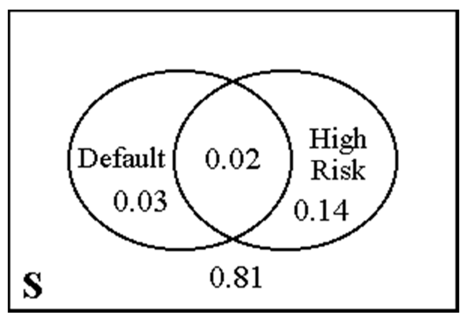
7. A bank classifies borrowers as "high risk" or "low risk," and 16% of its loans are made to those in the "high risk" category. Of all the bank's loans, 5% are in default. It is also known that 40% of the loans in default are to high-risk borrowers.

P(High risk) = 0.16, P(Default) = 0.05, P(High risk | Default) = 0.40.

a) What is the probability that a randomly selected loan is in default and issued to a high-risk borrower?

P(Default $\cap$ High risk) = P(Default) · P(High risk | Default) = 0.05 · 0.40 = **0.02**.

	High Risk	Low Risk	
Default	**0.02**	0.03	**0.05**
Default′	0.14	0.81	0.95
	0.16	0.84	1.00



b) What is the probability that a loan will default, given that it is issued to a high-risk borrower?

$$\text{P(Default} \mid \text{High risk)} = \frac{\text{P(Default} \cap \text{High risk)}}{\text{P(High risk)}} = \frac{0.02}{0.16} = \mathbf{0.125}.$$

c) What is the probability that a randomly selected loan is either in default or issued to a high-risk borrower, or both?

$$P(\text{Default} \cup \text{High risk}) = P(\text{Default}) + P(\text{High risk}) - P(\text{Default} \cap \text{High risk})$$

$$= 0.05 + 0.16 - 0.02 = \mathbf{0.19}.$$

d) A loan is being issued to a borrower who is not high-risk. What is the probability that this loan will default?

$$P(\text{ Default} \mid \text{High risk}') = \frac{P(\text{ Default} \cap \text{High risk}')}{P(\text{ High risk}')} = \frac{0.03}{0.84} \approx \mathbf{0.0357}.$$

8. A family that owns two automobiles is selected at random. Suppose that the probability that the older car is American is 0.70, the probability that the newer car is American is 0.50, and the probability that both the older and the newer cars are American is 0.40.

	New American	New Foreign	
Old American	**0.40**	0.30	**0.70**
Old Foreign	0.10	0.20	0.30
	0.50	0.50	1.00

a) Find the probability that at least one car is American (i.e. that either the older car or the newer car, or both cars are American).

$$P(\text{ OA} \cup \text{NA}) = P(\text{ OA}) + P(\text{ NA}) - P(\text{ OA} \cap \text{NA}) = 0.70 + 0.50 - 0.40 = \mathbf{0.80}.$$

b) Find the probability that neither car is American.

$$P(\text{ OF} \cap \text{NF}) = 1 - P(\text{ OA} \cup \text{NA}) = 1 - 0.80 = \mathbf{0.20}.$$

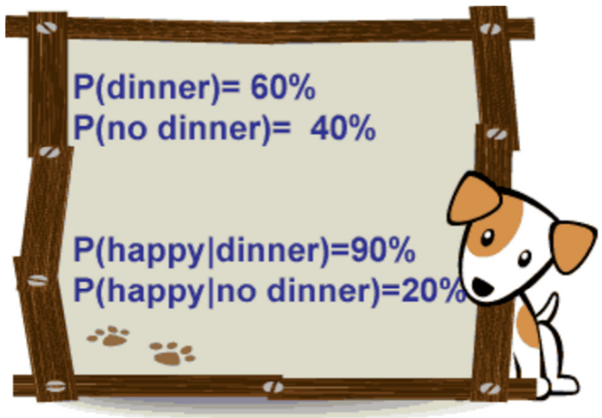
c) Suppose that the older car is American. What is the probability that the newer car is also American?

$$P(\text{ NA} \mid \text{OA}) = \frac{P(\text{ OA} \cap \text{NA})}{P(\text{ OA})} = \frac{0.40}{0.70} = \mathbf{{}^{4}/_{7}}.$$

d) What is the probability that the older car is American, given that the newer car is American?

$$P(OA \mid NA) = \frac{P(OA \cap NA)}{P(NA)} = \frac{0.40}{0.50} = \mathbf{0.80}.$$

9.



Find …

a) $P(\text{dinner} \mid \text{happy})$.

b) $P(\text{no dinner} \mid \text{not happy})$.

a) $$P(\text{dinner} \mid \text{happy}) = \frac{0.60 \times 0.90}{0.60 \times 0.90 + 0.40 \times 0.20} = \frac{0.54}{0.62} = \mathbf{\frac{27}{31}} \approx 0.871.$$

b) $$P(\text{no dinner} \mid \text{not happy}) = \frac{0.40 \times 0.80}{0.40 \times 0.80 + 0.60 \times 0.10} = \frac{0.32}{0.38} = \mathbf{\frac{16}{19}} \approx 0.842.$$

	happy	not happy	Total
dinner	0.54	0.06	0.60
no dinner	0.08	0.32	0.40
Total	0.62	0.38	1.00

From the textbook:

1.2-4 (1.2-2)

a)

H H H H	H T H H	T H H H	T T H H
H H H T	H T H T	T H H T	T T H T
H H T H	H T T H	T H T H	T T T H
H H T T	H T T T	T H T T	T T T T

b)

A	B	C	D
H H H H	H H T T	H H H H	H T T T
H H H T	H T H T	H H H T	T H T T
H H T H	H T T H	H T H H	T T H T
H T H H	H T T T	H T H T	T T T H
T H H H	T H H T	T H H H	
	T H T H	T H H T	
	T H T T	T T H H	
	T T H H	T T H T	
	T T H T		
	T T T H		
	T T T T		

(i) $P(A) = \mathbf{5/16}$

(ii) $A \cap B = \varnothing$ $\quad P(A \cap B) = \mathbf{0}$

(iii) $P(B) = \mathbf{11/16}$

(iv) $A \cap C = \{ HHHH, HHHT, HTHH, THHH \}$

$P(A \cap C) = \mathbf{4/16}$

(v) $P(D) = \mathbf{4/16}$

(vi) $P(A \cup C) = P(A) + P(C) - P(A \cap C) = 5/16 + 8/16 - 4/16 = \mathbf{9/16}$

(vii) $D \subset B$ $\qquad$ $B \cap D = D$ $\qquad$ $P(B \cap D) = P(D) = \mathbf{4/16}$

1.2-14 (1.2-12)

a) $\mathbf{1/3}$ $\qquad$ b) $\mathbf{2/3}$ $\qquad$ c) $\mathbf{0}$ $\qquad$ d) $\mathbf{1/2}$

1.2-16 (1.2-14)

a) $(1,2)$ $\quad (1,3)$ $\quad (1,4)$ $\quad (1,5)$ $\quad (2,3)$

$(2,4)$ $\quad (2,5)$ $\quad (3,4)$ $\quad (3,5)$ $\quad (4,5)$

b) (i) $\mathbf{1/10}$ $\qquad$ (ii) $\mathbf{5/10}$

1.4-4 (same for the 8th and the 7th editions)

(a) $P(H\,H) = \frac{13}{52} \cdot \frac{12}{51} = \mathbf{\frac{1}{17}}$;

(b) $P(H\,C) = \frac{13}{52} \cdot \frac{13}{51} = \mathbf{\frac{13}{204}}$;

(c) $P(\text{Non-Ace Heart, Ace}) + P(\text{Ace of Hearts, Non-Heart Ace})$

$$= \frac{12}{52} \cdot \frac{4}{51} + \frac{1}{52} \cdot \frac{3}{51} = \frac{51}{52 \cdot 51} = \mathbf{\frac{1}{52}}.$$

1.4-18

$$P(R_B) = P(R_A R_B) + P(W_A R_B) = \frac{3}{5} \cdot \frac{5}{8} + \frac{2}{5} \cdot \frac{4}{8} = \mathbf{\frac{23}{40}}.$$

1.6-2

(a) $$P(G) = P(A \cap G) + P(B \cap G)$$

$$= P(A)\,P(G \mid A) + P(B)\,P(G \mid B)$$

$$= (0.40)(0.85) + (0.60)(0.75) = \mathbf{0.79}.$$

(b) $$P(A \mid G) = \frac{P(A \cap G)}{P(G)}$$

$$= \frac{(0.40)(0.85)}{0.79} = \mathbf{0.43038}.$$