Examples for 10/10/2011

4.2 Covariance and Correlation Coefficient

Covariance of X and Y

$$\sigma_{XY} = \mathrm{Cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = E(XY) - \mu_X \mu_Y$$

(a) $\mathrm{Cov}(X,X) = \mathrm{Var}(X)$;

(b) $\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$;

(c) $\mathrm{Cov}(aX+b, Y) = a\,\mathrm{Cov}(X,Y)$;

(d) $\mathrm{Cov}(X+Y, W) = \mathrm{Cov}(X,W) + \mathrm{Cov}(Y,W)$.

$$\mathrm{Cov}(aX+bY, cX+dY)$$
$$= ac\,\mathrm{Var}(X) + (ad+bc)\,\mathrm{Cov}(X,Y) + bd\,\mathrm{Var}(Y).$$

$$\mathrm{Var}(aX+bY) = \mathrm{Cov}(aX+bY, aX+bY)$$
$$= a^2\,\mathrm{Var}(X) + 2ab\,\mathrm{Cov}(X,Y) + b^2\,\mathrm{Var}(Y).$$

0. Find in terms of σ_X^2, σ_Y^2, and σ_{XY}:

a) $\mathrm{Cov}(2X+3Y, X-2Y)$,

$$\mathrm{Cov}(2X+3Y, X-2Y) = 2\,\mathrm{Var}(X) - \mathrm{Cov}(X,Y) - 6\,\mathrm{Var}(Y).$$

b) $\mathrm{Var}(2X+3Y)$,

$$\mathrm{Var}(2X+3Y) = \mathrm{Cov}(2X+3Y, 2X+3Y)$$
$$= 4\,\mathrm{Var}(X) + 12\,\mathrm{Cov}(X,Y) + 9\,\mathrm{Var}(Y).$$

c) $\mathrm{Var}(X-2Y)$.

$$\mathrm{Var}(X-2Y) = \mathrm{Cov}(X-2Y, X-2Y)$$
$$= \mathrm{Var}(X) - 4\,\mathrm{Cov}(X,Y) + 4\,\mathrm{Var}(Y).$$

Correlation coefficient of X and Y

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)} \cdot \sqrt{\mathrm{Var}(Y)}} = E\left[\left(\frac{X-\mu_X}{\sigma_X}\right), \left(\frac{Y-\mu_Y}{\sigma_Y}\right)\right]$$

(a) $-1 \le \rho_{XY} \le 1$;

(b) ρ_{XY} is either $+1$ or -1 if and only if X and Y are linear functions of one another.

If random variables X and Y are independent, then

$$E(g(X) \cdot h(Y)) = E(g(X)) \cdot E(h(Y)).$$

$$\Rightarrow \qquad \mathrm{Cov}(X, Y) = \sigma_{XY} = 0, \qquad \mathrm{Corr}(X, Y) = \rho_{XY} = 0.$$

2. Consider the following joint probability distribution $p(x, y)$ of two random variables X and Y:

	y			
x	0	1	2	$p_X(x)$
1	0.15	0.10	0	0.25
2	0.25	0.30	0.20	0.75
$p_Y(y)$	0.40	0.40	0.20	1.00

Recall:

$E(X) = 1.75$,

$E(Y) = 0.8$,

$E(XY) = 1.5$.

Find $\mathrm{Cov}(X, Y) = \sigma_{XY}$ and $\mathrm{Corr}(X, Y) = \rho_{XY}$.

$\mathrm{Cov}(X, Y) = \sigma_{XY} = 1.5 - 1.75 \cdot 0.8 = \mathbf{0.10}$.

$E(X^2) = 1 \times 0.25 + 4 \times 0.75 = 3.25$.

$\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = 3.25 - 1.75^2 = 0.1875$.

$E(Y^2) = 0 \times 0.40 + 1 \times 0.40 + 4 \times 0.20 = 1.2$.

$\mathrm{Var}(Y) = E(Y^2) - [E(Y)]^2 = 1.2 - 0.8^2 = 0.56$.

$$\mathrm{Corr}(X, Y) = \rho_{XY} = \frac{0.10}{\sqrt{0.1875} \cdot \sqrt{0.56}} \approx \mathbf{0.3086}.$$

1. Let the joint probability density function for (X, Y) be

$$f(x,y)=\begin{cases} 60\,x^2\,y & 0 \le x \le 1,\ 0 \le y \le 1,\ x+y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $\operatorname{Cov}(X,Y) = \sigma_{XY}$ and $\operatorname{Corr}(X,Y) = \rho_{XY}$.

Recall: $f_X(x) = 30\,x^2\,(1-x)^2$, $0 < x < 1$, $E(X) = \frac{1}{2}$,

$f_Y(y) = 20\,y\,(1-y)^3$, $0 < y < 1$, $E(Y) = \frac{1}{3}$, $E(XY) = \frac{1}{7}$.

$\operatorname{Cov}(X,Y) = \frac{1}{7} - \frac{1}{2}\cdot\frac{1}{3} = -\mathbf{\frac{1}{42}}$. $\operatorname{Var}(X) = \frac{9}{252}$. $\operatorname{Var}(Y) = \frac{8}{252}$.

$$\rho_{XY} = \frac{-1/42}{\sqrt{9/252}\cdot\sqrt{8/252}} = -\frac{\mathbf{1}}{\sqrt{\mathbf{2}}} = -\frac{\sqrt{\mathbf{2}}}{\mathbf{2}} \approx \mathbf{-0.7071}.$$

4. Let the joint probability density function for (X, Y) be

$$f(x,y)=\begin{cases} 12\,x\,(1-x)\,e^{-2y} & 0 \le x \le 1,\ y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Find $\operatorname{Cov}(X,Y) = \sigma_{XY}$ and $\operatorname{Corr}(X,Y) = \rho_{XY}$.

Recall: $f_X(x) = 6\,x\,(1-x)$, $0 < x < 1$. $f_Y(y) = 2\,e^{-2y}$, $y > 0$.

Since X and Y are independent,

$\operatorname{Cov}(X,Y) = \sigma_{XY} = \mathbf{0}$, $\operatorname{Corr}(X,Y) = \rho_{XY} = \mathbf{0}$.

3. Let the joint probability density function for (X, Y) be

$$f(x,y)=\begin{cases} x+y & 0\le x\le 1, 0\le y\le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $\text{Cov}(X,Y) = \sigma_{XY}$ and $\text{Corr}(X,Y) = \rho_{XY}$.

Recall: $f_X(x) = x+\frac{1}{2}, \quad 0<x<1.$ $\qquad f_Y(y) = y+\frac{1}{2}, \quad 0<y<1.$

$$\mu_X = \int_0^1 x\left(x+\frac{1}{2}\right)\,dx = \left[\frac{1}{3}x^3+\frac{1}{4}x^2\right]_0^1 = \frac{7}{12};$$

$$\mu_Y = \int_0^1 y\left(y+\frac{1}{2}\right)\,dy = \frac{7}{12};$$

$$E(X^2) = \int_0^1 x^2\left(x+\frac{1}{2}\right)\,dx = \left[\frac{1}{4}x^4+\frac{1}{6}x^3\right]_0^1 = \frac{5}{12},$$

$$\sigma_X^2 = E(X^2)-\mu_X^2 = \frac{5}{12}-\left(\frac{7}{12}\right)^2 = \frac{11}{144}.$$

Similarly, $\sigma_Y^2 = \dfrac{11}{144}$.

$$E(XY) = \int_0^1\int_0^1 xy(x+y)\,dx\,dy = \int_0^1\int_0^1 \left(x^2y+xy^2\right)dx\,dy$$

$$= \int_0^1\left(\frac{y}{3}+\frac{y^2}{2}\right)dy = \frac{1}{3}.$$

$$\text{Cov}(X,Y) = \frac{1}{3}-\frac{7}{12}\cdot\frac{7}{12} = -\mathbf{\frac{1}{144}}. \qquad \rho_{XY} = \frac{-1/144}{\sqrt{11/144}\cdot\sqrt{11/144}} = -\mathbf{\frac{1}{11}}.$$