Exercises: 1 below and Judson: 6.5bd, 6.11ade, 6.16, 6.18
Due date: Friday, 10/24

1. Prove or disprove the following:

(a) There exists a group G of order $|G| = 8$ with an element $g \in G$ of order $|g| = 3$.
(b) If H and K are subgroups of a group G with $|H| = 2$ and $|K| = 3$, then $|G| \geq 6$.
(c) Every subgroup of the integers has finite index.
(d) Every subgroup of the integers has finite order.

Solution:

(a) False. Note that $|g| = |\langle g \rangle|$, and by Lagrange's Theorem $|\langle g \rangle|$ divides $|G|$. So the order of every element of a group must divide the order of the group. Since $3 \nmid 8$, there is no element of order 3 in a group of order 8.

(b) True. If $H, K \leq G$ then $|H| \bigm| |G|$ and $|K| \bigm| |G|$, by Lagrange's Theorem. In case $|H| = 2$ and $|K| = 3$, we have $2 \bigm| |G|$ and $3 \bigm| |G|$, so $|G| \geq \operatorname{lcm}(2, 3) = 6$.

(c) False. The subgroup $\{0\} \leq \mathbb{Z}$ has infinite index, $[\mathbb{Z} : \{0\}] = \infty$. The cosets of $\{0\}$ in $\mathbb{Z}$ have the form $k + \{0\} = \{k\}$, where $k \in \mathbb{Z}$. That is, for each $k \in \mathbb{Z}$, the set $\{k\}$ is the coset of $\{0\}$ containing k.

(d) False. The subgroup $H = 2\mathbb{Z} \leq \mathbb{Z}$ has infinite order.

6.5. In each case below, list the left cosets of H in G.

b. $G = U(8)$, $H = \langle 3 \rangle$.
c. $G = S_4$, $H = A_4$.

Solution:

b. If $G = U(8) = \{1, 3, 5, 7\}$ and $H = \langle 3 \rangle = \{1, 3\}$, then there are $[G : H] = |G|/|H| = 4/2 = 2$ cosets, namely, $H = \{1, 3\}$ and $5H = \{5, 7\}$. That is,

$$G/H = \{gH : g \in G\} = \{H, 5H\} = \{\{1, 3\}, \{5, 7\}\}.$$

c. If $G = S_4$ and $H = A_4$, then there are $[G : H] = |G|/|H| = 4!/(4!/2) = 2$ cosets, namely, A_4 and gA_4, where g is any element of S_4 that does not belong to A_4. (For example, let $g = (12)$, which is odd.) That is, $G/H = \{gH : g \in G\} = \{A_4, (12)A_4\}$.

6.11. Let H be a subgroup of a group G and suppose that $g_1, g_2 \in G$. Prove that the following conditions are equivalent:

(a) $g_1 H = g_2 H$

(d) $g_2 \in g_1 H$

(e) $g_1^{-1} g_2 \in H$

Solution: This was proved in class. Please come to office hours if you don't understand it.

6.16. If $|G| = 2n$, prove that the number of elements of order 2 is odd. Use this result to show that G must contain a subgroup of order 2.

Solution: Suppose $|G| = 2n$. Let

$$X = \{x \in G | x^2 = e\} = \{x \in G | x = x^{-1}\},$$
$$X^c = \{x \in G | x^2 \neq e\} = \{x \in G | x \neq x^{-1}\}.$$

Then G is the disjoint union $G = X \coprod X^c$, so

$$|G| = |X| + |X^c|. \tag{1}$$

Note that X includes the identity element, which has order 1, so $X \setminus \{e\}$ is the set of all elements of G of order 2. There are $|X| - 1$ such elements, so the goal is to show that $|X| - 1$ is odd, or, equivalently, that $|X|$ is even. Now, $|X^c|$ is clearly even since, for each $x \in X^c$, there is a corresponding element x^{-1} (distinct from x) that also belongs to X^c. That is, elements of X^c come in pairs. Therefore, $|X^c| = 2k$ for some k, so by (1),

$$|X| = |G| - |X^c| = 2n - 2k = 2(n - k),$$

which is an even number. $\square$

6.18. If $[G : H] = 2$, prove that $gH = Hg$.

Solution: If $g \in H$, then $gH = H = Hg$ and we are done. So assume $g \notin H$. Then $H \neq gH$. Since $[G : H] = 2$, there are exactly two left cosets of H in G, namely H and gH. Similarly, there are two right cosets of H in G, namely H and Hg. Since G is the disjoint union of H and Hg, and also the disjoint union of H and gH, we have $gH = Hg$.