

Theorem 1. Let g be an element of a group G and write

$$\langle g \rangle = \{g^k : k \in \mathbb{Z}\}.$$

Then $\langle g \rangle$ is a subgroup of G .

Proof. Since $e = g^0$, $e \in \langle g \rangle$. Suppose $a, b \in \langle g \rangle$. Then $a = g^k$, $b = g^m$ and $ab = g^k g^m = g^{k+m}$. Hence $ab \in \langle g \rangle$ (note that $k + m \in \mathbb{Z}$). Moreover, $a^{-1} = (g^k)^{-1} = g^{-k}$ and $-k \in \mathbb{Z}$, so that $a^{-1} \in \langle g \rangle$. Thus, we have checked the three conditions necessary for $\langle g \rangle$ to be a subgroup of G . \square

Definition 2. If $g \in G$, then the subgroup $\langle g \rangle = \{g^k : k \in \mathbb{Z}\}$ is called the **cyclic subgroup of G generated by g** . If $G = \langle g \rangle$, then we say that G is a **cyclic group** and that g is a **generator** of G .

Examples 3. 1. If G is any group then $\{e\} = \langle e \rangle$ is a cyclic subgroup of G .

2. The group $G = \{1, -1, i, -i\} \subseteq \mathbb{C}^*$ (the group operation is multiplication of complex numbers) is cyclic with generator i . In fact $\langle i \rangle = \{i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i\} = G$. Note that $-i$ is also a generator for G since $\langle -i \rangle = \{(-i)^0 = 1, (-i)^1 = -i, (-i)^2 = -1, (-i)^3 = i\} = G$. Thus a cyclic group may have more than one generator. However, not all elements of G need be generators. For example $\langle -1 \rangle = \{1, -1\} \neq G$ so -1 is not a generator of G .

3. The group $G = U(7)$ = the group of units in \mathbb{Z}_7 is a cyclic group with generator 3. Indeed,

$$\langle 3 \rangle = \{1 = 3^0, 3 = 3^1, 2 = 3^2, 6 = 3^3, 4 = 3^4, 5 = 3^5\} = G.$$

Note that 5 is also a generator of G , but that $\langle 2 \rangle = \{1, 2, 4\} \neq G$ so that 2 is not a generator of G .

4. $G = \langle \pi \rangle = \{\pi^k : k \in \mathbb{Z}\}$ is a cyclic subgroup of \mathbb{R}^* .

5. The group $G = U(8)$ is not cyclic. Indeed, since $U(8) = \{1, 3, 5, 7\}$ and $\langle 1 \rangle = \{1\}$, $\langle 3 \rangle = \{1, 3\}$, $\langle 5 \rangle = \{1, 5\}$, $\langle 7 \rangle = \{1, 7\}$, it follows that $U(8) \neq \langle a \rangle$ for any $a \in U(8)$.

If a group G is written additively, then the identity element is denoted 0, the inverse of $a \in G$ is denoted $-a$, and the powers of a become na in additive notation. Thus, with this notation, the cyclic subgroup of G generated by a is $\langle a \rangle = \{na : n \in \mathbb{Z}\}$, consisting of all the multiples of a . Among groups that are normally written additively, the following are two examples of cyclic groups.

6. The integers \mathbb{Z} are a cyclic group. Indeed, $\mathbb{Z} = \langle 1 \rangle$ since each integer $k = k \cdot 1$ is a multiple of 1, so $k \in \langle 1 \rangle$ and $\langle 1 \rangle = \mathbb{Z}$. Also, $\mathbb{Z} = \langle -1 \rangle$ because $k = (-k) \cdot (-1)$ for each $k \in \mathbb{Z}$.

7. \mathbb{Z}_n is a cyclic group under addition with generator 1.

Theorem 4. Let g be an element of a group $\mathbf{G} = \langle G, \cdot, {}^{-1}, e \rangle$. Then there are two possibilities for the cyclic subgroup $\langle g \rangle$.

Case 1: The cyclic subgroup $\langle g \rangle$ is finite. In this case, there exists a smallest positive integer n such that $g^n = e$ and we have

(a) $g^k = e$ if and only if $n \mid k$.

(b) $g^k = g^m$ if and only if $k \equiv m \pmod{n}$.

(c) $\langle g \rangle = \{e, g, g^2, \dots, g^{n-1}\}$ and the elements $e, g, g^2, \dots, g^{n-1}$ are distinct.

Case 2: The cyclic subgroup $\langle g \rangle$ is infinite. Then

(d) $g^k = e$ if and only if $k = 0$.

(e) $g^k = g^m$ if and only if $k = m$.

(f) $\langle g \rangle = \{\dots, g^{-3}, g^{-2}, g^{-1}, e, g, g^2, g^3, \dots\}$ and all of these powers of g are distinct.

Proof. **Case 1.** Since $\langle g \rangle$ is finite, the powers g, g^2, g^3, \dots are not all distinct, so let $g^k = g^m$ with $k < m$. Then $g^{m-k} = e$ where $m - k > 0$. Hence there is a positive integer l with $g^l = e$. Hence there is a smallest such positive integer. We let n be this smallest positive integer, i.e., n is the smallest positive integer such that $g^n = e$.

(a) If $n \mid k$ then $k = qn$ for some $q \in \mathbb{Z}$. Then $g^k = g^{qn} = (g^n)^q = e^q = e$. Conversely, if $g^k = e$, use the division algorithm to write $k = qn + r$ with $0 \leq r < n$. Then $g^r = g^k(g^n)^{-q} = ee^{-q} = e$. Since $r < n$, this contradicts the minimality of n unless $r = 0$. Hence $r = 0$ and $k = qn$ so that $n \mid k$.

(b) $g^k = g^m$ if and only if $g^{k-m} = e$. Now apply Part (a).

(c) Clearly, $\{e, g, g^2, \dots, g^{n-1}\} \subseteq \langle g \rangle$. To prove the other inclusion, let $a \in \langle g \rangle$. Then $a = g^k$ for some $k \in \mathbb{Z}$. As in Part (a), use the division algorithm to write $k = qn + r$, where $0 \leq r < n$. Then

$$a = g^k = g^{qn+r} = (g^n)^q g^r = e^q g^r = g^r \in \{e, g, g^2, \dots, g^{n-1}\}$$

which shows that $\langle g \rangle \subseteq \{e, g, g^2, \dots, g^{n-1}\}$, and hence that

$$\langle g \rangle = \{e, g, g^2, \dots, g^{n-1}\}.$$

Finally, suppose that $g^k = g^m$ where $0 \leq k < m \leq n-1$. Then $g^{m-k} = e$ and $0 < m-k < n$. This implies that $m-k = 0$ because n is the smallest positive power of g which equals e . Hence all of the elements $e, g, g^2, \dots, g^{n-1}$ are distinct.

Case 2. (d) Certainly, $g^k = e$ if $k = 0$. If $g^k = e$, $k \neq 0$, then $g^{-k} = (g^k)^{-1} = e^{-1} = e$, also. Hence $g^n = e$ for some $n > 0$, which implies that $\langle g \rangle$ is finite by the proof of Part (c), contrary to our hypothesis in Case 2. Thus $g^k = e$ implies that $k = 0$.

(e) $g^k = g^m$ if and only if $g^{k-m} = e$. Now apply Part (d).

(f) $\langle g \rangle = \{g^k : k \in \mathbb{Z}\}$ by definition of $\langle g \rangle$, so all that remains is to check that these powers are distinct. But this is the content of Part (e). \square

Recall that if g is an element of a group G , then the **order** of g is the smallest positive integer n such that $g^n = e$, and it is denoted $|g| = n$. If there is no such positive integer, then we say that g has **infinite order**, denoted $|g| = \infty$. By Theorem 4, the concept of order of an element g and order of the cyclic subgroup generated by g are the same.

Corollary 5. If g is an element of a group G , then $|g| = |\langle g \rangle|$.

Proof. This is immediate from Theorem 4, Part (c). \square

If G is a cyclic group of order n , then it is easy to compute the order of all elements of G . This is the content of the following result.

Theorem 6. Let $G = \langle g \rangle$ be a cyclic group of order n , and let $0 \leq k \leq n-1$. If $m = \gcd(k, n)$, then $|g^k| = \frac{n}{m}$.

Proof. Let $k = ms$ and $n = mt$. Then $(g^k)^{n/m} = g^{kn/m} = g^{msn/m} = (g^n)^s = e^s = e$. Hence n/m divides $|g^k|$ by Theorem 4 Part (a). Now suppose that $(g^k)^r = e$. Then $g^{kr} = e$, so by Theorem 4 Part (a), $n \mid kr$. Hence

$$\frac{n}{m} \mid \frac{k}{m}r$$

and since n/m and k/m are relatively prime, it follows that n/m divides r . Hence n/m is the smallest power of g^k which equals e , so $|g^k| = n/m$. \square

Theorem 7. Let $G = \langle g \rangle$ be a cyclic group where $|g| = n$. Then $G = \langle g^k \rangle$ if and only if $\gcd(k, n) = e$.

Proof. By Theorem 6, if $m = \gcd(k, n)$, then $|g^k| = n/m$. But $G = \langle g^k \rangle$ if and only if $|g^k| = |G| = n$ and this happens if and only if $m = 1$, i.e., if and only if $\gcd(k, n) = 1$. \square

Example 8. If $G = \langle g \rangle$ is a cyclic group of order 12, then the generators of G are the powers g^k where $\gcd(k, 12) = 1$, that is g, g^5, g^7 , and g^{11} . In the particular case of the additive cyclic group \mathbb{Z}_{12} , the generators are the integers 1, 5, 7, 11 (mod 12).

Now we ask what the subgroups of a cyclic group look like. The question is completely answered by Theorem 10. Theorem 9 is a preliminary, but important, result.

Theorem 9. Every subgroup of a cyclic group is cyclic.

Proof. Suppose that $G = \langle g \rangle = \{g^k : k \in \mathbb{Z}\}$ is a cyclic group and let H be a subgroup of G . If $H = \{e\}$, then H is cyclic, so we assume that $H \neq \{e\}$, and let $g^k \in H$ with $g^k \neq e$. Then, since H is a subgroup, $g^{-k} = (g^k)^{-1} \in H$. Therefore, since k or $-k$ is positive, H contains a positive power of g , not equal to e . So let m be the smallest positive integer such that $g^m \in H$. Then, certainly all powers of g^m are also in H , so we have $\langle g^m \rangle \subseteq H$. We claim that this inclusion is an equality. To see this, let g^k be any element of H (recall that all elements of G , and hence H , are powers of g since G is cyclic). By the division algorithm, we may write $k = qm + r$ where $0 \leq r < m$. But $g^k = g^{qm+r} = g^{qm}g^r = (g^m)^qg^r$ so that

$$g^r = (g^m)^{-q}g^k \in H.$$

Since m is the smallest positive integer with $g^m \in H$ and $0 \leq r < m$, it follows that we must have $r = 0$. Then $g^k = (g^m)^q \in \langle g^m \rangle$. Hence we have shown that $H \subseteq \langle g^m \rangle$ and hence $H = \langle g^m \rangle$. That is H is cyclic with generator g^m where m is the smallest positive integer for which $g^m \in H$. \square

Theorem 10 (Fundamental Theorem of Finite Cyclic Groups). Let $G = \langle g \rangle$ be a cyclic group of order n .

1. If H is any subgroup of G , then $H = \langle g^d \rangle$ for some $d \mid n$.
2. If H is any subgroup of G with $|H| = k$, then $k \mid n$.
3. If $k \mid n$, then $\langle g^{n/k} \rangle$ is the unique subgroup of G of order k .

Proof. 1. By Theorem 9, H is a cyclic group and since $|G| = n < \infty$, it follows that $H = \langle g^m \rangle$ where $m > 0$. Let $d = \gcd(m, n)$. Since $d \mid n$ it is sufficient to show that $H = \langle g^d \rangle$. But $d \mid m$ also, so $m = qd$. Then $g^m = (g^d)^q$ so $g^m \in \langle g^d \rangle$. Hence $H = \langle g^m \rangle \subseteq \langle g^d \rangle$. But $d = rm + sn$, where $r, s \in \mathbb{Z}$, so

$$g^d = g^{rm+sn} = g^{rm}g^{sn} = (g^m)^r(g^n)^s = (g^m)^r(e)^s = (g^m)^r \in \langle g^m \rangle = H.$$

This shows that $\langle g^d \rangle \subseteq H$ and hence $\langle g^d \rangle = H$.

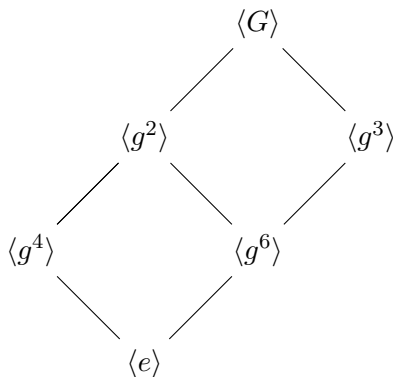
2. By Part (a), $H = \langle g^d \rangle$ where $d \mid n$. Then $k = |H| = n/d$ so $k \mid n$.
3. Suppose that K is any subgroup of G of order k . By Part (a), let $K = \langle g^m \rangle$ where $m \mid n$. Then Theorem 6 gives $k = |K| = |g^m| = n/m$. Hence $m = n/k$, so $K = \langle g^{n/k} \rangle$. This proves (c). □

Remark 11. Part (b) of Theorem 10 is actually true for *any* finite group G , whether or not it is cyclic. This result is Lagrange's Theorem (Theorem 6.5, Page 86 of Judson).

The subgroups of a group G can be diagrammatically illustrated by listing the subgroups, and indicating inclusion relations by means of a line directed upward from H to K if H is a subgroup of K . Such a scheme is called the **lattice diagram** for the subgroups of the group G . We will illustrate by determining the lattice diagram for all the subgroups of a cyclic group $G = \langle g \rangle$ of order 12. Since the order of g is 12, Theorem 10 (c) shows that there is exactly one subgroup $\langle g^d \rangle$ for each divisor d of 12. The divisors of 12 are 1, 2, 3, 4, 6, 12. Then the unique subgroup of G of each of these orders is, respectively,

$$\{e\} = \langle g^{12} \rangle, \quad \langle g^6 \rangle, \quad \langle g^4 \rangle, \quad \langle g^3 \rangle, \quad \langle g^2 \rangle, \quad \langle g \rangle = G.$$

Note that $\langle g^m \rangle \subseteq \langle g^k \rangle$ if and only if $k \mid m$. Hence the lattice diagram of G is:



Finally, here is one more result about cyclic groups that is sometimes useful (for example, in the proof that $U(4n)$ is cyclic—see Homework 5 solutions).

Lemma 12. *A cyclic group contains at most one element of order 2.*

Put another way, an involution¹ of a cyclic group, if it exists, is unique.

Proof. Let $G = \langle a \rangle$ be a cyclic group.

If G is infinite, then there are no elements of order 2. So, assume the order of G is finite: $|G| = n < \infty$. If $n = 1$, then $G = \langle e \rangle$; if $n = 2$, then $G = \{e, a\}$ and $a^2 = e$. In both cases, there is nothing to prove.

Suppose $n > 2$, and let $x, y \in G$ be two non-identity elements of G , say, $x = a^j$ and $y = a^k$, where $1 < j, k < n$. If $x^2 = e$, then $a^{2j} = e$. Therefore n divides $2j$ (by Theorem 4(a)). But $j < n$ implies $2j < 2n$, so the only way to have $n \mid 2j$ is $n = 2j$. If $y^2 = e$, then the same argument applied to k yields $n = 2k$. It follows that if $x^2 = e = y^2$, then $j = k$ and so $x = a^j = a^k = y$. Hence involutions of cyclic groups are unique. □

¹Recall, an *involution* is an element of order 2.