MATH 301 - Section B **Fall 2014**

Midterm Exam

RULES

1. All phones and other electronic devices must be silenced for the duration of the exam.
2. No books, notes, or calculators allowed.
3. Out of consideration for your classmates, do not make disturbing noises during the exam. If you need a tissue, please ask for one.
4. Are you still reading the rules? Did you read rule number 1? If you haven't yet taken out your phone to turn it off, read rule number 1 a few more times.

Cheating will not be tolerated. If there are any indications that a student may have given or received unauthorized aid on this exam, the case will be brought to the ISU Office of Academic Integrity.

After finishing the exam, please sign the following statement acknowledging that you understand and accept this policy:

"On my honor as a student I, ________________________, have neither given nor received unauthorized aid on this exam." (print name clearly)

Signature: ____________________________________ Date: _______________

Do not sign the pledge until after you have finished the exam.

1. Give precise definitions of the following:

(a) **semigroup**

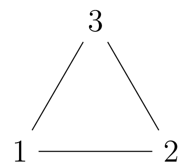
(b) **monoid**

(c) **group**

(d) **abelian group**

(e) **cyclic group**

2. Let G denote the symmeteries of an equilateral triangle.



(a) List the elements of this group using cycle notation.

(b) What is the order of this group?

(c) Is this group cyclic? Is it abelian?
(Give a brief justification for your answers. You may cite 4(b).)

3. Suppose H and K are subgroups of a group G. Prove or disprove the following:

(a) $H \cap K$ is a subgroup of G.

(b) $H \cup K$ is a subgroup of G.

4. Let G be a group. Prove the following:

(a) G is abelian if and only if $(gh)^2 = g^2h^2$ holds for all $g, h \in G$.
Proof:

(b) If G is cyclic, then G is abelian.
Proof:

(c) Give a specific example of a group G and elements $g, h \in G$ for which $(gh)^2 \neq g^2h^2$.
(Justify your answer.)

(d) Give a specific example of a group that is abelian but not cyclic.
(No justification necessary.)

5. This problem has several parts. First, state the following (without proof):

(a) *The Well Ordering Principle.* (about subsets of natural numbers)

(b) *The Division Algorithm.* (about integers a and b where $b > 0$)

Prove that every subgroup of a cyclic group is cyclic by following the steps below. First, let $G = \langle a \rangle$ be a cyclic group and fix an arbitrary subgroup $H \leq G$.

(a) Suppose H contains only the identity, e. Say why H must be cyclic in this case. (one line/sentence)

(b) Suppose instead that H contains more than just the identity element. Let m be the smallest positive integer such that $a^m \in H$. Why does such a number m exists? (Hint: consider the set $\{m \in \mathbb{N} : a^m \in H\}$; cite a well known principle; say why it applies here.)

(c) Finally, prove the **Claim** stated on the **next page**. $\rightarrow$
(which says that a^m generates H, where m is the number from part (b)).

Claim: If $H \leq G = \langle a \rangle$ and m is the smallest positive integer such that $a^m \in H$, then $H = \langle a^m \rangle$.

Proof:

6. Answer either (a) or (b). Only one answer will be graded. (If you answer both, then clearly mark which should be graded.)

(a) Show that, for any cyclic group $G = \langle a \rangle$, the subgroup $\langle a^j, a^k \rangle$ generated by a^j and a^k is equal to $\langle a^d \rangle$, where $d = \gcd(j, k)$. (Hint: Recall that there exist integers r and s such that $d = rm + sk$; you may use this fact without proving it.)

(b) Show that for any group G, and any fixed element $g \in G$, the map $\lambda_g : G \to G$ defined by $\lambda_g(a) = ga$ is a permutation of G. Then show that the order of the alternating group on n letters is $|A_n| = n!/2$.

7. (a) Give a precise definition of *equivalence relation*, then give an example.

(b) Give a precise definition of *partial order relation*, then give an example.

(c) Let $f : X \to Y$ be a function and define the relation $\sim$ on the set X as follows:

$$m \sim n \quad \text{if and only if} \quad f(m) = f(n)$$

What kind of relation is $\sim$? (Justify your answer by checking the properties.)

Extra Credit

EC 1. (6 points) Below I have drawn the subgroup lattice diagrams for the groups $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_7$, $\mathbb{Z}_{12}$, $\mathbb{Z}_{16}$, $\mathbb{Z}_{30}$, and S_3, but I've forgotten which diagram go with which group. I was able to label the first diagram correctly. If you think you can help me label the others, go for it. But don't guess!

$+1$ *point for each correct answer,* $-1/2$ *point for each incorrect answer.*

$G = \mathbb{Z}_2$

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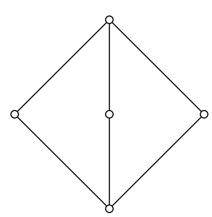

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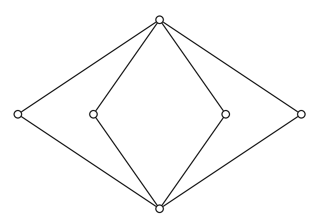

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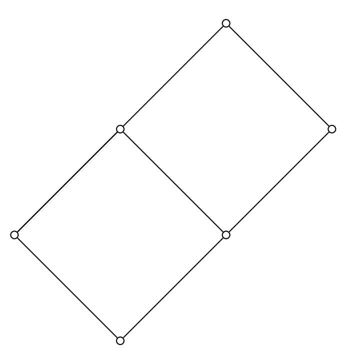

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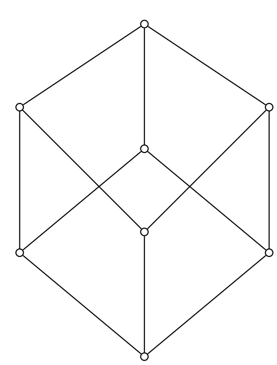

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EC 2. (1/2 point) Which group appears in William DeMeo's GitHub gravatar? (William DeMeo the mathematician, not the actor.)