MATH 611 NOTES
GROUPS ACTING ON SETS
OR
G–SETS

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The notion of a group acting on a set is an old one. The groups studied by Galois consisted of groups of permutations of roots of a polynomial. The notion of a group acting on a set is a slightly more general notion. We will study this notion from the point of view of universal algebra. Some of the associated notions are clearer this way.

Definition. Let $\mathbf{G} = \langle G; \cdot\,, ^{-1}, 1\rangle$ be a group. A $\mathbf{G}$–set is a unary algebra $\mathbf{A} = \langle A; O\rangle$ where $O = (\overline{g} : g \in G)$ and $\overline{1_{\mathbf{G}}} = \iota_A$ and $\overline{g} \circ \overline{h} = \overline{g \cdot h}$ for all $g, h \in G$. Often we will use the more suggestive notation $\mathbf{A} = \langle A; G\rangle$ or $\mathbf{A} = \langle A; \overline{G}\rangle$ for a $\mathbf{G}$–set.

So given a group $\mathbf{G}$, the class of all $\mathbf{G}$–sets is a variety.

Proposition. *If* $\mathbf{G}$ *is a group and* $\mathbf{A} = \langle A; \overline{G}\rangle$ *is a* $\mathbf{G}$*–set, then:*

1. *for each* $g \in G$*, the mapping* $\overline{g} : A \longrightarrow A$ *is a bijection (or permutation);*
2. *the mapping which sends* $g \longrightarrow \overline{g}$ *is a homomorphism from* $\mathbf{G}$ *onto* $\langle \overline{G}\,; \circ\,, ^{-1}, \iota_A\rangle$.

Definition. When $\mathbf{G}$ is a group and $\mathbf{A} = \langle A; \overline{G}\rangle$ is a $\mathbf{G}$–set and the homomorphism of the above Proposition is 1–1, we say that *the action of* $\mathbf{G}$ *on* A *is effective or faithful.*

Examples. Suppose $\mathbf{G} = \langle G; \cdot\,, ^{-1}, 1\rangle$ is a group.

1. Cayley's representation or "$\mathbf{G}$ acting on itself by left multiplication" – $\langle G; (\lambda_g : g \in G)\rangle$ is a $\mathbf{G}$–set. (Recall that $\lambda_g(x) = g \cdot x$ for all $x \in G$.)
2. $\mathbf{G}$ acting on itself by conjugation – for $g \in G$ the mapping $\tau_g : G \longrightarrow G$ is defined by $\tau_g(x) = gxg^{-1}$. $\langle G; (\tau_g : g \in G)\rangle$ is a $\mathbf{G}$–set.
3. $\mathbf{G}$ acting on $\mathrm{Sub}(\mathbf{G})$ by conjugation – For $g \in G$ and $H \in \mathrm{Sub}(\mathbf{G})$ we let $\hat{\tau}_g(H) = gHg^{-1}$. Then $\langle \mathrm{Sub}(\mathbf{G})\,; (\hat{\tau}_g : g \in G)\rangle$ is a $\mathbf{G}$–set.
4. $\mathbf{G}$ acting on G/H by left multiplication – Let H be a subgroup of $\mathbf{G}$. We take G/H to be the set of left cosets of $\mathbf{G}$ by H; that is, $G/H = \{aH : a \in G\}$. For $g \in G$ we define the function $\hat{\lambda}_g$ on G/H by $\hat{\lambda}_g(aH) = (ga)H$. Then $\langle G/H\,; (\hat{\lambda}_g : g \in G)\rangle$ is a $\mathbf{G}$–set.
5. $\mathbf{S}_n$ – If $\mathbf{G} = \mathbf{S}_n$, the full symmetric group on n, then $\langle n\,; S_n\rangle$ is a $\mathbf{G}$–set.

Cayley's representation gives a faithful (or effective) action of $\mathbf{G}$ on itself, while "$\mathbf{G}$ acting on itself by conjugation" is a faithful action iff the center of $\mathbf{G}$ is $\{1\}$.

By an abuse of notation, we will refer to the $\mathbf{G}$–set $\langle G/H\,;(\hat{\lambda}_g : g \in G)\rangle$ in (4) as G/H.

Proposition on Reducts. *Suppose that* $\mathbf{G} = \langle G; \cdot\,, {}^{-1}, 1\rangle$ *is a group and* H *is a subgroup of* $\mathbf{G}$ *and* $\mathbf{A} = \langle A; (\overline{g}; g \in G)\rangle$ *is a* $\mathbf{G}$*–set. Then* $\mathbf{A} = \langle A; (\overline{g}; g \in H)\rangle = \langle A; \overline{H}\rangle$ *is an* $\mathbf{H}$*–set (where* $\mathbf{H} = \langle H\,; \cdot\,, {}^{-1}, 1\rangle$*).*

So each of the above examples produces more examples by applying this Proposition.

Exercises

Suppose $\mathbf{G}$ is a group and $\mathbf{A} = \langle A; \overline{G}\rangle$ is a $\mathbf{G}$–set.

(1) If $a \in A$, then the subalgebra generated by a $(= [a]) = \{\overline{g}(a) : g \in G\}$.
(2) If $a, b \in A$ and $b \in [a]$, then $[b] = [a]$.
(3) The one generated subalgebras of $\mathbf{A}$ form a partition of A.
(4) Suppose $A = B \cup C$ and B and C are distinct one generated subalgebras of $\mathbf{A}$. (We let $\mathbf{B} = \langle B; (\overline{g}\big|_B : g \in G)\rangle$ and similarly for $\mathbf{C}$.) If $\Phi \in \mathrm{Con}(\mathbf{B})$ and $\Psi \in \mathrm{Con}(\mathbf{C})$, then $\Phi \cup \Psi \in \mathrm{Con}(\mathbf{A})$.
(5) State and prove a generalization of (4).

Definitions. Suppose $\mathbf{G}$ is a group and $\mathbf{A} = \langle A; \overline{G}\rangle$ is a $\mathbf{G}$–set. The one generated subalgebras of $\mathbf{A}$ are called *orbits* or *the orbits of* $\mathbf{A}$ or *orbits of the action of* $\mathbf{G}$ *on* A. If $\mathbf{A}$ has only one orbit, then $\mathbf{A}$ is said to be *transitive*, or the action of $\mathbf{G}$ on A is said to be *transitive*, or $\overline{G}$ is said to be a *transitive permutation group.*

Let $\sigma \in S_n$, and consider the $\mathbf{S}_n$–set $\langle n; S_n\rangle$. Then an orbit of σ is the same thing as an orbit of the reduct $\langle n; [\sigma]\rangle$, where $[\sigma]$ denotes the subgroup of $\mathbf{S}_n$ generated by σ.

Definition. Suppose $\mathbf{G}$ is a group and $\mathbf{A} = \langle A; \overline{G}\rangle$ is a $\mathbf{G}$–set and $a \in A$.

$$\mathrm{Stab}(a) = \{g \in G : \overline{g}(a) = a\}.$$

$\mathrm{Stab}(a)$ is called the *stabilizer* of a.

Stabilizer Proposition 1. *Suppose* $\mathbf{G}$ *is a group and* $\mathbf{A} = \langle A; \overline{G}\rangle$ *is a* $\mathbf{G}$*–set and* $a \in A$*. Then* $\mathrm{Stab}(a)$ *is a subgroup of* $\mathbf{G}$*.*

Stabilizer Proposition 2. *Suppose* $\mathbf{G}$ *is a group and* $\mathbf{A} = \langle A; \overline{G}\rangle$ *is a* $\mathbf{G}$*–set and* $a \in A$ *and* $b = \overline{g}(a)$*. Then*

$$\mathrm{Stab}(b) = g(\mathrm{Stab}(a))g^{-1}.$$

That is, elements belonging to the same orbit have conjugate stabilizers.

Some authors call isomorphic $\mathbf{G}$–sets *equivalent.*

Theorem 1. *Suppose* $\mathbf{G}$ *is a group and* $\mathbf{A} = \langle A; \overline{G}\rangle$ *is a* $\mathbf{G}$*–set. If* $\mathbf{A}$ *is transitive, then* $\mathbf{A}$ *is isomorphic to the* $\mathbf{G}$*–set* $G/\,\mathrm{Stab}(a)$ *for any* $a \in A$*.*

Corollary 1. *Suppose* $\mathbf{G}$ *is a group and* $\mathbf{A} = \langle A; \overline{G} \rangle$ *is a* $\mathbf{G}$*–set. If* $\mathbf{A}$ *is transitive, then* $|A| = [G : \mathrm{Stab}(a)]$*, the index of the stabilizer of* a *in* $\mathbf{G}$*.*

Corollary 2. *Suppose* $\mathbf{G}$ *is a group and* $\mathbf{A} = \langle A; \overline{G} \rangle$ *is a* $\mathbf{G}$*–set. Then*

$$|A| = \sum_{a \in R} [G : \mathrm{Stab}(a)]$$

where R *is a set containing exactly one element from each orbit.*

Corollary 3. *Suppose* $\mathbf{G}$ *is a group. Then*

$$|G| = |C| + \sum_{g \in T} [G : C(g)]$$

where C *denotes the center of* $\mathbf{G}$ *and* $C(g)$ *denotes the centralizer of* g *(which* $= \{x : xg = gx\}$*) and* T *contains one element from each non trivial conjugacy class of* $\mathbf{G}$*.*

The equation in Corollary 3 is called the *class equation* of $\mathbf{G}$.

Corollary 4. *Let* p *be a prime number. Any finite group of prime power order has a center* $C \neq \{1\}$*.*

Theorem 2. *Suppose* $\mathbf{G}$ *is a group and* $\mathbf{A} = \langle A; \overline{G} \rangle$ *is a* $\mathbf{G}$*–set. If* $\mathbf{A}$ *is transitive, then* $\mathbf{Con}(\mathbf{A})$ *is isomorphic to*

$$\langle \{H \in \mathrm{Sub}(\mathbf{G}) : H \supseteq \mathrm{Stab}(a)\}\, ; \subseteq \rangle$$

for any $a \in A$*.*

Some authors call a $\mathbf{G}$–set *primitive* just in case it is simple. Recall that a simple algebra is one that has exactly two congruences.

Corollary. *Suppose* $\mathbf{G}$ *is a group and* $\mathbf{A} = \langle A; \overline{G} \rangle$ *is a transitive* $\mathbf{G}$*–set.* $\mathbf{A}$ *is primitive iff for any* $a \in A$,* $\mathrm{Stab}(a)$ *is a maximal subgroup of* $\mathbf{G}$*.*

References

Nathan Jacobson, *Basic Algebra I*, Second Edition, W. H. Freeman and Co., New York, 1985.

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