Exercises: 1–7 of the file GeneralAlgebraNotes.pdf.
Submit: Solutions to Exercises 5 and 7.
Due: Friday, 12/5.

Ex. 5. Suppose K and L are normal subgroups of G. Prove that $G/K \cap L$ is isomorphic to a subgroup of $G/K \times G/L$ (the external direct product),[1] and compute the index of this subgroup in $G/K \times G/L$, in terms of $[G : K]$, $[G : L]$, and $[G : KL]$.

Solution: Define $\varphi : G \to G/K \times G/L$ by $\varphi(g) = (gK, gL)$. First we show φ is a homomorphism. By the definition of coset multiplication and the definition of multiplication in Cartesian products,

$$\begin{aligned}\varphi(g_1g_2) &= (g_1g_2K, g_1g_2L) = (g_1Kg_2K, g_1Lg_2L)\\ &= (g_1K, g_1L)(g_2Kg_2L) = \varphi(g_1)\varphi(g_2).\end{aligned}$$

for all $g_1, g_2 \in G$, which proves that φ is a homomorphism from G to $G/K\times G/L$. Therefore, by the First Isomorphism Theorem,

$$G/N_\varphi \cong \varphi(G),$$

where N_φ is the kernel subgroup associated with φ, and where $\varphi(G) = \{\varphi(g) : g \in G\}$ is the image of G under the map φ. Since φ is a homomorphism, and since homomorphisms map subgroups of G to subgroups of the codomain, it follows that $\varphi(G)$ is a subgroup of $G/K \times G/L$. Thus we have shown that G/N_φ is isomorphic to a subgroup of $G/K \times G/L$.

Finally, the identity element of $G/K \times G/L$ is (K, L), so the kernel subgroup is

$$\begin{aligned}N_\varphi &= \{g \in G : \varphi(g) = (K, L)\}\\ &= \{g \in G : (gK, gL) = (K, L)\}\\ &= \{g \in G : gK = K \text{ and } gL = L\}\\ &= \{g \in G : g \in K \text{ and } g \in L\}\\ &= K \cap L.\end{aligned}$$

By assumption $K \cap L = \{e\}$. Therefore, $G = G/\{e\} = G/N_\varphi \cong \varphi(G)$, which is a subgroup of $G/K \times G/L$.[2]

The index of $G/K \cap L$ in $G/K \times G/L$, in terms of $[G : K], [G : L]$ and $[G : K \cap L]$, is

$$|(G/K \times G/L)|/|G/K \cap L| = \frac{(|G|/|K|)(|G|/|L|)}{|G|/|K \cap L|} = \frac{[G : K][G : L]}{[G : K \cap L]}.$$

[1] Note that the expression $G/K \cap L$ can only be interpreted as $G/(K \cap L)$, since $(G/K) \cap L$ doesn't make sense.

[2] The equality $G = G/\{e\}$ is technically an isomorphism $G \cong G/\{e\}$, since $G/\{e\}$ is a collection of cosets, namely $G/\{e\} = \{g\{e\} : g \in G\}$. However, since $g\{e\} = \{g\}$, it's common practice to identify the elements of $G/\{e\} = \{\{g\} : g \in G\}$ with the elements of G, and say that the quotient group $G/\{e\}$ *is* the group G.

Note to students: The above solution illustrates a **standard method** of applying the First Isomorphism Theorem to prove that a factor group G/N is isomorphic to a subgroup of H. Let us pause to abstract the essence of this method, so you will know exactly how to apply it in future situations.

To begin with, we are presented with the following data: G, N, and H, are groups, and N is a normal subgroup of G (so G/N is a factor group). Our objective is to show that G/N is isomorphic to some subgroup of H. (The specific subgroup $K \leq H$ may or may not be specified in advance.)

Standard method: To show $G/N \cong K$ for some subgroup $K \leq H$,

(a) Find a function $\varphi : G \to H$ with the following properties (which you must check):

  (i) φ is a homomorphism from G to H

  (ii) the image of φ is $K = \varphi(G)$

  (iii) the kernel subgroup of φ is $N = \{g \in G : \varphi(g) = e_H\}$

(b) Conclude by citing the First Isomorphism Theorem, which implies

$$G/N \cong \varphi(G) = K \leq H.$$

Notice that even though our initial goal is to find an isomorphism $G/N \cong K$, we don't start by searching around for a one-to-one and onto homomorphisms defined on G/N. Rather, we start with the (seemingly) more modest (but equivalent) goal of finding a homomorphism from G to H with kernel subgroup N and image K. Once we accomplish this, the First Isomorphism Theorem does the rest of the work for us.

Ex. 7. Let G be a nonabelian simple group. Let S_n be the symmetric group of all permutations on an n-element set, and let A_n be the alternating group.

a. Show that if G is a subgroup of S_n, n finite, then G is a subgroup of A_n.

b. Let H be a proper subgroup of G, and, for $g \in G$, let λ_g be the map of the set of left cosets of H onto themselves defined by $\lambda_g(xH) = gxH$. Show that the map $g \mapsto \lambda_g$ is a monomorphism (injective homomorphism) of G into the group of permutations of the set of left cosets of H.

c. Let H be a subgroup of G of finite index n and assume $n > 1$ (so $H \neq G$). Show that G can be embedded in A_n. (That is, show that G is isomorphic to a subgroup of A_n.

d. If G is infinite, it has no proper subgroup of finite index.

Solution:

(a) Suppose G is a subgroup of S_n, n finite, and assume G is nonabelian and simple. Clearly $G \neq S_n$, since S_n is not simple (for example, $A_n \lhd S_n$). Consider $G \cap A_n$. This is a normal subgroup of G. To see this, note that if $\sigma \in G \cap A_n$ and $g \in G$, then $g\sigma g^{-1} \in G$ and $g\sigma g^{-1} \in A_n$, since $A_n \lhd S_n$, so $g\sigma g^{-1} \in G \cap A_n$.

So, $G \cap A_n \lhd G$ and G is simple, so $G \cap A_n = (e)$ or $G \cap A_n = G$. In case $G \cap A_n = G$, we have $G \leq A_n$ and we are done. The other case, $G \cap A_n = (e)$, in fact never occurs under the given hypotheses. Two alternative proofs of this fact are given below.

Proof 1: Suppose $G \cap A_n = (e)$. Then G contains only e and odd permutations. Let $\zeta \in G$ be an odd permutation. Then ζ^2 is an even permutation in G, so it must be e. Thus, every nonidentity element of G has order 2. Suppose η is another odd permutation in G. Then $\zeta\eta$ is an even permutation in G, so $\zeta\eta = e$. Therefore, $\zeta\zeta = e = \zeta\eta$, so $\eta = \zeta$. This shows G has only two elements e and ζ. But then $G \cong \mathbb{Z}_2$ is abelian, contradicting our hypothesis.
Proof 2: If G were not a subgroup of A_n, then the group $GA_n \leq S_n$ would have order larger than $|A_n| = |S_n|/2$. By Lagrange's theorem, then, it would have order $|S_n|$. Therefore, $GA_n = S_n$, and, by the second isomorphism theorem,

$$G/(G \cap A_n) \cong GA_n/A_n = S_n/A_n.$$

This rules out $G \cap A_n = (e)$, since that would give $G \cong S_n/A_n \cong \mathbb{Z}_2$ (abelian), contradicting the hypothesis. $\square$
Remark: Although the first proof above is completely elementary, the second is worth noting since it reveals useful information even when we don't assume G is simple and nonabelian. For example, if G is a subgroup of S_n containing an odd permutation, then we can show that exactly half of the elements of G are even and half are odd. Indeed, under these conditions we have $GA_n = S_n$ (as in the second proof above), and then the second isomorphism theorem implies $|G/(G \cap A_n)| = |S_n/A_n| = 2$. Therefore, $|G \cap A_n| = |G|/2$, which says that half the elements of G belong to A_n, as claimed.

(b) Let $H \leq G$, and let G/H denote the set of left cosets of G. Let $\mathrm{Sym}(G/H)$ denote the group of permutations of the set G/H. Then the map $\lambda : G \to \mathrm{Sym}(G/H)$—defined by $\lambda(g) = \lambda_g$, where $\lambda_g(xH) = gxH$—is a monomorphism of G into $\mathrm{Sym}(G/H)$. To prove this, we first show that $\lambda(g) = \lambda_g$ is indeed a bijection of G/H—that is, $\lambda_g \in \mathrm{Sym}(G/H)$—then we show that λ is a homomorphism of G, and finally we show that the kernel subgroup of λ is $\{e\}$ (so λ is a monomorphism).
To see that λ_g is one-to-one, observe that

$$\begin{aligned} \lambda_g(xH) = \lambda_g(yH) \quad &\Longrightarrow \quad gxH = gyH \quad \Longrightarrow \quad (gy)^{-1}gx \in H \\ &\Longrightarrow \quad y^{-1}x \in H \quad \Longrightarrow \quad xH = yH. \end{aligned}$$

To see that λ_g is onto, fix $xH \in G/H$, and note that $\lambda_g(g^{-1}xH) = gg^{-1}xH = xH$.
To see that λ is a homomorphism, note that for any $xH \in G/H$ we have

$$\lambda_{g_1 g_2}(xH) = g_1 g_2 xH = \lambda_{g_1}(g_2 xH) = \lambda_{g_1} \circ \lambda_{g_2}(xH).$$

That is, $\lambda(g_1 g_2) = \lambda(g_1)\lambda(g_2)$, so λ is a homomorphism.

Finally, note that the kernel subgroup of λ, which we denote by N_λ, is a normal subgroup of G. Therefore, as G is simple, either $N_\lambda = \{e\}$, or $N_\lambda = G$. But

$$\begin{aligned} N_\lambda &= \{g \in G : \lambda_g = \mathrm{id}_{G/H}\} \qquad (\mathrm{id}_{G/H} = \text{the identity map on } G/H) \\ &= \{g \in G : \lambda_g(xH) = xH \text{ for all } x \in G\} \\ &= \{g \in G : gxH = xH \text{ for all } x \in G\} \\ &= \{g \in G : x^{-1}gxH = H \text{ for all } x \in G\} \\ &= \{g \in G : x^{-1}gx \in H \text{ for all } x \in G\}. \end{aligned}$$

If we let $x = e$ in the last expression, we see that N_λ must be a subgroup of H, and since H is properly contained in G, we have $N_\lambda \neq G$. Therefore, $N_\lambda = \{e\}$, which proves that λ is one-to-one, hence a monomorphism.

(c) Let H be a subgroup of G of finite index $n > 1$, so $H \neq G$. We must show that G can be embedded into A_n.

In the previous part of the exercise we saw that λ embeds G into $\mathrm{Sym}(G/H)$, where G/H is the set of left cosets of H in G. Now $[G : H] = |G/H| = n$, which implies that $\mathrm{Sym}(G/H) \cong S_n$, the group of permutations of n elements. Therefore, G is isomorphic to a subgroup of S_n. By the first part above, then, G is isomorphic to a subgroup of A_n.

(d) Suppose G is infinite. We must show that G has no proper subgroup of finite index. If, on the contrary, $H \leq G$ with $[G : H] = n$, then, G would be isomorphic to a subgroup of the finite group A_n, which is clearly impossible, since G is infinite.