Olivier Vallée
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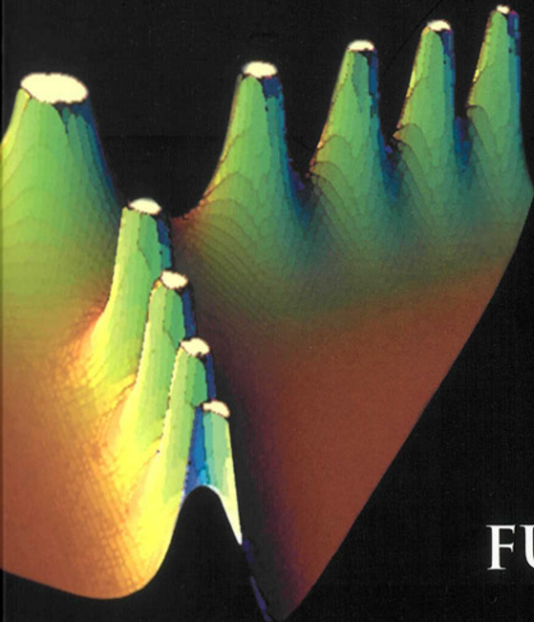

AIRY FUNCTIONS AND APPLICATIONS TO PHYSICS

Imperial College Press

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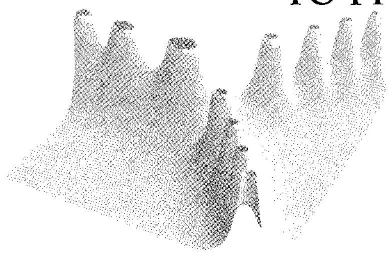

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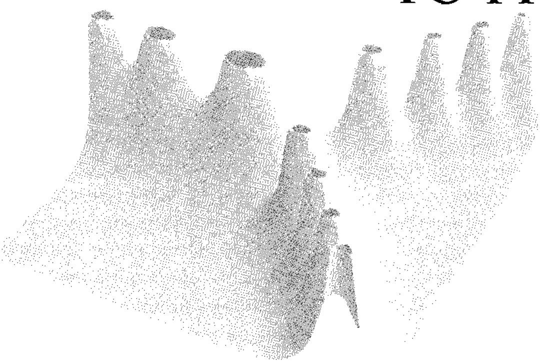

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Preface

The use of special functions, and in particular of Airy functions, is rather common in physics. The reason may be found in the need, and even in the necessity, to express a physical phenomenon in terms of an effective and comprehensive analytical form for the whole scientific community. However, for almost the last twenty years, many physical problems have been resolved by computers. This trend is now becoming the norm as the importance of computers continues to grow. As a last resort, the special functions employed in physics will have, indeed, to be calculated numerically, even if the analytic formulation of physics is of first importance.

The knowledge on Airy functions was periodically the subject of many review articles. Generally these were about their tabulations for the numerical calculation of these functions which is particularly difficult. We shall quote the most known works in this field: the tables of J.C.P. Miller which are from 1946 and the chapter in the *Handbook of Mathematical Functions* by Abramowitz and Stegun whose first version appeared in 1954. No noteworthy compilation on Airy functions has been published since that time, in particular about the calculus implying these functions. For example, in the last editions of the tables of Gradshteyn and Ryzhik, they are hardly evoked. At the same time, many accumulated results in the scientific literature, remain extremely dispersed and fragmentary.

The Airy functions are used in many fields of physics, but the analytical outcomes that have been obtained are not (or weakly) transmitted between the various fields of research which after all remain isolated. Moreover the tables of Abramowitz and Stegun are still the only common reference to all the authors using these functions. Thus many of the results have been rediscovered, sometimes extremely old findings are the subject of publications and consequently a useless effort for researchers.

In this work, we would like to make a rather exhaustive compilation of the current knowledge on the analytical properties of Airy functions. In particular, the calculus implying the Airy functions is developed with care. This is, actually, one of the major objectives of this book. We are however aware of making a great number of repetitions regarding the previous compilations, but, it seemed necessary to ensure coherence. This book is addressed mainly to physicists (from undergraduate students to researchers). For the mathematical demonstrations, as one will see, we do not have any claim about the rigour.[1] The aim is the outcome, or the fastest way to reach it. Finally, in the second part of this work, the reader will find some applications to various fields of physics. These examples are not exhaustive. They are only given to succinctly illustrate the use of Airy functions in classical or in quantum physics. For instance, we point out to the physicist in fluid mechanics, that he can find what he is looking for, in the works of molecular physics or in physics of surfaces, and *vice versa.*

The authors would like to warmly thank Nick Rowswell who considerably improved the content of this book.

O. Vallée *&* M. Soares, Fall 2003

[1]As a matter of fact, the Airy function can be considered as a distribution (generalised function) whose Fourier transform is an imaginary exponential. Also most of the integrals evoked in this work should be evaluated with the help of a convergence factor.

Contents

Chapter 1

A Historical Introduction : Sir George Biddell Airy

George Biddell Airy was born July 27, 1801 at Alnwick in Northumberland (North of England). His family was rather modest, but thanks to the generosity of his uncle Arthur Biddell, he went to study at Trinity College (University of Cambridge). He was a brilliant student although being a sizar,[1] and finally graduated in 1823 as a senior wrangler. Three years later, he was elected to occupy the celebrated Lucasian chair of mathematics. Nevertheless, his salary as Lucasian professor was too small to marry Richarda, as her father said. So he applied for a new position. In 1828, Airy obtained the Plumian chair becoming professor of Astronomy and director of the new observatory at Cambridge. His first works at this time were about the mass of Jupiter and also about the irregular motions of Earth and Venus.

In 1834, Airy started his first mathematical studies on the diffraction phenomenon and optics. Due to the diffraction phenomenon, the image of a point through a telescope is actually a spot surrounded by rings of smaller intensity, this spot is now called the "Airy spot", the associated Airy function has nothing to do with the purpose of the present book.

In June 1835, Airy became the 7^{th} Royal Astronomer and director of the Greenwich observatory, succeeding John Pond. Under his administration, modern equipment was installed, leading the observatory to its worldwide fame assisted by the quality of its published data. Airy also introduced the study of sun spots and the magnetism of Earth, he built a new apparatus for the observation of the Moon, and also for cataloging the stars. The question of absolute time was also a broad challenge, Airy defined the "Airy Transit Circle", that became in 1884: Greenwich Mean Time. But the

[1]With the meaning that he paid a reduced fee but worked as a servant to richer students.

renown of Airy is also due to the "Neptune affair". During the decade 1830–40, astronomers were interested in the perturbations of Uranus that were discovered in 1781. In France, François Arago suggested to Urbain Le Verrier finding a new planet that might cause the perturbations of Uranus. In England, the young John Adams was doing the same calculations with a slight advance, however Airy was doubtful on the issue of such a work. Adams tried twice to meet Airy in 1845 but was unsuccessful: the first time Airy was away, the second time Airy was taking dinner and did not like to be disturbed. Finally, Airy entrusted the astronomer James Challis with the observation of the new planet from the calculations of Adams. Unfortunately, Challis failed in his task. At the same time, Le Verrier asked the German astronomer Johann Galle in Berlin to locate the planet from his data: the new planet was discovered on September 20, 1846. A polemic started then between Airy and Arago, between France and England, and also against Airy himself. The polemic spread out with the name of the planet itself: Airy wanting to name the new planet Oceanus. The name of Neptune was finally given. The story goes that in the end, Adams and Le Verrier became good friends.

In 1854 Airy attempted to determine the mean density of the Earth. The experiment stood in the comparison of gravity forces on a single pendulum at the entrance of a pit and at its ground. This experiment was carried out near South Shields in a mine of 1250 feet in depth. Taking into account the elliptical form and the rotation of the Earth, Airy found a density of 6.56, which is not so far – considering the epoch – from the usually admitted density 5.42.

Airy was knighted in 1872, and so became Sir George Biddell.[2] At this time, Airy started a lunar theory. The results were published in 1886, but in 1890 he found an error in his calculations. The author was eighty-nine years old and was unwilling to revise his calculations. Late in 1881, Sir George left his astronomer position at Greenwich for retirement. He died January 2, 1892.

The autobiography of Sir George, edited by his son Wilfred, was published in 1896 ("*Autobiography of G.B.* AIRY", W. Airy ed., 1896). The name of Airy is associated with many phenomena such as the Airy spiral (optical phenomenon visible in quartz crystals), the Airy spot in diffraction phenomena or the Airy stress function he introduced in his work on elasticity, different as well from the Airy functions that we shall discuss in this

[2]After he declined the offer on three occasions, arguing the fees.

Fig. 1.1 Sir George Biddell Airy (after the Daily Graphic, January 6, 1892).

book. Among of the most-known books he wrote, we may quote "*Mathematical tracts on physical astronomy*" (1826) and "*Popular astronomy*" (1849).

Airy was particularly involved in optics, for instance he made special glasses to correct his own astigmatism. For the same reason, he was also interested by the calculation of light intensity in the neighbourhood of a caustic [Airy (1838), (1849)]. For this purpose, he introduced the function defined by the integral

$$W(m) = \int_0^\infty \cos\left[\frac{\pi}{2}\left(\omega^3 - m\omega\right)\right] d\omega,$$

which is now called the Airy function. This is the object of the present book. W is the solution of the differential equation

$$W'' = -\frac{\pi^2}{12} mW.$$

The numerical calculation of Airy functions is somewhat tricky, even today!

However in 1838, Airy gave a table of the values of W for m varying from -4.0 to $+4.0$. Thence in 1849, he published a second table for m varying from -5.6 to $+5.6$, for which he employed the ascending series. The problem is that this series is slowly convergent as m increases. A few years after, Stokes (1851, 1858) introduced the asymptotic series of $W(m)$, of its derivative and of the zeros. Practically no reseach was endeavoured on Airy function until the work of Nicholson (1909), Brillouin (1916) and Kramers (1926) who contributed broadly to a better knowledge of this function.

In 1928 Jeffreys introduced the notation used nowadays

$$Ai(x) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) \mathrm{d}t,$$

which is the solution of the homogeneous differential equation, called the Airy's equation

$$y'' = xy.$$

Clearly, this equation may be considered as an approximation of the differential equation of the second order

$$y'' + F(x)y = 0,$$

where F is any function of x. If $F(x)$ is expanded in a neighbourhood of a point $x = x_0$, we have to the first order ($F'(x_0) \neq 0$)

$$y'' + [F(x_0) + (x - x_0)F'(x_0)]\, y = 0.$$

Then with a change of variable, we find the Airy's equation. This method is particularly useful in a neighbourhood of a zero of $F(x)$. The point x_0 defined by the relation $F(x_0) = 0$ is called a transition point by mathematicians and a turning point by physicists. Turning points are involved in the asymptotic solutions of linear differential equations of the second order [Jeffreys (1942)], such as the stationary Schrödinger equation.

Finally we can note that Airy functions are Bessel functions (or linear combinations of these functions) of order 1/3. The relation between both of the Airy's equation and the Bessel equation is done with the change of variable $\xi = \frac{2}{3}x^{3/2}$, yielding Jeffreys (1942) to say: "*Bessel functions of order* 1/3 *seem to have no application except to provide an inconvenient way of expressing this function*"!

Chapter 2

Definitions and Properties

This chapter is devoted to general definitions and properties of Airy functions as they can be, at least partially, found in the chapter concerning these functions in the "*Handbook of Mathematical Functions*" by Abramowitz & Stegun (1965).

2.1 The Homogeneous Airy Functions

2.1.1 *The Airy's equation*

We consider the following homogeneous second order differential equation called the Airy's equation

$$y'' - xy = 0. \tag{2.1}$$

This differential equation may be solved by the method of Laplace, *i.e.* in seeking a solution as an integral

$$y = \int_{\mathrm{C}} \mathrm{e}^{xz} v(z) \mathrm{d}z,$$

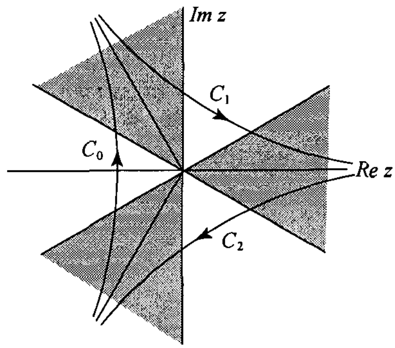
this is equivalent to solve the first order differential equation

$$v' + z^2 v = 0.$$

We thus obtain the solution to the equation (2.1), except a normalisation constant,

$$y = \int_{\mathrm{C}} \mathrm{e}^{xz - z^3/3} \mathrm{d}z.$$

The integration path C is chosen such that the function $v(z)$ must vanish at the boundaries. This is the reason why the extremities of the path must go into the regions of the complex plane z, where the real part of z^3 is positive (shading regions of the complex plane).



From symmetry considerations, it is useful to work with the paths $\mathrm{C_0}$, $\mathrm{C_1}$ and $\mathrm{C_2}$. Clearly the integration paths $\mathrm{C_1}$ and $\mathrm{C_2}$ lead to solutions that tend to infinity when x goes to infinity. When we consider the path $\mathrm{C_0}$ and the associated solution, we can deform this curve until it joins the imaginary axis. Now we define the Airy function Ai by

$$Ai(x) = \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{e}^{xz - z^3/3} \mathrm{d}z. \tag{2.2}$$

If 1, j, j^2 are the cubic roots of unity (that is to say $\mathrm{j} = \mathrm{e}^{\mathrm{i}2\pi/3}$) the functions defined by the paths $\mathrm{C_1}$ and $\mathrm{C_2}$ are respectively the functions $Ai(\mathrm{j}x)$ and $Ai(\mathrm{j}^2x)$. We have between these solutions, two by two linearly independents for they satisfy the same second order differential equation, the relation

$$Ai(x) + \mathrm{j}Ai(\mathrm{j}x) + \mathrm{j}^2Ai(\mathrm{j}^2x) = 0. \tag{2.3}$$

Now, in place of the functions $Ai(\mathrm{j}x)$ and $Ai(\mathrm{j}^2x)$, we define the function $Bi(x)$, linearly independent of $Ai(x)$, which has the interesting property to be real when x is real

$$Bi(x) = \mathrm{i}\mathrm{j}^2Ai(\mathrm{j}^2x) - \mathrm{i}\mathrm{j}Ai\,(\mathrm{j}x)\,. \tag{2.4}$$

Similarly to $Ai(x)$ (cf. formula (2.3)), we have the relation

$$Bi(x) + \mathrm{j}Bi(\mathrm{j}x) + \mathrm{j}^2 Bi(\mathrm{j}^2 x) = 0. \tag{2.5}$$

On Figs. 2.1 and 2.2, the plots of the functions $Ai(x)$, $Bi(x)$, and of their derivatives $Ai'(x)$ and $Bi'(x)$ are given.

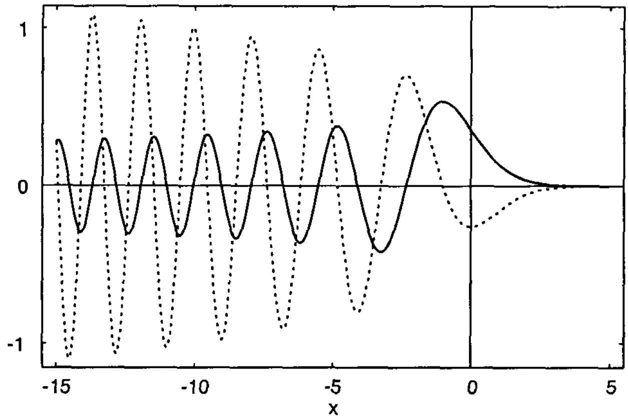


Fig. 2.1 Plot of the Airy function Ai (full line) and its derivative (dotted line).

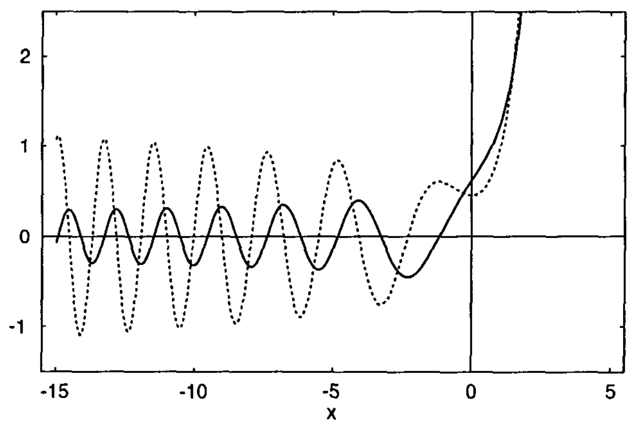


Fig. 2.2 Plot of the Airy function Bi (full line) and its derivative (dotted line).

2.1.2 *Elementary properties*

2.1.2.1 *Wronskians of homogeneous Airy functions*

The Wronskian $W\{f,g\}$ of two functions $f(x)$ and $g(x)$ is defined by

$$W\{f,g\} = f(x)\frac{\mathrm{d}g(x)}{\mathrm{d}x} - \frac{\mathrm{d}f(x)}{\mathrm{d}x}g(x).$$

For the Airy functions Ai and Bi, we have the following Wronskians [Abramowitz & Stegun (1965)]

- $$W\{Ai(x),\ Bi(x)\} = \frac{1}{\pi} \tag{2.6}$$
- $$W\left\{Ai(x),\ Ai\left(x\mathrm{e}^{\mathrm{i}2\pi/3}\right)\right\} = \frac{\mathrm{e}^{-\mathrm{i}\pi/6}}{2\pi} \tag{2.7}$$
- $$W\left\{Ai(x),\ Ai\left(x\mathrm{e}^{-\mathrm{i}2\pi/3}\right)\right\} = \frac{\mathrm{e}^{\mathrm{i}\pi/6}}{2\pi} \tag{2.8}$$
- $$W\left\{Ai\left(x\mathrm{e}^{\mathrm{i}2\pi/3}\right),\ Ai\left(x\mathrm{e}^{-\mathrm{i}2\pi/3}\right)\right\} = \frac{\mathrm{i}}{2\pi}. \tag{2.9}$$

2.1.2.2 *Particular values of Airy functions*

The values at the origin of homogeneous Airy functions are

$$Ai(0) = \frac{Bi(0)}{\sqrt{3}} = \frac{1}{3^{2/3}\Gamma\left(\frac{2}{3}\right)} = 0.355\,028\,053\,887\,817\,239 \tag{2.10}$$

$$-Ai'(0) = \frac{Bi'(0)}{\sqrt{3}} = \frac{1}{3^{1/3}\Gamma\left(\frac{1}{3}\right)} = 0.258\,819\,403\,792\,806\,798 \tag{2.11}$$

and therefore

$$Ai(0)Ai'(0) = \frac{-1}{2\pi\sqrt{3}}. \tag{2.12}$$

More generally, we have for the higher derivatives [Crandal (1996)]

$$Ai^{(n)}(0) = (-1)^n c_n \sin(\pi(n+1)/3), \tag{2.13}$$

and

$$Bi^{(n)}(0) = c_n[1 + \sin(\pi(4n+1)/6)], \tag{2.14}$$

where the coefficient c_n is

$$c_n = \frac{1}{\pi}3^{(n-2)/3}\Gamma\left(\frac{n+1}{3}\right).$$

2.1.2.3 *Relations between Airy functions*

The following relations are deduced from the formulae (2.3), (2.4) and (2.5) [Miller (1946); Abramowitz & Stegun (1965)]

- $$Ai\left(x\mathrm{e}^{\pm \mathrm{i}2\pi/3}\right) = \frac{\mathrm{e}^{\pm \mathrm{i}\pi/3}}{2}\left[Ai(x) \mp \mathrm{i}Bi(x)\right] \tag{2.15}$$
- $$Ai'\left(x\mathrm{e}^{\pm \mathrm{i}2\pi/3}\right) = \frac{\mathrm{e}^{\mp \mathrm{i}\pi/3}}{2}\left[Ai'(x) \mp \mathrm{i}Bi'(x)\right] \tag{2.16}$$
- $$Bi\,(x) = \mathrm{e}^{\,\mathrm{i}\pi/6}Ai\left(x\mathrm{e}^{\,\mathrm{i}2\pi/3}\right) + \mathrm{e}^{-\mathrm{i}\pi/6}Ai\left(x\mathrm{e}^{-\mathrm{i}2\pi/3}\right) \tag{2.17}$$
- $$Bi'\,(x) = \mathrm{e}^{\,\mathrm{i}5\pi/6}Ai'\left(x\mathrm{e}^{\,\mathrm{i}2\pi/3}\right) + \mathrm{e}^{-\mathrm{i}5\pi/6}Ai'\left(x\mathrm{e}^{-\mathrm{i}2\pi/3}\right). \tag{2.18}$$

2.1.3 *Integral representations*

An integral definition of $Ai(x)$ was given by the formula (2.2). This function can also be defined by the following formulae [Abramowitz & Stegun (1965)]

- $$Ai(x) = \frac{1}{\pi}\int_0^{\infty} \cos\left(\frac{z^3}{3} + xz\right)\mathrm{d}z \tag{2.19}$$
- $$Ai(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{\,\mathrm{i}\left(z^3/3+xz\right)}\mathrm{d}z \tag{2.20}$$
- $$Ai(x) = \frac{x^{1/2}}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{\,\mathrm{i}x^{3/2}\left(z^3/3+z\right)}\mathrm{d}z, \quad x > 0 \tag{2.21}$$
- $$Ai(x) = \frac{\mathrm{e}^{-\xi}}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-xz^2+\mathrm{i}z^3/3}\mathrm{d}z, \quad x > 0, \ \xi = \frac{2}{3}x^{3/2} \tag{2.22}$$
- $$Ai(x) = \frac{\mathrm{e}^{-\xi}}{\pi}\int_0^{\infty} \mathrm{e}^{-xz^2}\cos\left(\frac{z^3}{3}\right)\mathrm{d}z, \quad x > 0, \ \xi = \frac{2}{3}x^{3/2}. \tag{2.23}$$

More generally, we have

$$Ai(a\,x) = \frac{1}{2\pi a}\int_{-\infty}^{+\infty} \exp\left[\mathrm{i}\left(\frac{u^3}{3\,a^3} + xu\right)\right]\mathrm{d}u. \tag{2.24}$$

We give also, the useful formula

$$\int_{-\infty}^{+\infty} \exp\left[\mathrm{i}\left(\frac{t^3}{3} + a\,t^2 + bt\right)\right] \mathrm{d}t = 2\pi\,\mathrm{e}^{\mathrm{i}a(2a^2/3-b)}\,Ai(b-a^2). \tag{2.25}$$

Olver (1974) gives the expressions

$$Ai(x) = \frac{1}{\mathrm{i}\pi}\int_0^{\mathrm{i}\infty} \cosh\left(\frac{z^3}{3} - xz\right)\mathrm{d}z, \tag{2.26}$$

and for $x > 0$

$$Ai(-x) = \frac{x^{1/2}}{\pi}\int_{-1}^{\infty} \cos\left[x^{3/2}\left(\frac{z^3}{3} + z^2 - \frac{2}{3}\right)\right]\mathrm{d}z. \tag{2.27}$$

Copson (1963) the expression

$$Ai(x) = \frac{\mathrm{e}^{-\xi}}{2\pi}\int_0^{\infty} \mathrm{e}^{-x^{1/2}z}\cos\left(\frac{z^{3/2}}{3}\right)\frac{\mathrm{d}z}{\sqrt{z}},\; x > 0,\; \xi = \frac{2}{3}x^{3/2}, \tag{2.28}$$

and Reid (1995) the following expression ($x > 0$)

$$\begin{aligned} Ai(x) &= \frac{\sqrt{3}}{2\pi}\int_0^{+\infty} \mathrm{e}^{-\frac{x^3t^3}{3}-\frac{1}{3t^3}}\frac{\mathrm{d}t}{t^2}, \qquad (2.29)\\ &= \frac{\sqrt{3}}{2\pi}\int_0^{+\infty} \mathrm{e}^{-\frac{t^3}{3}-\frac{x^3}{3t^3}}\mathrm{d}t. \end{aligned}$$

For the function Bi, we have the integral representation

$$Bi(x) = \frac{1}{\pi}\int_0^{\infty}\left[\mathrm{e}^{-z^3/3+xz} + \sin\left(\frac{z^3}{3} + xz\right)\right]\mathrm{d}z. \tag{2.30}$$

Other formulae [Gordon (1969); Schulten et al. (1979)] having a great interest for the numerical computation of Airy functions are obtained by setting

$$\rho(x) = \frac{1}{\pi^{1/2}\,2^{11/6}\,3^{2/3}\,x^{2/3}}\mathrm{e}^{-x}Ai\left[\left(\frac{3x}{2}\right)^{2/3}\right],\; x > 0. \tag{2.31}$$

In fact, the Bessel function $K_\nu(x)$ verifies the relation [Gradshetyn & Ryzhik (1965)]

$$\int_0^\infty \frac{e^{-u}K_\nu(u)}{u+t}\frac{du}{\sqrt{u}} = \pi\frac{e^t K_\nu(t)}{\sqrt{t}\cos(\pi\nu)}, \quad \Re(\nu) < \frac{1}{2}, \ \arg(t) < \pi.$$

In particular, for $K_{1/3}(x) = \frac{\pi\sqrt{3}}{\left(\frac{3x}{2}\right)^{1/3}} Ai\left[\left(\frac{3x}{2}\right)^{2/3}\right]$ (cf. §2.2.4), we obtain

$$Ai(x) = \frac{e^{-\xi}}{2\pi^{1/2}x^{1/4}}\int_0^{+\infty}\frac{\rho(z)}{1+\frac{z}{\xi}}dz, \tag{2.32}$$

with ρ defined as above and $\xi = \frac{2}{3}x^{3/2}$. In a similar fashion, we shall have, for $x > 0$

$$Bi(x) = \frac{e^{\xi}}{\pi^{1/2}x^{1/4}}\int_0^{+\infty}\frac{\rho(z)}{1-\frac{z}{\xi}}dz \tag{2.33}$$

$$Ai(-x) = \frac{1}{\pi^{1/2}x^{1/4}}\int_0^{+\infty}\frac{\cos\left(\xi-\frac{\pi}{4}\right)+\frac{z}{\xi}\sin\left(\xi-\frac{\pi}{4}\right)}{1+\left(\frac{z}{\xi}\right)^2}\rho(z)dz \tag{2.34}$$

$$Bi(-x) = \frac{1}{\pi^{1/2}x^{1/4}}\int_0^{+\infty}\frac{\frac{z}{\xi}\cos\left(\xi-\frac{\pi}{4}\right)-\sin\left(\xi-\frac{\pi}{4}\right)}{1+\left(\frac{z}{\xi}\right)^2}\rho(z)dz. \tag{2.35}$$

It should be noted that the integral representations (2.19) and (2.20) are the most frequently used.

2.1.4 *Ascending and asymptotic series*

2.1.4.1 *Expansion of Ai near the origin*

The expansion of Ai near the origin $x = 0$ is [Copson (1967)]

$$Ai(x) = \frac{1}{\pi 3^{2/3}}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{n+1}{3}\right)}{n!}\sin\left[\frac{2}{3}(n+1)\pi\right]\left(3^{1/3}x\right)^n. \tag{2.36}$$

2.1.4.2 *Ascending series of Ai and Bi*

The ascending series of $Ai(x)$ and $Bi(x)$ are defined [Miller (1946); Abramowitz & Stegun (1965)] by the following chain rule

$$Ai(x) = c_1 f(x) - c_2 g(x) \tag{2.37}$$

$$Bi(x) = \sqrt{3}\left[c_1 f(x) + c_2 g(x)\right], \tag{2.38}$$

with $c_1 = Ai(0)$ and $c_2 = Ai'(0)$, and the series

$$f(x) = \sum_{k=0}^{\infty} 3^k \left(\frac{1}{3}\right)_k \frac{x^{3k}}{(3k)!} = 1 + \frac{1}{3!}x^3 + \frac{1.4}{6!}x^6 + \frac{1.4.7}{9!}x^9 + \ldots$$

$$g(x) = \sum_{k=0}^{\infty} 3^k \left(\frac{2}{3}\right)_k \frac{x^{3k+1}}{(3k+1)!} = x + \frac{2}{4!}x^4 + \frac{2.5}{7!}x^7 + \frac{2.5.8}{10!}x^{10} + \ldots$$

where the Pochhammer symbol $(a)_n$ is defined by

$$(a)_0 = 1,\ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)(a+2)\ldots(a+n-1). \tag{2.39}$$

The ascending series of the derivatives $Ai'(x)$ and $Bi'(x)$ are obtained from the differentiation of the series $f(x)$ and $g(x)$ term–by–term. We obtain therefore

$$Ai'(x) = c_1 f'(x) - c_2 g'(x) \tag{2.40}$$

$$Bi'(x) = \sqrt{3}\left[c_1 f'(x) + c_2 g'(x)\right], \tag{2.41}$$

and the series

$$f'(x) = \frac{x^2}{2} + \frac{1}{2.3}\frac{x^5}{5} + \frac{1}{2.3.5.6}\frac{x^8}{8} + \ldots$$

$$g'(x) = 1 + \frac{1}{1.3}\frac{x^3}{3} + \frac{1}{1.3.4.6}\frac{x^6}{6} + \frac{1}{1.3.4.6.7.9}\frac{x^9}{9} + \ldots$$

2.1.4.3 *Asymptotic series of Ai and Bi*

The asymptotic series of Ai and Bi are calculated with the steepest descent method [Olver (1954); Chester et al. (1957)]. We will calculate the asymptotic series of $Ai(x)$ for $x > 0$. The definition (2.2) of Ai allows us to write

$$Ai(x) = \frac{1}{2\pi\mathrm{i}} \int\limits_{\mathrm{C_0}} \mathrm{e}^{t^3/3 - xt} \mathrm{d}t,$$

where $\mathrm{C_0}$ is the contour defined in §2.1.1. Setting

$$t = \sqrt{x} + \mathrm{i}u, \ -\infty < u < \infty,$$

we obtain

$$\begin{aligned}\pi \mathrm{e}^{\xi} Ai(x) &= \int\limits_0^{\infty} \mathrm{e}^{-u^2\sqrt{x}} \cos\left(\frac{u^3}{3}\right) \mathrm{d}u \\ &= \frac{1}{2x^{1/4}} \int\limits_{-\infty}^{+\infty} \mathrm{e}^{-v^2} \cos\left(\frac{v^3}{3x^{3/4}}\right) \mathrm{d}v,\end{aligned}$$

with $\xi = \frac{2}{3}x^{3/2}$. The *cosine* function may be replaced by its expansion:

$$\pi \mathrm{e}^{\xi} Ai(x) = \frac{1}{2x^{1/4}} \int\limits_{-\infty}^{+\infty} \mathrm{e}^{-v^2} \left(1 - \frac{v^6}{2!3^2x^{3/2}} + \frac{v^{12}}{4!3^4x^3} - \ldots\right) \mathrm{d}v.$$

Integrating term–by–term

$$\pi \mathrm{e}^{\xi} Ai(x) \approx \frac{\pi^{1/2}}{2x^{1/4}} \left(1 - \frac{3.5}{1!144x^{3/2}} + \frac{5.7.9.11}{2!144^2x^3} - \ldots\right),$$

we obtain the formula given below (2.44). The other series are evaluated similarly. For $x \gg 1$ and $s \geq 1$, one defines

$$u_s = \frac{\Gamma(3s + 1/2)}{54^s s!\Gamma(s + 1/2)} = \frac{(2s+1)(2s+3)\ldots(6s-1)}{216^s s!} \tag{2.42}$$

$$v_s = -\frac{6s+1}{6s-1} u_s, \tag{2.43}$$

and the other series, according to the notation of Olver (1954)

$$L(z)=\sum_{s=0}^{\infty}\frac{u_s}{z^s}=1+\frac{3.5}{1!216}\frac{1}{z}+\frac{5.7.9.11}{2!216^2}\frac{1}{z^2}+\frac{7.9.11.13.15.17}{3!216^3}\frac{1}{z^3}+\dots$$

$$M(z)=\sum_{s=0}^{\infty}\frac{v_s}{z^s}=1-\frac{3.7}{1!216}\frac{1}{z}-\frac{5.7.9.13}{2!216^2}\frac{1}{z^2}-\frac{7.9.11.13.15.19}{3!216^3}\frac{1}{z^3}-\dots$$

$$P(z)=\sum_{s=0}^{\infty}(-1)^s\frac{u_{2s}}{z^{2s}}=1-\frac{5.7.9.11}{2!216^2}\frac{1}{z^2}+\frac{9.11.13.15.17.19.21.23}{4!216^4}\frac{1}{z^4}-\dots$$

$$Q(z)=\sum_{s=0}^{\infty}(-1)^s\frac{u_{2s+1}}{z^{2s+1}}=\frac{3.5}{1!216}\frac{1}{z}-\frac{7.9.11.13.15.17}{3!216^3}\frac{1}{z^3}+\dots$$

$$R(z)=\sum_{s=0}^{\infty}(-1)^s\frac{v_{2s}}{z^{2s}}=1+\frac{5.7.9.13}{2!216^2}\frac{1}{z^2}-\frac{9.11.13.15.17.19.21.25}{4!216^4}\frac{1}{z^4}+\dots$$

$$S(z)=\sum_{s=0}^{\infty}(-1)^s\frac{v_{2s+1}}{z^{2s+1}}=-\frac{3.7}{1!216}\frac{1}{z}+\frac{7.9.11.13.15.19}{3!216^3}\frac{1}{z^3}-\dots$$

Hence we obtain the asymptotic series of the Airy functions and of their derivatives (with $\xi=\frac{2}{3}x^{3/2}$)

$$Ai(x)\approx\frac{1}{2\pi^{1/2}x^{1/4}}\mathrm{e}^{-\xi}L(-\xi) \tag{2.44}$$

$$Ai'(x)\approx-\frac{x^{1/4}}{2\pi^{1/2}}\mathrm{e}^{-\xi}M(-\xi) \tag{2.45}$$

$$Bi(x) \approx \frac{1}{\pi^{1/2}x^{1/4}} e^{\xi} L(\xi) \tag{2.46}$$

$$Bi'(x) \approx \frac{x^{1/4}}{\pi^{1/2}} e^{\xi} M(\xi) \tag{2.47}$$

$$Ai(-x) \approx \frac{1}{\pi^{1/2}x^{1/4}} \left[\sin\left(\xi - \frac{\pi}{4}\right) Q(\xi) + \cos\left(\xi - \frac{\pi}{4}\right) P(\xi)\right] \tag{2.48}$$

$$Ai'(-x) \approx \frac{x^{1/4}}{\pi^{1/2}} \left[\sin\left(\xi - \frac{\pi}{4}\right) R(\xi) - \cos\left(\xi - \frac{\pi}{4}\right) S(\xi)\right] \tag{2.49}$$

$$Bi(-x) \approx \frac{1}{\pi^{1/2}x^{1/4}} \left[-\sin\left(\xi - \frac{\pi}{4}\right) P(\xi) + \cos\left(\xi - \frac{\pi}{4}\right) Q(\xi)\right] \tag{2.50}$$

$$Bi'(-x) \approx \frac{x^{1/4}}{\pi^{1/2}} \left[\sin\left(\xi - \frac{\pi}{4}\right) S(\xi) + \cos\left(\xi - \frac{\pi}{4}\right) R(\xi)\right]. \tag{2.51}$$

2.2 Properties of Airy Functions

2.2.1 *Zeros of Airy functions*

Zeros of the Airy function $Ai(x)$ are located on the negative part of the real axis. According to the notation of Miller (1946), we define a_s and a'_s, the s^{th} zeros of $Ai(x)$ and $Ai'(x)$, b_s and b'_s the real zeros of $Bi(x)$ and $Bi'(x)$, β_s and β'_s the complex zeros of $Bi(x)$ and $Bi'(x)$ in the region defined by $\frac{\pi}{3} < \arg(x) < \frac{\pi}{2}$. The complex zeros of $Bi(x)$ and $Bi'(x)$ in the region $-\frac{\pi}{2} < \arg(x) < -\frac{\pi}{3}$ are the conjugates of β_s and β'_s. We thus obtain

$$a_s = -f\left[\frac{3\pi}{8}(4s-1)\right] \tag{2.52}$$

$$a'_s = -g\left[\frac{3\pi}{8}(4s-3)\right] \tag{2.53}$$

$$b_s = -f\left[\frac{3\pi}{8}(4s-3)\right] \tag{2.54}$$

$$b'_s = -g\left[\frac{3\pi}{8}(4s-1)\right] \tag{2.55}$$

$$\beta_s = \mathrm{e}^{\mathrm{i}\pi/3} f\left[\frac{3\pi}{8}(4s-1) + \frac{3\mathrm{i}}{4}\ln(2)\right] \tag{2.56}$$

$$\beta'_s = \mathrm{e}^{\mathrm{i}\pi/3} g\left[\frac{3\pi}{8}(4s-3) + \frac{3\mathrm{i}}{4}\ln(2)\right]. \tag{2.57}$$

We also have the relations:

$$Ai'(a_s) = (-1)^{s-1} f_1\left[\frac{3\pi}{8}(4s-1)\right] \tag{2.58}$$

$$Ai(a'_s) = (-1)^{s-1} g_1\left[\frac{3\pi}{8}(4s-3)\right] \tag{2.59}$$

$$Bi'(b_s) = (-1)^{s-1} f_1\left[\frac{3\pi}{8}(4s-3)\right] \tag{2.60}$$

$$Bi(b'_s) = (-1)^{s} g_1\left[\frac{3\pi}{8}(4s-1)\right] \tag{2.61}$$

$$Bi'(\beta_s) = (-1)^s\sqrt{2}\mathrm{e}^{-\mathrm{i}\pi/6} f_1\left[\frac{3\pi}{8}(4s-1) + \frac{3\mathrm{i}}{4}\ln(2)\right] \tag{2.62}$$

$$Bi(\beta_s) = (-1)^{s-1}\sqrt{2}\mathrm{e}^{\mathrm{i}\pi/6} g_1\left[\frac{3\pi}{8}(4s-3) + \frac{3\mathrm{i}}{4}\ln(2)\right]. \tag{2.63}$$

The distribution of the zeros of Ai and Bi in the complex plane is given on Figs. 2.3 and 2.4.

The functions $f(x)$, $g(x)$, $f_1(x)$ and $g_1(x)$ are defined, with $|x| \gg 1$, by the relations

$$f(x) \approx x^{2/3}\left(1 + \frac{5}{48x^2} - \frac{5}{36x^4} + \frac{77\,125}{82\,944x^6} - \ldots\right) \tag{2.64}$$

$$g(x) \approx x^{2/3}\left(1 - \frac{7}{48x^2} + \frac{35}{288x^4} - \frac{181\,223}{207\,360x^6} + \ldots\right) \tag{2.65}$$

$$f_1(x) \approx \frac{x^{1/6}}{\pi^{1/2}}\left(1 + \frac{5}{48x^2} - \frac{1\,525}{4\,608x^4} + \frac{2\,397\,875}{663\,552x^6} - \ldots\right) \tag{2.66}$$

$$g_1(x) \approx \frac{x^{-1/6}}{\pi^{1/2}}\left(1 - \frac{7}{96x^2} + \frac{1\,673}{6\,144x^4} - \frac{84\,394\,709}{26\,552\,080x^6} + \ldots\right). \tag{2.67}$$

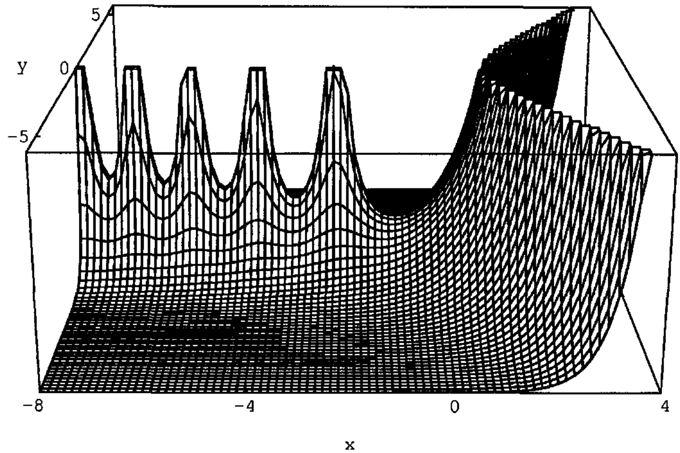


Fig. 2.3 Plot of $1/|Ai(x+iy)|$. Zeros of $|Ai(x+iy)|$ are located on the negative part of the real axis. The modulus of the Airy function $|Ai(z)|$ blows up outside this axis, except in the sector defined by $-\frac{\pi}{3} < \arg(z) < \frac{\pi}{3}$ where it goes to 0.

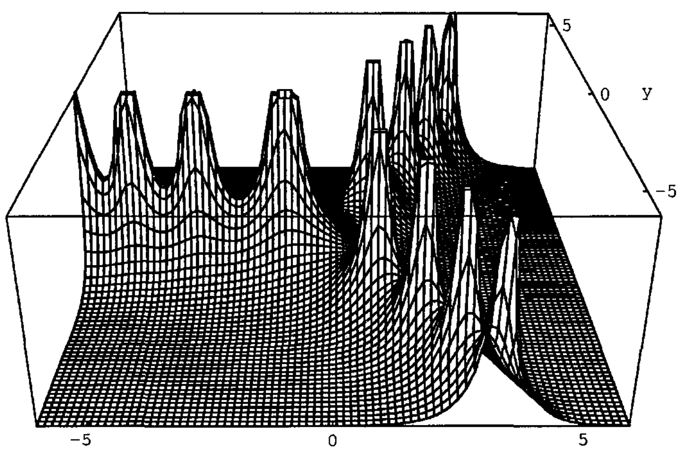


Fig. 2.4 Plot of $1/|Bi(x+iy)|$. The real zeros of $|Bi(x+iy)|$ can be discerned on the negative part of the real axis, and the conjugated complex pair of zeros in the sectors $\frac{\pi}{3} < \arg(z) < \frac{\pi}{2}$ and $-\frac{\pi}{2} < \arg(z) < -\frac{\pi}{3}$.

2.2.2 The spectral zeta function

The Airy function Ai is an entire function, we can then make use of the Weierstrass infinite product

$$Ai(z) = Ai(0)\, \mathrm{e}^{-\kappa z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{|a_n|}\right) \mathrm{e}^{-z/|a_n|}, \tag{2.68}$$

where a_n are the zeros of the Airy function Ai and

$$\kappa = \left|\frac{Ai'(0)}{Ai(0)}\right| = \frac{3^{5/6}\,\Gamma(2/3)^2}{2\pi} = 0.72901113\ldots.$$

We now take the logarithmic derivative of this product

$$\frac{\mathrm{d}}{\mathrm{d}z}\ln Ai(z) = \frac{Ai'(z)}{Ai(z)} = -\kappa + \sum_{n=1}^{\infty} \frac{1}{z+|a_n|} - \frac{1}{|a_n|}. \tag{2.69}$$

One more derivation of this result provides

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln Ai(z) = z - \left(\frac{Ai'(z)}{Ai(z)}\right)^2 = -\sum_{n=1}^{\infty} \frac{1}{(z+|a_n|)^2}. \tag{2.70}$$

Therefore, we obtain the following properties of the zeros of the Airy function

$$\sum_{n=1}^{\infty} \frac{1}{a_n^2} = \kappa^2. \tag{2.71}$$

More generally, we may define the Airy zeta function constructed with the zeros of this function

$$Z^-(s) = \sum_{n=1}^{\infty} \frac{1}{|a_n|^s}. \tag{2.72}$$

According to the asymptotic property of the zeros a_n (2.52), (2.64) and (2.65), the convergence of this series is ensured for $s > \frac{3}{2}$. Other particular cases of this zeta function may be obtained by taking the successive derivatives of Eq. (2.69). For instance, we have

$$Z^-(3) = \sum_{n=1}^{\infty} \frac{1}{|a_n|^3} = \frac{1}{2} - \kappa^3. \tag{2.73}$$

The same can be done for the derivative of the Airy function ($Ai''(0) = 0$)

$$Ai'(z) = Ai'(0) \prod_{n=1}^{\infty} \left(1 + \frac{z}{|a'_n|}\right) \mathrm{e}^{-z/|a'_n|}, \tag{2.74}$$

for which we define the zeta function[1] ($s > \frac{3}{2}$)

$$Z^+(s) = \sum_{n=1}^{\infty} \frac{1}{|a'_n|^s}. \tag{2.75}$$

Some particular cases are given in Table 2.1 [Crandal (1996); Voros (1999)]. The values for $n = 1$ are obtained by analytic continuations.

Table 2.1 Some particular values of the Airy zeta functions $Z^{\pm}(s)$.

n	1	2	3	4
$Z^-(n)$	$-\kappa$	κ^2	$\frac{1}{2} - \kappa^3$	$\frac{1}{3}\kappa - \kappa^4$
$Z^+(n)$	0	$\frac{1}{\kappa}$	1	$\frac{1}{\kappa^2}$

In this table, we can see the noteworthy result: *the sum of the inverse cubes of the zeros of $Ai'(z)$ is the unity.*

Flajolet & Louchard (2001) have studied the area Airy distribution which occurs in a number of combinatorial structures, like path length in trees, area below random walks, displacement in parking sequence, ... They found interesting results for the spectral function $Z^-(s)$ which is closely related to this area distribution. In particular, they extended the definition Eq. (2.72) by analytic continuation, and from the Mellin transform

$$\mathcal{M}\left[\frac{Ai'(z)}{Ai(z)} + \kappa\right](s) = Z^-(1-s)\,\frac{\pi}{\sin \pi s},$$

where $\mathcal{M}$ has the meaning of a Mellin transform, they showed that

$$Z^-(-1) = Z^-(-2) = Z^-(-4) = \ldots = 0.$$

Other results may be found in their paper [Flajolet & Louchard (2001)].

[1]The notation: $Z^{\pm}(s)$ comes from Voros (1999).

2.2.3 *Inequalities*

Some of the previous results may be used to prove inequalities for the Airy function Ai which is concave in the domain $(-a_1, \infty)$ [Salmassi (1999)]

$$x\,Ai^2(x) \leq Ai'^2(x). \tag{2.76}$$

For two different arguments

$$Ai(x)Ai(y) \leq Ai^2\left(\frac{x+y}{2}\right), \tag{2.77}$$

and for the scaling

$$Ai(x)^\alpha Ai(0)^{1-\alpha} \leq Ai(\alpha x); \quad 0 \leq \alpha \leq 1. \tag{2.78}$$

Similar inequalities may be found for the function Bi.

2.2.4 *Connection with Bessel functions*

As we said in the introduction, Airy functions Ai and Bi may be alternatively written in terms of Bessel functions I and J of order 1/3, and of order 2/3 for their derivatives, $\xi = \frac{2}{3}x^{3/2}$ [Jeffreys (1942); Olver (1974)]

$$\begin{aligned} Ai(x) &= \frac{\sqrt{x}}{3}\left[I_{-1/3}(\xi) - I_{1/3}(\xi)\right] \\ &= \frac{1}{\pi}\sqrt{\frac{x}{3}}K_{1/3}(\xi) \end{aligned} \tag{2.79}$$

$$\begin{aligned} Ai(-x) &= \frac{\sqrt{x}}{3}\left[J_{-1/3}(\xi) + J_{1/3}(\xi)\right] \\ &= \sqrt{\frac{x}{3}}\,\Re\left[e^{i\frac{\pi}{6}}H^{(1)}_{1/3}(\xi)\right] \end{aligned} \tag{2.80}$$

$$\begin{aligned} Ai'(x) &= -\frac{x}{3}\left[I_{-2/3}(\xi) - I_{2/3}(\xi)\right] \\ &= -\frac{1}{\pi}\frac{x}{\sqrt{3}}K_{2/3}(\xi) \end{aligned} \tag{2.81}$$

$$\begin{aligned} Ai'(-x) &= -\frac{x}{3}\left[J_{-2/3}(\xi) - J_{2/3}(\xi)\right] \\ &= \frac{x}{\sqrt{3}}\,\Re\left[e^{-i\frac{\pi}{6}}H^{(1)}_{2/3}(\xi)\right] \end{aligned} \tag{2.82}$$

$$\begin{aligned} Bi(x) &= \sqrt{\frac{x}{3}}\left[I_{-1/3}(\xi) + I_{1/3}(\xi)\right] \\ &= \sqrt{\frac{x}{3}}\,\Re\left[e^{i\frac{\pi}{6}} H^{(1)}_{1/3}(-i\xi)\right] \end{aligned} \tag{2.83}$$

$$\begin{aligned} Bi(-x) &= \sqrt{\frac{x}{3}}\left[J_{-1/3}(\xi) - J_{1/3}(\xi)\right] \\ &= -\sqrt{\frac{x}{3}}\,\Im\left[e^{i\frac{\pi}{6}} H^{(1)}_{1/3}(\xi)\right] \end{aligned} \tag{2.84}$$

$$\begin{aligned} Bi'(x) &= \frac{x}{\sqrt{3}}\left[I_{-2/3}(\xi) + I_{2/3}(\xi)\right] \\ &= \frac{x}{\sqrt{3}}\,\Re\left[e^{i\frac{\pi}{6}} H^{(1)}_{2/3}(-i\xi)\right] \end{aligned} \tag{2.85}$$

$$\begin{aligned} Bi'(-x) &= \frac{x}{\sqrt{3}}\left[J_{-2/3}(\xi) + J_{2/3}(\xi)\right] \\ &= -\frac{x}{\sqrt{3}}\,\Im\left[e^{-i\frac{\pi}{6}} H^{(1)}_{2/3}(\xi)\right]. \end{aligned} \tag{2.86}$$

The modified Bessel function $K_\nu(z)$ is defined by

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\pi\nu)},$$

and the Hankel functions $H^{(1)}_\nu(z)$, $H^{(2)}_\nu(z)$ by

$$H^{(1)}_\nu(z) = J_\nu(z) + iY_\nu(z),$$

$$H^{(2)}_\nu(z) = J_\nu(z) - iY_\nu(z),$$

with the Weber function

$$Y_\nu(z) = \frac{J_\nu(z) - J_{-\nu}(z)}{\sin(\pi\nu)}.$$

2.2.5 *Modulus and phase of Airy functions*

2.2.5.1 *Definitions*

We define the modulus $M(x)$ and the phase $\theta(x)$ of the functions $Ai(x)$ and $Bi(x)$, and the modulus $N(x)$ and the phase $\phi(x)$ of the functions $Ai'(x)$

and $Bi'(x)$, for any $x > 0$ by the relations [Miller (1946); Abramowitz & Stegun, (1965)]

$$Ai(-x) = M(x)\cos[\theta(x)] \tag{2.87}$$
$$Bi(-x) = M(x)\sin[\theta(x)] \tag{2.88}$$
$$Ai'(-x) = N(x)\cos[\phi(x)] \tag{2.89}$$
$$Bi'(-x) = N(x)\sin[\phi(x)], \tag{2.90}$$

and the inverse relations

$$M(x) = \left[Ai^2(-x) + Bi^2(-x)\right]^{1/2} \tag{2.91}$$
$$\theta(x) = \arctan\left[\frac{Bi(-x)}{Ai(-x)}\right] \tag{2.92}$$
$$N(x) = \left[Ai'^2(-x) + Bi'^2(-x)\right]^{1/2} \tag{2.93}$$
$$\phi(x) = \arctan\left[\frac{Bi'(-x)}{Ai'(-x)}\right]. \tag{2.94}$$

2.2.5.2 *Differential equations*

Moduli and phases are solutions to the following differential equations, for $x > 0$ [Miller (1946); Abramowitz & Stegun (1965)]

$$M^2\theta' = -\frac{1}{\pi} \tag{2.95}$$
$$N^2\phi' = -\frac{x}{\pi} \tag{2.96}$$
$$N^2 = M'^2 + M^2\theta'^2 \tag{2.97}$$
$$NN' = -xMM' \tag{2.98}$$

$$\tan(\phi - \theta) = \frac{M\theta'}{M'} \tag{2.99}$$
$$\sin(\phi - \theta) = \frac{1}{\pi MN} \tag{2.100}$$

$$M'' + xM - \frac{1}{\pi^2 M^3} = 0 \tag{2.101}$$
$$\left(M^2\right)''' + 4x\left(M^2\right)' + 2M^2 = 0 \tag{2.102}$$
$$\theta'^2 + \frac{1}{2}\frac{\theta'''}{\theta'} - \frac{3}{4}\left(\frac{\theta''}{\theta'}\right)^2 = x. \tag{2.103}$$

This last expression may be alternatively written $\theta'^2 + \frac{1}{2}\{\theta, x\} = x$, where $\{\theta, x\}$ is the Schwarzian derivative of θ with respect to x, as we shall see in §5.2.

2.2.5.3 *Asymptotic expansions*

For $x \gg 1$, the asymptotic series for the moduli and phases of Airy functions are [Miller (1946); Abramowitz & Stegun (1965)]

$$\begin{aligned} M^2(x) &\approx \frac{1}{\pi x^{1/2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{12^k k!} 2^{3k} \left(\frac{1}{2}\right)_{3k} (2x)^{-3k} \\ &\approx \frac{1}{\pi x^{1/2}} \left(1 - \frac{1.3.5}{1!96x^3} + \frac{1.3.5.7.9.11}{2!96^2x^6} \right. \\ &\qquad \left. - \frac{1.3.5.7.9.11.13.15.17}{3!96^3x^9} + \dots \right) \end{aligned} \tag{2.104}$$

$$\begin{aligned} N^2(x) &\approx \frac{x^{1/2}}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{12^k k!} \frac{6k+1}{6k-1} 2^{3k} \left(\frac{1}{2}\right)_{3k} (2x)^{-3k} \\ &\approx \frac{x^1/2}{\pi} \left(1 + \frac{1.3}{1!96} \frac{7}{x^3} - \frac{1.3.5.7.9}{2!96^2} \frac{13}{x^6} \right. \\ &\qquad \left. + \frac{1.3.5.7.9.11.13.15}{3!96^3} \frac{19}{x^9} - \dots \right) \end{aligned} \tag{2.105}$$

and

$$\begin{aligned} \theta(x) \approx \frac{\pi}{4} - \frac{2}{3} x^{3/2} &\left[1 - \frac{5}{4} \frac{1}{(2x)^3} + \frac{1\,105}{96} \frac{1}{(2x)^6} \right. \\ &\left. - \frac{82\,825}{128} \frac{1}{(2x)^9} + \frac{1\,282\,031\,525}{14\,336} \frac{1}{(2x)^{12}} - \dots \right] \end{aligned} \tag{2.106}$$

$$\begin{aligned} \phi(x) \approx \frac{3\pi}{4} - \frac{2}{3} x^{3/2} &\left[1 + \frac{7}{4} \frac{1}{(2x)^3} - \frac{1\,463}{96} \frac{1}{(2x)^6} \right. \\ &\left. + \frac{495\,271}{640} \frac{1}{(2x)^9} - \frac{206\,530\,429}{2\,048} \frac{1}{(2x)^{12}} - \dots \right], \end{aligned} \tag{2.107}$$

where $(\alpha)_k$ is the Pochhammer symbol defined in §2.1.4, formula (2.39).

2.2.5.4 *Functions of positive arguments*

When the argument of the Airy functions is positive, they do not oscillate, but increase or decrease exponentially. It is then convenient to "exponentially" normalise these functions [Alexander & Manolopoulos (1987)]

$$ai(x) = \mathrm{e}^{\xi} Ai(x) \tag{2.108}$$

$$ai'(x) = \mathrm{e}^{\xi} Ai'(x) \tag{2.109}$$

$$bi(x) = \mathrm{e}^{-\xi} Bi(x) \tag{2.110}$$

$$bi'(x) = \mathrm{e}^{-\xi} Bi'(x) \tag{2.111}$$

with $\xi = \frac{2}{3}x^{3/2}$, $x > 0$. We can then, as above, define the moduli $\overline{M}(x)$ and $\overline{N}(x)$, and the hyperbolic phases $\chi(x)$ and $\eta(x)$ of Airy functions by

$$ai(x) = \overline{M}(x) \cosh\left[\chi(x)\right] \tag{2.112}$$

$$bi(x) = \overline{M}(x) \sinh\left[\chi(x)\right] \tag{2.113}$$

$$ai'(x) = \overline{N}(x) \cosh\left[\eta(x)\right] \tag{2.114}$$

$$bi'(x) = \overline{N}(x) \sinh\left[\eta(x)\right], \tag{2.115}$$

and the inverse relations

$$\overline{M}(x) = \left[bi^2(x) - ai^2(x)\right]^{1/2} \tag{2.116}$$

$$\chi(x) = \operatorname{arctanh}\left[\frac{ai(x)}{bi(x)}\right] \tag{2.117}$$

$$\overline{N}(x) = \left[bi'^2(x) - ai'^2(x)\right]^{1/2} \tag{2.118}$$

$$\eta(x) = \operatorname{arctanh}\left[\frac{ai'(x)}{bi'(x)}\right]. \tag{2.119}$$

We can also calculate the asymptotic expansion of these functions. For $x \gg 1$, we have

$$\overline{M}^2(x) \approx \frac{3}{4\pi x^{1/2}}\left[1 + \frac{25}{72x^{3/2}} + \ldots\right] \tag{2.120}$$

$$\overline{N}^2(x) \approx \frac{3x^{1/2}}{4\pi}\left[1 + \frac{7}{24x^{3/2}} + \ldots\right] \tag{2.121}$$

$$\chi(x) \approx \frac{1}{2}\ln 3 - \frac{5}{36x^{3/2}} + \ldots \tag{2.122}$$

$$\eta(x) \approx -\frac{1}{2}\ln 3 + \ldots \tag{2.123}$$

2.3 The Inhomogeneous Airy Functions

2.3.1 *Definitions*

In this section, we consider the inhomogeneous differential equation of the second order [Scorer (1950); Abramowitz & Stegun (1965)]

$$y'' - xy = \pm\pi^{-1}, \tag{2.124}$$

the resolution of which is done by a similar method to the homogeneous one. The solutions are the inhomogeneous functions $Gi(x)$ and $Hi(x)$ (also called *Scorer functions*), according to the sign of the right member $-$ or $+$ respectively. The integral representations of these functions are given by

$$Gi(x) = \frac{1}{\pi}\int_0^\infty \sin\left(\frac{t^3}{3} + xt\right)\mathrm{d}t \tag{2.125}$$

and:

$$Hi(x) = \frac{1}{\pi}\int_0^\infty \mathrm{e}^{-t^3/3+xt}\mathrm{d}t. \tag{2.126}$$

We can alternatively define these functions from the primitives of the homogeneous functions [Scorer (1950); Olver (1974)]

$$Gi(x) = Bi(x)\int_x^{+\infty} Ai(t)\mathrm{d}t + Ai(x)\int_0^x Bi(t)\mathrm{d}t \tag{2.127}$$

$$Hi(x) = Bi(x)\int_{-\infty}^x Ai(t)\mathrm{d}t - Ai(x)\int_{-\infty}^x Bi(t)\mathrm{d}t. \tag{2.128}$$

These functions are related to the homogeneous Airy function $Bi(x)$ by the important relation

$$Bi(x) = Gi(x) + Hi(x). \tag{2.129}$$

We have plotted on Figs. 2.5 and 2.6 the inhomogeneous Airy functions $Gi(x)$, $Hi(x)$ and their derivatives $Gi'(x)$ and $Hi'(x)$.

The general solutions of Eq. (2.124) are then

$$y(x) = aAi(x) + bBi(x) + Gi(x), \tag{2.130}$$

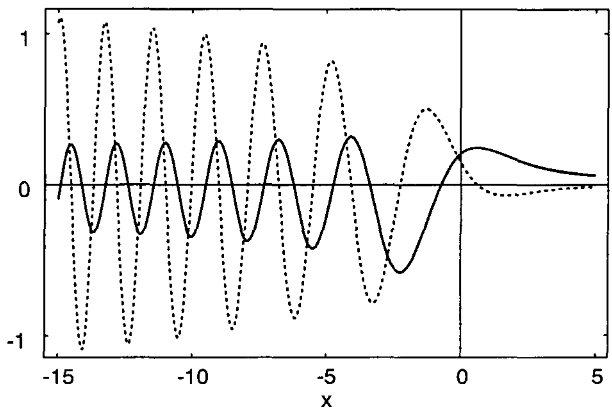


Fig. 2.5 Plot of the inhomogeneous Airy function Gi (full line) and its derivative Gi' (dotted line).

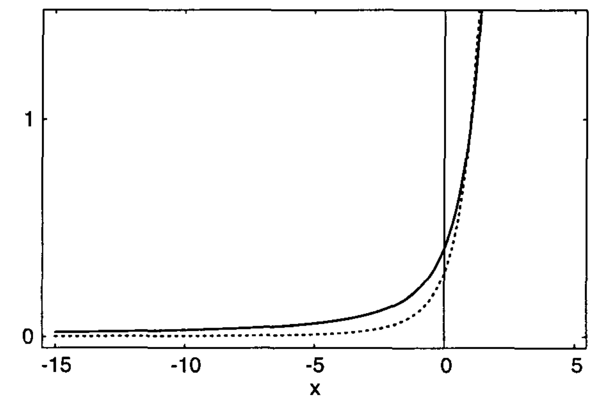


Fig. 2.6 Plot of the inhomogeneous Airy function Hi (full line) and its derivative Hi' (dotted line).

or

$$y(x) = cAi(x) + dBi(x) + Hi(x), \tag{2.131}$$

according to the equation (2.129); a, b, c and d being integration constants.

2.3.2 Properties of inhomogeneous Airy functions

2.3.2.1 Values at the origin

The values at the origin of inhomogeneous Airy functions and of their derivatives are given by [Scorer (1950); Gordon (1970)]

$$Gi\,(0) = \frac{Hi\,(0)}{2} = \frac{Ai\,(0)}{\sqrt{3}} = \frac{1}{3^{7/6}\Gamma\left(\frac{2}{3}\right)}, \tag{2.132}$$

$$Gi'\,(0) = \frac{Hi'\,(0)}{2} = -\frac{Ai'\,(0)}{\sqrt{3}} = \frac{1}{3^{5/6}\Gamma\left(\frac{1}{3}\right)}. \tag{2.133}$$

2.3.2.2 Other integral representations

Besides the definitions (2.125) and (2.126) of Gi and Hi, these functions may be given in terms of the following integrals, for $x > 0$ [Gordon (1970)]

$$Gi(x) = \frac{4x^2}{3\sqrt{3}\pi^2}\int_0^\infty \frac{K_{1/3}(t)}{\xi^2 - t^2}\mathrm{d}t, \tag{2.134}$$

$$Hi(-x) = \frac{4x^2}{3\sqrt{3}\pi^2}\int_0^\infty \frac{K_{1/3}(t)}{\xi^2 + t^2}\mathrm{d}t, \tag{2.135}$$

where $K_{1/3}(t)$ is the modified Bessel function and $\xi = \frac{2}{3}x^{3/2}$. We also have for $Gi(x)$ the following integral representation [Lee (1980)]

$$Gi(x) = -\frac{1}{\pi}\int_0^\infty \mathrm{e}^{-t^3/3 - tx/2}\cos\left(\frac{\sqrt{3}}{2}tx + \frac{2\pi}{3}\right)\mathrm{d}t. \tag{2.136}$$

It should be noted that the function $Hi(x)$ is a particular case of the Faxén integral (1921),

$$Fi\,(\alpha, \beta; y) = \int_0^\infty e^{t^\alpha y - t}t^{\beta - 1}\mathrm{d}t,\ 0 \leq \Re(\alpha) < 1,\ \Re(\beta) > 0,$$

that is to say

$$Hi(x) = \frac{1}{3^{2/3}\pi}Fi\left(\frac{1}{3}, \frac{1}{3}; 3^{1/3}x\right). \tag{2.137}$$

2.3.3 *Ascending and asymptotic series*

2.3.3.1 *Ascending series*

We can write the integral representation of the function $Hi(x)$ (formula (2.126)) [Scorer (1950); Lee (1980)], as

$$Hi(x) = \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{x^k}{k!}\int_0^{\infty} t^k \mathrm{e}^{-t^3/3}\mathrm{d}t.$$

We recognise, on this relation, the expression of the gamma function $\Gamma(z) = \int_0^\infty \mathrm{e}^{-u}u^{z-1}\mathrm{d}u$. From which we deduce the ascending series

$$Hi(x) = \frac{1}{\pi}\sum_{k=0}^{\infty} 3^{(k-2)/3}\Gamma\left(\frac{k+1}{3}\right)\frac{x^k}{k!}. \tag{2.138}$$

This result may be alternatively written

$$Hi(x) = c_3 f(x) + c_4 g(x) + \frac{1}{\pi}h(x), \tag{2.139}$$

with: $c_3 = Hi(0)$, $c_4 = Hi'(0)$, and the series

$$f(x) = 1 + \frac{1}{3!}x^3 + \frac{1.4}{6!}x^6 + \frac{1.4.7}{9!}x^9 + \ldots$$

$$g(x) = x\left(1 + \frac{2}{4!}x^3 + \frac{2.5}{7!}x^6 + \frac{2.5.8}{10!}x^9 + \ldots\right)$$

$$h(x) = x^2\left(\frac{1}{2} + \frac{3}{5!}x^3 + \frac{3.6}{8!}x^6 + \frac{3.6.9}{11!}x^9 + \ldots\right).$$

Note that the series of f and g are identical to the series defined in §2.1.4.2. The ascending series of $Gi(x)$ can be deduced from the ones of $Hi(x)$ and $Bi(x)$ (formula (2.38)) thanks to the relation (2.129). We obtain

$$Gi(x) = \frac{1}{3}Bi(x) - \frac{1}{\pi}h(x). \tag{2.140}$$

For the derivatives, we have

$$Hi'(x) = \frac{1}{\pi}\sum_{k=0}^{\infty} 3^{(k-1)/3}\Gamma\left(\frac{k+2}{3}\right)\frac{x^k}{k!}. \tag{2.141}$$

The ascending series of $Gi'(x)$ is deduced (like for $Gi(x)$) thanks to the ascending series of $Bi'(x)$ and $Hi'(x)$.

2.3.3.2 *Asymptotic series*

The asymptotic expansions of the functions $Gi(x)$ and $Hi(-x)$ are, for $x \gg 1$ [Scorer (1950); Lee (1980)]:

$$Gi(x) \approx \frac{1}{\pi x} \sum_{n=0}^{\infty} \frac{(3n)!}{3^n n!} \frac{1}{x^{3n}} \tag{2.142}$$

$$\approx \frac{1}{\pi x} \left(1 + \frac{2!}{x^3} + \frac{5!}{3x^6} + \frac{8!}{3.6x^9} + \dots\right)$$

$$Hi(-x) \approx \frac{1}{\pi x} \sum_{n=0}^{\infty} \frac{(-1)^n (3n)!}{3^n n!} \frac{1}{x^{3n}}. \tag{2.143}$$

$$\approx \frac{1}{\pi x} \left(1 - \frac{2!}{x^3} + \frac{5!}{3x^6} - \frac{8!}{3.6x^9} + \dots\right).$$

From the two preceding formulae and from the relation (2.129), we can obtain the expansion of $Gi(-x)$ and $Hi(x)$. So we have

$$Gi(-x) = Bi(-x) - Hi(-x),$$

and:

$$Hi(x) = Bi(x) - Gi(x),$$

the asymptotic expansions of $Bi(x)$ and $Bi(-x)$ being given by the formulae (2.46) and (2.50). Olver (1954) gives the asymptotic series under an equivalent form

$$Gi(x) \approx \frac{1}{\pi x} \left[1 + \frac{1}{x^3} \sum_{s=0}^{\infty} \frac{(3s+2)!}{s!\,(3x^3)^s}\right], \quad x \to +\infty \tag{2.144}$$

$$Hi(x) \approx -\frac{1}{\pi x} \left[1 + \frac{1}{x^3} \sum_{s=0}^{\infty} \frac{(3s+2)!}{s!\,(3x^3)^s}\right], \quad x \to -\infty. \tag{2.145}$$

2.3.4 *Zeros of the Scorer functions*

In an interesting paper on the zeros of the Scorer functions, Gil, Segura and Temme (2003) gave several important results on the subject. Here, we limit ourselves to two:

- The Scorer function Hi has no real zero, but infinite many complex zeros on the half line ph$z = \pi/3$, and at the complex conjugated values.
- The derivative Gi' has exactly one positive zero at $g' = 0.609075417\ldots$, as it is seen on Fig. (2.5).

For the other values and properties of these zeros, we leave the reader to the paper of Gil et al. (2003).

2.4 Squares and Products of Airy Functions

2.4.1 *Differential equation and integral representation*

The homogeneous differential equation of the third order

$$y''' - 4xy' - 2y = 0 \tag{2.146}$$

has three linearly independent solutions [Aspnes (1966); Reid (1995)]: $Ai^2(x)$, $Ai(x)Bi(x)$ and $Bi^2(x)$, whose Wronskian is

$$W\left\{Ai^2(x), Ai(x)Bi(x), Bi^2(x)\right\} = \frac{2}{\pi^3}\,. \tag{2.147}$$

The solution $Ai^2(x)$ may be written as the integral

$$Ai^2(x) = \frac{1}{4\pi^2} \iint_{\mathbb{R}^2} \mathrm{e}^{\,\mathrm{i}\left[u^3/3+v^3/3+(u+v)x\right]} \mathrm{d}u \mathrm{d}v.$$

Changing the variables

$$s = \frac{1}{2}(v-u)\,; \quad t = v+u,$$

and integrating the variable s, we obtain an integral representation of $Ai^2(x)$,

$$Ai^2(x) = \frac{1}{2\pi^{3/2}} \int_0^{\infty} \cos\left(\frac{t^3}{12} + tx + \frac{\pi}{4}\right) \frac{\mathrm{d}t}{\sqrt{t}}. \tag{2.148}$$

Similarly, we have an integral representation of $Ai(x)Bi(x)$

$$Ai(x)Bi(x) = \frac{1}{2\pi^{3/2}} \int_0^{\infty} \sin\left(\frac{t^3}{12} + tx + \frac{\pi}{4}\right) \frac{\mathrm{d}t}{\sqrt{t}}. \tag{2.149}$$

An interesting formula for $Ai^2(x)$ is given by

$$Ai^2(x) = \frac{1}{4\pi\sqrt{3}} \int_0^{\infty} J_0\left(\frac{t^3}{12} + tx\right) t\,\mathrm{d}t. \tag{2.150}$$

We can generalise the preceding result in the case of the product of Airy functions with different arguments [Vallée et al. (1997)]:

$$\begin{aligned} &Ai(u)Ai(v) \\ &= \frac{1}{2^{1/3}\pi} \int_{-\infty}^{+\infty} Ai\left[2^{2/3}\left(t^2 + \frac{u+v}{2}\right)\right] \mathrm{e}^{\mathrm{i}(u-v)t} \mathrm{d}t \\ &= \frac{1}{2\pi^{3/2}} \int_0^{\infty} \cos\left(\frac{t^3}{12} + \frac{u+v}{2}t - \frac{(u-v)^2}{4t} + \frac{\pi}{4}\right) \frac{\mathrm{d}t}{\sqrt{t}}. \end{aligned} \tag{2.151}$$

This result allows us to obtain

$$Ai^2(x) = \frac{1}{2^{1/3}\pi} \int_{-\infty}^{+\infty} Ai\left[2^{2/3}\left(t^2 + x\right)\right] \mathrm{d}t \tag{2.152}$$

$$Ai(x)Ai(-x) = \frac{1}{2^{1/3}\pi} \int_{-\infty}^{+\infty} Ai\left[2^{2/3}t^2\right] \mathrm{e}^{2\mathrm{i}xt} \mathrm{d}t \tag{2.153}$$

$$\begin{aligned} &Ai(x)Ai(x^*) = \left|Ai(x)\right|^2 \\ &= \frac{1}{2^{1/3}\pi} \int_{-\infty}^{+\infty} Ai\left[2^{2/3}\left(t^2 + \Re(x)\right)\right] \cosh\left[2t\ \Im(x)\right] \mathrm{d}t. \end{aligned} \tag{2.154}$$

The following relations can also be established [Aspnes (1966): cf. §3.5.2]

$$Ai^2(x) = \frac{1}{2^{2/3}\pi} \int_0^{\infty} Ai\left(2^{2/3}x + t\right) \frac{\mathrm{d}t}{\sqrt{t}} \tag{2.155}$$

$$Ai(x)Bi(x) = \frac{1}{2^{2/3}\pi} \int_0^{\infty} Ai\left(2^{2/3}x - t\right) \frac{\mathrm{d}t}{\sqrt{t}} \tag{2.156}$$

$$Ai(x)Ai'(x) = \frac{1}{2\pi} \int_0^{\infty} Ai'\left(2^{2/3}x + t\right) \frac{\mathrm{d}t}{\sqrt{t}}, \tag{2.157}$$

and [Aspnes (1967): cf. §3.5.2]

$$Ai^2(x) = -\frac{1}{2^{2/3}\pi}\int_0^\infty Gi\left(2^{2/3}x - t\right)\frac{\mathrm{d}t}{\sqrt{t}} \tag{2.158}$$

$$Ai(x)Bi(x) = \frac{1}{2^{2/3}\pi}\int_0^\infty Gi\left(2^{2/3}x + t\right)\frac{\mathrm{d}t}{\sqrt{t}}. \tag{2.159}$$

Finally, Reid (1995) gives the relations

$$Ai^2(x) + Bi^2(x) = \frac{1}{\pi^{3/2}}\int_0^\infty \mathrm{e}^{xt - t^3/12}\frac{\mathrm{d}t}{\sqrt{t}} \tag{2.160}$$

$$Ai\left(x\mathrm{e}^{-\mathrm{i}\pi/6}\right)Ai\left(x\mathrm{e}^{\mathrm{i}\pi/6}\right) = \frac{1}{4\pi^{3/2}}\int_0^{+\infty} \mathrm{e}^{-t^3/12 - x^2/t}\frac{\mathrm{d}t}{\sqrt{t}} \tag{2.161}$$

$$\begin{aligned} Ai(x)Ai(y) &= \frac{\sqrt{3}}{2\pi}\int_0^\infty \mathrm{e}^{-(x^3+y^3)\frac{t^3}{3} - \frac{1}{3t^3}} Ai\left(xyt^2\right)\frac{\mathrm{d}t}{t^2} \\ &= \frac{\sqrt{3}}{2\pi}\int_0^\infty \mathrm{e}^{-\frac{t^3}{3} - \frac{x^3+y^3}{3t^3}} Ai\left(\frac{xy}{t^2}\right)\mathrm{d}t, \quad x + y > 0. \end{aligned} \tag{2.162}$$

2.4.2 *A remarkable identity*

Similar to the relations in the complex plane (*cf.* Eqs. (2.3), (2.5)), Voros (1999) proved the noteworthy result

$$D_0^2 + D_1^2 + D_2^2 - 2(D_0D_1 + D_1D_2 + D_2D_0) + 4 = 0, \tag{2.163}$$

where

$$D_0 = -2\pi\left(Ai^2(z)\right)', \quad D_1 = -2\pi\left(Ai^2(\mathrm{j}^2 z)\right)', \quad D_2 = -2\pi\left(Ai^2(\mathrm{j}z)\right)'.$$

2.4.3 *The product $Ai(x)Ai(-x)$: Airy wavelets*

As we shall see in §4.2, the Airy function allows the definition of a semi-group of transformation. Unfortunately the weak decreasing of the Airy function for the negative values of the variable (as $x^{-1/4}$ in average) deters the use of this transformation as a numerical filter.

However, as we have seen in the preceding paragraph an integral representation of the product $Ai(x)Ai(-x)$ corresponds to the Fourier transform

$$Ai(x)Ai(-x) = \frac{1}{2^{4/3}\pi} \int_{-\infty}^{+\infty} Ai\left[2^{-4/3}u^2\right] \mathrm{e}^{\mathrm{i}xu} \mathrm{d}u. \tag{2.164}$$

This product and its Fourier transform are well localized with a fast decreasing, for we have

$$2^{-4/3} Ai\left[2^{-4/3}u^2\right] \to \frac{1}{8\sqrt{\pi\,|u|}} e^{-\frac{1}{6}|u|^3}$$

when $u \to \pm\infty$. Then, this product and its Fourier transform form a couple of square integrable functions.

The decreasing of the Fourier transform being faster than a Gaussian, we can consider this product as a continuous basis of wavelets [Holschneider (1995)]. As a matter of fact, when we take twice the derivative of $Ai(x)Ai(-x)$, we obtain

$$Ai'(x)Ai'(-x) = \frac{1}{2^{4/3}\pi} \int_{-\infty}^{+\infty} Ai\left[2^{-4/3}u^2\right] \mathrm{e}^{\mathrm{i}xu} u^2 \mathrm{d}u. \tag{2.165}$$

The three first moments of this function cancel:

$$\int_{-\infty}^{+\infty} Ai'(x)Ai'(-x)\, x^n \,\mathrm{d}x = 0, \qquad n = 0, 1, 2, \tag{2.166}$$

then its Fourier transform behaves parabolically near the origin.

On Figs. 2.7 and 2.8, we plotted the wavelet $Ai'(x)Ai'(-x)$ and its Fourier transform (for the positive values of the Fourier variable). The maximum value of the Fourier transform is given by the solution of the equation: $Ai(x_0) + x_0 Ai'(x_0) = 0$, *i.e.* $x_0 = 0.88405$, and $u_0 = 1.4925$ for the Fourier variable.

This continuous basis of wavelets is very close to what is called the "Mexican hat", that is to say the Maar wavelets, but with a specificity of a faster decreasing of the Fourier transform for the Airy wavelets.

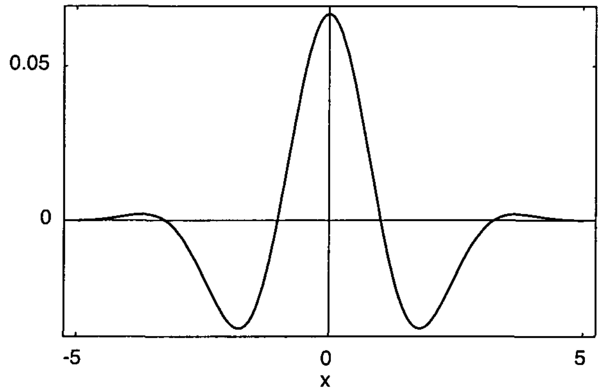


Fig. 2.7 Plot of the wavelets $Ai'(x)Ai'(-x)$.

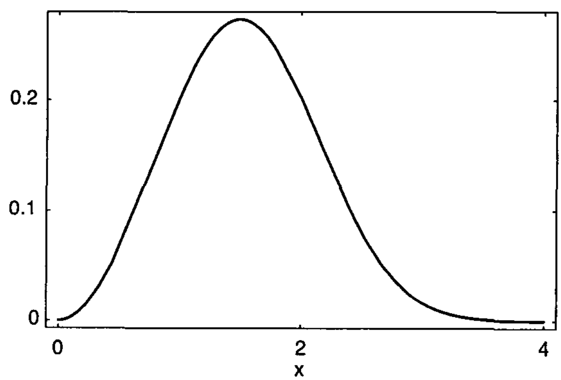


Fig. 2.8 Fourier transform of the wavelets $Ai'(x)Ai'(-x)$.

Finally, we give the normalisation of the mother wavelet (see Eq. (2.164))

$$\frac{1}{\alpha}\int_{-\infty}^{+\infty} Ai^2\left(\frac{x-a}{\alpha}\right) Ai^2\left(\frac{a-x}{\alpha}\right) \mathrm{d}x = \frac{1}{12\pi}, \tag{2.167}$$

whose demonstration will be given in §3.6.4 (see also in Reid (1995)).

Exercises

1. Prove, with an appropriate change of function on the Airy differential equation (Eq. (2.1)), that the logarithmic derivative of $Ai(x)$ satisfies the Riccati equation $u' + u^2 = x$. Find a differential equation of the first order of which the solution is $\frac{Ai(x)}{Ai'(x)}$. Conclude in relation with §2.2.2.

2. Find the solution of the differential equation

$$u\frac{\mathrm{d}u}{\mathrm{d}x} = 2xu + 1$$

in terms of Airy functions (choose a convenient boundary condition). *Hint*: see Davis (1962) in connection with the period of the van der Pol oscillator.

3. Show that if $\xi = \frac{2}{3}\, x^{3/2}$

$$Ai(x) = \frac{1}{2\sqrt{\pi}} x^{-1/4} W_{0,1/3}(2\xi),$$

and

$$Bi(x) = \frac{1}{2^{1/3}} \left(-\frac{2}{3}\right) x^{-1/4} M_{0,-1/3}(2\xi),$$

where $W_{\lambda,\mu}(.)$ and $M_{\lambda,\mu}(.)$ are the Whittaker functions. Find similar relations for the derivatives of Airy functions. Find relations between Airy functions and the confluent hypergeometric functions.

4. Find a particular solution to the differential equation

$$y'' - xy = Ai(x).$$

Hint: Calculate the derivatives of $Ai(x)$ up to the third order.

5. From the Wronskian relation

$$W\left\{Ai(x),\ Bi(x)\right\} = \frac{1}{\pi},$$

prove that the Wronskian of squares and product of Airy functions is

$$W\left\{Ai^2(x), Ai(x)Bi(x), Bi^2(x)\right\} = \frac{2}{\pi^3},$$

see Eq. (2.147). *Hint*: The Wronskian of a canonical differential equation of the third order is given in §6.2.1.

Chapter 3

Primitives and Integrals of Airy Functions

3.1 Primitives Containing One Airy Function

3.1.1 *In terms of Airy functions*

From the formulae (2.127) and (2.128), we deduce the expressions of the primitives of $Ai(x)$ and $Bi(x)$ [Abramowitz & Stegun (1965)]

$$\int_0^x Ai(t)\mathrm{d}t = \frac{1}{3} + \pi\left[Ai'(x)\,Gi(x) - Ai(x)\,Gi'(x)\right] \tag{3.1}$$
$$= -\frac{2}{3} - \pi\left[Ai'(x)\,Hi(x) - Ai(x)\,Hi'(x)\right]$$

$$\int_0^x Ai(-t)\mathrm{d}t = -\frac{1}{3} - \pi\left[Ai'(-x)\,Gi(-x) - Ai(-x)\,Gi'(-x)\right] \tag{3.2}$$
$$= \frac{2}{3} + \pi\left[Ai'(-x)\,Hi(-x) - Ai(-x)\,Hi'(-x)\right]$$

$$\int_0^x Bi(t)\mathrm{d}t = \pi\left[Bi'(x)\,Gi(x) - Bi(x)\,Gi'(x)\right] \tag{3.3}$$
$$= -\pi\left[Bi'(x)\,Hi(x) - Bi(x)\,Hi'(x)\right]$$

$$\int_0^x Bi(-t)\mathrm{d}t = -\pi\left[Bi'(-x)\,Gi(-x) - Bi(-x)\,Gi'(-x)\right] \tag{3.4}$$
$$= \pi\left[Bi'(-x)\,Hi(-x) - Bi(-x)\,Hi'(-x)\right].$$

3.1.2 *Ascending series*

The ascending series of the primitives of Airy functions are [Abramowitz & Stegun (1965)]

$$\int_0^x Ai\,(t)\mathrm{d}t = c_1 F\,(x) - c_2 G\,(x) \tag{3.5}$$

$$\int_0^x Ai\,(-t)\mathrm{d}t = -c_1 F\,(-x) + c_2 G\,(-x) \tag{3.6}$$

$$\int_0^x Bi\,(t)\mathrm{d}t = \sqrt{3}\,[c_1 F\,(x) + c_2 G\,(x)] \tag{3.7}$$

$$\int_0^x Bi\,(-t)\mathrm{d}t = -\sqrt{3}\,[c_1 F\,(-x) + c_2 G\,(-x)]\,, \tag{3.8}$$

the series $F(x)$ and $G(x)$ being defined by integration term–by–term of the series f and g (cf. §2.1.4)

$$\begin{aligned} F\,(x) &= \sum_{k=0}^{\infty} 3^k \left(\frac{1}{3}\right)_k \frac{x^{3k+1}}{(3k+1)!} \\ &= x + \frac{1}{4!}x^4 + \frac{1.4}{7!}x^7 + \frac{1.4.7}{10!}x^{10} + \ldots \end{aligned}$$

$$\begin{aligned} G\,(x) &= \sum_{k=0}^{\infty} 3^k \left(\frac{2}{3}\right)_k \frac{x^{3k+2}}{(3k+2)!} \\ &= \frac{1}{2!}x^2 + \frac{2}{5!}x^5 + \frac{2.5}{8!}x^8 + \frac{2.5.8}{11!}x^{11} + \ldots \end{aligned}$$

where the constants c_1 and c_2 are defined in §2.1.4.2 $c_1 = Ai(0)$ and $c_2 = Ai'(0)$.

3.1.3 *Asymptotic series*

For $x \gg 1$ (and $\xi = \frac{2}{3}x^{3/2}$), the first terms of the asymptotic series of the primitives of the homogeneous Airy functions are [Abramowitz & Stegun

(1965)]

$$\int_0^x Ai(t)\mathrm{d}t \approx \frac{1}{3} - \frac{\mathrm{e}^{-\xi}}{2\sqrt{\pi}x^{3/4}}\left(1 - \frac{41}{48x^{3/2}} + \frac{9\,241}{4\,608x^3} - \dots\right) \tag{3.9}$$

$$\int_0^x Ai(-t)\mathrm{d}t \approx \frac{2}{3} + \frac{1}{\sqrt{\pi}x^{3/4}}\left[\left(\frac{7}{48x^{3/2}} - \frac{5}{48x^3} + \dots\right)\sin\left(\xi - \frac{\pi}{4}\right)\right. \tag{3.10}$$
$$\left. - \left(1 + \frac{1}{x^{3/2}} - \frac{8\,761}{4\,608x^3} - \dots\right)\cos\left(\xi - \frac{\pi}{4}\right)\right]$$

$$\int_0^x Bi(t)\mathrm{d}t \approx \frac{\mathrm{e}^{\xi}}{\sqrt{\pi}x^{3/4}}\left(1 + \frac{41}{48x^{3/2}} + \frac{9\,241}{4\,608x^3}\dots\right) \tag{3.11}$$

$$\int_0^x Bi(-t)\mathrm{d}t \approx \frac{1}{\sqrt{\pi}x^{3/4}}\left[\left(\frac{7}{48x^{3/2}} - \frac{5}{48x^3} + \dots\right)\cos\left(\xi - \frac{\pi}{4}\right)\right. \tag{3.12}$$
$$\left. + \left(1 + \frac{1}{x^{3/2}} - \frac{8\,761}{4\,608x^3} + \dots\right)\sin\left(\xi - \frac{\pi}{4}\right)\right].$$

These series are obtained by integrating term–by–term the series defined in §2.1.4.

3.1.4 *Primitive of Scorer functions*

Gordon (1970) also gives some primitives implying the inhomogeneous function $Gi(x)$. The primitive $\int Gi\left[\alpha(x+\beta)\right]\mathrm{d}x$ seems unable to be expressed simply in terms of Airy functions. Nevertheless, we can calculate

- $$\int xGi\left[\alpha(x+\beta)\right]\mathrm{d}x = \frac{x}{\alpha\pi} + \frac{1}{\alpha^2}Gi'\left[\alpha(x+\beta)\right] - \beta\int Gi\left[\alpha(x+\beta)\right]\mathrm{d}x \tag{3.13}$$

- $$\int x^2 Gi\left[\alpha(x+\beta)\right]\mathrm{d}x \tag{3.14}$$
$$= \frac{x}{\alpha\pi}\left(\frac{x}{2} - \beta\right) + \frac{x-\beta}{\alpha^2}Gi'\left[\alpha(x+\beta)\right] - \frac{1}{\alpha^3}Gi\left[\alpha(x+\beta)\right] + \beta^2\int Gi\left[\alpha(x+\beta)\right]\mathrm{d}x$$

$$\bullet \int x^3 Gi\left[\alpha(x+\beta)\right]\mathrm{d}x = \frac{x}{\alpha\pi}\left(\frac{x^2}{3} - \beta\frac{x}{2} + \beta^2\right) + \frac{x^2 - \beta x + \beta^2}{\alpha^2} Gi'\left[\alpha(x+\beta)\right] + \frac{\beta - 2x}{\alpha^3} Gi\left[\alpha(x+\beta)\right] + \left(2\alpha^{-3} - \beta^3\right)\int Gi\left[\alpha(x+\beta)\right]\mathrm{d}x. \tag{3.15}$$

3.1.5 *Repeated primitives*

For all the primitives given below, the integration constant has been omitted. If y is any linear combination of Airy function and y' its derivative, we note y_1 its primitive. Then we have

$$\bullet \int y_1 \mathrm{d}x = x\, y_1 - y' \tag{3.16}$$

$$\bullet \int x\, y_1 \mathrm{d}x = \frac{1}{2}(x^2 y_1 - x\, y' + y) \tag{3.17}$$

$$\bullet \int x^2\, y_1 \mathrm{d}x = \frac{1}{3}(x^3 y_1 - x\, y' + 2(x\, y - y_1)). \tag{3.18}$$

From which we find

$$\int^x \int^{x'} y_1 \mathrm{d}x' \mathrm{d}x'' = \frac{1}{2}(x^2 y_1 - x\, y' - y), \quad \text{etc.} \tag{3.19}$$

3.2 Product of Airy Functions

It is sometimes possible to easily calculate the primitive of a product of Airy function. For example, we can calculate

$$I = \int_x^\infty Ai^2(x)\mathrm{d}x,$$

from an integration by parts

$$I = \left[xAi^2(x)\right]_x^\infty - 2\int_x^\infty xAi(x)Ai'(x)\mathrm{d}x.$$

Thanks to the Airy equation (2.1), we can write

$$I = \left[xAi^2(x)\right]_x^\infty - 2\int_x^\infty Ai'(x)Ai''(x)\mathrm{d}x$$
$$= \left[xAi^2(x)\right]_x^\infty - \left[Ai'^2(x)\right]_x^\infty .$$

Since $\lim_{x\to\infty} xAi^2(x) = 0$ and $\lim_{x\to\infty} Ai'(x) = 0$, we finally obtain

$$I = -xAi^2(x) + Ai'^2(x).$$

However, this kind of calculation is not always so straightforward. This is the reason why we are going to detail the method of Albright (1977) in the next section. This method allows us to calculate the primitives of linear combinations of homogeneous Airy functions Ai and Bi.

3.2.1 *The method of Albright*

We want to calculate integrals of the kinds

$$\int x^n y^2 \mathrm{d}x, \quad \int x^n y' y \,\mathrm{d}x \text{ and } \int x^n y'^2 \mathrm{d}x, \tag{3.20}$$

where y is a linear combination of the functions $Ai(x)$ and $Bi(x)$ (*i.e.* $y(x) = \alpha Ai(x) + \beta Bi(x)$), the sign *prime* ($'$) stands for the differentiation with respect to x.

Albright builds the following table, where D stands for the operator $\mathrm{d}/\mathrm{d}x$, and where y'' is replaced by xy (according to the Airy equation (2.1))

	y^2	$y'y$	y'^2	xy^2	$xy'y$	xy'^2	x^2y^2	$x^2y'y$	$x^2y'^2$
$\mathrm{D}y^2$		2							
$\mathrm{D}y'y$			1	1					
$\mathrm{D}y'^2$					2				
$\mathrm{D}xy^2$	1				2				
$\mathrm{D}xy'y$		1				1	1		
$\mathrm{D}xy'^2$			1					2	
$\mathrm{D}x^2y^2$				2				2	
$\mathrm{D}x^2y'y$					2				1
$\mathrm{D}x^2y'^2$						2			

The properties of this table are such, that we are able to calculate the primitives of the kinds (3.20). For example, to calculate

$$F = \int y^2 \mathrm{d}x,$$

we just have to subtract lines (4) and (5) of the table. So we obtain

$$\mathrm{D}\left(xy^2 - y'^2\right) = y^2,$$

and then the result Eq. (3.24)

$$F = xy^2 - y'^2,$$

except the integration constant. For the following primitive

$$G = \int y'^2 \mathrm{d}x,$$

three lines of the table have to be considered, these can be written into the matrix form

$$D\begin{pmatrix} y'y \\ xy'^2 \\ x^2y^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} y'^2 \\ xy^2 \\ x^2y'y \end{pmatrix} \tag{3.21}$$

It is then sufficient to inverse the system

$$\begin{pmatrix} y'^2 \\ xy^2 \\ x^2y'y \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -4 & -2 & 2 \\ -2 & 2 & -2 \\ 2 & -2 & -1 \end{pmatrix} D \begin{pmatrix} y'y \\ xy'^2 \\ x^2y^2 \end{pmatrix} \tag{3.22}$$

leading immediately to the result

$$G = \frac{1}{3}\left(2y'y + xy'^2 - x^2y^2\right).$$

In the next section we set out some particular cases of primitives obtained by the method of Albright and with the help of the primitive

$$L = \int x^n y' y \mathrm{d}x,$$

calculated from an integration by parts

$$L = x^n y^2 - \int n x^{n-1} y^2 \mathrm{d}x - L,$$

$$L = \frac{1}{2}\left(x^n y^2 - n\int x^{n-1} y^2 \mathrm{d}x\right). \tag{3.23}$$

3.2.2 *Some primitives*

If y is a linear combination of $Ai(x)$ and $Bi(x)$ and n a positive integer.

- $$\int y^2 \mathrm{d}x = xy^2 - y'^2 \tag{3.24}$$

- $$\int xy^2 \mathrm{d}x = \frac{1}{3}\left(y'y - xy'^2 + x^2y^2\right) \tag{3.25}$$

- $$\int x^2y^2 \mathrm{d}x = \frac{1}{5}\left[2\left(xy'\,y - \frac{1}{2}y^2\right) - x^2y'^2 + x^3y^2\right] \tag{3.26}$$

- $$\int y'y \mathrm{d}x = \frac{1}{2}y^2 \tag{3.27}$$

- $$\int xy'y \mathrm{d}x = \frac{1}{2}y'^2 \tag{3.28}$$

- $$\int x^2y'y \mathrm{d}x = \frac{1}{6}\left(x^2y^2 - 2y'y + 2xy'^2\right) \tag{3.29}$$

- $$\int y'^2 \mathrm{d}x = \frac{1}{3}\left(2y'y + xy'^2 - x^2y^2\right) \tag{3.30}$$

- $$\int xy'^2 \mathrm{d}x = \frac{1}{5}\left[3\left(xy'y - \frac{1}{2}y^2\right) + x^2y'^2 - x^3y^2\right] \tag{3.31}$$

- $$\int x^2y'^2 \mathrm{d}x = \frac{1}{7}\left(4x^2y'y - 4y'^2 + x^3y'^2 - x^4y^2\right) \tag{3.32}$$

- $$\int x^ny^2 \mathrm{d}x = \frac{1}{2n+1}(nx^{n-1}y'y - n(n-1)\int x^{n-2}y'y\mathrm{d}x - x^ny'^2 + x^{n+1}y^2) \tag{3.33}$$

- $$\int x^ny'y \mathrm{d}x = \frac{1}{2}\left(x^ny^2 - n\int x^{n-1}y^2\mathrm{d}x\right) \tag{3.34}$$

or

- $$\int x^ny'y \mathrm{d}x = \frac{1}{2n-1}\left[\frac{1}{2}nx^{n-1}y'^2 + \frac{1}{2}(n-1)x^ny^2 - \frac{1}{2}n(n-1)\left(x^{n-2}y'y - (n-2)\int x^{n-3}y'y\mathrm{d}x\right)\right] \tag{3.35}$$

- $$\int x^n y'^2 \mathrm{d}x = \frac{1}{2n+3}\left[x^{n+1}y'^2 - x^{n+2}y^2 + (n+2)\left(x^n y' y - n\int x^{n-1}y'y\mathrm{d}x\right)\right]$$

If $A(x)$ and $B(x)$ are any two linear combinations of $Ai(x)$ and $Bi(x)$, and n is a positive integer

- $$\int A(x)B(x)\mathrm{d}x = xA(x)B(x) - A'(x)B'(x) \tag{3.36}$$
- $$\int A'(x)B(x)\mathrm{d}x = \frac{1}{2}\left[A(x)B(x) + xA'(x)B(x) - xA(x)B'(x)\right] \tag{3.37}$$
- $$\int A'(x)B'(x)\mathrm{d}x = \frac{1}{3}\left[A'(x)B(x) + A(x)B'(x) + xA'(x)B'(x) - x^2A(x)B(x)\right] \tag{3.38}$$
- $$\int xA(x)B(x)\mathrm{d}x = \frac{1}{6}\left[A'(x)B(x) + A(x)B'(x) - 2xA'(x)B'(x) + 2x^2A(x)B(x)\right] \tag{3.39}$$
- $$\int xA'(x)B(x)\mathrm{d}x = \frac{1}{4}\left[2A'(x)B'(x) + x^2A'(x)B(x) - x^2A(x)B'(x)\right] \tag{3.40}$$
- $$\int xA'(x)B'(x)\mathrm{d}x = \frac{1}{5}\left[\frac{3}{2}\left(xA'(x)B(x) + xA(x)B'(x) - A(x)B(x)\right) + x^2A'(x)B'(x) - x^3A(x)B(x)\right] \tag{3.41}$$
- $$\int x^n A(x)B(x)\mathrm{d}x = \frac{1}{2(2n+1)}\left[nx^{n-1}\left(A'(x)B(x) + A(x)B'(x)\right) - 2x^nA'(x)B'(x) + 2x^{n+1}A(x)B(x) - n(n-1)\int x^{n-2}\left(A'(x)B(x) + A(x)B'(x)\right)\mathrm{d}x\right] \tag{3.42}$$

- $$\int x^n A(x)B(x)\mathrm{d}x = \frac{1}{2}\left[x^{n-1}\left(A'(x)B(x) + A(x)B'(x)\right) - 2\int x^{n-1}A'(x)B'(x)\mathrm{d}x - (n-1)\int x^{n-2}\left(A'(x)B(x) + A(x)B'(x)\right)\mathrm{d}x\right],\ n \geq 2 \tag{3.43}$$

$$\bullet \int x^n A'(x)B(x)\mathrm{d}x = \frac{1}{2}\left[x^n A(x)B(x) - n\int x^{n-1}A(x)B(x)\mathrm{d}x \right. \tag{3.44}$$

$$\left. + \frac{x^{n+1}}{n+1}\left(A'(x)B(x) - A(x)B'(x)\right)\right]$$

$$\bullet \int x^n A'(x)B'(x)\mathrm{d}x = \frac{1}{2(2n+3)}\Big\{2x^{n+1}\Big(A'(x)B'(x) \tag{3.45}$$

$$-xA(x)B(x)\Big) + (n+2)\Big[x^n\left(A'(x)B(x) + A(x)B'(x)\right)$$

$$-n\int x^{n-1}\left(A'(x)B(x) + A(x)B'(x)\right)\mathrm{d}x\Big]\Big\}.$$

The formulae (3.36) to (3.45) are true for any $A(x)$ and $B(x)$. In particular, if $A(x) = Ai(x)$ and $B(x) = Bi(x)$, we can simplify some of these expressions by using the Wronskian relationship (formula (2.6)),

$$AiBi' - Ai'Bi = \frac{1}{\pi}.$$

For example, the formula (3.37) becomes

$$\int Ai'Bi\,\mathrm{d}x = \frac{1}{2}\left(AiBi - \frac{x}{\pi}\right).$$

Some of the preceding primitives can be calculated for a more general form [Gordon (1969, 1970, 1971)]. If A is a linear combination of $Ai(x)$ and $Bi(x)$, we have the following primitives

$$\bullet \int A\left[\alpha(x+\beta)\right]\mathrm{d}x = \frac{\pi}{\alpha}\left\{A'\left[\alpha(x+\beta)\right]Gi\left[\alpha(x+\beta)\right]\right. \tag{3.46}$$

$$\left. -A\left[\alpha(x+\beta)\right]Gi'\left[\alpha(x+\beta)\right]\right\}$$

$$\bullet \int xA\left[\alpha(x+\beta)\right]\mathrm{d}x \tag{3.47}$$

$$= \frac{1}{\alpha^2}A'\left[\alpha(x+\beta)\right] - \beta\int A\left[\alpha(x+\beta)\right]\mathrm{d}x$$

$$\bullet \int x^2 A\left[\alpha(x+\beta)\right] \mathrm{d}x = \frac{x-\beta}{\alpha^2} A'\left[\alpha(x+\beta)\right] - \frac{1}{\alpha^3} A\left[\alpha(x+\beta)\right] + \beta^2 \int A\left[\alpha(x+\beta)\right] \mathrm{d}x \tag{3.48}$$

$$\bullet \int x^3 A\left[\alpha(x+\beta)\right] \mathrm{d}x = \frac{x^2-\beta x+\beta^2}{\alpha^2} A'\left[\alpha(x+\beta)\right] - \frac{\beta-2x}{\alpha^3} A\left[\alpha(x+\beta)\right] + \left(2\alpha^{-3}-\beta^3\right) \int A\left[\alpha(x+\beta)\right] \mathrm{d}x \tag{3.49}$$

When A and B are any linear combinations of $Ai(x)$ and $Bi(x)$, we have the primitives

$$\bullet \int A\left[\alpha(x+\beta)\right] B\left[\alpha(x+\beta)\right] \mathrm{d}x = (x+\beta) A\left[\alpha(x+\beta)\right] B\left[\alpha(x+\beta)\right] - \frac{1}{\alpha} A'\left[\alpha(x+\beta)\right] B'\left[\alpha(x+\beta)\right] \tag{3.50}$$

$$\bullet \int x A\left[\alpha(x+\beta)\right] B\left[\alpha(x+\beta)\right] \mathrm{d}x = \frac{1}{3}\left(x^2-x\beta-2\beta^2\right) A\left[\alpha(x+\beta)\right] B\left[\alpha(x+\beta)\right] + \frac{1}{6\alpha^2}\left\{A'\left[\alpha(x+\beta)\right] B\left[\alpha(x+\beta)\right] + A\left[\alpha(x+\beta)\right] B'\left[\alpha(x+\beta)\right]\right\} + \frac{2\beta-x}{3\alpha} A'\left[\alpha(x+\beta)\right] B'\left[\alpha(x+\beta)\right] \tag{3.51}$$

$$\bullet \int x^2 A\left[\alpha(x+\beta)\right] B\left[\alpha(x+\beta)\right] \mathrm{d}x = \frac{1}{15}\left(3x^3-x^2\beta+4x\beta^2+8\beta^3-3\alpha^{-3}\right) \times A\left[\alpha(x+\beta)\right] B\left[\alpha(x+\beta)\right] + \frac{3x-2\beta}{15\alpha^2}\left\{A'\left[\alpha(x+\beta)\right] B\left[\alpha(x+\beta)\right] + A\left[\alpha(x+\beta)\right] B'\left[\alpha(x+\beta)\right]\right\} - \frac{3x^2-4x\beta+8\beta^2}{15\alpha} A'\left[\alpha(x+\beta)\right] B'\left[\alpha(x+\beta)\right] \tag{3.52}$$

and for different arguments

- $$\int A\left[\alpha(x+\beta_1)\right]B\left[\alpha(x+\beta_2)\right]\mathrm{d}x \tag{3.53}$$
$$= \frac{1}{\alpha^2(\beta_1-\beta_2)}\left\{A'\left[\alpha(x+\beta_1)\right]B\left[\alpha(x+\beta_2)\right] - A\left[\alpha(x+\beta_1)\right]B'\left[\alpha(x+\beta_2)\right]\right\}$$

- $$\int xA\left[\alpha(x+\beta_1)\right]B\left[\alpha(x+\beta_2)\right]\mathrm{d}x \tag{3.54}$$
$$\begin{aligned}= &-\frac{\beta_1+\beta_2+2x}{\alpha^3(\beta_1-\beta_2)^2}A\left[\alpha(x+\beta_1)\right]B\left[\alpha(x+\beta_2)\right] \\ &+\left[\frac{x}{\alpha^2(\beta_1-\beta_2)}+\frac{2}{\alpha^5(\beta_1-\beta_2)^3}\right] \\ &\times\ \left\{A'\left[\alpha(x+\beta_1)\right]B\left[\alpha(x+\beta_2)\right] - A\left[\alpha(x+\beta_1)\right]B'\left[\alpha(x+\beta_2)\right]\right\} \\ &+\frac{2}{\alpha^4(\beta_1-\beta_2)^2}A'\left[\alpha(x+\beta_1)\right]B'\left[\alpha(x+\beta_2)\right].\end{aligned}$$

For completeness, we also give the integral[1]

- $$\int x^2\,A[\alpha(x+\beta_1)]\,B[\alpha(x+\beta_2)]\,\mathrm{d}x = \frac{4}{(\beta_2-\beta_1)^2} \tag{3.55}$$
$$\begin{aligned}\times&\left\{-\frac{1}{\alpha^3}\left[x^2+c\,x+\frac{3(\beta_1+\beta_2)}{\alpha^3(\beta_2-\beta_1)^2}\right]A[\alpha(x+\beta_1)]\,B[\alpha(x+\beta_2)]\right. \\ &-\left[\frac{(\beta_2-\beta_1)\,x^2}{4\,\alpha^2}+\frac{3\,x+\beta_2+c}{\alpha^5(\beta_2-\beta_1)}\right]A'[\alpha(x+\beta_1)]\,B[\alpha(x+\beta_2)] \\ &+\left[\frac{(\beta_2-\beta_1)\,x^2}{4\,\alpha^2}+\frac{3\,x+\beta_1+c}{\alpha^5(\beta_2-\beta_1)}\right]A[\alpha(x+\beta_1)]\,B'[\alpha(x+\beta_2)] \\ &\left.+\frac{1}{\alpha^4}\left[x+\frac{6}{\alpha^3(\beta_2-\beta_1)^2}\right]A'[\alpha(x+\beta_1)]\,B'[\alpha(x+\beta_2)]\right\},\end{aligned}$$

with

$$c = \frac{(\beta_2-\beta_1)^2\,(\beta_2+\beta_1)+12/\alpha^3}{2\,(\beta_2-\beta_1)^2}.$$

Finally, we give some primitives involving the primitive y_1 of any linear

[1] It is not given in the work of Gordon.

combination of Airy function y,

- $$\int y_1^2 \,\mathrm{d}x = x\, y_1^2 - 2y' y_1 + y^2 \tag{3.56}$$
- $$\int y_1 y \,\mathrm{d}x = \frac{1}{2}\, y_1^2 \tag{3.57}$$
- $$\int y_1 y' \,\mathrm{d}x = y_1 y - x\, y^2 + y'^2, \tag{3.58}$$

where y' is the derivative of y.

3.3 Other Primitives

Albright & Gavathas (1986) give other kinds of primitives of Airy functions. The expression

$$\int \frac{Ai(x)Bi(x)}{\left[Ai^2(x) + Bi^2(x)\right]^2}\mathrm{d}x = \frac{\pi}{2}\frac{Bi^2(x)}{Ai^2(x) + Bi^2(x)}, \tag{3.59}$$

is obtained by differentiation and thanks to the Wronskian of Ai and Bi (formula (2.6)). More generally, we have

$$\int \frac{Ai^{n-1}(x)Bi^{n-1}(x)}{\left[Ai^n(x) + Bi^n(x)\right]^2}\mathrm{d}x = \frac{\pi}{n}\frac{Bi^n(x)}{Ai^n(x) + Bi^n(x)}, \tag{3.60}$$

and from a similar method, we obtain

$$\int \frac{\mathrm{d}x}{Ai^2(x)} = \pi \frac{Bi(x)}{Ai(x)} \tag{3.61}$$

$$\int \frac{\mathrm{d}x}{Bi^2(x)} = -\pi \frac{Ai(x)}{Bi(x)} \tag{3.62}$$

$$\int \frac{\mathrm{d}x}{Ai(x)Bi(x)} = \pi \ln \frac{Bi(x)}{Ai(x)} \tag{3.63}$$

$$\int \frac{Bi^n(x)}{Ai^{n+2}(x)}\mathrm{d}x = \frac{\pi}{n+1}\left(\frac{Bi(x)}{Ai(x)}\right)^{n+1}. \tag{3.64}$$

Albright & Gavathas (1986) build more general results by considering two functions: namely f and F such that $f = F'$. We have then

$$\int \frac{1}{Ai^2(x)} f\left(\frac{Bi(x)}{Ai(x)}\right) \mathrm{d}x = \pi F\left(\frac{Bi(x)}{Ai(x)}\right), \tag{3.65}$$

and

$$\int \frac{1}{Bi^2(x)} f\left(\frac{Ai(x)}{Bi(x)}\right) \mathrm{d}x = -\pi F\left(\frac{Ai(x)}{Bi(x)}\right). \tag{3.66}$$

These last results are easily verified. The equations (3.59) to (3.64) are nothing but particular cases of this general result. There are also some interesting particular cases, such as

$$\int \frac{\mathrm{d}x}{Ai^2(x) + Bi^2(x)} = \pi \tan^{-1} \frac{Bi(x)}{Ai(x)}, \tag{3.67}$$

and

$$\int \frac{Bi^n(x)}{Ai^{n+2}(x)} \exp\left(\frac{Bi(x)}{Ai(x)}\right)^{n+1} \mathrm{d}x = \frac{\pi}{n+1} \exp\left(\frac{Bi(x)}{Ai(x)}\right)^{n+1}. \tag{3.68}$$

3.4 Miscellaneous

Isolated values of primitives involving Airy functions, may be given here as well. For instance, we can quote

$$\int \frac{Ai'^2}{x^2} \mathrm{d}x = Ai^2 - \frac{Ai'^2}{x}. \tag{3.69}$$

There are also some results related to a solution of the heat equation

$$u(x,t) = \exp(2t^3/3 - xt)\, Ai(t^2 - x).$$

This function has particular and interesting properties. In the case where $x = 0$, we have for example

$$\int \exp(2\,t^3/3)\, Ai(t^2)\, \mathrm{d}t = \exp(2\,t^3/3)\, [t\, Ai(t^2) - Ai'(t^2)], \tag{3.70}$$

$$\int 2t \exp(2\,t^3/3)\, [t\, Ai(t^2) + Ai'(t^2)]\, \mathrm{d}t = \exp(2\,t^3/3)\, Ai(t^2), \tag{3.71}$$

$$\int 2t^2 \exp(2\,t^3/3)\, [t\, Ai(t^2) + Ai'(t^2)]\, \mathrm{d}t = \exp(2\,t^3/3)\, Ai'(t^2). \tag{3.72}$$

We shall return to this solution in §7.4.

3.5 Elementary Integrals

3.5.1 *Particular integrals*

From the asymptotic expansion of §3.1.3, we may deduce the values of the defined integrals

$$\int_{-\infty}^{0} Ai(t)\,dt = \frac{2}{3}, \qquad \int_{0}^{\infty} Ai(t)\,dt = \frac{1}{3}, \quad \text{then} \int_{-\infty}^{+\infty} Ai(t)\,dt = 1 \tag{3.73}$$

and

$$\int_{-\infty}^{0} Bi(t)\,dt = 0, \qquad \int_{0}^{\infty} Bi(t)\,dt \approx \infty. \tag{3.74}$$

The Airy function Ai is not square integrable on $\mathbb{R}$ but

$$\int_{0}^{\infty} Ai^2(t)\,dt = \frac{1}{3^{2/3}\Gamma^2(\frac{1}{3})}. \tag{3.75}$$

For the cube of the Airy function, we have [Reid (1997a)]

$$\int_{-\infty}^{+\infty} Ai^3(t)\,dt = \frac{1}{4\pi^2}\Gamma^2(\frac{1}{3}), \tag{3.76}$$

and

$$\int_{-\infty}^{+\infty} Ai^2(t)Bi(t)\,dt = \frac{1}{4\pi^2\sqrt{3}}\Gamma^2(\frac{1}{3}). \tag{3.77}$$

Reid (1997b) gives integrals with fourth power, for example

$$\int_{0}^{+\infty} Ai^4(t), dt = \frac{1}{24\pi^2}\ln 3. \tag{3.78}$$

The reader may find other integrals in the papers by Reid (1997a,b).

3.5.2 Integrals containing a single Airy function

3.5.2.1 Integrals involving algebraic functions

According to the notation of Aspnes (1966), we set

$$Ai_1(x) = \int_x^\infty Ai(t)\mathrm{d}t,$$

which can be expressed in terms of inhomogeneous functions (formulae (3.1) and (3.73)),

$$Ai_1(x) = \pi \left[Ai(x)Gi'(x) - Ai'(x)Gi(x)\right].$$

We have now, for $n > 0$

$$\begin{aligned} \int_0^\infty t^n Ai'(t+x)\mathrm{d}t &= -n \int_0^\infty t^{n-1} Ai(t+x)\mathrm{d}t \qquad (3.79) \\ &= \frac{\mathrm{d}}{\mathrm{d}x} \int_0^\infty t^n Ai(t+x)\mathrm{d}t, \end{aligned}$$

and for $n > -1$

$$\begin{aligned} \int_0^\infty t^n Ai_1(t+x)\mathrm{d}t &= \frac{1}{n+1} \int_0^\infty t^{n+1} Ai(t+x)\mathrm{d}t \qquad (3.80) \\ &= \int_x^\infty \int_0^\infty t^n Ai(t+x)\mathrm{d}t\mathrm{d}x, \end{aligned}$$

with the particular case

$$\int_0^\infty Ai_1(t+x)\mathrm{d}t = Ai'(x) + x\, Ai_1(x). \qquad (3.81)$$

These two kinds of integrals can be treated from the calculation of $\int_0^\infty t^n Ai(t+x)\mathrm{d}t$. Now from the equation of Airy (formula (2.1)), we can deduce

$$\int_0^\infty t^n Ai(t+x)\mathrm{d}t \tag{3.82}$$

$$= \int_0^\infty t^{n-1}\frac{\mathrm{d}^2 Ai(t+x)}{\mathrm{d}x^2}\mathrm{d}t - x\int_0^\infty t^{n-1}Ai(t+x)\mathrm{d}t$$

$$= \left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} - x\right]\int_0^\infty t^{n-1}Ai(t+x)\mathrm{d}t.$$

In the two last equations, the exponent of the variable t is reduced of one unity until the iteration leads to $0 \geq n > -1$.

It should be noted that for $x = 0$ the integral (3.82) can be explicitly written [Olver (1974)]

$$\int_0^\infty t^n Ai(t)\mathrm{d}t = \frac{\Gamma(n+1)}{3^{(n+3)/3}\Gamma\left(\frac{n+3}{3}\right)}. \tag{3.83}$$

We can obtain the moments of the Airy function in calculating [Gislason (1973)]

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty} Ai(x)x^n \mathrm{e}^{-\varepsilon x^2}\mathrm{d}x,\ \ n\in\mathbb{N},$$

that is to say

$$\int_{-\infty}^{+\infty} Ai(x)\,x^{3k}\mathrm{d}x = \frac{(3k)!}{3^k\,k!} \tag{3.84}$$

$$\int_{-\infty}^{+\infty} Ai(x)\,x^{3k+1}\mathrm{d}x = \int_{-\infty}^{+\infty} Ai(x)\,x^{3k+2}\mathrm{d}x = 0. \tag{3.85}$$

We can also explicitly write the integrals (3.79) to (3.82) for $n = -1/2$, with the help of the definition (2.20) of $Ai(x)$, [2]

$$\int_0^\infty Ai(x+t)\frac{dt}{\sqrt{t}} = \frac{1}{\pi^{1/2}}\int_0^\infty \cos\left(\frac{u^3}{3}+xu+\frac{\pi}{4}\right)\frac{du}{\sqrt{u}}.$$

We obtain, by comparison to Eq. (2.148)

$$\int_0^\infty Ai(x+t)\frac{dt}{\sqrt{t}} = 2^{2/3}\pi Ai^2\left(\frac{x}{2^{2/3}}\right). \tag{3.86}$$

We retrieve here the formula (2.155). In a similar manner, we shall have

$$\int_0^\infty Ai(x-t)\frac{dt}{\sqrt{t}} = 2^{2/3}\pi Ai\left(\frac{x}{2^{2/3}}\right) Bi\left(\frac{x}{2^{2/3}}\right). \tag{3.87}$$

Differentiating and integrating the equation (3.82), we obtain

$$\int_0^\infty Ai'(x+t)\frac{dt}{\sqrt{t}} = 2\pi Ai\left(\frac{x}{2^{2/3}}\right) Ai'\left(\frac{x}{2^{2/3}}\right) \tag{3.88}$$

$$\int_0^\infty Ai_1(x+t)\frac{dt}{\sqrt{t}} \tag{3.89}$$

$$= 2^{4/3}\pi\left\{Ai'^2\left(\frac{x}{2^{2/3}}\right) - \frac{x}{2^{2/3}}Ai^2\left(\frac{x}{2^{2/3}}\right)\right\}.$$

We can also quote the integrals [Aspnes (1967)]

$$\int_0^{+\infty} Gi(x+t)\frac{dt}{\sqrt{t}} = 2^{2/3}\pi Ai\left(\frac{x}{2^{2/3}}\right) Bi\left(\frac{x}{2^{2/3}}\right) \tag{3.90}$$

$$\int_0^{+\infty} Gi(x-t)\frac{dt}{\sqrt{t}} = -2^{2/3}\pi Ai^2\left(\frac{x}{2^{2/3}}\right), \tag{3.91}$$

that are obtained thanks to the causal relations between the Airy functions (cf. §4.1.1). It should be noted we can encounter the formula (3.86) under

[2]Some of the following integrals may be found thanks to the relationship between the Fredholm equations $\int_0^\infty f(x+t)\,\frac{dt}{\sqrt{t}} = g(x) \iff f(x) = -\frac{1}{\pi}\int_0^\infty g'(x+t)\,\frac{dt}{\sqrt{t}}$.

a different form [Berry (1977a)]

$$L = \int_{-y}^{\infty} \frac{Ai(x)}{\sqrt{x+y}} \mathrm{d}x = \int_{-\infty}^{+\infty} Ai\left(u^2 - y\right) \mathrm{d}u. \tag{3.92}$$

Thanks to the definition (2.20) of the Airy function Ai, we can express this function as

$$L = \frac{1}{2\pi} \iint_{\mathbb{R}^2} \mathrm{e}^{\,\mathrm{i}\left[v^3/3 + (u^2 - y)v\right]} \mathrm{d}u \mathrm{d}v.$$

With the following change of variables

$$\begin{cases} u = 2^{-2/3}\,(Y - X) \\ v = 2^{-2/3}\,(Y + X)\,, \end{cases}$$

we obtain

$$L = \frac{1}{2\pi 2^{1/3}} \int_{-\infty}^{+\infty} \mathrm{e}^{\,\mathrm{i}\left[X^3/3 + Y^3/3 - 2^{-2/3} y (X+Y)\right]} \mathrm{d}X \mathrm{d}Y,$$

in other words

$$L = 2^{2/3} \pi Ai^2 \left(\frac{y}{2^{2/3}}\right). \tag{3.93}$$

3.5.2.2 *Integrals involving transcendental functions*

Considering first the integral

$$M = \int_{-\infty}^{+\infty} Ai\left(x^2 + a\right) \mathrm{e}^{\,\mathrm{i}kx} \mathrm{d}x, \tag{3.94}$$

that can be written using again (2.20) for the Ai function

$$M = \frac{1}{2\pi} \iint_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}\left[t^3/3 + t\left(x^2 + a\right)\right]} \mathrm{e}^{\,\mathrm{i}kx} \mathrm{d}x \mathrm{d}t.$$

The calculation of this integral is not so easy if one integrates first on the variable x. However, it becomes simpler if we employ the method described in §3.6.5, that is to say if we canonise the cubics $t^3/3 + x^2 t$. When we make the following change of variable

$$\begin{cases} X = 2^{-1/3}(t-x) \\ Y = 2^{-1/3}(t+x), \end{cases}$$

we immediately obtain the result (see formulae (2.151) to (2.154))

$$M = 2^{2/3}\pi Ai\left[2^{-2/3}(a-k)\right] Ai\left[2^{-2/3}(a+k)\right]. \tag{3.95}$$

We can also, thanks to the integral representation (2.20) of $Ai(x)$, obtain the relation [Widder (1979)]

$$\int_{-\infty}^{+\infty} \mathrm{e}^{u\,t} Ai\,(t)\,\mathrm{d}t = \mathrm{e}^{u^3/3}. \tag{3.96}$$

We can also quote the following integral

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-t^2/4\,u} Ai\,(t)\,\mathrm{d}t = 2\sqrt{\pi u}\,\mathrm{e}^{2u^3/3} Ai\left(u^2\right). \tag{3.97}$$

In this last expression, we find, thanks to the asymptotic formula (2.44), the relation (3.73) $\int_{-\infty}^{+\infty} Ai(t)\,\mathrm{d}t = 1$, for $u \to \infty$.

Now we give some other integrals involving one Airy function and transcendentals without demonstration

- $$\int_0^\infty Ai(-x\,t)\,\ln t\,\mathrm{d}t = \frac{2}{9x}\left[\psi\left(\frac{2}{3}\right) + \psi\left(\frac{1}{3}\right) - \ln\left(\frac{x^3}{9}\right)\right], \tag{3.98}$$

where $\psi(.)$ stands for the logarithmic derivative of the gamma function.

- $$\int_0^\infty \mathrm{e}^{-2t^3/3x^3} Ai(t^2)\,\mathrm{d}t \tag{3.99}$$

$$= \frac{x^3}{6\sqrt{1-x^6}}\left[\frac{1}{x}\left(1+\sqrt{1-x^6}\right)^{1/3} - x\left(1+\sqrt{1-x^6}\right)^{-1/3}\right].$$

- $$\int_0^\infty \mathrm{e}^{-t^3/3} Ai\left(-\frac{x^2}{t^2}\right)\,\mathrm{d}t = \frac{2\pi}{\sqrt{3}} Ai(x)Ai(-x). \tag{3.100}$$

- $$\int_0^\infty \mathrm{e}^{-t^3/12} Ai'\,(xt)\,\frac{\mathrm{d}t}{\sqrt{t}} = -\sqrt{\frac{4\pi}{3}}\,\mathrm{e}^{2x^3/3} Ai(x^2). \tag{3.101}$$

- $$\int_0^{\infty} e^{-t^3/12} Ai\,(xt)\,\frac{dt}{t\sqrt{t}} = -\sqrt{\frac{4\pi}{3}}\, e^{2x^3/3}\,\left(x\,Ai(x^2) - Ai'(x^2)\right). \tag{3.102}$$

- $$\int_0^{\infty} e^{-t^3/12} Ai\,(xt)\; t\,\sqrt{t}\,dt = -2\sqrt{\frac{4\pi}{3}}\, e^{2x^3/3}\,\left(x\,Ai(x^2) + Ai'(x^2)\right). \tag{3.103}$$

3.5.3 *Integrals of products of two Airy functions*

In a similar manner as in §3.5.2, according to the method of Aspnes, we can establish the relation [Aspnes (1966)]

$$\int_0^{\infty} t^n Ai^2\,(t+x)\,dt \tag{3.104}$$
$$= \frac{n}{2n+1}\left[\frac{1}{2}\frac{d^2}{dx^2} - 2x\right]\int_0^{\infty} t^{n-1} Ai^2\,(t+x)\,dt,\ \ n > 0.$$

We extend this relation to the one containing Ai' for we have $Ai(x)Ai'(x) = \frac{1}{2}\frac{d}{dx}Ai^2(x)$ and $Ai'^2(x) = \frac{1}{2}\left[\frac{d^2}{dx^2} - x\right]Ai^2(x)$.

In particular for $n = -1/2$ we find

$$\int_0^{\infty} Ai^2(t+x)\frac{dt}{\sqrt{t}} = \frac{1}{2}Ai_1\left(2^{2/3}x\right) \tag{3.105}$$

$$\int_0^{\infty} Ai(t+x)Ai'(t+x)\frac{dt}{\sqrt{t}} = -2^{-4/3}\,Ai\left(2^{2/3}x\right) \tag{3.106}$$

$$\int_0^{\infty} Ai'^2(t+x)\frac{dt}{\sqrt{t}} \tag{3.107}$$
$$= -\frac{2^{-2/3}}{4}\left\{3Ai'\left(2^{2/3}x\right) + 2^{2/3}x Ai_1\left(2^{2/3}x\right)\right\}.$$

- We can also quote the important integrals

$$\frac{1}{|\alpha\beta|}\int_{-\infty}^{+\infty} Ai\left[\frac{x+a}{\alpha}\right] Ai\left[\frac{x+b}{\beta}\right] \mathrm{d}x \tag{3.108}$$

$$= \begin{cases} \delta(b-a) & \text{if } \beta = \alpha \\ \frac{1}{|\beta^3-\alpha^3|^{1/3}} Ai\left[\frac{b-a}{(\beta^3-\alpha^3)^{1/3}}\right] & \text{if } \beta > \alpha, \end{cases}$$

and

$$\frac{1}{|\alpha\beta|}\int_{-\infty}^{+\infty} Ai\left[\frac{x+a}{\alpha}\right] Gi\left[\frac{x+b}{\beta}\right] \mathrm{d}x \tag{3.109}$$

$$= \begin{cases} \frac{1}{\pi}\wp\frac{1}{b-a} & \text{if } \beta = \alpha \\ \frac{1}{|\beta^3-\alpha^3|^{1/3}} Gi\left[\frac{b-a}{(\beta^3-\alpha^3)^{1/3}}\right] & \text{if } \beta > \alpha, \end{cases}$$

This last is calculated thanks to the integral representation (4.7) of $Ai+\mathrm{i}Gi$, $\wp$ being the Cauchy principal value.

• In order to calculate the following integral [Bienniek (1977)], which is a generalisation of formula (3.108)

$$I_n = \int_{-\infty}^{+\infty} x^n Ai\left[\frac{x+a}{\alpha}\right] Ai\left[\frac{x+b}{\beta}\right] \mathrm{d}x, \ \beta > \alpha, \ \beta > 0, \tag{3.110}$$

the general method consists in taking for the Airy functions their integral representation (2.20), namely

$$Ai\left[\frac{x+a}{\alpha}\right] = \frac{|\alpha|}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{\,\mathrm{i}\left[(\alpha t)^3/3+(x+a)t\right]}\mathrm{d}t. \tag{3.111}$$

Then we obtain

$$I_n = \iint_{\mathbb{R}^2} \mathrm{d}t\mathrm{d}t' \ \frac{|\alpha\beta|}{4\pi^2}\mathrm{e}^{\,\mathrm{i}\left[(\alpha t)^3/3+(\beta t')^3/3+at+bt'\right]} \int_{-\infty}^{+\infty} x^n \mathrm{e}^{\,\mathrm{i}x(t+t')}\mathrm{d}x,$$

with

$$\delta^{(n)}(t+t') = \frac{\mathrm{i}^n}{2\pi}\int_{-\infty}^{+\infty} x^n \mathrm{e}^{\,\mathrm{i}x(t+t')}\mathrm{d}x$$

and

$$\iint f(x)g(y)\delta^{(n)}(x+y)\mathrm{d}x\mathrm{d}y = \int f(x)g^{(n)}(-x)\mathrm{d}x,$$

where δ is the Dirac delta function and $\delta^{(n)}$ its n^{th} derivative. We finally find the relation

$$I_n = \frac{|\alpha\beta|}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{\,\mathrm{i}\left[(\alpha t)^3/3+at\right]}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left\{\mathrm{i}^{-n}\mathrm{e}^{-\mathrm{i}\left[(\beta t)^3/3+bt\right]}\right\}\mathrm{d}t. \tag{3.112}$$

Another method starts from the equation (3.110) and from the Airy equation (2.1) leading to the recurrence relation

$$I_{n+1} = \beta^3\frac{\mathrm{d}^2 I_n}{\mathrm{d}b^2} - bI_n. \tag{3.113}$$

I_0 being given by the relation (3.108), we can then deduce

$$I_0 = \frac{|\alpha\beta|}{(\beta^3-\alpha^3)^{1/3}}Ai\left[\frac{b-a}{(\beta^3-\alpha^3)^{1/3}}\right] \tag{3.114}$$

$$I_1 = \frac{b\alpha^3 - a\beta^3}{\beta^3-\alpha^3}I_0 \tag{3.115}$$

$$I_2 = \left[\frac{b\alpha^3 - a\beta^3}{\beta^3-\alpha^3}\right]^2 I_0 + \frac{2\alpha^3\beta^3}{\beta^3-\alpha^3}I_0' \tag{3.116}$$

$$\vdots$$

• We also quote the following integral [Balazs & Zipfel (1973)], which appears, in particular, in the semiclassical calculation of the Wigner function (cf. §8.3)

$$F = \int_{-\infty}^{+\infty} Ai(x+t)Ai(x-t)\mathrm{e}^{\,\mathrm{i}2pt}\mathrm{d}t. \tag{3.117}$$

To calculate this integral, we use again the integral representation of Ai (2.20) to thus obtain

$$F = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \mathrm{e}^{\,\mathrm{i}\left(u^3/3+v^3/3+ux+vx\right)} \mathrm{e}^{\,\mathrm{i}(2p+u-v)t} \mathrm{d}u\mathrm{d}v\mathrm{d}t.$$

The integration on the variable t leads to a Dirac function, allowing us to derive the relation

$$F = \frac{\mathrm{e}^{\,\mathrm{i}\left(8p^3/3+2px\right)}}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{\,\mathrm{i}\left[2u^3/3+2pu^2+(2x+4p^2)u\right]} \mathrm{d}u,$$

and after the change of variable $\omega^3 = 2u^3$

$$F = \frac{\mathrm{e}^{\,\mathrm{i}\left(8p^3/3+2px\right)}}{2\pi} 2^{-1/3} \int_{-\infty}^{+\infty} \mathrm{e}^{\,\mathrm{i}\left[\omega^3/3+2^{1/3}p\omega^2+2^{2/3}(x+2p^2)\omega\right]} \mathrm{d}\omega.$$

The integral

$$G = \int_{-\infty}^{+\infty} \mathrm{e}^{\,\mathrm{i}\left(t^3/3+at^2+bt\right)} \mathrm{d}t, \tag{3.118}$$

is simply calculated by putting $u = t - a$, providing

$$G = \mathrm{e}^{\,\mathrm{i}a\left(2a^2/3-b\right)} \int_{-\infty}^{+\infty} \mathrm{e}^{\,\mathrm{i}\left(u^3/3+(b-a^2)u\right)} \mathrm{d}u,$$

in other terms (see formula (2.25))

$$G = 2\pi \mathrm{e}^{\,\mathrm{i}a\left(2a^2/3-b\right)} Ai\left(b - a^2\right). \tag{3.119}$$

In our case, we obtain, after simplification

$$F = 2^{-1/3} Ai\left[2^{2/3}\left(x + p^2\right)\right]. \tag{3.120}$$

More generally the same method yields for $\beta > \alpha$

$$
\begin{aligned}
I_\lambda &= \frac{1}{|\alpha\beta|} \int_{-\infty}^{+\infty} Ai\left[\frac{x+a}{\alpha}\right] Ai\left[\frac{x+b}{\beta}\right] \mathrm{e}^{\mathrm{i}\lambda x} \mathrm{d}x \\
&= \frac{1}{\left(\beta^3-\alpha^3\right)^{1/3}} \exp\left[-\mathrm{i}\frac{\lambda}{\beta^3-\alpha^3}\left(\lambda^2\alpha^3\beta^3/3 + a\beta^3 - b\alpha^3\right)\right] \\
&\quad \times Ai\left[\frac{1}{\left(\beta^3-\alpha^3\right)^{1/3}}\left(b - a - \frac{\lambda^2\alpha^3\beta^3}{\beta^3-\alpha^3}\right)\right],
\end{aligned}
\tag{3.121}
$$

and for $\beta = \alpha$

$$
\begin{aligned}
I_\lambda &= \frac{1}{2\sqrt{\pi}\,|\lambda|^{1/2}\,|\alpha|^{3/2}} \\
&\quad \times \exp\left\{-\mathrm{i}\left[\frac{\alpha^3\lambda^3}{12} - \frac{(a-b)^2}{4\alpha^3\lambda} + \frac{\lambda(a+b)}{2} + \frac{\pi}{4}\mathrm{sgn}(\alpha\lambda)\right]\right\}.
\end{aligned}
\tag{3.122}
$$

Finally, we give the following integral

$$
\int_0^\infty Ai\left(\frac{x}{t}\right) Ai(t)\sqrt{t}\,\mathrm{d}t = \frac{-1}{2^{2/3}\sqrt{3}} Ai'(2^{2/3}\sqrt{x}), \ x > 0. \tag{3.123}
$$

3.6 Other Integrals

3.6.1 *Integrals involving the Volterra μ-function*

The Volterra μ–function is relatively intriguing in the bestiary of special functions, for it has received only a few considerations, despite its remarkable properties. In this section, we would like to add some properties linked to Airy functions [Vallée (2002)].

In his work on integral equations with logarithmic kernel, Volterra introduced a function that can be generalised by [Erdélyi et al. (1981)]

$$
\mu(u,\beta,\alpha) = \int_0^\infty \frac{t^\beta}{\Gamma(\beta+1)} \frac{u^{\alpha+t}}{\Gamma(t+\alpha+1)} \,\mathrm{d}t. \tag{3.124}
$$

We are going to calculate the following integral in using the definition of the μ–function.

$$\int_0^\infty Ai(u)\,\mu(\gamma u,\beta,\alpha)\,\mathrm{d}u = \\ = \int_0^\infty \mathrm{d}t \frac{t^\beta}{\Gamma(\beta+1)} \int_0^\infty Ai(u) \frac{(\gamma u)^{\alpha+t}}{\Gamma(t+\alpha+1)}\,\mathrm{d}u. \tag{3.125}$$

Using the Mellin transform of the Airy function (see Eq. (3.83)):

$$\int_0^\infty Ai(u)\,u^n\,\mathrm{d}u = \frac{\Gamma(n+1)}{3^{(n+3)/3}\Gamma(\frac{n+3}{3})}.$$

We then find for the right member of Eq. (3.125)

$$\int_0^\infty \frac{\mathrm{d}t}{3} \frac{t^\beta}{\Gamma(\beta+1)} \frac{(\gamma^3/3)^{(\alpha+t)/3}}{\Gamma(\frac{\alpha+t}{3}+1)}.$$

Making the change $t = 3\,z$, we obtain

$$\int_0^\infty Ai(u)\,\mu(\gamma u,\beta,\alpha)\,\mathrm{d}u = 3^\beta \int_0^\infty \frac{z^\beta}{\Gamma(\beta+1)} \frac{(\gamma^3/3)^{\alpha/3+z}}{\Gamma(\frac{\alpha}{3}+z+1)}\,\mathrm{d}z,$$

from which we deduce the result

$$\int_0^\infty Ai(u)\,\mu(\gamma u,\beta,\alpha)\,\mathrm{d}u = 3^\beta \mu(\frac{\gamma^3}{3},\beta,\frac{\alpha}{3}). \tag{3.126}$$

This can also be written

$$\frac{1}{(3t)^{1/3}} \int_0^\infty Ai\left(\frac{x}{(3t)^{1/3}}\right)\,\mu(x,\beta,\alpha)\,\mathrm{d}x = 3^\beta \mu(t,\beta,\frac{\alpha}{3}). \tag{3.127}$$

Therefore we have found that $\mu(x,\beta,0)$ is an eigenfunction of the above integral equation for the kernel

$$\frac{1}{(3t)^{1/3}} Ai\left(\frac{x}{(3t)^{1/3}}\right). \tag{3.128}$$

This kernel is easily recognised to satisfy (as a similarity solution) the evolution equation

$$\frac{\partial f(x,t)}{\partial t} + \frac{\partial^3 f(x,t)}{\partial x^3} = 0, \tag{3.129}$$

which is the linearised Korteweg–de Vries equation (l-KdV) [Ablowitz (1981)].

Volterra functions may be the eigenfunctions of many other integral equations involving the Airy function, by using the recurrence relations between Volterra functions

$$\frac{\mathrm{d}}{\mathrm{d}x}\mu(x,\beta,\alpha) = \mu(x,\beta,\alpha-1) \tag{3.130a}$$

$$x\mu(x,\beta,\alpha) = \mu(x,\beta+1,\alpha+1) + (\alpha+1)\,\mu(x,\beta,\alpha+1). \tag{3.130b}$$

From the Airy equation : $Ai'' - u\,Ai = 0$, we can write

$$\int_0^\infty u\,Ai(u)\,\mu(\gamma u,\beta,\alpha)\,\mathrm{d}u = \int_0^\infty Ai''(u)\,\mu(\gamma u,\beta,\alpha)\,\mathrm{d}u \tag{3.131}$$

We then make an integration by parts:

$$\begin{aligned}\int_0^\infty Ai''(u)\,\mu(\gamma u,\beta,\alpha)\,\mathrm{d}u &= \\ Ai'(u)\,\mu(\gamma u,\beta,\alpha)\big|_0^\infty &- \gamma\int_0^\infty Ai'(u)\,\mu'(\gamma u,\beta,\alpha)\,\mathrm{d}u\end{aligned} \tag{3.132}$$

The first term of the right member cancel, for $u=0$ due to the μ function and for $u\to\infty$ thanks to the Airy function. Now we use the relationship (3.130a) and we find

$$\int_0^\infty u\,Ai(u)\,\mu(\gamma u,\beta,\alpha)\,\mathrm{d}u = -\gamma\int_0^\infty Ai'(u)\,\mu(\gamma u,\beta,\alpha-1)\,\mathrm{d}u. \tag{3.133}$$

Making a second integration by parts:

$$\int_0^\infty u\,Ai(u)\,\mu(\gamma u,\beta,\alpha)\,\mathrm{d}u = \gamma^2\int_0^\infty Ai(u)\,\mu(\gamma u,\beta,\alpha-2)\,\mathrm{d}u.$$

Thus using the result from Eq. (3.126), we obtain

$$\int_0^\infty u\,Ai(u)\,\mu(\gamma u,\beta,\alpha)\,\mathrm{d}u = 3^\beta\gamma^2\,\mu(\frac{\gamma^3}{3},\beta,\frac{\alpha-2}{3}). \tag{3.134}$$

There are other ways to prove this result. The first one is to take the derivative of Eq. (3.126)

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}\int_0^\infty Ai(u)\,\mu(\gamma u,\beta,\alpha)\,\mathrm{d}u = \int_0^\infty u\,Ai(u)\,\mu(\gamma u,\beta,\alpha-1)\,\mathrm{d}u,$$

which is equal to

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}\left[3^\beta\mu(\frac{\gamma^3}{3},\beta,\frac{\alpha}{3})\right] = 3^\beta\gamma^2\,\mu(\frac{\gamma^3}{3},\beta,\frac{\alpha}{3}-1),$$

leading to the same result.

Another method is to use the recurrence relation between Volterra μ-functions (3.130b). Eq. (3.134) may be written into the other form

$$\frac{1}{(3t)^{4/3}}\int_0^\infty x\,Ai\left[\frac{x}{(3t)^{1/3}}\right]\mu(x,\beta,\alpha)\,\mathrm{d}x = 3^\beta\gamma^2\,\mu(t,\beta,\frac{\alpha-2}{3}). \quad (3.135)$$

Then $\mu(x,\beta,-1)$ is deduced to be the eigenfunction of an integral equation with the kernel

$$\frac{x}{(3t)^{4/3}}Ai\left(\frac{x}{(3t)^{1/3}}\right), \quad (3.136)$$

which is again a solution to the Eq. (3.129).

By the same way, we can find other interesting results like

$$\int_0^\infty Ai'(u)\,\mu(\gamma u,\beta,\alpha)\,\mathrm{d}u = -3^\beta\gamma\,\mu(\frac{\gamma^3}{3},\beta,\frac{\alpha-1}{3}), \quad (3.137)$$

and

$$\int_0^\infty Ai_1(u)\,\mu(\gamma u,\beta,\alpha)\,\mathrm{d}u = -3^\beta\frac{1}{\gamma}\,\mu(\frac{\gamma^3}{3},\beta,\frac{\alpha+1}{3}), \quad (3.138)$$

where $Ai_1(u) = \int_u^\infty Ai(t)\,\mathrm{d}t$.

More generally, we have a formula for the n^{th} derivative of the Airy function

$$\int_0^\infty Ai^{(n)}(u)\,\mu(\gamma u,\beta,\alpha)\,\mathrm{d}u = 3^\beta(-\gamma)^n\,\mu(\frac{\gamma^3}{3},\beta,\frac{\alpha-n}{3}). \quad (3.139)$$

Each of the equations (3.137)–(3.139), may be put into a form, which satisfies an eigenfunction equation with a kernel that is a similarity solution to the l-KdV equation. Finally, we give a result for Ai^2 from its Mellin transform [Reid (1995)]

$$\int_0^\infty Ai^2(u)\,u^n\,\mathrm{d}u = \frac{2\Gamma(n+1)}{\sqrt{\pi}12^{(2n+7)/6}\Gamma(\frac{2n+7}{6})},$$

which reads

$$\int_0^\infty Ai^2(u)\,\mu(\gamma u,\beta,\alpha)\,\mathrm{d}u = \frac{3^\beta}{2\sqrt{\pi\gamma}}\,\mu(\frac{\gamma^3}{12},\beta,\frac{2\alpha+1}{6}). \quad (3.140)$$

This equation may alternatively be written as

$$\frac{1}{(12t)^{1/6}}\int_0^\infty Ai^2\left[\frac{x}{(12t)^{1/3}}\right]\mu(x,\beta,\alpha)\,\mathrm{d}x = \frac{3^\beta}{2\sqrt{2}}\mu(t,\beta,\frac{2\alpha+1}{6}). \quad (3.141)$$

It is then found that $\mu(x,\beta,\frac{1}{4})$ is an eigenfunction of this equation, with the kernel

$$\frac{1}{(12t)^{1/6}}\,Ai^2\left[\frac{x}{(12t)^{1/3}}\right], \tag{3.142}$$

which again satisfies the l-KdV equation.

3.6.2 *Canonisation of cubic form*

In this section, we are going to derive a canonisation method of cubic forms as described by Turnbull [Turnbull (1960)]. With the help of the following linear transformation

$$\begin{cases} X = \alpha x + \beta y \\ Y = \gamma x + \delta y \end{cases}$$

with

$$\Delta = \alpha\delta - \beta\gamma \neq 0.$$

We are looking for the transformation of the cubic form:

$$F(x,y) = Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3$$

into its canonical form $F(X,Y) = X^3 + Y^3$. For this purpose, we consider the covariants and the invariant of the cubics:

- The Hessian, *i.e.* the determinant of the second derivatives

$$\mathscr{H} = \left(AC - B^2\right)x^2 + (AD - BC)\,xy + \left(BD - C^2\right)y^2\ ;$$

- The cubic covariant is obtained by taking the Jacobian of the cubics and of the Hessian

$$\begin{aligned}\mathscr{C} = {} & \left(A^2D - 3ABC + 2B^3\right)x^3 + 3\left(ABD - 2AC^2 + B^2C\right)x^2y \\ & +3\left(2B^2D - ACD - BC^2\right)xy^2 + \left(3BCD - AD^2 - 2C^3\right)y^3\end{aligned}$$

- The invariant of the cubics is built by taking the Hessian of the Hessian, which is the discriminant of the cubics F,

$$\mathscr{D} = (AD - BC)^2 - 4\left(AC - B^2\right)\left(BD - C^2\right).$$

There is a noteworthy relation (which is a so called *syzygy*) between the invariant and the covariants

$$F^2\mathscr{D} = \mathscr{C}^2 + 4\mathscr{H}^3.$$

For similar quantities of the canonical cubics $F(X,Y) = X^3 + Y^3$, we have

$$\begin{cases} \mathscr{H} = \Delta^2 XY, \\ \mathscr{C} = \Delta^3 \left(X^3 - Y^3\right), \\ \mathscr{D} = \Delta^6. \end{cases}$$

By comparison between the cubics F and the cubic covariant $\mathscr{C}$, identifying the terms x^3 and y^3 with the terms X^3 and Y^3 respectively, we finally obtain

$$\begin{cases} \alpha^3 = \frac{1}{2}\left[A + \frac{1}{\sqrt{\mathscr{D}}}\left(A^2D - 3ABC + 2B^3\right)\right], \\ \gamma^3 = \frac{1}{2}\left[A - \frac{1}{\sqrt{\mathscr{D}}}\left(A^2D - 3ABC + 2B^3\right)\right], \\ \beta^3 = \frac{1}{2}\left[D - \frac{1}{\sqrt{\mathscr{D}}}\left(AD^2 - 3BCD + 2C^3\right)\right], \\ \delta^3 = \frac{1}{2}\left[D + \frac{1}{\sqrt{\mathscr{D}}}\left(AD^2 - 3BCD + 2C^3\right)\right], \end{cases}$$

with the condition

$$\mathscr{D} = (AD - BC)^2 - 4\left(AC - B^2\right)\left(BD - C^2\right) = \Delta^6 \neq 0.$$

It should be noted, in the case where $\mathscr{D} = 0$, one cannot use the general canonical form $F(X,Y) = X^3 + Y^3$. In this case, the canonical form of $F(x,y)$ is either X^3 or X^2Y.

3.6.3 *Integrals with three Airy functions*

We consider the following integral (where a, c and e are different from zero) [Vallée (1982)]

$$L = \int_{-\infty}^{+\infty} Ai(ax+b)Ai(cx+d)Ai(ex+f)\mathrm{d}x. \tag{3.143}$$

Using once again the integral representation (2.20) of the Airy function

$$L = \frac{1}{8\pi^3} \iiiint_{\mathbb{R}^4} \mathrm{e}^{\,\mathrm{i}(u^3+v^3+w^3)/3} \mathrm{e}^{\,\mathrm{i}(ax+b)u} \times \mathrm{e}^{\,\mathrm{i}(cx+d)v} \mathrm{e}^{\,\mathrm{i}(ex+f)w} \mathrm{d}u\mathrm{d}v\mathrm{d}w\mathrm{d}x.$$

An integration on the variable x results in a Dirac delta function allowing us to express w as a function of u and v

$$L = \frac{1}{4\pi^2} \iiint_{\mathbb{R}^3} \mathrm{e}^{\,\mathrm{i}(u^3+v^3+w^3)/3} \mathrm{e}^{\,\mathrm{i}(bu+dv+fw)} \delta(au + cv + ew)\mathrm{d}u\mathrm{d}v\mathrm{d}w.$$

We then have an integral of the kind

$$L = \frac{1}{4\pi^2 |e|} \iint_{\mathbb{R}^2} \mathrm{e}^{\,\mathrm{i}\left(Ax^3/3+Bx^2y+Cxy^2+Dy^3/3+Vx+Wy\right)} \mathrm{d}x\mathrm{d}y,$$

with

$$\begin{cases} A = 1 - a^3/\mathrm{e}^3, \\ B = -a^2c/\mathrm{e}^3, \\ C = -ac^2/\mathrm{e}^3, \\ D = 1 - c^3/\mathrm{e}^3, \\ V = b - af/e, \\ W = d - cf/e. \end{cases}$$

The calculation of this integral is carried out in §3.6.5, formula (3.148), and the result reads

$$L = \frac{1}{K^{1/6}} Ai\left(\frac{(be-af)\delta - (de-cf)\gamma}{K^{1/6}}\right) Ai\left(\frac{(de-cf)\alpha - (be-af)\beta}{K^{1/6}}\right),$$

with

$$\begin{cases} K = e^6 + a^6 + c^6 - 2a^3c^3 - 2a^3e^3 - 2c^3e^3, \\ \alpha^3 = \frac{1}{2}\left[A + \frac{1}{\sqrt{K}}\left(A^2D - 3ABC + 2B^3\right)\right] \\ \beta^3 = \frac{1}{2}\left[D - \frac{1}{\sqrt{K}}\left(AD^2 - 3BCD + 2C^3\right)\right], \\ \gamma^3 = \frac{1}{2}\left[A - \frac{1}{\sqrt{K}}\left(A^2D - 3ABC + 2B^3\right)\right], \\ \delta^3 = \frac{1}{2}\left[D + \frac{1}{\sqrt{K}}\left(AD^2 - 3BCD + 2C^3\right)\right]. \end{cases}$$

We give, as a particular case, the integral with the product of three Airy functions [Vallée et al. (1997)]

$$\int_{-\infty}^{+\infty} Ai(x)Ai(y-x)Ai(z-x)\mathrm{d}x \tag{3.144}$$

$$= 5^{-1/6} Ai\left[5^{-1/3}\left(y/\varepsilon - \varepsilon z\right)\right] Ai\left[5^{-1/3}\left(z/\varepsilon - \varepsilon y\right)\right]$$

where ε is the golden mean $\varepsilon = \frac{\sqrt{5}-1}{2}$.

3.6.4 *Integrals with four Airy functions*

Let us consider the integral [Aspnes (1966)]

$$I = \int_{-\infty}^{+\infty} Ai^2\left(\frac{a+x}{\alpha}\right) Ai^2\left(\frac{b-x}{\beta}\right) \mathrm{d}x \tag{3.145}$$

with $\alpha > 0$ and $\beta > 0$. With the help of the formula (2.155), this integral reads

$$I = \frac{1}{2^{4/3}\pi^2} \int_{-\infty}^{+\infty} \mathrm{d}x \int_0^{\infty} \frac{\mathrm{d}u}{\sqrt{u}} \int_0^{\infty} \frac{\mathrm{d}v}{\sqrt{v}} Ai\left[\frac{2^{2/3}}{\alpha}(a+x)+u\right] \times Ai\left[\frac{2^{2/3}}{\beta}(b-x)+v\right].$$

By now using the formula (3.108), we have

$$I = \frac{\alpha\beta}{4\pi^2\left(\alpha^3+\beta^3\right)^{1/3}} \int_0^{\infty} \frac{\mathrm{d}u}{\sqrt{u}} \int_0^{\infty} \frac{\mathrm{d}v}{\sqrt{v}}\, Ai\left[\frac{2^{2/3}(a+b)+\alpha u+\beta v}{\left(\alpha^3+\beta^3\right)^{1/3}}\right].$$

Using again the formula (2.155), this expression becomes

$$I = \frac{\sqrt{\alpha\beta}}{2\pi} \int_0^{\infty} \frac{\mathrm{d}w}{\sqrt{w}}\, Ai^2\left[\frac{(a+b)}{\left(\alpha^3+\beta^3\right)^{1/3}}+w\right].$$

We find here the primitive of the Airy function Ai (cf. formula (3.105))

$$Ai_1(x) = 2\int_0^{\infty} Ai^2\left(t+2^{-2/3}x\right) \frac{\mathrm{d}t}{\sqrt{t}}.$$

We finally obtain

$$\int_{-\infty}^{+\infty} Ai^2\left(\frac{a+x}{\alpha}\right) Ai^2\left(\frac{b-x}{\beta}\right) \mathrm{d}x = \frac{\sqrt{\alpha\beta}}{4\pi} Ai_1\left[2^{2/3}\frac{(a+b)}{\left(\alpha^3+\beta^3\right)^{1/3}}\right]. \tag{3.146}$$

3.6.5 *Double integrals*

Let us consider the following double integral [Vallée (1982)]

$$G = \iint_{\mathbb{R}^2} Ai(ax+by+p)Ai(cx+dy+q) \times Ai(ex+fy+r)Ai(gx+hy+w)\mathrm{d}x\mathrm{d}y, \tag{3.147}$$

where a, c, e, g are non-zero constants. We make use of the integral representation (2.20) of $Ai(x)$, and we rearrange the terms, producing

$$G = \frac{1}{16\pi^4} \iiiint_{\mathbb{R}^4} \mathrm{d}t\mathrm{d}t'\mathrm{d}s\mathrm{d}s'\ \mathrm{e}^{\,\mathrm{i}\left[\left(t^3+t'^3+s^3+s'^3\right)/3+pt+qt'+rs+ws'\right]} \times \iint_{\mathbb{R}^2} \mathrm{e}^{\,\mathrm{i}\left[x\left(at+ct'+es+gs'\right)+y\left(bt+dt'+fs+hs'\right)\right]}\mathrm{d}x\mathrm{d}y.$$

From the integral representation of the Dirac delta function $\delta(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{\,\mathrm{i}zw}\mathrm{d}w$, we have

$$G = \frac{1}{16\pi^4} \iiiint_{\mathbb{R}^4} \mathrm{d}t\mathrm{d}t'\mathrm{d}s\mathrm{d}s'\ \mathrm{e}^{\,\mathrm{i}\left[\left(t^3+t'^3+s^3+s'^3\right)/3+pt+qt'+rs+ws'\right]} \times \delta\left(at+ct'+es+gs'\right)\delta\left(bt+dt'+fs+hs'\right).$$

An integration over s et s', then gives us

$$G = \frac{K^3}{4\pi^2} \iint_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}y\ \mathrm{e}^{\,\mathrm{i}\left(Ax^3/3+Bx^2y+Cxy^2+Dy^3/3+Vx+Wy\right)}, \tag{3.148}$$

with

$$\begin{cases} K = eh - fg \neq 0, \\ A = (eh-fg)^3 + (bg-ah)^3 + (af-be)^3, \\ D = (eh-fg)^3 + (dg-ch)^3 + (cf-de)^3, \\ B = (bg-ah)^2(dg-ch) + (af-be)^2(cf-de), \\ V = v(bg-ah) + w(af-be) + p(eh-fg), \\ W = v(dg-ch) + w(cf-de) + q(eh-fg). \end{cases}$$

The next step is to canonise the cubic form (cf. §3.6.2), leading to

$$G = \frac{K^3}{4\pi^2\Delta} \iint\limits_{\mathbb{R}^2} \mathrm{d}X\mathrm{d}Y \, \exp\left[\mathrm{i}\left(\frac{X^3}{3} + \frac{Y^3}{3} + \frac{V\delta - W\gamma}{\Delta}X + \frac{W\alpha - V\beta}{\Delta}\right)Y\right], \tag{3.149}$$

with

$$\Delta^6 = \mathscr{D} = (AD - BC)^2 - 4\left(AC - B^2\right)\left(BD - C^2\right).$$

Finally the integral (3.147) may be written as

$$G = \frac{K^3}{\Delta} Ai\left(\frac{V\delta - W\gamma}{\Delta}\right) Ai\left(\frac{W\alpha - V\beta}{\Delta}\right). \tag{3.150}$$

Exercises

1. Generalise the table of Albright (see §3.2.1) to calculate the following primitives (n=1, 2)

$$\int x^n y_1^2 \, dx, \quad \int x^n y_1 y \, dx, \quad \int x^n y_1 y' \, dx.$$

2. Calculate the following integrals in terms of Ai and Gi functions

$$\int_0^\infty \cos\left(a\,t^3 + b\,t + c\right)\,dt, \quad \int_0^\infty \sin\left(a\,t^3 + b\,t + c\right)\,dt.$$

3. Calculate

$$\int\limits_0^\infty \cos\left(\frac{t^3}{12} + \alpha t - \frac{\beta}{t} + \frac{\pi}{4}\right)\frac{\mathrm{d}t}{\sqrt{t}}.$$

4. Use the relations between Bessel and Airy functions (cf. §2.2.4) to express the following integral in terms of Airy functions

$$\int\limits_0^\infty x\mathrm{e}^{-\frac{x^2}{2a}}\left[I_\nu(x) + I_{-\nu}(x)\right]K_\nu(x)\,dx = a\mathrm{e}^a\,K_\nu(a),$$

when $\nu = \frac{1}{3}$ and $\nu = \frac{2}{3}$, a > 0.

5. Check the formula (3.144), with the method described in §3.6.3.

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Chapter 4

Transformations of Airy Functions

4.1 Causal Properties of Airy Functions

4.1.1 *Causal relations*

With the help of the integral representation of $Ai(x)$, $Gi(x)$, $Ai^2(x)$ and $Ai(x)Bi(x)$ respectively given by the formulae (2.20), (2.125), (2.148) and (2.149), we can write [Scorer (1950); Aspnes (1967)]

$$Ai(x) + \mathrm{i}Gi(x) = \frac{1}{\pi}\int_0^\infty \mathrm{e}^{\mathrm{i}\left(t^3/3+tx\right)}\mathrm{d}t \tag{4.1}$$

$$Ai^2(x) + \mathrm{i}Ai(x)Bi(x) = \frac{1}{2\pi^{3/2}}\int_0^\infty \mathrm{e}^{\mathrm{i}\left(t^3/12+tx+\pi/4\right)}\frac{\mathrm{d}t}{\sqrt{t}}. \tag{4.2}$$

These functions being analytic in the complex plane, the real and imaginary parts may be written as a Hilbert transform. We then have

$$Ai(x) = \frac{1}{\pi}\wp\int_{-\infty}^{+\infty}\frac{Gi(x')}{x'-x}\mathrm{d}x', \tag{4.3}$$

and conversely

$$Gi(x) = -\frac{1}{\pi}\wp\int_{-\infty}^{+\infty}\frac{Ai(x')}{x'-x}\mathrm{d}x', \tag{4.4}$$

where $\wp$ is the Cauchy principal value. From the relation (4.2), we can also deduce

$$Ai^2(x) = \frac{1}{\pi}\wp \int_{-\infty}^{+\infty} \frac{Ai(x')Bi(x')}{x'-x}\mathrm{d}x', \tag{4.5}$$

and conversely

$$Ai(x)Bi(x) = -\frac{1}{\pi}\wp \int_{-\infty}^{+\infty} \frac{Ai^2(x')}{x'-x}\mathrm{d}x'. \tag{4.6}$$

With the help of the relation (4.1), we can write

$$\frac{1}{|\alpha|}\left[Ai\left(\frac{x}{\alpha}\right) + \mathrm{i}Gi\left(\frac{x}{\alpha}\right)\right] = \frac{1}{\pi}\int_0^{\infty} \mathrm{e}^{\mathrm{i}\left(\alpha^3u^3/3+ux\right)}\mathrm{d}u. \tag{4.7}$$

So that when $\alpha \to 0$, this last relation becomes

$$\lim_{\alpha\to 0}\frac{1}{|\alpha|}\left[Ai\left(\frac{x}{\alpha}\right) + \mathrm{i}Gi\left(\frac{x}{\alpha}\right)\right] = \frac{1}{\pi}\int_0^{\infty} \mathrm{e}^{\mathrm{i}ux}\mathrm{d}u = \delta(x) + \mathrm{i}\frac{1}{\pi}\wp\frac{1}{x},$$

that is to say

$$\lim_{\alpha\to 0}\frac{1}{|\alpha|}Ai\left(\frac{x}{\alpha}\right) = \delta(x), \tag{4.8}$$

and

$$\lim_{\alpha\to 0}\frac{1}{|\alpha|}Gi\left(\frac{x}{\alpha}\right) = \frac{1}{\pi}\wp\frac{1}{x}. \tag{4.9}$$

It can be seen that these last two expressions are the beginning of the expansions given by Lee (1980)

$$\frac{1}{|\alpha|}Ai\left(\frac{x}{\alpha}\right) = \delta(x) + \frac{\alpha^3}{3}\delta^{(3)}(x) - \frac{\alpha^6}{18}\delta^{(6)}(x) + \ldots \tag{4.10}$$

and

$$\begin{aligned}&\frac{1}{|\alpha|}Gi\left(\frac{x}{\alpha}\right) \\ &= \frac{1}{\pi}\left[\wp\left(\frac{1}{x}\right) + \frac{\alpha^3}{3}\frac{\mathrm{d}^3}{\mathrm{d}x^3}\wp\left(\frac{1}{x}\right) - \frac{\alpha^6}{18}\frac{\mathrm{d}^6}{\mathrm{d}x^6}\wp\left(\frac{1}{x}\right) + \ldots\right].\end{aligned} \tag{4.11}$$

4.1.2 *Green function of the Airy equation*

In this section, we are going to build the Green function satisfying the differential equation [Moyer (1973); Burnett & Belsley (1983)]

$$\left(\frac{\partial^2}{\partial x^2} - x\right) G(x, x') = \delta(x - x'). \tag{4.12}$$

The integral expression of this function given by Lukes & Somaratna (1969), is

$$G(x, x') = -\mathrm{i} \int_0^\infty U(x, x', t) \mathrm{d}t, \tag{4.13}$$

with

$$U(x, x', t) = \left(\frac{1}{4\pi t}\right)^{1/2} \exp\left\{\mathrm{i}\left[-\frac{\pi}{4} + \left(\frac{x - x'}{4}\right)^2 \frac{1}{t} - \left(\frac{x + x'}{2}\right) t - \frac{t^3}{12}\right]\right\}.$$

We can write, thanks to the integral representation (2.20) of the function $Ai(x)$

$$Ai\left(z'\mathrm{e}^{\mathrm{i}2\pi/3}\right) Ai\left(z\right) = \frac{\mathrm{e}^{\mathrm{i}\pi/12}}{4\pi^{3/2}} \int_0^\infty \mathrm{e}^{\mathrm{i}\left[-\left(\frac{z+z'}{2}\right)v + \left(\frac{z-z'}{2}\right)^2 \frac{1}{v} - \frac{v^3}{12}\right]} \frac{\mathrm{d}v}{\sqrt{v}}, \tag{4.14}$$

for $0 \leq \arg(z - z') \leq \pi/2$, and

$$Ai\left(z\mathrm{e}^{\mathrm{i}2\pi/3}\right) Ai\left(z'\right) = \frac{\mathrm{e}^{\mathrm{i}\pi/12}}{4\pi^{3/2}} \int_0^\infty \mathrm{e}^{\mathrm{i}\left[-\left(\frac{z+z'}{2}\right)v + \left(\frac{z-z'}{2}\right)^2 \frac{1}{v} - \frac{v^3}{12}\right]} \frac{\mathrm{d}v}{\sqrt{v}}, \tag{4.15}$$

for $0 \leq \arg(z' - z) \leq \pi/2$. Comparing Eqs. (4.13–4.15), we obtain

$$G(x, x') = 2\pi \mathrm{i} \mathrm{e}^{\mathrm{i}2\pi/3} \times \begin{cases} Ai\left(x'\mathrm{e}^{\mathrm{i}2\pi/3}\right) Ai\left(x\right) \text{ if } x \geq x' \\ \\ Ai\left(x'\right) Ai\left(x\mathrm{e}^{\mathrm{i}2\pi/3}\right) \text{ if } x \leq x'. \end{cases}$$

Thanks to the relation

$$\mathrm{e}^{\mathrm{i}2\pi/3} Ai\left(z\mathrm{e}^{\mathrm{i}2\pi/3}\right) = -\frac{1}{2}\left[Ai(z) - \mathrm{i}Bi(z)\right],$$

we can then write $G(x, x')$ as a function of Ai and Bi

$$G(x,x') = 2\pi \mathrm{i} e^{\mathrm{i}2\pi/3} \times \begin{cases} Ai(x)Bi(x') + \mathrm{i}Ai(x)Ai(x') \text{ if } x \geq x' \\ \\ Ai(x')Bi(x) + \mathrm{i}Ai(x')Ai(x) \text{ if } x \leq x'. \end{cases}$$

4.2 The Airy Transform

4.2.1 *Definitions and elementary properties*

We define the family of functions

$$\omega_\alpha(x) = \frac{1}{|\alpha|} Ai\left(\frac{x}{\alpha}\right), \quad \alpha \in \mathbb{R}. \tag{4.16}$$

On the one hand, one of the most important properties of this family is (formula (4.8))

$$\lim_{\alpha \to 0} \{\omega_\alpha(x)\} = \delta(x), \tag{4.17}$$

where $\delta(x)$ is the Dirac delta function and $\alpha \in \mathbb{R}$. On the other hand, from the formula (3.108), we can set the relation giving the convolution product of two functions $\omega_\alpha(x)$ and $\omega_\beta(x)$

$$\omega_\alpha * \omega_\beta(x) = \omega_\gamma(x), \tag{4.18}$$

with $\gamma^3 = \alpha^3 + \beta^3$. In other words, the family of functions defined by Eq. (4.16) is stable under the convolution product and then leads to a semi-group of convolution. Note also the functions ω_α are normalised (formula (3.73))

$$\int_{-\infty}^{+\infty} Ai(x)\mathrm{d}x = \int_{-\infty}^{+\infty} \omega_\alpha(x)\mathrm{d}x = 1. \tag{4.19}$$

Therefore if f is a function of x, and $\hat{f}$ its Fourier transform, we write φ_α, the Airy transform of f, as the convolution product

$$\varphi_\alpha(x) = f * \omega_\alpha(x) = \mathcal{A}_\alpha [f(x)], \tag{4.20}$$

which is also written

$$\varphi_\alpha(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}(\alpha^3\xi^3/3+\xi x)} \hat{f}(\xi)\mathrm{d}\xi. \tag{4.21}$$

Consequently, $f \mapsto \varphi_\alpha$ is a particular class of functional transform that can be reversed by the formula

$$f(x) = \varphi_\alpha * \omega_{-\alpha}(x) = \overline{\mathcal{A}}_\alpha \left[\varphi_\alpha(x)\right]. \tag{4.22}$$

In order to sum up the results, Eqs. (4.20) and (4.22) define the *Airy transform*, where the inverse satisfies $\mathcal{A}_\alpha \overline{\mathcal{A}}_\alpha = \mathcal{A}_\alpha \mathcal{A}_{-\alpha} = \mathcal{I}$.

As physical applications of this transform, we can quote a work by Hunt [Hunt (1981)] concerning molecular physics, and a work [Bertoncini et al. (1990)] using the Airy transform, as a tool in the calculation of the Green function of non-equilibrium high field quantum transport.

As a matter of fact, the Airy transform was mathematically introduced by Widder [Widder (1979)] from another approach, *i.e.* by considering the self adjoint Schrödinger operator: $\left\{x - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right\}$. The solutions of the eigenvalue equation are $Ai(\xi - x)$. These functions form a continuous set of eigenfunctions, with the eigenvalues ξ. Indeed, from the relation (3.108), we have the orthogonality condition

$$\int_{-\infty}^{+\infty} Ai(\xi - x)Ai(\xi' - x)dx = \delta(\xi - \xi').$$

The definition of the Airy transform then follows

$$\varphi(\xi) = \int_{-\infty}^{+\infty} f(x)Ai(\xi - x)\mathrm{d}x, \tag{4.23}$$

and the inverse transform

$$f(x) = \int_{-\infty}^{+\infty} \varphi(\xi)Ai(\xi - x)\mathrm{d}\xi. \tag{4.24}$$

Therefore, in the transformation defined by Eqs. (4.20) and (4.22), α appears as a scaling parameter, relatively to the basic transform defined by the above Eqs. (4.23) and (4.24), but in this case with $\alpha > 0$.

From the definition of the Airy transform, some elementary properties can be derived:

(i) Translation

If $\varphi_\alpha(x)$ is the transform of $f(x)$, then $\varphi_\alpha(x+s)$ is the transform of $f(x+s)$.

(ii) Scaling

If $\varphi_\alpha(x)$ is the transform of $f(x)$, then $\varphi_{\alpha k}(kx)$ is the transform of $f(kx)$.

(iii) Derivative

If $\varphi_\alpha(x)$ is the transform of $f(x)$, then $\varphi'_\alpha(x)$ is the transform of $f'(x)$.

(iv) Iteration

If $\varphi_\alpha(x)$ is the transform of $f(x)$ by the function $\omega_\alpha(x)$, then the transform of $\varphi_\alpha(x)$ by the function $\omega_\beta(x)$ is $\varphi_\gamma(x)$, where $\gamma^3 = \alpha^3 + \beta^3$. In other words

$$\varphi_\gamma = (f * \omega_\alpha) * \omega_\beta = f * (\omega_\alpha * \omega_\beta) = f * \omega_\gamma. \tag{4.25}$$

(v) Convolution

If $\varphi_\alpha(x)$ is the transform of $f(x)$ by the function $\omega_\alpha(x)$ and $\psi_\beta(x)$ is the transform of $g(x)$ by the function $\omega_\beta(x)$, then the convolution product $\varphi_\alpha * \psi_\beta$ is the transform of $f * g$ by the function $\omega_\gamma(x)$, where $\gamma^3 = \alpha^3 + \beta^3$

$$\varphi_\alpha * \psi_\beta = (f * \omega_\alpha) * (g * \omega_\beta) = (f * g) * (\omega_\alpha * \omega_\beta) = (f * g) * \omega_\gamma. \tag{4.26}$$

As the Airy functions $\{Ai(\xi - x),\ \xi \in \mathbb{R}\}$ form a continuous basis of orthogonal functions, we have:

(vi) The Plancherel–Parceval rule

If $\varphi_\alpha(x)$ and $\psi_\alpha(x)$ are the transforms of the real functions $f(x)$ and $g(x)$ by $\omega_\alpha(x)$, then for all real α

$$\int_{-\infty}^{+\infty} f(x)g(x)\mathrm{d}x = \int_{-\infty}^{+\infty} \varphi_\alpha(x)\psi_\alpha(x)\mathrm{d}x. \tag{4.27}$$

The proof of all these properties of the Airy transform can be shown rigourously, with the methods described in the paper of Widder. It will be the same for the examples that are presented in the next section.

4.2.2 *Some examples*

Let us start with the transform of a constant. From Eq. (4.19), we immediately deduce that the transform of $f(x) = 1$ is $\varphi_\alpha(x) = 1$.

From these, we can examine the periodic functions by putting

$$f(x) = \mathrm{e}^{\mathrm{i}\xi x}.$$

We then obtain, with the integral representation (2.20) of Ai and the Dirac delta representation

$$\varphi_\alpha(x) = \frac{1}{|\alpha|}\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\xi y} Ai\left(\frac{x-y}{\alpha}\right) \mathrm{d}y = \mathrm{e}^{\mathrm{i}\left(\alpha^3\xi^3/3+\xi x\right)}.$$

Now from this equation, we can obtain the transform of periodic functions

$$\sin(\xi x) \xrightarrow{\mathcal{A}_\alpha} \sin\left(\xi x + \frac{\alpha^3\xi^3}{3}\right) \tag{4.28}$$

$$\cos(\xi x) \xrightarrow{\mathcal{A}_\alpha} \cos\left(\xi x + \frac{\alpha^3\xi^3}{3}\right) \tag{4.29}$$

This result may be employed to transform a periodic function of which we have the Fourier expansion

$$f(x) = \sum c_n \mathrm{e}^{\mathrm{i}\pi n x}.$$

The Airy transform of this function is then

$$\varphi_\alpha(x) = \sum b_n(\alpha)\mathrm{e}^{\mathrm{i}\pi n x},$$

with

$$b_n = c_n \mathrm{e}^{\mathrm{i}(\pi n \alpha)^3}.$$

We now consider the case of a normalised Gaussian function

$$f(x) = \frac{1}{\sqrt{\pi}}\,\mathrm{e}^{-x^2}. \tag{4.30}$$

Few algebra gives the result of the transform

$$\varphi_\alpha(x) = \frac{1}{|\alpha|}\, \mathrm{e}^{\frac{1}{4\alpha^3}\left(x+\frac{1}{24\alpha^3}\right)} Ai\left(\frac{x}{\alpha} + \frac{1}{16\alpha^4}\right). \tag{4.31}$$

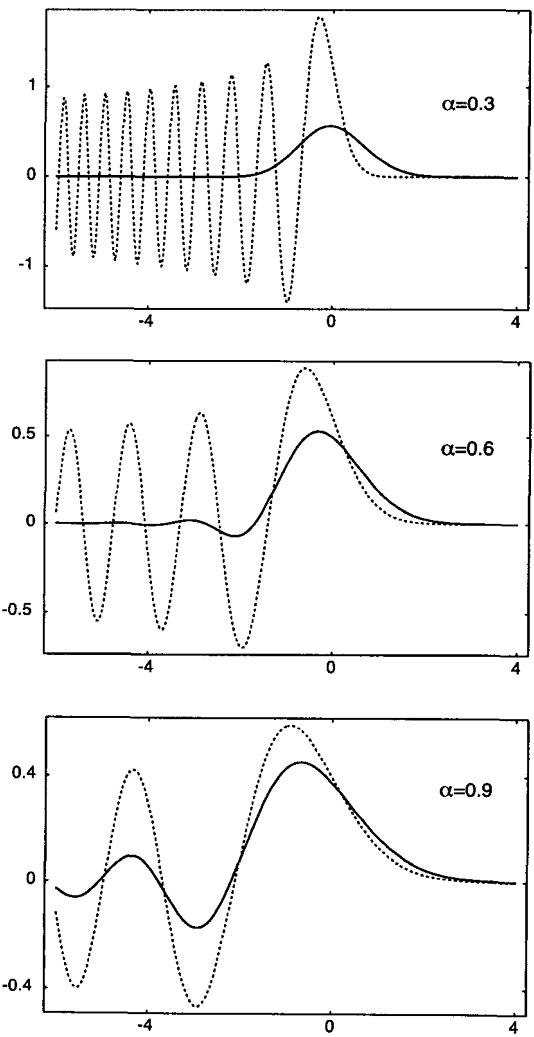


Fig. 4.1 Airy transform of a Gaussian: the transform (full lines) given from the formula (4.31) is compared to the normalised Airy function $\omega_\alpha(x)$ (formula (4.16)), for $\alpha = 0.3$, $\alpha = 0.6$, $\alpha = 0.9$.

On Fig. 4.1, the Airy function $\omega_\alpha(x)$ (dotted lines) and the transform of the Gaussian function $\varphi_\alpha(x)$ (full lines) are plotted for different values of the parameter α. For the smaller values of α, the transform resembles a Gaussian, but it behaves rapidly, as an Airy function, when α increases, losing its Gaussian character even asymptotically.

The last example concerns the step function $\theta(x)$, its Airy transform is

$$\varphi_\alpha(x) = \frac{1}{|\alpha|}\int_{-\infty}^{+\infty} \theta(y) Ai\left(\frac{x-y}{\alpha}\right) \mathrm{d}y = \int_{-\infty}^{x/\alpha} Ai(u)\mathrm{d}u. \tag{4.32}$$

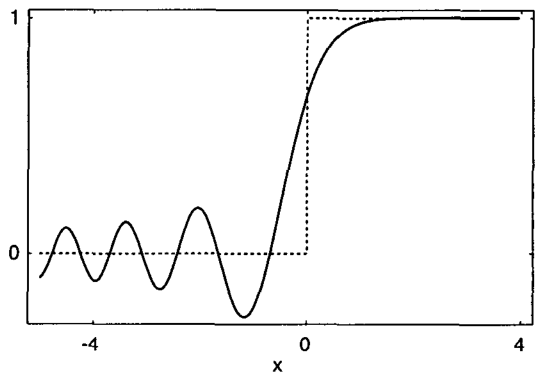


Fig. 4.2 Airy transform of the step function $\theta(x)$.

The transformed function oscillates for $x < 0$ and goes exponentially to 1 for $x > 0$ (Fig. 4.2), as can be seen on the following asymptotic expansions

$$\varphi_\alpha(x) \approx 1 - \frac{1}{2\sqrt{\pi}}\left(\frac{x}{\alpha}\right)^{-3/4} \mathrm{e}^{-\frac{2}{3}\left(\frac{x}{\alpha}\right)^{3/2}},$$

when $x \to +\infty$, and

$$\varphi_\alpha(x) \approx \frac{1}{\sqrt{\pi}}\left(\frac{x}{\alpha}\right)^{-3/4} \cos\left[\frac{2}{3}\left(\frac{x}{\alpha}\right)^{3/2} + \frac{\pi}{4}\right],$$

when $x \to -\infty$.

We can also give an important result allowing the transform of many functions to be calculated analytically.

Lemma 4.1 *If the Airy transform of a function f is φ_α, then the Airy transform of xf is*

$$\mathcal{A}_\alpha\left[xf(x)\right] = x\varphi_\alpha - \alpha^3\varphi''_\alpha. \tag{4.33}$$

Actually the derivation rule implies

$$\varphi''_\alpha = f'' * \omega_\alpha = f * \omega''_\alpha,$$

but ω_α is a solution of the Airy equation $\omega''_\alpha - \frac{x}{\alpha^3}\omega_\alpha = 0$. So we readily have:

$$\varphi''_\alpha = \frac{1}{\alpha^3} f * (x\omega_\alpha) = \frac{1}{\alpha^3}(x(f * \omega_\alpha) - (xf) * \omega_\alpha),$$

which proves the result.

As we have seen, the Airy transform of 1 is 1, the transform of x is x,[1] and the Airy transform of x^2 is x^2.

This chain of transforms may be continued in order to form a family of polynomials: the *Airy polynomials* $Pi_n(x)$. Some properties of these polynomials are given in the next section. Airy polynomials may be used to calculate the Airy transform of functions that can be written as a power–series

$$f(x) = \sum_{n=0}^{\infty} c_n x^n. \tag{4.34}$$

The result in the transformation is a power–series

$$\varphi_\alpha(x) = \sum_{n=0}^{\infty} \alpha^n c_n Pi_n\left(\frac{x}{\alpha}\right). \tag{4.35}$$

We can also use the preceding lemma 4.1 to calculate the Airy transform of a function f, knowing the transform of xf. We have just to solve an inhomogeneous differential equation

$$\varphi'' - \frac{x}{\alpha^3}\varphi = \psi.$$

[1] To prove this result, and the one given by Eq. (4.35), we should have to introduce a converging factor.

For example, we know that the Airy transform of 1 is 1. We can therefore calculate the transform of $f(x) = 1/x$. So that $\psi = A_\alpha\left[x\frac{1}{x}\right] = 1$), and:

$$\varphi_\alpha(y) = \frac{1}{\alpha}\wp \int_{-\infty}^{+\infty} \frac{1}{x} Ai\left(\frac{y-x}{\alpha}\right) \mathrm{d}x = \frac{\pi}{\alpha} Gi\left[\frac{y}{\alpha}\right],$$

where $\wp$ is the Cauchy principal value and Gi is the inhomogeneous function (*cf.* §2.3.1) and $\alpha > 0$. This last relation is nothing else but formula (4.4).

Another result is useful to calculate Airy transforms, it consists of a generalisation of the Plancherel–Parceval rule

Lemma 4.2 *If the Airy transform of the function $f(x)$ is $\varphi_\alpha(x)$, then these two functions have the same autocorrelation function*

$$F(x) = \int_{-\infty}^{+\infty} f(y) f(x+y) \mathrm{d}y = \int_{-\infty}^{+\infty} \varphi_\alpha(u) \varphi_\alpha(x+u) \mathrm{d}u. \tag{4.36}$$

The proof stems from the substitution of the value of φ_α into the right member of Eq. (4.36),

$$F(x) = \frac{1}{\alpha^2} \int_{\mathbb{R}^2} f(y) f(y') \mathrm{d}y \mathrm{d}y' \int_{-\infty}^{+\infty} Ai\left(\frac{u-y}{\alpha}\right) Ai\left(\frac{x+u-y}{\alpha}\right) \mathrm{d}u.$$

The formula (3.108) then leads to the announced result.

If $\hat{F}(\xi)$ is the Fourier transform of $F(x)$, we can see that the relation (4.36) may be reversed according to

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sqrt{\hat{F}(\xi)} \mathrm{e}^{\mathrm{i}\left(\alpha^3\xi^3/3+\xi x\right)} \mathrm{d}\xi. \tag{4.37}$$

This relation has the meaning of an Airy transform in terms of the Fourier variable ξ (cf. Eq. (4.21)).[2] However, the inversion can also be performed in putting $\alpha = 0$. These results are easily checked in the particular case where f is a Gaussian or an Airy function. This inversion is an example of an ill–posed inverse problem.

We close this paragraph, with a result that will be used in §7.5. In the case where the scaling parameter depends on another parameter: $\alpha = \alpha(t)$, we have the following lemma:

[2]This result is a particular case of a more general result of Wiener, see for instance Davis (1962), page 434.

Lemma 4.3 *The Airy transform of the partial derivative with respect to the parameter t is given by*

$$\mathcal{A}_\alpha\left[\frac{\partial f}{\partial t}\right] = \frac{\partial \varphi_\alpha}{\partial t} + \dot{\alpha}\alpha^2 \frac{\partial^3 \varphi_\alpha}{\partial x^3} \tag{4.38}$$

where $\mathcal{A}_\alpha[f] = \varphi_\alpha$ *and* $\dot{\alpha} = \frac{\mathrm{d}\alpha}{\mathrm{d}t}$.

The proof is given by taking the Fourier transform of both members of the equality. We readily see all the interest of such a result, in the Airy transform of an evolution equation. In particular, the case: $\dot{\alpha}\alpha^2 = \mathrm{C}^{\mathrm{te}}$ was studied by Widder [Widder (1979)]. We shall employ this lemma in the section §7.5.1.1.

4.2.3 *Airy polynomials*

We define an Airy polynomial of degree n, $Pi_n(x)$, from the Airy transform of x^n (this is actually a convolution product of two distributions)

$$Pi_n(x) = \int_{-\infty}^{+\infty} y^n Ai(x-y)\mathrm{d}y. \tag{4.39}$$

When we take twice the derivative of this formula, we directly obtain the recurrence formula

$$Pi_{n+1}(x) = xPi_n(x) - Pi_n''(x). \tag{4.40}$$

Taking once again the derivative of this expression, we obtain the following third order differential equation satisfied by Airy polynomials

$$Pi_n'''(x) - xPi_n'(x) + nPi_n(x) = 0. \tag{4.41}$$

This equation has to be compared with formulae (2.102) and (2.146).[3] From the last two equations, we deduce the recurrence relation

$$Pi_{n+1}(x) = xPi_n(x) - n(n-1)Pi_{n-2}(x). \tag{4.42}$$

There is also an addition theorem for Airy polynomials. From the defi-

[3] We shall return to that kind of equation in §6.2.

nition (4.39), we can write

$$\sum_{k=0}^{n}\binom{n}{k}Pi_{n-k}(x)Pi_k(y)=\sum_{k=0}^{n}\binom{n}{k}\iint_{\mathbb{R}^2}(x')^{n-k}(y')^kAi(x-x')$$
$$\times\ Ai(y-y')\mathrm{d}x'\mathrm{d}y'.$$

Then from the binomial relation $\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k=(x+y)^n$, we get

$$\sum_{k=0}^{n}\binom{n}{k}Pi_{n-k}(x)Pi_k(y)$$
$$=\iint_{\mathbb{R}^2}(x'+y')^nAi(x-x')Ai(y-y')\mathrm{d}x'\mathrm{d}y'.$$

After a change of variables and the use of formula (3.108), we find the addition theorem of these polynomials

$$\sum_{k=0}^{n}\binom{n}{k}Pi_{n-k}(x)Pi_k(y)=2^{n/3}Pi_n\left(\frac{x+y}{2^{1/3}}\right). \tag{4.43}$$

Other properties of Airy polynomials can be quoted: for instance their values at the origin (cf. formula (3.84))

$$\begin{cases} Pi_{3n}(0)=(-1)^n\,\frac{(3n)!}{3^n n!} \\ Pi_{3n+1}(0)=Pi_{3n+2}(0)=0, \end{cases}$$

and a generating function

$$\mathrm{e}^{-t^3/3+xt}=\sum_{n=0}^{\infty}\frac{t^n}{n!}Pi_n(x). \tag{4.44}$$

Airy polynomials have similar properties to those of Hermite $He_n(x)$ [Abramowitz & Stegun (1965)]. As a matter of fact, Airy polynomials belong (like the Hermite's $He_n(x)$) to a family of polynomials – somewhat related to the so called Appel set [Erdélyi et al. (1981)] – sharing, in particular, the property

$$\frac{\mathrm{d}}{\mathrm{d}x}P_n(x)=n\,P_{n-1}(x).$$

4.2.4 Summary of Airy transform

In Table 4.1, we have summed up the Airy transform of some functions.

Table 4.1 Some Airy transforms

$f(x)$	$\varphi_\alpha(y) = \frac{1}{\alpha}\int_{-\infty}^{+\infty} f(x) Ai\left(\frac{y-x}{\alpha}\right) \mathrm{d}x,\ \alpha > 0$
$f(x+k)$	$\varphi_\alpha(y+k)$
$f(kx)$	$\varphi_{\alpha k}(ky)$
$f'(x)$	$\varphi'_\alpha(y)$
$xf(x)$	$y\varphi_\alpha(y) - \alpha^3\varphi''_\alpha(y)$
1	1
x	y
x^2	y^2
x^n	$\alpha^n Pi_n\left(\frac{y}{\alpha}\right)$
$\mathrm{e}^{\mathrm{i}kx}$	$\exp\left[\mathrm{i}\left(ky + \alpha^3 k^3/3\right)\right]$
$\delta(x)$	$\frac{1}{\alpha} Ai\left(\frac{y}{\alpha}\right)$
$\mathrm{sgn}(x)$	$1 - 2\int_{y/\alpha}^{+\infty} Ai(u)\mathrm{d}u$
$\theta(x)$	$1 - \int_{y/\alpha}^{+\infty} Ai(u)\mathrm{d}u$
$\frac{1}{x}$	$\frac{\pi}{\alpha} Gi\left[\frac{y}{\alpha}\right]$
$\frac{1}{\sqrt{\lvert x\rvert}}$	$\frac{2^{2/3}\pi}{\alpha^{1/2}}\left[Ai^2\left(\frac{y}{2^{2/3}\alpha}\right) + Ai\left(\frac{y}{2^{2/3}\alpha}\right) Bi\left(\frac{y}{2^{2/3}\alpha}\right)\right]$
e^{-x^2}	$\frac{\sqrt{\pi}}{\alpha}\exp\left[\frac{1}{4\alpha^3}\left(y + \frac{1}{24\alpha^3}\right)\right] Ai\left(\frac{y}{\alpha} + \frac{1}{16\alpha^4}\right)$
$Ai(x)$	$\left(\alpha^3+1\right)^{-1/3} Ai\left[\left(\alpha^3+1\right)^{-1/3} y\right]$
$Gi(x)$	$\left(\alpha^3+1\right)^{-1/3} Gi\left[\left(\alpha^3+1\right)^{-1/3} y\right]$
$Ai^2(x)$	$\left(4\alpha^3+1\right)^{-1/6} Ai^2\left[\left(4\alpha^3+1\right)^{-1/3} y\right]$
$Ai(x)Bi(x)$	$\left(4\alpha^3+1\right)^{-1/6} Ai\left[\left(4\alpha^3+1\right)^{-1/3} y\right] Bi\left[\left(4\alpha^3+1\right)^{-1/3} y\right]$

4.2.5 *Airy averaging*

Some years ago Englert and Schwinger (1984) introduced a method they called the Airy averaging, in the context of an improvement of the Thomas–Fermi statistical model. The definition of the Airy averaging of a function $f(x)$ is

$$\langle f(x) \rangle_{Ai} = \int_{-\infty}^{\infty} Ai(x) f(x)\, \mathrm{d}\, x. \tag{4.45}$$

In their paper [Englert and Schwinger (1984)], these authors gave several properties of Airy averaging, such as

$$\begin{aligned} \langle x \rangle_{Ai} = 0, \quad \langle x^2 \rangle_{Ai} = 0, \quad & \langle x^3 \rangle_{Ai} = 2 \\ \langle x^{n+1} \rangle_{Ai} &= n(n-1) \langle x^{n-2} \rangle_{Ai} \\ \langle x f(x) \rangle_{Ai} &= \langle \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) \rangle_{Ai} \end{aligned}$$

Clearly, from the definition and properties, the Airy averaging is a particular case of an inverse Airy transform for we have

$$\langle f(x) \rangle_{Ai} = \overline{\mathcal{A}}_1 [f(x)]. \tag{4.46}$$

In particular, we have $\langle x^{3n} \rangle_{Ai} = (-1)^n Pi_n(0)$, and all other properties of Airy averaging can be deduced from those of Airy transform.

4.3 Other Kinds of Transformations

4.3.1 *Laplace transform of Airy functions*

In 1982 Davison & Glasser calculated the Laplace transform of the functions $Ai(-x)$ and $Bi(-x)$, $x > 0$, for their application in surface physics (Schrödinger equation in a uniform electric field). In 1983, Leach calculated the Laplace transform of $Ai(\pm x)$ and $Bi(-x)$, $x > 0$, for their application to magnetohydrodynamics, from their integral representation. Exton (1985) and Wille (1986) calculated the Laplace transform of the Meijer function $G = G^{m,n}_{p,q}\left(z \left| \begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix} \right. \right)$. We shall not detail the definition of this function [see Gradsteyn & Ryzhik (1965)], but we are concerned by a particular case of G $G^{2,0}_{0,2}\left(x^3 \left| a, b \right.\right) = 2x^{3(a+b)/2} K_{a-b}\left(2x^{3/2}\right)$, with $(a,b) = \left(\frac{1}{3}, 0\right)$, $\left(\frac{2}{3}, 0\right)$ or $\left(\frac{2}{3}, \frac{1}{3}\right)$, K_ν being, the modified Bessel function (cf. §2.2.4). We therefore obtain the Laplace transforms of Airy functions

$$I_{Ai_-}(p) = \int_0^\infty \mathrm{e}^{-px} Ai(-x)\mathrm{d}x \tag{4.47}$$
$$= \frac{\mathrm{e}^a}{3}\left[2 - \frac{\gamma\left(\frac{2}{3}, a\right)}{\Gamma\left(\frac{2}{3}\right)} - \frac{\gamma\left(\frac{1}{3}, a\right)}{\Gamma\left(\frac{1}{3}\right)}\right]$$

$$I_{Bi_-}(p) = \int_0^\infty \mathrm{e}^{-px} Bi(-x)\mathrm{d}x \tag{4.48}$$
$$= \frac{\mathrm{e}^a}{\sqrt{3}}\left[-\frac{\gamma\left(\frac{2}{3}, a\right)}{\Gamma\left(\frac{2}{3}\right)} + \frac{\gamma\left(\frac{1}{3}, a\right)}{\Gamma\left(\frac{1}{3}\right)}\right]$$

$$I_{Ai_+}(p) = \int_0^\infty \mathrm{e}^{-px} Ai(x)\mathrm{d}x \tag{4.49}$$
$$= \frac{\mathrm{e}^{-a}}{3}\left[1 + \frac{\phi\left(\frac{2}{3}, a\right)}{\Gamma\left(\frac{2}{3}\right)} - \frac{\phi\left(\frac{1}{3}, a\right)}{\Gamma\left(\frac{1}{3}\right)}\right],$$

with $a = \frac{p^3}{3}$. $\Gamma(x)$ is the gamma function, $\gamma(\alpha, x)$ is the incomplete gamma function $\gamma(\alpha, x) = \int_0^x \mathrm{e}^{-u} u^{\alpha-1}\mathrm{d}u$, and $\phi(\alpha, x) = \int_0^x \mathrm{e}^{u} u^{\alpha-1}\mathrm{d}u$. We can see that it is possible to express these transforms in terms of the confluent hypergeometric function F, thanks to the relations

$$\gamma(\alpha, x) = \frac{x^\alpha}{\alpha}\mathrm{e}^{-x}\, F(1, 1+\alpha; x) \quad \text{and} \quad \phi(\alpha, x) = \frac{x^{1-\alpha}}{1-\alpha}\, F(1-\alpha, 2-\alpha; x).$$

4.3.2 *Mellin transform of Airy function*

The Mellin transform of a function $f(x)$, with the notation $f^*(s)$, is defined by the integral:

$$f^*(s) = \int_0^{+\infty} f(x) x^{s-1}\mathrm{d}x.$$

We shall not detail the calculus of the following transforms, but rather exhibit the following Table 4.2 of Mellin transforms, mostly determined by Reid (1995).

Table 4.2 Mellin transform of Airy functions [Reid (1995)]. We have defined $\beta = \frac{1}{6}(1-2s)$, $\gamma = \frac{1}{6}(1+s)$.

$f(x)$	$f^*(s) = \int_0^{+\infty} f(x) x^{s-1} \mathrm{d}x$
$Ai(x)$	$3^{-(s+2)/3} \frac{\Gamma(s)}{\Gamma((s+2)/3)}$, $\Re(s) > 0$
$Ai^2(x)$	$\pi^{-1/2} 12^{-(2s+5)/6} \frac{2\Gamma(s)}{\Gamma((2s+5)/6)}$, $\Re(s) > 0$
$Ai(x)Bi(x)$	$2\pi^{-3/2} 12^{\beta-1} \cos(\beta\pi)\Gamma(s)\Gamma(\beta)$, $0 < \Re(s) < \frac{1}{2}$
$Ai^2(-x) + Bi^2(-x)$	$4\pi^{-3/2} 12^{\beta-1} \Gamma(s)\Gamma(\beta)$, $0 < \Re(s) < \frac{1}{2}$
$Ai(x)Ai(-x)$	$\pi^{-3/2} 12^{\gamma-1} \sin(\gamma\pi)\Gamma(s/2)\Gamma(\gamma)$, $\Re(s) > 0$
$Ai(x)Bi(-x)$	$\pi^{-3/2} 12^{\gamma-1} \cos(\gamma\pi)\Gamma(s/2)\Gamma(\gamma)$, $\Re(s) > 0$
$Ai(x\mathrm{e}^{\mathrm{i}\pi/6})Ai(x\mathrm{e}^{-\mathrm{i}\pi/6})$	$\frac{1}{2}\pi^{-3/2} 12^{\gamma-1} \Gamma(s/2)\Gamma(\gamma)$, $\Re(s) > 0$

Table 4.3 Fourier transform of Airy functions

$f(x)$	$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(x) \mathrm{e}^{-\mathrm{i}\omega x} \mathrm{d}x$
$Ai(x)$	$\exp\left(\mathrm{i}\omega^3/3\right)$
$Ai_1(x)$	$\left[\pi\delta(\omega) + \mathrm{i}\wp\frac{1}{\omega}\right] \exp\left(\mathrm{i}\omega^3/3\right)$
$Gi(x)$	$\exp\left[\mathrm{i}\left(\omega^3/3 - \frac{\pi}{2}\mathrm{sgn}\omega\right)\right]$
$Ai(x^2)$	$2^{2/3}\pi Ai\left(-2^{-2/3}\omega\right) Ai\left(2^{-2/3}\omega\right)$
$Ai_1(x^2)$	$\frac{2\pi}{\omega}\left[Ai'\left(-2^{-2/3}\omega\right) Ai\left(2^{-2/3}\omega\right) - Ai\left(-2^{-2/3}\omega\right) Ai'\left(2^{-2/3}\omega\right)\right]$
$Ai^2(x)$	$\frac{1}{2\sqrt{\pi\lvert\omega\rvert}} \exp\left[-\mathrm{i}\left(\omega^3/12 + \frac{\pi}{4}\mathrm{sgn}\,\omega\right)\right]$
$Ai(x)Bi(x)$	$\frac{1}{2\sqrt{\pi\lvert\omega\rvert}} \exp\left[-\mathrm{i}\left(\omega^3/12 - \frac{\pi}{4}\mathrm{sgn}\,\omega\right)\right]$
$Ai(x)Ai(-x)$	$2^{-1/3} Ai\left(2^{-4/3}\omega^2\right)$
$Ai'(x)Ai'(-x)$	$2^{-1/3}\omega^2 Ai\left(2^{-4/3}\omega^2\right)$

4.3.3 *Fourier transform of Airy functions*

The Fourier transform of a function $f(x)$, with the notation $\hat{f}(\omega)$, is defined by the integral

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(x) \mathrm{e}^{-\mathrm{i}\omega x} \mathrm{d}x.$$

In Table 4.3, we have given some Fourier transform of Airy functions.

4.4 Expansion into Fourier–Airy Series

We recall that the zeros of the Airy function Ai, $\{a_n, n = 1, 2, \dots\}$ are placed on the real negative axis of the complex plane. (cf. §2.2.1). Let us consider the integral

$$I_{nn'} = \int_0^\infty Ai\,(x + a_n)\,Ai\,(x + a_{n'})\,\mathrm{d}x. \tag{4.50}$$

In the case where $n \neq n'$ the formula (3.53) allows us to obtain

$$I_{nn'} = \frac{Ai\,(a_n)\,Ai'\,(a_{n'}) - Ai'\,(a_n)\,Ai\,(a_{n'})}{a_n - a_{n'}} = 0, \tag{4.51}$$

whereas if $n = n'$, the formula (3.50) gives:

$$I_{nn} = \int_0^\infty Ai^2\,(x + a_n)\,\mathrm{d}x = Ai'^2\,(a_n)\,. \tag{4.52}$$

In both cases, we have used the property of the Airy function to decrease exponentially toward infinity (see the asymptotic expansions (2.44) and (2.45)). Therefore the functions $\{Ai\,(x + a_n)\,/Ai'\,(a_n)\,,\ n \in \mathbb{N}\}$ form an orthonormal basis on the interval $[0, \infty[$ [Titchmarsh (1962)]. Then for any integrable function $f(x)$, piecewise continuous, we can write the expansion

$$f(x) = \sum_{n=1}^{\infty} c_n \frac{Ai\,(x + a_n)}{Ai'\,(a_n)}. \tag{4.53}$$

The coefficients c_n of this expansion are determined by the relation

$$\begin{aligned} c_n &= \frac{1}{Ai'\,(a_n)} \int_0^\infty f(x) Ai\,(x + a_n)\,\mathrm{d}x \\ &= \sum_{n=1}^{\infty} \frac{c_{n'}}{Ai'^2\,(a_n)} \int_0^\infty Ai\,(x + a_n)\,Ai\,(x + a_{n'})\,\mathrm{d}x. \end{aligned} \tag{4.54}$$

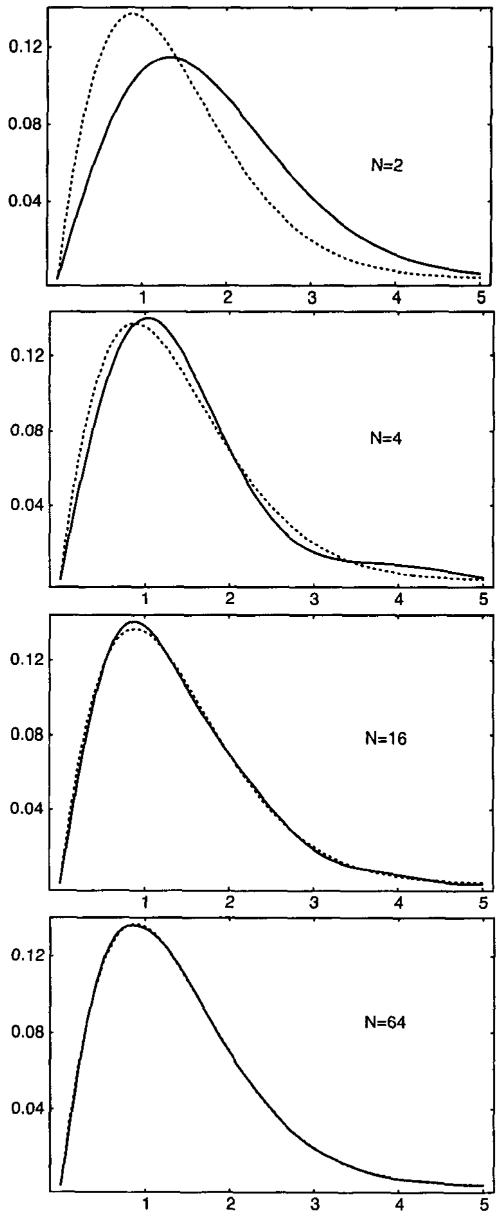


Fig. 4.3 Reconstitution of the function: $xAi(x)$ (dotted lines) by a Fourier–Airy series (full lines) bound by 2, 4, 16 and 64 terms.

As an example, we can consider the function $f(x) = xAi(x)$. The above integral Eq. (4.54) gives (see formula (3.54)) for the coefficient c_n

$$c_n = \frac{2}{a_n^3}\left[Ai(0) - a_n Ai'(0)\right].$$

Fig. 4.3 illustrates the reconstitution of $f(x)$ for partial sums with 2, 4, 16 and 64 terms of the series.

Exercises

1. Calculate the Airy transform of $\mathrm{e}^{\mathrm{i}\lambda x}\, Ai(x)$ *Hint*: See formula (3.121)

2. Calculate the Airy transform of the Airy wavelet $Ai'(x)Ai'(-x)$ in the case $\alpha = 1$. Is it still a wavelet? *Hint*: Use the integral (3.144), and then the property of the derivative of Airy transform (see §4.2.1).

3. Show that the heat equation $\partial_t u = \partial_{xx} u$, is invariant under an Airy transform. Solve the transformed equation with the initial condition $\tilde{u}_\alpha(y, 0) = \delta(y)$ *i.e.* the fundamental solution. Then take the inverse transform of this solution. What is the initial condition in the inverse transform $u(x, 0)$? *Hint*: See §7.4.

4. Calculate the Airy transform of Hermite polynomials. Are they still orthogonal?

5. Find the coefficients, in a Fourier–Airy expansion, of the function $f(x) = x^2 Ai(x)$. Plot the reconstitution of $f(x)$ for partial sums with 2, 4, 16 and 64 terms of the series.

Chapter 5

The Uniform Approximation

5.1 Oscillating Integrals

5.1.1 *The method of stationary phase*

In this paragraph, we are going to present the stationary phase method, which was studied during the early XX$^{\text{th}}$ by Stokes, Kelvin and Brillouin. Erdélyi (1956) and later Copson (1967) detailed this method. A more recent review paper by Knoll & Schaeffer (1977), can also be quoted.

Let us consider an integral of the form

$$I = \int_a^b g(z)\mathrm{e}^{\mathrm{i}\lambda f(z)}\mathrm{d}z,$$

where λ is an arbitrary large parameter.

If the function occurring in this integral is analytic, in a given domain of the complex plane, the integration contour can be deformed. The aim is to obtain an approximation of I in the limit $\lambda \to \infty$, it is then advantageous to keep the contour in regions where the integrand is as small as possible. If a topography of the complex plane with $\left|\mathrm{e}^{\mathrm{i}\lambda f(z)}\right|$ as an altitude is introduced, it is then easier to speak, by analogy, of valley and top. Thus the most favourable contours will be those that remain as far as possible in the valleys, except for the transitions from one valley to another, *i.e.* at points like z_i such that: $f'(z_i) = 0$. This is the reason why this method is also called the steepest descent method. The points z_i are called the stationary points, they are the points where the integrand is maximum giving the most important contribution to the value of the integral I. Let us consider a function f with only one stationary point z_0 such that: $f'(z_0) = 0$. We can expand $g(z)$ to the zero order (other terms are neglected) and $f(z)$ to the second order

$$\begin{cases} g(z) \approx g(z_0) \\ f(z) \approx f(z_0) + \frac{(z-z_0)^2}{2} f''(z_0). \end{cases}$$

The integral I may be written

$$I \approx \int_a^b g(z_0) \mathrm{e}^{\frac{\mathrm{i}}{\hbar} f(z_0)} \mathrm{e}^{\mathrm{i}\lambda \frac{(z-z_0)^2}{2} f''(z_0)} \mathrm{d}z.$$

The integration can now be extended from $-\infty$ to ∞, because we take into account only the neighbourhood of the stationary point

$$I \approx g(z_0) \mathrm{e}^{\mathrm{i}\lambda f(z_0)} \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\lambda \frac{(z-z_0)^2}{2} f''(z_0)} \mathrm{d}z.$$

When doing the change of variable

$$u = (z - z_0) \left(\frac{\lambda f''(z_0)}{2\mathrm{i}} \right)^{1/2},$$

we obtain

$$I \approx \left(\frac{2\mathrm{i}}{\lambda f''(z_0)} \right)^{1/2} g(z_0) \mathrm{e}^{\mathrm{i}\lambda f(z_0)} \int_{-\infty}^{+\infty} \mathrm{e}^{-u^2} \mathrm{d}u$$

with $\int_{-\infty}^{+\infty} \mathrm{e}^{-u^2} \mathrm{d}u = \sqrt{\pi}$. This yields finally to the expression

$$I \approx \left(\frac{2\pi\mathrm{i}}{\lambda f''(z_0)} \right)^{1/2} g(z_0) \mathrm{e}^{\mathrm{i}\lambda f(z_0)},$$

which can be generalised in the case where $f(z)$ admits several stationary points z_i

$$I \underset{\lambda\to\infty}{\to} \sum_{[z_i]} g(z_i) \left[\frac{2\pi\mathrm{i}}{\lambda f''(z_i)} \right]^{1/2} \mathrm{e}^{\mathrm{i}\lambda f(z_i)}.$$

This last expression can also be written

$$I \underset{\lambda\to\infty}{\to} \sum_{[z_i]} g(z_i) \sqrt{\frac{2\pi}{|f''(z_i)|}} \, \mathrm{e}^{\mathrm{i}\lambda \left[f(z_i) - \frac{\pi}{4} \mathrm{sgn}\left(f''(z_i) \right) \right]}.$$

5.1.2 *The uniform approximation of oscillating integral*

The limit when $\lambda \to \infty$ of an integral of the kind [Knoll & Schaeffer (1977)]

$$I = \int g(z) \mathrm{e}^{\mathrm{i}\lambda f(z)} \mathrm{d}z \tag{5.1}$$

is given in the frame of the stationary phase approximation as

$$I \underset{\lambda \to \infty}{\to} \sum_{[i]} G_i \mathrm{e}^{\mathrm{i}\lambda f(z_i)} = \sum_{[f'(z_i)=0]} g(z_i) \left[\frac{2\pi \mathrm{i}}{\lambda f''(z_i)} \right]^{1/2} \mathrm{e}^{\mathrm{i}\lambda f(z_i)}, \tag{5.2}$$

where the stationary points z_i are defined by

$$f'(z_i) = 0.$$

But this approximation is no more valid if two stationary points z_1 and z_2 are coalescing, *i.e.* if $|f(z_1) - f(z_2)|$ is of the order $1/\lambda$. The uniform approximation gives, however uniformly valid solutions, even in a neighbourhood of coalescing stationary points. We make the change of variable

$$\tilde{f}(y) = f(z(y)),$$

such that the integrand is transformed into a simpler form allowing an analytic evaluation of the integral. The stationary points z_i become

$$\left. \frac{\mathrm{d}\tilde{f}(y)}{\mathrm{d}y} \right|_{y_i} = 0 \quad \Leftrightarrow \quad y_i = y(z_i),$$

and the amplitude

$$\tilde{g}(y) = g(y(z)) \frac{\mathrm{d}z(y)}{\mathrm{d}y}.$$

The integral Eq. (5.1) is now written in terms of the new variables

$$I = \int \tilde{g}(y) \mathrm{e}^{\mathrm{i}\lambda \tilde{f}(y)} \mathrm{d}y,$$

which is tantamount to the integral (5.1), for it does not employ any approximation. The change of variables is chosen in such a manner that

$$\tilde{g}(y_i) = g(z_i) \frac{\mathrm{d}z}{\mathrm{d}y} = g(z_i) \left[\frac{\mathrm{d}^2 \tilde{f}}{\mathrm{d}y^2} \Big/ \frac{\mathrm{d}^2 f}{\mathrm{d}z^2} \right]^{1/2}.$$

So we recover far from the stationary points the approximation (5.2) given by the stationary phase method.

Now, we are going to study a particular case of the uniform treatment of the integral (5.1): the uniform approximation by Airy function.

5.1.3 *The Airy uniform approximation*

Numerous variations on this method can be found in scientific literature, for instance Child (1974) or Knoll & Schaeffer (1977). Let us choose for $\tilde{f}$ a cubic form

$$\tilde{f}(y) = f\left(z(y)\right) = \lambda\left(\eta - \xi^2 y - \frac{y^3}{3}\right), \tag{5.3}$$

with the stationary points $y_{1,2} = \pm\xi$. When we take Eq. (5.3) in z_1 and z_2, we obtain

$$\begin{cases} \eta = \lambda f\left(z_1\right) + f\left(\frac{z_2}{2}\right) \\ \xi = \lambda\left[\frac{3}{4}\left(f\left(z_1\right) - f\left(z_2\right)\right)\right]^{1/3}. \end{cases}$$

Now if we take for the amplitude a linear form, we have

$$\tilde{g}(y) = r_1 + \mathrm{i} r_2,$$

so that we obtain

$$I \to I_{\mathrm{A}i} = \left[r_1 Ai\left(-\xi^2\right) + r_2 Ai'\left(-\xi^2\right)\right] \mathrm{e}^{\mathrm{i}\eta}, \tag{5.4}$$

where r_1 and r_2 are defined in such a manner that the integrals (5.1) and (5.4) have the same asymptotic behaviour when $\lambda \to \infty$, that is to say

$$\begin{cases} r_1 + \mathrm{i}\xi r_2 = 2\sqrt{\pi\xi}\, G_1 \mathrm{e}^{-\mathrm{i}\pi/4} \\[2ex] r_1 - \mathrm{i}\xi r_2 = 2\sqrt{\pi\xi}\, G_2 \mathrm{e}^{\mathrm{i}\pi/4}, \end{cases}$$

amplitudes G_1 and G_2 being defined by Eq. (5.2)

$$G_i = g(z_i)\left[\frac{2\pi \mathrm{i}}{f''(z_i)}\right]^{1/2}.$$

5.2 Differential Equation of the Second Order

5.2.1 *The JWKB method*

Liouville (1837) and Green (1837) were looking for a solution of the second order differential equation [Morse & Feshbach (1953); Berry & Mount (1972); Nayfeh (1973); Olver (1974)]

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left[\lambda^2 q_1(x) + q_2(x)\right] y = 0, \tag{5.5}$$

for large values of λ, where $q_1(x)$ is a positive, twice differentiable function on a given interval. The function $q_2(x)$ is continuous on the same interval. We now make the following transforms on the dependent and the independent variables

$$\begin{cases} z = \phi(x) \\ v = \psi(x) y(x). \end{cases}$$

The differential equation (5.5) becomes

$$\begin{aligned} &\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} + \frac{1}{\phi'^2}\left(\phi'' - \frac{2\phi'\psi'}{\psi}\right)\frac{\mathrm{d}v}{\mathrm{d}z} \\ &\qquad + \frac{1}{\phi'^2}\left[\lambda^2 q_1(x) + q_2(x) - \psi\left(\frac{\psi'}{\psi^2}\right)'\right] v = 0. \end{aligned} \tag{5.6}$$

In order to have a self–similar equation in the transformation, we have to cancel the second term, namely

$$\phi'' - \frac{2\phi'\psi'}{\psi} = 0,$$

or

$$\psi = \sqrt{\phi'}. \tag{5.7}$$

Eq. (5.6) then becomes

$$\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} + \left[\lambda^2 \frac{q_1}{\phi'^2} + \frac{q_2}{\phi'^2} - \frac{1}{2\phi'^2}\{\phi, x\}\right] v = 0, \tag{5.8}$$

where $\{\phi, x\}$ is the Schwarzian derivative of ϕ

$$\{\phi, x\} = \frac{\phi'''}{\phi'} - \frac{3}{2}\left(\frac{\phi''}{\phi'}\right)^2. \tag{5.9}$$

We first notice that Eq. (5.8) is strictly equivalent to Eq. (5.5), because no approximation was necessary to obtain the result. Now if we suppose that λ is large enough, we can neglect the two last terms inside the bracket $[\ldots]$. Consequently Eq. (5.8) becomes in a first approximation

$$\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} + \lambda^2 \frac{q_1}{\phi'^2} v = 0. \tag{5.10}$$

Then we choose ϕ such that

$$\frac{q_1}{\phi'^2} = 1, \tag{5.11}$$

that is to say

$$\phi = \int \sqrt{q_1(t)}\mathrm{d}t. \tag{5.12}$$

Eq. (5.10) now reads:

$$\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} + \lambda^2 v = 0. \tag{5.13}$$

The solutions to the differential equation (5.13) are

$$v = a\cos(\lambda z) + b\sin(\lambda z),$$

so in making the inverse transform, we find [1]

$$y = \frac{1}{[q_1(x)]^{1/4}} \left\{ a\cos\left(\lambda \int \sqrt{q_1(x)}\mathrm{d}x\right) + b\sin\left(\lambda \int \sqrt{q_1(x)}\mathrm{d}x\right) \right\}. \tag{5.14}$$

But if the function $q_1(x)$ is negative, the solution is

$$y = \frac{1}{[-q_1(x)]^{1/4}} \left\{ a e^{\lambda \int \sqrt{-q_1(x)}\mathrm{d}x} + b e^{-\lambda \int \sqrt{-q_1(x)}\mathrm{d}x} \right\}. \tag{5.15}$$

We should mention that this approximation is no longer valid in the neighbourhood of the zeros of $q_1(x)$, *i.e.* in the neighbourhood of the so called turning points. The transformations (5.7) and (5.12) are called the Green–Liouville transformations. The solutions (5.14) and (5.15) are called JWKB approximated solutions, after Jeffreys (1924), Wentzel (1926), Kramers (1926) and Brillouin (1926).

[1] A convenient phase factor is introduced usually leading to the solution: $y = N[q_1(x)]^{-1/4} \cos\left(\lambda \int \sqrt{q_1(x)}\mathrm{d}x - \frac{\pi}{4}\right)$, where N is a normalisation constant.

5.2.2 *The generalisation of Langer*

The factor $q_1^{-1/4}(x)$, in the solutions (5.14) and (5.15) of Eq. (5.5), yields to the divergence of these solutions at the turning points. In order to have uniformly valid solutions in the whole neighbourhood of a turning point x_t, we put [Langer (1931), (1934)]

$$\frac{q_1}{\phi'^2} = \phi, \tag{5.16}$$

in other words

$$\begin{cases} \frac{2}{3}\phi^{3/2} = \int\limits_{x_t}^{x} \sqrt{q_1(t)}\,\mathrm{d}t & \text{if } x > x_t \\ \frac{2}{3}(-\phi)^{3/2} = \int\limits_{x}^{x_t} \sqrt{-q_1(t)}\,\mathrm{d}t & \text{if } x < x_t. \end{cases} \tag{5.17}$$

The differential equation (5.10) then becomes

$$\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} + \lambda^2 z v = 0, \tag{5.18}$$

whose solutions are

$$v = aAi\left(-\lambda^{2/3}z\right) + bBi\left(-\lambda^{2/3}z\right),$$

where Ai and Bi are the homogeneous Airy functions. The uniform solution to the equation (5.5), valid even in the neighbourhood of the turning point, is

$$y = \frac{1}{\sqrt{\phi'(x)}}\left\{aAi\left[-\lambda^{2/3}\phi(x)\right] + bBi\left[-\lambda^{2/3}\phi(x)\right]\right\}. \tag{5.19}$$

We can see that, in the asymptotic limit, the expression (5.19) goes to the JWKB approximation given by formulae (5.14) and (5.15), thanks to (2.44) and (2.46).

Olver (1954) generalises the transformation of Langer, setting

$$\begin{cases} \zeta = \zeta(z) = \int\limits_{x} \sqrt{q_1(t)}\,\mathrm{d}t \\ \chi = \frac{q_1(x)}{\zeta'^2}, \quad y = \chi^{-1/4} v, \end{cases}$$

where the independent variable z is an arbitrary function of x. Eq. (5.5) then reads

$$\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} + \lambda^2 \zeta'^2 v = 0.$$

The solutions of this equation are asymptotically equivalent to the solutions (5.14) and (5.15). Consequently, $\zeta(z)$ has to be chosen in such a manner that ζ'^2 has the same number of zeros as $q_1(x)$. For example, if $q_1(x)$ owns two turning points, we have [Pike (1964)]

$$\zeta'^2 = 4a^2\left(1 - z^2\right),$$

such that $z = -1$ corresponds to $x = x_1$, and $z = +1$ to $x = x_2$. We then obtain

$$\zeta = 2a \int_{-1}^{z} \sqrt{1 - t^2}\mathrm{d}t = \int_{x_1}^{x} \sqrt{q_1(t)}\mathrm{d}t,$$

that is to say

$$a = \frac{1}{\pi} \int_{x_1}^{x_2} \sqrt{q_1(t)}\mathrm{d}t.$$

Eq. (5.8) reads now:

$$\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} + 4a^2\lambda^2\left(1 - z^2\right) v = 0,$$

with the solution

$$v = D_\mu\left(2\sqrt{a\lambda}z\right),\ \mu + \frac{1}{2} = a\lambda,$$

where D_μ is the parabolic cylinder function of order μ [Abramowitz & Stegun (1965)].

5.3 Inhomogeneous Differential Equations

We consider the following inhomogeneous differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left[\lambda^2 q_1(x) + q_2(x)\right] y = \lambda^2 G(x), \tag{5.20}$$

where $q_1(x)$ owns a simple zero $x = x_t$. When the parameter λ goes to infinity, we obtain a particular approximate solution as

$$y = \frac{G(x)}{q_1(x)}.$$

This solution is singular for $x = x_t$, except if $G(x)$ also has a zero for $x = x_t$. In order to determine a particular solution in the case where $G(x_t) \neq 0$, we use the transformation

$$\begin{cases} z = \phi(x) \\ \frac{2}{3} z^{3/2} = \int\limits_{x_t}^{x} \sqrt{q_1(t)} \mathrm{d}t \; y = \frac{v}{\sqrt{\phi'}}. \end{cases}$$

Eq. (5.20) now reads

$$\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} + \left[\lambda^2 z - \delta\right] v = \lambda^2 g(z), \tag{5.21}$$

where δ is defined by the relation

$$\delta = \frac{1}{2\phi'^2} \{\phi, x\} - \frac{q_2}{\phi'^2},$$

with $g(z) = \{\phi'[x(z)]\}^{-3/2} G[x(z)]$, $\{\phi, x\}$ being the Schwarzian derivative of ϕ relatively to x. If δ is negligible in front of $\lambda^2 z$, Eq. (5.21) becomes:

$$\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} + \lambda^2 z v = \lambda^2 g(z). \tag{5.22}$$

Therefore, we can write $g(z)$ as a sum of two terms

$$g(z) = g(0) + [g(z) - g(0)],$$

and hence, we have to determine a particular solution for each of those terms. A particular solution for the second term is given in a first approximation by [Nayfeh (1973)]

$$v_1 = \frac{g(z) - g(0)}{z}.$$

In order to find a particular solution corresponding to the first term, we put $\xi = \lambda^{2/3} z$. Eq. (5.22) now reads

$$\frac{\mathrm{d}^2 v}{\mathrm{d}\xi^2} + \xi v = \lambda^{2/3} g(0),$$

with the particular solution

$$v_2 = \pi \lambda^{2/3} g(0) Wi(\xi),$$

where Wi is an inhomogeneous Airy function, either Hi or Gi, according to the sign of $g(0)$ is either positive or negative. A particular solution of

(5.22) is then

$$v = \pi\lambda^{2/3} g(0) Wi\,(\xi) + \frac{g(z) - g(0)}{z}.$$

From the inverse transform, we obtain

$$y = \pi\lambda^{2/3} \frac{G(x_t)}{[\phi'(x) f(x_t)]^{1/2}} Wi\left(\lambda^{2/3} z\right) + \frac{1}{q_1(x)} \left[G(x) - \frac{G(x_t)\phi^{3/2}}{\sqrt{f(x_t)}} \right] \tag{5.23}$$

with $f(x) = \frac{q(x_t)}{x - x_t}$.

Exercises

1. Consider the following nonlinear third order differential equation

$$\dddot{x} + a\,\ddot{x} - x\dot{x} + x = 0.$$

Make a scaling on x and t to transform this equation into

$$\dddot{x} + \lambda^2(\ddot{x} - x\dot{x} + x) = 0.$$

In the limit $\lambda \to \infty$, prove that a solution $x(t) + \mathcal{O}(1/\lambda)$ satisfies the second order differential equation, inside the parenthesis of the above equation.
Find a particular solution $x_p(t)$ of the second order equation. Check that

$$I = (y - 1) \exp\left(y - \frac{x^2}{2a} \right)$$

is a first integral of this equation ($y = \dot{x}$). Plot in the phase space this first integral for different values of $I =$ constant.
Using the transformation $x \to t + x_0 + u$ where x_0 is a constant and u is a perturbed solution searched near the particular solution, find the third order linear equation satisfied by u from the initial third order equation. Solve this equation in terms of Airy functions.
What is the condition on the integration constants, for which the solution is a true perturbed solution? *Hint*: The solution to this problem is given in [Letellier & Vallée (2003)].

Chapter 6

Generalisation of Airy Functions

6.1 Generalisation of the Airy Integral

Airy functions are solutions to the Airy differential equation $y'' - xy = 0$. A generalisation of these functions may be made in considering the solutions of the second order differential equation of the kind: $y'' + Cx^k y = 0$.

Since the Airy function is defined by the integral (2.19)

$$Ai(x) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{z^3}{3} + xz\right) \mathrm{d}z,$$

we can generalise this integral as follows. We put, according to Watson (1966),

$$T_n(t, \alpha) = t^n \, F\left(-\frac{n}{2}, \frac{1-n}{2}; 1-n; -\frac{4\alpha}{t^2}\right), \tag{6.1}$$

where $n \in \mathbb{N}$, $n \geq 2$, and F is the hypergeometric function defined by [Abramowitz & Stegun (1965)]

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}.$$

Thus we obtain

$$\begin{aligned} T_2(t, \alpha) &= t^2 + 2\alpha \\ T_3(t, \alpha) &= t^3 + 3\alpha t \\ T_4(t, \alpha) &= t^4 + 4\alpha t^2 + 2\alpha^2 \\ T_5(t, \alpha) &= t^5 + 5\alpha t^3 + 5\alpha^2 t \\ \vdots \quad & \quad \vdots \end{aligned} \tag{6.2}$$

So the generalisation of the Airy integral is given by

$$Ci_n(\alpha) = \int_0^\infty \cos\left[T_n(t,\alpha)\right] \mathrm{d}t \tag{6.3}$$

$$Si_n(\alpha) = \int_0^\infty \sin\left[T_n(t,\alpha)\right] \mathrm{d}t \tag{6.4}$$

$$Ei_n(\alpha) = \int_0^\infty \exp\left[-T_n(t,\alpha)\right] \mathrm{d}t. \tag{6.5}$$

As particular cases, we have $Ai(x) = \frac{3^{1/3}}{\pi} Ci_3\left(\frac{x}{3^{2/3}}\right)$ (formula (2.19)) and $Gi(x) = \frac{3^{1/3}}{\pi} Si_3\left(\frac{x}{3^{2/3}}\right)$ (formula (2.125)). [1]

Like in the case of Airy functions (cf. §2.2.4), we can express the integrals (6.3), (6.4) and (6.5), thanks to the Bessel functions I, J and K, according to the parity of n.

We shall not detail the calculations (see for instance Watson, 1966), but if n is even, $Ci_n(\alpha)$ and $Si_n(\alpha)$ are solutions of the differential equation

$$\frac{\mathrm{d}^2 Wi}{\mathrm{d}\alpha^2} + n^2\alpha^{n-2} Wi = 0, \tag{6.6}$$

with $Wi = Ci_n(\alpha)$, $Si_n(\alpha)$. The function $Ei_n(\alpha)$ is a solution to the equation

$$\frac{\mathrm{d}^2 Ei_n}{\mathrm{d}\alpha^2} - n^2\alpha^{n-2} Ei_n = 0. \tag{6.7}$$

These three functions may be expressed in the following form, with $\alpha > 0$,

$$Ci_n(\alpha) = \frac{\pi\alpha^{1/2}}{2n\sin\left(\frac{\pi}{2n}\right)}\left[J_{-1/n}\left(2\alpha^{n/2}\right) - J_{1/n}\left(2\alpha^{n/2}\right)\right] \tag{6.8}$$

$$Si_n(\alpha) = \frac{\pi\alpha^{1/2}}{2n\sin\left(\frac{\pi}{2n}\right)}\left[J_{-1/n}\left(2\alpha^{n/2}\right) + J_{1/n}\left(2\alpha^{n/2}\right)\right] \tag{6.9}$$

$$Ei_n(\alpha) = \frac{2\alpha^{1/2}}{n} K_{1/n}\left(2\alpha^{n/2}\right), \tag{6.10}$$

[1] Note that the functions Ci, Si and Ei (so defined) have nothing to do with the functions: cosine integral, sine integral and exponential integral.

and

$$Ci_n(-\alpha) = \frac{\pi\alpha^{1/2}}{2n\sin\left(\frac{\pi}{2n}\right)}\left[J_{-1/n}\left(2\alpha^{n/2}\right) + J_{1/n}\left(2\alpha^{n/2}\right)\right] \tag{6.11}$$

$$Si_n(-\alpha) = \frac{\pi\alpha^{1/2}}{2n\sin\left(\frac{\pi}{2n}\right)}\left[J_{-1/n}\left(2\alpha^{n/2}\right) - J_{1/n}\left(2\alpha^{n/2}\right)\right] \tag{6.12}$$

$$Ei_n(-\alpha) = \frac{\pi\alpha^{1/2}}{n\sin\left(\frac{\pi}{n}\right)}\left[I_{-1/n}\left(2\alpha^{n/2}\right) + I_{1/n}\left(2\alpha^{n/2}\right)\right]. \tag{6.13}$$

For example, in the case $n = 4$, we obtain

$$P(x,0) = 2\mathrm{e}^{-\mathrm{i}x^2/8}\left[Ci_4\left(\frac{x}{4}\right) + \mathrm{i}Si_4\left(\frac{x}{4}\right)\right],$$

where $P(x,y)$ is the integral of Pearcey [Pearcey (1946); Connor & Farrelly (1980)], given by

$$P(x,y) = \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\left(u^4+xu^2+yu\right)}\mathrm{d}u.$$

If n is odd, we obtain similarly, for $\alpha > 0$,

$$Ci_n(\alpha) = \frac{\pi\alpha^{1/2}}{2n\sin\left(\frac{\pi}{2n}\right)}\left[I_{-1/n}\left(2\alpha^{n/2}\right) - I_{1/n}\left(2\alpha^{n/2}\right)\right] \tag{6.14}$$

$$= \frac{2\alpha^{1/2}\cos\left(\frac{\pi}{2n}\right)}{n}K_{1/n}\left(2\alpha^{n/2}\right) \tag{6.15}$$

$$Ci_n(-\alpha) = \frac{\pi\alpha^{1/2}}{2n\sin\left(\frac{\pi}{2n}\right)}\left[J_{-1/n}\left(2\alpha^{n/2}\right) + J_{1/n}\left(2\alpha^{n/2}\right)\right]. \tag{6.16}$$

The function $Ci_n(\alpha)$ (n odd) verifies then, the differential equation

$$\frac{\mathrm{d}^2 Ci_n}{\mathrm{d}\alpha^2} - n^2\alpha^{n-2}Ci_n = 0. \tag{6.17}$$

In particular for $n = 3$, we find the Airy equation (2.1).

The expressions for Si and Ei are more complicated. $Si_n(-\alpha)$ can be written under the form ($\alpha > 0$)

$$Si_n(\alpha) = -\frac{\pi\alpha^{\frac{n+1}{2}}}{n\cos\left(\frac{\pi}{2n}\right)} \sum_{m=0}^{\infty} \frac{\alpha^{mn}}{\Gamma\left(m+\frac{3}{2}-\frac{1}{2n}\right)\Gamma\left(m+\frac{3}{2}+\frac{1}{2n}\right)} \tag{6.18}$$

$$+\frac{\pi\alpha^{1/2}}{2n\cos\left(\frac{\pi}{2n}\right)}\left[I_{-1/n}\left(2\alpha^{n/2}\right)+I_{1/n}\left(2\alpha^{n/2}\right)\right]$$

$$Si_n(-\alpha) = \frac{(-1)^{\frac{n-1}{2}}\pi\alpha^{\frac{n+1}{2}}}{n\cos\left(\frac{\pi}{2n}\right)} \sum_{m=0}^{\infty} \frac{(-1)^m\alpha^{mn}}{\Gamma\left(m+\frac{3}{2}-\frac{1}{2n}\right)\Gamma\left(m+\frac{3}{2}+\frac{1}{2n}\right)}$$

$$+\frac{\pi\alpha^{1/2}}{2n\cos\left(\frac{\pi}{2n}\right)}\left[J_{-1/n}\left(2\alpha^{n/2}\right)-J_{1/n}\left(2\alpha^{n/2}\right)\right]. \tag{6.19}$$

$Si_n(\alpha)$ (n odd) satisfies the differential equation

$$\frac{\mathrm{d}^2 Si_n}{\mathrm{d}\alpha^2} - n^2\alpha^{n-2}Si_n = -n\alpha^{\frac{n-3}{2}}. \tag{6.20}$$

In particular for $n = 3$, we find again the Airy inhomogeneous differential equation (2.124).

The function $Ei_n(\alpha)$ (n odd) may be written as the series

$$Ei_n(\alpha) = \frac{\pi\alpha^{\frac{n+1}{2}}}{n\cos\left(\frac{\pi}{2n}\right)} \sum_{m=0}^{\infty} \frac{(-1)^m\alpha^{mn}}{\Gamma\left(m+\frac{3}{2}-\frac{1}{2n}\right)\Gamma\left(m+\frac{3}{2}+\frac{1}{2n}\right)} \tag{6.21}$$

$$+\frac{\pi}{n\sin\left(\frac{\pi}{n}\right)}\left\{\sum_{m=0}^{\infty}\frac{(-1)^m\alpha^{mn}}{m!\Gamma\left(m+1-\frac{1}{n}\right)}\right.$$

$$\left.-\alpha\sum_{m=0}^{\infty}\frac{(-1)^m\alpha^{mn}}{m!\Gamma\left(m+1+\frac{1}{n}\right)}\right\}.$$

$Ei_n(\alpha)$ (n odd) satisfies the differential equation

$$\frac{\mathrm{d}^2 Ei_n}{\mathrm{d}\alpha^2} + n^2\alpha^{n-2}Ei_n = n\alpha^{\frac{n-3}{2}}. \tag{6.22}$$

6.2 Third Order Differential Equations

6.2.1 *The linear third order differential equation*

In this section, some generalities about the linear third order differential equation are given

$$y''' + P(x)y'' + Q(x)y' + R(x)y = 0, \tag{6.23}$$

that can be written in the canonical form [Ince (1956)]

$$z''' + f(x)z' + g(x)z = 0, \tag{6.24}$$

with the help of the following change of function

$$y(x) = z(x)\exp\left(-\frac{1}{3}\int^x P(t)\mathrm{d}t\right).$$

If L is the operator associated to the above equation

$$L[z] = z''' + f(x)z' + g(x)z, \tag{6.25}$$

then the adjoint operator $\overline{L}$ reads

$$\overline{L}[y] = -y''' - \frac{\mathrm{d}}{\mathrm{d}x}(f(x)y) + g(x)y. \tag{6.26}$$

Now we have the following theorem

Theorem 6.1 *If z_i $(i = 1,2,3)$ are the three linearly independent solutions of the equation $L[z] = 0$, then the adjoint equation $\overline{L}[z] = 0$ admits as solutions the minors of the Wronskian of the equation $L[z] = 0$: $y_i = \epsilon_{ijk}(z_j z_k' - z_j' z_k)$, where ϵ_{ijk} is the completely antisymmetric tensor.*

The Wronskian of the equation $L[z] = 0$ is defined by the determinant

$$W(z_1, z_2, z_3) = \begin{vmatrix} z_1 & z_2 & z_3 \\ z_1' & z_2' & z_3' \\ z_1'' & z_2'' & z_3'' \end{vmatrix} = \mathrm{C}^{\mathrm{t}}.$$

The proof of this theorem is given by a simple substitution. More likely the result may be generalised to the canonical differential equation of any order. In particular, it is clearly found in the case of the second order differential equation $z''+h(x)z = 0$ (which is self-adjoint) where the minors are written this time $y_i = \delta_{ij}z_j$, $(i,j = 1,2)$.

6.2.2 Asymptotic solutions

Let us consider the differential equation [Langer (1955a,b)]

$$z''' + \lambda^2 \left(f(x)z' + g(x)z\right) = 0, \tag{6.27}$$

where λ is a large parameter. We are looking for the solutions to this equation when $\lambda \to \infty$, and when $f(x)$ has a transition point x_0, *i.e.* when $f(x_0) = 0$. Moreover, we suppose that: $f'(x_0) \neq 0$ and $g(x_0) \neq 0$. We start with a change of the dependent and of the independent variables, as was done for the second order differential equation (cf. §5.2)

$$\begin{cases} u = u(x) \\ z = \frac{1}{u'}\phi(u). \end{cases}$$

The equation (6.27) so becomes

$$\phi''' + \left[\lambda^2 \frac{f\left(x(u)\right)}{u'^2} + S(u)\right]\phi' + \left[\lambda^2 \frac{g\left(x(u)\right)}{u'^3} + \frac{1}{2}\frac{\mathrm{d}S(u)}{\mathrm{d}u}\right]\phi = 0, \tag{6.28}$$

where ϕ' is the derivative of ϕ with respect to the variable u and u' the derivative of u with respect to the variable x. The quantity $S(u)$ is proportional to the Schwarzian derivative $\{u, x\}$

$$S(u) = -\frac{1}{u'^2}\{u, x\} = -\frac{1}{u'^2}\left[\frac{u'''}{u'} - \frac{3}{2}\left(\frac{u''}{u'}\right)^2\right].$$

Now we complete the choice of the change of variable, setting $u = f(x)/u'^2$, namely

$$u = \left[\frac{3}{2}\int_{x_0}^{x} f(x')^{\frac{1}{2}}\mathrm{d}x'\right]^{\frac{2}{3}}. \tag{6.29}$$

The equation (6.28) thus becomes

$$\phi''' + \left[\lambda^2 u + S(u)\right]\phi' + \left[\lambda^2 h(u) + \frac{1}{2}\frac{\mathrm{d}S(u)}{\mathrm{d}u}\right]\phi = 0, \tag{6.30}$$

with: $h(u) = g\left(x(u)\right)/u'^3$. However, we have not used the asymptotic limit $\lambda \to \infty$, which allows the problem to be simplified. In particular, we may compare Eq. (6.30) to the following reference (comparison) equation, where $\mu =$ constant,

$$\Phi''' + \lambda^2 u\Phi' + \lambda^2 \mu\Phi = 0. \tag{6.31}$$

For this purpose, we carry out the asymptotic expansion $\phi(u) = \varpi(u)\Phi(u) + \mathcal{O}(\frac{1}{\lambda})$, which is introduced into Eq. (6.31), leading (up to terms $\mathcal{O}(\frac{1}{\lambda})$) to the identification

$$u\varpi'(u) = (\mu - h(u))\,\varpi(u). \tag{6.32}$$

After an integration, we find

$$\varpi(u) = \exp\left[-\int_0^u \frac{h(v)-\mu}{v}\mathrm{d}v\right]. \tag{6.33}$$

The convergence of the integral in the relation (6.33) is ensured by setting $h(0) = \mu$, allowing the asymptotic solution of Eq. (6.27) to be written as

$$\lim_{\lambda\to\infty} z(x,\lambda) = \frac{1}{u'}\exp\left[-\int_0^u \frac{h(v)-h(0)}{v}dv\right]\Phi(u), \tag{6.34}$$

where $\Phi(u)$ is the solution of Eq. (6.31) and u is given by the expression (6.29).

Moreover, we have obtained a uniform solution at the transition point x_0. Especially, in a neighbourhood of x_0, $f(x) \approx f'(x_0)(x - x_0)$, and according to the relation Eq. (6.29)

$$u \approx \left[\frac{3}{2}f(x_0)^{\frac{1}{2}}\int_{x_0}^{x}\sqrt{(x'-x_0)}\mathrm{d}x'\right]^{\frac{2}{3}} = f'(x_0)^{\frac{1}{3}}(x-x_0).$$

Hence $u'(x_0) = f'(x_0)^{\frac{1}{3}}$. Consequently, as $h(u) = g(x(u))/u'^3$, we obtain $h(0) = g(x_0)/f'(x_0) = \mu$.

These results lead naturally to the study of the comparison equation.

6.2.3 *The comparison equation*

The aim of this section is to give some analytic solutions to the equation [Langer (1955a,b)]

$$y''' - xy' - \mu y = 0. \tag{6.35}$$

A first remark about this equation is that a scaling of the x variable leads to the comparison equation (6.31) in putting $x = -\lambda^{2/3}\overline{x}$.

A solution of this equation can be obtained with the Laplace method [Ince (1956)]. In writing

$$y = \int_{\mathrm{C}} \mathrm{e}^{zx} f(z)\mathrm{d}z,$$

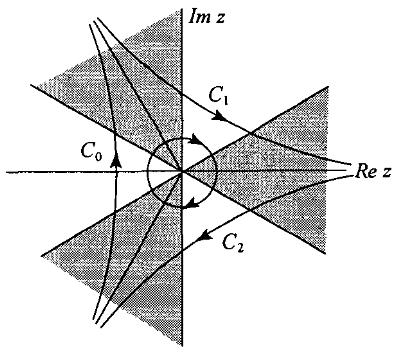
with the condition

$$\mathrm{e}^{-z^3/3}\Big|_{\mathrm{C}} = 0, \tag{6.36}$$

we obtain the integral representation

$$y = \int_{\mathrm{C}_i} \mathrm{e}^{-z^3/3+zx} z^{\mu-1}\mathrm{d}z, \tag{6.37}$$

where the integration paths C_i are given on the figure below.



We consider now some particular cases of the comparison equation (6.35)

- The case where $\mu = 0$ leads to the Airy equation (2.1): $(y')'' - xy' = 0$, producing the general solution (cf. §2.1)

$$y_0 = a\int Ai(x)\mathrm{d}x + b\int Bi(x)\mathrm{d}x + c, \tag{6.38}$$

where a, b, c are integration constants;
- The case where $\mu = 1$ corresponds to the inhomogeneous Airy equation (2.124) (cf. §2.3) $y'' - xy = c/\pi$, giving the solution[2]

$$y_1 = aAi(x) + bBi(x) + cHi(x). \tag{6.39}$$

[2] It should be noted that if y is a solution of $y''' - xy' - \mu y = 0$, then $y' = z$ is a solution of $z''' - xz' - (\mu + 1)z = 0$.

- The case where $\mu = 1/2$ involves the square of Airy functions (cf. §2.4)

$$y_{1/2} = aAi^2\left(2^{-2/3}x\right) + bBi^2\left(2^{-2/3}x\right) + cAi\left(2^{-2/3}x\right)Bi\left(2^{-2/3}x\right). \tag{6.40}$$

Note that this case is self-adjoint.

- The case where $\mu = -1/2$ involves also the square of Airy functions

$$\begin{aligned} y_{-1/2} = a\left[Ai'^2\left(2^{-2/3}x\right) - 2^{-2/3}x\,Ai^2\left(2^{-2/3}x\right)\right] + \\ +b\left[Bi'^2\left(2^{-2/3}x\right) - 2^{-2/3}x\,Bi^2\left(2^{-2/3}x\right)\right] + \\ +c\left[Ai'\left(2^{-2/3}x\right)Bi'\left(2^{-2/3}x\right) - 2^{-2/3}x\,Ai\left(2^{-2/3}x\right)Bi\left(2^{-2/3}x\right)\right]. \end{aligned} \tag{6.41}$$

Now we consider the more general case, where μ is any relative integer. We set $A(x) = aAi(x) + bBi(x)$, $A'(x)$ its derivative and $A_1(x)$ its primitive.[3] It can be seen (according to the above results) that $y_0(x) = A_1(x)$. So if we set $z_n = y_{-n}$ for the parameter $\mu = -n$, $n \in \mathbb{N}$, we obtain the recurrence relation: $z'_n = nz_{n-1}$. Thus we can verify that the first solutions of Eq. (6.35) are

$$\begin{aligned} z_1 &= xA_1(x) - A'(x) \\ z_2 &= x^2A_1(x) - A(x) - xA'(x) \\ z_3 &= (x^3-2)A_1(x) - xA(x) - x^2A'(x). \end{aligned} \tag{6.42}$$

We can then generate the solutions of Eq. (6.35), thanks to the recurrence relation

$$z_n = xz_{n-1} - (n-1)(n-2)z_{n-3}, \tag{6.43}$$

allowing these solutions to be written explicitly as

$$z_n(x) = Pi_n(x)A_1(x) - Qi_{n-2}(x)A(x) - Ri_{n-1}(x)A'(x), \tag{6.44}$$

where the polynomials Pi_n, Qi_n and Ri_n obey the recurrence relations

$$Pi_n = xPi_{n-1} - (n-1)(n-2)Pi_{n-3} \tag{6.45}$$

$$Qi_n = xQi_{n-1} - n(n+1)Qi_{n-3} \tag{6.46}$$

$$Ri_n = xRi_{n-1} - n(n-1)Ri_{n-3}. \tag{6.47}$$

The values of these polynomials [4] for $n = 0, 1, 2$, are respectively $1, x, x^2$.

[3] This primitive includes the additive integration constant c

[4] The properties of the polynomial Pi_n were already given in §4.2.3.

It is interesting to note that if the integration constants a and b cancel, the solutions are only given by the polynomial $z_n = Pi_n(x)$, which is nothing else but the Airy polynomial (cf. §4.2.3). These solutions also allow us to obtain the ones corresponding to the adjoint equation, *i.e.* for $\mu = n+1$, $n \in \mathbb{N}$,

$$y''' - xy' - (n+1)y = 0. \tag{6.48}$$

This equation has the following noteworthy recurrence properties

$$y_n = y'_{n-1}, \tag{6.49a}$$
$$y_n = xy_{n-1} - (n-2)y_{n-3}. \tag{6.49b}$$

According to the theorem given in §6.2.1, when we have the solutions for $\mu = -n$, we consequently obtain the solutions of Eq. (6.48). For example, in the case where $\mu = 0$, the solution

$$y_0(x) = a \int Ai(x)\mathrm{d}x + b \int Bi(x)\mathrm{d}x + c,$$

allows us to find the solution in the case where $\mu = 1$

$$y_1(x) = \alpha Ai(x) + \beta Bi(x) + \gamma Hi(x),$$

with $\alpha = ca' - c'a$, $\beta = c'b - cb'$, $\gamma = ab' - a'b$, thanks to the relation (2.128)

$$Hi(x) = Bi(x) \int_{-\infty}^{x} Ai(t)\mathrm{d}t - Ai(x) \int_{-\infty}^{x} Bi(t)\mathrm{d}t.$$

Another example is found from the first recurrence relation (6.49a), it allows the n^{th} order derivative of an Airy function to be calculated. As a matter of fact, the first derivatives are Ai', $Ai'' = xAi$, $Ai''' = Ai + xAi'$, and for $n \geq 3$ one has

$$\begin{aligned} Ai^{(n)}(x) = \frac{1}{(n-1)!} &\left[\left(Pi_{n-1}Qi_{n-2} - Pi_nQi_{n-3} \right) Ai(x) \right. \\ &\left. + \left(Pi_{n-1}Ri_{n-1} - Pi_nRi_{n-2} \right) Ai'(x) \right]. \end{aligned} \tag{6.50}$$

With the same method, we find the derivatives of the Scorer function $Hi(x)$

with the integral representation (for $n \geq 3$)

$$Hi^{(n)}(x) = \frac{1}{\pi}\int_0^\infty e^{-t^3/3+xt}t^n dt \tag{6.51}$$

$$= \frac{1}{(n-1)!}\left[(Pi_{n-1}Qi_{n-2} - Pi_nQi_{n-3})\,Hi(x)\right.$$
$$+ (Pi_{n-1}Ri_{n-1} - Pi_nRi_{n-2})\,Hi'(x)$$
$$\left.+ \frac{1}{\pi}\,(Qi_{n-2}Ri_{n-2} - Qi_{n-3}Ri_{n-1})\right].$$

6.3 Differential Equation of the Fourth Order

Langer has used the comparison equation (6.35), as a reference equation, for the study of the asymptotic solutions to the Orr-Sommerfeld equation (a fourth order differential equation), which describes the hydrodynamic instabilities of a Poiseuille flow. In this case the parameter λ^2 corresponds to the Reynolds number of the fluid. A somewhat more elaborated asymptotic method [Drazin & Reid (1981)] involves, as a comparison equation, a fourth order differential with two parameters

$$y'''' - xy'' - \alpha y' - \beta y = 0. \tag{6.52}$$

In the case where $\beta = 0$, we find again Eq. (6.35).

In the general case, a method of Laplace yields to solutions of (6.52) with the following integral representations

$$y_k(\alpha,\beta;x) = \int_{C_k} e^{-z^3/3+xz-\beta/z} z^{\alpha-2} dz, \tag{6.53}$$

where the contours C_k are conveniently chosen (cf. §6.2.3). These solutions, which constitute generalisations of Airy functions, were studied in detail by Rabenstein (1958). We refer the reader to this article for further information. However, we notice that the adjoint equation of (6.52) may be written

$$z'''' - \frac{d^2}{dx^2}(xz) + \alpha z' - \beta z = 0, \tag{6.54}$$

or

$$z'''' - xz'' + (\alpha - 2)z' - \beta z = 0. \tag{6.55}$$

We see consequently the noteworthy result, for $\alpha = 1$, Eq. (6.52) is selfadjoint. This result (6.55) allows us to generate the solutions of $y(2-\alpha, \beta; x)$ as soon as we know the solutions $y(\alpha, \beta; x)$ with $\alpha > 0$, as previously for Eq.(6.35).

The equation (6.35), as well as Eq. (6.52), deserve a remark concerning their Airy transform (see §4.2). In fact, this transform allows these equations to be rewritten in a particularly simpler form.

We start with the third order equation (6.35). Taking into account the properties of the Airy transform (§4.2), we have

$$\overline{\mathcal{A}}\,[xf] = x\varphi + \varphi''. \tag{6.56}$$

Hence, in this transform, Eq. (6.35) becomes a first order differential equation

$$x\frac{\mathrm{d}\varphi}{\mathrm{d}x} + \mu\varphi = 0. \tag{6.57}$$

A solution of equation (6.35) is then

$$y = \mathcal{A}\,[\varphi] = \int_{-\infty}^{+\infty} Ai(x - x')\frac{\mathrm{d}x'}{x'^{\mu}}. \tag{6.58}$$

We can easily recognise this is the solution $Pi_n(x)$ of Eq. (6.35) in the case $\mu = -n$, namely the Airy polynomials. This method also works for the equation (6.52). The formula (6.56) gives

$$x\varphi'' + \alpha\varphi' + \beta\varphi = 0, \tag{6.59}$$

which is a Bessel equation of which solutions are for $\beta > 0$

$$\varphi = x^{(1-\alpha)/2} Z_{1-\alpha}\left(\frac{1}{2}\sqrt{\frac{x}{\beta}}\right), \tag{6.60}$$

where $Z_{1-\alpha}$ is a Bessel function which is chosen to ensure the convergence (and the continuity at $x = 0$) of the integral in the inverse Airy transform, so a solution of (6.52) reads

$$y = \int_{-\infty}^{+\infty} Ai(x - x')x'^{(1-\alpha)/2} Z_{1-\alpha}\left(\frac{1}{2}\sqrt{\frac{x'}{\beta}}\right) \mathrm{d}x'. \tag{6.61}$$

For instance, in the selfadjoint case $\alpha = 1$, one has J_0 for $x' > 0$ and I_0 for $x' < 0$.

In order to close this section on the generalisation of Airy functions, we quote the functions defined by the integrals [see Drazin & Reid (1981)]

$$A_k(p,q;x) = \frac{1}{2i\pi} \int_{C_k} e^{-z^3/3+xz} (\ln z)^q \frac{dz}{z^p}. \qquad (6.62)$$

They have noteworthy recurrence properties that were used recently in search of analytic solutions to non-linear equations [Laurenzi (1993)].

Exercises

1. Find three linearly independent solutions to the equations

$$y''' - xy' \pm \frac{3}{2} y = 0.$$

Hints: See footnote p. 108 and examples in §6.2.3, Eq. (6.41).

2. Find a third order differential equation for the following functions

$$y(x) = \int_0^\infty Ai'(x+t) \frac{dt}{\sqrt{t}},$$

$$z(x) = \int_0^\infty Ai_1(x+t) \frac{dt}{\sqrt{t}}.$$

3. Find a differential equation of the fourth order of which a solution is

$$y(x) = \int_0^\infty \cos\left(\frac{t^3}{12} + xt - \frac{\beta^2}{t} + \frac{\pi}{4}\right) \frac{dt}{\sqrt{t}}.$$

Then, using a scaling, compare the result with Eq. (6.55). *Hint*: Express $y(x)$ in term of Airy functions.

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Chapter 7

Applications to Classical Physics

7.1 Optics and Electromagnetism

Airy functions were introduced by G.B. Airy in 1838 in his article about the calculation of the light intensity in the neighbourhood of a caustic. We shall establish here the expression of this intensity (not in the way followed by Airy), but by the more "modern" approach of Landau & Lifchitz (1964).

Let us consider a monochromatic source and an aperture in an opaque screen. According to the laws of geometrical optics, beyond this screen, space is shared in two zones: the dark zone presenting a clear border with the enlightened zone. However the phenomenon of diffraction, as intense as the wavelength of the source, is large compared to the dimension of the opening, complicates the distribution of the light intensity in the neighbourhood of this border. According to the Huygens principle, we consider that each element of the surface $\mathrm{d}S$ of the opening is the source of a spherical wave. That is to say, u being the amplitude of the field on $\mathrm{d}S$ and k the wave number of the source of light, the electromagnetic field in a point P located at a distance R from the opening, is proportional to the sum of these spherical waves

$$u_P \propto \int u \frac{\mathrm{e}^{\mathrm{i}kR}}{R} \mathrm{d}S_{\mathrm{n}}, \tag{7.1}$$

where $\mathrm{d}S_{\mathrm{n}}$ is the projection of $\mathrm{d}S$ on the normal plane, corresponding to the direction of the ray resulting from the source, and arriving on the surface of the opening (cf. Fig. 7.1). If moreover we consider that u is constant on the surface of the opening, the field in P is

$$u_P \propto \int \frac{\mathrm{e}^{\mathrm{i}kR}}{R} \mathrm{d}S_{\mathrm{n}}. \tag{7.2}$$

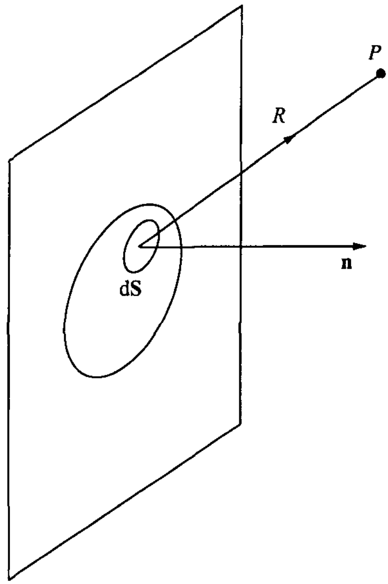

Fig. 7.1 Diffraction by an opening in an opaque screen.

Let us consider now the particular case where the point P is located in the neighbourhood of a caustic, which is the line separating the shaded zone from the lit zone, in the ideal case of geometrical optics. Let us note that the caustic is also defined, in geometrical optics, as being the surface where the light intensity becomes infinite.

We consider, at first, only one section $\mathscr{C}$ in the plane (x, y) of the surface defined previously. The caustic $\mathscr{C}'$ is then the envelope of $\mathscr{C}$. We look to establish the light intensity in the neighbourhood of the point O, which is the contact point between the ray $M'O$ and the caustic. D indicates the length of the segment $[M'O]$. P is the point located in the neighbourhood of O where we shall calculate the intensity of the electromagnetic field: this point is located by its ordinate y (cf. figure 7.2). We note C the curvature center of the caustic and ρ the radius of curvature.

The variable R in the expression (7.2) is then the distance from a point M of the wave surface $\mathscr{C}$ at the point P. The geometrical properties of the envelope make it possible to establish the equality of the angles (M, H, M') and (O, C, O'), which is noted α. If we suppose that we are located at a point P far from the wave surface, *i.e.* α is small enough to be expanded up to the third order: $\sin\alpha \approx \theta - \theta^3/6$, and that the points O and O' are sufficiently close to consider that the radius of curvature ρ is constant on the arc OO', we can establish the relations

$$R = MP \approx MO - y\sin\alpha$$
$$MO \approx MO' + \rho\sin\alpha$$
$$MO' \approx M'O' - \alpha\rho.$$

Thus we get

$$R \approx D - y\alpha - \frac{\rho\alpha^3}{6}, \tag{7.3}$$

and the formula (7.2) is reduced to

$$u_P \propto \int \frac{\mathrm{e}^{\mathrm{i}kR}}{R}\mathrm{d}\alpha.$$

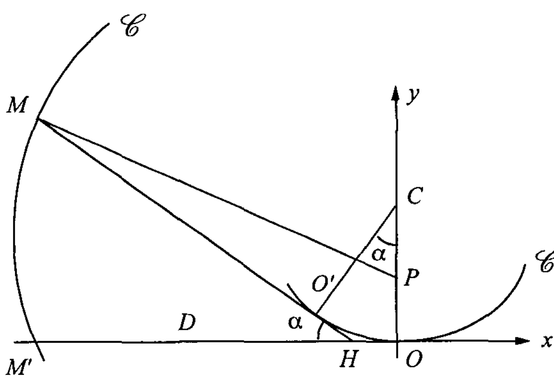

Fig. 7.2 Calculation of the light intensity at the point P in the neighbourhood of the caustic $\mathscr{C}'$.

The factor $1/R$ varies slowly according to α, so we will neglect this variation in front of the exponential $\frac{1}{R} \approx \frac{1}{D}$ (this is the principle of the stationary phase method, *cf.* §5.1). We obtain finally the expression of the amplitude at the point P

$$u_P \propto \int \mathrm{e}^{-\mathrm{i}k\left(y\alpha+\rho\alpha^3/6\right)}\mathrm{d}\alpha.$$

Let us note that we find here the expression given by Hochstadt (1973), which carries out the same type of calculation by considering plane waves reflected by a concave surface. By carrying out the change of variable $t = \left(\frac{k\rho}{2}\right)^{1/3} \alpha$, this expression becomes

$$u_P \propto \int \mathrm{e}^{\mathrm{i}\left(t^3/3+\left(2k^2/\rho\right)^{1/3} yt\right)} \mathrm{d}t.$$

We recognise the integral definition (2.20) of the Airy function Ai. The amplitude can thus be written

$$u_P \propto Ai\left[\left(\frac{2k^2}{\rho}\right)^{1/3} y\right].$$

By reintroducing a proportionality factor, the light intensity in the vicinity of a caustic becomes

$$I_P = A\ Ai^2\left[\left(\frac{2k^2}{\rho}\right)^{1/3} y\right]. \tag{7.4}$$

It is noticeable, in particular, that the intensity is not maximum on the caustic, but in the lit zone, at a position determined by the first maximum of the function Ai: $\left(\frac{2k^2}{\rho}\right)^{1/3} y \simeq -1.02$. For "large" values of y, the expression (7.4) becomes (cf. §2.1.4.3)

$$I_P = \frac{1}{\sqrt{y}} \exp\left[-\frac{4}{3}\left(2k^2/\rho\right)^{1/2} y^{3/2}\right]. \tag{7.5}$$

In the shaded zone, the intensity decreases exponentially. For the negative values of y, the formula (7.4) gives the asymptotic expression

$$I_P \propto \frac{1}{\sqrt{-y}} \cos^2\left[\frac{2}{3}\left(\frac{2k^2}{\rho}\right)^{1/2} (-y)^{3/2} - \frac{\pi}{4}\right]. \tag{7.6}$$

In the lit zone, the intensity oscillates quickly according to y. Its average value is

$$\langle I_P \rangle \propto \frac{1}{\sqrt{-y}},$$

i.e. we find here the intensity given by the geometrical optics.

7.2 Fluid Mechanics

7.2.1 *The Tricomi equation*

The stationary, two-dimensional flow (in a (x, y) plane) of a compressible gas obeys the Tchaplyguine equation [Landau & Lifchitz (1971)]

$$\frac{\partial^2\Phi}{\partial\theta^2} + \frac{v^2}{1 - v^2/c^2}\frac{\partial^2\Phi}{\partial v^2} + v\frac{\partial\Phi}{\partial v} = 0, \tag{7.7}$$

where v is the velocity of gas, θ the angle between $\vec{v}$ and the x axis, c the speed of sound in gas. Φ is a function of speed defined by

$$\Phi = -\phi + xv_x + yv_y, \tag{7.8}$$

where ϕ is the velocity potential $\vec{v} = \vec{\nabla}\phi$. The relation (7.8) defines, in fact, a Legendre transformation which brings back the non-linear equation of the motion to Eq. (7.7), which is linear. The price to be paid for this linearisation, is that the boundary conditions become non-linear [Landau & Lifchitz (1971)].

At the transonic limit, the velocity of gas approaching the speed of sound, the third term of the equation (7.7) becomes negligible in front of the second term. Moreover, without going into the details of this system, we can introduce a new variable η, known as "variable of Tchaplyguine" [Hayasi (1971)], which depends on the velocity v (as well as the critical speed of the sound and other parameters), such that the equation (7.7) becomes

$$\frac{\partial^2\Phi}{\partial\eta^2} - \eta\frac{\partial^2\Phi}{\partial\theta^2} = 0. \tag{7.9}$$

This partial derivative equation is called the Tricomi equation, or Euler-Tricomi. Gramtcheff pursues a mathematical study of this equation and its relationship to the Airy equation [Gramtcheff (1981)].

Let us note first of all, that for $\eta > 0$, Eq. (7.9) is a hyperbolic equation, and for $\eta < 0$, an elliptic one.

Then, let us consider the Fourier transform $\tilde{\Phi}(\eta, \omega)$ of the function $\Phi(\eta, \theta)$

$$\tilde{\Phi}(\eta, \omega) = \int e^{-i\omega\theta}\Phi(\eta, \theta)d\theta.$$

By applying this transformation to Eq. (7.9), we obtain

$$\frac{\partial^2 \tilde{\Phi}}{\partial \eta^2} + \eta\omega^2 \tilde{\Phi} = 0.$$

By the change of variable $\xi = \eta\omega^{2/3}$, we find then the Airy equation

$$\frac{\partial^2 \tilde{\Phi}}{\partial \xi^2} + \xi \tilde{\Phi} = 0. \tag{7.10}$$

The solution of this equation being of the type $\tilde{\Phi} = Ai(-\xi) = Ai(-\eta\omega^{2/3})$, the solution of the Tricomi equation, is determined by the inverse Fourier transform

$$\Phi(\eta, \theta) = \int \mathrm{e}^{\mathrm{i}\omega\theta} Ai(-\eta\omega^{2/3}) \tilde{f}(\omega) \mathrm{d}\omega,$$

where $\tilde{f}(\omega)$ is the Fourier transform of the initial angular profile.[1]

In fact, the general integral of the Tricomi equation can be written

$$\Phi(\eta, \theta) = \int\limits_{\mathrm{C}} g\left(\frac{z^3}{3} - \eta z + \theta\right) \mathrm{d}z, \tag{7.11}$$

where g is an arbitrary function such that g' takes the same values at the ends of the integration path C. Assuming $\tilde{g}$ the Fourier transform of g, the relation (7.11) becomes

$$\Phi(\eta, \theta) = \int\limits_{\mathrm{C}} \int\limits_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}(z^3/3 - \eta z + \theta)t} \tilde{g}(t) \mathrm{d}t \mathrm{d}z.$$

By gathering the terms in z, this last relation can also be written

$$\Phi(\eta, \theta) = \int\limits_{-\infty}^{+\infty} \mathrm{d}t \; \mathrm{e}^{\mathrm{i}\theta t} \tilde{g}(t) \int\limits_{\mathrm{C}} \mathrm{d}z \; \mathrm{e}^{\mathrm{i}(z^3/3 - \eta z)t}. \tag{7.12}$$

Under the condition of choosing conveniently the integration path C, we find here the integral expression of the Airy function Ai.

[1] Notice that the separation of the variables method leads to the same kind of solutions.

7.2.2 *The Orr-Sommerfeld equation*

Plane flow of an incompressible viscous fluid

The Navier equation for an incompressible viscous fluid comes from the general equations of the motion of a fluid

$$\begin{cases} \frac{\partial \vec{u}}{\partial t} + \vec{u}.\vec{\nabla}\vec{u} + \vec{\nabla}\left(\frac{p}{\rho}\right) = \nu \Delta \vec{u} + \frac{\vec{f}}{\rho} \\ \vec{\nabla}.\vec{u} = 0, \end{cases}$$

where the density $\rho = \rho_0$ is a constant. We will also suppose that the external force $\vec{f}$ comes from a potential. By introducing the vorticity

$$\vec{\omega} = \frac{1}{2}\vec{\nabla} \wedge \vec{u},$$

and by noticing that

$$\vec{u}.\vec{\nabla}\vec{u} = 2\vec{\omega} \wedge \vec{u} + \frac{1}{2}\vec{\nabla}\left|\vec{u}\right|^2,$$

we can transform the Navier equation (7.2.2) into the vorticity equation

$$\frac{\partial \vec{\omega}}{\partial t} + \vec{\nabla} \wedge (\vec{\omega} \wedge \vec{u}) = \nu \Delta \vec{\omega}. \tag{7.13}$$

In the particular case of the plane flow $\vec{\omega} = \omega \vec{k}$, the velocity field can be obtained from the current function $\psi(x, y, t)$ by the relations

$$\begin{cases} u_x = \frac{\partial \psi}{\partial y} \\ u_y = -\frac{\partial \psi}{\partial x}. \end{cases}$$

For the vorticity component, we obtain

$$\begin{aligned} \omega &= \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) \\ &= -\frac{1}{2}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) \\ &= -\frac{1}{2}\Delta \psi. \end{aligned} \tag{7.14}$$

Taking the curl of the first member of Eq. (7.13) yields to

$$\vec{\nabla} \wedge (\vec{\omega} \wedge \vec{u}) = \left(\frac{\partial \omega}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial \omega}{\partial y}\frac{\partial \psi}{\partial x}\right)\vec{k}. \tag{7.15}$$

Consequently, taking into account Eqs. (7.14) and (7.15), the vorticity equation is reduced to the quasilinear, fourth order equation for the current function

$$\left[\frac{\partial}{\partial t}+\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}-\frac{1}{\mathrm{Re}}\Delta\right]\Delta\psi=0. \tag{7.16}$$

In this equation, we scale the variables, so that the Reynolds number Re appears instead of the viscosity.

The equation (7.16), in the extreme cases Re $=0$ and Re $=\infty$, leads to remarkable equations of the fluid. For instance, the case Re $=0$ produces the biharmonic equation

$$\Delta^2\psi=0,$$

which presents the disadvantage (in the nonstationary case) to lead to an ill–posed problem. As for the cases Re $=\infty$, which gives the Euler equation in the stationary case

$$\frac{\partial\psi}{\partial y}\frac{\partial\Delta\psi}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial y}=0,$$

yielding to the resolution of

$$\Delta\psi=F(\psi),$$

where F is an arbitrary function of the current ψ. The case Re $\gg 1$ leads to the Prandtl equation for the boundary layer, but also to the study of the stability of a plane flow and to the Orr-Sommerfeld equation.

Stability of an almost parallel flow

In this paragraph, we concern oursleves with the perturbation of the velocity field along the x-axis, parallel to the flow. We suppose thus that we have upstream

$$\begin{cases} u_x=U(y)=\frac{\partial\psi_0}{\partial y}\neq 0\\ u_x=0. \end{cases}$$

The velocity field $U(y)$ should not be unspecified for being strictly parallel. In particular, the pressure scalar field should depend only on x, and the pressure gradient $\frac{\mathrm{d}p}{\mathrm{d}x}$ should be constant, because of the equation of motion,

$$\nu\frac{\mathrm{d}^2U}{\mathrm{d}y^2}=\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x}=\text{constant}.$$

Thus $U(y)$ should be a quadratic function of y. Then two particular cases emerge

- the Couette flow where $U(y) = y$;
- the Poiseuille flow where $U(y) = 1 - y^2$,

that we wrote by standardising the variables conveniently. Hence, we seek the perturbed solution $\psi_1(t, x, y)$ by writing that

$$\psi(t, x, y) = \psi_0(y) + \eta\psi_1(t, x, y),$$

is the solution of Eq. (7.16), where η is a "small" parameter. We find thus ψ_1, by keeping only the dominant terms according to η, satisfying the linear equation

$$\frac{\partial \Delta\psi_1}{\partial t} + U\frac{\partial \Delta\psi_1}{\partial x} - U''\frac{\partial \psi_1}{\partial x} - \frac{1}{\mathrm{Re}}\Delta^2\psi_1 = 0. \tag{7.17}$$

Because we are seeking a perturbation solution of the speed distribution $U(y)$, it is natural to look for a solution ψ_1 in the form of a wave that propagates along to the x axis with a speed c

$$\psi_1(t, x, y) = \phi(y)\, \mathrm{e}^{\mathrm{i}\alpha(x-ct)}. \tag{7.18}$$

By introducing the relation (7.18) into the perturbed equation, we obtain

$$\frac{1}{\mathrm{i}\alpha\mathrm{Re}}\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \alpha^2\right]^2 \phi + (c - U)\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \alpha^2\right]\phi + U''\phi = 0, \tag{7.19}$$

which is nothing else but the Orr-Sommerfeld equation. It is necessary to add to this equation the boundary conditions $\phi(y) = \phi'(y) = 0$ when $y = y_1$ or $y = y_2$. Thus to obtain a non–trivial solution to this problem, it is necessary to find a relation between the various parameters

$$\mathcal{S}(\alpha, c; \mathrm{Re}) = 0,$$

this relation defining the eigenvalues of the problem. The reader will find a detailed analysis of the Orr-Sommerfeld equation in the work by Drazin & Reid (1981).

Among the cases where the Airy functions are involved as solutions of the Orr-Sommerfeld equation (7.19), we shall see only two of them.

At first, let us consider the case of a Couette flow. We have $U(y) = y$, and thus $U''(y) = 0$. The equation is simplified by writing

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \alpha^2\right)\phi = \psi. \tag{7.20}$$

So we obtain

$$\varepsilon^2\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \alpha^2\right)\psi - z\psi = 0, \tag{7.21}$$

where we note $z = y - c$ and $\varepsilon = (\mathrm{i}\alpha\mathrm{Re})^{-1/3}$. The boundary conditions are $\phi' = \frac{\mathrm{d}\phi}{\mathrm{d}z} = 0$ at $z = \pm 1 - c$. The solution of Eq. (7.21) is expressed in terms of Airy functions and the solution of Eq. (7.20) can then be obtained by the use of the method of variation of constants [Drazin & Reid (1981)].

For the second case, we consider the case of large Reynolds numbers, or more precisely, we look for asymptotic solutions according to the parameter

$$\varepsilon^3 = \frac{1}{\mathrm{i}\alpha\mathrm{Re}} \ll 1.$$

The general analysis of the problem was carried out by Langer (1955) as we announced in §6.2, but also by Rabenstein (1958) (as well as of other authors that are quoted in Drazin & Reid (1981)). We shall just evoke the method that consists in taking as starting equation, the Orr-Sommerfeld equation which is "truncated" of its less important terms

$$\frac{1}{\mathrm{i}\alpha\mathrm{Re}}\phi'''' + (c - U)\phi'' = 0. \tag{7.22}$$

We are then in a case where the uniform approximation method for a differential equation can be applied (cf. §5.2). Indeed the unspecified velocity field $U(y)$ having a turning point $U(y_t) = c$, the solution expressed by this method will be uniformly valid in the neighbourhood of this turning point.

7.3 Elasticity

Let us consider the system formed by a homogeneous rod of small circular section which is in equilibrium. This rod is a one dimensional system, because its section Σ_p can be comparable to a geometrical point P, located by its curvilinear coordinate s. The vectorial equations of equilibrium are

[Landau & Lifchitz (1967)]

$$\frac{\mathrm{d}\vec{T}}{\mathrm{d}s} + \vec{f} = 0 \tag{7.23}$$

$$\frac{\mathrm{d}\vec{M}}{\mathrm{d}s} + \vec{u} \wedge \vec{T} + \vec{m} = 0, \tag{7.24}$$

where $\vec{u}$ is a normal unit vector to the section, and directed towards the increasing values of s, where $\vec{T}(s)$ and $\vec{M}(s)$ are the reduced element of the torque (*i.e.* the resultant and the moment of the torque) of the interior constraints in P, and where $\vec{f}(s)$ and $\vec{m}(s)$ are the reduced elements of the torque of external constraints.

In the case we are studying, *i.e.* the weak inflexion without torsion of a rod, we have

$$\begin{cases} \vec{m}(s) = 0 \\ T_T = 0 \\ M_u = 0 \end{cases}$$

where $\vec{m}$ is the linear distribution of torques, $\vec{T}$ is the normal component to $\vec{u}$ (there is no shearing stress), and $M_u = \vec{M}.\vec{u}$ is the torsion moment. The interior and external constraints are thus reduced to the tension (or normal compression) $T = T_u = \vec{T}.\vec{u}$ and the bending moment (normal component with $\vec{u}$) $M = M_F$.

A rod of length L is placed vertically in the gravity field. This rod is free at its top and embedded, at its bottom, in a slab of concrete (cf. Fig. 7.3). We choose the referential (x, y, z) so that the deformation takes place in the (x, z) plane $s(x, y, z)$ is reduced to $s(x, 0, z)$. Moreover we consider a "weak" flexion (the radius of curvature of the rod is at any point much lower than its length), so that the vector $\vec{u}$ is directed along the z axis and that its norm depends only on x.

If we note I the moment of inertia of the rod and E its Young modulus (the quantity IE is the rigidity flexion of the rod), the flexion momentum is written [Landau & Lifchitz (1967)]

$$\vec{M} = (0, IEx'', 0).$$

Then Eq. (7.24) gives the relation:

$$IEx''' = T_x x'. \tag{7.25}$$

The relation (7.23) allows us to determine T_x : q being the weight per unit length of the rod, we obtain, under the condition $T(L) = 0$ (the top of the rod being free),

$$T_x = -q(L - z). \tag{7.26}$$

The equilibrium equation of the rod is thus

$$IEx''' = -q(L - z)x'. \tag{7.27}$$

By writing $u = x'$ and $\eta = \left(\frac{q}{EI}\right)^{1/3}(z - L)$, we obtain the Airy equation (2.1) $u'' = \eta u$, with the solution $u = aAi(\eta) + bBi(\eta)$.

Hence we obtain the form of the arrow x of the rod (its horizontal variation compared to the vertical position), according to the primitives of Ai and Bi. The constants a and b are determined from some limit conditions. In particular, at the upper end of the rod, we know that the bending moment is zero: $\left.\frac{\mathrm{d}^2x}{\mathrm{d}z^2}\right|_{z=L} = 0$. Since the lower end of the rod is fixed, we also have $\left.\frac{\mathrm{d}x}{\mathrm{d}z}\right|_{z=0} = 0$. These relations determine the critical length L of the rod as a function of the parameters of the material $L = 1.986\,(IE/q)^{1/3}$.

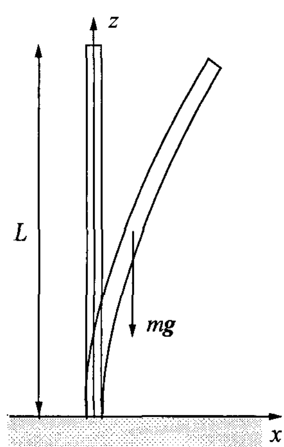

Fig. 7.3 Flexion of a rod placed vertically in the gravity field.

7.4 The Heat Equation

In this section, we shall see that the heat equation

$$\partial_t u = \partial_{xx} u, \tag{7.28}$$

may have a solution which can be expressed in terms of Airy functions. Actually, a Lie group analysis of this equation [Olver (1991)] produce the following solution

$$u(x,t) = \exp(2t^3/3 - xt)\, Ai(t^2 - x), \quad t \geq 0. \tag{7.29}$$

This result can be alternatively found by using the general solution of the heat equation as the convolution integral

$$u(x,t) = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{+\infty} \psi(y) \exp\left[-\frac{(x-y)^2}{4\nu t}\right] dy, \tag{7.30}$$

where the initial condition $\psi(x)$ is chosen as $\psi(x) = Ai(x)$. In other words, this solution is the Airy transform of the Gaussian (see Eq. (4.31))

$$\frac{1}{\sqrt{4\pi t}} \exp\left[-\frac{x^2}{4t}\right],$$

leading to the solution Eq. (7.29).

Surprisingly, Eq. (7.29) appears in a number of very different domains of sciences. Except the case of the heat equation itself, we can first quote the use Eq. (7.29) in probability calculus. A considerable number of asymptotic distributions arising in random combinatorics and analysis of algorithms are of exponential–quadratic type, *i.e.* Gaussian. But when confluences of critical points and singularities occur, the Airy function immediately appears. Recently, a new probability distribution, called the "map–Airy distribution" was introduced [Banderier et al. (2000)]. This distribution concerns the statistical properties of random maps, *i.e.* the question of what such random maps typically look like. It is defined by the probability density[2]

$$\mathcal{A}(s) = -2\exp(2s^3/3)\left[sAi(s^2) + Ai'(s^2)\right], \quad s \in \mathbb{R}. \tag{7.31}$$

In the solution (7.29), if we set $x = 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(u(0,s)\right) = -s\mathcal{A}(s).$$

[2]We have change $s \to -s$ on the contrary of the work of Banderier et al.

Moreover, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\exp(2s^3/3)\,Ai'(s^2)\right) = -s^2\mathcal{A}(s),$$

and the following integral representation

$$\mathcal{A}(s) = \sqrt{\frac{3}{4\pi}}\int_0^{\infty} \mathrm{e}^{-u^3/12}\,Ai(us)\,u^{3/2}\,\mathrm{d}u \tag{7.32}$$

In Fig. 7.4, we have plotted the density $\mathcal{A}(s)$ where we can see its slow decreasing when $s \to \infty$, as $s^{-5/2}$ and fast increasing for the negative values as $\exp(s^3)$.

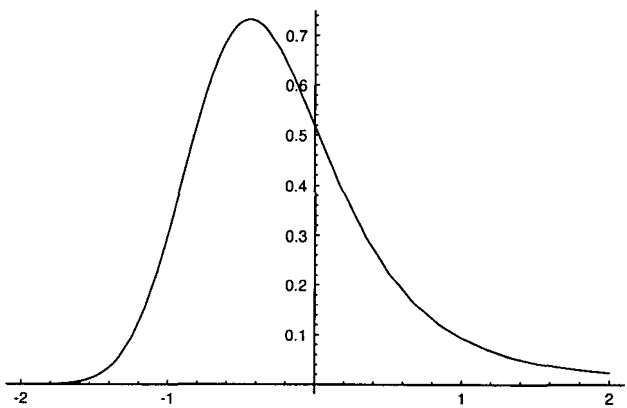

Fig. 7.4 The probability density $\mathcal{A}(s)$.

Another intriguing result related to the solution of the heat equation (7.29) appeared in research on the rational solutions of the Painlevé equation. The result is that the function $\theta(x,t) = \exp(2t^3/3)\,Ai(t^2 - x)$ gives rise to an asymptotic expansion

$$\frac{\partial}{\partial t}\ln\theta(x,t) \sim \sum_{n=0}^{\infty} a_n(x)\,(-2t)^{-n}\,,$$

of which coefficients $a_n(x)$ are involved in the rational solution of the Painlevé equation [Iwasaki et al. (2002)]. For further details, we refer the reader to this paper. The Painlevé equation, in relation with Airy functions, will be discussed in the next section.

7.5 Nonlinear Physics

7.5.1 *Korteweg–de Vries equation*

The history of solitons starts around 1838, when John Scott Russel, a Scottish engineer, observed on a channel close to Edinburgh, a phenomenon which astounded him. Where a barge stopped in this channel, a wave occurred from its prow. The observer followed it with his horse for several kilometers. This wave was propagating identical to itself, at constant speed, as opposed to what we are used to seeing. Russel multiplied the experiments and deduced from them that the propagation velocity of the wave is dependent of its amplitude η

$$v = \sqrt{g(h+\eta)}\,,$$

where g is the gravitational constant and h the depth of the channel. In 1844, he published a report of his observations.

Some time after, in 1845, G. B. Airy published a report on "tides and waves". He established (by the theory) the existence of waves propagating with a speed close to that already determined by Russel's experiments. Airy concluded however that the "solitary waves" do not exist, while its theory described the experiments correctly. A polemic started then between the two men. Only three years after the death of Airy, did this controversy die out. In 1895, Korteweg and his student de Vries deduced an equation describing the solitary waves in shallow water. But the irony of the story was that the solutions of the Korteweg-de Vries equation are closely related to Airy functions!

7.5.1.1 *The linearised Korteweg–de Vries equation*

The Korteweg–de Vries equation [Ablowitz & Segur (1981)]

$$\frac{\partial f}{\partial t} - 6f\frac{\partial f}{\partial x} + \frac{\partial^3 f}{\partial x^3} = 0, \tag{7.33}$$

and the modified Korteweg–de Vries equation

$$\frac{\partial f}{\partial t} - 6f^2\frac{\partial f}{\partial x} + \frac{\partial^3 f}{\partial x^3} = 0, \tag{7.34}$$

can be both linearised, for the amplitudes, giving the equation

$$\frac{\partial f}{\partial t} + \frac{\partial^3 f}{\partial x^3} = 0, \tag{7.35}$$

which is the starting equation of the work of Widder (1979). Indeed an Airy transform upon the variable x (t being regarded as a parameter), yields to the equation (see lemma 4.3)

$$\frac{\partial \varphi_a}{\partial t} + \left(1 + \dot{a}a^2\right) \frac{\partial^3 \varphi_a}{\partial x^3} = 0.$$

We set $a = -(3t)^{1/3}$ so that the term with the third order derivative disappears, and the solution of the transformed equation is an arbitrary function of the space (*i.e.* the initial condition). The inverse transform leads then to the solution

$$f(x,t) = \frac{1}{(3t)^{1/3}} \int_{-\infty}^{+\infty} Ai\left(\frac{x-y}{(3t)^{1/3}}\right) \phi(y)\mathrm{d}y. \tag{7.36}$$

This solution is alternatively obtained by carrying out on Eq. (7.35) a Fourier transform as proposed by Ablowitz & Segur (1981).

The solution (7.36) does not represent however a solitary wave, since the amplitude is a decreasing function of time. It is the competition between the dispersive term $\partial^3 f/\partial x^3$ and the non-linear term which generates the soliton. To create a soliton, it is necessary to look for a solution as a wave $f(x-ct)$ travelling with the constant speed c. In this case, the Korteweg–de Vries equation (7.33) is equivalent to the ordinary differential equation

$$f''' - 6ff'' - cf' = 0.$$

If we look for the solutions of this equation under the condition that f and its successive derivatives f' and f'' decrease towards zero when $z = x - ct$ tends towards infinity, we obtain

$$f(x-ct) = f(z) = -\frac{c}{2} \frac{1}{\operatorname{ch}^2\left(\frac{\sqrt{c}}{2}(z - z_0)\right)}. \tag{7.37}$$

Except for the sign (which can be restored by changing the sign of the non-linear term), this solution corresponds to the observation of Russel: a solitary wave of which the propagation velocity is as large as the amplitude is.

7.5.1.2 *Similarity solutions*

The search for a group of similarity for a differential equation leads to what is called, a similarity solution. If for the variables x and t, and the function f, we carry out the transformations ($a \in \mathbb{R}^+$)

$$\begin{cases} x = a^{\chi}\overline{x} \\ t = a^{\tau}\overline{t} \\ f = a^{\varphi}\overline{f}, \end{cases}$$

then the Korteweg–de Vries equation is invariant if there are the relations

$$\begin{cases} \varphi = \chi - \tau \\ \tau = 3\chi. \end{cases}$$

It can be seen therefore, that the variables: $z = x(3t)^{-1/3}$ and $\phi = (3t)^{2/3}f(z)$, are also invariant in this group of similarity. A similarity solution of Eq. (7.33) can then be written

$$f(x,t) = \frac{1}{(3t)^{2/3}}\,\phi\left(\frac{x}{(3t)^{1/3}}\right),$$

where the function ϕ is a solution of the ordinary differential equation

$$\phi''' - 6\phi\phi' - z\phi' - 2\phi = 0.$$

For the low amplitudes of ϕ, the linearisation leads to a solution in terms of the derivative of the Airy function, Ai'

$$f_L(x,t) = \frac{1}{(3t)^{2/3}}\,Ai'\left(\frac{x}{(3t)^{1/3}}\right),$$

It is noteworthy, that the linearised similarity solution has the same similarity group as the group of the linearised solutions (7.36) [Ablowitz & Segur (1981)].

We can carry out the same work on the modified Korteweg-de Vries equation (7.34). The similarity group is given by

$$\begin{cases} 2\varphi = \chi - \tau \\ \tau = 3\chi. \end{cases}$$

The invariants are now given by the variables $z = x(3t)^{-1/3}$ and $\phi = (3t)^{1/3}f(z)$. The similarity solution of the modified Korteweg–de Vries equation is then written

$$f(x,t) = \frac{1}{(3t)^{1/3}}\,\phi\left(\frac{x}{(3t)^{1/3}}\right),$$

where ϕ is the solution of the ordinary differential equation

$$\phi''' - 6\phi^2\phi' - z\phi' - \phi = 0. \tag{7.38}$$

Among the linearised solutions of this equation, we find this time the solution

$$f_L(x,t) = \frac{1}{(3t)^{1/3}}\,Ai\left(\frac{x}{(3t)^{1/3}}\right),$$

which belongs to the family defined by Eq. (7.36).

Let us consider now Eq. (7.38). An integration leads to

$$\phi'' - 2\phi^3 - z\phi + a = 0, \tag{7.39}$$

where a is an integration constant. Eq. (7.39) is called the "second Painlevé equation", or P_{II}. This equation is treated in the following section. For more details on the theory of solitons and on the solutions of the Korteweg–de Vries equation, we refer the reader to the book by Ablowitz & Segur (1981), [see also Ablowitz & Clarkson (1991)].

7.5.2 *The second Painlevé equation*

7.5.2.1 *The Painlevé equations*

At the end of XIX$^{\mathrm{th}}$ century and at the beginning of XX$^{\mathrm{th}}$ century mathematicians started the classification of the differential equations [Ince (1956)]. For this purpose, they considered the class of the rational equations in the unknown function of which only singular points, depending on the initial conditions are poles, or in other words of which the critical points (branching points, essentials singularities ...) are fixed. By definition therefore, the solutions are said to satisfy the Painlevé property. Thus Fuchs demonstrated in 1884 that for the first order equation

$$\frac{\mathrm{d}y}{\mathrm{d}z} = F(y,z),$$

where F is rational in y, and at least locally analytical in z, the only equation which does not have a moving critical point (*i.e.* a critical point that depends on the initial conditions) is the Riccati equation

$$y' = P(z)y^2 + Q(z)y + R(z).$$

Painlevé and his co-workers (Gambier, Boutroux) were interested in the second order equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} = F(y', y, z).$$

These mathematicians found fifty canonical equations, rational in y and y', and locally analytical in z, satisfying the property of Painlevé. Six of these equations cannot be solved in terms of known functions, we indicate these last equations under the name of "Painlevé transcendents".

The second Painlevé equation P_{II} is the object of the present paragraph

$$y'' = zy + 2y^3 + a, \tag{7.40}$$

where a is a parameter. This equation can be considered as a *non-linear generalisation of the Airy equation.* Indeed, the solutions of Eq. (7.40) are closely related to the Airy functions, as we shall see now.

The case $a = 1/2$ is remarkable in the sense that a solution of Eq. (7.40) can be expressed simply with the Airy functions. At first let us consider the Riccati equation

$$y' = y^2 + \frac{z}{2}. \tag{7.41}$$

If we assume: $y = -w'/w$, this equation becomes

$$w'' + \frac{z}{2}w = 0, \tag{7.42}$$

which is nothing but an Airy equation. By derivating Eq. (7.41), we obtain

$$y'' = 2yy' + \frac{1}{2}.$$

Then we can replace y' with its value from Eq. (7.41), and we obtain Eq. (7.40)

$$y'' = 2y\left(y^2 + \frac{z}{2}\right) + \frac{1}{2}.$$

The Painlevé equation has the following interesting property that, if $y(x;a)$ is a solution of P_{II}, then

$$\tilde{y}(x; a+1) = -y(x;a) - \frac{1+2a}{2y^2 + 2y' + x}, \tag{7.43}$$

is also a solution of P_{II}. Therefore, we can deduce a solution for the case $a = n+\frac{1}{2}$ successively from the case $a = \frac{1}{2}$. For example, we have [Ablowitz & Clarkson (1991)]

$$y(x;\frac{3}{2}) = \frac{2(w'/w)^3 + x(w'/w) - 1}{2(w'/w)^2 + x}, \tag{7.44}$$

$$y(x;\frac{5}{2}) = \frac{4x(w'/w)^4 + 6(w'/w)^3 + x^2(w'/w)^2 + 3x(w'/w) + x^3 - 1}{[2(w'/w)^2 + x][4(w'/w)^3 + 2x(w'/w) - 1]}, \tag{7.45}$$

where w is any solution of Eq. (7.42) and w' its derivative.

7.5.2.2 *An integral equation*

A noteworthy result occured more recently, with the discovery by Ablowitz & Segur of an exact linearisation of P_{II} by an integral equation [Ablowitz & Segur (1977)]. This result comes from research of these authors about a connection between the partial derivative equations which are solved by the inverse scattering transform and the Painlevé equations [Ablowitz & Segur (1981)].

Let us consider the linear integral equation

$$\begin{aligned} K(x,y) &= r\, Ai\left(\frac{x+y}{2}\right) \\ &\quad + \sigma\frac{r^2}{4}\int_x^\infty\int_x^\infty K(x,z) Ai\left(\frac{z+s}{2}\right) Ai\left(\frac{s+y}{2}\right) dz ds, \end{aligned} \tag{7.46}$$

where $y \geq x$, $\sigma = \pm 1$, r a parameter. Then the function $W(z;r) = K(z,z)$ is a solution of the Painlevé equation (case $a = 0$ for Eq. (7.40)),

$$W'' = zW + 2\sigma W^3,$$

under the condition that when $z \to \infty$, W behaves like the Airy function

$$W(z;r) \approx r\, Ai(z).$$

It can also be seen that the kernel of the integral equation can be written differently. Indeed, by using the formula (3.54), we obtain [Ablowitz &

Clarkson (1991)]

$$\int_x^\infty Ai\left(\frac{z+s}{2}\right) Ai\left(\frac{s+y}{2}\right) \mathrm{d}s$$
$$= \frac{4}{z-y}\left[Ai\left(\frac{x+z}{2}\right) Ai'\left(\frac{x+y}{2}\right) - Ai'\left(\frac{x+z}{2}\right) Ai\left(\frac{x+y}{2}\right)\right].$$

In relation to the Painlevé equation, this kernel was studied in a thorough way by Tracy & Widom (1994).

7.5.2.3 *Rational solutions*

Kajiwata and Otha (1996) determined a determinantal representation of rational solutions of P_{II} [Clarkson (2003)].

Theorem 7.1 *Let $p_k(z)$ be the polynomial defined by*

$$\sum_{k=0}^{\infty} p_k(z)\, \lambda^k = \exp\left(z\lambda - \frac{4}{3}\lambda^3\right), \tag{7.47}$$

with $p_k(z) = 0$ for $k < 0$, and $\tau_n(z)$ be the $n \times n$ determinant ($n \geq 1$)

$$\tau_n(z) = \begin{vmatrix} p_n(z) & p_{n+1}(z) & \cdots & p_{2n-1}(z) \\ p_{n-2}(z) & p_{n-1}(z) & \cdots & p_{2n-3}(z) \\ \vdots & \vdots & \ddots & \vdots \\ p_{-n+2}(z) & p_{-n+3}(z) & \cdots & p_1(z) \end{vmatrix}. \tag{7.48}$$

Then

$$w(z;\, n) = \frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\left[\frac{\tau_{n-1}(z)}{\tau_n(z)}\right]\right\}, \quad n \geq 1$$

satisfies P_{II} (Eq.(7.40)) with $a = n$.

We clearly see, on this important result, that again Airy functions are involved for the polynomials $p_n(z)$ are (except a scaling), the Airy polynomials (cf. §4.2.3 Eq. (4.44)).

Exercises

1. Prove the relation $L = 1.986(IE/q)^{1/3}$ given on page 126. *Hint*: The first zero of the Bessel function $J_{-1/3}(u)$ is $u = 1.866$.

2. What is the solution to the heat equation with the initial condition $\psi(x) = Ai'(x)$?

3. Prove that $\int_{-\infty}^{\infty} \mathcal{A}(s)\,\mathrm{d}s = 1$, where $\mathcal{A}(s)$ is defined by Eq. (7.31).

4. Prove the integral representation of the probability density (7.32). *Hint*: See integrals (3.101) and (3.103).

5. Check that the following function is a similarity solution of the l-KdV equation
$$\frac{1}{(12t)^{1/6}} Ai^2 \left[\frac{x}{(12t)^{1/3}}\right],$$
Hint: See Eq.(3.142).

6. Explore the problem of convective-diffusive mass transfer with chemical reaction in a Couette flow. In particular at steady state, neglecting diffusion in the direction of convective transport, the problem is governed by the following partial differential equation
$$(u_0 + ay)\frac{\partial c}{\partial x} = D\frac{\partial^2 c}{\partial y^2} - kc,$$
with the boundary conditions
$$c(0, y) = 0, \quad y > 0;$$
$$c(x, 0) = c_0, \quad x \leq 0;$$
$$\lim_{y\to\infty, x>0} c(x, y) < \infty.$$
Hint: See the following articles [Apelblat (1980, 1982); Chen et al. (1996); Chen & Arce (1997)].

Chapter 8

Applications to Quantum Physics

8.1 The Schrödinger Equation

8.1.1 *Particle in a uniform field*

Let us consider a free, q charged particle, moving on the $\vec{x}$ axis plunged into a uniform electric field $\overrightarrow{\mathcal{E}}$. This particle is submitted to the force $\overrightarrow{F} = q\,\overrightarrow{\mathcal{E}}$ and its potential energy is $U = -Fx$. So the Schrödinger equation is checked by the wave function of the particle

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \frac{2m}{\hbar^2}\left(E + Fx\right)\psi = 0, \tag{8.1}$$

where E is the total energy of the particle. Let us perform the change of variable

$$\xi = \left(x + \frac{E}{F}\right)\left(\frac{2mF}{\hbar^2}\right)^{1/3},$$

where ξ is a one dimensional variable. Then, the Schrödinger equation is reduced to the Airy equation (2.1)

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\xi^2} + \xi\psi = 0. \tag{8.2}$$

The solution of this equation is

$$\psi\left(\xi\right) = N\,Ai\left(-\xi\right) + N'\,Bi\left(-\xi\right),$$

where Ai and Bi are the homogeneous Airy functions. But $Bi(x)$ goes to infinity for $x > 0$. This solution is not relevant, so $N' = 0$, then the solution of the equation (8.2) is reduced to

$$\psi\left(\xi\right) = N\,Ai\left(-\xi\right).$$

The constant N is determined by the energy normalisation condition for the wave functions of the continuum spectrum [Landau & Lifchitz (1966)]

$$\int_{-\infty}^{+\infty} \psi(\xi)\,\psi^*(\xi')\,\mathrm{d}x = \delta(E - E').$$

Then we obtain

$$N = \frac{(2m)^{1/3}}{\hbar^{2/3} F^{1/6}},$$

and the exact solution of Eq. (8.1) is

$$\psi(x) = \frac{(2m)^{1/3}}{\hbar^{2/3} F^{1/6}} Ai\left[-\frac{(2mF)^{1/3}}{\hbar^{2/3}}\left(x + \frac{E}{F}\right)\right]. \tag{8.3}$$

Of course, we can obtain this result by seeking the wave function $\phi(p)$ in the momentum representation [Landau & Lifchitz (1966)]. With the previous notations, the Hamiltonian operator in the momentum representation is

$$\hat{H} = \frac{p^2}{2m} - \mathrm{i}\hbar F \frac{\mathrm{d}}{\mathrm{d}p}.$$

Then, the Schrödinger equation for the wave function $\phi(p)$ is

$$-\mathrm{i}\hbar F \frac{\mathrm{d}\phi(p)}{\mathrm{d}p} + \left(\frac{p^2}{2m} - E\right)\phi(p) = 0.$$

Solving this equation, we get

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar F}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar F}\left(Ep - \frac{p^3}{6m}\right)},$$

which is energy normalised by the condition

$$\int_{-\infty}^{+\infty} \phi^*_E(p)\phi_{E'}(p)\mathrm{d}p = \delta(E - E').$$

The wave function in the position space, is the Fourier transform of the wave function in the momentum space, we thus obtain the solution of (8.3)

$$\psi(x) = \int_{-\infty}^{+\infty} \phi(p) \mathrm{e}^{\frac{\mathrm{i}}{\hbar} px} \mathrm{d}x,$$

where $p = u\,(2m\hbar F)^{1/3}$.

The case of a particle in the potential

$$V(r) = \begin{cases} r & \text{if } r > 0 \\ \infty & \text{if } r \leq 0, \end{cases}$$

is similar to the previous one. Nevertheless, in this potential well, the states are bounded and the energy spectrum is discrete: the energy E_n corresponds to the level n. The Schrödinger equation of the n^{th} state of the particle wave function is

$$\frac{\mathrm{d}^2\psi_n(r)}{\mathrm{d}r^2} + \frac{2m}{\hbar^2}\,(E_n - r)\,\psi_n(r) = 0. \tag{8.4}$$

Performing the change of variable

$$\xi = (E_n - r)\left(\frac{2m}{\hbar^2}\right)^{1/3},$$

we get the Airy equation (cf. (8.2))

$$\frac{\mathrm{d}^2\psi_n}{\mathrm{d}\xi^2} + \xi\psi_n = 0.$$

The solution is (excluding the Bi term)

$$\psi_n\,(\xi) = N\,Ai\,(-\xi)\,.$$

The energy levels are determined by the condition $\psi_n(0) = 0$

$$Ai\left[-\left(\frac{2m}{\hbar^2}\right)^{1/3} E_n\right] = 0.$$

Then, the E_n are determined by the zeros a_n of the Airy function (cf.

§2.2.1)[1]

$$E_n = -a_{n+1}\left(\hbar^2/2m\right)^{1/3},$$

i.e.

$$\begin{aligned} E_0 &\simeq 2.33811\left(\hbar^2/2m\right)^{1/3} \\ E_1 &\simeq 4.08795\left(\hbar^2/2m\right)^{1/3} \\ E_2 &\simeq 5.52056\left(\hbar^2/2m\right)^{1/3} \\ E_3 &\simeq 6.78671\left(\hbar^2/2m\right)^{1/3} \\ &\vdots \end{aligned}$$

The normalisation factor N is determined by the orthonormalisation condition of the wave functions

$$\int_0^{+\infty} \psi_n(r)\psi_m^*(r)\mathrm{d}r = \delta\left(E_n - E_m\right),$$

with the wave function ψ_n which is written

$$\psi_n(r) = N\,Ai\left[r\left(\frac{2m}{\hbar^2}\right)^{1/3} + a_n\right].$$

Then we get (cf. §4.4) $N = \left(\frac{2m}{\hbar^2}\right)^{1/6}\frac{1}{Ai'(a_n)}$. The solution of Eq. (8.4) is finally given by

$$\psi_n(r) = \left(\frac{2m}{\hbar^2}\right)^{1/6}\frac{1}{Ai'\left(a_n\right)}Ai\left[\left(\frac{2m}{\hbar^2}\right)^{1/3}\left(r - E_n\right)\right]. \tag{8.5}$$

We can see on Fig. 8.1 the potential $V(r) = r$ and the first energy levels, with $m = 1/2$ and $\hbar = 1$.

8.1.2 *The $|x|$ potential*

Let us consider the one-dimensional system with a particle in the potential $V(x) = |x|$, $x \in \left]-\infty, +\infty\right[$. The bound states are determined by solving

[1]We note: $E_n \propto -a_{n+1}$, and not: $E_n \propto -a_n$, because the numbering of the levels begins from zero for the ground state, meanwhile a_n is the n$^{\text{th}}$ zero of Ai.

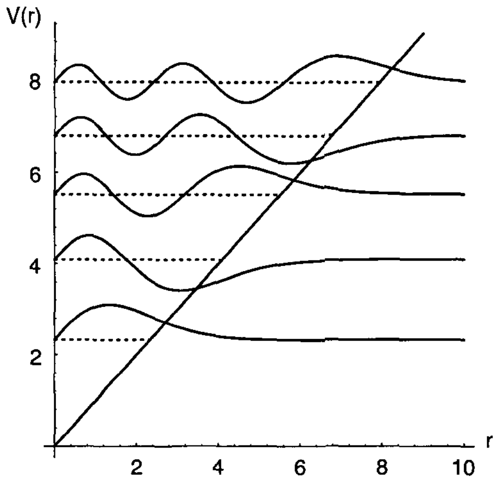

Fig. 8.1 The linear potential: in this figure are the first energy levels and the corresponding wave functions.

the Schrödinger equation for the wave function $\psi_n(x)$ of the particle in the state n

$$\frac{\mathrm{d}^2\psi_n(x)}{\mathrm{d}x^2} + \frac{2m}{\hbar^2}\left(E_n - |x|\right)\psi_n(x) = 0. \tag{8.6}$$

While proceeding in the same way as above for the particle in the potential $V(r) = r$, we have, for $x > 0$, the wave function

$$\psi_n(x) = N\,Ai\left[\left(\frac{2m}{\hbar^2}\right)^{1/3}(x - E_n)\right], \tag{8.7}$$

where the normalisation constant N and the energy levels E_n remain to be determined.

The energy levels are defined by matching two wave functions $\psi_n^{(1)}(x)$ and $\psi_n^{(2)}(x)$ at $x = 0$. At this step, the respective wave functions and their derivatives must be equal (except for the sign)

$$\psi_n^{(1)}(0) = \psi_n^{(2)}(0) \tag{8.8}$$

$$\psi_n^{'(1)}(0) = \pm\psi_n^{'(2)}(0). \tag{8.9}$$

It is then necessary to distinguish two cases, according to the parity of the quantum number n

▶ n is even: the wave function has an even number of nodes, the axis $x = 0$ is a symmetry axis, and consequently the derivative of $\psi_n(x)$ is zero for $x = 0$ (*i.e.* $\psi_n(x)$ presents a local extremum in $x = 0$, cf. Fig. 8.2). We thus have (with the atomic units $2m = \hbar = 1$ to simplify) $\psi_n^{(1)}(x) = N\,Ai\,(-x - E_n)$, $x < 0$, and $\psi_n^{(2)}(x) = N\,Ai\,(x - E_n)$, $x > 0$. The relation (8.9) enables us to obtain

$$Ai'\,[-E_n] = 0,$$

i.e.

$$E_n = -a'_{n+1},$$

where a'_n indicates the n$^{\text{th}}$ zero of the Ai' function (cf §2.2.1).
▶ n is odd: the wave function has an odd number of nodes, which are located on the $x = 0$ axis. There is no longer an axial symmetry, but a central symmetry at the origin (cf. Fig. 8.2). The wave functions are $\psi_n^{(1)}(x) = -N\,Ai\,(-x - E_n)$, $x < 0$, et $\psi_n^{(2)}(x) = N\,Ai\,(x - E_n)$, $x > 0$. The relation (8.8) gives us

$$Ai\,[-e_n] = 0,$$

i.e.

$$E_n = -a_{n+1},$$

where a_n is the n$^{\text{th}}$ zero of the Ai function, that we already met in the case of the potential $V(r) = r$.

The coefficient of normalisation N is determined by the condition

$$\int_{-\infty}^{+\infty} \psi_n(r)\psi_m^*(r)dr = \delta\,(E_n - E_m),$$

which is reduced to

$$N^2 \int_0^{+\infty} Ai\,(x - E_n)\,Ai\,(x - E_m)\,\mathrm{d}x = \delta\,(E_n - E_m).$$

Then we obtain (cf §4.4 and formula (3.50))

$$N = \frac{1}{\sqrt{-a'_n}\,Ai(a'_n)},$$

if n is even, and

$$N = \frac{1}{Ai'(a_n)},$$

if n is odd.

Finally, we reach the solution of the Schrödinger Eq. (8.6) (by restoring the constant $(2m/\hbar^2)^{1/3}$)

► n even:

$$\psi_n(x) = \left(\frac{2m}{\hbar^2}\right)^{1/6} \frac{1}{\sqrt{-a'_n Ai(a'_n)}} Ai\left[\left(\frac{2m}{\hbar^2}\right)^{1/3} (|x| - E_n)\right], \tag{8.10}$$

that is to say

$$E_n = -a'_{n+1}\left(\frac{\hbar^2}{2m}\right)^{1/3}.$$

► n odd:

$$\psi_n(x) = \operatorname{sgn}(x)\left(\frac{2m}{\hbar^2}\right)^{1/6} \frac{1}{Ai'(a_n)} Ai\left[\left(\frac{2m}{\hbar^2}\right)^{1/3} (|x| - E_n)\right], \tag{8.11}$$

that is to say

$$E_n = -a_{n+1}\left(\frac{\hbar^2}{2m}\right)^{1/3}.$$

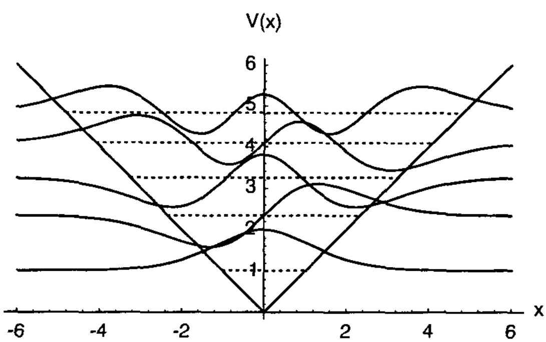

Fig. 8.2 The $|x|$ potential. On this figure, the first energy levels and the corresponding wave functions are represented.

We show on Fig. 8.2 the potential $V(x) = |x|$ and the first energy levels, with $m = 1/2$ and $\hbar = 1$.

8.1.3 *Uniform approximation of the Schrödinger equation*

The one-dimensional Schrödinger equation (or the radial equation in the three-dimensional case) can be written [Berry & Mount (1972); Eu (1984)]

$$y'' + \frac{p^2(r)}{\hbar^2} y = 0, \; r \in [0, +\infty[, \tag{8.12}$$

where $p(r)$ is the momentum of the particle. We find here the equation (5.5) where $q_1(x) = p^2(r)$, $q_2(x) = 0$ and $\lambda = 1/\hbar$. By the same way that for the §5.2, we carry out the changes of variable and function according to

$$\begin{cases} x = x(r) \\ y = z\Omega(x). \end{cases}$$

The equation (8.12) then becomes

$$\begin{aligned} z'' &+ \left[2\frac{\mathrm{d}}{\mathrm{d}x} \ln \Omega - \frac{\mathrm{d}^2 r}{\mathrm{d}x^2}\frac{\mathrm{d}x}{\mathrm{d}r} \right] z' \\ &+ \left[\frac{1}{\Omega}\frac{\mathrm{d}^2\Omega}{\mathrm{d}x^2} - \frac{\mathrm{d}^2 r}{\mathrm{d}x^2}\frac{\mathrm{d}x}{\mathrm{d}r}\left(\frac{\mathrm{d}}{\mathrm{d}x}\ln\Omega \right) + \frac{p^2(r)}{\hbar^2}\left(\frac{\mathrm{d}x}{\mathrm{d}r} \right)^2 \right] z = 0, \end{aligned} \tag{8.13}$$

where $z' = \mathrm{d}z/\mathrm{d}x$ and $z'' = \mathrm{d}^2 z/\mathrm{d}x^2$. We choose the change of function so that the term z' disappears. We then obtain

$$\Omega = \left(\frac{\mathrm{d}x}{\mathrm{d}r} \right)^{-1/2}.$$

Thus we are able to transform Eq. (8.13) into an equation depending only on x and p where the Schwarzian derivative appears $\{x, r\}$:

$$\{x, r\} = \frac{x'''}{x'} - \frac{3}{2}\left(\frac{x''}{x'} \right)^2,$$

where x', x'', x''' are the successive derivatives of x as a function of r. We thus obtain

$$z'' + \left[\frac{p^2(r)}{\hbar^2 x'^2(r)} - \frac{1}{x'^2(r)}\{x, r\} \right] z = 0. \tag{8.14}$$

In the semiclassical limit (the Planck constant $\hbar$ leads formally towards 0), the term containing the Schwarzian derivative can be neglected in front of the other term. The generalised JWKB approximation consists in preserving only the first term in z, with a suitable choice of change of variable

$$z'' + \frac{p^2(r)}{\hbar^2 x'^2} z = 0. \tag{8.15}$$

This choice will be driven by considerations under the topological nature (in particular the turning points) of the momentum $p(r)$. The general method consists in choosing

$$\xi(x) = \frac{p^2(r)}{\hbar^2 x'^2}, \tag{8.16}$$

so that we know the solutions of Eq. (8.15). Then the integration of Eq. (8.16) determines the change of variable

$$\int_{x_t}^{x} \xi^{1/2}(x)\mathrm{d}x = \frac{1}{\hbar}\int_{r_t}^{r} p(u)\mathrm{d}u, \tag{8.17}$$

where r_t is a zero of the momentum $p(r)$, or turning point. So, according to the change of function (8.1.3), the approximate solution of the Schrödinger equation is now

$$y = \left(\frac{\mathrm{d}x}{\mathrm{d}r}\right)^{-1/2} z(x). \tag{8.18}$$

8.1.3.1 *The JWKB approximation*

This approximation consists to consider Eq. (8.15) as a the harmonic oscillator and consequently setting $\xi(x) = 1$. The integration of Eq. (8.16) gives us

$$x(r) = \frac{1}{\hbar}\int_{r_t}^{r} p(u)\mathrm{d}u,$$

if $p^2(r) > 0$. The wave function, which is the approximate solution of the Schrödinger equation, is thus written

$$y(r) = \frac{C_1}{\sqrt{p(r)}}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int_{r_t}^{r} p(u)\mathrm{d}u} + \frac{C_2}{\sqrt{p(r)}}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_{r_t}^{r} p(u)\mathrm{d}u}.$$

The constants C_1 and C_2 are determined by the conditions of normalisation and by the connection of the exact solutions to the Schrödinger equation

[Landau & Lifchitz (1966)] in the semiclassical limit. This last condition gives us

$$y(r) = \frac{C}{\sqrt{p(r)}} \sin\left(\frac{1}{\hbar}\int_{r_t}^{r} p(u)\mathrm{d}u + \frac{\pi}{4}\right) \tag{8.19}$$

in the classically allowed region.

In the classically forbidden region ($p^2(r) < 0$), we deduce from Eq. (5.15)

$$y(r) = \frac{1}{\sqrt{-p(r)}}\left\{C_3 \mathrm{e}^{\frac{1}{\hbar}\int_{r_t}^{r} p(u)\mathrm{d}u} + C_4 \mathrm{e}^{-\frac{1}{\hbar}\int_{r_t}^{r} p(u)\mathrm{d}u}\right\}.$$

In this equation, we cannot physically keep the term going towards infinity when $r \to \infty$. Then the JWKB solution of Eq. (8.12) is

$$y(r) = \frac{C'}{\sqrt{-p(r)}}\mathrm{e}^{-\frac{1}{\hbar}\int_{r_t}^{r} p(u)\mathrm{d}u}. \tag{8.20}$$

Physically, the criterion of validity of the JWKB approximation can be expressed in two forms

i) $\frac{1}{2\pi}\left|\frac{\mathrm{d}\lambda}{\mathrm{d}r}\right| \ll 1$, *i.e.* λ, the de Broglie wavelength of the particle, must vary slowly at a distance about this same wavelength;
ii) $\frac{\hbar m|F|}{p^3} \ll 1$, where $F = -\mathrm{d}V/\mathrm{d}x$ is the classical force acting on the particle. This shows, in particular, that the JWKB approximation is no more valid at the turning points, *i.e.* at the points where the momentum $p(r)$ is null.

8.1.3.2 *The Airy uniform approximation*

We present in this section an approximation to the solutions of Eq. (8.12) when the momentum $p(r)$ has a single turning point r_t, for this purpose we now make the change of variable $\xi(x) = x$. Under this condition, the solutions of the differential equation (8.15) are Airy functions. Taking into account the behaviour at infinity (cf. Figs. 2.1 and 2.2), the $Ai(x)$ function is the only admissible function. The integration of Eq. (8.16) then gives for the change of variable

$$x(r) = \left[\frac{3}{2\hbar} \int_{r_t}^{r} p(u) \mathrm{d}u \right]^{2/3} . \tag{8.21}$$

Therefore, except a multiplicative constant, the approximate solution of the Schrödinger equation uniformly valid far or close to the turning point, may be written

$$y(r) = \left[\hbar^2 \frac{x(r)}{p^2(r)} \right]^{1/4} Ai \left[\varepsilon x(r) \right], \tag{8.22}$$

where ε is -1 in the classically allowed region and $+1$ in the classically forbidden region. If there are several turning points, a good connection between wave functions like (8.22) can approach the exact solution of the Schrödinger equation with accuracy [Miller (1968)]. Let us note in addition, that by taking the asymptotic limit of the expression (8.22), with the formulae (2.44) and (2.49), we find the usual JWKB semiclassical limit given by the formulae (8.19) and (8.20).

Also let us note that $x(r)$ checks the relation:

$$\frac{p(r)}{\hbar\sqrt{x(r)}} = x'(r).$$

Consequently, the expression (8.22) of the uniform solution to the Schrödinger equation can be written

$$y(r) = \frac{1}{\sqrt{x'(r)}} Ai \left[x(r) \right]. \tag{8.23}$$

In addition, the expansion of $x'(r)$ in the neighbourhood of the turning point r_t

$$x'^2(r) = \frac{p^2(r)}{\hbar^2 x(r)} \approx \frac{0 + (r - r_t) \left(p^2(r_t) \right)'}{0 + \hbar^2 (r - r_t) \, x'(r_t)} = \frac{2mV'(r_t)}{\hbar^2 x'(r_t)},$$

leads for the wave function, to the following important expression

$$y(r) = \frac{1}{\sqrt{\alpha}} Ai \left[\alpha (r - r_t) \right], \tag{8.24}$$

(for $r \approx r_t$), where α is a constant depending on the slope of the potential at the turning point

$$\alpha = \left[2mV'(r_t) \right]^{1/3} \hbar^{-2/3},$$

m being the reduced mass of the system.

8.1.3.3 *Exact vs approximate wave functions*

To highlight the validity of the Airy uniform approximation, we shall consider two well-known potentials for which we can solve analytically the Schrödinger equation. We shall be able therefore, to compare the exact wave functions and the uniform ones.

The case of the exponential potential
Let us consider a wave function $\psi_E(r)$, satisfying the Schrödinger equation

$$\frac{\mathrm{d}^2\psi_E(r)}{\mathrm{d}r^2} + \frac{p^2(r)}{\hbar^2}\psi_E(r) = 0, \tag{8.25}$$

where the momentum $p(r)$ is defined by

$$p^2(r) = 2m\left[E - V(r)\right].$$

If we choose a purely repulsive exponential potential

$$V(r) = V_0\ \mathrm{e}^{-r/r_m},$$

the exact solution of the Schrödinger equation is [Child (1974)]

$$\psi_E(r) = \frac{2}{\pi}\left[\frac{mr_m}{\hbar^2}\sinh(2\pi a)\right]^{1/2} K_{\mathrm{i}2a}\left(2b\mathrm{e}^{-r/2r_m}\right), \tag{8.26}$$

where $K_\nu(z)$ is the Bessel modified function, and

$$\begin{cases} a = \frac{r_m}{\hbar}\sqrt{2mE} \\ b = \frac{r_m}{\hbar}\sqrt{2mV_0}\,. \end{cases}$$

This system admits a single turning point, so we can apply the uniform approximation method to the equation (8.25). The solution is therefore

$$\psi_E^{\mathrm{unif.}}(r) = \sqrt{2m}\left[\frac{\hbar^2 x(r)}{p^2(r)}\right]^{1/4} Ai\left[\varepsilon x(r)\right], \tag{8.27}$$

where $\varepsilon = +1$ in the classically forbidden regions and $\varepsilon = -1$ in the classically allowed regions, $x(r)$ being defined by the relation (8.21).

In Figs. 8.3 to 8.4, we compare the exact wave function, formula (8.26) (full line), with the JWKB wave function, formulae (8.19) and (8.20) (dotted lines), and with the uniform wave function, formula (8.27) (bullets •).

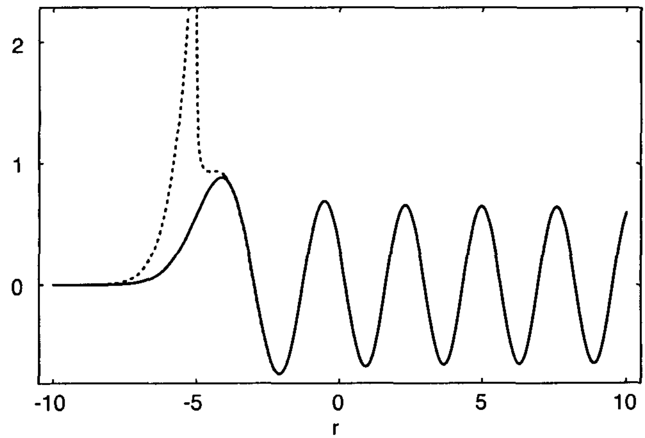

Fig. 8.3 Exponential potential: comparison between the exact wave function (full line) and the JWKB wave function (dotted line).

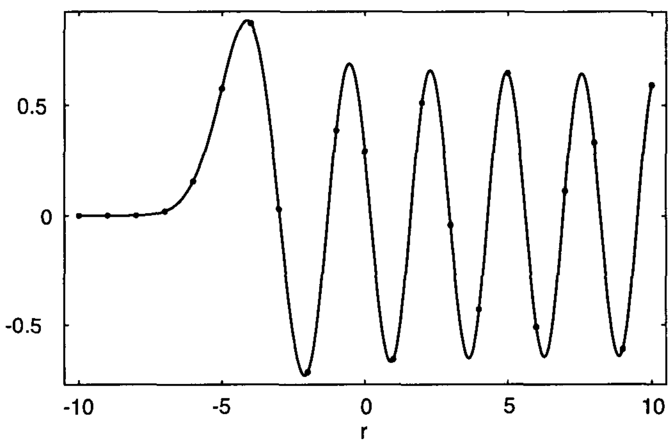

Fig. 8.4 Exponential potential: comparison between exact wave function (full line) and uniform wave function (•).

In Figs. 8.3, we can see the JWKB wave function becoming divergent at the turning point. On the figure 8.4, the Airy wave function is continuous, even in the neighbourhood of this turning point, the agreement with the exact function being of the order of the percent.

Case of the Coulomb potential.
The Schrödinger radial equation associated with an electron in the Coulomb potential $V(r) = 1/r$ of a hydrogen atom is, for the discrete spectrum ($E = -1/n^2$) (in atomic units) [Landau & Lifchitz (1966)]

$$\frac{\mathrm{d}^2\psi_{n,\ell}(r)}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}\psi_{n,\ell}(r)}{\mathrm{d}r} + \left(\frac{2}{r} - \frac{(\ell+1/2)^2}{r^2} - \frac{1}{n^2}\right)\psi_{n,\ell}(r) = 0. \qquad (8.28)$$

The exact solution of this equation is

$$\psi_{n,\ell}(r) = -\frac{2}{n^2}\sqrt{\frac{(n-\ell-1)!}{[(n+\ell)!]^3}}\ \mathrm{e}^{-r/n}\left(\frac{2r}{n}\right)^{\ell} L_{n+\ell}^{2\ell+1}\left(\frac{2r}{n}\right), \qquad (8.29)$$

where $L_\beta^\alpha(x)$ is the generalised Laguerre polynomials [Abramowitz & Stegun (1965)]. The classical turning points r_+ and r_- are defined by

$$r_\pm = n^2 \pm n\sqrt{n^2 - (\ell+1/2)^2}.$$

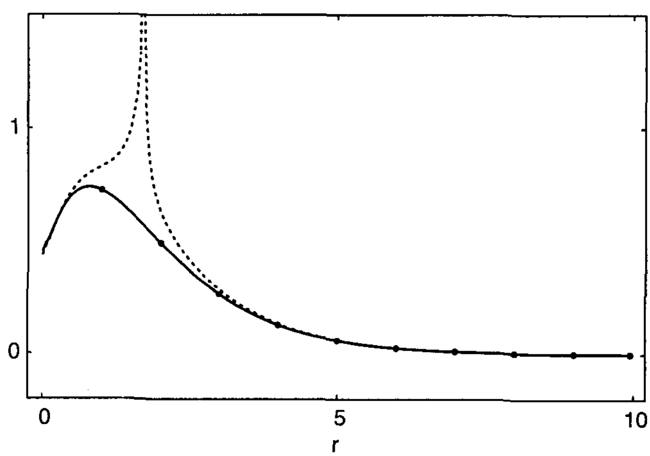

Fig. 8.5 Coulomb potential: comparison between the exact (full line), JWKB (dotted line) and uniform wave functions (•) with $n = 1$, $\ell = 0$.

If we apply the uniform approximation to Eq. (8.28), we obtain the solution [Fonck & Tracy (1980)]

$$\psi_{n,\ell}^{\text{unif.}}(r) = N_n \left(\frac{6\pi}{p_s(r)}\right)^{1/2} \left(\frac{2}{3}\right)^{1/3} [x_s(r)]^{1/6} \times Ai\left\{(-1)^s \left[\frac{3}{2}x_s(r)\right]^{2/3}\right\}, \tag{8.30}$$

with $r > r_-$, and

- $s = 1$ for $r_- < r \le r_+$ and $s = 2$ with $r \ge r_+$;
- $p_1^2(r) = -p_2^2(r) = \frac{2}{r} - \frac{(\ell+1/2)^2}{r^2} - \frac{1}{n^2}$;
- $x_1(r) = \int_r^{r_+} p_1(r)\mathrm{d}r = -rp_1(r) - n\arcsin\left(\frac{r-n^2}{n\sqrt{n^2-(\ell+1/2)^2}}\right)$
$+ \left(\ell+\frac{1}{2}\right)\arcsin\left(\frac{n}{r}\frac{r-(\ell+1/2)^2}{\sqrt{n^2-(\ell+1/2)^2}}\right) + \left(n-\ell-\frac{1}{2}\right)\frac{\pi}{2}$;
- $x_2(r) = \int_{r_+}^{r} p_2(r)\mathrm{d}r = rp_2(r) - n\ln\left|rp_2(r) + \frac{r}{n} - n\right|$
$+\frac{n}{2}\ln\left|n^2 - \left(\ell+\frac{1}{2}\right)^2\right| - \left(\ell+\frac{1}{2}\right)\ln\left|\frac{rp_2(r)+(\ell+1/2)}{r} - \frac{1}{(\ell+1/2)}\right|$
$+\frac{1}{2}\left(\ell+\frac{1}{2}\right)\ln\left|\frac{1}{(\ell+1/2)^2} - \frac{1}{n^2}\right|$;
- $N_n = (-1)^{n-\ell-1}\left(2\pi n^3\right)^{-1/2}$, normalisation coefficient given by Bethe & Salpeter (1957).

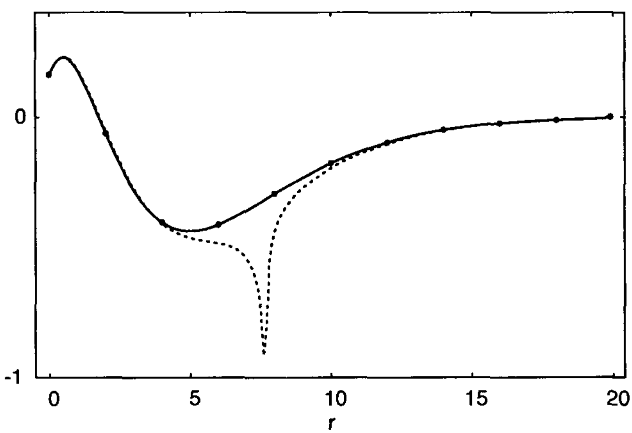

Fig. 8.6 Coulomb potential: comparison between the exact (full line), JWKB (dotted line) and uniform wave functions (•) pour $n = 2$, $\ell = 0$.

Moreover, from Eqs. (8.19) and (8.20), we can deduce the JWKB solutions of Eq. (8.28)

$$\begin{cases} \psi_{n,\ell}(r) = N_n \frac{2}{\sqrt{p_1(r)}} \cos\left[x_1(r) - \frac{\pi}{4}\right] & \text{for} \quad r_- < r \leq r_+ \\ \psi_{n,\ell}(r) = N_n \frac{1}{\sqrt{p_2(r)}} \mathrm{e}^{-x_2(r)} & \text{for} \quad r > r_+ \,, \end{cases}$$

where p_1, p_2, x_1 et x_2 were previously defined.

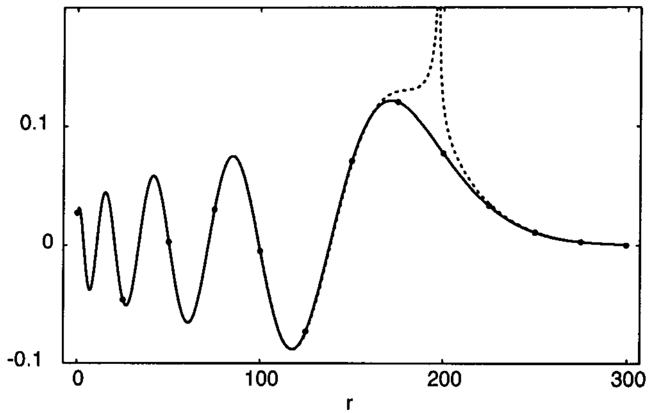

Fig. 8.7 Coulomb potential: comparison between the exact (full line), JWKB (dotted line) and uniform wave functions (•) pour $n = 10$, $\ell = 1$.

In Figs. 8.5 to 8.7, we compare the exact wave function, formula (8.29) (full line), the JWKB wave function, formula (8.1.3.3) (dotted lines) and the uniform wave function, formula (8.30) (bullets •). As in the case of the exponential potential, on the first two figures we can see JWKB function becoming divergent at the turning point r_+, whereas the uniform wave function is continuous.

8.2 Evaluation of the Franck-Condon Factors

In this section, we shall establish the JWKB semiclassical expression, and the uniform semiclassical expression, of

$$F_{if} = \langle \psi_i(r) \left| D(r) \right| \psi_f(r) \rangle = \int \psi_i(r) D(r) \psi_f(r) \mathrm{d}r, \tag{8.31}$$

where $\psi_i(r)$ is the wave function describing an initial quantum state and $\psi_f(r)$ a final state, $D(r)$ being the dipole operator.

F_{if} is the Franck-Condon factor, named after the work of J. Franck and E.U. Condon (Condon (1928); also see Child (1974); Tellinghuisen (1985). This factor is used, in particular, to calculate line profiles or transition probabilities between two levels: i (initial) and f (final). For example, within the framework of an adiabatic theory, the line profile corresponding to the transition between a level i (state described by the wave function ψ_i) and a level f (state described by the wave function ψ_f) is given by the average of the square of the dipole matrix element of the considered transition. This is nothing other than the application of the Fermi golden rule, *i.e.*

$$J(\omega) = \int_0^\infty \mathrm{d}p_i \ \rho(p_i) \sum_\ell (2\ell+1) \left|\langle\psi_i(r)\left|D(r)\right|\psi_f(r)\rangle\right|^2, \tag{8.32}$$

where the wave functions check the Schrödinger radial equations ($\alpha = i, f$)

$$\frac{\mathrm{d}^2\psi_\alpha(r)}{\mathrm{d}r^2} + p_\alpha^2(r)\psi_\alpha(r) = 0,$$

with $p_\alpha^2(r) = 2m\left(\varepsilon_\alpha - V_\alpha(r) - \frac{\ell(\ell+1)}{2mr^2}\right)^{1/2}$, and where ℓ is the angular quantum number, $\rho(p_i)$ being the statistical distribution of the initial states.

Let us note that F_{if} is *stricto sensu* the Franck-Condon amplitude. The Franck-Condon factor is the square of F_{if}, but the use makes that we employ indifferently, by misuse of language, one or the other term to indicate F_{if}.

We shall use in this section the atomic units $\hbar = m_e = e = 1$.

8.2.1 *The Franck-Condon principle*

The Franck-Condon principle was established to describe the transition between two electronic states of a molecule. For example, the phenomenon of bound-free transition *e.g.* the photodissociation of the molecule I_2 in the visible spectrum. This principle stipulates that the relative positions and the momenta of the atoms are preserved during the electronic transition. Physically, this is due to the fact that the electronic velocities are large in front of the nuclear velocities. The Franck-Condon principle also stipulates that it is possible to evaluate the transition probabilities with the help of overlap integrals, containing the classically allowed and forbidden regions.

The dipole operator varies slowly with the internuclear distance r, meanwhile, the electronic motion varies very quickly, so that we can replace the dipole operator $D(r)$ by its average value $D(r_0)$, where r_0 is the interatomic distance at the instant of the transition. This is as realistic as the molecule is massive. The Franck-Condon principle is equivalent to the principle of stationary phase. A Taylor expansion of the dipole operator D is

$$D(r) = D(r_0) + (r - r_0)\, D'(r_0) + \frac{(r - r_0)^2}{2} D''(r_0) + \dots \tag{8.33}$$

So we have to the zero order

$$\langle \psi_i(r)\, | D(r) |\, \psi_f(r) \rangle \approx D(r_0) \langle \psi_i(r) \mid \psi_f(r) \rangle ,$$

where r_0 is a stationary phase point.

8.2.2 *The JWKB approximation of Franck-Condon factors*

The calculation of F_{if} requires the knowledge of the wave functions of both involved states. If it is possible, we can use the JWKB approximation (see the specific criteria of validity for this method, Landau & Lifchitz (1966)). In this case, we obtain for $\psi_i(r)$ and $\psi_f(r)$ the expressions, in the classically allowed region (cf. §5.2.1 and §8.1.3.1)

$$\psi_\alpha(r) = \frac{1}{\sqrt{p_\alpha(r)}} \cos\left(\int_{r_\alpha}^{r} p_\alpha(u)\mathrm{d}u - \frac{\pi}{4} \right) , \quad \alpha = i, f, \tag{8.34}$$

where r_i and r_f are the turning points of the two considered levels (*i.e.* $p_\alpha(r_\alpha) = 0$). In order to simplify the writing, the normalisation factor of the wave functions has not been introduced in this formula. For the Franck-Condon factor, we obtain then [Jablonski (1945); Landau & Lifchitz (1966)]

$$F_{if} = \int_{r_m}^{+\infty} \frac{D(r)\mathrm{d}r}{\left(p_i(r) p_f(r)\right)^{1/2}}$$
$$\times \cos\left(\int_{r_i}^{r} p_i(u)\mathrm{d}u - \frac{\pi}{4} \right) \cos\left(\int_{r_f}^{r} p_f(u)\mathrm{d}u - \frac{\pi}{4} \right) ,$$

where $r_m = \max(r_i, r_f)$.

This product of *cosine* functions can be changed into a sum of *cosine* where the sum and the difference of the phases appear. Only the term containing the difference of the phases contributes significantly to the integral, the other term oscillating very quickly with r [Jablonski (1945)]. This yields to

$$F_{if} \approx \frac{1}{2} \int_{r_m}^{+\infty} \frac{D(r)\mathrm{d}r}{\left(p_i(r) p_f(r)\right)^{1/2}} \times \cos\left(\int_{r_i}^{r} p_i(u)\mathrm{d}u - \int_{r_f}^{r} p_f(u)\mathrm{d}u \right). \tag{8.35}$$

In order to calculate this integral, we shall apply the stationary phase method exposed in §5.1. Let us consider a single stationary phase point r_0, and assume

$$x_\alpha(r) = \int_{r_\alpha}^{r} p_\alpha(u)\mathrm{d}u. \tag{8.36}$$

The second order Taylor series in the neighbourhood of r_0 can be written

$$\begin{aligned} x_i(r) - x_f(r) &= x_i(r_0) - x_f(r_0) \\ &+ (r - r_0)\left(x_i'(r_0) - x_f'(r_0)\right) \\ &+ \frac{(r-r_0)^2}{2}\left(x_i''(r_0) - x_f''(r_0)\right) + \dots, \end{aligned} \tag{8.37}$$

where $x_i'(r_0) - x_f'(r_0) = p_i(r_0) - p_f(r_0)$. As r_0 is a stationary phase point, the equality of the moments gives us

$$p_i(r_0) = p_f(r_0),$$

i.e. using the definition of $p_\alpha(r)$

$$\omega = V_i(r_0) - V_f(r_0) = \Delta V(r_0), \quad (\hbar = 1), \tag{8.38}$$

where $\omega = \varepsilon_i - \varepsilon_f$. Then r_0 is a function depending only on ω. The expression (8.35) can be now written

$$F_{if} \approx \frac{1}{2} \frac{D(r_0)}{p_i(r_0)} \int_{-\infty}^{+\infty} \cos\left(\beta + \gamma\xi^2\right) \mathrm{d}\xi,$$

where $\xi = r - r_0$, and β, γ being functions of r_0:

$$\beta = \frac{mr_0^2}{2p_0}\Delta V'(r_0) \tag{8.39}$$

$$\gamma = \frac{m}{2p_0}\Delta V'(r_0). \tag{8.40}$$

The lower limit of integration can be extended until $-\infty$ since the contribution of the region lower than $r_m - r_0$ is negligible. Moreover, in agreement with the Franck-Condon principle, we replaced $D(r)$ by its value at the stationary phase point $D(r_0)$. We finally obtain the expression of the Franck-Condon factor [Landau & Lifchitz (1966)]

$$F_{if} \approx \frac{1}{2}\frac{D(r_0)}{p_i(r_0)}\sqrt{\frac{\pi}{\gamma}}\cos\left(\beta + \frac{\pi}{4}\right). \tag{8.41}$$

The square of the modulus of this factor is written

$$|F_{if}|^2 \approx \frac{1}{8}\frac{D^2(r_0)}{p_i^2(r_0)}\frac{\pi}{\gamma}, \tag{8.42}$$

where we made use of the random phase approximation: the square of the *cosine* is replaced by its average value 1/2. Then we can calculate the profile (8.32) according to the method detailed in the next paragraph (§8.2.3), *i.e.* by taking as distribution of the initial states $\rho(\varepsilon_i) = \mathrm{e}^{-\varepsilon_i/kT}$ and by changing the discrete summation over the angular quantum numbers into a continuous summation.

This calculation leads to the so-called quasi-static profile [Jablonski (1945)]

$$J(\omega) = \pi D^2(r_0) r_0^2 \, \mathrm{e}^{-V_i(r_0)/kT} \frac{1}{|\Delta V'(r_0)|}, \tag{8.43}$$

except a multiplicative factor, since we did not normalise the wave functions. Let us recall that ω is related to r_0 by the expression (8.38).

Sando & Wormhoudt continue the development (8.37) up to the third order [Sando & Wormhoudt (1973)]. They obtained an expression involving the Airy functions for the profile $J(\omega)$, however we shall not detail this calculation, since the uniform approximation will also lead us to a similar expression.

8.2.3 *The uniform approximation of Franck-Condon factors*

We shall now expose the complete analytical calculation of the profile using the Airy uniform approximation, starting from the expression of $J(\omega)$

$$J(\omega) = \int_0^\infty \mathrm{d}\varepsilon_i \; \mathrm{e}^{-\varepsilon_i/kT} \sum_\ell (2\ell+1) \left| \langle \psi_i(r) \left| D(r) \right| \psi_f(r) \rangle \right|^2. \tag{8.44}$$

We consider as wavefunctions (cf §8.1.3.2):

$$\psi_\alpha(r) = \left(\frac{\pi^2 \sigma_\alpha(r)}{p_\alpha^2(r)} \right)^{1/4} Ai\left[\sigma_\alpha(r)\right], \tag{8.45}$$

where $\alpha = i, f$, and where σ_α is defined by:

$$\sigma_\alpha(r) = \left[\frac{3}{2} \int_{r_\alpha}^{r} p_\alpha(u)\mathrm{d}u \right]^{2/3} = \left[\frac{3}{2} x_\alpha(r) \right]^{2/3}.$$

This approach, which avoids the problems of divergence involved by the JWKB wave functions in the neighbourhood of the turning points, is very close to the method of Bienniek (1977) which also uses the uniform approximation. We have to calculate the matrix element

$$\begin{aligned} F_{if} &= \langle \psi_i(r) \left| D(r) \right| \psi_f(r) \rangle \\ &= \int_{-\infty}^{+\infty} \pi \left(\frac{\sigma_i(r)\sigma_f(r)}{p_i^2(r)p_f^2(r)} \right)^{1/4} D(r) Ai\left[\sigma_i(r)\right] Ai\left[\sigma_f(r)\right] \mathrm{d}r. \end{aligned}$$

We then employ the stationary phase method, since the preceding integral is written, with the definition (2.20) of $Ai(x)$

$$\begin{aligned} F_{if} = \frac{1}{4\pi^2} \iiint_{\mathbb{R}^3} \pi \left(\frac{\sigma_i(r)\sigma_f(r)}{p_i^2(r)p_f^2(r)} \right)^{1/4} D(r) \\ \times \exp\left\{ \mathrm{i} \left[\frac{t^3 + t'^3}{3} + t\sigma_i(r) + t\sigma_f(r) \right] \right\} \mathrm{d}r\mathrm{d}t\mathrm{d}t'. \end{aligned}$$

The stationary phase point is given by

$$p_i(r_0) = p_i(r_0) = p_0, \tag{8.46}$$

so we can develop σ_α into the neighbourhood of this point

$$\sigma_\alpha(r) = \sigma_\alpha(r_0) + (r - r_0)\,\sigma'_\alpha(r_0) + \dots \tag{8.47}$$

We shall not detail the calculation, but taking into account the relation $\sigma'_\alpha(r) = p_\alpha(r)/\sqrt{\sigma_\alpha(r)}$, we obtain, without difficulty, the uniform expression of the Franck-Condon factor

$$F_{if} = \frac{3\pi}{2p_0^2} \frac{\left[x_i(r_0)x_f(r_0)\right]^{1/2}}{\left[\frac{3}{2}\left(x_i(r_0) - x_f(r_0)\right)\right]^{1/3}} \times D(r_0) Ai\left\{\left[\frac{3}{2}\left(x_i(r_0) - x_f(r_0)\right)\right]^{2/3}\right\}. \tag{8.48}$$

This result is the same as the one obtained by Bienniek (1977).

With this analytical expression of the Franck-Condon factor, we can now calculate the profile $J(\omega)$. First of all, we limit the angular average to the value $\ell_{\max}$ beyond which the penetration at the distance r_0 is no more classically allowed $p_0(\ell_{\max}) = 0$. Then we obtain

$$J(\omega) = \int_0^\infty \mathrm{d}\varepsilon_i\; \mathrm{e}^{-\varepsilon_i/kT} \sum_{\ell=0}^{\ell_{\max}} (2\ell+1)\,|F_{if}|^2. \tag{8.49}$$

Consequently, we can change the discrete sum over the angular momenta into an integration with the help of the Poisson summation formula. [Morse & Feshbach (1953); Berry & Mount (1972)]

$$\sum_{\ell=0}^{\infty} f(\ell) = \frac{1}{h} \sum_{M=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}M\pi} \int_0^\infty f\left[\left(\frac{L}{h}\right) - \left(\frac{1}{2}\right)\right] \mathrm{e}^{\mathrm{i}2\pi ML/h} \mathrm{d}L.$$

Thereafter we shall limit the summation to the first term. In addition, assuming we can develop $p_\alpha(r)$ in the neighbourhood of r_0, we get

$$\begin{cases} x_i(r_0) - x_f(r_0) = \frac{mr_0^2}{2p_0}\Delta V'(r_0) \\ x_i(r_0)x_f(r_0) = r_0^2 p_0^2, \end{cases}$$

where $\delta V'(r_0)$ is the derivative of the difference of the potentials at the Franck-Condon point r_0. The expression (8.48) is now written

$$F_{if} = \frac{3\pi}{2} D(r_0) \left(\frac{4r_0}{3mp_0^2\Delta V'(r_0)}\right)^{1/3} Ai\left\{\left[\frac{3mr_0^2}{4p_0}\Delta V'(r_0)\right]^{2/3}\right\}. \tag{8.50}$$

Changing the angular quantum number ℓ into the continuous variable $\xi = \ell(\ell+1)$, $J(\omega)$ becomes

$$J(\omega) = \frac{9\pi^2}{4} D^2(r_0) \left(\frac{4r_0}{3mp_0^2 \Delta V'(r_0)} \right)^{2/3} \int_0^\infty \mathrm{d}\varepsilon_i \, \mathrm{e}^{-\varepsilon_i/kT} \tag{8.51}$$

$$\times \int_0^{\xi_{max}} \mathrm{d}\xi \, p_0^{-4/3}(\xi) Ai^2 \left\{ \left[\frac{3mr_0^2 \Delta V'(r_0)}{4p_0(\xi)} \right]^{2/3} \right\},$$

and with the relation

$$\frac{\mathrm{d}p_0(\xi)}{\mathrm{d}\xi} = -\frac{1}{2r_0^2 p_0(\xi)},$$

we actually obtain

$$J(\omega) = \frac{9\pi^2}{2} D^2(r_0) r_0^2 \left(\frac{4r_0}{3mp_0^2 \Delta V'(r_0)} \right)^{2/3} \int_{V_i(r_0)}^\infty \mathrm{d}\varepsilon_i \, \mathrm{e}^{-\varepsilon_i/kT} \tag{8.52}$$

$$\times \int_0^{2m[\varepsilon_i - V_i(r_0)]^{1/2}} \mathrm{d}p_0 \, p_0^{-1/3} Ai^2 \left\{ \left[\frac{3mr_0^2 \Delta V'(r_0)}{4p_0} \right]^{2/3} \right\}.$$

For the average over the initial states, the integration is done not from the zero value of the energy, but from the value $V_i(r_0)$, a value that cancels the upper limit of the other integral. Indeed, the perturbation is unable to reach the point r_0 if it has an energy lower than $V_i(r_0)$, which supposes that the potential curve of the initial state is repulsive.

We use an integration by parts for the integral on the energies ε_i

$$\int_{V_i(r_0)}^\infty \mathrm{d}\varepsilon_i \, \mathrm{e}^{-\varepsilon_i/kT} \mathcal{F}(\varepsilon_i) = -kT \mathrm{e}^{-\varepsilon_i/kT} \mathcal{F}(\varepsilon_i) \Big|_{V_i(r_0)}^\infty$$

$$+ \int_{V_i(r_0)}^\infty \mathrm{d}\varepsilon_i \, \mathrm{e}^{-\varepsilon_i/kT} \mathcal{F}'(\varepsilon_i).$$

The first term vanishes, because of the exponential for the upper limit, and the $\{(\varepsilon_i)$ quantity for the lower limit. The second term eliminates the integration on p_0. Then we obtain for the profile

$$J(\omega) = \frac{27\pi^2}{2} kT \, D^2(r_0) r_0^4 \, \mathrm{e}^{-V_i(r_0)/kT} \mathcal{L}(z), \tag{8.53}$$

where $z = \left(\frac{3}{4}mr_0^2\Delta V'(r_0)\right)^2/kT$, and the universal function [Sando & Wormhoudt (1973)]

$$\mathcal{L}(z) = \int_0^\infty \mathrm{e}^{-1/t^3} Ai^2(-zt)\frac{\mathrm{d}t}{t^2}. \tag{8.54}$$

This result is valid all along the spectrum. In particular, in the quasi-static approximation, we have to take the asymptotic form of (z):

$$\mathcal{L}(z) \approx \frac{1}{\sqrt{36\pi z}},$$

leading to the quasi-static profile (8.43).

Up to now, we have limited the development of $D(r)$ to zero order in the neighbourhood of r_0: $D(r) \approx D(r_0)$. If we develop now $D(r)$ to the second order (cf. Eq. (8.33)), the Franck-Condon amplitude becomes

$$\begin{aligned}
&\langle \psi_i(r)\,|D(r)|\,\psi_f(r)\rangle \\
&= \left[D(r_0) - r_0D'(r_0) + \frac{r_0^2}{2}D''(r_0)\right]\langle \psi_i(r) \mid \psi_f(r)\rangle \\
&\quad + [D'(r_0) - r_0D''(r_0)]\,\langle \psi_i(r)\,|r|\,\psi_f(r)\rangle \\
&\quad + \frac{D''(r_0)}{2}\left\langle \psi_i(r)\,\left|r^2\right|\,\psi_f(r)\right\rangle .
\end{aligned} \tag{8.55}$$

The calculation of the overlap integral $\langle \psi_i(r) \mid \psi_f(r)\rangle$ having already been carried out (formula (8.50)), it remains to evaluate the integrals $\langle \psi_i(r)\,|r|\,\psi_f(r)\rangle$ and $\left\langle \psi_i(r)\,\left|r^2\right|\,\psi_f(r)\right\rangle$. With this intention, we shall again use the uniform wave function developed in the neighbourhood of the stationary phase point r_0

$$\psi_\alpha(r) = \frac{\sqrt{\pi}}{\sqrt{\sigma'_\alpha(r_0)}}Ai\left[\sigma_\alpha(r_0) + (r - r_0)\sigma'_\alpha(r_0)\right], \ \alpha = i, f. \tag{8.56}$$

At first, let us calculate the integral

$$\begin{aligned}
&\langle \psi_i(r)\,|r|\,\psi_f(r)\rangle \\
&= \frac{\pi}{\left[\sigma'_i(r_0)\sigma'_f(r_0)\right]^{1/2}}\int_{-\infty}^{+\infty} Ai\left[\sigma_i(r_0) + (r - r_0)\sigma'_i(r_0)\right] \\
&\qquad \times Ai\left[\sigma_f(r_0) + (r - r_0)\sigma'_f(r_0)\right].
\end{aligned} \tag{8.57}$$

We find the formula (3.110), where $n = 1$ and

$$\begin{cases} \alpha = \frac{1}{\sigma_i'(r_0)} \; a = \frac{\sigma_i(r_0)}{\sigma_i'(r_0)} - r_0 \\ \beta = \frac{1}{\sigma_f'(r_0)} \; b = \frac{\sigma_f(r_0)}{\sigma_f'(r_0)} - r_0. \end{cases}$$

The formula (3.115) gives the result

$$\langle \psi_i(r) \left| r \right| \psi_f(r) \rangle = A r_0 I_0,$$

where $A = \pi \left[\sigma_i'(r_0)\sigma_f'(r_0)\right]^{-1/2}$ and $I_0 = \langle \psi_i(r) \mid \psi_f(r) \rangle$. So we obtain

$$\langle \psi_i(r) \left| r \right| \psi_f(r) \rangle = r_0 \langle \psi_i(r) \mid \psi_f(r) \rangle ,$$

yielding

$$\langle \psi_i(r) \left| r - r_0 \right| \psi_f(r) \rangle = 0. \tag{8.58}$$

Then the Franck-Condon amplitude (8.55) is limited to

$$\begin{aligned} &\langle \psi_i(r) \left| D(r) \right| \psi_f(r) \rangle \\ &\quad = \left[D(r_0) + \frac{r_0^2}{2} D''(r_0) \right] \langle \psi_i(r) \mid \psi_f(r) \rangle \\ &\qquad + \frac{D''(r_0)}{2} \left\langle \psi_i(r) \left| r^2 \right| \psi_f(r) \right\rangle . \end{aligned} \tag{8.59}$$

Let us now calculate the term $\left\langle \psi_i(r) \left| r^2 \right| \psi_f(r) \right\rangle$. With the same notations as previously, we have

$$\left\langle \psi_i(r) \left| r^2 \right| \psi_f(r) \right\rangle = A I_2,$$

where I_2 is defined by the formula (3.116)

$$I_2 = r_0^2 I_0 + \frac{2}{\sigma_i'^3(r_0) - \sigma_f'^3(r_0)} I_0'.$$

We obtain finally the expression of the uniform Franck-Condon amplitude, developed to the second order

$$\begin{aligned} \langle \psi_i(r) \left| D(r) \right| \psi_f(r) \rangle &= D(r_0) \langle \psi_i(r) \mid \psi_f(r) \rangle \\ + D''(r_0) \frac{\pi}{p_0^4} \left(\frac{3}{2} \right)^{4/3} &\frac{\left[x_i(r_0) x_f(r_0)\right]^{3/2}}{\left[x_i(r_0) - x_f(r_0)\right]^{5/3}} \\ \times \; Ai' &\left\{ \left[\frac{3}{2} \left(x_f(r_0) - x_i(r_0) \right) \right]^{2/3} \right\} \end{aligned} \tag{8.60}$$

It can be seen in particular that, in this expression, we do not have any more term in $D'(r_0)$. Then, for a better approximation to the Franck-Condon approximation, it is necessary to develop $D(r)$ up to the second order.

8.3 The Semiclassical Wigner Distribution

For several years, scientists have sought to establish a correspondence between chaotic classical systems and their quantum correspondent, as well as criteria allowing to decide if a quantum system expresses, or not, a chaotic behaviour: irregularity of the spectra [Percival (1973); Berry (1977)], sensitivity of the spectrum over the small disturbances [Pomphrey (1974); Grémaud & al (1993)], distributions of the energy levels [Tabor (1977); Mcdonald & Kaufmann (1979)], or structures of the stationary states [Heller (1984); Mcdonald & Kaufmann (1979)...] However other authors have doubts about the existence of quantum chaos [Ford (1989); Ford & Ilg (1992); Ford & Mantica (1992)], and point out the fact that even if quantum systems express a particular behaviour when their traditional equivalent is chaotic, there are serious reasons to think that quantum chaos does not exist.

Indeed, it has already been established that the finished and closed quantum systems do not express a chaotic behaviour [Ford & Ilg (1992)]. However the principal objection about the existence of quantum chaos remains the linearity of the Schrödinger equation, which makes it insensitive over "the exponential instability of initial conditions" [Ford & Ilg (1992); Dando & Monteiro (1994)], *sine qua non* condition of chaos. In addition, criteria of recognition of traditional chaos stand in the concept of trajectory. However the Heisenberg principle of uncertainty excludes the possibility of defining a trajectory in quantum mechanics (see in particular the objections of Ford & Mantica (1992)).

In addition, we lack analytical tools for the study of quantum chaos [Berry (1977)], and more generally our knowledge is poor on the correspondence principle between the classically chaotic systems and their quantum equivalents, as well as the quantification of the latter (determination of the eigenvalues and eigenfunctions in terms of classical quantities). However, Berry has shown that the distribution of Wigner can be a means of connecting quantum and traditional descriptions for an integrable system (see also Ozorio of Almeida & Hannay (1982); Meredith (1992)).

We thus propose to study a quantum probability distribution here, the

Wigner distribution, within the classical phase space, in the semiclassical limit $\hbar \to 0$, using the tool of semiclassical methods: the Airy uniform approximation.

8.3.1 *The Weyl–Wigner formalism*

We shall first present the function $W(q,p)$ succinctly, introduced by Wigner in 1932. This function makes it possible to represent a quantum state $|\psi\rangle$ and is interpreted like the quantum equivalent of a "density" of the traditional phase space.

Wigner has shown that in terms of operators, and in a similar way as in classical mechanics, we have

$$\langle\hat{A}\rangle = \int_{-\infty}^{+\infty} \mathrm{d}^N q \int_{-\infty}^{+\infty} \mathrm{d}^N p \; A(q,p)\, W(q,p),$$

where $W(q,p)$ is the Wigner distribution, defined by

$$W(q,p) = \frac{1}{(\pi\hbar)^N} \int_{-\infty}^{+\infty} \psi(q+\eta)\psi^*(q-\eta)\mathrm{e}^{\mathrm{i}2p\eta/\hbar}\mathrm{d}\eta, \tag{8.61}$$

for a state represented by the wave function $\psi(r)$, here independent of time.

This distribution has many properties (see for example Berry (1977); O'Connel (1983)). Without entering into too much detail, $W(q,p)$ is a real function, but which is not always positive. Consequently, $W(q,p)$ cannot be considered as a true density in the phase space. However, it has the usual properties of a probability distribution. Indeed, the Wigner distribution $W(q,p)$ makes it possible to find by projection the usual densities of the position space

$$\int_{-\infty}^{+\infty} \mathrm{d}^N p \; W(q,p) = |\langle q \mid \psi\rangle|^2,$$

as well as of the momentum space

$$\int_{-\infty}^{+\infty} \mathrm{d}^N p \; W(q,p) = |\langle p \mid \psi\rangle|^2.$$

So let us note that if $|\psi\rangle$ is normalised to 1, then

$$\int_{-\infty}^{+\infty} \mathrm{d}^N q \int_{-\infty}^{+\infty} \mathrm{d}^N p \, W(q,p) = \langle \psi \mid \psi \rangle = 1.$$

In the semiclassical limit $\hbar \to 0$, if the system is classically integrable, $W(q,p)$ is reduced to the classical density and its equation of motion is reduced to the Liouville equation, which rules the time evolution of a density of the classical phase space [Heller (1976)]. Indeed, the semiclassical approach developed by Heller makes possible to determine the evolution in the course of the time of a quantum state while following classical trajectories. The link is thus established between the classical field and the quantum one: the wave function is represented by a quantum distribution which moves on a classical path.

However if the system considered is not integrable, then the dynamics of this system, as well as the nature of the stationary states and the distribution of the energy levels, are still unknown to us. However, a conjecture was proposed independently by Voros (1976) and Berry (1977): the microcanonical hypothesis. Even in the case of an ergodic system (*i.e.* irregular), the Wigner distribution would be reduced, to the classical limit, into the microcanonical distribution, that is to say

$$W_m(q,p) = \frac{\delta(E - H(q,p))}{\iint \mathrm{d}p\mathrm{d}q\delta(E - H(q,p))}, \tag{8.62}$$

where δ is the Dirac function.

8.3.2 *The one-dimensional Wigner distribution*

The one-dimensional Wigner distribution $W_1(r,p)$, independent of time, function of the position r and the momentum p, is defined by

$$W_1(r,p) = \frac{1}{\pi\hbar} \int_{-\infty}^{+\infty} \psi(r+\eta)\, \psi^*(r-\eta)\, \mathrm{e}^{\mathrm{i}2p\eta/\hbar} \mathrm{d}\eta. \tag{8.63}$$

In this expression we shall replace the wave function $\psi(r)$ of the particle with the mass $m = 1/2$ and the energy E into a potential $V(r)$: $p(r) = \sqrt{E - V(r)}$, by the Airy uniform wave function developed in the neighbourhood of a turning point r_t (cf. §8.1.3.2, formula (8.24)), *i.e.*

$$\psi(r) = \frac{\sqrt{\pi}}{\hbar^{1/6}\sqrt{\alpha}} Ai\left[\frac{\alpha}{\hbar^{2/3}}(r - r_t)\right], \tag{8.64}$$

where the constant α is defined by:

$$\alpha^3 = \left[\frac{\mathrm{d}V(r)}{\mathrm{d}r}\right]_{r=r_t}.$$

Let us note, that the definitions of $\psi(r)$ and α are not the same ones as those given in §8.1.3.2 because we chose here to highlight the constant $\hbar$.

At this step, let us note that this method of the turning point cannot be employed in the neighbourhood of a local minimum or maximum of the potential, since we would have in this case $\alpha = 0$. We have now to apply the stationary phase method. The one-dimensional Wigner semiclassical distribution is written now

$$W_1(r,p) = \frac{1}{\hbar^{4/3}} \int_{-\infty}^{+\infty} \frac{1}{\alpha} Ai\left[\frac{\alpha}{\hbar^{2/3}}(r - r_t - \eta)\right] \tag{8.65}$$

$$\times Ai\left[\frac{\alpha}{\hbar^{2/3}}(r - r_t + \eta)\right] \mathrm{e}^{\,\mathrm{i}2p\eta/\hbar}\mathrm{d}\eta.$$

The calculation of this integral is given in the §3.5.3, formula (3.120), hence the result is

$$W_1(r,p) = \frac{1}{2^{1/3}\hbar^{2/3}\alpha^2} Ai\left\{\frac{2^{2/3}}{\hbar^{2/3}\alpha^2}\left[\alpha^3(r - r_t) + p^2(r)\right]\right\}. \tag{8.66}$$

This expression of the one-dimensional, local and semiclassical Wigner distribution must be reduced, to the classical limit $\hbar \to 0$, to the microcanonical distribution. Let us take again the expression (8.66) of the Wigner distribution into the neighbourhood of a turning point r_t and consider the argument of the Airy function, where $\alpha^3 = V'(r_t)$

$$A = (r - r_t)V'(r_t) + p^2,$$

which can be written, by neglecting the higher order terms of the development of the potential:

$$A = V(r) - V(r_t) + p^2(r) = (E - H)_{r=r_t},$$

where H is the classical Hamiltonian of the particle with energy E and mass $m = 1/2$, in a potential $V(r)$. The Wigner distribution developed into the neighbourhood of a turning point becomes consequently

$$W_1(r,p) = \frac{1}{2^{1/3}\hbar^{2/3}\alpha^2} Ai\left\{\frac{2^{2/3}}{\hbar^{2/3}\alpha^2}\left[(E-H)_{r=r_t}\right]\right\}.$$

Assuming $\beta = \frac{2^{2/3}}{\hbar^{2/3}\alpha^2}$, we obtain

$$W_1(r,p) = \frac{\beta}{2} Ai\left\{\beta\left[(E-H)_{r=r_t}\right]\right\}. \tag{8.67}$$

The passage to the classical limit is obtained formally in making the Planck constant $\hbar$ towards 0, *i.e.* here in making β towards infinity. However the Airy function $Ai(x)$ checks the relation (formula (4.8))

$$\lim_{\beta\to\infty} \beta Ai(\beta x) = \delta(x). \tag{8.68}$$

Therefore, in the classical limit, the Wigner local distribution (8.66) becomes

$$\lim_{\hbar\to 0} W_1(r,p) = \frac{1}{2}\delta(H-E)_{r=r_t}. \tag{8.69}$$

In the semiclassical limit, the Wigner distribution (8.66) developed into the neighbourhood of a turning point is reduced, as it should be, to the microcanonical distribution.

Let us note that the factor 1/2 results from the fact, that by carrying out the development in the neighbourhood of a turning point, we took into account only half of the phase space.

8.3.3 *The two-dimensional Wigner distribution*

With the aim to determine the analytical expression of the two-dimensional Wigner distribution, we shall proceed in the same way as in the previous section, *i.e.* we define the quantities $\sigma(x,y) = \left(\frac{3}{2}\int_{x_t}^{x} p_1(u,y)\mathrm{d}u\right)^{2/3}$, $\xi(x,y) = \left(\frac{3}{2}\int_{y_t}^{y} p_2(x,u)\mathrm{d}u\right)^{2/3}$, where p_1 and p_2 are defined by separating the momentum in two parts, such that $p_1^2 + p_2^2 = E - V$. [2] The two-

[2]This separation in p_1 and p_2 can appear arbitrary, but it is usual to distinguish one of the variables compared to the other, while defining for example $p_1(x)$ and $p_2(x,y)$ [Martens et al. (1988)].

dimensional uniform wave function can thus be written

$$\psi(x,y) = \frac{\pi}{\hbar^{1/3}\left[\sigma_x'(x,y)\xi_y'(x,y)\right]^{1/2}} Ai\left[\frac{\xi(x,y)}{\hbar^{2/3}}\right] Ai\left[\frac{\sigma(x,y)}{\hbar^{2/3}}\right]. \qquad (8.70)$$

As previously, we develop this wave function in the neighbourhood of a turning point $r_t = (x_t, y_t)$, to obtain

$$\begin{aligned}\psi(x,y) = \frac{\pi}{\hbar^{1/3}\sqrt{ad}} & Ai\left[\frac{a}{\hbar^{2/3}}(x-x_t) + \frac{b}{\hbar^{2/3}}(y-y_t)\right] \\ & \times Ai\left[\frac{c}{\hbar^{2/3}}(x-x_t) + \frac{d}{\hbar^{2/3}}(y-y_t)\right],\end{aligned} \qquad (8.71)$$

where a, b, c and d are the two-dimensional equivalents of the preceding defined constant α, leading to

$$\begin{cases} a = \left[\frac{\mathrm{d}\sigma(x,y)}{\mathrm{d}x}\right]_{(x,y)=(x_t,y_t)} & b = \left[\frac{\mathrm{d}\sigma(x,y)}{\mathrm{d}y}\right]_{(x,y)=(x_t,y_t)} \\ c = \left[\frac{\mathrm{d}\xi(x,y)}{\mathrm{d}x}\right]_{(x,y)=(x_t,y_t)} & d = \left[\frac{\mathrm{d}\xi(x,y)}{\mathrm{d}y}\right]_{(x,y)=(x_t,y_t)} \end{cases}.$$

A rather long calculation, though without any particular difficulties, leads to the analytical expression of the two-dimensional Wigner semiclassical distribution developed in the neighbourhood of a turning point (x_t, y_t)

$$\begin{aligned} & W_2(x,y,p_x,p_y) \\ & = \frac{1}{2^{2/3}\hbar^{4/3}a^2d^2} Ai\left\{\frac{2^{2/3}}{\hbar^{2/3}}\left[(ax+by) + \left(\frac{p_xd - p_yc}{\Delta}\right)^2\right]\right\} \\ & \quad \times Ai\left\{\frac{2^{2/3}}{\hbar^{2/3}}\left[(cx+dy) + \left(\frac{p_xb - p_ya}{\Delta}\right)^2\right]\right\},\end{aligned} \qquad (8.72)$$

where we posed, to reduce the writing and without loss of generality, $x = x - x_t$ and $y = y - y_t$.

Let us note that, as in the one-dimensional case, this method is not applicable in the neighbourhood of a local minimum or maximum of the potential ($\delta = 0$).

At first, let us examine the case where the Hamiltonian variables are separable, *i.e.* where the Hamiltonian can be written according to $H(x,y) = h_1(x) + h_2(y)$. The functions $\xi(x,y)$ and $\sigma(x,y)$ are reduced then to

$$\begin{cases} \sigma(x,y) = \sigma(x) \\ \xi(x,y) = \xi(y). \end{cases}$$

We thus have, in this case $b = c = 0$ (cf formula (8.3.3)) and the discriminant δ is reduced to $\delta = ad$. Consequently, the expression (8.72) of the Wigner distribution becomes

$$\begin{aligned} W_2^{\text{sep.}}&(x, p_x; y, p_y) \\ &= \frac{1}{2^{2/3}\hbar^{4/3}a^2d^2} Ai\left\{\frac{2^{2/3}}{\hbar^{2/3}}\left[ax + \left(\frac{p_x}{a}\right)^2\right]\right\} \\ &\quad \times Ai\left\{\frac{2^{2/3}}{\hbar^{2/3}}\left[dy + \left(\frac{p_y}{d}\right)^2\right]\right\}. \end{aligned} \tag{8.73}$$

Thus, in the case of the separable variables (integrable case), $W_2^{\text{sep.}}$ is reduced to the product of two one-dimensional Wigner distributions, and consequently, to the classical limit, the two-dimensional semiclassical Wigner distribution developed in the neighbourhood of a turning point is reduced again to the microcanonical distribution.

Let us return now to the more general case where the variables are not *a priori* separable. The two-dimensional Wigner distribution is then given by the expression (8.72). Let us carry out the linear changes of variable

$$\begin{cases} X = ax + by \\ Y = cx + dy\,. \end{cases}$$

As the discriminant does not cancel ($\Delta = ad - bc \neq 0$), the inverse transformation is

$$\begin{cases} x = \frac{1}{\Delta}\,(dX - bY) \\ y = \frac{1}{\Delta}\,(-cX + aY)\;. \end{cases}$$

However the conjugate variables x and p_x check the relation $p_x = \frac{\partial L}{\partial \dot{x}}$, where $L(\dot{x}, x, \dot{y}, y)$ is the Lagrangian of the system. We shall thus have after the transformation the new combined moment

$$P_X = \frac{\partial L}{\partial \dot{X}} = \frac{\partial L}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial \dot{X}} + \frac{\partial L}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \dot{Y}}$$

i.e.

$$P_X = p_x\frac{d}{\Delta} + p_y\frac{-c}{\Delta}. \tag{8.74}$$

Similarly, we have

$$P_Y = p_x \frac{-b}{\Delta} + p_y \frac{a}{\Delta}. \tag{8.75}$$

The expression of the Wigner distribution then becomes

$$W_2\left(X, P_X; Y, P_Y\right) = \frac{1}{2^{2/3}\hbar^{4/3}a^2d^2} Ai\left\{\frac{2^{2/3}}{\hbar^{2/3}}\left(X + P_X^2\right)\right\} \times Ai\left\{\frac{2^{2/3}}{\hbar^{2/3}}\left(Y + P_Y^2\right)\right\}. \tag{8.76}$$

The linear transformation (8.3.3) is thus a canonical transformation, which transforms the Hamiltonian $H\left(x, y, p_x, p_y\right)$ into the Hamiltonian $\mathcal{H}\left(X, P_X; Y, P_Y\right)$. As previously, we can consequently pass to the classical limit and write

$$W_2\left(X, P_X; Y, P_Y\right) = \frac{1}{4}\delta\left(\mathcal{H} - \mathcal{E}\right)_{X_t, Y_t}.$$

Thus, the reverse linear transformation (8.3.3) makes it possible, at least locally, to obtain the microcanonical distribution

$$\delta\left(H\left(x, y, p_x, p_y\right) - E\right). \tag{8.77}$$

Even if we are not in the case of a system whose variables are separable, the fact of developing the function into the neighbourhood of a turning point, *i.e.* of linearising the potential in this neighbourhood, allows, by means of a linear and canonical transformation, to reduce the problem to the case of separable variables.

8.3.4 *Configuration of the Wigner distribution in the phase space*

Since we now have an analytical expression of the Wigner distribution, it is natural to represent this one in the phase space. On the one hand we already know, at least qualitatively, the behaviour of this function for an integrable system [Berry (1977b)]. On the other hand, we are unaware of its behaviour in the case of a non-integrable system. However, Berry put forth the hypothesis that the Wigner distribution would be like a series of minima and maxima randomly distributed in the phase space. We give on the figure 8.8 the representation in a cut of the phase space for an integrable system, then for a non-integrable system on the figure 8.9. We also give, in the first case, the configuration of the Wigner distribution, and in the second one the distribution of Berry's conjecture. The boxes materialise the fact that each time we shall consider only half of the phase space.

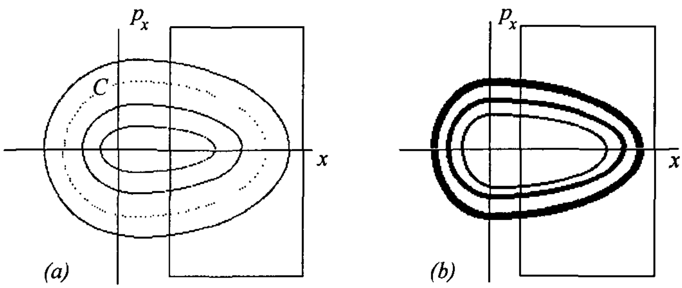

Fig. 8.8 (a): Section S of the phase space (classical case) for a particle in a bound state energy E_m. The curve C is, in classical terms, the intersection of the torus I_m with S. (b): The corresponding Wigner distribution is concentrated in the neighbourhood of C by forming fringeset. (Berry, 1977b).

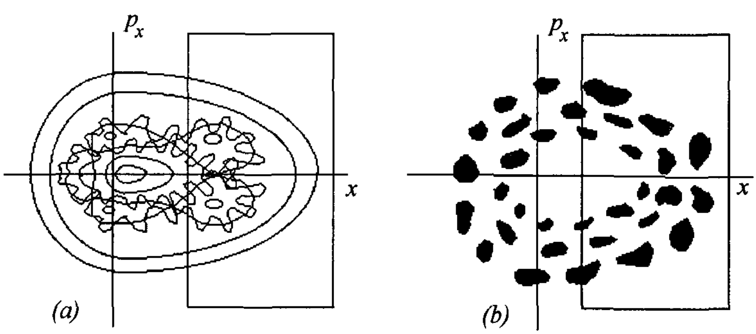

Fig. 8.9 (a): Section corresponding to the figure 8.8, with a non-integrable perturbation. Some tori are still present, but irregular trajectories appear. (b): The corresponding Wigner distribution in a state where the classical motion would be irregular. W would take the form of a series of random minima and maxima covering the areas occupied by these irregular trajectories (Berry, 1977b).

The expression of the Wigner distribution given by the formula (8.76), except for the constants, can be written

$$W_2 = Ai\left(X + P_X^2\right). \tag{8.78}$$

We represented this function on the figure 8.10, in the phase space (x, p), reduced here to (x, p_x).

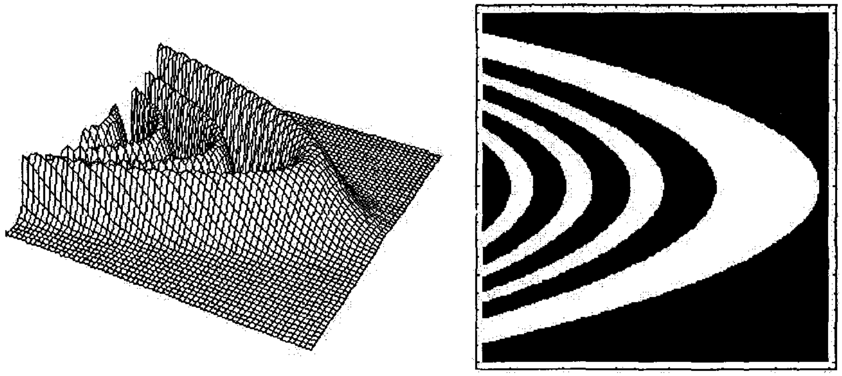

Fig. 8.10 The two-dimensional Wigner distribution, in the case of separable variables, represented in the plane defined by (X, P_X). At the right-hand side a cut of the distribution showing regular minima and maxima is represented.

As we already mentioned, we developed into the neighbourhood of only one turning point, consequently Fig. 8.10 represents only half of the distribution.

As we can see on this figure, we obtain a regular distribution, identical with Fig. 8.8, having the shape called "Airy fringeset" by Berry.

The expression of the two-dimensional Wigner distribution given by the formula (8.72) lends itself particularly well to our aim. It consists in replacing the constants a, b, c and d with the elements of an unimodular transformation matrix.

It is *a priori* possible to choose any matrix whose determinant is 1, this in order to ensure the conservation of the volume of the phase space. Indeed, the volume occupied by a dynamical system in the phase space remains unchanged during the evolution of this system. We thus go, by means of a transformation matrix, to perturb the system by locally deforming the space, and to observe the reaction on the distribution of these constraints.

Among the unimodular transformation matrices, we consider the shearing matrix

$$\begin{bmatrix} 1 & 0 \\ \gamma & 1 \end{bmatrix},$$

of which the deformation effects are more sensitive than with the rotation, or hyperbolic rotation matrices. For various values of the parameter γ, we shall represent the Wigner distribution into the plane (x, p_x) of the phase space. The formula (8.72) gives the expression (except for the constants)

$$W_2 = Ai\left[x + p_x^2\right] Ai\left[\gamma x\right]. \tag{8.79}$$

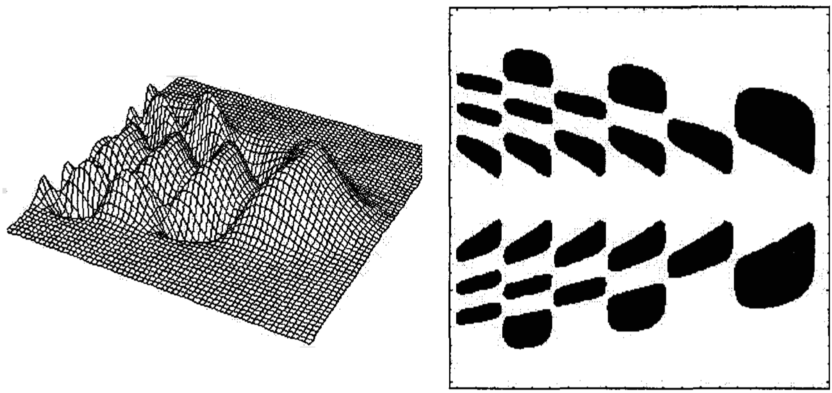

Fig. 8.11 The two-dimensional Wigner distribution ($\gamma = 1$) in the (x, p_x) plane. On the right the cut of the distribution shows the distribution of peaks formed by minima and maxima.

For $\gamma = 0$, W_2 is reduced to the one-dimensional distribution in the (x, p_x) plane. For other values of the parameter γ, the distribution undergoes very significant deformations. In particular for the value $\gamma = 1$, we observe the figure 8.11 and the distribution of the minima and maxima of W_2. This distribution, in qualitative terms, resembles Berry's conjecture, on Fig. 8.9, where he supposed that the minima and maxima of the semi-classical distribution are randomly dispatched in the ergodic system case. However, we obtain a qualitatively similar distribution whereas we are in an integrable case, since the variables are separable.

It would seem consequently, that it is not necessary to be in the case of an ergodic system leading to a relatively irregular form for the Wigner distribution. Let us note that the figure 8.11 seems still "regular" enough, but we are in the particular case of a development in the neighbourhood of a turning point. But, even within the framework of a very simple model, where the variables are separable and the trajectories regular, the distribution can have no simple structure, at least locally, which seems to invalidate the reciprocal of Berry's conjecture.

Recently Tomsovic & Heller (1993) built "chaotic" wave functions *i.e.* the wave functions of a system where the dynamics of the corresponding classical system are chaotic, from purely semiclassical methods. Their results were applied to the stadium billiard problem (classically chaotic) and stationary states were obtained, which seem to be the same for the Wigner distribution as the Berry conjecture, in other words the same we obtained by semiclassical methods, but where all possibility of chaotic behaviour is excluded.

8.4 Airy Transform of the Schrödinger Equation

A few years ago Berry & Balazs (1979), built a wave packet which moves without dispersion and which accelerates, without being subjected to an external force, thus seeming to contradict the Ehrenfest theorem. Let us consider the Schrödinger equation for a free particle with a mass m

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}, \tag{8.80}$$

and the wave packet defined at $t = 0$ by the relation $\psi(x, 0) = Ai(Bx/\hbar^{2/3})$, where B is an arbitrary positive constant. The solution of Eq. (8.80) is the *a priori* strange wave packet

$$\psi(x,t) = \exp\left[i\left(\frac{B^3 t}{2m\hbar}\right)\left(x - \frac{B^3t^2}{6m^2}\right)\right]\ Ai\left[\frac{B}{\hbar^{2/3}}\left(x - \frac{B^3t^2}{4m^2}\right)\right], \tag{8.81}$$

for it is apparently in contradiction with quantum mechanics. Indeed, this "Airy wave packet" (or more exactly the density of probability $|\psi(x,t)|^2$) does not disperse with time and, moreover, accelerates without any force acting on it.

However, Berry & Balazs show that this solution satisfies the Ehrenfest theorem, because the Airy wave packet does not have a centre of mass that is perfectly localised. Indeed, the Airy function $Ai(x)$ is not square integrable. The wave packet (8.81) cannot thus represent the density of probability corresponding to a single particle. Consequently the Airy wave packet represents a set of particles in an infinite number, *i.e.* a family of semiclassical orbits in the phase space (this is the analogue of the plane wave for the theory of diffraction).

Greenberger (1980) has shown by a Gallilean transformation dependent on time, that this solution is equivalent to the stationary state of a particle in a uniform gravitational potential. The system in free fall is described by the same type of function, except for argument and phase shift.

Recent work [Besieris et al. (1994) ; Nassar et al. (1995)] shows that a change of variables depending on time, can lead to alternatives of the Airy wave packet of (8.81), in more general cases than a null or constant force. We can thus build dispersive stationary wave packets, or not accelerating if they are not subjected to an external force...

We shall show, using the Airy transform (cf. §4.2), that connection between the Schrödinger equation for the free particle and the one for the free fall does not require a transformation that depends on time. Let us consider the Schrödinger equation. The atomic units ($\hbar = m = e = 1$) will be used

$$\mathrm{i}\frac{\partial \psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi. \tag{8.82}$$

The Airy transform of this equation is

$$\mathrm{i}\frac{\partial \varphi_a}{\partial t} = -\frac{1}{2}\frac{\partial^2 \varphi_a}{\partial y^2} + \int W_a(y,z)\varphi_a(z)\mathrm{d}z, \tag{8.83}$$

where $\varphi_a(y,t)$ is the Airy transform of $\psi(x,t)$, and where the potential $V(x)$ is transformed into a non local potential $W_a(y,z)$

$$W_a(y,z) = \frac{1}{a^2}\int V(x)Ai\left(\frac{y-x}{a}\right)Ai\left(\frac{z-x}{a}\right)\mathrm{d}x. \tag{8.84}$$

Let us examine some particular cases. If the potential $V(x)$ is null or constant, we can see, according to the equations (8.82) and (8.83), that the Schrödinger equation is invariant under the Airy transform. Now let us consider the linear potential $V(x) = Fx$. Thanks to the properties of the Airy transform, the Schrödinger equation (8.82) is written into the self-similar form

$$\mathrm{i}\frac{\partial \varphi_a}{\partial t} = -\left(\frac{1}{2} + Fa^3\right)\frac{\partial^2 \varphi_a}{\partial y^2} + Fy\varphi_a(y). \tag{8.85}$$

Thus a scaling of space and time leaves the equation invariant. Let us choose as parameter $a = -(1/2F)^{1/3}$ in Eq. (8.85). We obtain the simple solution

$$\varphi_a(y,t) = \mathrm{e}^{-\mathrm{i}tFy}\phi(y),$$

where $\phi(y)$ is the Airy transform of the initial wave function. The interpretation of this result is that Airy transform neutralises the effect of the uniform field in such a way that, except a phase factor, the wave function remains the same one as at initial time. Let us suppose that $\phi(y)$ is square integrable, the Plancherel-Parceval rule gives us

$$\int |\phi|^2 \mathrm{d}y = \int |\varphi_a|^2 \mathrm{d}y = \int |\psi|^2 \mathrm{d}x < \infty,$$

i.e. the Airy inverse transform is also square integrable. We can then calculate explicitly the wave function

$$\psi(x,t) = \frac{1}{2\pi} \int \exp\left[\mathrm{i}\left(\frac{\xi^3}{6F} + \xi x\right)\right] \hat{\varphi}\left(\xi + Ft\right) \mathrm{d}\xi,$$

where $\hat{\varphi}$ is the Fourier transform of φ.

So now we consider the solution of null energy as initial wave function $\hat{\varphi}(\xi) = \mathrm{e}^{\mathrm{i}\xi^3/6F}$, which is not square integrable, we obtain the wave packet

$$\psi(x,t) = \mathrm{e}^{-\mathrm{i}\frac{1}{2}Ftx} Ai\left[F^{1/3}\left(x + \frac{Ft^2}{4}\right)\right], \tag{8.86}$$

which is the solution of the Schrödinger equation for a particle in free fall. We can compare this solution with

$$\psi_0(x,t) = \mathrm{e}^{\mathrm{i}\left(\frac{F^2t^3}{12} + \frac{1}{2}Ftx\right)} Ai\left[F^{1/3}\left(x - \frac{Ft^2}{4}\right)\right], \tag{8.87}$$

which is nothing but Eq. (8.81) where $m = \hbar = 1$ and $F = B^3$. The only difference (except the phase factor containing the term in t^3) between these two last equations is the direction of propagation: the relation (8.86) describes a backwards propagation and Eq. (8.87) a forwards propagation.

Exercises

1. With an appropriate change of variable and function, find the general solution of the Schrödinger equation in a time-dependent electric field

$$\mathrm{i}\frac{\partial \psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \psi}{\partial x^2} + F(t)\, x\psi.$$

   Hint: See the paper by [Feng (2001)].

2. Find the energy eigenvalues of a particle in a one-dimensional box submitted to an electric field $V(x) = Fx$. See Fig. 8.12 (a).

3. Quantum wells formed in semiconductor heterostructures, have been given considerable theoretical consideration for their optical device applications. A simple model of tunneling in such devices, is the study of the energy spectrum of a particle in a square well submitted to an electric field $V(x) = Fx$. See Fig. 8.12 (b). *Hint*: For further discussion, see the following paper [Panda & Panda (2001)].

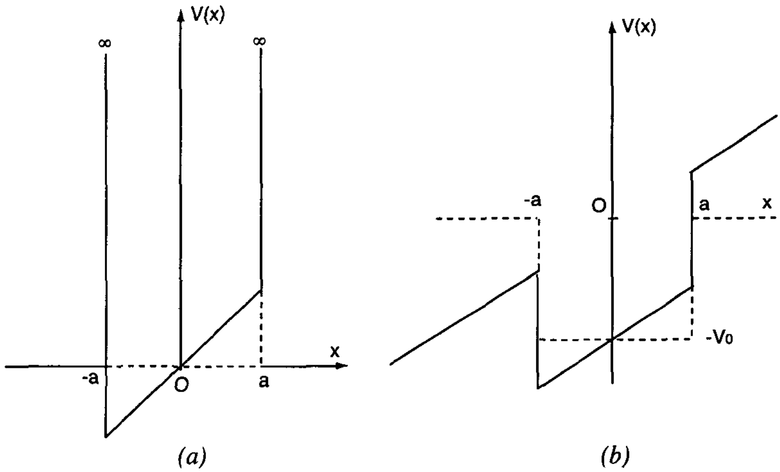

Fig. 8.12 (a) Particle in a box submitted to an electric field. (b) Particle in a square well submitted to an electric field.

Appendix A

Numerical Computation of the Airy Functions

A.1 The Homogeneous Functions

Except the numerical tables of Miller (1946) and Abramowitz & Stegun (1965), the calculation of the Airy functions can be carried out by means of the algorithms of Gordon (1969, 1970) and Lee (1980). Indeed, if we use a Gauss method of quadrature, the Airy function $Ai(x)$ should be replaced by the following sums (here we highlight the formulae (2.32) and (2.34)), where $\xi = \frac{2}{3}x^{3/2}$, $x > 0$

$$Ai(x) \approx \frac{e^{-\xi}}{2\pi^{1/2}x^{1/4}} \sum_{i=1}^{n} \frac{w_i}{1+\left(\frac{x_i}{\xi}\right)} \tag{A.1}$$

$$Ai(-x) \approx \frac{1}{\pi^{1/2}x^{1/4}} \sum_{i=1}^{n} w_i \frac{\cos\left(\xi - \frac{\pi}{4}\right) + \frac{x_i}{\xi}\sin\left(\xi - \frac{\pi}{4}\right)}{1+\left(\frac{x_i}{\xi}\right)^2}, \tag{A.2}$$

and the $Bi(x)$ function by (formulae (2.33) and (2.35))

$$Bi(x) \approx \frac{e^{\xi}}{\pi^{1/2}x^{1/4}} \sum_{i=1}^{n} \frac{w_i}{1-\left(\frac{x_i}{\xi}\right)} \tag{A.3}$$

$$Bi(-x) \approx \frac{1}{\pi^{1/2}x^{1/4}} \sum_{i=1}^{n} w_i \frac{\sin\left(\xi - \frac{\pi}{4}\right) - \frac{x_i}{\xi}\cos\left(\xi - \frac{\pi}{4}\right)}{1+\left(\frac{x_i}{\xi}\right)^2}. \tag{A.4}$$

w_i are the weight factors corresponding to the integration points x_i [Gordon (1970)]. In Table A.1, we give the partition of $\mathbb{R}$ for the calculation of the homogeneous Airy functions. We give in Table A.2 the values of w_i and x_i for integration in ten points. In all the cases, the neighbourhood of the origin will be calculated with the help of the ascending series, given by the formulae (2.37) and (2.38).

Table A.1 Partition of $\mathbb{R}$ for the calculation of the homogeneous Airy functions

x	$Ai(x)$	$Bi(x)$
$x < -3.7$	integration A.2	integration A.4
$-3.7 < x < 2.35$	series 2.37	series 2.38
$2.35 < x < 8.5$	integration A.1	series 2.38
$8.5 < x$	integration A.1	integration A.3

Table A.2 Weight factors w_i and integration abscissas x_i for the Gauss quadrature method with 10 points.

	x_i	w_i
1	1.408308107218096D + 01	3.154251576296478D − 14
2	1.021488547919733D + 01	6.639421081958493D − 11
3	7.441601845045093	1.758388906134567D − 08
4	5.307094306178192	1.371239237043582D − 06
5	3.634013502913246	4.435096663928435D − 05
6	2.331065230305245	7.155501091771825D − 04
7	1.344797082460927	6.488956610333538D − 03
8	6.418885836956729D − 01	3.644041587577328D − 02
9	2.010034599812105D − 01	1.439979241859100D − 01
10	8.059435917205284D − 03	8.123114133626148D − 01

It should be noted, that for the large values of the argument of Airy functions ($|x| > 10$), we can carry out the Gauss quadrature with less integration points and use the asymptotic series (cf. §2.1.4.3), without loss of precision. We can easily test the validity of these various methods, in particular at the points of connection, with the help of the Wronskian of the Airy functions (formula (2.6))

$$Ai(x)Bi'(x) - Ai'(x)Bi(x) = \frac{1}{\pi}$$

There are other methods to numerically calculate the Airy functions. Among them, Krüger (1981), proposed the following method for the calculation of $Ai(x)$. Let us consider the definition of the Airy function

$$Ai(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\left(t^3/3+tx\right)} \mathrm{d}t.$$

By carrying out the change of variable $t \to t + \mathrm{i}\alpha$, in the complex plane, we obtain

$$Ai(x) = \frac{1}{2\pi} \mathrm{e}^{\alpha^3/3-\alpha x} \int_{-\infty+\mathrm{i}\alpha}^{+\infty+\mathrm{i}\alpha} \mathrm{e}^{-\alpha t^2+\mathrm{i}\left[t^3/3+\left(x-\alpha^2\right)t\right]} \mathrm{d}t.$$

We can then use the Poisson summation formula to get

$$Ai(x) = h \sum_{k=0}^{\infty} \psi \left[h \left(n + \frac{1}{2} \right) \right] - \Delta(p), \ p = \frac{2\pi}{h}, \ h > 0, \qquad \text{(A.5)}$$

and

$$\psi(t) = \frac{1}{\pi} \mathrm{e}^{\alpha^3/3-\alpha x-\alpha t^2} \cos\left[\frac{t^3}{3} + t\left(x - \alpha^2 1\right) \right],$$

$$\Delta(p) = \sum_{r=1}^{\infty} (-1)^r R(rp),$$

$$R(p) = R(-p) = \mathrm{e}^{\alpha p} Ai(x+p) + \mathrm{e}^{-\alpha p} Ai(x-p).$$

The expression (A.5) is an exact trapezoidal summation formula. Under some conditions, detailed by Krüger (1981), we can neglect the term $\Delta(p)$, and limit the summation to a range N, so that we obtain for $\alpha = 1$

$$Ai(x) \approx h \sum_{k=0}^{N} \psi \left[h \left(k + \frac{1}{2} \right) \right]$$

where

$$\psi(t) = \frac{1}{\pi} \mathrm{e}^{-x-t^2+1/3} \cos\left[\frac{t^3}{3} + t(x-1) \right].$$

If we choose $h = \frac{2\pi}{26}$ and $N = 24$, these two formulae give the value of $Ai(x)$ for $-10 < x < 0$ with a relative error lower than 3.10^{-8}, and for $x > 0$, an error of about 10^{-12}.

A.2 The Inhomogeneous Functions

As previously, the functions $Gi(x)$ and $Hi(x)$ can be calculated using the sums, for $x > 0$ (cf. formulae (2.136) and (2.126))

$$Gi\,(x) = -\frac{1}{\pi}\sum_{i=1}^{n} w_i \mathrm{e}^{-x_i x/2}\cos\left(\frac{\sqrt{3}}{2}x_i x + \frac{2\pi}{3}\right) \tag{A.6}$$

$$Hi\,(-x) = \frac{1}{\pi}\sum_{i=1}^{n} w_i \mathrm{e}^{-x_i x}. \tag{A.7}$$

$Gi(-x)$ and $Hi(x)$, result from the two preceding equations and of the formulae (A.3) and (A.4) thanks to the relation between Bi, Gi and Hi (formula (2.129)) $Bi(x) = Gi(x) + Hi(x)$. The neighbourhood of the origin will be calculated by using the ascending series defined in the §2.3.3: formulae (2.139) for Hi, (2.38) (series of Bi) and the relation (2.129) for $Gi(x)$.

In Table A.3, we give the partition of $\mathbb{R}$ for the calculation of the inhomogeneous Airy functions, and in Table A.4 the weight factors of integration w_i and the corresponding abscissae x_i, calculated thanks to the algorithm of Gordon (1968).

Exercises

Check your favourite computer library for the computation of Airy functions with the following tests:

1. Check the Wronskian relationships
   $W\{Ai(x),\ Bi(x)\} = \frac{1}{\pi}$ and $W\{Ai^2(x), Ai(x)Bi(x), Bi^2(x)\} = \frac{2}{\pi^3}$.

2. Compare the logarithmic derivative of Airy functions with a numerical solution to the Riccati equation $u' + u^2 = x$.

3. In the complex plane: check the relation $Ai(x) + \mathrm{j}Ai(\mathrm{j}x) + \mathrm{j}^2 Ai(\mathrm{j}^2 x) = 0$, and the corresponding one for Bi. Check the relation (2.163) of §2.4.2.

Table A.3 Partition of $\mathbb{R}$ for the calculation of the Airy inhomogeneous functions.

x	$Gi(x)$	$Hi(x)$
$x < -6$	A.7 and 2.129	integration A.7
$-6 < x < -4$	integration A.7 and 2.129	integration A.7
$-4 < x < 6$	series 2.139 and 2.129	series 2.139
$6 < x < 8$	integration A.6	series 2.139
$8 < x$	integration A.6	A.6 and 2.129

Table A.4 Weight factors w_i and integration abscissas x_i for the Gauss method with 15 points.

	x_i	w_i
1	1.4576978176136D − 02	3.8053986078615D − 02
2	8.1026698767654D − 02	9.6220284128805D − 02
3	2.0814345959022D − 01	1.5721761605002D − 01
4	3.9448412556694D − 01	2.0918953325833D − 01
5	6.3156478398822D − 01	2.3779904013329D − 01
6	9.0760339986136D − 01	2.2713825749406D − 01
7	1.2106768087608	1.7328458073252D − 01
8	1.5309839772429	9.869554247686D − 02
9	1.8618445873124	3.893631493517D − 02
10	2.1997121656815	9.812496327697D − 03
11	2.5438398040282	1.4391914183288D − 03
12	2.8961730431054	1.0889100255168D − 04
13	3.2620667311773	3.5468667194632D − 06
14	3.6533718875065	3.5907188198098D − 08
15	4.1023767739755	5.1126116783291D − 11

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