Karlsruhe Institute of Technology

Lecture notes
Theoretische Teilchenphysik I

von

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Contents

1. Introduction

1.1 Quarks and Leptons

Particles of matter:

- electrons (e^-) and other leptons are elementary particles.
- protons and neutrons ($|p\rangle = |uud\rangle$, $|n\rangle = |udd\rangle$) are combinations of elementary quarks and gluons. The binding energy of the quarks is very large in comparison to the absolute energy of the proton and neutron ($m_p c^2 = 938$ MeV) if you compare this to the binding energies of Atoms (~ 1 Ry) and their absolute energies ($\sim 10^9$ Ry). Because the proton and the neutron are similar/symmetric in the strong interaction (not in the electroweak interaction though) we can combine them into a isospin dublett $\begin{pmatrix} p \\ n \end{pmatrix}$.

There are many more particles/boundstates of quarks and gluons for different combination of quarks. Another example are the Δ baryons. These are spin $\frac{3}{2}$ particles and have masses of $m_\Delta c^2 \approx 1230$ MeV:

- Δ^- : $|ddd\rangle$
- Δ^0 : $|ddu\rangle$
- Δ^+ : $|duu\rangle$
- Δ^{++} : $|uuu\rangle$

Because the Δ baryons are spin $\frac{3}{2}$ particles all of the quarks spins must be aligned, so the spin wavefunction is symmetric. Also the orbital wavefunction is symmetric for the Δ^{++} baryon because it consists of thrice the same quark. However, the total wavefunction of the baryons must be antisymmetric because it is a fermion.
This is the reason a color charge was introduced to interpret the Dirac statistics correctly and characterize the strong interaction with a new quantum number.
Alltogether one can describe the four Δ baryons in an isospin quartet with $I = \frac{3}{2}$.

Another group of particles are the mesons, they consist of one quark and an anti quark. The lightest examples are the pions:

- π^+ : $|u\bar{d}\rangle$
- π^0 : $\frac{1}{\sqrt{2}}\left(|u\bar{u}\rangle - |d\bar{d}\rangle\right)$
- π^- : $|d\bar{u}\rangle$

They have masses of $m_\pi c^2 \approx 140$ MeV and are spin 0 particles. Together they form the isospin triplet $I = 1$.

Another group of mesons with spin 0 are the kaons. These have another type of quark, the strange quark. For this new quark a new quantum number (next to isospin) was introduced, the strangeness.
We can summarize the kaons and the pions in a meson octett depicted in figure 1.1. The kaons have masses of $m_K c^2 \approx 495$ MeV. Additionally to the four kaons and the three pions there is an η meson with the same strangeness and isospin as the π^0. It is like the π^0 but has additional strange quarks: $|\eta\rangle = \frac{1}{\sqrt{6}}\left(|u\bar{u}\rangle + |d\bar{d}\rangle - 2|s\bar{s}\rangle\right)$.

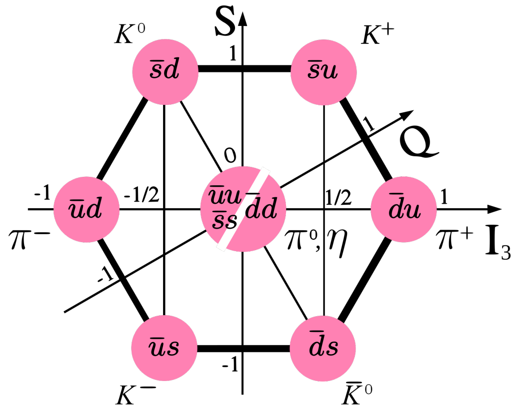

Figure 1.1: meson octett for spin 0

The quarks and leptons are probably the fundamental layer of particles; mesons and baryons are complex bound states described through nuclear physics. The quarks and leptons are described by the dirac equation

$$\left(i\partial\!\!\!/ - \frac{mc}{\hbar}\right)\psi = 0 + \text{interactions}$$

One interaction is for example the electromagnetism: $\partial_\mu \to \partial_\mu + iqA_\mu$

In table 1.1 and 1.2 all quarks and leptons are summarized with their electric charge and mass. The charge is given in units of elementary charge as $q = Q \cdot e$.

quark	mc^2 [MeV]	Q
u	$2.2^{+0.6}_{-0.7}$	$+\frac{2}{3}$
d	4.7	$-\frac{1}{3}$
c	1270	$+\frac{2}{3}$
s	96	$-\frac{1}{3}$
t	173200	$+\frac{2}{3}$
b	4180	$-\frac{1}{3}$

Table 1.1: quarks

lepton	mc^2 [MeV]	Q
e^-	0.511	-1
μ^-	105.66	-1
τ^-	1777	-1
ν_e		0
ν_μ		0
ν_τ		0

Table 1.2: leptons

It is important, that the down quark is slightly heavier than the up quark; because then the down quark more likely decays into the up quark than vice versa and therefore the proton is much more stable than the neutron. This way also atoms remain stable and charged.

These leptons and quarks are all known matter fields save the bosons:

- Higgs boson H

- γ, $W^\pm$, Z which are carriers of the electromagnetic and weak force
- gluon g which is the carrier of the strong force

Additionaly it is known, from observing the Higgs coupling, that there are no more generations of quarks which behave similarly to the three existing generations. Additional generations might exist but must behave fundamentally different.

1.2 Course Contents

In this course of theoretical particle physics the following topics will be discussed:

- theoretical description of interactions of quarks and leptons
  $\rightarrow$ gauge theories (*Eichtheorien*)
- pair production of particles and γ, $W^\pm$, Z, g emission
  $\rightarrow$ changing particle number and content
  $\rightarrow$ quantum field theory (QFT) which is relativistic for particle physics
- development of pertubation theory for QFT
- calculation of cross sections and decay rates
- symmetries: Lorentz invariance and internal symmetries like isospin and color

1.3 Natural Units

In particle physics it is not practical to use the usual units. It is much more practicable to factor out constants like ϵ_0, $\hbar$, c and k_B such that the remaining quantities have dimensions of energy to a power.
The unit of energy will be electron volts (eV). In table 1.3 some important quantities and their dimensions in natural units are shown.

quantity	SI units	natural units	dimension
velocity	$\tilde{v}$	$v \cdot c$	$[v] = 1$
length	$\tilde{L}$	$L \cdot \hbar c$	$[L] = 1/\text{MeV}$
time	$\tilde{t}$	$t \cdot \hbar$	$[t] = 1/\text{MeV}$
electric field	$\tilde{E}$	$\frac{1}{\sqrt{\epsilon_0 (\hbar c)^3}} \vec{E}$	$[\vec{E}] = \text{MeV}^2$
magnetic field	$\tilde{B}$	$\frac{1}{\sqrt{\epsilon_0 c^2 (\hbar c)^3}} \vec{B}$	$[\vec{B}] = \text{MeV}^2$

Table 1.3: natural units

An example of the simplification is the Hamiltionan for radiation:

$$H_{rad} = \frac{\epsilon_0}{2} \int \mathrm{d}^3 \tilde{\vec{x}} \left[\tilde{\vec{B}}^2 + c^2 \tilde{\vec{E}}^2 \right] \rightarrow \frac{1}{2} \int \mathrm{d}^3 \vec{x} \left[\vec{B}^2 + \vec{E}^2 \right]$$

Another useful thing are translations from the normal system to the natural units and vice versa. For example:

- $\hbar c = 197$ MeVfm
- $\frac{1}{\text{GeV}^2} = \frac{3.89 \cdot 10^{-4}\ \text{b}}{(\hbar c)^2}$ where a barn is 10^{-28} m^2
- $\tilde{e} = 1.6 \cdot 10^{-19}\ \text{C} \quad \rightarrow \quad e = \frac{\tilde{e}}{\sqrt{\epsilon_0 \hbar c}} = \sqrt{4\pi\alpha} = 0.3028$

1.3.1 Klein Gordon and Dirac Equations in Natural Units

The Klein-Gordon equation in SI units is given as

$$\left[\tilde{\Box} + \left(\frac{mc}{\hbar}\right)^2\right]\phi(\tilde{x}) = 0$$

where x is a four vector $\tilde{x}^\mu = (x\tilde{t}, \vec{\tilde{x}}) = \hbar c(t, \vec{x})$. Also the d'Alembert operator in SI units is given as

$$\tilde{\Box} = \frac{1}{c^2}\frac{\partial^2}{\partial\tilde{t}^2} - \vec{\tilde{\nabla}}^2 = \frac{1}{(\hbar c)^2}\Box = \frac{1}{(\hbar c)^2}\left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)$$

So in natural units the equation simplifies to

$$(\Box + m^2)\phi(x) = 0$$

Similarly the Dirac equation simplifies when using natural units

$$\left(i\gamma^\mu\tilde{\partial}_\mu - \frac{mc}{\hbar}\right)\psi(\tilde{x}) = 0 \quad \rightarrow \quad (i\gamma^\mu\partial_\mu - m)\psi(x) = 0$$

1.4 Lagrange Density and Equations of Motion

1.4.1 Lagrangian Field Theory

First we take a look at a classical point particle. Its trajectory is given by $x_i(t)$ for $i = 1, 2, 3$. For this particle we can define an action

$$\mathcal{S}([x_i], t_1, t_2) = \int_{t_1}^{t_2} \mathrm{d}t \left(\frac{1}{2}m\left(\sum_i \frac{\mathrm{d}x_i}{\mathrm{d}t}\right)^2 - V(x_i(t))\right)$$

The action is a functional of the trajectory. Now we can find an extremum of $\mathcal{S}$ for the classical path by adding a inifinitesimal variation δx_i to the trajectory: $x_i(t) + \delta x_i(t)$. Then, the extremal condition is given by $\Delta S = 0$ where ΔS is given by

$$\Delta\mathcal{S} = \mathcal{S}([x_i + \delta x_i]) - \mathcal{S}([x_i]) = 0$$

where the boundary condition is set, so the variation δx_i vanishes at the endpoints

$$\delta x_i(t_1) = \delta x_i(t_2) = 0$$

Calculating the action for the changed trajectory leads to

$$\mathcal{S}([x_i + \delta x_i]) = \int_{t_1}^{t_2} \mathrm{d}t \left(\frac{1}{2}m\left(\frac{\mathrm{d}x_i}{\mathrm{d}t} + \frac{\mathrm{d}(\delta x_i)}{\mathrm{d}t}\right)^2 - V(x_i + \delta x_i)\right)$$

with

$$\left(\frac{\mathrm{d}x_i}{\mathrm{d}t} + \frac{\mathrm{d}(\delta x_i)}{\mathrm{d}t}\right) = \left(\frac{\mathrm{d}x_i}{\mathrm{d}t}\right)^2 + 2\frac{\mathrm{d}x_i}{\mathrm{d}t}\frac{\mathrm{d}(\delta x_i)}{\mathrm{d}t} = \left(\frac{\mathrm{d}x_i}{\mathrm{d}t}\right)^2 + 2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}x_i}{\mathrm{d}t}\delta x_i\right) - 2\frac{\mathrm{d}^2x_i}{\mathrm{d}t^2}\delta x_i$$

where second order terms in δx_i were neglected. The first term also appears in the action for the original trajectory and the total derivative in the second term cancels the integral. Therefore

$$\mathcal{S}([x_i + \delta x_i]) = S([x_i]) + \left[\int_{t_1}^{t_2} \mathrm{d}t \sum_i \left(-m\frac{\mathrm{d}^2x_i}{\mathrm{d}t^2} - \frac{\partial V}{\partial x_i}\right)\delta x_i\right] + m\sum_i \frac{\mathrm{d}x_i}{\mathrm{d}t}\delta x_i\Bigg|_{t_1}^{t_2}$$

Because the last term vanishes due to the boundary conditions and δx_i is chosen arbitrarily $\Delta\mathcal{S}$ can only vanish if the term inside the integral is zero. Therefore

$$m\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\partial V}{\partial x_i}$$

This equation of motion is true for all δx_i.

Symmetries

Lets assume the system has a rotational invariance $V = V(r)$ with $r = \sqrt{\sum_i x_i^2}$. If a transformation O_{ij} orthogonal to the rotational invariance is applied to the trajectory x_j the action remains the same

$$\mathcal{S}[\sum_j O_{ij}x_j(t)] = S[x_j(t)]$$

Also the equation of motions is invariant

$$m\frac{\mathrm{d}^2(O_{ij}x_j)}{\mathrm{d}t^2} = -\frac{O_{ij}x_j}{r}\frac{\mathrm{d}V}{\mathrm{d}r}$$

1.4.2 Field Theory Lagrangian

In quantum field theory the action is given by

$$\mathcal{S}([\phi_r]) = \int \mathrm{d}^4x\ \mathcal{L}(\phi_r, \partial_\mu\phi_r)$$

with $\phi_r = \phi_r(\vec{x}, t)$. $\mathcal{L}$ is the Lagrange density. It is not directly dependant on x, because it should be invariant in the whole four dimensional space.
The integral here is over all four dimensions because time and space are treated equally in field theory.
There are some requirements to the Lagrange density:

1. $\mathcal{L}$ is local - there are no connections or interactions between two arbitrary space points. Also there can not be any instantaneus interaction of two spacepoints because information travels at finite speeds.
2. $\mathcal{L}$ is real - this is necessary to conserve probability
3. $\mathcal{L}$ is Lorentz invariant - $x'^\mu = \Lambda^\mu{}_\nu x^\nu \ \rightarrow\ \mathrm{d}^4x' = (\det\Lambda)\mathrm{d}^4x = \mathrm{d}^4x$. Therefore also the action is Lorentz invariant.
4. there is no need for derivatives higher than the first, this is implied by causality (?)

In natural units the action is dimensionless (whereas in SI units it has the same unit as $\hbar$). Also d^4x has units of $\frac{1}{\mathrm{GeV}^4}$ in natural units, therefore $\mathcal{L}$ has to have units of GeV^4

Extremal of Action

Same as before we can calculate the extremal of the action $\mathcal{S}$ for variations $\delta\phi_r$. Here the boundary condition has to be $\delta\phi_r(x) = 0$ for $x \in \partial\Omega$ where $\partial\Omega$ is the surface of the integrated space.
It follows

$$0 = \Delta\mathcal{S} = \int_\Omega \mathrm{d}^4x \sum_r \left[\frac{\partial\mathcal{L}}{\partial\phi_r}\delta\phi_r + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\partial_\mu(\delta\phi_r)\right]$$

In the second term the equality of $\delta(\partial_\mu\phi_r) = \partial_\mu(\delta\phi_r)$ was used. Also we can rewrite the partial derivative in the second term to an absolute derivative

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\delta\phi_r\right) - \delta\phi_r\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}$$

Therefore

$$\Delta\mathcal{S} = \int_\Omega \mathrm{d}^4x \left[\sum_r \delta\phi_r \left(\frac{\partial\mathcal{L}}{\partial\phi_r} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\right) + \partial_\mu \left(\sum_r \delta\phi_r \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\right)\right]$$

The last term is rewritable into a surface integral via Gauss' theorem, therefore it vanishes due to the boundary conditions. Similar to the classical approach $\delta\phi_r$ can be chosen arbitrarily and therefore the action only vanishes if the first term is equal to zero

$$\frac{\partial\mathcal{L}}{\partial\phi_r} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)} = 0$$

These are the Euler-Lagrange equations.

Example
Consider a scalar field $\phi(x)$. The Lagrange density is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\alpha\phi)(\partial^\alpha\phi) - V(\phi) = \frac{1}{2}\left[(\partial_0\phi)^2 - \sum_i(\partial_i\phi)^2\right]$$

So the equation of motion calculates to

$$\frac{\partial\mathcal{L}}{\partial\phi} = -V'(\phi) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = \partial_0\partial_0\phi - \vec{\nabla}(\vec{\nabla}\phi) = \Box\phi$$

$$\rightarrow \Box\phi + V'(\phi) = 0$$

Different potentials then lead to different equations of motion

- $V(\phi) = \frac{m^2}{2}\phi^2 \;\rightarrow\; V' = m^2\phi$ leads to $\Box\phi + m^2\phi = 0$ which is the Klein-Gordon equation. Its Lagrange density is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{m^2}{2}\phi^2$$

- $V(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4$ leads to $\Box\phi + m^2\phi + \lambda\phi^3 = 0$
- $V(\phi) = A\cos\frac{\phi}{M}$ leads to $\Box\phi - \frac{A}{M}\sin\frac{\phi}{M}$ which is called the sine-Gordon equation.

1.4.3 Dirac Lagrangian

For spin $\frac{1}{2}$ particles the Lagrangian is connected to the Dirac equation. It is given by

$$\mathcal{L} = \bar{\psi}(x)(i\partial\!\!\!/ - m)\psi(x), \quad \bar{\psi} = \psi^\dagger\gamma^0 = (\psi^*)^T\gamma^0$$

where ψ has four complex and eight real components.
The components of ψ and $\bar{\psi}$ are treated as independant fields. This leads to the following equations of motion

$$\frac{\partial\mathcal{L}}{\partial\bar{\psi}} = (i\partial\!\!\!/ - m)\psi = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\right) = 0 \qquad \rightarrow \quad (i\partial\!\!\!/ - m)\psi = 0$$

$$\frac{\partial\mathcal{L}}{\partial\psi} = -m\bar{\psi} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right) = i\partial_\mu(\bar{\psi}\gamma^\mu) \qquad \rightarrow \quad i\partial_\mu(\bar{\psi}\gamma^\mu) + m\bar{\psi} = 0$$

$$\mathcal{L} = \bar{\psi}\gamma^\mu\partial_\mu\psi + iq\bar{\psi}\gamma^\mu A_\mu\psi + m\bar{\psi}\psi + \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}$$

2. Groups and Symmetries

2.1 Representation of Groups

A group is an object of the form $G = \{g_i | i = 1, \ldots\}$ where g_i are the elements of the group. In the group a multiplication exists so, that $g_1 g_2 = g_3 \in G$. This also has the following traits:

- it has to be associative: $g_1(g_2 g_3) = (g_1 g_2) g_3$
- there is an identity element $e \equiv 1$ so that $g \cdot e = e \cdot g = g \;\; \forall g \in G$
- $\forall g \in G$ there is an inverse $g^{-1} \in G$: $gg^{-1} = g^{-1}g = e$

The representation of G is a mapping $r : G \to \mathbb{C}^{(n,n)}$ where $r(g_i) = M_i$ is a $n \times n$-matrix. With this representation the multiplication rules are being preserved: $r(g_1 g_2) = r(g_1) r(g_2)$ or $M_3 = M_1 M_2$. Also $r(e) = \mathbb{1}_n$ and $r(g^{-1}g) = MM^{-1} = \mathbb{1}$.
A reducable representation is a representation such that a single unitary $n \times n$ matrix U exists such that

$$UM_iU^{-1} = \begin{pmatrix} M_i' & 0 \\ 0 & M_i'' \end{pmatrix}$$

So instead of working with M_i we could have worked with the block diagonal matrices M_i' and M_i''.
An irreducable representation then is a representation where no unitary matrix exists that splits the matrix M_i into block diagonal matrices.
For finite groups $G = \{g_i | i = 1, \ldots, n\}$ the dimensions d_r of all the irreducible representations are bounded by

$$n = \sum_r d_r^2$$

Therefore an infinite group has infinte number of different finite dimensional irreducible representations.

2.2 Lie Groups and Lie Algebra

Lie groups are a special case of groups. They are parametrizised by $G = \{U(\theta) | \theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n\}$ with $U(0) = e$.
$U(\theta)$ is analytic in all its components; it is infinitely differentiable.
The simplest example of a Lie group is the three dimensional rotation group $SO(3)$ with the rotation matrices $R(\phi, \psi, \theta)$. The three rotations are
the rotation around the z-axis:

$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the rotation around the y-axis:

$$R_y(\psi) = \begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix}$$

the rotation around the x-axis:

$$R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\psi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}$$

Lie algebra

As the elements of the Lie group are infinitely differentiable we can apply the definition of the Taylor expansion onto an element of the group. This leads to the generators of the group

$$L_a \equiv \frac{1}{i} \frac{\partial U}{\partial \phi_a}\bigg|_{\theta=0}$$

These generators completely describe the groups properties.
The Taylor expansion also leads to infinitesimal transformations:

$$U(\theta) = 1 + i \sum_a \theta_a L_a + \dots$$

In summation convention this also can be written as $\sum_a \theta_a L_a = \theta_a L_a = \theta \cdot L$.
As one general group property is the formation of the 1-element: $U(\theta)U^{-1}(\theta) = 1$ this also has to be applicable for inifitiesimal transformations. This leads to

$$G \ni U(\theta)U(\psi)U^{-1}(\theta)U^{-1}(\psi) \neq 1$$

which is not neccesarily the 1-element for non commuting groups. The Taylor expanison of this expression leads to

$$1+i^2\theta_a\psi_b(L_aL_b+L_aL_b-L_aL_b-L_bL_a)+\ldots = 1-\theta_a\psi_b[L_a,L_b]+\ldots = 1+i\sum_c \theta_a\psi_b(-L_cf^{abc})$$

In the first step the only terms of first order are either linear in ψ or θ so they cancel. In the second order all terms quadratic in θ or ψ also cancel and the only things left are mixed terms. In the second step the definition of the commutator was applied and in the last step we used, that the resulting element had to be in the group G again, so it must be able to be written as a taylor expansion again. This therefore leads to the identity

$$[L_a, L_b] = -iL_cf^{abc}$$

Where f^{abc} are group specific structure constants.
For the $SO(3)$ group this is for example the known ϵ-tensor. The generators are the components of angular momentum:

$$L_x = \begin{pmatrix} & & \\ & & -i \\ & i & \end{pmatrix}, \quad L_y = \begin{pmatrix} & & i \\ & & \\ -i & & \end{pmatrix}, \quad L_z = \begin{pmatrix} & -i & \\ i & & \\ & & \end{pmatrix}$$

The identity here is $[L_x, L_y] = iL_z$.
These generators form the basis of the Lie algebra.

Another trait of the group can be shown for a finite group with $\theta = (\theta_1, \dots, \theta_n)$. If we use $\epsilon = \frac{1}{N}\theta$ then for high N, ϵ becomes small. So a Taylor expansion can be applied:

$$U(\epsilon) = 1 + i\epsilon_a L_a = 1 + i\frac{\theta_a L_a}{N}$$

So the original θ can be written as the following

$$U(\theta) = U(\epsilon)^N \to \lim_{N\to\infty} U(\epsilon)^N = \lim_{N\to\infty}\left(1 + i\frac{\theta_a L_a}{N}\right)^N = e^{i\theta_a L_a} = \sum_{k=0}^{\infty}\frac{1}{k!}(i\theta_a L_a)^k$$

The i was introduced in the definition of the generators, so the generators would be hermitian operators. If we look at $U(\theta)$, which is unitary it follows

$$U(\theta) = e^{i\theta L} \quad \to \quad U^{-1}(\theta) = e^{-i\theta L}$$

and thus

$$e^{-i\theta_a L_a^\dagger} = U^\dagger(\theta) = U^{-1}(\theta) = e^{-i\theta_a L_a}$$

From this we can see, that L_a and $L_a^\dagger$ must be the same.

Special unitary groups

Another group of Lie groups are the special unitary groups $SU(N) = \{U \in \mathbb{C}^{(N\times N)} | U^{-1} = U^\dagger, \det(U) = 1\}$. Its group elements are unitary and have a determinante of one. The number of generators L_a can be expressed formally for every N:
L_a has to be a hermitian $N \times N$ matrix with trace $\mathrm{tr}(L_a) = 0$. This follows from

$$\det(U) = \det\left(e^{i\theta_a L_a}\right) = e^{i\mathrm{tr}(\theta_a L_a)} \overset{!}{=} 1 \quad \to \quad \mathrm{tr}(L_a) = 0$$

Now because there are N^2 matrices but one of them neccesarily is $\mathbb{1}$ which is not traceless, there are always $N^2 - 1$ generators in $SU(N)$.
For $N = 2$ the generators are $L_a = \frac{\sigma_a}{2}$ where σ_a are the Pauli matrices.
For $N = 3$ the generators are the eight Gell-Mann matrices. They have the following forms

$$\lambda_1 = \begin{pmatrix} & 1 & \\ 1 & & \\ & & \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} & -i & \\ i & & \\ & & \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} & & 1 \\ & & \\ 1 & & \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} & & -i \\ & & \\ i & & \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} & & \\ & & 1 \\ & 1 & \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} & & \\ & & -i \\ & i & \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

These matrices have normalization conditions which also apply for abitrary N:

$$\mathrm{tr}(\lambda^a \lambda^b) = 2\delta^{ab}, \quad L_a = \frac{\lambda_a}{2}$$

Rank of the Lie algebra
The Lie algebra also has a rank. The rank is defined as the maximum number of commuting generators in the algebra. $SU(N)$ has $N-1$ diagonal generators, which naturally all commute. So the rank of an arbitraty $SU(N)$ algebra is always $N-1$.
These $N-1$ eigenvalues of these L_a then specify the basis states in a $SU(N)$ multiplet.
For example for $N=2$ the rank is one, so it has only one invariant, which is $\vec{J}^2$ so that $[\vec{J}^2, J_i] = 0$.
Generally, if we have any polynomial such that $C = \eta_{ab} L_a L_b + \eta_{abc} L_a L_b L_c + \dots$ that commutes with all generators, $[C, L_a] = 0 \; \forall a$ then it is called a Casimir invariant.
For $SU(2)$ the only Casimir invariant is $\vec{J}^2$, for $SU(3)$ there are two different Casimir invariants.
For any $SU(N)$ there is a Casimir invariant $C_2 = L_a L_a$ which is also referred to as the quadratic Casimir of the $SU(N)$ group.
Furhtermore, the eigenvalues of all independant Casimir invariants do specify an irreducible representation.

Adjoint irreducible representation
If we take a look at the Jacobi identity

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$$

and take $A = L_a$, $B = L_b$, $C = L_j$ we get

$$[A,B] = i f_{abm} L_m \quad \to \quad [C,[A,B]] = i f_{abm} [L_j, L_m] = i f_{abm} i f_{jmc} L_c$$

from this follows the relationship

$$(-i f_{aim}(-i f_{bmj}) - (-i f_{bim})(-i f_{amj}) = i f_{abc}(-i f_{aij})$$

If we now write $i f_{abc} = (F^a)_{bc}$ as a matrix element of a matrix F^a we get

$$(F^a)_{im}(F^b)_{mj} - (F^b)_{im}(F^a)_{mj} = i f_{abc}(F^c)_{ij}$$

due to the summation convention this is equivalent with

$$(F^a F^b)_{ij} - (F^b F^a)_{ij} = i f_{abc}(F^c)_{ij}$$

So this in it self satisfies the commutation relation of the Lie algebra:

$$[F^a, F^b] = i f^{abc} F^c$$

So these objects for an irreducible representation of the Lie algebra called the adjoint irreducible representation.

In summary, for any $SU(N)$ group there are three irreducible representations:

- the trivial representation: $U(\theta) = 1$
- the fundamental representation: $U(\theta) = \exp\left(i \frac{\lambda^a}{2} \theta^a\right)$
- the adjoint representation: $U(\theta) = \exp\left(i F^a \theta^a\right)$

2.3 The Lorentz Group and Relativistic Invariance

Lorentz transformations
Consider two inertial frames with common origin at $t = 0$ and moving with relative velocity v along the x-axis. Lets assume that in the first frame an event is taking place at $x^\mu = (t, \vec{x})^T$ and is seen at $x'^\mu = (t', \vec{x}')^T$ in the primed frame. The transformation is as follows

$$x'^\mu = \begin{pmatrix} t' \\ \vec{x}' \end{pmatrix} = \begin{pmatrix} \gamma t - \gamma v t \\ \gamma v t + \gamma x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot x^\mu$$

The transformation matrix is called Λ so that $x'^\mu = \Lambda^\mu{}_\alpha x^\alpha \equiv Lx$.

A Lorentz transformation now is any linear transformation Λ which keeps the relative length invariant:

$$s^2 = x'^\mu x'^\nu g_{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta x^\alpha x^\beta g_{\mu\nu} \stackrel{!}{=} x^\alpha x^\beta g_{\alpha\beta}$$

This must hold for all possible x^μ.
From this follows

$$\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta g_{\mu\nu} = g_{\alpha\beta}$$

If we now regard x^μ as a column vector x and $\Lambda^\mu{}_\nu$ as the elements of some 4×4 matrix L then follows

$$x' = L \cdot x, \qquad s^2 = x^T g x$$

Then the condition on the Λ's reads

$$g^\alpha{}_\beta = \Lambda_\mu{}^\alpha g^\mu{}_\nu \Lambda^\nu{}_\beta \quad \rightarrow \quad g_{\alpha\beta} = \Lambda^\mu{}_\alpha g_{\mu\nu} \Lambda^\nu{}_\beta = (L^T g L)_{\alpha\beta}$$

Now, because g is symmetric ($g_{\alpha\beta} = g_{\beta\alpha}$) also $L^T g L$ must be symmetric. Therefore there are only ten conditions on L (ten independent elements).
If we now take the determinante of that expression we find

$$\det(g) = \det(L^T)\det(g)\det(L) \quad \rightarrow \quad \det(L) = \pm 1$$

and with the Jacobi determinant follows

$$\int \mathrm{d}^4 x' = \int \mathrm{d}^4 x |\det(L)| = \int \mathrm{d}^4 x$$

We call Lorentz transformations with $\det(L) = 1$ proper Lorentz transformations and Lorentz transformations with $\det(L) = -1$ improper.
Also, if we take $\alpha = \beta = 0$ in the equation, we find

$$g_{00} = 1 = \Lambda^\mu{}_0 g_{\mu\nu} \Lambda^\nu{}_0 = (\Lambda^0{}_0)^2 - (\Lambda^i{}_0)^2$$

So necessarily $|\Lambda^0{}_0| \geq 1$.
We then call Lorentz transformations with $\Lambda^0{}_0 \geq 1$ orthochronous and Lorentz transformations with $\Lambda^0{}_0 \leq 1$ non orthonormous.

Lorentz group
All Lorentz transformations form a group, the Lorentz group. It has the following attributes

- the product of two Lorentz transformations is again a Lorentz transformation
- the inverse exists. In the example of the beginning it would replace $v \to -v$.
- there is a unitary element ($L = \mathbb{1}_4$)

Because a product of Lorentz transformations is again a Lorentz transformation we can reduce the types of different transformations to four:

1. time inversions: $x'^0 = -x^0, x'^i = x^i$ (non orthochronous, improper)
2. parity transformation: $x'^0 = x^0, x'^i = -x^i$ (orthochronous, improper)
3. rotations: $x'^0 = x^0, x'^i = a^{ij}x^j$ with a rotation matrix a^{ij}, $L = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$ where a is the 3×3 rotation matrix with $\det(a) = 1$. (orthochronous, proper)
4. boosts (here in one direction): $x'^0 = x^0 \cosh\eta + x^1 \sinh\eta$, $x'^1 = x^0 \sinh\eta x^1 \cosh\eta$, $x'^{(2,3)} = x^{(2,3)}$ where η is the rapidity. (orthochronous, proper)

The number of possible rotations is three (for example the Euler angles), the number of boosts also is three (for example the three boost directions or two angles plus v). Therefore, any proper, orthochronous Lorentz transformation can be described by six real parameters. The time inversion and party transformation do not have infinitesimal representations, because they are discrete transformations.

Inifitesimal generators of proper and orthochronous Lorentz transformations
For any adequate description of the Lorentz group we need to study infinitesimal generators. Therefore we consider the infinitesimal Lorentz transformation:

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \epsilon^\mu{}_\nu \equiv g^\mu{}_\nu + \epsilon^\mu{}_\nu$$

Now, the condition $g = L^T g L$ can be rewritten as

$$\delta^\alpha{}_\beta = g^\alpha{}_\beta = \Lambda_\mu{}^\alpha g^\mu{}_\nu \Lambda^\nu{}_\beta = \Lambda_\mu{}^\alpha \Lambda^\mu{}_\beta = (g_\mu{}^\alpha + \epsilon_\mu{}^\alpha)(g^\mu{}_\beta + \epsilon^\mu{}_\beta) = g_\beta{}^\alpha \epsilon_\beta{}^\alpha + \epsilon^\alpha{}_\beta + \mathcal{O}(\epsilon^2)$$

From there follows $\epsilon_\beta{}^\alpha + \epsilon^\alpha{}_\beta = 0$, for example $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$. So this is antisymmetric with six independant real elements.

We now introduce $L_{\mu\nu} = i(x_\mu \partial_\mu - x_\nu \partial_\mu)$ with $\partial_\mu = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)^T$ as a generalization of the angular momentum operator $J^i - i\epsilon^{ijk}x^j\partial_k$. It comes as as surprise, that these $L_{\mu\nu}$ exactly are the generators of the infinitesimal Lorentz transformation:

$$J^i = i\epsilon^{ijk}x_j\partial_k = \frac{1}{2}\epsilon^{ijk}L_{jk}$$

With $\delta x^\mu = \Lambda^\mu{}_\nu x^\nu - x^\mu = \epsilon^\mu{}_\nu x^\nu$ it follows

$$\frac{i}{2}e^{\rho\sigma}L_{\rho\sigma}x^\mu = \frac{i}{2}i\epsilon^{\rho\sigma}(x_\rho g_\sigma{}^\mu - x_\sigma g_\rho{}^\mu) = -\frac{1}{2}(e^{\rho\mu}x_\rho - \epsilon^{\mu\sigma}x_\sigma) = \epsilon^{\mu\sigma}x_\sigma = \delta x^\mu$$

where we used $\epsilon^{\rho\mu} = -\epsilon^{\mu\rho}$. Therefore

$$\frac{i}{2}\epsilon^{\rho\sigma}L_{\rho\sigma}x^\mu = \epsilon^\mu{}_\nu x^\nu$$

So the $L_{\rho\sigma}$ are indeed the generators of rotations in Minkovski space, explicitly, the $SO(3,1)$. The Lie algebra is

$$[L_{\mu\nu}, L_{\rho\sigma}] = i(g_{\nu\rho}L_{\mu\sigma} - g_{\mu\rho}L_{\nu\sigma} - g_{\nu\sigma}L_{\mu\rho} + g_{\mu\sigma}L_{\nu\rho})$$

As a generalization, $L_{\mu\nu}$ is analogous to orbital angular momentum. We can add a spin term, which of course commutes with L and forms some algebra similar to the L's among themselves.
The most general representation of $SO(3,1)$ generators is by $M_{\mu\nu} \equiv i(x_\mu\partial_\nu - x_\nu\partial_\mu) + S_{\mu\nu}$. The $M_{\mu\nu}$ space compppnents, or more familiarly, the $J^i = \frac{1}{2}\epsilon^{ijk}M_{jk}$ components are the generators of inifinetismal rotations with $[J_i, J_j] = i\epsilon^{ijk}J^k$ (the commutation relation from $SO(2)$).
The $M^{0i} \equiv K^i$ are space time components and generate the boosts.
The commutation relation of these K^i and the known J^i are

$$[K^i, K^j] = -\epsilon^{ijk}J^k, \quad [J^i, K^j] = i\epsilon^{ijk}K^k$$

These are much simpler than the commutation relation of the $L_{\mu\nu}$.
Also, there usually is a trick to separate two $SO(2)$'s from each other by taking the linear combinations

$$N^i \equiv \frac{1}{2}(J^i + iK^i), \qquad N^{i\dagger} \equiv \frac{1}{2}(J^i - iK^i)$$

For these we find the commutation relations

$$[N^i, N^{i\dagger}] = 0, \quad [N^i, N^j] = i\epsilon^{ijk}N^k, \quad [N^{i\dagger}, N^{j\dagger}] = i\epsilon^{ijk}N^{k\dagger}$$

Because the first one is zero, $N^\dagger$ and N are decoupled.
Finally, we find, that the finite dimensional representations of the restriced Lorentz group is characterized by (n, m) where $n, m = 0, \frac{1}{2}, 1, \ldots$ and are given by the eigenvalues of the Casimir operators:

$$N^iN^i|n, n_3\rangle = n(n+1)|n, n_3\rangle, \qquad N^{i\dagger}N^{i\dagger}|m, m_3\rangle = m(m+1)|m, m_3\rangle$$

These m and n are related by parity: $J_i \xrightarrow{P} J_i$, $K_i \xrightarrow{P} -K_i$. This follows, because J_i has two spacial directions and K_i only has one. Now, the parity transformation changes the sign of the spacial components, so for J_i this cancels. The N transform as

$$N^i \xrightarrow{P} N^{i\dagger}, \quad \rightarrow \quad (n, m) \xrightarrow{P} (m, n)$$

Hence n and m are related. Additionaly we can identify the spin of the representation as $n + m$, this follows, because $J^i = N^i + N^{i\dagger}$ (?).
Some special cases are for example:

- The scalar representation with spin 0 is $(0, 0)$.
- $\left(\frac{1}{2}, 0\right)$ and $\left(0, \frac{1}{2}\right)$ are representations of spin $\frac{1}{2}$ where the first one represents right handed spinors and the second one left handed spinors. Both are Weyl spinors.
- Combining $\left(0, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, 0\right)$ as a direct sum this leads to Dirac spinors. This is for example needed, when a parity invariant theory is wanted.

With J and K we can describe finite Lorentz transformations by exponenciating the infinitesimal ones in a non covariant form:

$$e^{-i(\vec{\omega}\vec{J} + \vec{\nu}\vec{K})}$$

Here $\vec{\omega}$ is the direction of the rotation axis and $|\vec{\omega}|$ is the rotation angle. The direction of the boost is given by $\vec{\nu}$ and $|\vec{\nu}|$ is the rapidity of the boost.

Poincaré group

The Poincaré group is obtaioned from the Lorentz group by adding translations $x^\mu \to x'^\mu = x^\mu + a^\mu$ where a^μ is a constant 4 vector. The most general translation is

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$$

This has ten real parameters, so we need four additional generators. These will be the momentum operators.
Infinitesimally the difference between the translated and untranslated vector can be expressed as $\delta x^\mu = \epsilon^\mu = -\epsilon^\rho P_\rho x^\mu$ with $P_\rho = i\partial_\rho$.
The commutation relations of the momentum operator are

$$[P_\mu, P_\nu] = 0, \quad [M_{\mu\nu}, P_\rho] = -ig_{\mu\nu} P_\nu + ig_{\nu\rho} P_\mu$$

2.4 Transformation of Fields under Lorentz Transformations

Lets take a look at a vector field $A_\mu(x)$ (for example the vector potential of the electromagnetic field). An observer in the primed frame describes the same physical situation by another field $A'_\mu(x)$. The two observed fields are of course related, for $x'_\mu = \Lambda_\mu{}^\nu x_\nu$ it follows

$$A'_\mu = \Lambda_\mu{}^\nu A_\nu(x)$$

This relation is, what defines A_μ as a vectorfield. We can generalize this:
Let x_μ and $x'_\mu = \Lambda_\mu{}^\nu x_\nu$. Thee should describe some point P in two inertial frames. Let $\Lambda_A{}^B$ $(A, B = 1, \dots d)$ be the corresponding d-dimensional representation matrixes of an irreducible representation of $SO(3,1)$. A d-component field $f_A(x)$ and $f'_A(x')$ in the two transformations is said to transform under this irreducible representation if $f'_A(x') = \lambda_A{}^B f_B(x)$. In the infinitesimal version this is $\Lambda_A{}^B = \delta_A{}^B - \frac{i}{2}\epsilon^{\alpha\beta}(S_{\alpha\beta})_A{}^B$.
Here are some examples

irr. rep. of $SO(3,1)$	field	transf. property
$(0,0)$, scalar	$\phi(x)$	$\phi'(x') = \phi(x)$
$\left(\frac{1}{2}, \frac{1}{2}\right)$, vector	$V^\mu(x)$	$V'^\mu(x') = \Lambda^\mu{}_\nu V^\nu(x)$
$\left(0, \frac{1}{2}\right)$, left handed Weyl spinor	$\psi_L(x)$	$\psi'_L(x') = \Lambda_L \psi_L(x)$
$\left(\frac{1}{2}, 0\right)$, right handed Weyl spinor	$\psi_R(x)$	$\psi'_R(x') = Lambda_R \psi_R(x)$

The transformation matrices Λ_R and Λ_L are given by:

$$\Lambda_{L,R} = e^{-i(\vec{\omega}\vec{J} + \vec{\nu}\vec{K})}$$

with $\vec{J} = \vec{N} + \vec{N}^\dagger$ and $\vec{K} = i(\vec{N}^\dagger - \vec{N})$. For $Lambda_L$ the N are $\vec{N}^\dagger = \frac{\vec{\sigma}}{2}$ and $\vec{N} = 0$. For Λ_R they are $\vec{N}^\dagger = 0$ and $\vec{N} = \frac{\vec{\sigma}}{2}$. This follows, because $\vec{N}$ are $SU(2)$ groups and we need a representation of $SU(2)$ which describes spin $\frac{1}{2}$ particles.
With this parametrization the Λ are given by

$$\Lambda_L = \exp\left(-i\frac{\vec{\sigma}}{2}(\vec{\omega} + i\vec{\nu})\right)$$

$$\Lambda_R = \exp\left(i\frac{\vec{\sigma}}{2}(\vec{\omega} - i\vec{\nu})\right)$$

Some properties of $\Lambda_{R,L}$ are

- $\Lambda_L^{-1} = \Lambda_R^\dagger$
- $\sigma^2 \Lambda_L \sigma^2 = \Lambda_R^*$
- $\sigma^2 \Lambda_L^{-1} \sigma^2 = \Lambda_L^T$

Now consider a Lorentz transformation of the field $\sigma^2\psi_R^*$:

$$\sigma^2\psi_R^* \quad \rightarrow \quad \sigma^2\Lambda_R^*\psi_R^* = \underbrace{\sigma^2\Lambda_R^*\sigma^2}_{\Lambda_L}\sigma^2\psi_R^* = \Lambda_L\sigma^2\psi_R^*$$

Thus $\sigma^2\psi_R^*$ is a left handed spinor because it transforms the same way. It is called the charge conjugate of ψ_R). Therefore:

- ψ_L and $\sigma^2\psi_R^*$ transform as $\left(0, \frac{1}{2}\right)$
- ψ_R and $\sigma^2\psi_L^*$ transform as $\left(\frac{1}{2}, 0\right)$

ψ_L and ψ_R are called Weyl spinors.

Construction of scalars and vectors
The Weyl spinors can be combined to construct scalars and vectors. First, consider products of two left handed spinors χ_L and ψ_L. Since $\left(0, \frac{1}{2}\right) \otimes \left(0, \frac{1}{2}\right) = (0,0) \oplus (0,1)$ we can construct a scalar from them:

$$\chi_L^T\sigma^2\psi_L \quad \stackrel{LT}{\rightarrow} \chi_L^T\Lambda_L^T\sigma^2\Lambda_L\psi_L = \chi_L^T\sigma^2\psi_L$$

In the second step we used, that $\Lambda_L^T\sigma^2\Lambda_L$ is equal to $\sigma^2\Lambda_L^{-1}\sigma^2\sigma^2\Lambda_L$ due to the properties of Λ_L and that $(\sigma^2)^2 = 1$ and $\Lambda_L^{-1}\Lambda_L = 1$.
We therefore know, that $\chi_L^T\sigma^2\psi_L$ is a scalar. With

$$\chi_L = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \quad \psi_L = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

follows

$$\chi_L^T\sigma^2\psi_L = \begin{pmatrix} \chi_1 & \chi_2 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = -i(\chi_1\psi_2 - \chi_2\psi_1)$$

This shows, that this scalar is antisymmetrical in the two fields. It is important to realize, that these are spin $\frac{1}{2}$ fields, so they must be represented by anitcommuting Grassmann numbers to conserve the properties demanded by spin $\frac{1}{2}$.
Hence even von $\chi_L = \psi_L$ we get

$$\psi_L^T\sigma^2\psi_L = -i(\psi_1\psi_2 - \psi_2\psi_1) = 2i\psi_2\psi_1 \neq 0$$

This is non zero because of the anticommuting Grassmann numbers. This would not be the case, if these scalars would commute like normal scalars.
This combination is used in Majorana mass terms.
We can also take $\chi_L = \sigma^2\psi_R^*$ because it also is a lefthanded spinor. Then we get

$$\chi_L^T\sigma^2\psi_L = \psi_R^\dagger(-\sigma^2)^2\psi_L = -\psi_R^\dagger\psi_L$$

Which is a particle of the Dirac mass term.

Vectors $\left(\frac{1}{2}, \frac{1}{2}\right)$ can be obtained from the product of ψ_L and $\sigma^2\psi_R^*$, $\left(\frac{1}{2}, 0\right) \otimes \left(0, \frac{1}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$.
We start from $\psi_L^\dagger\psi_L$ which is invariant under rotations. Under boosts this transforms as

$$\psi_L^\dagger\psi_L \quad \rightarrow \quad \psi_L^\dagger\Lambda_L^\dagger\Lambda_L\psi_L = \psi_L^\dagger e^{\vec{\sigma}\vec{\nu}}\psi_L = \psi_L^\dagger\psi_L + \vec{\nu}\psi_L^\dagger\vec{\sigma}\psi_L + \dots$$

Similarly for $\psi_L^\dagger \sigma^i \psi_L$:

$$\psi_L^\dagger \sigma^i \psi_L \quad \to \quad \psi_L^\dagger e^{\vec{\sigma}\vec{\nu}/2} \sigma^i e^{\vec{\sigma}\vec{\nu}/2} \psi_L = \psi_L^\dagger \sigma^i \psi_L - \nu^i \psi_L^\dagger \psi_L$$

In the second step we used $e^{\vec{\sigma}\vec{\nu}/2} \sigma^i e^{\vec{\sigma}\vec{\nu}/2} = \sigma^i - \frac{1}{2}\nu^j \{\sigma^j, \sigma^i\} = \sigma^i - \nu^i \delta^{ji}$.
Summarizing the differnce between transformed and untransformed vectors δ is given by

$$\delta(\psi_L^\dagger \psi_L) = -\nu^i \psi_L^\dagger (-\sigma^i) \psi_L$$

and

$$\delta(\psi_L(-i\sigma^i)\psi_L) = -\nu^i \psi_L^\dagger \psi_L$$

If we compare this with the general transformation of 4 vectors under boosts which is given by

$$\frac{1}{2}\epsilon^{\mu\nu} M_{\mu\nu} = \epsilon^{0i} M_{0i} = -\epsilon^{0i} K^i = \vec{\nu}\vec{K}$$

where of course only boosts were considered. The difference here is given by

$$\delta V^\mu = -\epsilon^\mu{}_\nu V^\nu$$

For the time component this leads to

$$\delta V^0 = \epsilon^{0i} V^i = -\nu^i V^i$$

and for the spacial components

$$\delta V^i = -\epsilon^{i0} V^0 = \epsilon^{0i} V^0 = -\nu^i V^0$$

Therefore $(\psi_L^\dagger \psi_L, -\psi_L^\dagger \vec{\sigma} \psi_L)$ transforms as a 4 vector. This is often times written as $\psi_L^\dagger \sigma_-^\mu \psi_L$ with $\sigma_-^\mu \equiv (\mathbb{1}, -\vec{\sigma})$.
Similarly for the right handed spinors $\psi_R^\dagger \psi_R$ with $\Lambda_R = e^{\vec{\sigma}\vec{\nu}/2}$ we get the vector

$$\psi_R^\dagger \sigma_+^\mu \psi_R = (\psi_R^\dagger \psi_R, \psi_R^\dagger \vec{\sigma} \psi_R), \quad \sigma_+^\mu \equiv (\mathbb{1}, \vec{\sigma})$$

Lorentz scalars now can be formed by contracting with Lorentz vectors in particular with ∂_μ: $(\partial_\mu \psi_L^\dagger)\sigma_-^\mu \psi_L, \psi_L^\dagger \sigma_-^\mu (\partial_\mu \psi_L))$, etc are thus candidates for kinetic energy terms in Lagrangians.

3. Klassiche Feldtheorie: Lagrangians

3.1 Bewegungsgleichungen

3.2 Symmetrien (Noether's Theorem)

3.3 Eichsymmetrie, Eichfelder

4. Kanonische (zweite) Quantisierung von Spin 0, 1/2, 1 Feldern

4.1 Erzeugungs- und Vernichtungsoperatoren

4.2 Fockraum

4.3 Propagatoren

4.4 Gupta Bleuler Quantisierung des Photons

5. S-Matrix, LSZ Reduktionsformel

6. Störungstheorie

6.1 Feynman Regeln der QED

6.2 Wirkungsquerschnitte und Zerfallsraten

6.3 radiative Korrekturen