Dynamic Pressure Threshold Optimization: A Rigorous Framework Using HJB and BSDE Methodologies

Contents

1 Introduction **2**

2 Problem Formulation **2**
2.1 System Dynamics . . . 2
2.2 Objective Function Derivation . . . 3
2.2.1 Derivation of Equation (2) . . . 3
2.2.2 Enhancement with Responsiveness (Equation 4) . . . 3
2.2.3 **Analytical Determination of Constants** . . . 3

3 Backward Stochastic Differential Equations (BSDEs) **4**
3.1 Definition and Context . . . 4

4 Hamilton-Jacobi-Bellman (HJB) Equation **5**
4.1 Derivation . . . 5
4.1.1 Derivation of Equation (13) . . . 5
4.2 Control Law Adaptation Using HJB Equation . . . 5

5 Dynamic Threshold Optimization **6**
5.1 Threshold Transition Criteria . . . 6
5.2 Optimization Problem . . . 6
5.3 Control States and Fast Response Cycle . . . 6

6 Conclusion **6**

1 Introduction

This document presents a highly detailed and rigorous framework for optimizing dynamic pressure thresholds in pressurant systems, particularly in scenarios where fuel tank pressures $P_{\text{fuel}}(t)$ must be dynamically regulated for stability and operational efficiency. The primary aim is to combine the methodologies of Hamilton-Jacobi-Bellman (HJB) equations, Backward Stochastic Differential Equations (BSDEs), and dynamic feedback control to achieve optimal stability over time.

We extend initial research to incorporate predictive modeling, dynamic threshold optimization, and detailed mathematical formulations, considering real-world system constraints like flow rate, tank volume changes, and stochastic disturbances.

2 Problem Formulation

2.1 System Dynamics

The modified system dynamics now consider two control inputs:

$$dP_{fuel}(t) = -\alpha P_{fuel}(t)dt + u_{up}(t)dt - u_{down}(t)dt + \sigma dW(t) \tag{1}$$

where:

- α represents the natural pressure loss rate.
- $u_{up}(t)$ is the control action of the upstream solenoid, regulating pressurant flow into the tank.
- $u_{down}(t)$ is the control action of the downstream solenoid, mitigating excessive pressure drops.
- $\sigma dW(t)$ accounts for stochastic disturbances.

To enhance adaptability, a moving threshold $P_{threshold,down}(t)$ is introduced:

$$P_{threshold,down}(t) = P_{threshold,base} + k\frac{dP_{fuel}}{dt} + c\int_0^t (P_{fuel} - P_{threshold})dt \tag{2}$$

where:

- $P_{threshold,base}$ is the nominal base threshold.
- k adjusts the threshold sensitivity to instantaneous pressure drop rate.
- c accounts for cumulative deviations to allow recovery over longer time horizons.

This moving threshold enables adaptive response, reducing premature solenoid engagement and improving pressure stability in fluctuating conditions.

2.2 Objective Function Derivation

The objective is to design a cost functional that balances pressure stability and control responsiveness, with an emphasis on controlled recovery. Starting with the goal to minimize deviations between $P_{\text{fuel}}(t)$ and $P_{\text{threshold}}(t)$, we include a term penalizing control effort:

$$J(u_{up}, u_{down}) = E\left[\int_0^T \left((P_{threshold} - P_{fuel})^2 + \beta u_{up}^2 + \delta u_{down}^2\right) dt\right] \tag{3}$$

where: - βu_{up}^2 penalizes excessive upstream solenoid control. - δu_{down}^2 penalizes excessive downstream solenoid intervention.

2.2.1 Derivation of Equation (2)

The cost functional is motivated by two key goals:

1. Minimizing deviations: The term $(P_{\text{threshold}}(t) - P_{\text{fuel}}(t))^2$ penalizes differences between the threshold and actual pressures.

2. Penalizing control effort: The quadratic term $\beta u_{up}(t)^2$ ensures that the pressure control action $u_{up}(t)$ remains within efficient limits. While δu_{down}^2 maintains that the recovery control action maintains combustion.

Using the expected value operator $\mathbb{E}[\cdot]$, we integrate over the time horizon $[0, T]$ to account for cumulative deviations and control efforts. This leads directly to the cost functional in Equation (2).

2.2.2 Enhancement with Responsiveness (Equation 4)

To encourage rapid adjustments, we introduce a term $-\gamma|\nabla u(t)|$ that rewards higher oscillatory control. The resulting modified cost functional:

$$J(u_{up}, u_{down}) = E\left[\int_0^T \left((P_{threshold} - P_{fuel})^2 - \gamma|\nabla u_{up}| - \gamma|\nabla u_{down}| + \beta u_{up}^2 + \delta u_{down}^2\right) dt\right] \tag{4}$$

where: - βu_{up}^2 penalizes excessive upstream solenoid control. - δu_{down}^2 penalizes excessive downstream solenoid intervention. - $\gamma|\nabla u_{up}|$ and $\gamma|\nabla u_{down}|$ encourages high-frequency oscillatory control for combustion continuity.

2.2.3 Analytical Determination of Constants

To derive optimal values for the control constants $\alpha, \beta, \gamma, \delta, \lambda$, we use the following analytical approaches:

Stability Analysis

- Linearizing the system dynamics, we express small perturbations around the steady-state as:

$$\frac{d}{dt}\Delta P_{fuel} = A\Delta P_{fuel} + BU \tag{5}$$

where A is the system matrix, B represents control inputs, and $U = [u_{up}, u_{down}]^T$.

- The system stability is determined by the eigenvalues of A. The characteristic equation $\det(A - \lambda I) = 0$ ensures eigenvalues with negative real parts for stability.

Quadratic Cost Minimization

- The Hamilton-Jacobi-Bellman (HJB) equation governs optimal control strategies:

$$\frac{\partial V}{\partial t} + \min_{U} \left[\frac{\partial V}{\partial P}(AP + BU) + L(P, U) \right] = 0 \tag{6}$$

where $L(P, U)$ is the cost function penalizing deviations and control effort.

- The optimal U^* is derived from:

$$U^* = -R^{-1}B^T P \tag{7}$$

where P is the solution to the Riccati equation:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \tag{8}$$

which determines β and δ for minimizing energy consumption and control smoothness.

Frequency Response Analysis

- Applying Laplace transform, the system transfer function is:

$$G(s) = C(sI - A)^{-1}B \tag{9}$$

where C maps state variables to output.

- Using Bode and Nyquist plots, we ensure system stability and desired response characteristics.

- The bandwidth and phase margin constraints determine γ for oscillatory control.

3 Backward Stochastic Differential Equations (BSDEs)

3.1 Definition and Context

BSDEs provide a framework to model future system behavior by incorporating stochastic dynamics backward in time. Consider a system state $Y(t)$ driven by:

$$dY(t) = f(t, Y(t), Z(t))dt + Z(t)dW(t), \tag{10}$$

where:

- $f(t, Y, Z)$ is the generator function modeling system dynamics.

- $Z(t)$ is an adapted process capturing stochastic influence.

For pressure dynamics, $Y(t)$ represents $P_{\text{fuel}}(t)$, and we solve for the backward trajectory given terminal conditions:

$$Y(T) = g(P_{\text{fuel}}(T)), \tag{11}$$

where $g(\cdot)$ encodes terminal objectives (e.g., final pressure thresholds).

4 Hamilton-Jacobi-Bellman (HJB) Equation

4.1 Derivation

The HJB equation provides the optimal control law by minimizing the cost functional. Starting from the dynamic programming principle:

$$V(t,P) = \min_u \mathbb{E}\left[\int_t^T \left((P_{\text{threshold}}(s) - P(s))^2 + \beta u(s)^2\right) ds \middle| P(t) = P\right], \tag{12}$$

we differentiate $V(t,P)$ with respect to time to account for changes in the value function:

$$\frac{\partial V}{\partial t} + \min_u \left[\alpha P \frac{\partial V}{\partial P} + u\frac{\partial V}{\partial P} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial P^2} + (P_{\text{threshold}}(t) - P)^2 + \beta u^2\right] = 0. \tag{13}$$

4.1.1 Derivation of Equation (13)

To find the optimal control $u^*(t)$, differentiate the Hamiltonian H with respect to u:

$$H = \alpha P \frac{\partial V}{\partial P} + u\frac{\partial V}{\partial P} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial P^2} + (P_{\text{threshold}}(t) - P)^2 + \beta u^2. \tag{14}$$

Setting $\frac{\partial H}{\partial u} = 0$ yields:

$$\frac{\partial H}{\partial u} = \frac{\partial V}{\partial P} + 2\beta u = 0, \tag{15}$$

from which:

$$u^*(t) = -\frac{1}{2\beta}\frac{\partial V}{\partial P}. \tag{16}$$

Substituting $u^*(t)$ back into the HJB equation provides the final form:

$$\frac{\partial V}{\partial t} + \alpha P \frac{\partial V}{\partial P} - \frac{1}{4\beta}\left(\frac{\partial V}{\partial P}\right)^2 + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial P^2} + (P_{\text{threshold}}(t) - P)^2 = 0. \tag{17}$$

4.2 Control Law Adaptation Using HJB Equation

Following the Hamilton-Jacobi-Bellman (HJB) framework, we define the value function:

$$V(t,P) = \min_{u_{up}, u_{down}} E\left[\int_t^T \left((P_{threshold} - P)^2 + \beta u_{up}^2 + \delta u_{down}^2\right) ds \mid P(t) = P\right] \tag{18}$$

The optimal control actions are derived from:

$$u_{up}^*(t) = -\frac{1}{2\beta}\frac{\partial V}{\partial P}, \quad u_{down}^*(t) = \frac{1}{2\delta}\frac{\partial V}{\partial P} \tag{19}$$

where u_{down} is engaged only when pressure descent exceeds $P_{threshold,down}$.

5 Dynamic Threshold Optimization

5.1 Threshold Transition Criteria

Dynamic thresholds $P_{\text{threshold}}(t)$ are determined in real time to stabilize $P_{\text{fuel}}(t)$ over specified durations Δt_n. The transition criteria to a new threshold $P_{\text{threshold},n+1}$ include:

- **Stability**: If $|P_{\text{fuel}}(t) - P_{\text{threshold}}(t)| > \epsilon$, transition is triggered.
- **Recovery potential**: Evaluate whether the pressurant tank can maintain the next threshold based on pressure differentials.
- **Predicted behavior**: Use BSDEs or numerical simulations to estimate the stability duration Δt_n.

5.2 Optimization Problem

At each time step t, the threshold $P_{\text{threshold}}(t)$ is updated by solving:

$$P_{\text{threshold}}(t) = \arg\min_{P} \int_{t}^{t+\Delta t_n} \left[(P_{\text{fuel}}(t) - P)^2 + \lambda \cdot \text{instability}(P_{\text{fuel}}, P) \right] dt, \tag{20}$$

where λ penalizes rapid transitions to ensure smooth operation and $\text{instability}(P_{\text{fuel}}, P)$ accounts for oscillatory behavior.

5.3 Control States and Fast Response Cycle

The control system transitions between different states based on real-time feedback, ensuring a rapid response cycle:

1. **Nominal Regulation State**: The upstream solenoid maintains the required tank pressure within the target range.
2. **Recovery Damping State**: When the pressure drops faster than $P_{threshold,down}$, the downstream solenoid engages to slow the descent.
3. **High-Frequency Oscillation State**: The system maintains pressure with minimal deviation using fast solenoid switching.

This fast response cycle is achieved through real-time execution of a control loop with minimal computation delay. The implemented code leverages adaptive step-size integration to ensure high-frequency response while avoiding numerical instability.

6 Conclusion

This framework combines HJB equations, BSDEs, and feedback control to dynamically regulate pressure thresholds in real-time systems. The approach accounts for coupled dynamics between $P_{\text{fuel}}(t)$ and $P_{\text{threshold}}(t)$, ensuring stability and efficiency over varying operational conditions.