1.

2. Write the following permutations in cycle notation.

(a)
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \end{pmatrix}$$

(b)
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 1 & 3 \end{pmatrix}$$

(c)
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 4 & 2 \end{pmatrix}$$

(d)
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 2 & 5 \end{pmatrix}$$

3. Compute each of the following.

(a) $(1345)(234)$
(b) $(12)(1253)$
(c) $(143)(23)(24)$
(d) $(1423)(34)(56)(1324)$
(e) $(1254)(13)(25)$
(f) $(1254)(13)(25)^2$
(g) $(1254)^{-1}(123)(45)(1254)$
(h) $(1254)^2(123)(45)$
(i) $(123)(45)(1254)^{-2}$
(j) $(1254)^{100}$
(k) $|(1254)|$
(l) $|(1254)^2|$
(m) $(12)^{-1}$
(n) $(12537)^{-1}$
(o) $[(12)(34)(12)(47)]^{-1}$
(p) $[(1235)(467)]^{-1}$

4. Express the following permutations as products of transpositions and identify them as even or odd.

(a) (14356)
(b) $(156)(234)$
(c) $(1426)(142)$
(d) $(17254)(1423)(154632)$
(e) (142637)

5. Find $(a_1, a_2, \dots, a_n)^{-1}$.

6. List all of the subgroups of S_4. Find each of the following sets.
(a) $\{\sigma \in S_4 : \sigma(1) = 3\}$
(b) $\{\sigma \in S_4 : \sigma(2) = 2\}$
(c) $\{\sigma \in S_4 : \sigma(1) = 3 \text{ and } \sigma(2) = 2\}$
Are any of these sets subgroups of S_4?

7. Find all of the subgroups in A_4. What is the order of each subgroup?

8. Find all possible orders of elements in S_7 and A_7.

9. Show that A_{10} contains an element of order 15.

10. Does A_8 contain an element of order 26?

11. Find an element of largest order in S_n for $n = 3, \dots, 10$.

12. What are the possible cycle structures of elements of A_5? What about A_6?

13. Let $\sigma \in S_n$ have order n. Show that for all integers i and j, $\sigma^i = \sigma^j$ if and only if $i \equiv j \pmod{n}$.

14. Let $\sigma = \sigma_1 \cdots \sigma_m \in S_n$ be the product of disjoint cycles. Prove that the order of σ is the least common multiple of the lengths of the cycles $\sigma_1, \dots, \sigma_m$.

15. Using cycle notation, list the elements in D_5. What are r and s? Write every element as a product of r and s.

16. If the diagonals of a cube are labeled as Figure **??**, to which motion of the cube does the permutation $(12)(34)$ correspond? What about the other permutations of the diagonals?

17. Find the group of rigid motions of a tetrahedron. Show that this is the same group as A_4.

18. Prove that S_n is nonabelian for $n \geq 3$.

19. Show that A_n is nonabelian for $n \geq 4$.

20. Prove that D_n is nonabelian for $n \geq 3$.

21. Let $\sigma \in S_n$. Prove that σ can be written as the product of at most $n-1$ transpositions.

22. Let $\sigma \in S_n$. If σ is not a cycle, prove that σ can be written as the product of at most $n-2$ transpositions.

23. If σ can be expressed as an odd number of transpositions, show that any other product of transpositions equaling σ must also be odd.

24. If σ is a cycle of odd length, prove that σ^2 is also a cycle.

25. Show that a 3-cycle is an even permutation.

26. Prove that in A_n with $n \geq 3$, any permutation is a product of cycles of length 3.

27. Prove that any element in S_n can be written as a finite product of the following permutations.

(a) $(12), (13), \ldots, (1n)$

(b) $(12), (23), \ldots, (n-1, n)$

(c) $(12), (12 \ldots n)$

28. Let G be a group and define a map $\lambda_g : G \to G$ by $\lambda_g(a) = ga$. Prove that λ_g is a permutation of G.

29. Prove that there exist $n!$ permutations of a set containing n elements.

30. Recall that the *center* of a group G is

$$Z(G) = \{g \in G : gx = xg \text{ for all } x \in G\}.$$

Find the center of D_8. What about the center of D_{10}? What is the center of D_n?

31. Let $\tau = (a_1, a_2, \ldots, a_k)$ be a cycle of length k.

(a) Prove that if σ is any permutation, then

$$\sigma\tau\sigma^{-1} = (\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_k))$$

is a cycle of length k.

(b) Let μ be a cycle of length k. Prove that there is a permutation σ such that $\sigma\tau\sigma^{-1} = \mu$.

32. For α and β in S_n, define $\alpha \sim \beta$ if there exists an $\sigma \in S_n$ such that $\sigma\alpha\sigma^{-1} = \beta$. Show that $\sim$ is an equivalence relation on S_n.

33. Let $\sigma \in S_X$. If $\sigma^n(x) = y$, we will say that $x \sim y$.

(a) Show that $\sim$ is an equivalence relation on X.

(b) If $\sigma \in A_n$ and $\tau \in S_n$, show that $\tau^{-1}\sigma\tau \in A_n$.

(c) Define the *orbit* of $x \in X$ under $\sigma \in S_X$ to be the set

$$\mathcal{O}_{x,\sigma} = \{y : x \sim y\}.$$

Compute the orbits of α, β, γ where

$$\begin{aligned} \alpha &= (1254) \\ \beta &= (123)(45) \\ \gamma &= (13)(25). \end{aligned}$$

(d) If $\mathcal{O}_{x,\sigma} \cap \mathcal{O}_{y,\sigma} \neq \varnothing$, prove that $\mathcal{O}_{x,\sigma} = \mathcal{O}_{y,\sigma}$. The orbits under a permutation σ are the equivalence classes corresponding to the equivalence relation $\sim$.

(e) A subgroup H of S_X is *transitive* if for every $x, y \in X$, there exists a $\sigma \in H$ such that $\sigma(x) = y$. Prove that $\langle \sigma \rangle$ is transitive if and only if $\mathcal{O}_{x,\sigma} = X$ for some $x \in X$.

34. Let $\alpha \in S_n$ for $n \geq 3$. If $\alpha\beta = \beta\alpha$ for all $\beta \in S_n$, prove that α must be the identity permutation; hence, the center of S_n is the trivial subgroup.

35. If α is even, prove that α^{-1} is also even. Does a corresponding result hold if α is odd?

36. Show that $\alpha^{-1}\beta^{-1}\alpha\beta$ is even for $\alpha, \beta \in S_n$.

37. Let r and s be the elements in D_n described in Theorem **??**.

(a) Show that $srs = r^{-1}$.

(b) Show that $r^k s = sr^{-k}$ in D_n.

(c) Prove that the order of $r^k \in D_n$ is $n/\gcd(k, n)$.