MATH 301 - Section B **Fall 2014**

Midterm Exam 2

RULES

1. All phones and other electronic devices must be silenced for the duration of the exam.
2. No books, notes, or calculators allowed.

Cheating will not be tolerated. If there are any indications that a student may have given or received unauthorized aid on this exam, the case will be brought to the ISU Office of Academic Integrity.

After finishing the exam, please sign the following statement acknowledging that you understand and accept this policy:

"On my honor as a student I, ________________________, have neither given nor received unauthorized aid on this exam." (print name clearly)

Signature: ____________________________________ Date: _______________

Do not sign the pledge until after you have finished the exam.

1. Give precise definitions of the following:

(a) n**-ary operation** on a set A

(b) n**-ary relation** on a set A

(c) **algebra** or **algebraic structure**

(d) **relational structure**

(e) **group homomorphism**

(f) **normal subgroup**

2. (a) State *Lagrange's Theorem* about the order of a group and its subgroups. (Be sure to state all assumptions that are needed in order for the theorem to hold.)

(b) Recall that G is the *internal direct product* of the subgroups H and K if $H \cap K = \{e\}$ and H and K centralize each other (i.e., $hk = kh$ for all $h \in H$ and $k \in K$).

Suppose G has order 28 and subgroups H and K of orders 4 and 7 respectively which centralize each other. Prove that G is the internal direct product of H and K.

3. Prove either (a) OR (b) OR (c). If you prove more than one, circle the letter you want graded.

(a) Prove that if $G \cong H$ and G is cyclic, then H is cyclic.

(b) If a group G has a subgroup H of index 2, then H is normal in G.
Conclude that $A_n \triangleleft S_n$ for $n \geq 3$.

(c) If a group G has exactly one subgroup H of order k, then H is normal in G.

4. The **center** of a group G is $Z(G) = \{x \in G : xg = gx \text{ for all } g \in G\}$.

(a) Show that the center of any group is a normal subgroup.

(b) The dihedral group D_4 (symmetries of the square) can be described as the permutation group with two generators $\rho = (1234)$ and $\mu = (13)$ satisfying $\rho^4 = e = \mu^2$. Therefore, the elements of D_4 are $\{e, \rho, \rho^2, \rho^3, \mu, \rho\mu, \rho^2\mu, \rho^3\mu\}$.

Calculate $Z(D_4)$, the center of D_4. [Hint: only one nonidentity element of D_4 commutes with all other elements of D_4, and finding this element should not require too much calculation.]

(c) Is $D_4/Z(D_4)$ cyclic? Explain. [Hint: Recall, we proved that G is abelian if $G/Z(G)$ is cyclic.]

5. State the First Isomorphism Theorem and use it to prove that every group G is isomorphic to a subgroup of $\mathrm{Sym}(G)$. [Hint: show that a certain function $\lambda : G \to \mathrm{Sym}(G)$ is a homomorphism with trivial kernel.]

6. Let $\mathbf{S} = \langle S, \cdot \rangle$ and $\mathbf{T} = \langle T, \circ \rangle$ be two semilattices.

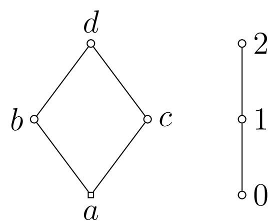
(a) Say what it means for a function $\varphi : S \to T$ to be a *semilattice homomorphism* $\varphi : \mathbf{S} \to \mathbf{T}$.

(b) Let $S = \{a, b, c, d\}$ and $T = \{0, 1, 2\}$, and suppose $\mathbf{S} = \langle S, \cdot \rangle$ and $\mathbf{T} = \langle T, \circ \rangle$ have the Cayley tables given below

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	a	b
c	a	a	c	c
d	a	b	c	d

$\circ$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

The Hasse diagrams of $\mathbf{S}$ and $\mathbf{T}$ are as follows:



Determine which of the functions φ_i defined below is a homomorphism. In case φ_i is not a homomorphism, give an example of a violation of the definition in Part (a).

x	$\varphi_1(x)$
a	0
b	1
c	0
d	1

x	$\varphi_2(x)$
a	0
b	1
c	1
d	2

EXTRA CREDIT

Below I have drawn the subgroup lattice diagrams for the groups $\mathbb{Z}_2$, $\mathbb{Z}_7$, $\mathbb{Z}_{12}$, $\mathbb{Z}_{16}$, $\mathbb{Z}_{30}$, S_3, and $D_4/Z(D_4)$ but I've forgotten which diagram go with which group. I was able to label the first diagram correctly. If you think you can help me label the others, go for it. But don't guess.

+1 *point for each correct answer,* −1/4 *point for each incorrect answer.*

$G = \mathbb{Z}_2$