## Review (3.4)

- Uniform distribution, its mean, variance, m.g.f.
- Exponential distribution and its mean, variance.
- Memoryless properties of exponential distribution.

## Today's Lecture (3.7, 3.5)

- Comparison between discrete and continuous random variables
- Cauchy distribution, GOT A MOMENT?
- Gamma distribution and its connection to Poisson Process
- Mean, variance and m.g.f. of Gamma distribution.
- $\chi^2$ distribution.

random variables

**discrete**

probability **mass** function

p.m.f.

$$p(x) = \mathrm{P}(\mathrm{X} = x)$$

$$\forall x \quad 0 \le p(x) \le 1$$

$$\sum_{\text{all } x} p(x) = 1$$

**continuous**

probability **density** function

p.d.f.

$$f(x)$$

$$\forall x \quad f(x) \ge 0$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

cumulative distribution function

c.d.f.

$$\mathrm{F}(x) = \mathrm{P}(\mathrm{X} \le x)$$

$$\mathrm{F}(x) = \sum_{y \le x} p(y) \qquad\qquad \mathrm{F}(x) = \int_{-\infty}^{x} f(y)\,dy$$

expected value

$$\mathrm{E}(\mathrm{X}) = \mu_{\mathrm{X}}$$

**discrete**

If $\sum_{\text{all } x} |x| \cdot p(x) < \infty$,

$$\mathrm{E}(\mathrm{X}) = \sum_{\text{all } x} x \cdot p(x)$$

**continuous**

If $\int_{-\infty}^{\infty} |x| \cdot f(x)\,dx < \infty$,

$$\mathrm{E}(\mathrm{X}) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

**discrete**

If $\sum_{\text{all } x} |g(x)| \cdot p(x) < \infty$,

$$\mathrm{E}(g(\mathrm{X})) = \sum_{\text{all } x} g(x) \cdot p(x)$$

**continuous**

If $\int_{-\infty}^{\infty} |g(x)| \cdot f(x)\,dx < \infty$,

$$\mathrm{E}(g(\mathrm{X})) = \int_{-\infty}^{\infty} g(x) \cdot f(x)\,dx$$

## variance

$$\mathrm{Var}(X) = \sigma_X^2 = E([X-\mu_X]^2) = E(X^2) - [E(X)]^2$$

discrete

$$\mathrm{Var}(X) = \sum_{\text{all } x} (x-\mu_X)^2 \cdot p(x)$$

$$= \sum_{\text{all } x} x^2 \cdot p(x) - [E(X)]^2$$

continuous

$$\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x-\mu_X)^2 \cdot f(x)\,dx$$

$$= \left[\int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx\right] - [E(X)]^2$$

## moment-generating function

$$M_X(t) = E(e^{tX})$$

discrete

$$M_X(t) = \sum_{\text{all } x} e^{tx} \cdot p(x)$$

continuous

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)dx$$

<u>Example</u>

(Standard) Cauchy distribution: $f_X(x) = \dfrac{1}{\pi\left(1+x^2\right)}, \quad -\infty < x < \infty.$

Even though $f_X(x)$ is symmetric about zero, $\mathrm{E}(\mathrm{X})$ is undefined since

$$\int_{-\infty}^{\infty} |x| \cdot \frac{1}{\pi\left(1+x^2\right)} dx = \infty.$$

$$\mathrm{F}_X(x) = \int_{-\infty}^{x} \frac{1}{\pi\left(1+y^2\right)} dy = \frac{1}{\pi}\arctan(x) + \frac{1}{2}, \qquad -\infty < x < \infty.$$

$\mathrm{M}_X(0) = 1.$ $\mathrm{M}_X(t)$ is undefined for all $t \neq 0$.

## Volcano eruption (again)

Let $N_t$ be the number of volcano eruptions to have occurred by time $t$, starting from now. Suppose that the volcano eruption forms a Poisson process with rate $\lambda$. Then $N_t \sim$ ____________.

Let $X$ be the waiting time until the 4-th volcano eruption occurs and find the distribution of $X$.

$$F_X(x) \ = \ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

In general, if $X$ is the waiting time until the $\alpha$-th volcano eruption, then

$$F_X(x) = 1 - \sum_{k=0}^{\alpha-1} \frac{(\lambda x)^k e^{-\lambda x}}{k!}, \ x > 0$$

and

$$f_X(x) \ = \ F'_X(x) = \frac{\lambda^\alpha x^{\alpha-1}}{(\alpha-1)!} e^{-\lambda x}, \quad x > 0$$

## Gamma function

$$\Gamma(t) = \int_0^\infty y^{t-1} e^{-y} dy, \quad t > 0$$

Properties of $\Gamma(t)$:

- $\Gamma(t) = (t-1)\Gamma(t-1)$, $t > 1$.
- $\Gamma(1) = \int_0^\infty e^{-y} dy = 1$.
- When $t = n$, a positive integer,

$$\Gamma(n) = \Gamma(n-1)(n-1) = \cdots = (n-1)!$$

## Gamma distribution

Definition: The random variable $X$ has a gamma distribution if its p.d.f. is defined by

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}, \quad 0 \leq x < \infty$$

Write Gamma$(\alpha, \theta)$, where $\theta = 1/\lambda$, $\alpha > 0$ (not necessarily an integer).

$$\begin{aligned} M(t) &= \frac{1}{(1-\theta t)^\alpha}, \ t < 1/\theta \\ \mu &= M'(0) = \alpha\theta \\ \sigma^2 &= M''(0) - [M'(0)]^2 = \alpha(\alpha+1)\theta^2 - \alpha^2\theta^2 = \alpha\theta^2 \end{aligned}$$

## Example

Telephone calls enter a college switchboard at a mean rate of $1/2$ call per minute according to a Poisson process. Let $X$ denote the waiting time until the second call arrives.

- What is the distribution of $X$?

- What is the average waiting time?

- What is the probability that the waiting time is longer than $3$ minutes?

# Chi-square distribution

Gamma distribution:

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-x/\theta}, \quad 0 \le x < \infty$$

Chi-square distribution: $\theta = 2$, $\alpha = r/2$, r is a positive integer.

$$f(x) = \frac{1}{\Gamma(r/2)2^{r/2}} x^{r/2-1} e^{-x/2}, \quad 0 \le x < \infty$$

$r$: degree of freedom. Write $X \sim \chi^2(r)$.

- Mean: $\mu = \alpha\theta = (r/2)2 = r$
- Variance: $\sigma^2 = \alpha\theta^2 = (r/2)2^2 = 2r$.
- m.g.f. $M(t) = (1-2t)^{-r/2}$, $t < 1/2$.

Plot: