Exercises: Judson 10.1abe, 10.5, 10.10, 10.11, 10.13acd, and Problem 6 below.
Due date: Wednesday, 11/05

10.1 For each of the following groups G, determine whether H is a normal subgroup of G. If H is a normal subgroup, write out a Cayley table for the factor group G/H.

(a) $G = S_4$ and $H = A_4$

(b) $G = A_5$ and $H = \{(1), (123), (132)\}$

(e) $G = \mathbb{Z}$ and $H = 5\mathbb{Z}$

Solution:

(a) The subgroup $H = A_4$ has index $[S_4 : A_4] = 2$. Therefore, by Exercise 10.10 (below), A_4 is normal in S_4. The elements of the factor group, that is, the cosets of A_4 in S_4, are $\{A_4, gA_4\}$, where g is any element of S_4 that is not contained in A_4. For example, $g = (23)$ works.

Recall that, for a normal subgroup $N \triangleleft G$, coset multiplication is defined by $g_1N * g_2N = (g_1 \cdot g_2)N$, where $g_1 \cdot g_2$ is the product in G. So one acceptable representation of the Cayley table of S_4/A_4 is

$*$	A_4	$(23)A_4$
A_4	A_4	$(23)A_4$
$(23)A_4$	$(23)A_4$	A_4

An acceptable alternative is

$*$	A_4	$(12)A_4$
A_4	A_4	$(12)A_4$
$(12)A_4$	$(12)A_4$	A_4

or even

$*$	$(123)A_4$	$(34)A_4$
$(123)A_4$	$(123)A_4$	$(34)A_4$
$(34)A_4$	$(34)A_4$	$(123)A_4$

or any other table involving two cosets g_0A_4 and g_1A_4, where $g_0 \in A_4$, so $g_0A_4 = A_4$ is the identity element, and $g \in S_4 - A_4$, so $g_1A_4 \neq A_4$ is the nonidentity element.

Note that $(123) \in A_4$ since it can be written as $(123) = (13)(12)$, which is a product of an *even* number of transpositions. Therefore, $(123)A_4 = A_4$. For this reason, we could have use $(123)A_4$ to represent the identity element of the factor group. Of course, we cannot use $(123)A_4$ as the nonidentity element, so the following table would be incorrect:

$*$	A_4	$(123)A_4$
A_4	A_4	$(123)A_4$
$(123)A_4$	$(123)A_4$	A_4

$\longleftarrow$ (not the Cayley table of S_4/A_4)

(b) The subgroup $H = \{(1), (123), (132)\}$ is not normal in $G = A_5$, as we will show using **a standard way to prove a subgroup H is not normal in G:**

find elements $g \in G$ and $h \in H$ such that $ghg^{-1} \notin H$.

In the present example, if we let $g = (234) \in A_5$ and $h = (123) \in H$, then

$$ghg^{-1} = (234)(123)(243) = (234)(124) = (134) \notin H.$$

(e) Certainly $H = 5\mathbb{Z}$ is normal in $G = \mathbb{Z}$, since G is abelian (so every subgroup of G is normal). The elements of the factor group are the cosets of $5\mathbb{Z}$ in $\mathbb{Z}$, and the Cayley table can be presented as follows:

$+$	$5\mathbb{Z}$	$1+5\mathbb{Z}$	$2+5\mathbb{Z}$	$3+5\mathbb{Z}$	$4+5\mathbb{Z}$
$5\mathbb{Z}$	$5\mathbb{Z}$	$1+5\mathbb{Z}$	$2+5\mathbb{Z}$	$3+5\mathbb{Z}$	$4+5\mathbb{Z}$
$1+5\mathbb{Z}$	$1+5\mathbb{Z}$	$2+5\mathbb{Z}$	$3+5\mathbb{Z}$	$4+5\mathbb{Z}$	$5\mathbb{Z}$
$2+5\mathbb{Z}$	$2+5\mathbb{Z}$	$3+5\mathbb{Z}$	$4+5\mathbb{Z}$	$5\mathbb{Z}$	$1+5\mathbb{Z}$
$3+5\mathbb{Z}$	$3+5\mathbb{Z}$	$4+5\mathbb{Z}$	$5\mathbb{Z}$	$1+5\mathbb{Z}$	$2+5\mathbb{Z}$
$4+5\mathbb{Z}$	$4+5\mathbb{Z}$	$5\mathbb{Z}$	$1+5\mathbb{Z}$	$2+5\mathbb{Z}$	$3+5\mathbb{Z}$

It is also acceptable to use the shorthand $[k]$ or (k) for the coset of $5\mathbb{Z}$ containing k, in which case, the Cayley table could be presented as follows:

$+$	[0]	[1]	[2]	[3]	[4]
[0]	[0]	[1]	[2]	[3]	[4]
[1]	[1]	[2]	[3]	[4]	[0]
[2]	[2]	[3]	[4]	[0]	[1]
[3]	[3]	[4]	[0]	[1]	[2]
[4]	[4]	[0]	[1]	[2]	[3]

which looks an awful lot like the group of integers with addition modulo 5 that we encountered earlier, and called $\mathbb{Z}_5$. In fact, the group $\mathbb{Z}_5$, whose universe is the set of integers $\{0, 1, 2, 3, 4\}$ and whose binary operation is addition modulo 5 is isomorphic to the group $\mathbb{Z}/5\mathbb{Z}$. While the elements of $\mathbb{Z}/5\mathbb{Z}$ are infinite sets of integers, the elements of $\mathbb{Z}_5$ are just the five integers $\{0, 1, 2, 3, 4\}$. Apart from this distinction, the group structure is the same in each case, as we can see from the Cayley tables.

10.5. Show that the intersection of two normal subgroups is a normal subgroup.

Solution: Let H and K be normal subgroups of a group G. We have proved in the past that the intersection $N = H \cap K$ of two subgroups is a subgroup. We will now prove that N is normal using

a standard way to prove a subgroup N is normal in G:

Pick arbitrary elements $g \in G$ and $n \in N$ and show that $gng^{-1} \in N$.

Fix $g \in G$ and $n \in N = H \cap K$. Since $n \in H$ and $H \triangleleft G$, we have $gng^{-1} \in H$. Since $n \in K$ and $K \triangleleft G$, we have $gng^{-1} \in K$. Therefore, $gng^{-1} \in H \cap K = N$. $\square$

10.10. Let H be a subgroup of index 2 of a group G. Prove that H must be a normal subgroup of G. Conclude that S_n is not simple for $n \geq 3$.

Solution: We will show that $[G : H] = 2$ implies $H \triangleleft G$ using

another standard way to prove a subgroup H is normal in G:

Pick an arbitrary element $g \in G$ and show that $gH = Hg$.

If $[G : H] = 2$, then there are two left cosets of H in G. Fix $g \in G$. If $g \in H$, then $gH = Hg$ and there is nothing to prove. Assume $g \notin H$. Then the two left cosets of H in G, are H and gH. Recall that a full set of left cosets partitions the group as a *disjoint* union $G = H \cup gH$. Similarly, the two right cosets of H in G must be H and Hg, and again we have a partition of G as a into *disjoint* union of sets $G = H \cup Hg$. It follows that $gH = G - H = Hg$. $\square$

10.11. If a group G has exactly one subgroup H of order k, prove that H is normal in G.

Solution: We will solve this using

another standard way to prove a subgroup H is normal in G:

Pick an arbitrary element $g \in G$ and show that $gHg^{-1} = H$.

First, given a subgroup $H \leq G$, and an arbitrary element $g \in G$, it is not hard to see that the *conjugate of H by g*, which is defined by

$$gHg^{-1} := \{ghg^{-1} | h \in H\},$$

is also a subgroup of G. Moreover, the function $h \mapsto ghg^{-1}$ is a bijection.[1] Therefore, $|H| = |gHg^{-1}|$. If $|H| = k$ and if H is the only subgroup of G of order k, then, since $|gHg^{-1}| = k$, we must have $H = gHg^{-1}$. Since g was arbitrary, this proves that H is normal in G. $\square$

10.13. Recall that the **center** of a group G is the set

$$Z(G) = \{x \in G : xg = gx \text{ for all } g \in G\ \}.$$

(a) Calculate the center of S_3.

(c) Show that the center of any group G is a normal subgroup of G.

(d) If $G/Z(G)$ is cyclic, show that G is abelian.[2]

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[1]In fact, as we will see later, $x \mapsto gxg^{-1}$ is an automorphism.

[2]Hint: Let $Z := Z(G)$. If G/Z is cyclic then there exists $x \in G$ such that for each $a \in G$ there exists $m \in \mathbb{N}$ such that $aZ = x^m Z$. Fix $a, b \in G$ and show $ab = ba$ using the fact that $aZ = x^m Z$ and $bZ = x^n Z$ for some m and n.

Solution:

(a) Recall the elements of S_3 are $\{(\ ), (12), (13), (23), (123), (132)\}$. The center of a group consists of those elements that commute with every other element in the group. In case $G = S_3$, the center is trivial. That is, the only element of S_3 that commutes with every other element is the identity permutation $(\ )$. This can be proved by direct calculation. For each element $x \in S_3$, there is at least one $y \in S_3$ such that $xy \neq yx$:

$$(12)(13) = (132) \neq (123) = (13)(12)$$

$$(23)(123) = (13) \neq (12) = (123)(23)$$

$$(132)(13) = (12) \neq (23) = (13)(132)$$

With the exception of the identity $(\ )$, every element of S_3 appears at least one of the noncommuting products above. Therefore, $Z(S_3) = \{(\ )\}$.

(c) *Claim:* The center of any group is normal: $Z(G) \triangleleft G$.

Proof 1: We use the standard method mentioned above in Problem 10.5. That is, we fix arbitrary $z \in Z(G)$ and $a \in G$, and show that $aza^{-1} \in Z(G)$. Indeed, since z belongs to $Z(G)$, it commutes with every element of G. Therefore, $aza^{-1} = aa^{-1}z = ez = z \in Z(G)$. □

Although the proof above is nice and short, there is another approach to this problem that is worth considering because it involves an automorphism, called an inner automorphism, that will come up again and again. An *inner automorphism* is an automorphism $\varphi_a : G \to G$ of the form $\varphi_a(x) = axa^{-1}$. The alternative proof is also worth studying because it employs an important idea that we will use again in Problem 6 below when we prove *Cayley's Representation Theorem*, which says that every group G is isomorphic to a subgroup of the permutation group $\mathrm{Sym}(G)$.

Proof 2: For $a \in G$, consider the function $\varphi_a : G \to G$ defined by $\varphi_a(g) = aga^{-1}$. It is not hard to show that the function $\varphi : G \to \mathrm{Sym}(G)$ defined by $\varphi(a) = \varphi_a$—that is, the function sending each $a \in G$ to the permutation $\varphi_a \in \mathrm{Sym}(G)$—is a group homomorphism. Moreover, the kernel of φ is

$$\begin{aligned}
\ker \varphi &= \{(a, b) \in G^2 : \varphi(a) = \varphi(b)\} = \{(a, b) \in G^2 : \varphi_a = \varphi_b\} \\
&= \{(a, b) \in G^2 : \varphi_a(g) = \varphi_b(g) \text{ for all } g \in G\} \\
&= \{(a, b) \in G^2 : aga^{-1} = bgb^{-1} \text{ for all } g \in G\}.
\end{aligned}$$

The equivalence class of $\ker\varphi$ that contains the identity element of G (what the book calls the "kernel" of φ) is the set

$$\begin{aligned} N_\varphi &= \{a \in G : \varphi_a = \varphi_e\} \\ &= \{a \in G : aga^{-1} = g \text{ for all } g \in G\} \\ &= \{a \in G : ag = ga \text{ for all } g \in G\} \\ &= Z(G). \end{aligned}$$

Finally, recall the following **important fact:**[3]

For any group homomorphism $\varphi : G \to H$, the subset

$$N_\varphi = \{g \in G : \varphi(g) = e_H\}$$

of elements mapped by φ to the identity of H is a normal subgroup of G.

By taking φ to be the conjugation homomorphism as above, we have $Z(G) = N_\varphi$. Therefore, $Z(G) \lhd G$. □

Remark: At this point, Proof 2 might seem harder and more complicated than Proof 1. However, once you become more comfortable with such arguments you may find that the easiest and quickest way to see that a certain subgroup is normal is to simply note that it is (the identity class of) the kernel of a homomorphism.

(d) *Claim:* If $G/Z(G)$ is cyclic, then G is abelian.

Proof. For ease of notation, let $Z := Z(G)$. Assume G/Z is cyclic. Then there exists $x \in G$ such that for each $g \in G$ we have $gZ = x^m Z$, for some $m \in \mathbb{N}$. Fix $a, b \in G$. We will show $ab = ba$. Let m and n be such that $aZ = x^m Z$ and $bZ = x^n Z$. Then $a = x^m z_1$ for some $z_1 \in Z$ and $b = x^n z_2$ for some $z_2 \in Z$. Therefore,

$$\begin{aligned} ab &= x^m z_1 x^n z_2 \\ &= x^m x^n z_2 z_1 && \text{(since } z_1 \in Z\text{)} \\ &= x^{m+n} z_2 z_1 \\ &= x^{n+m} z_2 z_1 \\ &= x^n x^m z_2 z_1 \\ &= x^n z_2 x^m z_1 && \text{(since } z_2 \in Z\text{)} \\ &= ba. \end{aligned}$$

□

[3] We proved this in class. It is also proved in the book.

Problem 6. Let $\mathbf{G} = \langle G, \cdot, ^{-1}, e\rangle$ be a finite group of order n. Take the set G (the elements of $\mathbf{G}$) and consider the group of all permutations of these elements. This group is sometimes denoted by $\mathrm{Sym}(G)$; note that it is isomorphic to the symmetric group S_n of permutations of an n-element set. Now fix an element $a \in G$ and recall that the function $\lambda_a : G \to G$, defined by $\lambda_a(g) = a \cdot g$, is a permutation of the set G. That is, λ_a belongs to the permutation group $\mathrm{Sym}(G)$.

(a) Prove that the function $\lambda : G \to \mathrm{Sym}(G)$ is a group homomorphism.

(b) What is the kernel of λ?[4]

(c) Let N denote the equivalence class of $\ker \lambda$ that contains the identity element e of G. Prove that N is a normal subgroup of G.

[4]Recall that the kernel of a function $f : X \to Y$ is the subset of $X \times X$ defined by

$$\ker f = \{(x_1, x_2) : f(x_1) = f(x_2)\}.$$

As you have already proved, the kernel is an equivalence relation on X.