Exercises: Chapter 14: 1 (except the $GL_2(\mathbb{R})$ example), 2, 3, 9, 11 (justify!).
Due date: Friday, 12/12

Recall, if G acts on a set X and $x, y \in X$, then x is said to be G-*equivalent* to y if there exists a $g \in G$ such that $gx = y$. We write $x \sim_G y$ or $x \sim y$ if x and y are G-equivalent. In class, we proved that the G-equivalent equivalence relation is reflexive and symmetric. You should check the transitive property on your own to complete the proof that $\sim$ is an equivalence relation on X.

14.1 Each of the examples below describes an action of a group G on a set X, which will give rise to the equivalence relation defined by G-equivalence. For each example, compute the equivalence classes of the G-equivalent equivalence relation.

Solution:

(a) Let $G = D_4$ be the symmetry group of a square, $X = \{1, 2, 3, 4\}$ the vertices of the square, and suppose D_4 consists of the following permutations:

$$\{(1), (13), (24), (1432), (1234), (12)(34), (14)(23), (13)(24)\}.$$

Then the D_4-equivalence classes are the orbits under the action of D_4 on X. For example, the orbit of 1 is

$$\mathcal{O}_1 = \{\bar{\sigma}(1) : \sigma \in D_4\} = \{\overline{(\ )}1, \overline{(1234)}1, \overline{(13)}1, \overline{(1432)}1\} = \{1, 2, 3, 4\} = X.$$

So, in this example, we have $X \subseteq \mathcal{O}_1$, and since orbits partition the set X into a union of disjoint orbits (i.e., disjoint G-equivalent equivalence classes), we see that all elements of X are in the same equivalence class. That is, $\mathcal{O}_1 = \mathcal{O}_2 = \mathcal{O}_3 = \mathcal{O}_4 = X$.

(b) Let $X = G$ and let G act on itself by the left regular action $\lambda_g(x) = gx$. Then the orbit of the identity element is all of G,

$$\mathcal{O}_e = \{\lambda_g(e) : g \in G\} = \{ge : g \in G\} = G.$$

So again we have only a single G-equivalent equivalence class, $\mathcal{O}_g = G$, for all $g \in G$.

(c) Let $X = G$, a group and let $H \leq G$. Then G is an H-set under the conjugation action: $\varphi : H \to \mathrm{Sym}(G)$ is defined by $\varphi_h(g) = hgh^{-1}$ for each $h \in H$. Then the orbits are conjugacy classes with respect to H, that is,

$$\mathcal{O}_g = \{\varphi_h(g) : h \in H\} = \{hgh^{-1} : h \in H\}.$$

(d) Let H be a subgroup of G and let $X = G/H$ be the set of left cosets of H. The set G/H is a G-set under the action $\lambda : G \to \mathrm{Sym}(G/H)$ given by $\lambda_g(xH) = gxH$. In this case, all elements of G/H are in the same orbit, so there is just one G-equivalent equivalence class—namely, for every $xH \in G/H$,

$$\mathcal{O}_{xH} = \{\lambda_g(xH) : g \in G\} = \{gxH : g \in G\} = \{gH : g \in G\} = G/H.$$

14.2 Compute all X_g and all G_x for each of the following permutation groups.

(a) $X = \{1, 2, 3\}$,
$G = S_3 = \{(\ ), (12), (13), (23), (123), (132)\}$

(b) $X = \{1, 2, 3, 4, 5, 6\}$,
$G = \{(\ ), (12), (345), (354), (12)(345), (12)(354)\}$

Solution:

(a) If $G = \{(\ ), (12), (13), (23), (123), (132)\}$, and if $X = \{1, 2, 3\}$ is a G-set under the left regular action, $\bar{g} : x \mapsto gx$, then

$$X_{(\ )} = \{x \in X : \overline{(\ )}x = x\} = X \qquad X_{(12)} = \{x \in X : \overline{(12)}x = x\} = \{3\}$$

$$X_{(123)} = \{x \in X : \overline{(123)}x = x\} = \varnothing \qquad X_{(13)} = \{x \in X : \overline{(13)}x = x\} = \{2\}$$

$$X_{(132)} = \{x \in X : \overline{(132)}x = x\} = \varnothing \qquad X_{(23)} = \{x \in X : \overline{(23)}x = x\} = \{1\}$$

$$G_1 = \{g \in G : \bar{g}(1) = 1\} = \{(\ ), (23)\}$$
$$G_2 = \{g \in G : \bar{g}(2) = 2\} = \{(\ ), (13)\}$$
$$G_3 = \{g \in G : \bar{g}(3) = 3\} = \{(\ ), (12)\}$$

(b) If $G = \{(\ ), (12), (345), (354), (12)(345), (12)(354)\}$, and $X = \{1, 2, 3, 4, 5, 6\}$ is a G-set under the left regular action, then

$$X_{(\ )} = X, \quad X_{(12)} = \{3, 4, 5, 6\},$$
$$X_{(345)} = \{1, 2, 6\} = X_{(345)}, \quad X_{(12)(345)} = \{6\} = X_{(12)(354)}.$$
$$G_1 = G_2 = \{(\ ), (345), (354)\}, \quad G_3 = G_4 = G_5 = \{(\ ), (12)\}, \quad G_6 = G.$$

14.3 Compute the G-equivalent equivalence classes of X for each of the G-sets in Exercise 14.2. For each $x \in X$ verify that $|G| = |\mathcal{O}_x| \cdot |G_x|$.

Solution:

(a) If $G = \{(\ ), (12), (13), (23), (123), (132)\}$ and $X = \{1, 2, 3\}$, then under the left regular action, $\mathcal{O}_1 = \mathcal{O}_2 = \mathcal{O}_3 = \{1, 2, 3\} = X$, so, for each $x \in X$, we have $|G| = 6 = 2 \cdot 3 = |\mathcal{O}_x| \cdot |G_x|$.

(b) If $G = \{(\ ), (12), (345), (354), (12)(345), (12)(354)\}$, and $X = \{1, 2, 3, 4, 5, 6\}$, then under the left regular action, $\mathcal{O}_1 = \mathcal{O}_2 = \{1, 2\}$, and $\mathcal{O}_3 = \mathcal{O}_4 = \mathcal{O}_5 = \{3, 4, 5\}$, and $\mathcal{O}_6 = \{6\}$. Therefore, $|\mathcal{O}_6| \cdot |G_6| = 6 \cdot 1 = |G|$, and

$$|\mathcal{O}_x| \cdot |G_x| = 2 \cdot 3 = |G| \text{ for } x \in \{1, 2\}, \text{ and}$$
$$|\mathcal{O}_x| \cdot |G_x| = 3 \cdot 2 = |G| \text{ for } x \in \{3, 4, 5\}.$$

14.9 How many ways can the vertices of an equilateral triangle be colored using three different colors?

Solution: Since the problem does not specify whether we should distinguish colorings that are equivalent up to rotations or reflections, there are three acceptable answers to this question. However, the best solution would consider all reasonable interpretations of the question and provide an answer for each interpretation.

(a) *Solution 1:* If we can distinguish the sides of the triangle, and we don't consider two colorings to be the same if they differ by reflection or rotation, then there are $3^3 = \mathbf{27}$ ways to color the triangle with 3 colors.

(b) *Solution 2:* If we take colorings that differ by a rotation to be equivalent, but we distinguish between colorings that differ by reflection, then we consider the rotation subgroup $\{(\ ), (123), (132)\}$ (which is a subgroup of the six-element group of symmetries of the triangle). Since $(\ ) = (1)(2)(3)$ has 3 cycles and (123) and (132) each have one cycle, by Proposition 14.8 of our textbook, the number of ways to color the sides of a triangle with 3 colors up to rotation is

$$\frac{1}{3}(3^3 + 2 \cdot 3) = 3^2 + 2 = \mathbf{11}.$$

(c) *Solution 3:* If we take colorings that differ by a rotation to be equivalent, and we also take colorings that differ by a reflection to be equivalent, then we factor out by the full group of symmetries of the triangle, $G = \{(\ ), (12), (13), (23), (123), (132)\}$. Since $(12) = (12)(3)$ has two cycles, as does (13) and (23), then by Proposition 14.8, the number of ways to color the sides of a triangle with 3 colors up to rotation and reflection is

$$\frac{1}{6}(3^3 + 2 \cdot 3 + 3 \cdot 3^2) = \frac{1}{6}(2 \cdot 3^3 + 2 \cdot 3) = 3^2 + 1 = \mathbf{10}.$$

14.11 Up to a rotation, how many ways can the faces of a cube be colored with three different colors? (Justify any formula you use.)

Solution: There are 24 rotations of the cube. They are

- one identity "rotation" $(1)(2)(3)(4)(5)(6)$, with 6 cycles in its decomposition,
- six $90°$ rotations of the form $(1234)(5)(6)$, each with 3 cycles,
- eight $120°$ rotations of the form $(145)(263)$, each with 2 cycles,
- six $180°$ rotations of the form $(12)(34)(56)$, each with 3 cycles,
- three $180°$ rotations of the form $(13)(24)(5)(6)$, each with 4 cycles.

Therefore, according to Proposition 14.8, the number of ways to color the faces of a cube with three different colors is

$$\frac{1}{24}(3^6 + 12 \cdot 3^3 + 8 \cdot 3^2 + 3 \cdot 3^4) = \mathbf{57}.$$