Exercises: 1, 2, 3 below and Judson: 9.22, 9.27, 9.31
Recommended: 9.19, 9.21, 9.23, 9.41, 9.42, 9.45
Due date: Friday, 10/31

The first few exercises require some definitions from lecture, repeated here for your convenience.

Let $\mathbf{A} = \langle A, F^{\mathbf{A}}\rangle$ and $\mathbf{B} = \langle B, F^{\mathbf{B}}\rangle$ be two algebras of the same *similarity type*. That is, to each operation symbol $f \in F$ there corresponds an operation $f^{\mathbf{A}}$ defined on $\mathbf{A}$ and an operation $f^{\mathbf{B}}$ defined on $\mathbf{B}$. Thus, the set of operations defined on $\mathbf{A}$ is the set $F^{\mathbf{A}} = \{f^{\mathbf{A}} : f \in F\}$; similarly $F^{\mathbf{B}} = \{f^{\mathbf{B}} : f \in F\}$.

For example, any two groups $\mathbf{G}$ and $\mathbf{H}$ have the same similarity type. To emphasize this, we could denote the operations of these groups using the precise (albeit somewhat awkward) notation of the previous paragraph, as follows:

$$\mathbf{G} = \langle G, \circ^{\mathbf{G}}, \text{inv}^{\mathbf{G}}, e^{\mathbf{G}}\rangle \quad \text{and} \quad \mathbf{H} = \langle H, \circ^{\mathbf{H}}, \text{inv}^{\mathbf{H}}, e^{\mathbf{H}}\rangle.$$

Here $\circ^{\mathbf{G}}$, $\text{inv}^{\mathbf{G}}$, and $e^{\mathbf{G}}$ represent the *interpretation in* $\mathbf{G}$ of the binary, unary (inverse), and nullary (identity) operations that a group must possess (similarly for $\mathbf{H}$).

An *algebra homomorphism* (or simply *homomorphism*), denoted by $\varphi : \mathbf{A} \to \mathbf{B}$, is a function φ with domain A and codomain B that satisfies the following conditions: for each $f \in F$, if f is an n-ary operation symbol, and if $a_1, \dots, a_n \in A$, then

$$\varphi(f^{\mathbf{A}}(a_1, \dots, a_n)) = f^{\mathbf{B}}(\varphi(a_1), \dots, \varphi(a_n)).$$

For example, a *group homomorphism* $\varphi : \mathbf{G} \to \mathbf{H}$ is a function φ with domain G and codomain H that satisfies, $\forall x, y \in G$,

(1) $\varphi(x \circ^{\mathbf{G}} y) = \varphi(x) \circ^{\mathbf{H}} \varphi(y)$,
(2) $\varphi(\text{inv}^{\mathbf{G}}(x)) = \text{inv}^{\mathbf{H}}(\varphi(x))$,
(3) $\varphi(e^{\mathbf{G}}) = e^{\mathbf{H}}$.

The textbook defines a group *isomorphism* to be a group homomorphism that is both one-to-one and onto. This definition is fine for algebraic structures (like groups). It does not work, however, for relational structures, like posets. (See Exercise 3 below). A definition that works for both algebraic and relational structures is the following: A homomorphism $\varphi : \mathbf{A} \to \mathbf{B}$ is an *isomorphism* if there exists a homomorphism $\psi : \mathbf{B} \to \mathbf{A}$ that composes with φ to give the identity, that is, $\varphi \circ \psi = \text{id}_B$ and $\psi \circ \varphi = \text{id}_A$. (Here, id_X denotes the identity function on the set X: $\text{id}_X(x) = x$.)

Exercises

1. When discussing two groups, like $\mathbf{G}$ and $\mathbf{H}$ above, our textbook uses more convenient notation, such as $(G, \cdot)$ and $(H, \circ)$ (or, even more simply, G and H). The book will then define a *homomorphism* to be a function $\varphi : G \to H$ satisfying $\varphi(x \cdot y) = \varphi(x) \circ \varphi(y)$. Prove that this is equivalent to the definition given above by showing that conditions (2) and (3) follow from condition (1).
[Hint: Assuming (1), derive (3), then derive (2).]

Solution: We will be a little bit sloppy and rewrite the three conditions of our definition as follows:
(1) $\varphi(x \cdot y) = \varphi(x) \circ \varphi(y)$,
(2) $\varphi(x^{-1}) = [\varphi(x)]^{-1}$,
(3) $\varphi(e^{\mathbf{G}}) = e^{\mathbf{H}}$.
since in each case the context makes clear whether we mean the operation in $\mathbf{G}$ or in $\mathbf{H}$.
[(1) $\Rightarrow$ (3)]
By (1),

$$\varphi(e^{\mathbf{G}}) = \varphi(e^{\mathbf{G}} \cdot e^{\mathbf{G}}) = \varphi(e^{\mathbf{G}}) \circ \varphi(e^{\mathbf{G}}).$$

Multiplying on the right of both sides by $[\varphi(e^{\mathbf{G}})]^{-1}$ yields

$$e^{\mathbf{H}} = \varphi(e^{\mathbf{G}}) \circ [\varphi(e^{\mathbf{G}})]^{-1} = \varphi(e^{\mathbf{G}}) \circ \varphi(e^{\mathbf{G}}) \circ [\varphi(e^{\mathbf{G}})]^{-1}.$$

Equivalently, $e^{\mathbf{H}} = \varphi(e^{\mathbf{G}})$.

[(1) and (3) $\Rightarrow$ (2)]
Assuming (1) and (3) hold,

$$e^{\mathbf{H}} = \varphi(e^{\mathbf{G}}) = \varphi(x \cdot x^{-1}) = \varphi(x) \circ \varphi(x^{-1}).$$

Similarly,

$$e^{\mathbf{H}} = \varphi(e^{\mathbf{G}}) = \varphi(x^{-1} \cdot x) = \varphi(x^{-1}) \circ \varphi(x),$$

which shows that, for each $x \in G$, the element $\varphi(x^{-1})$ is the inverse of $\varphi(x)$. That is, (2) holds.

2. Define a lattice homomorphism. Then consider a lattice $\mathbf{L} = \langle L, \wedge, \vee \rangle$ and a poset $\mathbf{P} = \langle P, \preccurlyeq \rangle$. Is it possible to define a homomorphism $\varphi : \mathbf{L} \to \mathbf{P}$? Explain.

Solution: A lattice homomorphism is a function $\varphi : \mathbf{A} \to \mathbf{B}$ from a lattice $\mathbf{A} = \langle A, \wedge, \vee \rangle$ to a lattice $\mathbf{B} = \langle B, \wedge, \vee \rangle$, that satisfies, for all $x, y \in A$,

$$\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b) \quad \text{and} \quad \varphi(a \vee b) = \varphi(a) \vee \varphi(b).$$

(Again, the interpretation of the operations, $\wedge$ and $\vee$, depend on the context.)

It's not possible to define a homomorphism from the lattice $\mathbf{L}$ to the poset $\mathbf{P}$ because these are not algebras of the same "similarity type." To define a homomorphism from an algebra $\mathbf{A}$ to an algebra $\mathbf{B}$, we first have to know which operations of $\mathbf{A}$ correspond to which operations of $\mathbf{B}$. In the case where $\mathbf{A}$ is a lattice and $\mathbf{B}$ is a poset, we don't even have the same *number* of operations. In fact, the poset has no *operations* at all—it only has the *relation* $\preccurlyeq$.

However, as we learned in lecture, it may be the case that the poset $\mathbf{P} = \langle P, \preccurlyeq \rangle$ has a special property—namely, that every pair $a, b \in P$ has a greatest lower bound and a least upper bound. In that case, we can define the binary operations $\wedge$ and $\vee$ on the poset as follows:

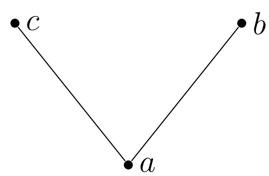
$$x \wedge y = \text{glb}(x, y) \quad \text{and} \quad x \vee y = \text{lub}(x, y),$$

and then the poset becomes a lattice and we can talk about homomorphisms from $\mathbf{L}$ to $\mathbf{P}$. Conversely, every lattice (with operations $\wedge$ and $\vee$) is already a poset with a partial order $\leq$ defined as follows:

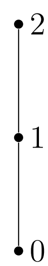
$$x \leq y \iff x \vee y = x \quad \text{and} \quad x \leq y \iff x \wedge y = x.$$

So, once we make this translation, it would then be possible to consider *poset homomorphisms* from $\mathbf{L}$ to $\mathbf{P}$. (Poset homomorphisms are defined and discussed in the next problem.)

3. A *poset homomorphism* is an order preserving map. That is, if $\mathbf{P} = \langle P, \leqslant \rangle$ and $\mathbf{Q} = \langle Q, \preccurlyeq \rangle$ are two partially ordered sets, then a homomorphism $\varphi : \mathbf{P} \to \mathbf{Q}$ is a function satisfying, for all $x, y \in P$, if $x \leqslant y$ then $\varphi(x) \preccurlyeq \varphi(y)$. Consider the two definitions of *isomorphism* given in the last paragraph on Page 1 above. Using the two posets shown below, explain why the first of these definitions is not appropriate for posets.



$\mathbf{P} = \langle \{a, b, c\}, \leqslant \rangle$



$\mathbf{Q} = \langle \{0, 1, 2\}, \preccurlyeq \rangle$

Solution: The first definition says that an isomorphism is simply a bijective homomorphism. For the two posets shown above, we can certainly find a bijective poset homomorphism, that is, an order preserving, one-to-one, and onto function. For example, let $\varphi(a) = 0$, $\varphi(b) = 1$, and $\varphi(c) = 2$. It's easy to see that this is an order preserving map: $\varphi(x) \preccurlyeq \varphi(y)$ in Q whenever $x \leq y$ in P. However, φ fails to be a poset isomorphism since the inverse map φ^{-1} is clearly not order preserving. For example, $1 \preccurlyeq 2$ while $\varphi^{-1}(1) = b \nleq c = \varphi^{-1}(2)$.

It should also be clear that any "reasonable" definition of isomorphism of posets should not characterize the two posets in the diagrams above as isomorphic. For one thing, the poset on the right is *totally* ordered while the one on the left is only partially ordered.

9.22 Let G be a group of order 20. If G has subgroups H and K of orders 4 and 5 respectively such that $hk = kh$ for all $h \in H$ and $k \in K$, prove that G is the internal direct product of H and K.

Solution: Assume $hk = kh$ for all $h \in H$ and $k \in K$. Suppose $x \in H \cap K$. Then $x \in H$ implies $\langle x \rangle$ is a subgroup of H so, by Lagrange's Theorem $|x|$ divides $|H| = 4$. Similarly $x \in K$ implies $\langle x \rangle$ is a subgroup of K so, by Lagrange's Theorem $|x|$ divides $|K| = 5$. Since $|x|$ divides both 4 and 5, we must have $|x| = 1$, so $x = e$. That is $H \cap K = \{e\}$.

To complete the proof, we must show $G = HK$. Since $[G : K] = |G|/|K| = 20/5 = 4$, there are four cosets of K in G and, since $H \cap K = \{e\}$, we can list these four cosets using as representatives the four distinct elements of H. That is, the cosets of K in G are K, h_1K, h_2K, h_3K. Since any group is the disjoint union of the cosets of any subgroup, we have $G = K \cup h_1K \cup h_2K \cup h_3K = HK$. □

9.27 Let $G \cong H$. Show that if G is cyclic, then so is H.

Solution: Suppose $G = \langle a \rangle \cong H$ and suppose $\varphi : G \to H$ is an isomorphism. Then $H = \langle \varphi(a) \rangle$. To see this, fix $h \in H$. We will show $h = (\varphi(a))^k$ for some $k \in \mathbb{N}$. Indeed, let $b = \varphi^{-1}(h)$. Since $b \in G = \langle a \rangle$, we must have $b = a^k$ for some $k \in \mathbb{N}$. Also, $\varphi(a^k) = (\varphi(a))^k$, since φ is a homomorphism. Therefore, $(\varphi(a))^k = \varphi(a^k) = \varphi(b) = \varphi(\varphi^{-1}(h)) = h$. □

9.31 Let $\varphi : G_1 \to G_2$ and $\psi : G_2 \to G_3$ be isomorphisms. Show that φ^{-1} and $\psi \circ \varphi$ are both isomorphisms. Using these results, show that the isomorphism of groups determines an equivalence relation on the class of all groups.

Solution: First we show that φ^{-1} and $\psi \circ \varphi$ are both homomorphisms. Fix $u, v \in G_2$. We must prove the homomorphism property:

$$\varphi^{-1}(u \cdot v) = \varphi^{-1}(u) \cdot \varphi^{-1}(v). \tag{1}$$

Since φ is onto, there exist $a, b \in G_1$ such that $\varphi(a) = u$ and $\varphi(b) = v$. Since φ is one-to-one, the inverse is well-defined and we have $\varphi^{-1}(u) = \varphi^{-1}(\varphi(a)) = a$ and $\varphi^{-1}(v) = \varphi^{-1}(\varphi(b)) = b$. Therefore,

$$\begin{aligned}
\varphi^{-1}(u \cdot v) &= \varphi^{-1}(\varphi(a) \cdot \varphi(b)) \\
&= \varphi^{-1}(\varphi(a \cdot b)) \qquad \text{(since } \varphi \text{ is a homomorphism)} \\
&= (\varphi^{-1} \circ \varphi)(a \cdot b) \qquad \text{(by definition of function composition)} \\
&= a \cdot b \\
&= \varphi^{-1}(u) \cdot \varphi^{-1}(v).
\end{aligned}$$

Fix $x, y \in G_1$. We must prove the homomorphism property:

$$(\psi \circ \varphi)(x \cdot y) = (\psi \circ \varphi)(x) \cdot (\psi \circ \varphi)(y). \tag{2}$$

Indeed,

$$\begin{aligned}
(\psi \circ \varphi)(x \cdot y) &= \psi(\varphi(x \cdot y)) \qquad \text{(by definition)} \\
&= \psi(\varphi(x) \cdot \varphi(y)) \qquad \text{(since } \varphi \text{ is a homomorphism)} \\
&= \psi(\varphi(x)) \cdot \psi(\varphi(y)) \qquad \text{(since } \psi \text{ is a homomorphism)} \\
&= (\psi \circ \varphi)(x) \cdot (\psi \circ \varphi)(y) \qquad \text{(by definition)}
\end{aligned}$$

To finish the proof we must show that φ^{-1} and $\psi \circ \varphi$ are one-to-one and onto.[1] For the inverse function, these properties are obvious. As for $\psi \circ \varphi$, suppose $x, y \in G_1$ and $\psi(\varphi(x)) = (\psi(\varphi(y))$. Then since ψ is one-to-one we must have $\varphi(x) = \varphi(y)$, and since φ is one-to-one, we have $x = y$, which proves that $\psi \circ \varphi$ is one-to-one. To prove that $\psi \circ \varphi$ is onto, let $z \in G_3$. Then since ψ is onto, there exists $y \in G_2$ such that $\psi(y) = z$. Since φ is onto, there exists $x \in G_1$ such that $\varphi(x) = y$. Therefore, $\psi\varphi(x) = \psi(y) = z$, so $\psi \circ \varphi$ is onto.

Finally, we must show that $\cong$ determines an equivalence relation on the class of all groups.

(a) (reflexive) For every group G, we have $G \cong G$, since the identity map, $x \mapsto x$, is easily seen to be an isomorphism.

(b) (symmetric) Assume $G \cong H$ and $\varphi : G \to H$ is an isomorphism. Then, by what we proved above, $\varphi^{-1} : H \to G$ is an isomorphism, so $H \cong G$.

(c) (transitive) Assume $G \cong H$ and $H \cong K$. We must show $G \cong K$. Let $\varphi : G \to H$ and $\psi : H \to K$ be isomorphisms. Then, by what we proved above $\psi \circ \varphi : G \to K$ is an isomorphism so $G \cong K$.

[1] Since we are working with groups, the two definitions of isomorphism we gave at the bottom of page 1 are equivalent, so for groups we can take "isomorphism" to mean bijective homomorphism.