Exercises: 1, 2, 3 below and Judson: 9.22, 9.27, 9.31
Recommended: 9.19, 9.21, 9.23, 9.41, 9.42, 9.45
Due date: Friday, 10/31

The first few exercises require some definitions from lecture, repeated here for your convenience.

Let $\mathbf{A} = \langle A, F^{\mathbf{A}}\rangle$ and $\mathbf{B} = \langle B, F^{\mathbf{B}}\rangle$ be two algebras of the same *similarity type*. That is, to each operation symbol $f \in F$ there corresponds an operation $f^{\mathbf{A}}$ defined on $\mathbf{A}$ and an operation $f^{\mathbf{B}}$ defined on $\mathbf{B}$. Thus, the set of operations defined on $\mathbf{A}$ is the set $F^{\mathbf{A}} = \{f^{\mathbf{A}} : f \in F\}$; similarly $F^{\mathbf{B}} = \{f^{\mathbf{B}} : f \in F\}$.

For example, any two groups $\mathbf{G}$ and $\mathbf{H}$ have the same similarity type. To emphasize this, we could denote the operations of these groups using the precise (albeit somewhat awkward) notation of the previous paragraph, as follows:

$$\mathbf{G} = \langle G, \circ^{\mathbf{G}}, \mathrm{inv}^{\mathbf{G}}, e^{\mathbf{G}}\rangle \quad \text{and} \quad \mathbf{H} = \langle H, \circ^{\mathbf{H}}, \mathrm{inv}^{\mathbf{H}}, e^{\mathbf{H}}\rangle.$$

Here $\circ^{\mathbf{G}}$, $\mathrm{inv}^{\mathbf{G}}$, and $e^{\mathbf{G}}$ represent the *interpretation in* $\mathbf{G}$ of the binary, unary (inverse), and nullary (identity) operations that a group must possess (similarly for $\mathbf{H}$).

An *algebra homomorphism* (or simply *homomorphism*), denoted by $\varphi : \mathbf{A} \to \mathbf{B}$, is a function φ with domain A and codomain B that satisfies the following conditions: for each $f \in F$, if f is an n-ary operation symbol, and if $a_1, \dots, a_n \in A$, then

$$\varphi(f^{\mathbf{A}}(a_1, \dots, a_n)) = f^{\mathbf{B}}(\varphi(a_1), \dots, \varphi(a_n)).$$

For example, a *group homomorphism* $\varphi : \mathbf{G} \to \mathbf{H}$ is a function φ with domain G and codomain H that satisfies, $\forall x, y \in G$,

(1) $\varphi(x \circ^{\mathbf{G}} y) = \varphi(x) \circ^{\mathbf{H}} \varphi(y)$,
(2) $\varphi(\mathrm{inv}^{\mathbf{G}}(x)) = \mathrm{inv}^{\mathbf{H}}(\varphi(x))$,
(3) $\varphi(e^{\mathbf{G}}) = e^{\mathbf{H}}$.

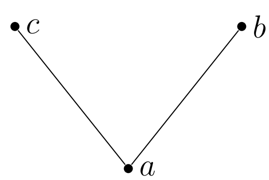
The textbook defines a group *isomorphism* to be a group homomorphism that is both one-to-one and onto. This definition is fine for algebraic structures (like groups). It does not work, however, for relational structures, like posets. (See Exercise 3 below). A definition that works for both algebraic and relational structures is the following: A homomorphism $\varphi : \mathbf{A} \to \mathbf{B}$ is an *isomorphism* if there exists a homomorphism $\psi : \mathbf{B} \to \mathbf{A}$ that composes with φ to give the identity, that is, $\varphi \circ \psi = \mathrm{id}_B$ and $\psi \circ \varphi = \mathrm{id}_A$. (Here, id_X denotes the identity function on the set X: $\mathrm{id}_X(x) = x$.)

Exercises

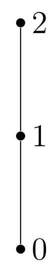
1. When discussing two groups, like **G** and **H** above, our textbook uses more convenient notation, such as $(G, \cdot)$ and $(H, \circ)$ (or, even more simply, G and H). The book will then define a *homomorphism* to be a function $\varphi : G \to H$ satisfying $\varphi(x \cdot y) = \varphi(x) \circ \varphi(y)$. Prove that this is equivalent to the definition given above by showing that conditions (2) and (3) are unnecessary. [Hint: Assuming (1), derive (3), then derive (2).]

2. Define a lattice homomorphism. Then consider a lattice $\mathbf{L} = \langle L, \wedge, \vee \rangle$ and a poset $\mathbf{P} = \langle P, \preccurlyeq \rangle$. Is it possible to define a homomorphism $\varphi : \mathbf{L} \to \mathbf{P}$? Explain.

3. A *poset homomorphism* is an order preserving map. That is, if $\mathbf{P} = \langle P, \leqslant \rangle$ and $\mathbf{Q} = \langle Q, \preccurlyeq \rangle$ are two partially ordered sets, then a homomorphism $\varphi : \mathbf{P} \to \mathbf{Q}$ is a function satisfying, for all $x, y \in P$, if $x \leqslant y$ then $\varphi(x) \preccurlyeq \varphi(y)$. Consider the two definitions of *isomorphism* given in the last paragraph on Page 1 above. Using the two posets shown below, explain why the first of these definitions is not appropriate for posets.



$\mathbf{P} = \langle \{a, b, c\}, \leqslant \rangle$



$\mathbf{Q} = \langle \{0, 1, 2\}, \preccurlyeq \rangle$

9.22 Let G be a group of order 20. If G has subgroups H and K of orders 4 and 5 respectively such that $hk = kh$ for all $h \in H$ and $k \in K$, prove that G is the internal direct product of H and K.

9.27 Let $G \cong H$. Show that if G is cyclic, then so is H.

9.31 Let $\phi : G_1 \to G_2$ and $\psi : G_2 \to G_3$ be isomorphisms. Show that ϕ^{-1} and $\psi \circ \phi$ are both isomorphisms. Using these results, show that the isomorphism of groups determines an equivalence relation on the class of all groups.