

Exercises: Judson 10.1abe, 10.5, 10.10, 10.11, 10.13ac, and Problem 6 below.

Due date: Wednesday, 11/05

10.1 For each of the following groups G , determine whether H is a normal subgroup of G . If H is a normal subgroup, write out a Cayley table for the factor group G/H .

- (a) $G = S_4$ and $H = A_4$
- (b) $G = A_5$ and $H = \{(1), (123), (132)\}$
- (c) $G = \mathbb{Z}$ and $H = 5\mathbb{Z}$

10.5. Show that the intersection of two normal subgroups is a normal subgroup.

10.10. Let H be a subgroup of index 2 of a group G . Prove that H must be a normal subgroup of G . Conclude that S_n is not simple for $n \geq 3$.

10.11. If a group G has exactly one subgroup H of order k , prove that H is normal in G .

10.13. Recall that the **center** of a group G is the set

$$Z(G) = \{x \in G : xg = gx \text{ for all } g \in G\}.$$

- (a) Calculate the center of S_3 .
- (c) Show that the center of any group G is a normal subgroup of G .

Problem 6. Let $\mathbf{G} = \langle G, \cdot, {}^{-1}, e \rangle$ be a finite group of order n . Take the set G (the elements of \mathbf{G}) and consider the group of all permutations of these elements. This group is sometimes denoted by $\text{Sym}(G)$; note that it is isomorphic to the symmetric group S_n of permutations of an n -element set. Now fix an element $a \in G$ and recall that the function $\lambda_a : G \rightarrow G$, defined by $\lambda_a(g) = a \cdot g$, is a permutation of the set G . That is, λ_a belongs to the permutation group $\text{Sym}(G)$.

- (a) Prove that the function $\lambda : G \rightarrow \text{Sym}(G)$ is a group homomorphism.
- (b) What is the kernel of λ ?¹
- (c) Let N denote the equivalence class of $\ker \lambda$ that contains the identity element e of G . Prove that N is a normal subgroup of G .

¹Recall that the kernel of a function $f : X \rightarrow Y$ is the subset of $X \times X$ defined by

$$\ker f = \{(x_1, x_2) : f(x_1) = f(x_2)\}.$$

As you have already proved, the kernel is an equivalence relation on X .