Study Guide: Best-Fit Subspaces and Singular Value Decomposition (SVD)

Main Heading: Best-Fit Subspaces

* **Definition**: The best-fit subspace is a k-dimensional subspace that minimizes the sum of the squares of the perpendicular distances of the points to the subspace, or equivalently, maximizes the sum of squares of the lengths of the projections of the points onto this subspace.
* **Characteristics**: The best-fit subspace is found by k applications of the best-fit line algorithm, where on the ith iteration, we find the best-fit line perpendicular to the previous i-1 lines.

Main Heading: Singular Value Decomposition (SVD))

* **Definition**: The Singular Value Decomposition (SVD) is the factorization of a matrix A into the product of three matrices, $A = UDV^T$, where the columns of U and V are orthonormal and the matrix D is diagonal with positive real entries.
* **Characteristics**:
+ The columns of V are the unit-length vectors defining the best-fit lines.
+ The coordinates of a row of U will be the fractions of the corresponding row of A along the direction of this basis, which produces the least possible total sum of squares error for that value of k.

Relationship between SVD and Eigenvalue Decomposition

* **Eigenvalue Decomposition**: A vector v such that $Av = \lambda v$ is called an eigenvector and λ the eigenvalue. For a symmetric matrix A can be expressed as $A = VDV^T$ where the eigenvectors are the columns of V and D is a diagonal matrix with the corresponding eigenvalues on its diagonal.
* **Relationship**: The singular value decomposition is defined for all matrices, whereas the more familiar eigenvector decomposition requires that the matrix A be

square and certain other conditions on the matrix to ensure orthogonality of the eigenvectors.

Applications

* **Best Rank-k Approximations**: The singular value decomposition can be used to find the best rank-k approximation of a matrix.

Examples and Diagram Suggestions

* **Example 1**: Consider a 2x3 matrix A = [[1, 2], [3, 4], [5, 6]]. Find the best-fit 1-dimensional subspace and its singular value decomposition.
* **Diagram Suggestion**: A 2D space with three points (1,2), (3,4), and (5,6) and a line representing the best-fit 1-dimensional subspace.

Summary of Key Points

* Best-fit subspaces are found by minimizing the sum of squares of perpendicular distances or maximizing the sum of squares of lengths of projections onto the subspace.
* Singular Value Decomposition (SVD) is a factorization of a matrix A into three matrices UDV^T.
* SVD is defined for all matrices, whereas eigenvalue decomposition requires a square matrix.
* Best rank-k approximations can be found using SVD.

Flashcards

Q1: What is the definition of a best-fit subspace?

A1: A k-dimensional subspace that minimizes the sum of squares of perpendicular distances of points to the subspace.

Q2: What is the Singular Value Decomposition (SVD) of a matrix A?

A2: A factorization of A into three matrices UDV^T, where U and V are orthonormal and D is diagonal.

Q3: What is the relationship between SVD and eigenvalue decomposition?

A3: SVD is defined for all matrices, whereas eigenvalue decomposition requires a square matrix and certain conditions for orthogonality of eigenvectors.