MATH 301 - Section B **Fall 2014**

Midterm Exam 2 – SOLUTIONS

1. Give precise definitions of the following:

(a) n-**ary operation** on a set A

An n-**ary operation** on a set A is a function, $f : A^n \to A$, with domain A^n and codomain A.

(b) n-**ary relation** on a set A

An n-**ary relation** on a set A is a subset of A^n.

(c) **algebra** or **algebraic structure**

An **algebra** or **algebraic structure** is a pair $\langle A, F\rangle$ consisting of a nonempty set A along with a set F of operations on A.

(d) **relational structure**

An **relational structure** is a pair $\langle A, R\rangle$ consisting of a nonempty set A along with a set R of relations on A.

(e) **group homomorphism**

A **group homomorphism** is a function $\varphi : G \to H$ from a group G to a group H satisfying, for all $x, y \in G$, $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$.

(f) **normal subgroup**

A **normal subgroup** of a group G is a subgroup $N \leq G$ such that any one (hence all) of the following equivalent conditions holds for all $g \in G$:

i. $gNg^{-1} = N$;
ii. $gN = Ng$;
iii. $gng^{-1} \in N$, for all $n \in N$.

2. (a) State *Lagrange's Theorem* about the order of a group and its subgroups. (Be sure to state all assumptions that are needed in order for the theorem to hold.)

Theorem. Let G be a finite group and let H be a subgroup of G. Then $|G|/|H| = [G : H]$ is the number of distinct left cosets of H in G. In particular, the number of elements in H must divide the number of elements in G.

(b) Recall that G is the *internal direct product* of the subgroups H and K if $G = HK$, and $H \cap K = \{e\}$, and H and K centralize each other (i.e., $hk = kh$ for all $h \in H$ and $k \in K$).[1]

Suppose G has order 28 and subgroups H and K of orders 4 and 7 respectively which centralize each other. Prove that G is the internal direct product of H and K.

Solution: Since we are already given that $hk = kh$ for all $h \in H$ and $k \in K$, we must show that $H \cap K = \{e\}$ and $G = HK$.[1]

Suppose $x \in H \cap K$, then $x \in H$ implies $\langle x \rangle$ is a subgroup of H so, by Lagrange's Theorem $|x|$ divides $|H| = 4$. Similarly $x \in K$ implies $\langle x \rangle$ is a subgroup of K so, by Lagrange's Theorem $|x|$ divides $|K| = 7$. Since $|x|$ divides both 4 and 7, we must have $|x| = 1$, so $x = e$. That is $H \cap K = \{e\}$.

Finally, we show that $G = HK$.[1] Since $[G : K] = |G|/|K| = 28/7 = 4$, there are four cosets of K in G and, since $H \cap K = \{e\}$, we can list these four cosets using as representatives the four distinct elements of H. That is, the cosets of K in G are K, h_1K, h_2K, h_3K. Since any group is the disjoint union of the cosets of any of its subgroups, we have $G = K \cup h_1K \cup h_2K \cup h_3K = HK$.

[1]Note to students: On the actual exam, I left out the condition that $G = HK$. (Points were not deducted from answers that didn't check this condition.)

3. Prove either (a) OR (b) OR (c). If you prove more than one, circle the letter of the one you want graded.

(a) If $G \cong H$ and G is cyclic, then H is cyclic.

(b) If a group G has a subgroup H of index 2, then H is normal in G. Conclude that $A_n \triangleleft S_n$ for $n \geq 3$.

(c) If a group G has exactly one subgroup H of order k, then H is normal in G.

Solution:

(a) **Claim:** If $G \cong H$ and G is cyclic, then H is cyclic.

Proof: Suppose $G = \langle a \rangle \cong H$ and suppose $\varphi : G \to H$ is an isomorphism. Then $H = \langle \varphi(a) \rangle$. To see this, fix $h \in H$. We will show $h = (\varphi(a))^k$ for some $k \in \mathbb{N}$. Indeed, let $b = \varphi^{-1}(h)$. Since $b \in G = \langle a \rangle$, we must have $b = a^k$ for some $k \in \mathbb{N}$. Also, $\varphi(a^k) = (\varphi(a))^k$, since φ is a homomorphism. Therefore, $(\varphi(a))^k = \varphi(a^k) = \varphi(b) = \varphi(\varphi^{-1}(h)) = h$.

(b) **Claim:** If a group G has a subgroup H of index 2, then H is normal in G. Conclude that $A_n \triangleleft S_n$ for $n \geq 3$.

Proof: If $[G : H] = 2$, then there are two left cosets of H in G. Pick a $g \in G$ with $g \notin H$. Then the two left cosets of H in G, are H and gH. Recall that the set of left cosets partitions the group into a *disjoint* union. In the present case, $G = H \cup gH$. Similarly, the two right cosets of H in G must be H and Hg, and again we have a partition of G as the *disjoint* union of sets $G = H \cup Hg$. It follows that $gH = G - H = Hg$.

Finally, since $[S_n : A_n] = 2$ for $n \geq 3$, we have $A_n \triangleleft S_n$.

(c) **Claim:** If a group G has exactly one subgroup H of order k, then H is normal in G.

Proof: Given a subgroup $H \leq G$, and an arbitrary element $g \in G$, the set $gHg^{-1} := \{ghg^{-1} | h \in H\}$ is also a subgroup of G. Moreover, the function $h \mapsto ghg^{-1}$ is a bijection. Therefore, $|H| = |gHg^{-1}|$. If $|H| = k$ and if H is the only subgroup of order k, then, since $|gHg^{-1}| = k$, we must have $H = gHg^{-1}$. Since g was arbitrary, this proves that H is normal in G.

4. The **center** of a group G is $Z(G) = \{x \in G : xg = gx \text{ for all } g \in G\}$.

(a) Show that the center of any group is a normal subgroup.

Solution: Fix arbitrary $z \in Z(G)$ and $a \in G$. Since z belongs to $Z(G)$, it commutes with every element of G. Therefore, $aza^{-1} = aa^{-1}z = ez = z \in Z(G)$, so $Z(G) \triangleleft G$.

(b) The dihedral group D_4 (symmetries of the square) can be described as the permutation group with two generators $\rho = (1234)$ and $\mu = (13)$ satisfying $\rho^4 = e = \mu^2$. Therefore, the elements of D_4 are $\{e, \rho, \rho^2, \rho^3, \mu, \rho\mu, \rho^2\mu, \rho^3\mu\}$.
Calculate $Z(D_4)$, the center of D_4. [Hint: only one nonidentity element of D_4 commutes with all other elements of D_4, and finding this element should not require too much calculation.]

Solution: For easy reference we make a table displaying the elements of the group.

e	ρ	ρ^2	ρ^3	μ	$\rho\mu$	$\rho^2\mu$	$\rho^3\mu$
( )	(1234)	(13)(24)	(1432)	(13)	(14)(23)	(24)	(12)(34)

Note that $\rho^2 = (13)(24) = (24)(13)$, and ρ^2 commutes with both generators. Indeed,

$$\rho^2\rho = \rho^3 = \rho\rho^2 \qquad \text{and} \qquad \rho^2\mu = (13)(24)(13) = (13)(13)(24) = \mu\rho^2.$$

Therefore, ρ^2 commutes with every element of D_4, so $\rho^2 \in Z(D_4)$.
From the following calculations, we see that none of the other nonidentity elements of D_4 belongs to the center:

$$\begin{aligned} \mu\rho &= (13)(1234) = (12)(34) \\ \rho\mu &= (1234)(13) = (14)(23) \quad \Rightarrow \quad \mu\rho \neq \rho\mu, \end{aligned}$$

so $\mu \notin Z(D_4)$ and $\rho \notin Z(D_4)$.

$$\begin{aligned} \rho^3\mu &= (1432)(13) = (12)(34) \\ \mu\rho^3 &= (13)(1432) = (14)(23) \quad \Rightarrow \quad \mu\rho^3 \neq \rho^3\mu, \end{aligned}$$

so $\rho^3 \notin Z(D_4)$. From this and the fact that ρ^2 commutes with everything, we also have $\rho^2\mu\rho = \mu\rho^2\rho = \mu\rho^3 \neq \rho^3\mu = \rho\rho^2\mu$, so $\rho^2\mu \notin Z(D_4)$.

$$\begin{aligned} \rho^3 \cdot \rho\mu &= \mu = (13) \qquad (\text{since } \rho^4 = (\,)) \\ \rho\mu \cdot \rho^3 &= (1234)(13)(1432) = (24) \quad \Rightarrow \quad \rho^3 \cdot \rho\mu \neq \rho\mu \cdot \rho^3, \end{aligned}$$

so $\rho\mu \notin Z(D_4)$. Finally,

$$\begin{aligned} \rho^3\mu \cdot \mu &= \rho^3 = (1432) \\ \mu \cdot \rho^3\mu &= (13)(12)(34) = (1234) \end{aligned}$$

so $\rho^3\mu \notin Z(D_4)$.

(c) Is $D_4/Z(D_4)$ cyclic? Explain. [Hint: Recall, G is abelian if $G/Z(G)$ is cyclic.]

Since D_4 is nonabelian, $D_4/Z(D_4)$ is not cyclic.

5. State the First Isomorphism Theorem and use it to prove that every group G is isomorphic to a subgroup of $\mathrm{Sym}(G)$. [Hint: show that a certain function $\lambda : G \to \mathrm{Sym}(G)$ is a homomorphism with trivial kernel.]

First Isomorphism Theorem.

Suppose $\varphi : G \to H$ is a group homomorphism and

$$N := \varphi^{-1}(\{e_H\}) = \{g \in G : \varphi(g) = e_H\}.$$

Then there is a one-to-one group homomrorphism $\psi : G/N \to H$ such that $\psi\pi = \varphi$.

Let $\mathbf{G} = \langle G, \cdot, ^{-1}, e\rangle$ be a finite group of order n. Take the set G (the elements of $\mathbf{G}$) and consider the group of all permutations of these elements. We will denote this group by $\mathrm{Sym}(G) = \langle S, \circ, ^{-1}, \mathrm{id}\rangle$. Fix an element $a \in G$ and recall that the function $\lambda_a : G \to G$, defined by $\lambda_a(g) = a \cdot g$, is a permutation of the set G. That is, λ_a belongs to the permutation group $\mathrm{Sym}(G)$. The function $\lambda : G \to \mathrm{Sym}(G)$ defined by $a \mapsto \lambda_a$ is a group homomorphism.[2] To see this, fix $a, b \in G$. We must show that the permutation $\lambda(a \cdot b) = \lambda_{ab}$ is the same as the permutation $\lambda(a) \circ \lambda(b) = \lambda_a \circ \lambda_b$. Indeed, for all $g \in G$,

$$\begin{aligned}
\lambda_{ab}(g) &= (a \cdot b) \cdot g = a \cdot (b \cdot g) \qquad \text{(associativity)}\\
&= a \cdot \lambda_b(g) = \lambda_a(\lambda_b(g))\\
&= (\lambda_a \circ \lambda_b)(g).
\end{aligned}$$

Therefore, by the First Isomorphism Theorem, there is a group homomorphism $\psi : G/N \to \mathrm{Sym}(G)$ such that $\psi\pi = \varphi$. Here, $N := \lambda^{-1}(\{\mathrm{id}\}) = \{a \in G : \lambda_a = \mathrm{id}\}$ where id is the identity permutation in $\mathrm{Sym}(G)$, that is $\mathrm{id}(g) = g$. Also note that, since ψ is a group homomorphism, the image of G/N under ψ is a subgroup of $\mathrm{Sym}(G)$.

Now $N = \{a \in G : \lambda_a = \mathrm{id}\} = \{a \in G : ax = x \text{ for all } x \in G\} = \{e\}$. Therefore, $N = \{e\}$, and so $G \cong G/N$. If $\pi : G \to G/N$ denotes this isomorphism, we have $\psi \circ \pi : G \to G/N \to \mathrm{Sym}(G)$. Since ψ and π are both one-to-one in this case, we have an isomorphism between G and the subgroup $(\psi \circ \pi)(G)$ of $\mathrm{Sym}(G)$. That is G is isomorphic to a subgroup of $\mathrm{Sym}(G)$.

[2] Note that the function is given, for each $a \in G$, by $\lambda(a) = \lambda_a$.

6. Let $\mathbf{S} = \langle S, \cdot \rangle$ and $\mathbf{T} = \langle T, \circ \rangle$ be two semilattices.

(a) Say what it means for a function $\varphi : S \to T$ to be a *semilattice homomorphism* $\varphi : \mathbf{S} \to \mathbf{T}$.

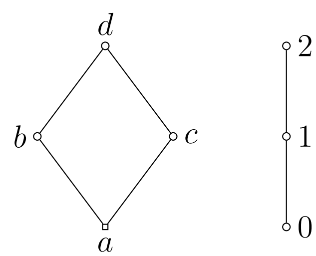
A function $\varphi : S \to T$ is a *semilattice homomorphism* if $\varphi(x \cdot y) = \varphi(x) \circ \varphi(y)$ holds for all $x, y \in S$.

(b) Let $S = \{a, b, c, d\}$ and $T = \{0, 1, 2\}$, and suppose $\mathbf{S} = \langle S, \cdot \rangle$ and $\mathbf{T} = \langle T, \circ \rangle$ have the Cayley tables given below

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	a	b
c	a	a	c	c
d	a	b	c	d

$\circ$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

The Hasse diagrams of $\mathbf{S}$ and $\mathbf{T}$ are as follows:



Determine which of the functions φ_i defined below is a homomorphism. In case φ_i is not a homomorphism, give an example of a violation of the definition in Part (a).

x	$\varphi_1(x)$
a	0
b	1
c	0
d	1

x	$\varphi_2(x)$
a	0
b	1
c	1
d	2

The function φ_1 is a homomorphism. For all $s \in S$, we have

$$\varphi_1(a \cdot s) = \varphi(a) = 0 = 0 \circ \varphi(s) = \varphi(a) \circ \varphi(s).$$

Also, $\varphi_1(b \cdot c) = \varphi(a) = 0 = \varphi_1(b) \circ 0 = \varphi_1(b) \circ \varphi_1(c)$, and

$$\varphi_1(b \cdot d) = \varphi(b) = 1 = 1 \circ 1 = \varphi_1(b) \circ \varphi_1(d), \text{ and}$$

$$\varphi_1(c \cdot d) = \varphi(c) = 0 = 1 \circ 1 = \varphi_1(b) \circ \varphi_1(d).$$

These facts, and commutativity, imply that $\varphi(x \cdot y) = \varphi(x) \circ \varphi(y)$ for all $x, y \in S$.

The function φ_2 is not a homomorphism. For example,

$$\varphi(b \cdot c) = \varphi(a) = 0 \neq 1 = \varphi(b) \circ \varphi(c).$$

EXTRA CREDIT

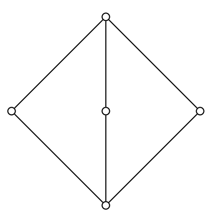
Below I have drawn the subgroup lattice diagrams for the groups $\mathbb{Z}_2$, $\mathbb{Z}_7$, $\mathbb{Z}_{12}$, $\mathbb{Z}_{16}$, $\mathbb{Z}_{30}$, S_3, and $D_4/Z(D_4)$ but I've forgotten which diagram go with which group. I was able to label the first diagram correctly. If you think you can help me label the others, go for it. But don't guess.

+1 *point for each correct answer,* −1/4 *point for each incorrect answer.*

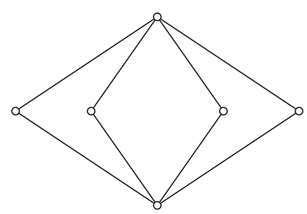
$G = \mathbb{Z}_2$

$\mathbb{Z}_7$

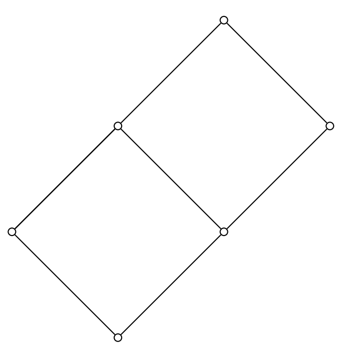
$\mathbb{Z}_{16}$



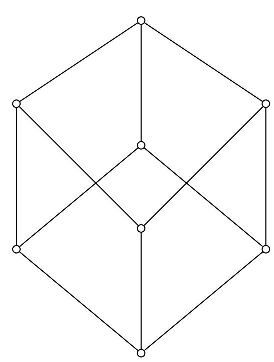
$D_4/Z(D_4)$



S_3



$\mathbb{Z}_{12}$



$\mathbb{Z}_{30}$