

**Exercises:** 1–7 of the file [GeneralAlgebraNotes.pdf](#). **Submit:** Solutions to Exercises 5 and 7.

**Due:** Friday, 12/5.

**Ex. 5.** Suppose  $K$  and  $L$  are normal subgroups of  $G$ . Prove that  $G/K \cap L$  is isomorphic to a subgroup of  $G/K \times G/L$  (the external direct product),<sup>1</sup> and compute the index of this subgroup in  $G/K \times G/L$ , in terms of  $[G : K]$ ,  $[G : L]$ , and  $[G : KL]$ .

**Solution:** Define  $\varphi : G \rightarrow G/K \times G/L$  by  $\varphi(g) = (gK, gL)$ . First we show  $\varphi$  is a homomorphism. By the definition of coset multiplication and the definition of multiplication in Cartesian products,

$$\begin{aligned}\varphi(g_1g_2) &= (g_1g_2K, g_1g_2L) = (g_1Kg_2K, g_1Lg_2L) \\ &= (g_1K, g_1L)(g_2Kg_2L) = \varphi(g_1)\varphi(g_2).\end{aligned}$$

for all  $g_1, g_2 \in G$ , which proves that  $\varphi$  is a homomorphism from  $G$  to  $G/K \times G/L$ . Therefore, by the First Isomorphism Theorem,

$$G/N_\varphi \cong \varphi(G),$$

where  $N_\varphi$  is the kernel subgroup associated with  $\varphi$ , and where  $\varphi(G) = \{\varphi(g) : g \in G\}$  is the image of  $G$  under the map  $\varphi$ . Since  $\varphi$  is a homomorphism, and since homomorphisms map subgroups to subgroups, it follows that  $\varphi(G)$  is a subgroup of  $G/K \times G/L$ . Thus  $G/N_\varphi$  is isomorphic to a subgroup of  $G/K \times G/L$ .

Finally, the identity element of  $G/K \times G/L$  is  $(K, L)$ , so the kernel subgroup is

$$\begin{aligned}N_\varphi &= \{g \in G : \varphi(g) = (K, L)\} = \{g \in G : (gK, gL) = (K, L)\} \\ &= \{g \in G : gK = K \text{ and } gL = L\} = \{g \in G : g \in K \text{ and } g \in L\} \\ &= K \cap L.\end{aligned}$$

By assumption  $K \cap L = \{e\}$ . Therefore,  $G = G/\{e\} = G/N_\varphi \cong \varphi(G)$ , which is a subgroup of  $G/K \times G/L$ .<sup>2</sup>

To find the index of  $\varphi(G)$  in  $G/K \times G/L$ , in terms of  $[G : K]$ ,  $[G : L]$  and  $[G : K \cap L]$ , note that  $G/K \cap L \cong \varphi(G)$  implies  $|\varphi(G)| = |G/K \cap L| = [G : K \cap L]$ . Also,

$$|G/K \times G/L| = |G/K||G/L| = [G : K][G : L].$$

Therefore, the index of  $\varphi(G)$  in  $G/K \times G/L$  is

$$[(G/K \times G/L) : \varphi(G)] = \frac{|G/K \times G/L|}{|\varphi(G)|} = \frac{[G : K][G : L]}{[G : K \cap L]}.$$

<sup>1</sup>Note that the expression  $G/K \cap L$  can only be interpreted as  $G/(K \cap L)$ , since  $(G/K) \cap L$  doesn't make sense.

<sup>2</sup>The equality  $G = G/\{e\}$  is technically an isomorphism  $G \cong G/\{e\}$ , since  $G/\{e\}$  is a collection of cosets, namely  $G/\{e\} = \{g\{e\} : g \in G\}$ . However, since  $g\{e\} = \{g\}$ , it's common practice to identify the elements of  $G/\{e\} = \{\{g\} : g \in G\}$  with the elements of  $G$ , and say that the quotient group  $G/\{e\}$  is the group  $G$ .

*Note to students:* The above solution illustrates a **standard method** of applying the First Isomorphism Theorem to prove that a factor group  $G/N$  is isomorphic to a subgroup of  $H$ . Let us pause to abstract the essence of this method, so you will know exactly how to apply it in future situations.

To begin with, we are presented with the following data:  $G$ ,  $N$ , and  $H$ , are groups, and  $N$  is a normal subgroup of  $G$  (so  $G/N$  is a factor group). Our objective is to show that  $G/N$  is isomorphic to some subgroup of  $H$ . (The specific subgroup  $K \leq H$  may or may not be specified in advance.)

**Standard method:** To show  $G/N \cong K$  for some subgroup  $K \leq H$ ,

- (a) Find a function  $\varphi : G \rightarrow H$  with the following properties (which you must check):
  - (i)  $\varphi$  is a homomorphism from  $G$  to  $H$
  - (ii) the image of  $\varphi$  is  $K = \varphi(G)$
  - (iii) the kernel subgroup of  $\varphi$  is  $N = \{g \in G : \varphi(g) = e_H\}$
- (b) Conclude by citing the First Isomorphism Theorem, which implies

$$G/N \cong \varphi(G) = K \leq H.$$

Notice that even though our initial goal is to find an isomorphism  $G/N \cong K$ , we don't start by searching around for a one-to-one and onto homomorphisms defined on  $G/N$ . Rather, we start with the (seemingly) more modest (but equivalent) goal of finding a homomorphism from  $G$  to  $H$  with kernel subgroup  $N$  and image  $K$ . Once we accomplish this, the First Isomorphism Theorem does the rest of the work for us.

- Ex. 7.** Let  $G$  be a nonabelian simple group. Let  $S_n$  be the symmetric group of all permutations on an  $n$ -element set, and let  $A_n$  be the alternating group.
- a. Show that if  $G$  is a subgroup of  $S_n$ ,  $n$  finite, then  $G$  is a subgroup of  $A_n$ .
  - b. Let  $H$  be a proper subgroup of  $G$ , and, for  $g \in G$ , let  $\lambda_g$  be the map of the set of left cosets of  $H$  onto themselves defined by  $\lambda_g(xH) = gxH$ . Show that the map  $g \mapsto \lambda_g$  is a monomorphism (injective homomorphism) of  $G$  into the group of permutations of the set of left cosets of  $H$ .
  - c. Let  $H$  be a subgroup of  $G$  of finite index  $n$  and assume  $n > 1$  (so  $H \neq G$ ). Show that  $G$  can be embedded in  $A_n$ . (That is, show that  $G$  is isomorphic to a subgroup of  $A_n$ .)
  - d. If  $G$  is infinite, it has no proper subgroup of finite index.

**Solution:**

- (a) Suppose  $G$  is a subgroup of  $S_n$ ,  $n$  finite, and assume  $G$  is nonabelian and simple. Clearly  $G \neq S_n$ , since  $S_n$  is not simple (for example,  $A_n \triangleleft S_n$ ). Consider  $G \cap A_n$ . This is a normal subgroup of  $G$ . To see this, note that if  $\sigma \in G \cap A_n$  and  $g \in G$ , then  $g\sigma g^{-1} \in G$  and  $g\sigma g^{-1} \in A_n$ , since  $A_n \triangleleft S_n$ , so  $g\sigma g^{-1} \in G \cap A_n$ . So,  $G \cap A_n \triangleleft G$  and  $G$  is simple, so  $G \cap A_n = (e)$  or  $G \cap A_n = G$ . In case  $G \cap A_n = G$ , we have  $G \leq A_n$  and we are done. The other case,  $G \cap A_n = (e)$ , in fact never occurs under the given hypotheses. Two alternative proofs of this fact are given below.

*Proof 1:* Suppose  $G \cap A_n = \{e\}$ . Then  $G$  contains only  $e$  and odd permutations. Let  $\zeta \in G$  be an odd permutation. Then  $\zeta^2$  is an even permutation in  $G$ , so it must be  $e$ . Thus, every nonidentity element of  $G$  has order 2. Suppose  $\eta$  is another odd permutation in  $G$ . Then  $\zeta\eta$  is an even permutation in  $G$ , so  $\zeta\eta = e$ . Therefore,  $\zeta\zeta = e = \zeta\eta$ , so  $\eta = \zeta$ . This shows  $G$  has only two elements  $e$  and  $\zeta$ . But then  $G \cong \mathbb{Z}_2$  is abelian, contradicting our hypothesis.

*Proof 2:* If  $G$  were not a subgroup of  $A_n$ , then the group  $GA_n \leq S_n$  would have order larger than  $|A_n| = |S_n|/2$ . By Lagrange's theorem, then, it would have order  $|S_n|$ . Therefore,  $GA_n = S_n$ , and, by the second isomorphism theorem,

$$G/(G \cap A_n) \cong GA_n/A_n = S_n/A_n.$$

This rules out  $G \cap A_n = \{e\}$ , since that would give  $G \cong S_n/A_n \cong \mathbb{Z}_2$  (abelian), contradicting the hypothesis.  $\square$

**Remark:** Although the first proof above is completely elementary, the second is worth noting since it reveals useful information even when we don't assume  $G$  is simple and nonabelian. For example, if  $G$  is a subgroup of  $S_n$  containing an odd permutation, then we can show that exactly half of the elements of  $G$  are even and half are odd. Indeed, under these conditions we have  $GA_n = S_n$  (as in the second proof above), and then the second isomorphism theorem implies  $|G/(G \cap A_n)| = |S_n/A_n| = 2$ . Therefore,  $|G \cap A_n| = |G|/2$ , which says that half the elements of  $G$  belong to  $A_n$ , as claimed.

- (b) Let  $H \leq G$ , and let  $G/H$  denote the set of left cosets of  $G$ . Let  $\text{Sym}(G/H)$  denote the group of permutations of the set  $G/H$ . Then the map  $\lambda : G \rightarrow \text{Sym}(G/H)$ —defined by  $\lambda(g) = \lambda_g$ , where  $\lambda_g(xH) = gxH$ —is a monomorphism of  $G$  into  $\text{Sym}(G/H)$ . To prove this, we first show that  $\lambda(g) = \lambda_g$  is indeed a bijection of  $G/H$ —that is,  $\lambda_g \in \text{Sym}(G/H)$ —then we show that  $\lambda$  is a homomorphism of  $G$ , and finally we show that the kernel subgroup of  $\lambda$  is  $\{e\}$  (so  $\lambda$  is a monomorphism).

To see that  $\lambda_g$  is one-to-one, observe that

$$\begin{aligned} \lambda_g(xH) = \lambda_g(yH) &\implies gxH = gyH \implies (gy)^{-1}gx \in H \\ &\implies y^{-1}x \in H \implies xH = yH. \end{aligned}$$

To see that  $\lambda_g$  is onto, fix  $xH \in G/H$ , and note that  $\lambda_g(g^{-1}xH) = gg^{-1}xH = xH$ .

To see that  $\lambda$  is a homomorphism, note that for any  $xH \in G/H$  we have

$$\lambda_{g_1g_2}(xH) = g_1g_2xH = \lambda_{g_1}(g_2xH) = \lambda_{g_1} \circ \lambda_{g_2}(xH).$$

That is,  $\lambda(g_1g_2) = \lambda(g_1)\lambda(g_2)$ , so  $\lambda$  is a homomorphism.

Finally, note that the kernel subgroup of  $\lambda$ , which we denote by  $N_\lambda$ , is a normal subgroup of  $G$ . Therefore, as  $G$  is simple, either  $N_\lambda = \{e\}$ , or  $N_\lambda = G$ . But

$$\begin{aligned} N_\lambda &= \{g \in G : \lambda_g = \text{id}_{G/H}\} \quad (\text{id}_{G/H} = \text{the identity map on } G/H) \\ &= \{g \in G : \lambda_g(xH) = xH \text{ for all } x \in G\} \\ &= \{g \in G : gxH = xH \text{ for all } x \in G\} \\ &= \{g \in G : x^{-1}gxH = H \text{ for all } x \in G\} \\ &= \{g \in G : x^{-1}gx \in H \text{ for all } x \in G\}. \end{aligned}$$

If we let  $x = e$  in the last expression, we see that  $N_\lambda$  must be a subgroup of  $H$ , and since  $H$  is properly contained in  $G$ , we have  $N_\lambda \neq G$ . Therefore,  $N_\lambda = \{e\}$ , which proves that  $\lambda$  is one-to-one, hence a monomorphism.

- (c) Let  $H$  be a subgroup of  $G$  of finite index  $n > 1$ , so  $H \neq G$ . We must show that  $G$  can be embedded into  $A_n$ .

In the previous part of the exercise we saw that  $\lambda$  embeds  $G$  into  $\text{Sym}(G/H)$ , where  $G/H$  is the set of left cosets of  $H$  in  $G$ . Now  $[G : H] = |G/H| = n$ , which implies that  $\text{Sym}(G/H) \cong S_n$ , the group of permutations of  $n$  elements. Therefore,  $G$  is isomorphic to a subgroup of  $S_n$ . By the first part above, then,  $G$  is isomorphic to a subgroup of  $A_n$ .

- (d) Suppose  $G$  is infinite. We must show that  $G$  has no proper subgroup of finite index. If, on the contrary,  $H \leq G$  with  $[G : H] = n$ , then,  $G$  would be isomorphic to a subgroup of the finite group  $A_n$ , which is clearly impossible, since  $G$  is infinite.