Review(2.1 Methods of Enumeration)

- Random variable, space.
- Discrete random variable, probability mass function and its properties.
- Bar graph, probability histogram.
- Uniform distribution, Hypergeometric distribution

Hypergeometric distribution

Example: Consider the urn model. Suppose we have 4 red balls and 6 blue balls in the urn. The balls are well mixed. Let X be the number of red balls drawn when taking 5 balls without replacement.

- What is S, the space of X?

$$S = \{x : x = 0, 1, 2, 3, 4\}$$

- What is $P(X = x)$, $x \in S$?

$$P(X=x) = \frac{\binom{4}{x}\binom{6}{5-x}}{\binom{10}{5}}$$

Suppose that an urn contains $N = N_1 + N_2$ balls, of which N_1 are red and N_2 are blue. Let X be the number of red balls drawn when taking n balls without replacement. Then

$$P(X = x) = \frac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N}{n}}$$

where $x \leq n$, $x \leq N_1$ and $n - x \leq N_2$.

Example

A lot of 200 semiconductor chips is inspected by selecting five at random and without replacement. If at least one of the five is defective, the lot is rejected. Find the probability of rejecting the lot if in the 200 chips, 10 are defective. (0.228)

Let X = # of defective chips

$N_1 = 10$

$N_2 = 190$ $\quad n = 5$

$$P(X \geq 1) = 1 - P(X=0)$$
$$= 1 - \frac{\binom{10}{0}\binom{190}{5}}{\binom{200}{5}} = 0.228$$

Today's Lecture(2.2,2.3)

- Mathematical expectation (mean,variance,standard deviation, moment)
- Properties of mathematical expectation (mean,variance)
- Sample mean, variance
- Discrete random variables

Example

Thirty-seven states, five Canadian provinces, and most European countries have government-sponsored lotteries. Here is a simple lottery wager, the Tri-State Pick 3 game that New Hampshire shares with Maine and Vermont. You choose a three-digit number, the state chooses a three-digit winning number at random and pays you \$500 if your number is chosen. Because there are 1000 three-digit numbers, you have probability $1/1000$ of winning. Taking X to be amount you win, the probability distribution of X is

outcome	\$0	\$500
Probability	0.999	0.001

What are your average winnings?

$0 \times 0.999 + 500 \times 0.001 = 0.5$

Mean of X

For a random variable X, which has space

$$S = \{u_1, u_2, \cdots, u_k\}$$

All these points have respective probabilities $P(X = u_i) = f(u_i) > 0$, where $f(x)$ is the p.m.f. The mean of X is defined as

$$\begin{aligned} \mu = \mu_X = E[X] &= \sum_{x \in S} x f(x) \\ &= u_1 f(u_1) + u_2 f(u_2) + \cdots + u_k f(u_k) \end{aligned}$$

Example: Roll one die. Let X be outcome of rolling one die. The p.m.f. is

$$P(X = x) = \frac{1}{6}, \quad x = 1, \cdots, 6$$

and hence

$$E(X) = \underline{\ 7/2\ }$$

Note: Mean is a measure of the middle (center) of the distribution.

Mathematical Expectation

Definition: If $f(x)$ is p.m.f. of the random variable X of the discrete type with space S and if the summation $\Sigma_{x \in S} u(x)f(x)$ exists, then the sum is called the *Mathematical Expectation* or the expected value of the function $u(X)$, and it is denoted by $E[u(X)]$, i.e.

$$E[u(X)] = \sum_{x \in S} u(x)f(x)$$

- A function of random variable is still a random variable, so $u(X)$ is a random variable.
- Mean of X corresponds to the special case $u(X) = X$.

Example: Let the random variable X has p.m.f.

$$f(x) = \begin{cases} 1/4 & x = -1 \\ 1/4 & x = 0 \\ 1/2 & x = 1 \end{cases}$$

- $E[X^3]$

X	$f(X)$	X^3
-1	$\frac{1}{4}$	-1
0	$\frac{1}{4}$	0
1	$\frac{1}{2}$	1

$E X^3 = (-1) \times \frac{1}{4} + 0 \times \frac{1}{4} + 1 \times \frac{1}{2} = \frac{1}{4}.$

Properties of Mathematical Expectation

- If c is a constant, $E(c) = c$.
- If c is a constant and u is a function,

$$E[cu(X)] = cE[u(X)]$$

- If c_1 and c_2 are constants and u_1 and u_2 are functions, then

linearity

$$E[c_1u_1(X) + c_2u_2(X)] = c_1E[u_1(X)] + c_2E[u_2(X)]$$

General version: $E[\Sigma_{i=1}^{k} c_iu_i(X)] = \Sigma_{i=1}^{k} c_iE[u_i(X)]$.

Properties of Mean

Let $g(b) = E[(X - b)^2]$, $b = \mu$ minimizes $g(b)$.

$$g(b) = E(X^2 - 2bX + b^2)$$
$$= EX^2 - 2b \cdot EX + b^2$$

$$g'(b) = 0$$
$$= -2EX + 2b$$
$$\Rightarrow b = EX = \mu$$

Variance of a Discrete R.V.

The *variance* of the r.v. X is defined as:

$$\sigma_X^2 = \sigma^2 = \mathsf{Var}(X) = E(X-\mu)^2 = E(X^2) - \mu^2.$$

Suppose X is a discrete random variable with space $S = \{u_1, u_2, \cdots, u_k\}$. The variance of X can be written as

$$\begin{aligned}\sigma_X^2 = \mathsf{Var}(X) &= \sum_{i=1}^{k} f(u_i)(u_i - \mu_X)^2 \\ &= \sum_{i=1}^{k} u_i^2 f(u_i) - \mu_X^2\end{aligned}$$

Definition: The *standard deviation* of X: $\sigma_X = \sqrt{\mathsf{Var}(X)}$.

Moment

Definition: Let r be a positive integer. If

$$E[X^r] = \sum_{x \in S} x^r f(x)$$

is finite, it is called the r-th moment of the distribution about the origin. Further,

$$E[(X-b)^r] = \sum_{x \in S} (x-b)^r f(x)$$

is called the r-th moment of the distribution about b.

$$\begin{cases} 1+2+\cdots+n = n(n+1)/2 \\ 1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6} \end{cases}$$

Example

Roll one fair die and X is the number of spots on the side that is up.

- $E(X) = \frac{7}{2}$
- $E(X^2) = \frac{1}{6}(1^2+2^2+3^2+4^2+5^2+6^2)$
  $= \frac{1}{6} \times \frac{6 \times 7 \times 13}{6} = \frac{91}{6}$
- $\text{Var}(X) = EX^2 - (EX)^2$
  $= 91/6 - (7/2)^2 = 2.917$
- $\sigma_X = \sqrt{\text{Var}(X)} = \sqrt{2.917} = 1.708$

Properties of Variances

- For constants a and b,

$$\text{Var}(aX+b) = a^2\text{Var}(X)$$

$$\sigma_{aX+b} = |a|\sigma_X$$

Note: Standard deviation measures the dispersion or spread of the points belonging to the space S.

Sample mean and sample variance

- Random experiments n times, observe $x_1, x_2, \cdots, x_n$.
- Sample: $x_1, x_2, \cdots, x_n$.
- Empirical distribution: $1/n$ mass on x_i, $i = 1, \cdots, n$.
- Mean of Empirical distribution (sample mean):

$$\bar{x} = \sum_{i=1}^{n} x_i \frac{1}{n}$$

- Variance of Empirical distribution:

$$v = \sum_{i=1}^{n} (x_i - \bar{x})^2 \frac{1}{n} = \frac{1}{n} \sum_{i=1}^{n} x_i^2 - \bar{x}^2$$

  Sample variance:

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

  Sample standard deviation: $s = \sqrt{s^2}$ measures the dispersion of the data.

Example: Roll a die 5 times and we get the sample

$$x_1 = 3, x_2 = 4, x_3 = 5, x_4 = 2, x_5 = 1$$

- $\bar{x} =$

- $s^2 =$

Discrete random variable

- *Bernoulli distribution*

Definition: A random experiment is called a set of Bernoulli trials if it consists of several trials such that

- Each trial has only 2 possible outcomes (usually called "Success" and "Failure").
- The probability of success p, remains constant for all trials;
- The trials are independent, i.e. the event "success in trial i" does not depend on the outcome of any other trials.

Examples: Repeated tossing of a fair die: success="6", failure="not 6". Each toss is a Bernoulli trial with

$$P(\text{success}) = \underline{\quad 1/6 \quad}$$

Definition: The random variable X is called a **Bernoulli random variable** if it takes only 2 values, 0 and 1.

The p.m.f. $f_X(x) = \begin{cases} p & \text{if } x = 1 \\ 1-p & \text{if } x = 0 \end{cases}$

- $E(X) = p$
- $Var(X) = p^2 - p$

- *Geometric distribution* : Consider the urn model. Suppose we have 10 red balls and 20 blue balls in the urn. The balls are well mixed. Draw one ball each time, record the color and then put it back (with replacement).
  - On each draw, a red ball occurs with probability $p = 1/3$.
  - Let X be the number of times up to the first red ball.

  What is the distribution of X?
  - Probability of no red balls in $x-1$ draws: $(2/3)^{x-1}$.
  - Probability of red ball in the x-th draw: $1/3$.

  The p.m.f. of X is
  $P(X = x) = \frac{1}{3}(2/3)^{x-1}, \quad x = 1, 2, 3, \cdots$
  X is geometrically distributed with parameter $p = 1/3$.

- *Negative Binomial distribution*: Consider the same urn model as in Geometric example. Draw one ball each time, record the color and then put it back (with replacement). Repeat this process until r red balls are picked up. Let X be the number of trials. Then

$$P(X = x) = \binom{x-1}{r-1} \left(\frac{2}{3}\right)^{x-r} \left(\frac{1}{3}\right)^{r}$$

for $x = r, r+1, r+2, \cdots$. The random variable X has a negative binomial distribution with parameter (r, p), where $p = 1/3$.

Logic: For general p, any particular such sequence has probability $p^r(1-p)^{k-r}$, from the independence assumption. The last draw is a red ball, and the remaining $r-1$ red balls can be assigned to the remaining $x-1$ draws in $\binom{x-1}{r-1}$ ways.