Let G be a group and suppose X is a nonempty set of elements of G. The **subgroup generated by** X is the smallest subgroup of G that contains X. For example, the subgroup of $\mathbb{Z}_{12}$ generated by the set $\{4, 6\}$ is $\{0, 2, 4, 6, 8, 10\}$ (explained below). For a single element $g \in G$, we often denote the subgroup generated by the set $\{g\}$ by $\langle g \rangle$ instead of $\langle \{g\} \rangle$. For finite sets, like $\{x, y\}$, we often write, $\langle x, y \rangle$ instead of $\langle \{x, y\} \rangle$.

A **one-generated** subgroup is a subgroup generated by one element, such as $\langle g \rangle$. A one-generated subgroup is also called a **cyclic** subgroup. A **two-generated** subgroup is a subgroup $\langle x, y \rangle$ generated by two elements, x and y. An n**-generated** subgroup is a subgroup of the form $\langle x_1, \ldots, x_n \rangle$, which is generated by the n-element set $\{x_1, \ldots, x_n\}$.

Let G be a group and let H be a subgroup of G. It is important to note the distinction between the following two statements:

1. "The cyclic subgroup H has two generators x and y."

2. "The subgroup H is generated by two elements x and y."

The first sentence means $H = \langle x \rangle = \langle y \rangle$. That is, you can take either x or y as the generator of H. The second sentence means something entirely different, namely, $H = \langle x, y \rangle$. This says that the smallest subgroup of G that contains both x and y is H. It may or may not be the case that H is cyclic. The notation $H = \langle x, y \rangle$ simply means that H can be generated by two elements. It's possible that we could find an element that generates H all by itself. That is, we may have $H = \langle g \rangle = \langle x, y \rangle$.

Examples

1. Consider the subgroup $H = \{e, (1, 2, 3), (1, 3, 2)\}$ of A_4, which can be generated by either one (or both) of its non-identity elements: $H = \langle (1, 2, 3) \rangle = \langle (1, 3, 2) \rangle$.

2. Continuing with the last example, we could write $H = \langle (1, 2, 3), (1, 3, 2) \rangle$. Here we have thrown in a redundant generator, which is harmless but not helpful because it doesn't call attention to an important feature of H–namely, that it is one-generated, i.e., cyclic.

3. As mentioned above, the subgroup of $\mathbb{Z}_{12}$ generated by the set $\{4, 6\}$ is $\{0, 2, 4, 6, 8, 10\}$. To see this, note that, if 4 and 6 belong to a subgroup of $\mathbb{Z}_{12}$, then so must $4 + 4 = 8$ and $4 + 6 = 10$ and $6 + 6 = 0$ and $4 + 4 + 6 = 2$.

4. Suppose $G = \langle a \rangle$ is a cyclic group, suppose $x = a^6$ and $y = a^8$. Then

$$H = \langle x, y \rangle = \langle a^6, a^8 \rangle = \langle a^2 \rangle.$$

5. Unlike all of the previous examples, the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not cyclic. You need at least two elements to generate this group. For example, you could use $(0, 1)$ and $(1, 1)$ (though there are other options):

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle (0, 1), (1, 1) \rangle.$$

6. The **Klein 4 group**, typically denoted by V_4, is a two-generated subgroup of A_4. It is generated by any pair of non-identity elements, say, $(1,2)(3,4)$ and $(1,3)(2,4)$:

$$V_4 = \langle (1,2)(3,4), (1,3)(2,4) \rangle = \{e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$$

   Again, we could have used a different pair, say, $(1,3)(2,4)$ and $(1,4)(2,3)$, to generate V_4. The point is that we need more than one element, so V_4 is not cyclic.

   It turns out that the Klein 4 group is essentially the same as the group $\mathbb{Z}_2 \times \mathbb{Z}_2$. As we will see later, these two groups are "isomorphic," which is an idea that we have surely thought about when comparing groups, but have left undefined for the moment.

See also the CyclicGroupSupplement.pdf document and CyclicGroupExercises.pdf, especially Exercise 6.