Math 301 **Homework 11 – Solutions** **Fall 2014**

Exercises: Judson 11.7, 11.11, 11.17, 11.18, 11.19
Due date: Friday, 11/21

11.7 In the group $\mathbb{Z}_{24}$, let $H = \langle 4 \rangle$ and $N = \langle 6 \rangle$.

(a) List the elements in HN (we usually write $H+N$ for these additive groups) and $H \cap N$.
(b) List the cosets in HN/N, showing the elements in each coset.
(c) List the cosets in $H/(H \cap N)$, showing the elements in each coset.
(d) Give the correspondence between HN/N and $H/(H \cap N)$ described in the proof of the Second Isomorphism Theorem.

Solution: First note that, as subgroups of $\mathbb{Z}_{24}$,

$$H = \langle 4 \rangle = \{0, 4, 8, 12, 16, 20\} \quad \text{and} \quad N = \langle 6 \rangle = \{0, 6, 12, 18\}.$$

(a) The elements in $H + N$ are $0, 2, 4, \cdots, 22$. The elements of $H \cap N$ are 0 and 12.

(b) The cosets in $(H + N)/N$ are

$$N = \{0, 6, 12, 18\}, \qquad 2 + N = \{2, 8, 14, 20\}, \qquad 4 + N = \{4, 10, 16, 22\}.$$

(c) The cosets in $H/H \cap N$ are $h + H \cap N$ for each $h \in H$. That is,

$$0 + H \cap N = \{0, 12\}, \qquad 4 + H \cap N = \{4, 16\}, \qquad 8 + H \cap N = \{8, 20\}.$$

(Note that $12 + H \cap N$, $16 + H \cap N$, and $20 + H \cap N$ already appear in the list.)

(d) The proof of the Second Isomorphism Theorem begins with a map that takes each $h \in H$ to $h + N \in H + N/N$; that is,

$$0 \mapsto 0 + N, \qquad 12 \mapsto 12 + N,$$
$$4 \mapsto 4 + N, \qquad 16 \mapsto 16 + N,$$
$$8 \mapsto 8 + N, \qquad 20 \mapsto 20 + N.$$

Then, since the kernel subgroup of this map is $H \cap N$, the one-to-one correspondence between $H/(H \cap N)$ and $H + N/N$ is given (by the First Isomorphism Theorem) as follows:

$$0 + H \cap N \longleftrightarrow 0 + N,$$
$$4 + H \cap N \longleftrightarrow 4 + N,$$
$$8 + H \cap N \longleftrightarrow 2 + N.$$

11.11 Show that a homomorphism defined on a cyclic group is completely determined by its action on the generator of the group.

Solution: Let $G = \langle a \rangle$ be a cyclic group. Let φ be a homomorphism from G to some other group. We want to show that, for any $x \in G$, we can write the image $\varphi(x)$ in terms of $\varphi(a)$. (That's what it means for φ to be "determined by its action on the generator.") Indeed, since $x = a^k$ for some k, and since φ is a homomorphism, we have $\varphi(x) = \varphi(a^k) = (\varphi(a))^k$.

11.17 If H and K are normal subgroups of G and $H \cap K = \{e\}$, prove that G is isomorphic to a subgroup of $G/H \times G/K$.

Solution: Define $\varphi : G \to G/H \times G/K$ by $\varphi(g) = (gH, gK)$. First we show φ is a homomorphism. By the definition of coset multiplication and the definition of multiplication in Cartesian products,

$$\begin{aligned}\varphi(g_1 g_2) &= (g_1 g_2 H, g_1 g_2 K) = (g_1 H g_2 H, g_1 K g_2 K)\\ &= (g_1 H, g_1 K)(g_2 H g_2 K) = \varphi(g_1)\varphi(g_2).\end{aligned}$$

for all $g_1, g_2 \in G$, which proves that φ is a homomorphism from G to $G/H \times G/K$.

Therefore, by the First Isomorphism Theorem, $G/N_\varphi \cong \varphi(G)$, where N_φ is the kernel subgroup associated with φ. Moreover, the image $\varphi(G) = \{\varphi(g) : g \in G\}$ is a subgroup of $G/H \times G/K$. Finally, note that the identity element of $G/H \times G/K$ is (H, K), so the kernel subgroup is

$$\begin{aligned}N_\varphi &= \{g \in G : \varphi(g) = (H, K)\}\\ &= \{g \in G : (gH, gK) = (H, K)\}\\ &= \{g \in G : gH = H \text{ and } gK = K\}\\ &= \{g \in G : g \in H \text{ and } g \in K\}\\ &= H \cap K.\end{aligned}$$

By assumption $H \cap K = \{e\}$. Therefore, $G = G/\{e\} = G/N_\varphi \cong \varphi(G)$, which is a subgroup of $G/H \times G/K$.[1]

[1] The equality $G = G/\{e\}$ is technically an isomorphism $G \cong G/\{e\}$, since $G/\{e\}$ is a collection of cosets, namely $G/\{e\} = \{g\{e\} : g \in G\}$. However, since $g\{e\} = \{g\}$, it's common practice to identify the elements of $G/\{e\} = \{\{g\} : g \in G\}$ with the elements of G, and say that the quotient group $G/\{e\}$ *is* the group G.

11.18 Let $\varphi : G_1 \to G_2$ be a surjective group homomorphism. Let H_1 be a normal subgroup of G_1 and suppose that $\varphi(H_1) = H_2$. Prove or disprove that $G_1/H_1 \cong G_2/H_2$.

Solution: That this statement is false can be seen by considering the First Isomorphism Theorem. Let e_1 and e_2 be the identity elements of G_1 and G_2, respectively. Since φ is surjective, $\varphi(G_1) = G_2$ so, by the First Isomorphism Theorem, $G_1/N_\varphi \cong \varphi(G_1) = G_2$, where $N_\varphi = \varphi^{-1}(\{e_2\})$ is the kernel subgroup.

Let $H_1 = \{e_1\}$, and suppose N_φ strictly contains H_1, so $G_1 \cong G_1/H_1 \ncong G_1/N_\varphi$. Since φ is a homomorphism, we have $\varphi(H_1) = \varphi(\{e_1\}) = \{e_2\} = H_2$, so

$$G_1 \cong G_1/H_1 \ncong G_1/N_\varphi \cong G_2 \cong G_2/H_2.$$

Alternatively, we could show that the statement is false by constructing a concrete counterexample, such as the following: Let $G_1 := \mathbb{Z}_9$ and $G_2 := \mathbb{Z}_9/\langle 3\rangle$ and let $\varphi : \mathbb{Z}_9 \to \mathbb{Z}_9/\langle 3\rangle$ be defined by $\varphi(x) = x + \langle 3\rangle$. If $H_1 := \{0\}$, then $\varphi(\{0\}) = \{0\} = H_2$, and

$$G_1/H_1 = \mathbb{Z}_9/\{0\} \ncong \mathbb{Z}_9/\langle 3\rangle \cong G_2/\{0\} = G_2/H_2.$$

11.19 Let $\phi : G \to H$ be a group homomorphism. Show that ϕ is one-to-one if and only if $\phi^{-1}(e) = \{e\}$.

Solution: ($\Rightarrow$) Suppose φ is one-to-one. Since φ is a homomorphism, $\varphi(e_G) = e_H$. Therefore, $\varphi(x) = e_H = \varphi(e_G)$ implies $x = e_G$, since φ is one-to-one. That is, $\varphi^{-1}(\{e_H\}) = \{e_G\}$.

($\Leftarrow$) Suppose $\varphi^{-1}(\{e_H\}) = \{e_G\}$, and suppose $x, y \in G$. We prove that $\varphi(x) = \varphi(y)$ implies $x = y$. Indeed, if $\varphi(x) = \varphi(y)$ then

$$e_H = \varphi(e_G) = \varphi(x^{-1}x) = \varphi(x^{-1})\varphi(x) = \varphi(x^{-1})\varphi(y) = \varphi(x^{-1}y).$$

Therefore, $x^{-1}y$ belongs to the set $\varphi^{-1}(\{e_H\}) = \{e_G\}$, so $x^{-1}y = e_G$. Equivalently, $x = y$, so φ is one-to-one.