IsomorphismTheorems

The First Isomorphism Theorem

Theorem 1. Let G and H be groups and let $\varphi : G \to H$ be a group homomorphism, that is, $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in G$. Then the kernel subgroup $K := \{x \in G : \varphi(x) = e_H\}$ is a normal subgroup of G and the factor group G/K is isomorphic to the subgroup $\varphi(G) = \{\varphi(x) : x \in G\} \leq H$.

Note to students: you must know Theorem 1 *and* its proof.

Proof: There are just two steps:

1. show K is a normal subgroup of G, so that G/K is a group;
2. construct a group isomorphism $\psi : G/K \to \varphi(G)$.

Step 1. Show K is normal in G.

Let $k \in K$ and $x \in G$. We want to show $xkx^{-1} \in K$. Indeed, $\varphi(xkx^{-1}) = \varphi(x)\varphi(k)\varphi(x^{-1}) = \cdots$

Exercise 1: Complete this part of the argument by verifying that $\varphi(xkx^{-1}) = e_H$.

Step 2. Show that G/K is isomorphic to $\varphi(G)$.

Define a function $\psi : G/K \to H$ as follows: for each coset $xK \in G/K$, let $\psi(xK) = \varphi(x)$.

Exercise 2:

a. Show that ψ is well defined by checking that if $xK = yK$, then $\psi(xK) = \psi(yK)$.
b. Show that ψ is a *monomorphism* (i.e., a one-to-one homomorphism). (You must verify two properties: one-to-one and homomorphism.)
c. Show that ψ maps G/K onto $\varphi(G)$.
d. Complete the proof by drawing an appropriate conclusion about the groups G/K and $\varphi(G)$. Cite the properties you proved in parts a--c.

Corollary: Suppose $\varphi : G \to H$ is a group homomorphism and K is the kernel subgroup as above. Then,

1. φ is one-to-one if and only if $K = \{e_G\}$.
2. $|G : K| = |\varphi(G)|$

Exercise 3: Prove the corollary.

Important: Item 1 in the above corollary describes the standard method for proving a group G is isomorphic to a group H:

To show $G \cong H$, find a homomorphism $\varphi : G \to H$ and check that the kernel subgroup is trivial: $N_\varphi = \{x \in G : \varphi(x) = e_H\} = \{e_G\}$.

The Canonical Epimorphism

Let G be a group and N a normal subgroup of G. Define the *canonical epimorphism* $\pi : G \to G/N$ as follows: for each $x \in G$, let $\pi(x) = xN$.

Exercise 4: Check that the function π so defined is a homomorphism, verify that it is onto (so it's an epimorphism), and show that π is one-to-one if and only if $N = \{e_G\}$.

Inner Automorphisms

To see the First Isomorphism Theorem in action, let's work through an example in which the codomain (H in the theorem above) is *the automorphism group of* G. So first we review the concept of automorphism group.

Recall that an *endomorphism* $\varphi : G \to G$ is a homomorphism from a group to itself. An *automorphism* is an isomorphism from a group onto itself. So, an automorphism is an endomorphism that is both one-to-one and onto.

Since automorphisms are one-to-one and onto, they are simply permutations of the elements of the group. Therefore, each automorphism has an inverse and, in fact, the set of all automorphisms of a group G is itself a group, where the binary operation is function composition. We let $\mathrm{Aut}(G)$ denote *the group of automorphisms* of G.

Example: For a fixed element $g \in G$, consider the function $\varphi_g : G \to G$ that takes each element x to its *conjugate* gxg^{-1}. That is, for each $x \in G$, let $\varphi_g(x) = gxg^{-1}$.

Exercise 4: Prove that the function φ_g (conjugation by g) is an automorphism of G.

(Hint: Check that $\varphi_g(xy) = \varphi_g(x)\varphi_g(y)$ and check that $\varphi_g : G \to G$ is one-to-one and onto.)

Now consider the function $\varphi : G \to \mathrm{Aut}(G)$ that takes each $g \in G$ to the automorphism φ_g. That is, for each $g \in G$, we let $\varphi(g) = \varphi_g$.

So, for each $g \in G$, for each $x \in G$, we have $\varphi(g)(x) = \varphi_g(x) = gxg^{-1}$.

Exercise 5: Prove that the map $\varphi : G \to \mathrm{Aut}(G)$ a homomorphism with kernel subgroup equal to the center of G, that is $K = \{g \in G : gx = xg \text{ for all } x \in G\}$.

(Hint: Check that $\varphi_{g_1 \cdot g_2} = \varphi_{g_1} \circ \varphi_{g_2}$, where $\cdot$ is multiplication in G and $\circ$ is function composition. Then check that φ_g is the identity map--i.e., the identity element of $\mathrm{Aut}(G)$--if and only if g belongs to the center of G.)

Exercise 6: Let $Z(G)$ denote the center of the group G. Use Exercise 5 and the First Isomorphism Theorem to conclude that $G/Z(G)$ is isomorphic to a subgroup of the automorphism group. What is the order of this subgroup (in terms of $|G|$ and $|Z(G)|$)?

Example: Let $G = D_4$, the dihedral group on four letters (symmetries of the square). Recall, D_4 can be generated by two elements, a rotation $\rho = (1234)$ and a reflection $\mu = (14)(23)$.

```
D4=DihedralGroup(4)
rho, mu = D4.gens()
print rho, mu
```

```
(1,2,3,4) (1,4)(2,3)
```

The center of D_4 is found in Sage as follows:

```
Z = D4.center()
list(Z)
```

```
[(), (1,3)(2,4)]
```

Note that $\rho^2 = (13)(24)$.

```
rho*rho
```

```
(1,3)(2,4)
```

So, $Z = \{e_G, (13)(24)\} = \{e_G, \rho^2\}$.

If $\varphi : D_4 \to \mathrm{Aut}(D_4)$ is the homomorphism described above, taking each $g \in D_4$ to the automorphism $\varphi_g : x \mapsto gxg^{-1}$, then the kernel is

$$Z = \{g \in D_4 : \varphi_g = \mathrm{id}_{D_4}\} = \{g \in D_4 : gxg^{-1} = x \text{ for all } x \in D_4\}$$

which is the center of D_4. So, the First Isomorphism Theorem tells us that D_4/Z is ismorphic to a subgroup of $\mathrm{Aut}(D_4)$.

We won't define the function φ in Sage, since this requires a bit more programming than we want to cover right now. However, we can easily compute the factor group D_4/Z.

```
D4modZ = D4.quotient(Z)
list(D4modZ)
```

```
[(), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)]
```

The Second Isomorphism Theorem

Theorem 2. Let H and N be subgroups of G and suppose that N is normal in G. Then

1. HN is a subgroup of G
2. $H \cap N$ is a normal subgroup of H
3. $HN/N \cong H/H \cap N$.

Exercise 7: Use the First Isomorphism to prove the Second Isomorphism Theorem.

The Correspondence Theorem

Theorem 3: Let N be a normal subgroup of G. Then the map $H \mapsto H/N$ is a one-to-one correspondence between the set of subgroups of G that contain N and the set of subgroups of G/N. That is,

$$\mathrm{Sub}(G) \ \ni \ H \ \longleftrightarrow \ H/N \ \in \ \mathrm{Sub}(G/N).$$

Moreover, the normal subgroups of G containing N correspond to normal subgroups of G/N.

The Third Isomorphism Theorem

Theorem 4: Let $N \leq H \leq G$ be a chain of groups and suppose N and H are both normal in G. Then,

$$G/H \cong \frac{G/N}{H/N}\,.$$

Subgroup lattices

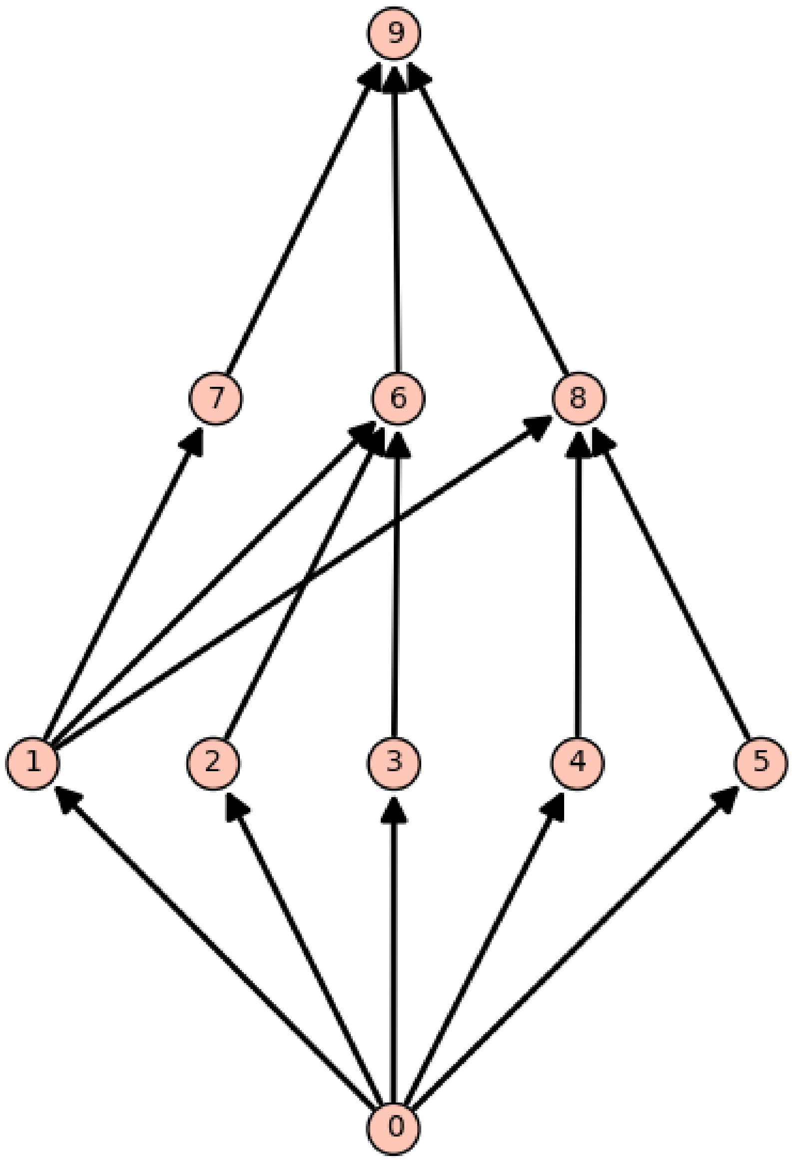
In this section, we will continue with the dihedral group example and draw some subgroup lattices to see the Correspondence Theorem in action.

The following is a utility function that will help us draw subgroup lattices. You don't have to understand it.

```
def draw_interval(G,H):
    interval_rec = G.IntermediateSubgroups(H)
    intSubgroups = gap.get_record_element(interval_rec,
'subgroups')
    intCoverings = gap.get_record_element(interval_rec,
'inclusions')
    minx = 0  # minx = min(min(intCoverings)) is always 0
    maxx = max(max(intCoverings))
    Poset([[minx..maxx],intCoverings.AsList().sage()]).show()
```

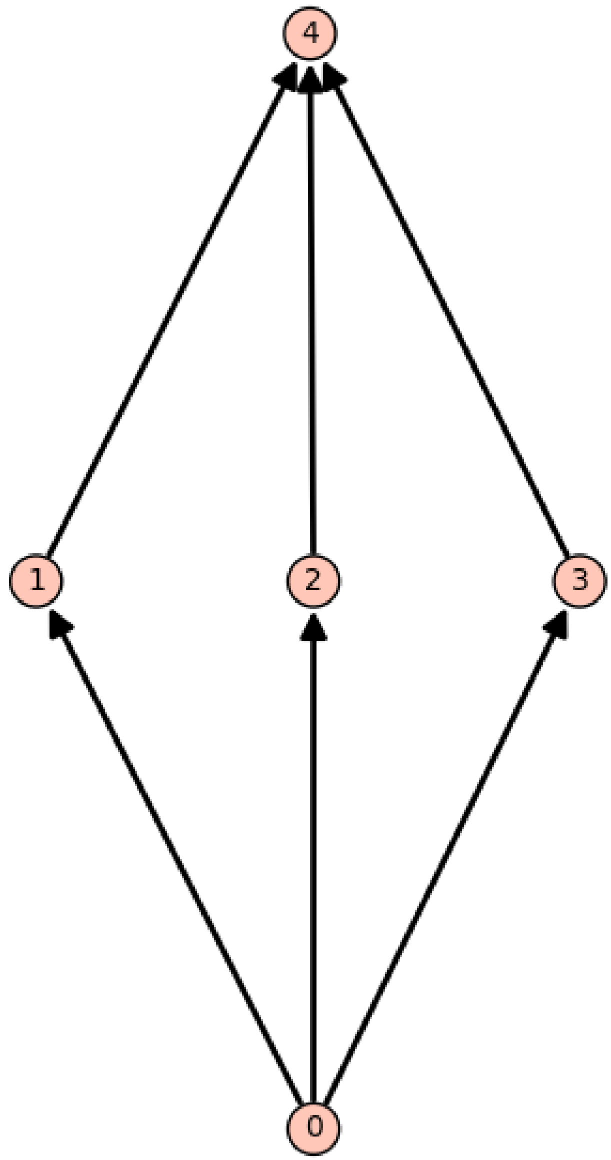
We first draw the subgroup lattice of D_4.

```
D4=gap.DihedralGroup(8);
IdD4=gap.Group([D4.Identity()]);
draw_interval(D4,IdD4);
```



For ease of notation, let us denote the center of D_4 by C. The center C is a subgroup of D_4 of order 2, which happens to be labeled 1 in the Hasse diagram above. Therefore, if we draw the subgroup lattice of the factor group D_4/C, then the Correspondence Theorem tells us what the resulting diagram should look like. It should be the same as the interval above 1 in the subgroup lattice of D_4. Let's check this.

```
D4=gap.DihedralGroup(8);
C=gap.Center(D4);
draw_interval(D4,C);
```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