

Let  $G$  be a group and suppose  $X$  is a nonempty set of elements of  $G$ . The **subgroup generated by  $X$**  is the smallest subgroup of  $G$  that contains  $X$ . For example, the subgroup of  $\mathbb{Z}_{12}$  generated by the set  $\{4, 6\}$  is  $\{0, 2, 4, 6, 8, 10\}$  (explained below). For a single element  $g \in G$ , we often denote the subgroup generated by the set  $\{g\}$  by  $\langle g \rangle$  instead of  $\langle \{g\} \rangle$ . For small finite sets, like  $\{x, y\}$ , we often write,  $\langle x, y \rangle$  instead of  $\langle \{x, y\} \rangle$ .

A **one-generated subgroup** is a subgroup generated by one element, such as  $\langle g \rangle$ . A one-generated subgroup is also called a **cyclic subgroup**. A **two-generated subgroup** is a subgroup  $\langle x, y \rangle$  that is generated by two elements,  $x$  and  $y$ . An  **$n$ -generated subgroup** is a subgroup of the form  $\langle x_1, \dots, x_n \rangle$ , generated by the  $n$  elements,  $x_1, \dots, x_n$ .

Let  $G$  be a group and let  $H$  be a subgroup of  $G$ . It is important to note the distinction between the following two statements:

1. “The cyclic subgroup  $H$  has two generators  $x$  and  $y$ .”
2. “The subgroup  $H$  is generated by two elements  $x$  and  $y$ .”

The first sentence means  $H = \langle x \rangle = \langle y \rangle$ . That is, you can take either  $x$  or  $y$  as the generator of  $H$ . The second sentence means something entirely different, namely,  $H = \langle x, y \rangle$ . This says that the smallest subgroup of  $G$  that contains both  $x$  and  $y$  is  $H$ . It may or may not be the case that  $H$  is cyclic in this case. The notation  $H = \langle x, y \rangle$  simply means that  $H$  can be generated by two elements. It’s possible that we could find an element that generates  $H$  all by itself. That is, we may have  $H = \langle g \rangle = \langle x, y \rangle$ .

### Examples

1. Consider the subgroup  $H = \{e, (1, 2, 3), (1, 3, 2)\}$  of  $A_4$ , which can be generated by either one (or both) of its non-identity elements:  $H = \langle (1, 2, 3) \rangle = \langle (1, 3, 2) \rangle$ .
2. Continuing with the last example, we could write  $H = \langle (1, 2, 3), (1, 3, 2) \rangle$ . Here we have thrown in a redundant generator, which is harmless, but not helpful because it doesn’t call attention to an important feature of  $H$ —namely, that it is one-generated, i.e., cyclic.
3. As mentioned above, the subgroup of  $\mathbb{Z}_{12}$  generated by the set  $\{4, 6\}$  is  $\{0, 2, 4, 6, 8, 10\}$ . To see this, note that, if 4 and 6 belong to a subgroup of  $\mathbb{Z}_{12}$ , then so must  $4 + 4 = 8$  and  $4 + 6 = 10$  and  $6 + 6 = 0$  and  $4 + 4 + 6 = 2$ .
4. Suppose  $G = \langle a \rangle$  is a cyclic group, suppose  $x = a^6$  and  $y = a^8$ . Then

$$H = \langle x, y \rangle = \langle a^6, a^8 \rangle = \langle a^2 \rangle.$$

See also the CyclicGroupSupplement.pdf document and CyclicGroupExercises.pdf, especially Exercise 6.