Chapter 5: 1bd, 3bd, 4, 6, 17, 18, 27.
Additional suggested exercises: 29, 31, 32, 33.
Due date: Friday, 10/10

(Exercise numbers correspond to the printed textbook, generated from 2013/08/16 source files.)

1. Write the following permutations in cycle notation.

(b)
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 1 & 3 \end{pmatrix}$$

(d)
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 2 & 5 \end{pmatrix}$$

Solution: (b) (14)(35) (d) (24)

3. Express the following permutations as products of transpositions; identify them as even or odd.

(b) (156)(234)

(d) (17254)(1423)(154632)

Solution:

(b) (16)(15)(24)(23) is even.

(d) (14)(15)(12)(17)(13)(12)(14)(12)(13)(16)(14)(15) is even.

4. Find $(a_1, a_2, \ldots, a_n)^{-1}$.

Solution: We will verify that the permutation $f = (a_1, a_2, \ldots, a_n)(a_n, a_{n-1}, \ldots, a_1)$ is the identity map $f(x) = x$, and thus the inverse of $(a_1, a_2, \ldots, a_n)$ is $(a_n, a_{n-1}, \ldots, a_1)$. Indeed, if x is not in the set $\{a_1, \ldots, a_n\}$, then clearly $f(x) = x$. If x is in the set $\{a_1, \ldots, a_n\}$, say $x = a_j$, then

$$\begin{aligned} f(x) = f(a_j) &= (a_1, a_2, \ldots, a_n)(a_n, a_{n-1}, \ldots, a_1)\, a_j \\ &= (a_1, a_2, \ldots, a_n)\, a_{j-1} \\ &= a_j = x \end{aligned}$$

6. Find all of the subgroups in A_4. What is the order of each subgroup?

Solution: First recall that the 12 elements of A_4 are

$$\{e, (12)(34), (13)(24), (14)(23), (123), (132), (124), (142), (134), (143), (234), (243)\}.$$

With the exception of the (improper) subgroup A_4, and the Klein 4 subgroup

$$V_4 = \{e, (12)(34), (13)(24), (14)(23)\} \quad \text{(of order 4)},$$

the other eight subgroups are cyclic. They are

$$\begin{aligned}
\langle(123)\rangle &= \{e, (123), (132)\} \quad \text{(order 3)}\\
\langle(124)\rangle &= \{e, (124), (142)\} \quad \text{(order 3)}\\
\langle(134)\rangle &= \{e, (134), (143)\} \quad \text{(order 3)}\\
\langle(234)\rangle &= \{e, (234), (243)\} \quad \text{(order 3)}\\
\langle(12)(34))\rangle &= \{e, (12)(34)\} \quad \text{(order 2)}\\
\langle(13)(24))\rangle &= \{e, (13)(24)\} \quad \text{(order 2)}\\
\langle(14)(23))\rangle &= \{e, (14)(23)\} \quad \text{(order 2)}\\
\langle e\rangle &= \{e\} \quad \text{(order 1)}
\end{aligned}$$

17. Prove that S_n is nonabelian for $n \geq 3$.

Solution: Note that, for every $n \geq 3$, the permutations (123) and (12) belong to S_n. Therefore, if we can show that these two elements do not commute, then we will have proved that S_n is nonabelian when $n \geq 3$. Indeed,

$$(123) \cdot (12) = (13) \neq (23) = (12) \cdot (123).$$

18. Prove that A_n is nonabelian for $n \geq 4$.

Solution: Note that, for every $n \geq 4$, the permutations (123) and (12)(34) belong to A_n. Therefore, if we can show that these two elements do not commute, then we will have proved that A_n is nonabelian when $n \geq 4$. Indeed,

$$(123) \cdot (12)(34) = (134) \neq (243) = (12)(34) \cdot (123).$$

27. Let G be a group and define a map $\lambda_g : G \to G$ by $\lambda_g(a) = ga$. Prove that λ_g is a permutation of G.

Solution: We must show that λ_g is a bijection (i.e., a one-to-one and onto map). The map λ_g is clearly one-to-one, since $\lambda_g(a) = \lambda_g(b)$ means $ga = gb$, so $g^{-1}ga = g^{-1}gb$, so $a = b$. The map λ_g is onto since, for any $x \in G$, we can find a $y \in G$ with $\lambda_g(y) = x$. Indeed, let $y = g^{-1}x$, which belongs to G. Then, $\lambda_g(y) = \lambda_g(g^{-1}x) = x$.

Additional suggested exercises: 29, 31, 32, 33.

29. Recall that the *center* of a group G is

$$Z(G) = \{g \in G : gx = xg \text{ for all } x \in G\}.$$

Find the center of D_8. What about the center of D_{10}? What is the center of D_n?

31. For α and β in S_n, define $\alpha \sim \beta$ if there exists an $\sigma \in S_n$ such that $\sigma\alpha\sigma^{-1} = \beta$. Show that $\sim$ is an equivalence relation on S_n.

32. Let $\sigma \in S_X$. If $\sigma^n(x) = y$, we will say that $x \sim y$.

(a) Show that $\sim$ is an equivalence relation on X.

(b) If $\sigma \in A_n$ and $\tau \in S_n$, show that $\tau^{-1}\sigma\tau \in A_n$.

(c) Define the *orbit* of $x \in X$ under $\sigma \in S_X$ to be the set

$$\mathcal{O}_{x,\sigma} = \{y : x \sim y\}.$$

Compute the orbits of α, β, γ where

$$\alpha = (1254)$$
$$\beta = (123)(45)$$
$$\gamma = (13)(25).$$

(d) If $\mathcal{O}_{x,\sigma} \cap \mathcal{O}_{y,\sigma} \neq \varnothing$, prove that $\mathcal{O}_{x,\sigma} = \mathcal{O}_{y,\sigma}$. The orbits under a permutation σ are the equivalence classes corresponding to the equivalence relation $\sim$.

(e) A subgroup H of S_X is *transitive* if for every $x, y \in X$, there exists a $\sigma \in H$ such that $\sigma(x) = y$. Prove that $\langle\sigma\rangle$ is transitive if and only if $\mathcal{O}_{x,\sigma} = X$ for some $x \in X$.

33. Let $\alpha \in S_n$ for $n \geq 3$. If $\alpha\beta = \beta\alpha$ for all $\beta \in S_n$, prove that α must be the identity permutation; hence, the center of S_n is the trivial subgroup.