Exercises: 1, 2 (below) and Judson 19.3, 19.14, 19.20.
Due date: Wednesday, 10/22

1. Let P with $\leq$ be a partially ordered set, let $S \subseteq P$ and let $u \in P$. We say that u is an *upper bound* for S iff $s \leq u$ for all $s \in S$. We say ℓ is the *least upper bound* of S iff ℓ is an upper bound of S and $\ell \leq u$ for every upper bound u of S. Prove that if ℓ is the least upper bound of the set $\{x, y\}$ and m is the least upper bound of the set $\{\ell, z\}$, then m is the least upper bound of the set $\{x, y, z\}$.

Solution: Since $\ell = \text{lub}\{x, y\}$ and $m = \text{lub}\{\ell, z\}$, we have

$$x \leq \ell, \qquad y \leq \ell \quad \text{and} \quad \ell \leq m, \qquad z \leq m.$$

Therefore, $x \leq \ell \leq m$, so $x \leq m$, by transitivity of $\leq$. Similarly, $y \leq \ell \leq m$, so transitivity implies $y \leq m$. It follows that m is an upper bound of the set $\{x, y, z\}$. We want to show that m is the *least* upper bound of $\{x, y, z\}$. That is, if n is another upper bound of $\{x, y, z\}$, we must show $m \leq n$.

If n is an upper bound of $\{x, y, z\}$, then it is also an upper bound of $\{x, y\}$, and since ℓ is the least upper bound of $\{x, y\}$, we have $\ell \leq n$. Therefore, n is an upper bound of $\{\ell, z\}$. Since m is the least upper bound of $\{\ell, z\}$, we have $m \leq n$. $\square$

2. Let $(P, \leq)$ be a partially ordered set with the property that every pair of elements $x, y \in P$ has a greatest lower bound. For $x, y \in P$, define $x \cdot y = \text{glb}(x, y)$. Prove that $(P, \cdot)$ is a semilattice.

Solution: Recall, a *semilattice* is a commutative idempotent semigroup, so we must check that $\langle P, \cdot \rangle$ has these three properties.

(i) To show $\langle P, \cdot \rangle$ is a semigroup, we must prove that $\cdot$ is associative. Indeed, for all $x, y, z \in P$, we have

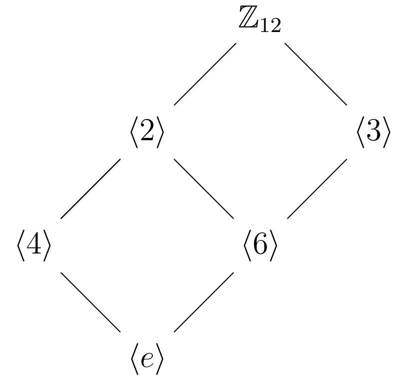
$$\begin{aligned} x \cdot (y \cdot z) &= \text{glb}\{x, \text{glb}\{y, z\}\} \\ &= \text{glb}\{x, y, z\} && \text{(by Problem 1)} \\ &= \text{glb}\{\text{glb}\{x, y\}, z\} && \text{(by Problem 1)} \\ &= (x \cdot y) \cdot z. \end{aligned}$$

(ii) To prove $\cdot$ is commutative, note that the set $\{x, y\}$ is the same as the set $\{y, x\}$, so $x \cdot y = \text{glb}\{x, y\} = \text{glb}\{y, x\} = y \cdot x$.

(iii) Recall the following definition from lecture: an operation $f : A^n \to A$ is called *idempotent* if it satisfies $f(a, \ldots, a) = a$ for all $a \in A$. Thus, to say that $\cdot$ is an idempotent operation is to say that $a \cdot a = a$, for all $a \in P$. Indeed, $a \cdot a = \text{glb}\{a, a\} = a$.

19.3. Draw a diagram of the lattice of subgroups of $\mathbb{Z}_{12}$.

Solution: How one determines that the diagram shown below represents the lattice of subgroups of $\mathbb{Z}_{12}$ was explained in lecture. If it is not clear to you, please attend office hours, or post a question on the class wiki, or email the instructor, or ask about it in lecture.



19.14. Let G be a group and X be the set of subgroups of G ordered by set-theoretic inclusion. If H and K are subgroups of G, show that the least upper bound of H and K is the subgroup generated by $H \cup K$.

Solution: This question is about the poset $\langle \mathrm{Sub}(G), \leq\rangle$, where the universe is the set $\mathrm{Sub}(G)$ of all subgroups of G, and the partial order is the relation $\leq$ defined as follows:

$$H \leq K \qquad \Longleftrightarrow \qquad H \text{ is a subgroup of } K.$$

Let X denote the least upper bound of H and K in $\langle \mathrm{Sub}(G), \leq\rangle$. Then, since X belongs to $\mathrm{Sub}(G)$, it is a subgroup of G, and since X is an upper bound of H and of K, we have $H \leq X$ and $K \leq X$. Therefore, all elements of $H \cup K$ are contained in X. Let Y be any other subgroup of G that contains $H \cup K$, then $H \leq Y$ and $K \leq Y$. Since X is the *least* upper bound of H and K, we have $X \leq Y$. What we have shown is that X is the smallest subgroup of G that contains the set $H \cup K$. This is the definition of *the subgroup of G generated by $H \cup K$*. $\square$

19.20. Let X and Y be posets. A map $\phi : X \to Y$ is *order-preserving* if $a \preceq b$ implies that $\phi(a) \preceq \phi(b)$. Let L and M be lattices. A map $\psi : L \to M$ is a *lattice homomorphism* if $\psi(a \vee b) = \psi(a) \vee \psi(b)$ and $\psi(a \wedge b) = \psi(a) \wedge \psi(b)$. Show that every lattice homomorphism is order-preserving, but that it is not the case that every order-preserving map is a lattice homomorphism.

Solution: Let $\psi : L \to M$ be a lattice homomorphism. We must show that ψ is order preserving. Let $a, b \in L$ be such that $a \leq b$. As we learned in lecture, if a and b have a greatest lower bound (as they must in a lattice), then the statement $a \leq b$ is equivalent to $a \wedge b = a$. Therefore,

$$\psi(a) = \psi(a \wedge b) = \psi(a) \wedge \psi(b).$$

(The second equality holds since ψ is a lattice homomorphism.) As mentioned, the equation $\psi(a) = \psi(a) \wedge \psi(b)$ is equivalent to $\psi(a) \leq \psi(b)$. We have thus proved that if ψ is a lattice homomorphism and if $a \leq b$, then $\psi(a) \leq \psi(b)$. That is, lattice homomorphisms are order preserving.

The converse is false. That is, an order preserving map need not be a lattice homomorphism. For example, consider the two lattice diagrams in the figure below. The map

$$\phi : \{0, 1, 2, 3\} \to \{a, b, c\} \quad \text{defined by}$$

$$\phi(0) = a, \quad \phi(2) = b = \phi(3), \quad \phi(1) = c$$

is clearly order preserving, but it is not a lattice homomorphism. For example,

$$a = \phi(0) = \phi(2 \wedge 3) \neq \phi(2) \wedge \phi(3) = b \wedge b = b.$$