Discrete Mathematics

Reference Books:

1. Discrete Mathematics and Its Applications 7^{th} ed, By Kenneth H. Rosen

This PDF contains the notes from the standards books and are only meant for GATE CSE aspirants.

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Discrete Mathematics

1.1 **Propositions**

A **proposition** is a declarative sentence (that is, a sentence that declares a fact) that is **either true or false, but not both**.

3. $x + 1 = 2$.
4. $x + y = z$.

Sentences 3 and 4 are **not propositions because they are neither true nor false**. Note that each of sentences 3 and 4 can be turned into a proposition if we assign values to the variables.

Let p be a proposition. The *negation of* p, denoted by $\neg p$ (also denoted by $\overline{p}$), is the statement

"It is not the case that p."

The proposition $\neg p$ is read "not p." The truth value of the negation of p, $\neg p$, is the opposite of the truth value of p.

Example: Find the negation of the proposition

"Vandana's smartphone has **at least** 32GB of memory"

Negation:

1. "Vandana's smartphone does not have at least 32GB of memory"
2. "Vandana's smartphone has less than 32GB of memory."

Let p and q be propositions. The *conjunction* of p and q, denoted by $p \wedge q$, is the proposition "p and q." The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

Note that in logic the word **"but" sometimes is used instead of "and"** in a conjunction. For example, the statement "The sun is shining, but it is raining" is another way of saying "The sun is shining and it is raining."

Let p and q be propositions. The *disjunction* of p and q, denoted by $p \vee q$, is the proposition "p or q." The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

Inclusive OR: "Students who have taken calculus or computer science can take this class."
Here, we mean that students who have taken both calculus and computer science can take the class, as well as the students who have taken only one of the two subjects.

Exclusive OR only those who have taken **exactly** one of the two courses can take the class. Students who have taken both calculus and a computer science course cannot take the class.

Let p and q be propositions. The *exclusive or* of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

Implication (→)

Let p and q be propositions. The *conditional statement* $p \to q$ is the proposition "if p, then q." The conditional statement $p \to q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \to q$, p is called the *hypothesis* (or *antecedent* or *premise*) and q is called the *conclusion* (or *consequence*).

Statement **p → q** is true when both p and q are true and when p is false (no matter what truth value q has).

"if p, then q"
"if p, q"
"p is sufficient for q"
"q if p"
"q when p"
"a necessary condition for p is q"
"q unless $\neg p$"
"p implies q"
"p only if q"
"a sufficient condition for q is p"
"q whenever p"
"q is necessary for p"
"q follows from p"

1. **If p, q** comma represents 'then'
2. **p is sufficient for q** p is sufficient for being q true
3. **q whenever p** q is true whenever p is true
4. **q when p** q is true when p is true
5. **q is necessary for p** q is necessary to be true for being p true
6. **q follows from p** the value of q as true follows the true value of p
7. **q if p** q is true if p is true
8. **p only if q** p only will be true if q is true or p can't be true if q is not true
9. **q unless ¬p** it means if ¬p is false then q must be true (if p is true then q must be true). q is true unless p is not true. q unless ¬p and p → q always have the same truth value.

p: Maria learns discrete mathematics
q: Maria will find a good job

a. "If Maria learns discrete mathematics, then she will find a good job."
b. "Maria will find a good job when she learns discrete mathematics."
c. "For Maria to get a good job, it is **sufficient** for her to learn discrete mathematics."
d. "Maria will find a good job **unless she does not** learn discrete mathematics."

Example:

(i) **I eat ice cream unless it rains**.

A = I eat ice cream
B = It rains
I eat ice cream unless it rain. Or if it doesn't rain then I eat ice cream

Note: "q unless ~p" means p → q, and "q unless p" means ~p → q
~B → A

(ii) If X then Y unless Z
= (x→Y) unless Z
= ~Z→(X→Y)
= Z V ~X V Y or ~X V Z V Y

a. **p → q (Implication)**
b. **q → p (Converse)**
c. **¬p → ¬q (Inverse)**
d. **¬q → ¬p (Contra-positive)**

Example: what are the contrapositive, the converse, and the inverse of the conditional statement?
"The home team wins whenever it is raining"

a. **Implication** if It is raining then home team wins (p →q)
b. **Converse** if home team wins then it is raining (q → p)
c. **Inverse** If it is not raining then home team doesn't win (**¬p → ¬q)**
d. **Contra-positive** if home team doesn't win then it is not raining (**¬q → ¬p)**

BICONDITIONALS

Let p and q be propositions. The *biconditional statement* $p \leftrightarrow q$ is the proposition "p if and only if q." The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called *bi-implications*.

The statement p ↔ q is true when both the **conditional statements p → q and q → p are true** and is false otherwise. **p ↔ q has exactly the same truth value as (p → q) ∧ (q → p)**.

"p is necessary and sufficient for q"
"if p then q, and conversely"
"p iff q."

IMPLICIT USE OF BICONDITIONALS

Biconditionals are not always explicit in natural language. In particular, the "if and only if" construction used in biconditionals is rarely used in common language. Instead, **biconditionals are often expressed using an "if, then" or an "only if" construction. The other part of the "if and only if" is implicit**. That is, the converse is implied, but not stated.

For example, consider the statement in English "If you finish your meal, then you can have dessert." What is really meant is "You can have dessert if and only if you finish your meal." This last statement is logically equivalent to the two statements "If you finish your meal, then you can have dessert" and "You can have dessert only if you finish your meal."

Precedence of Logical Operators

TABLE 8 Precedence of Logical Operators.	
Operator	*Precedence*
$\neg$	1
$\land$ $\lor$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

Bitwise Operators

A *bit string* is a sequence of zero or more bits. The *length* of this string is the number of bits in the string.

101010011 is a bit string of length nine.

We define the **bitwise OR**, **bitwise AND**, and **bitwise XOR** of two strings of the **same length** to be the strings that have as their bits the OR, AND, and XOR of **the corresponding bits** in the two strings, respectively.

Example-

```
01 1011 0110
11 0001 1101
------------
11 1011 1111   bitwise OR
01 0001 0100   bitwise AND
10 1010 1011   bitwise XOR
```

1.2 Applications of Propositional Logic

Translating English Sentences

a. **"You can access the Internet from campus only if you are a computer science major or you are not a freshman."**

"p only if q"
Let
a: You can access the Internet from campus
c: you are a computer science major
f: you are a freshman

$a \rightarrow (c \vee \sim f)$

b. **"You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old."**
q: you can ride the roller coaster
r: you are under 4 feet tall
s: you are older than 16 years old

Formula: q if p, unless z

If you are under 4 feet tall and you're not older than 16 years, then you cannot ride roller coaster.

$(r \wedge \sim s) \rightarrow \sim q$

$\sim s \rightarrow (r \rightarrow \sim q) = s \vee \sim r \vee \sim q = (\sim s \wedge r) \rightarrow \sim q$

System Specifications

Example: **Express the specification "The automated reply cannot be sent when the file system is full"** using logical connectives.
p: the automated reply can be sent
q: the file system is full

Our specification can be represented by the conditional statement $\mathbf{q \rightarrow \sim p}$.
System specifications should be consistent, that is, they should not contain conflicting requirements that could be used to derive a contradiction.

Example: Determine whether these system specifications are consistent:
"The diagnostic message is stored in the buffer or it is retransmitted."
"The diagnostic message is not stored in the buffer."

"If the diagnostic message is stored in the buffer, then it is retransmitted."

p: The diagnostic message is stored in the buffer.
q: The diagnostic message is retransmitted.
The specifications can then be written as
$p \vee q$, $\sim p$, $p \rightarrow q$

An assignment of truth values **that makes all three specifications true** must have p false to make ¬p true. Because we want p V q to be true but p must be false, q must be true. We conclude that these **Specifications are consistent, because they are all true when p is false and q is true**.
Example in the last example if we add "The diagnostic message is not retransmitted", does it still remain consistent?
$p \vee q$
$\sim p$
$P \rightarrow q$
$\sim q$
The three specifications from that example are true only in the case when p is false and q is true. However, this new specification is ¬q, which is false when q is true. **Consequently, these four specifications are inconsistent**.

1.2 Exercises

1. You cannot edit a protected Wikipedia entry unless you are an administrator. Express your answer in terms of e: "You can edit a protected Wikipedia entry" and a: "You are an administrator."

$\sim a \rightarrow \sim e$ or $e \rightarrow a$

3. You can graduate only if you have completed the requirements of your major and you do not owe money to the university and you do not have an overdue library book. Express your answer in terms of g: "You can graduate," m: "You owe money to the university," r: "You have completed the requirements of your major," and b: "You have an overdue library book."

$g \rightarrow (r \wedge (\sim m) \wedge (\sim b))$

7. Express these system specifications using the propositions p "The message is scanned for viruses" and q "The message was sent from an unknown system" together with logical connectives (including negations).

a) "The message is scanned for viruses whenever the message was sent from an unknown system."

b) "The message was sent from an unknown system but it was not scanned for viruses."

c) "It is necessary to scan the message for viruses whenever it was sent from an unknown system."

d) "When a message is not sent from an unknown system it is not scanned for viruses."

a. $q \rightarrow p$ (q whenever p)

b. q ∧ ~p
c. q → p (q whenever p)
d. ~q → ~p

8. Express these system specifications using the propositions p "The user enters a valid password," q "Access is granted," and r "The user has paid the subscription fee" and logical connectives (including negations).

a) "The user has paid the subscription fee, but does not enter a valid password."

b) "Access is granted whenever the user has paid the subscription fee and enters a valid password."

c) "Access is denied if the user has not paid the subscription fee."

d) "If the user has not entered a valid password but has paid the subscription fee, then access is granted."

a. r ∧ ~p
b. (r ∧ p) →q (q whenever p)
c. ~r → ~q
d. (~p ∧ r) → q

9. Are these system specifications consistent? "The system is in multiuser state if and only if it is operating normally. If the system is operating normally, the kernel is functioning. The kernel is not functioning or the system is in interrupt mode. If the system is not in multiuser state, then it is in interrupt mode. The system is not in interrupt mode."

m: ''The system is in multiuser state"
n: "The system is operating normally"
k: "The kernel is functioning"
i: "The system is in interrupt mode"

m ←→n
n → k
~k V i
~m → i
~i
For being system is not in interrupt mode, ~i will be true, **i will be false**.
~I → m, hence the system is in multiuser state. **m is true**.
~k V i, i is false hence ~k has to be true, **k is false**
If k is false then n will be false, **n is false.**
if n is false then m will be false too. **m is false.**

There is a contradiction for m, hence system is not consistent.
From book:

In order for this to happen, clearly i must be false. In order for $\neg m \rightarrow i$ to be true when i is false, the hypothesis $\neg m$ must be false, so m must be true. Since we want $m \leftrightarrow n$ to be true, this implies that n must also be true. Since we want $n \rightarrow k$ to be true, we must therefore have k true. But now if k is true and i is false, then the third specification, $\neg k \vee i$ is false. Therefore we conclude that this system is not consistent.

10. Are these system specifications consistent? "Whenever the system software is being upgraded, users cannot access the file system. If users can access the file system, then they can save new files. If users cannot save new files, then the system software is not being upgraded."

p: software is being upgraded
q: user can access the file system
r: users can save new files.

[q, whenever p] or [whenever p, q]

p → ~q [~p V q~] if P is true then q has to be false.
q → r [~q V r] Suppose r = T, q can be T or F
~r → ~p [r V ~p] p can be either true or false, if r is false then p has to be false.

Conclusion: r=T, p=T, q=F or r=T, p=F, q=F

11. Are these system specifications consistent? "The router can send packets to the edge system only if it supports the new address space. For the router to support the new address space it is necessary that the latest software release be installed. The router can send packets to the edge system if the latest software release is installed, The router does not support the new address space."

s: The router can send packets to the edge systems
a: router supports the new address space
r: the latest software release is installed

s → a (a is false, s must be false)
a→r (if r is false then a must be false) [q is necessary for p]
r → s (s is false, hence r must be false too)
~a (for being ~a True, a must be false)

System is **consistent**, for a = F, r=F, s=F

12. Are these system specifications consistent? "If the file system is not locked, then new messages will be queued. If the file system is not locked, then the system is functioning normally, and conversely. If new messages are not queued, then they will be sent to the message buffer. If the file system is not locked, then new messages will be sent to the message buffer. New messages will not be sent to the message buffer."

s: the file system is locked
q: new message will be queued
n: system is functioning normally
b: messages will be sent to message buffer
x
~s →q, ~s ←→ n, ~q → b, ~s →b, ~b
b=F, s=T, q=T, n=F, System is **consistent**.

***15.** Each inhabitant of a remote village always tells the truth or always lies. A villager will give only a "Yes" or a "No" response to a question a tourist asks. Suppose you are a tourist visiting this area and come to a fork in the road. One branch leads to the ruins you want to visit; the other branch leads deep into the jungle. A villager is standing at the fork in the road. What one question can you ask the villager to determine which branch to take?

“"**If I were to ask you whether the right branch leads to the ruins, would you say 'yes'?**"

Case 1: right branch leads to the ruins.

a. Villager always speak truth, if we ask him
   "Whether the right branch leads to the ruins" – Yes
   "If I were to ask you whether the right branch leads to the ruins, would you say 'yes'?" - Yes

b. Villager always speaks lie. If we ask him
   "Whether the right branch leads to the ruins" – No
   "If I were to ask you whether the right branch leads to the ruins, would you say 'yes'?" – Yes
   Because his actual answer is "No" and he always speak lie so will say "Yes".

Case 2: If right branch doesn't lead to the ruins.

c. Villager always speak truth, if we ask him
   "Whether the right branch leads to the ruins" – No
   "If I were to ask you whether the right branch leads to the ruins, would you say 'yes'?" - No

d. Villager always speaks lie. If we ask him
   "Whether the right branch leads to the ruins" – Yes
   "If I were to ask you whether the right branch leads to the ruins, would you say 'yes'?" – No
   Because his actual answer is "Yes" and he always speak lie so will say "No".

16. An explorer is captured by a group of cannibals. There are two types of cannibals—those who always tell the truth and those who always lie. The cannibals will barbecue the explorer unless he can determine whether a particular cannibal always lies or always tells the truth. He is allowed to ask the cannibal exactly one question.

a) Explain why the question "Are you a liar?" does not work.

b) Find a question that the explorer can use to determine whether the cannibal always lies or always tells the truth.

a) **Case 1:** Cannibal speaks truth.
   Are you a liar? He will say "No"
   Case 2: Cannibal speaks lie.
   Are you a liar? He will say "No"

We reached no conclusion!

b) "If I were to ask you whether you speak truth, would you say yes?"
   Case 1: Cannibal speaks truth, he will say "Yes", because his answer would be Yes.

Case 2: Cannibal speaks lie, He will say "No" because he would "yes" if that ques is asked to him.

17. When three professors are seated in a restaurant, the hostess asks them: "Does everyone want coffee?" The first professor says: "I do not know." The second professor then says: "I do not know." Finally, the third professor says: "No, not everyone wants coffee." The hostess comes back and gives coffee to the professors who want it. How did she figure out who wanted coffee?

The question was "Does everyone want coffee?" **If the first professor did not want coffee, then he would know that the answer to the hostess's question was "no."** Therefore we-and the hostess and the remaining professors-know that the first professor does want coffee. The same argument applies to the second professor, so she, too, must want coffee. The third professor can now answer the question. Because she said "no,'' we conclude that she does not want coffee. Therefore the hostess knows to bring coffee to the first two professors but not to the third.

Exercises 19–23 relate to inhabitants of the island of knights and knaves created by Smullyan, where knights always tell the truth and knaves always lie. You encounter two people, A and B. Determine, if possible, what A and B are if they address you in the ways described. If you cannot determine what these two people are, can you draw any conclusions?

19. A says "At least one of us is a knave" and B says nothing.

20. A says "The two of us are both knights" and B says "A is a knave."

21. A says "I am a knave or B is a knight" and B says nothing.

22. Both A and B say "I am a knight."

19. If A is a knight, then he is telling the truth, in which case B must be a knave. Since B said nothing, that is certainly possible. If A is a knave, then he is lying, which means that his statement that at least one of them is a knave is false; hence they are both knights. That is a contradiction. **So we can conclude that A is a knight and B is a knave**.

20. If A is knight then he is telling the truth, in which case B must be a knight. But B says A is a knave, but as per the statement of A, B must be telling truth there is a contradiction here.
If A is knave, then he is telling the lie, it means either B is knave or knight, then he is only lying. B says A is knave, **which is a truth, hence B is knight and A is a knave**.

21. If A is a knight, then he is telling the truth, in which case B must be a knight as well, since A is not a knave. (**If p V q and p is false, then q must be true**.) Since B said nothing, that is certainly possible. If A is a knave, then his statement is patently true, but that is a contradiction to the behavior of knaves. **So we can conclude that A is a knight and B is a knight**.

22. Both A and B say "I am a knight"

if A is:	***if B is;***	***Truth values of statements***	***Is assertion possible?***
knight	***knight***	***A(T), B(T)***	***Yes***
knight	***knave***	***A(T), B(F)***	***Yes***
knave	***knight***	***A(F), B(T)***	***Yes***
knave	***knave***	***A(F), B(F)***	***Yes***

It is possible for either A or B to be either a knight or a knave.

23. A says “We are both knaves” and B says nothing.
If A is knave then B must be knight. As A is lying.
If A is a knight then we are both knaves not possible because it’s a contradiction to his behaviour.

A is knave and B is knight.

36. Four friends have been identified as suspects for an unauthorized access into a computer system. They have made statements to the investigating authorities. Alice said “Carlos did it.” John said “I did not do it.” Carlos said “Diana did it.” Diana said “Carlos lied when he said that I did it.”

a) If the authorities also know that exactly one of the four suspects is telling the truth, who did it? Explain your reasoning.

b) If the authorities also know that exactly one is lying, who did it? Explain your reasoning.

The authorities know that only one is telling the truth
This indicates the solution is revealed after figuring out which suspect is telling the truth.

Alice said “Carlos did it.”
1. Is Alice's statement true?
If Alice's satement were considered to be true it would mean Carlos did it, but John said he didn't do it which would make John's statement also true if Carlos did it. There can only be one true statement. So Alice is lying.

Carlos said “Diana did it.”
2. Is Carlos's statement true?
If Carlos's satement were considered to be true it would mean Diana did it, but John said he didn't do it which would make John's statement also true if Diana did it. There can only be one true statement. So Carlos is lying.

Diana said “Carlos lied when he said that I did it.”
3. Is Diana's statement true?
We already proved Carlos lied. So when Diana said “Carlos lied when he said that I did it.” Diana is telling the truth

John said “I did not do it.”
4. Is John's statement true?
We proved Diana is telling the truth. We know only one statement is true. So when John said “I did not do it.” he was lying and guilty.

1.3 Propositional Equivalences

A **compound proposition** that is always true, no matter what the truth values of the propositional variables that occur in it, is called a **tautology**. E.g. p V ~p

A compound proposition that is always false is called a **contradiction**. E.g. p∧~p

A compound proposition that is neither a tautology nor a contradiction is called a **contingency**.

Logical Equivalences

Compound propositions that have the same truth values in all possible cases are called **logically equivalent**.

The compound propositions p and q are called *logically equivalent* if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

Example: Show that **¬(p V q)** and **¬p Λ¬q** are logically equivalent.
Because the truth values of the compound propositions ¬(p V q) and ¬p Λ¬q agree for all possible combinations of the truth values of p and q, it follows that**¬(p V q) ↔ (¬p Λ¬q)** is a tautology and that these compound propositions are logically equivalent.

TABLE 3 Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

EXAMPLE 8 Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution: To show that this statement is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to $\mathbf{T}$. (*Note:* This could also be done using a truth table.)

$$
\begin{aligned}
(p \wedge q) \rightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && \text{by Example 3} \\
&\equiv (\neg p \vee \neg q) \vee (p \vee q) && \text{by the first De Morgan law} \\
&\equiv (\neg p \vee p) \vee (\neg q \vee q) && \text{by the associative and commutative laws for disjunction} \\
&\equiv \mathbf{T} \vee \mathbf{T} && \text{by Example 1 and the commutative law for disjunction} \\
&\equiv \mathbf{T} && \text{by the domination law}
\end{aligned}
$$

◀

Propositional Satisfiability

Satisfiability:
A compound proposition is **satisfiable** if there is an assignment of truth values to its variables that makes it true.

Unsatisfiability:
When the compound **proposition is false for all assignments of truth values to its variables**, the compound proposition is **unsatisfiable**. Note that a compound proposition is unsatisfiable if and only if its negation is true for all assignments of truth values to the variables, that is, if and only if **its negation is a tautology**.

When we find a particular assignment of truth values that makes a compound proposition true, we have shown that it is **satisfiable**; **such an assignment is called a solution of this particular satisfiability problem.**
However, to show that a compound proposition is **unsatisfiable**, we need to show that every assignment of truth values to its variables makes it false.

Validity:
A compound proposition is **valid** if for all assignments of truth values to its variables makes it true. Its negation will be **unsatisfiable**.

Functionally complete
A collection of logical operators is called **functionally complete** if every compound proposition is logically equivalent to a compound proposition involving only these logical operators.

1.4 <u>Predicates and Quantifiers</u>

Statements involving variables, such as
"$x > 3$," "$x = y + 3$," "$x + y = z$,"

And

"Computer x is under attack by an intruder,"

And

"Computer x is functioning properly,"
These statements are **<u>neither true nor false</u>** when the values of the variables are not specified.

The statement **"x is greater than 3"** has two parts.

The variable x, **<u>is the subject of the statement</u>**.

The **predicate**, "**<u>is greater than 3</u>**"refers to a **<u>property</u>** that the **<u>subject</u>** <u>of the statement can have</u>.

We can denote the statement "x is greater than 3" by P(x).

P(x): **<u>P denotes the predicate "is greater than 3" and x is the variable.</u>**

The statement **<u>P(x) is also said to be the value of the propositional function P at x</u>**. Once a value has been assigned to the variable x, the statement P(x) <u>becomes a proposition and has a truth value</u>.

<u>We can also have statements that involve more than one variable</u> For instance, consider the statement "$x = y + 3$.
We can denote this statement by **Q(x, y), <u>where x and y are variables and Q is the predicate</u>**.

<u>Quantifiers</u>
Quantification expresses the <u>extent to which a predicate is true</u> over a range of elements.

<u>Universal quantification</u> which tells us that a predicate is true for every element under consideration.
The universal quantification of P(x) for a **<u>particular domain</u>** is the proposition that asserts that P(x) is true for **<u>all values of x in this domain</u>**.

> The *universal quantification* of $P(x)$ is the statement
>
> "$P(x)$ for all values of x in the domain."
>
> The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the **universal quantifier.** We read $\forall x P(x)$ as "for all $x P(x)$" or "for every $x P(x)$." An element for which $P(x)$ is false is called a **counterexample** of $\forall x P(x)$.

<u>Example:</u> Let P(x) be the statement "$x + 1 > x$." What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real numbers?
Because P(x) is true for all real numbers x, the quantification **$\forall x P(x)$ is true**.

Note: - Generally, an implicit assumption is made that all domains of discourse for quantifiers are nonempty. Note that if the domain is empty, then **∀xP(x) is true for any propositional function P(x) because there are no elements x in the domain for which P(x) is false**.

TABLE 1 **Quantifiers.**

Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

A statement ∀xP(x) is false, where P(x) is a propositional function, if and only if P(x) is not always true when x is in the domain.

When all the elements in the domain can be listed—say, x1, x2. . . xn—it follows that the universal quantification ∀xP(x) is the same as the conjunction

P(x1) ∧ P(x2) ∧ · · · ∧ P(xn),

because this conjunction is true if and only if P(x1), P(x2), . . . , P (xn) are all true.

Existential quantification

Which tells us that there is **one or more** element under consideration for which the predicate is true.

The area of logic that deals with predicates and quantifiers is called the **predicate calculus**.

With existential quantification, we form a proposition that is true if and only if P(x) is true for **at least** one value of x in the domain.

The *existential quantification* of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$."

We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the *existential quantifier*.

Observe that the statement ∃xP(x) is false if and only if there is no element x in the domain for which P(x) is true. That is, ∃xP(x) is false if and only if P(x) is false for every element of the domain.

Note:

Generally, an implicit assumption is made that all domains of discourse for quantifiers are **nonempty**. **If the domain is empty, then ∃xQ(x) is false** whenever Q(x) is a propositional function because when the domain is empty, there can be no element x in the domain for which Q(x) is true.

When all elements in the domain can be listed—say, $x_1, x_2, \ldots, x_n$— the existential quantification $\exists x P(x)$ is the same as the disjunction

$$P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n),$$

<u>THE UNIQUENESSQUANTIFIER (∃! or ∃1)</u>

The notation **∃!xP(x)** [or **∃1xP(x)**] states "**<u>There exists a unique x(exactly one) such that P(x) is true</u>**."

<u>Quantifiers with Restricted Domains</u>

An abbreviated notation is often used to restrict the domain of a quantifier. In this notation, <u>a condition a variable must satisfy is included after the quantifier</u>.

EXAMPLE 17 What do the statements $\forall x < 0\,(x^2 > 0)$, $\forall y \neq 0\,(y^3 \neq 0)$, and $\exists z > 0\,(z^2 = 2)$ mean, where the domain in each case consists of the real numbers?

Solution: The statement $\forall x < 0\ (x^2 > 0)$ states that for every real number x with $x < 0$, $x^2 > 0$. That is, it states "The square of a negative real number is positive." This statement is the same as $\forall x(x < 0 \to x^2 > 0)$.

The statement $\forall y \neq 0\,(y^3 \neq 0)$ states that for every real number y with $y \neq 0$, we have $y^3 \neq 0$. That is, it states "The cube of every nonzero real number is nonzero." Note that this statement is equivalent to $\forall y(y \neq 0 \to y^3 \neq 0)$.

Finally, the statement $\exists z > 0\,(z^2 = 2)$ states that there exists a real number z with $z > 0$ such that $z^2 = 2$. That is, it states "There is a positive square root of 2." This statement is equivalent to $\exists z(z > 0 \wedge z^2 = 2)$. ◀

Note that the restriction of a universal quantification is the same as the universal quantification of a conditional statement. For instance, $\forall x < 0\ (x^2 > 0)$ is another way of expressing $\forall x(x < 0 \to x^2 > 0)$. On the other hand, the restriction of an existential quantification is the same as the existential quantification of a conjunction. For instance, $\exists z > 0\ (z^2 = 2)$ is another way of expressing $\exists z(z > 0 \wedge z^2 = 2)$.

<u>Precedence of Quantifiers</u>

The quantifiers ∀ and ∃ have **<u>higher precedence</u>** than all logical operators from propositional calculus.
For example, **∀xP(x) ∨ Q(x)** is the disjunction of ∀xP(x) and Q(x). In other words, it means (∀xP(x)) ∨ Q(x) <u>rather than</u> ∀x(P(x) ∨ Q(x)).

<u>Binding Variables</u>

When a quantifier is used on the variable x, we say that this <u>occurrence of the variable is bound</u>.
An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be **<u>free</u>**.
All the variables that occur in a propositional function **<u>must be bound or set equal to a particular value to turn it into a proposition</u>**. This can be done using a combination of universal quantifiers, existential quantifiers, and value assignments.

The part of a logical expression to which a quantifier is applied is called the **<u>scope</u>** of this quantifier. Consequently, a variable is **<u>free</u>** <u>if it is outside the scope of all quantifiers</u> in the formula that specify this variable.

In the statement **$\exists x(x + y = 1)$**, the variable x is bound by the existential quantification $\exists x$, but the variable y is free because it is not bound by a quantifier and no value is assigned to this variable. This illustrates that in the statement $\exists x(x + y = 1)$, x is bound, but y is free.

In the statement **$\exists x(P(x) \land Q(x)) \lor \forall x R(x)$**, all variables are bound. The scope of the first quantifier, $\exists x$, is the expression $P(x) \land Q(x)$ because $\exists x$ is applied only to $P(x) \land Q(x)$, and not to the rest of the statement. Similarly, the scope of the second quantifier, $\forall x$, is the expression $R(x)$.

That is, the existential quantifier binds the variable x in $P(x) \land Q(x)$ and the universal quantifier $\forall x$ binds the variable x in $R(x)$. Observe that **we could have written our statement using two different variables x and y**, as $\exists x(P(x) \land Q(x)) \lor \forall y R(y)$, because the scopes of the two quantifiers do not overlap. The reader should be aware that in common usage, the same letter is often used to represent variables bound by different quantifiers with scopes that do not overlap.

Logical Equivalences Involving Quantifiers

Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions. We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

$$\forall x(P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x).$$

$$\exists x(P(x) \lor Q(x)) = \exists x P(x) \lor \exists x Q(x)$$

Note that,

1. We can distribute a universal quantifier over a conjunction.
2. We can distribute an existential quantifier over a disjunction.

Negating Quantified Expressions

"Every student in your class has taken a course in calculus."

This statement is a universal quantification, namely $\forall x P(x)$.

Where $P(x)$ is the statement "x has taken a course in calculus" and the domain consists of the students in your class. The negation of this statement is "It is not the case that every student in your class has taken a course in calculus." This is equivalent to "There is a student in your class who has not taken a course in calculus."

$$\exists x \neg P(x).$$

$$\neg\forall x P(x) \equiv \exists x \neg P(x).$$

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Note:
1. $\neg\forall xP(x)$ is the same as $\neg(P(x1) \wedge P(x2) \wedge \cdots \wedge P(xn))$, which is equivalent to $\neg P(x1) \vee \neg P(x2) \vee$ ……. $\vee \neg P(xn)$
2. $\neg\exists xP(x)$ is the same as $\neg(P(x1) \vee P(x2) \vee \cdots \vee P(xn))$, which by De Morgan's laws is equivalent to $\neg P(x1) \wedge \neg P(x2) \wedge \cdots \wedge \neg P(xn)$, and this is the same as $\forall x \neg P(x)$.

Example What are the negations of the statements $\forall x(x2 > x)$ and $\exists x(x2 = 2)$?

Solution: The negation of $\forall x(x^2 > x)$ is the statement $\neg\forall x(x^2 > x)$, which is equivalent to $\exists x\neg(x^2 > x)$. This can be rewritten as $\exists x(x^2 \leq x)$. The negation of $\exists x(x^2 = 2)$ is the statement $\neg\exists x(x^2 = 2)$, which is equivalent to $\forall x\neg(x^2 = 2)$. This can be rewritten as $\forall x(x^2 \neq 2)$. The truth values of these statements depend on the domain. ◀

EXAMPLE 22 Show that $\neg\forall x(P(x) \rightarrow Q(x))$ and $\exists x(P(x) \wedge \neg Q(x))$ are logically equivalent.

$P(x) \rightarrow Q(x) = \sim P(x) \vee Q(x)$

$= \neg\forall x(\sim P(x) \vee Q(x))$

$= \exists x(P(x) \wedge \neg Q(x))$

Translating from English into Logical Expressions

Express the statement "Every student in this class has studied calculus" using predicates and quantifiers.
We introduce **C(x)**, which is the statement "x has studied calculus."

1. If the domain for x consists of the students in the class, we can translate our statement as **AxC(x)**
2. If we change the domain to consist of all people, we will need to express our statement as

   "For every person x, if person x is a student in this class then x has studied calculus."

If S(x) represents the statement that person x is in this class, we see that our statement can be expressed as $\forall x(S(x) \rightarrow C(x))$.

Note: Our statement cannot be expressed as $\forall x(S(x) \wedge C(x))$ because this statement says that all people are students in this class and have studied calculus!

We may prefer to use the **two-variable quantifier** **Q(x, y)** for the statement "student x has studied subject y." Then we would replace C(x) by **Q(x, calculus)** in both approaches to obtain
$\forall xQ(x, \text{calculus})$ or $\forall x(S(x) \rightarrow Q(x, \text{calculus}))$.

Example express the statements "Some student in this class has visited Mexico" and "Every student in this class has visited either Canada or Mexico" using predicates and quantifiers.

Solution the statement "Some student in this class has visited Mexico" means that

"There is a student in this class with the property that the student has visited Mexico."

We can introduce a variable x, so that our statement becomes

"There is a student x in this class having the property that x has visited Mexico."

We introduce **M(x),** which is the statement "**x has visited Mexico**."

If the domain for x consists of the students in this class, we can translate this first statement as **∃xM(x)**.

However, if we are interested in people other than those in this class, we look at the statement a little differently. Our statement can be expressed as

"There is a person x having the properties that x is a student in this class and x has visited Mexico."

In this case, the domain for the variable x consists of all people. We introduce **S(x)** to represent "x is a student in this class."

$$\exists x(S(x) \land M(x))$$

Note: Our statement **cannot be expressed** as **∃x(S(x) → M(x)),** which is true when there is someone not in the class because, in that case, for such a person x, **S(x) → M(x)** becomes either **F→T** or **F→F**, both of which are true.

Similarly, the second statement can be expressed as

"For every x in this class, x has the property that x has visited Mexico or x has visited Canada."

We see that if the domain for x consists of the students in this class, this second statement can be expressed as **∀x(C(x) ∨ M(x))**.

However, if the domain for x consists of all people, our statement can be expressed as

"For every person x, if x is a student in this class, then x has visited Mexico or x has visited Canada."

In this case, the statement can be expressed as **∀x(S(x) → (C(x) ∨M(x)))**.

Instead of using M(x) and C(x) to represent that x has visited Mexico and x has visited Canada, respectively, we could use a two-place predicate V (x, y) to represent "**x has visited country y**." In this case, V (x, Mexico) and V (x, Canada) would have the same meaning as M(x) and C(x) and could replace them in our answers.

Example Use predicates and quantifiers to express the system specifications "**Every mail** message larger than one megabyte will be compressed" and "If a user is active, at least one network link will be available."

Let **S(m, y)** be "Mail message m is larger than y megabytes," where the variable m has the domain of all mail messages and the variable y is a positive real number, and let **C(m)** denote "Mail message m will be compressed." Then the specification "Every mail message larger than one megabyte will be compressed" can be represented as ∀

$$\forall m(S(m, 1) \rightarrow C(m))$$

Let A(u) represent "User u is active," where the variable u has the domain of all users, let S(n, x) denote "Network link n is in state x," where n has the domain of all network links and x has the domain of all possible states for a network link. "If a user is active, at least one network link will be available" can be represented by **$\exists u A(u) \rightarrow \exists n S(n, \text{available})$**.

Note: Here we are using $\rightarrow$ with existential quantifier, because there is a difference b/w "There exists a user who is active" and "if an active user exists".

Example Consider these statements. The first two are called premises and the third is called the conclusion. The entire set is called an argument. Assuming that the domain consists of all creatures.

"All lions are fierce." $\forall x(P(x) \rightarrow Q(x))$.
"Some lions do not drink coffee." $\exists x(P(x) \land \neg R(x))$.
"Some fierce creatures do not drink coffee." $\exists x(Q(x) \land \neg R(x))$.

P(x): x is a lion
Q(x): x is fierce
R(x): x drinks coffee

Example Consider these statements, of which the first three are premises and the fourth is a valid conclusion.
"All hummingbirds are richly colored."
"No large birds live on honey."
"Birds that do not live on honey are dull in color."
"Hummingbirds are small."

Let P(x), Q(x), R(x), and S(x) be the statements "x is a hummingbird," "x is large," "x lives on honey," and "x is richly colored," respectively. Assuming that the domain consists of all birds, express the statements in the argument using quantifiers and P(x), Q(x), R(x), and S(x).

Solution: We can express the statements in the argument as

$\forall x(P(x) \rightarrow S(x))$.
$\neg\exists x(Q(x) \land R(x))$.
$\forall x(\neg R(x) \rightarrow \neg S(x))$.
$\forall x(P(x) \rightarrow \neg Q(x))$.

______________________________________Exercise_________________________________

43. Determine whether $\forall x(P(x) \rightarrow Q(x))$ and $\forall x P(x) \rightarrow \forall x Q(x)$ are logically equivalent. Justify your answer.

Solution: No, both are not logically equivalent.

$\forall x(P(x) \rightarrow Q(x)) \leftrightarrow \forall xP(x) \rightarrow \forall xQ(x)$?

	Physics	Chemistry
Raju	X	✓
Rani	✓	X

P(x) = x is passed in physics
Q(x) = x is passed in chemistry

$\forall xP(x) \rightarrow \forall xQ(x)$ is true LHS is false all people are not passed in physics, hence over all proposition is true.

$\forall x(P(x) \rightarrow Q(x))$ is false because Rani is passed in physics but failed in chemistry. **Both are not equivalent as their truth values are different**.

$$\forall x(P(x) \rightarrow Q(x)) \rightarrow \forall xP(x) \rightarrow \forall xQ(x)$$

44. Determine whether $\forall x(P(x) \leftrightarrow Q(x))$ and $\forall x\ P(x) \leftrightarrow \forall x Q(x)$ are logically equivalent. Justify your answer.

$(\forall x(P(x) \leftrightarrow Q(x))) \leftrightarrow (\forall x\, P(x) \leftrightarrow \forall xQ(x))$?

P(x) = x is passed in physics
Q(x) = x is passed in chemistry

	Physics	Chemistry
Raju	✓	✓
Rani	X	X
Sudha	✓	X

LHS is false for Sudha as she passed in physics but failed in chemistry.
RHS is true because either not all people passed in physics or all people failed in chemistry.
LHS = False
RHS = True
Hence, not equivalent.

$$(\forall x(P(x) \leftrightarrow Q(x))) \rightarrow (\forall x\, P(x) \leftrightarrow \forall xQ(x))$$

45. Show that $\exists x(P(x) \vee Q(x))$ and $\exists x P(x) \vee \exists x Q(x)$ are logically equivalent.

P(x): x passed in physics
Q(x): x passed in chemistry
LHS = there is someone who is passed in physics or passed in chemistry
RHS = there are some people who are passed in physics or some people are passed in chemistry.

Problem $\forall x\ ((P(x) \wedge Q(x)) \Leftrightarrow (\forall x P(x) \wedge \forall x Q(x))$?

Yes, both are equivalent, domain for x is all the students in the class.
LHS: All students are passed in physics and chemistry
RHS: All people are passed in physics and all people are passed in chemistry.

Note: - $\exists x P(x) \vee Q(x) \mathrel{!=} \exists x[P(x) \vee Q(x)]$ because LHS contains a free variable, x is not bounded for Q(x).

Problem $\forall x\ ((P(x) \vee Q(x)) \Leftrightarrow (\forall x P(x) \vee \forall x Q(x))$?

Both are not equivalent!
LHS: All students are passed either in physics or chemistry.
RHS: All students are passed in physics or all students are passed in chemistry.

	Physics	Chemistry
Raju	✓	X
Rani	X	✓

LHS= True
RHS= False

$$(\forall x P(x) \vee \forall x Q(x)) \rightarrow \forall x\ ((P(x) \vee Q(x))$$

Problem $(\exists x P(x) \wedge \exists x\, Q(x)) \Leftrightarrow (\exists x[P(x) \wedge Q(x)])$?

No, both are not logically equivalent.

LHS: some people are passed in physics and some people are passed in chemistry.
RHS: there is someone who is passed in physics and chemistry both.

$$(\exists x[P(x) \wedge Q(x)]) \rightarrow (\exists x P(x) \wedge \exists x\, Q(x))$$

Null quantification – which we can use when a quantified variable does not appear in part of a statement.

46. Establish these logical equivalences, where x does not occur as a free variable in A. Assume that the domain is nonempty.

a) $(\forall x P(x)) \vee A \equiv \forall x(P(x) \vee A)$

b) $(\exists x P(x)) \vee A \equiv \exists x(P(x) \vee A)$

a) Suppose that A is true. Then the left-hand side is logically equivalent to $\forall P(x)$, since the conjunction of any proposition with a true proposition has the same truth value as that proposition. By similar reasoning the right-hand side is equivalent to $\forall P(x)$.
On the other hand, suppose that A is false. Then again both LHS and RHS are same.

b) This problem is similar to part (a). If A is true, then both sides are logically equivalent to $\exists x P(x)$. If A is true or false.

47. Establish these logical equivalences, where x does not occur as a free variable in A. Assume that the domain is nonempty.

a) $(\forall x P(x)) \wedge A \equiv \forall x(P(x) \wedge A)$

b) $(\exists x P(x)) \wedge A \equiv \exists x(P(x) \wedge A)$

a) If A is true then both sides will be equivalent. If A is false then both sides will be false.
b) This problem is similar to part (a). If A is true, then both sides are logically equivalent to ∃xP(x). If A is false, then both sides are false.

48. Establish these logical equivalences, where x does not occur as a free variable in A. Assume that the domain is nonempty.

a) $\forall x(A \rightarrow P(x)) \equiv A \rightarrow \forall x P(x)$

b) $\exists x(A \rightarrow P(x)) \equiv A \rightarrow \exists x P(x)$

a) **Suppose A is false**. Then A → p(x) is trivially true because if hypothesis is false then conditional statement is trivially true.
   Second case if A is true. Then there are two sub-cases.
   a. P(x) is true for every x, then left hand side is true, because if hypothesis and conclusion both are true then conditional proposition is true. Same reasoning can be given for right hand side also, right-hand side is also true as p(x) is true for every x.
   b. P(x) is true for some x, left-hand side is false, because for those objects that do not have property P, the conditional A→P(x) is false, and hence it is not true that for all objects in the domain A→P(x) is true.
      For right hand side it will always be false because A is true and ∀xP(x) is false

Hence, both propositions are equivalent.

b) If A is false, then both left-hand and right-hand sides are trivially true as hypothesis is false.
   If A is true, then there are two sub-cases.
   a. P(x) is true for every x, then left-hand side is true, and same reasoning can be given for right hand-side, and right-hand side is also true.
   b. If P(x) is true for some x, left hand side is true and right hand side is also true.

Hence, both propositions are equivalent.

49. Establish these logical equivalences, where x does not occur as a free variable in A. Assume that the domain is nonempty.

a) $\forall x(P(x) \rightarrow A) \equiv \exists x P(x) \rightarrow A$

b) $\exists x(P(x) \rightarrow A) \equiv \forall x P(x) \rightarrow A$

We can establish these equivalences by arguing that one side is true if and only if the other side is true. For both parts, we will look at the two cases: either A is true or A is false.

a) Suppose A is true. Then for each x, P(x) → A is true, because a conditional statement with a true conclusion is always true; therefore the left-hand side is always true in this case.
   By similar reasoning the right-hand side is always true in this case (here we used the fact that the domain is nonempty). Therefore the two propositions are logically equivalent when A is **true**.

On the other hand, suppose that A is false. There are two sub cases.

a. If p(x) is false for every x, then p(x) → A is trivially true (a conditional statement with a false hypothesis is true) so the left-hand side is trivially true.
   The same reasoning shows that the right-hand side is also true, because in this subcase ∃xP(x) is false.
b. For the second subcase, suppose that P(x) is true for some x, then for that x, p(x) →A is false as p(x) is true and A is false, so the left hand side is false.
   The right hand side is also false as in this case ∃xP(x) is true but A is false. Thus in all cases, **the two propositions have the same truth value**.

b) This problem is similar to A part. If A is true then both sides are trivially true, because the conditional statements are true with true conclusions.
   If A is **false,** then there are two subcases.
   a. If p(x) is false for some x, then p(x) →A is trivially true for that x. So the left hand side is true. The same reasoning shows that the right-hand side is true. Because in this sub case ∀xP(x) is false.
   b. For the second subcase, suppose that P(x) is true for each x, then for every x p(x) → A will be false, so left-hand side is false (there is no x making conditional statement true).
      The right hand side is also false, because it is a conditional statement with a true hypothesis and false conclusion.

Thus, in all cases, the two propositions have the same truth value.

59. Let $P(x)$, $Q(x)$, and $R(x)$ be the statements "x is a professor," "x is ignorant," and "x is vain," respectively. Express each of these statements using quantifiers; logical connectives; and $P(x)$, $Q(x)$, and $R(x)$, where the domain consists of all people.

a) No professors are ignorant.

b) All ignorant people are vain.

c) No professors are vain.

d) Does (c) follow from (a) and (b)?

a)
b) ∀x(Q(x) → R(x))
c) ~∃x(P(x) ∧ R(x)) or ∀x(P(x) → ~R(x))
d) The conclusion **(part (c)) does not follow**. There may well be vain professors, since the premises do not rule out the possibility that there are vain people besides the ignorant ones.

60. Let $P(x)$, $Q(x)$, and $R(x)$ be the statements "x is a clear explanation," "x is satisfactory," and "x is an excuse," respectively. Suppose that the domain for x consists of all English text. Express each of these statements using quantifiers, logical connectives, and $P(x)$, $Q(x)$, and $R(x)$.

a) All clear explanations are satisfactory.

b) Some excuses are unsatisfactory.

c) Some excuses are not clear explanations.

***d)** Does (c) follow from (a) and (b)?

a) $\forall x(P(x) \rightarrow Q(x))$
b) $\exists x(R(x) \wedge \sim R(x))$
c) $\exists x(R(x) \wedge \sim P(x))$
d) Yes, (c) follow from (a) and (b)

61. Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be the statements "x is a baby," "x is logical," "x is able to manage a crocodile," and "x is despised," respectively. Suppose that the domain consists of all people. Express each of these statements using quantifiers; logical connectives; and $P(x)$, $Q(x)$, $R(x)$, and $S(x)$.

a) Babies are illogical.
b) Nobody is despised who can manage a crocodile.
c) Illogical persons are despised.
d) Babies cannot manage crocodiles.
***e)** Does (d) follow from (a), (b), and (c)? If not, is there a correct conclusion?

a) $\forall x(P(x) \rightarrow \sim Q(x))$
b) If a person can manage a crocodile, then that person is not despised
$\forall x(R(x) \rightarrow \sim S(x))$
c) $\forall x(\sim P(x) \rightarrow S(x))$
d) $\forall x(P(x) \rightarrow \sim R(x))$
e) Yes, it follows.

Nested Quantifiers

Nested quantifiers, where one quantifier is within the scope of another, such as $\forall x \exists y(x + y = 0)$.

Note that everything within the scope of a quantifier can be thought of as a propositional function. For example,

$$\forall x \exists y(x + y = 0)$$

Is same thing as **$\forall x Q(x)$ where $Q(x)$ is $\exists y P(x, y)$, where $P(x, y)$ is $x + y = 0$**.

Understanding Statements Involving Nested Quantifiers

Example Assume that the domain for the variables x and y consists of all real numbers. Statement

$$\forall x \forall y(x + y = y + x)$$

Says that x+y = y+x for all real number x and y. this is commutative law for addition of real numbers. Likewise the statement

$\forall x \exists y(x + y = 0)$ says that for every real number x there is a real number y such that x + y = 0.

(x+y = 0) = for some specific x there is a specific y such that x+y = 0
$\exists y(x + y = 0)$ for some specific x, there exists some y such that x+y = 0
$\forall x \exists y(x + y = 0)$ for all x, there exists some y, such that x+y = 0

Example Translate into English the statement

$$\forall x \forall y((x > 0) \wedge (y < 0) \rightarrow (xy < 0)),$$

This statement says that for every real number x and for every real number y, If x > 0 and y < 0, then xy < 0. That is, this statement says that for real numbers x and y, if x is positive and y is negative, then xy is negative.

The Order of Quantifiers

$$\forall x \forall y P(x, y) \Leftrightarrow \forall y \forall x P(x, y)$$

Let P(x, y) be the statement "x + y = y + x." **∀x∀yP(x, y)** says "For all real numbers x, for all real numbers y, x + y = y + x."

Example Let Q(x, y) denote "x + y = 0." What are the truth values of the quantifications ∃y∀xQ(x, y) and ∀x∃yQ(x, y), where the domain for all variables consists of all real numbers?

Q(x, y) denote "x + y = 0."

∀x∃yQ(x, y), = "For every real number x there is a real number y such that x+y = 0;

∃y∀xQ(x, y) = There is a real number y such that for every real number x, x+y=0.

$$\exists y \forall x Q(x, y) \rightarrow \forall x \exists y Q(x, y))$$

Example Let Q(x, y, z) be the statement "x + y = z." What are the truth values of the statements ∀x∀y∃zQ(x, y, z) and ∃z∀x∀yQ(x, y, z), where the domain of all variables consists of all real numbers?

Solution

Q(x, y, z): x+ y = z

∃zQ(x, y, z): for x and y, there exists some real number z such that x + y = z

∀x∀y[∃zQ(x, y, z)]: for all real numbers x and for all real numbers y there is a real number z such that x + y = z.

TABLE 1 Quantifications of Two Variables.

Statement	*When True?*	*When False?*
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

$\exists z \forall x \forall y Q(x, y, z)$
"There is a real number z such that for all real numbers x and for all real numbers y it is true that x + y = z,"
Is false because there is no value of z that satisfies the equation x + y = z for all values of x and y.

Translating Mathematical Statements into Statements Involving Nested Quantifiers

Example translate the statement "The sum of two positive integers is always positive" into a logical expression.

To translate this statement into a logical expression, we first rewrite it so that the implied quantifiers and a domain are shown:
"For every two integers, if these integers are both positive, then the sum of these integers is positive."

Next, we introduce the variables x and y to obtain "For all positive integers x and y, x + y is positive."

$$\forall x \forall y((x > 0) \land (y > 0) \to (x + y > 0)),$$

Where the domain for both variables consists of all integers.
Note that we could also translate this using the positive integers as the domain. Then the statement "The sum of two positive integers is always positive" becomes "For every two positive integers, the sum of these integers is positive. We can express this as

$$\forall x \forall y(x + y > 0),$$

Where the domain for both variables consists of all positive integers.

Example Translate the statement "Every real number except zero has a multiplicative inverse." (A multiplicative inverse of a real number x is a real number y such that xy = 1.)
Solution
We can rewrite this as "For every real number x, if x != 0, then there exists a real number y such that xy = 1." This can be rewritten as

$$\forall x((x = 0) \to \exists y(xy = 1)).$$

Translating from Nested Quantifiers into English

Example translate the statement

$$\forall x(C(x) \lor \exists y(C(y) \land F(x, y)))$$

into English, where C(x) is "x has a computer," F(x, y) is "x and y are friends," and the domain for both x and y consists of all students in your school.

"Every student in your school has a computer or has a friend who has a computer"

Example translate the statement

$$\exists x \forall y \forall z((F(x, y) \land F(x, z) \land (y != z)) \to \neg F(y,z))$$

into English, where F(a,b) means a and b are friends and the domain for x, y, and z consists of all students in your school.

Solution
If students x and y are friends, and students x and z are friends, and furthermore, if y and z are not the same student, then y and z are not friends.
In other words, there is a student none of whose friends are also friends with each other.

Translating English Sentences into Logical Expressions

Example
Express the statement "If a person is female and is a parent, then this person is someone's mother" as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.

Solution
"For every person x, if person x is female and person x is a parent, then there exists a person y such that person x is the mother of person y."
We introduce the propositional functions
F(x): x is female,"
P(x): x is a parent,"
M(x, y) to represent "x is the mother of y.
The original statement can be represented as

$$\forall x((F(x) \wedge P(x)) \rightarrow \exists y M(x, y))$$

We can move $\exists y$ to the left so that it appears just after $\forall x$, because y does not appear in $F(x) \wedge P(x)$.
We obtain the logically equivalent expression

$$\forall x \exists y((F(x) \wedge P(x)) \rightarrow M(x, y))$$

Example express the statement "Everyone has exactly one best friend" as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.
Solution
The statement "Everyone has exactly one best friend" can be expressed as "For every person x, person x has exactly one best friend."
This statement is the same as "$\forall x$(person x has exactly one best friend),"

To say that x has exactly one best friend means that there is a person y who is the best friend of x, and furthermore, that for every person z, if person z is not person y, then z is not the best friend of x.
When we introduce the predicate **B(x, y) to be the statement "y is the best friend of x,"** the statement that x has exactly one best friend can be represented as

$$\exists y(B(x, y) \wedge \forall z((z \neq y) \rightarrow \neg B(x, z)))$$

Consequently, our original statement can be expressed as

$$\forall x \exists y(B(x, y) \wedge \forall z((z \neq y) \rightarrow \neg B(x, z))).$$

Note:- we can write this statement as $\forall x \exists! y B(x, y)$, where $\exists!$ is the "uniqueness quantifier.

Example "There is a woman who has taken a flight on every airline in the world."
Solution Let P(w, f) be "w has taken flight f " and Q(f, a) be "f is a flight on airline a." We can express the statement as

$$\exists w \forall a \exists f P(w, f) \wedge Q(f, a)$$

Where the domains of discourse for w, f , and a consist of all the women in the world, all airplane flights, and all airlines, respectively.

Negating Nested Quantifiers

Statements involving nested quantifiers can be negated by successively applying the rules for negating statements involving a single quantifier.

Example Express the negation of the statement ∀x∃y(xy = 1) so that no negation precedes a quantifier.

Solution

We can move the negation in ¬∀x∃y(xy = 1) inside all the quantifiers.

¬∀x∃y(xy = 1) = ∃x¬∃y(xy = 1)= ∃x∀y¬(xy = 1) = **∃x∀y(xy != 1)**

Example use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."

Solution

¬∃w∀a∃fP(w, f)∧Q(f, a) = ∀w∃a∀f(¬P(w, f) V ¬Q(f, a))

3. Let $Q(x, y)$ be the statement "x has sent an e-mail message to y," where the domain for both x and y consists of all students in your class. Express each of these quantifications in English.

a) $\exists x \exists y Q(x, y)$
b) $\exists x \forall y Q(x, y)$
c) $\forall x \exists y Q(x, y)$
d) $\exists y \forall x Q(x, y)$
e) $\forall y \exists x Q(x, y)$
f) $\forall x \forall y Q(x, y)$

Solution

a) There exist students x and y such that x has sent a message to y. In other words, there is some student in your class who has sent a message to some student in your class.
b) There is some student x in your class who has sent message to everyone in your class.
c) Everyone in your class has sent a mail to someone in your class.
d) There is a student in your class who has been sent a message by every student in your class.
e) There is someone in your class who has sent mail to everyone in your class or every student in your class has been sent a message from at least one student in your class.
f) Every student in the class has sent a message to every student in the class.

5. Let $W(x, y)$ mean that student x has visited website y, where the domain for x consists of all students in your school and the domain for y consists of all websites. Express each of these statements by a simple English sentence.

a) W(Sarah Smith, www.att.com)
b) $\exists x W(x,$ www.imdb.org$)$
c) $\exists y W$(José Orez, y)
d) $\exists y(W(\text{Ashok Puri}, y) \wedge W(\text{Cindy Yoon}, y))$
e) $\exists y \forall z(y \neq (\text{David Belcher}) \wedge (W(\text{David Belcher}, z) \rightarrow W(y, z)))$
f) $\exists x \exists y \forall z((x \neq y) \wedge (W(x, z) \leftrightarrow W(y, z)))$

a) Sarah Smith has visited www.att.com
b) At least person in your class has visited www.imdb.org
c) Jose Orez has visited at least one website/ some website.
d) Ashok Puri and Cindy Yoon have both visited the same website.

e) There is a person (y) other than David Belcher who has visited all the websites that David has visited. Note that **it is not saying that this person has visited only websites that David has visited** (that would be the converse conditional statement) – person may have visited other sites as well.

f) There are two different people who have visited exactly the same websites.

Rules of Inference

Valid Arguments in Propositional Logic

Consider the following argument involving propositions
"If you have a current password, then you can log onto the network."

"You have a current password."

Therefore

"You can log onto the network."

We would like to determine whether this is a valid argument. That is, we would like to determine whether the conclusion "**You can log onto the network" must be true** when the premises "If you have a current password, then you can log onto the network" and "You have a current password" **are both true**.

p: "You have a current password"
q: "You can log onto the network."

The argument has the form

$$\begin{array}{rl} & p \to q \\ & p \\ \hline \therefore & q \end{array}$$

Where ∴ is the symbol that denotes "**therefore**."

Valid Argument:
When both p → q and p are true, **we know that q must also be true**. We say this form of argument is **valid** because **whenever all its premises are true, the conclusion must also be true**.

An *argument* in propositional logic is a sequence of propositions. All but the final proposition in the argument are called *premises* and the final proposition is called the *conclusion*. An argument is *valid* if the truth of all its premises implies that the conclusion is true.

An *argument form* in propositional logic is a sequence of compound propositions involving propositional variables. An argument form is *valid* no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.

From the definition of a valid argument form we see that the argument form with premises $p_1, p_2, \ldots, p_n$ and conclusion q is valid, when $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \to q$ is a tautology.

Rules of Inference for Propositional Logic

TABLE 1 Rules of Inference.

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \to p$	Simplification
p q $\therefore p \land q$	$((p) \land (q)) \to (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution

Using Rules of Inference to Build Arguments

Example Show that the premises “It is not sunny this afternoon and it is colder than yesterday,” “We will go swimming only if it is sunny,” “If we do not go swimming, then we will take a canoe trip,” and “If we take a canoe trip, then we will be home by sunset” lead to the conclusion “We will be home by sunset.”

Solution:

Let p be the proposition “It is sunny this afternoon,” q the proposition “It is colder than yesterday,” r the proposition “We will go swimming,” s the proposition “We will take a canoe trip,” and t the proposition “We will be home by sunset.”

1. ~p ∧ q (premise)
2. ~p (~p ∧ q → ~p)
3. r → p (premise)
4. ~r (using 2 and 3 r→p ∧ ~p → ~r
5. ~r → s (premise)
6. s (using 4 and 5)
7. s → t (premise)
8. t (using 6 and 7)

Example Show that the premises "If you send me an e-mail message, then I will finish writing the program," "If you do not send me an e-mail message, then I will go to sleep early," and "If I go to sleep early, then I will wake up feeling refreshed" lead to the conclusion "If I do not finish writing the program, then I will wake up feeling refreshed."

Solution

p: You send me an e-mail message

q: I will finish writing the program

r: I will go to sleep early

s: I will wake up feeling refreshed

1. p → q premise
2. ~q → ~p (contrapositive)
3. ~p → r premise
4. ~q → r (using 2 and 3)
5. r → s premise
6. ~q → s (using 4 and 5)

Rules of Inference for Quantified Statements

TABLE 2 Rules of Inference for Quantified Statements.

Rule of Inference	*Name*
$\dfrac{\forall x P(x)}{\therefore P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$	Universal generalization
$\dfrac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$	Existential generalization

Universal instantiation is the rule of inference used to conclude that P(c) is true, where c is a particular member of the domain, given the premise **$\forall x P(x)$**.
Universal instantiation is used when we conclude from the statement "All women are wise" that "Lisa is wise," where Lisa is a member of the domain of all women.

Universal generalization is the rule of inference that states that **$\forall x P(x)$ is true**, given the premise that P(c) is true for all elements c in the domain.
Universal generalization is used when we show that $\forall x P(x)$ is true by taking an arbitrary element c from the domain and showing that P(c) is true. The element c that we select must be an arbitrary, and not a specific, element of the domain.

Existential instantiation is the rule that allows us to conclude that there is an element c in the domain for which P(c) is true if we know that $\exists x P(x)$ is true. We cannot select an arbitrary value of c here, but rather it must be a c for which P(c) is true.

Existential generalization is the rule of inference that is used to conclude that $\exists x P(x)$ is true when a particular element c with P(c) true is known. That is, if we know one element c in the domain for which P(c) is true, then we know that $\exists x P(x)$ is true.

Example show that the premises "Everyone in this discrete mathematics class has taken a course in computer science" and "Marla is a student in this class" imply the conclusion "Marla has taken a course in computer science."
Solution let
D(x): "x is in this discrete mathematics class,"
C(x): "x has taken a course in computer science."

Then the premises are $\forall x(D(x) \rightarrow C(x))$ and D(Marla). The conclusion is C(Marla).

1. $\forall x(D(x) \rightarrow C(x))$ premise
2. D(Marla) → C(Marla) Universal instantiation from (1)
3. D(Marla) premise
4. C(Marla) from 2 and 3 (**conclusion**)

Example show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."
Solution
Let
C(x): x is in the class
B(x): x has read the book
P(x): x passed the first exam

1. $\exists x C(x) \wedge \sim B(x)$ premise
2. $C(a) \wedge \sim B(a)$ existential instantiation
3. C(a) using 2
4. $\forall x(C(x) \rightarrow P(x))$. Premise

5. $C(a) \rightarrow P(a)$ Universal instantiation
6. $P(a)$ from 3 and 5
7. $\sim B(a)$ from 2
8. $P(a) \land \sim B(a)$ from 6 and 7 conjunction
9. $\exists x(P(x) \land \sim B(x))$ Existential generalization

SET & FUNCTIONS

SET

A set is an **unordered collection** of objects, called elements or members of the set. A set is said to contain its elements. We write a ∈ A to denote that 'a' is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A.

These sets, each denoted using a boldface letter, play an important role in discrete mathematics:

$\mathbf{N} = \{0, 1, 2, 3, \ldots\}$, the set of **natural numbers**
$\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of **integers**
$\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$, the set of **positive integers**
$\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, \text{ and } q \neq 0\}$, the set of **rational numbers**
$\mathbf{R}$, the set of **real numbers**
$\mathbf{R}^+$, the set of **positive real numbers**
$\mathbf{C}$, the set of **complex numbers**.

Note that some people do not consider 0 a natural number.

Two sets are *equal* if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write $A = B$ if A and B are equal sets.

Example

The sets {1, 3, 5} and {3, 5, 1} are equal, because they have the same elements. Note that the order in which the elements of a set are listed does not matter. Note also that it does not matter if an element of a set is listed more than once, so {1, 3, 3, 3, 5, 5, 5, 5} is the same as the set {1, 3, 5} because they have the same elements.

Sets inside a Set

Sets can have other sets as members
The set **{N, Z, Q, R}** is a set containing four elements, each of which is a set. The four elements of this set are N, the set of natural numbers; Z, the set of integers; Q, the set of rational numbers; and R, the set of real numbers.

THE EMPTY SET

There is a special set that has no elements. This set is called the **empty set**, or **null set**, and is denoted by **∅**. The empty set can also be denoted by **{ }** (that is, we represent the empty set with a pair of braces that encloses all the elements in this set).
Example The set of all positive integers that are greater than their squares is the **null set**.

Singleton Set

A set with one element is called a **singleton set**.
∅ is empty set.
{∅} is singleton set. The single element of the set {∅} is the empty set itself!

Subsets

The set A is a *subset* of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.

Showing that A is a Subset of B To show that $A \subseteq B$, show that if x belongs to A then x also belongs to B.

Showing that A is Not a Subset of B To show that $A \not\subseteq B$, find a single $x \in A$ such that $x \notin B$.

Every nonempty set S is guaranteed to have at least two subsets, the empty set and the set S itself, that is, $\emptyset \subseteq S$ and $S \subseteq S$

For every set S, $(i)\ \emptyset \subseteq S$ and $(ii)\ S \subseteq S$.

Showing that two sets are equal

Showing Two Sets are Equal To show that two sets A and B are equal, show that $A \subseteq B$ and $B \subseteq A$.

Sets may have other sets as members. For instance, we have the sets

A = {∅, {a}, {b}, {a, b}} and B = {x | x is a subset of the set {a, b}}.

Note that these two sets are equal, that is, A = B. Also note that $\{a\} \in A$, but $a \notin A$

The Size of a Set

Let S be a set. If there are exactly n **distinct** elements in S where n is a **nonnegative integer**, we say that S is a **finite set** and that n is the **cardinality** of S. The cardinality of S is denoted by **|S|**.

Example

a. Let A be the set of odd positive integers less than 10. Then |A| = 5
b. Let S be the set of letters in the English alphabet. Then |S| = 26
c. Because the null set has no elements, it follows that |∅| = 0.

Infinite Set

A set is said to be **infinite** if it is not finite.

Example The set of positive integers is infinite.

Power Sets

Given a set S, **the power set of S is the set of all subsets of the set S**. The power set of S is denoted by **P(S)**.

Example What is the power set of the set {0, 1, 2}?

P({0, 1, 2}) = {{}, {0}, {1}, {2}, {0, 1}, {0, 2, {1, 2}, {0, 1, 2}}}

Note that the empty set and the set itself are members of this set of subsets.

Example What is the power set of the empty set? What is the power set of the set {∅}?

P({ }) = {{}} or {∅} because empty set has only one subset, that is itself.

P({∅}) = {{ }, {∅}} or {∅, {∅}}

If a set has n elements, then its power set has **2^n elements**.

Cartesian Products

Ordered collection

The *ordered n-tuple* $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element a_2 as its second element, ..., and a_n as its nth element.

We say that two ordered n-tuples are equal if and only if each corresponding pair of their elements is equal. In other words, (a1, a2, . . . , an) = (b1, b2, . . . , bn) if and only if **ai = bi** , for i = 1, 2, . . . , n. In particular, ordered 2-tuples are called **ordered pairs**. The ordered pairs (a, b) and (c, d) are equal if and only if **a = c and b = d**. Note that **(a, b) and (b, a) are not equal unless a = b**.

Let A and B be sets. The *Cartesian product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

Example What is the Cartesian product of A = {1, 2} and B = {a, b, c}?
The Cartesian product A × B is

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$
$$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}.$$

Note: - the Cartesian products A×B and B×A are not equal, unless A = ∅ or B = ∅ (so that A × B = ∅) or A = B

The Cartesian product of **more than two sets** can also be defined

The *Cartesian product* of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \ldots, n$. In other words,

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}.$$

Example What is the Cartesian product A × B × C, where A = {0, 1}, B = {1, 2}, and C = {0, 1, 2}.

Note that when A, B, and C are "sets", (A × B) × C is not the same as A × B × C

We use the notation A^2 to denote $A \times A$, the Cartesian product of the set A with itself. Similarly, $A^3 = A \times A \times A$, $A^4 = A \times A \times A \times A$, and so on. More generally

Example Suppose that A = {1, 2}.

A^2 = {(1, 1), (1, 2), (2, 1), (2, 2)}
A^3 = {(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)}

A subset R of the Cartesian product A × B is called a **relation** from the set A to the set B. The elements of R are ordered pairs, where the first element belongs to A and the second to B.

A relation from a set A to itself is called a relation on A.

Example What are the ordered pairs in the less than or equal to relation, which contains (a, b) if a ≤ b, on the set {0, 1, 2, 3}?
Solution
R = {(0, 0), (0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)}

Using Set Notation with Quantifiers
Sometimes we restrict the domain of a quantified statement explicitly by making use of a particular notation.
For example, **∀x∈S(P(x))** denotes the **universal quantification** of P(x) over all elements in the set S. In other words, ∀x∈S(P(x)) is shorthand for ∀x(x ∈ S → P(x)). If x belongs to S then x satisfies the property P.
Similarly, **∃x∈S(P(x))** denotes the **existential quantification** of P(x) over all elements in S. That is, ∃x∈S(P(x)) is shorthand for **∃x(x ∈ S ∧ P(x))**.

Example What do the statements ∀x∈R (x^2 ≥ 0) and ∃x∈Z (x^2 = 1) mean?
Solution
The statement $\forall x \in \mathbf{R}(x^2 \geq 0)$ states that for every real number x, x^2 >=0 this statement can be expressed as "The square of every real number is nonnegative." is a true statement.

The statement ∃x∈Z(x^2 = 1) states that there exists an integer x such that x2 = 1. This statement can be expressed as "There is an integer whose square is 1." This is also a true statement because x = 1 is such an integer (as is −1).

3. For each of these pairs of sets, determine whether the first is a subset of the second, the second is a subset of the first, or neither is a subset of the other.
   a) the set of airline flights from New York to New Delhi, the set of nonstop airline flights from New York to New Delhi
   b) the set of people who speak English, the set of people who speak Chinese
   c) the set of flying squirrels, the set of living creatures that can fly

Solution
a) Second is the subset of first.
b) Neither set is a subset of the first.
c) Frist set is the subset of second

9. Determine whether each of these statements is true or false.
   a) $0 \in \emptyset$
   b) $\emptyset \in \{0\}$
   c) $\{0\} \subset \emptyset$
   d) $\emptyset \subset \{0\}$
   e) $\{0\} \in \{0\}$
   f) $\{0\} \subset \{0\}$
   g) $\{\emptyset\} \subseteq \{\emptyset\}$

a) False b) false c) false, empty set has no proper subset d) true e)False
f) false, For one set to be a **proper subset** of another, the two sets cannot be equal
g)True, every set is subset of itself.

10. Determine whether these statements are true or false.

a) $\emptyset \in \{\emptyset\}$ **b)** $\emptyset \in \{\emptyset, \{\emptyset\}\}$
c) $\{\emptyset\} \in \{\emptyset\}$ **d)** $\{\emptyset\} \in \{\{\emptyset\}\}$
e) $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$ **f)** $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$
g) $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$

a) True, b) True c)False d)True e)True f) True g) False, not proper subset

21. Find the power set of each of these sets, where a and b are distinct elements.

a) $\{a\}$ **b)** $\{a, b\}$ **c)** $\{\emptyset, \{\emptyset\}\}$

a) P({a}) = {Ø, {a}}
b) P({a, b}) = {Ø, {a}, {b}, {a, b}}
c) P({Ø, {Ø}}) = {Ø, {Ø}, {{Ø}}, {Ø, {Ø}}}

23. How many elements does each of these sets have where a and b are distinct elements?

a) $\mathcal{P}(\{a, b, \{a, b\}\})$
b) $\mathcal{P}(\{\emptyset, a, \{a\}, \{\{a\}\}\})$
c) $\mathcal{P}(\mathcal{P}(\emptyset))$

a) 8 b) 16 c)2 $P(\emptyset) = \{\emptyset\}$ $P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$

41. Translate each of these quantifications into English and determine its truth value.

a) $\forall x \in \mathbf{R}\ (x^2 \neq -1)$ **b)** $\exists x \in \mathbf{Z}\ (x^2 = 2)$
c) $\forall x \in \mathbf{Z}\ (x^2 > 0)$ **d)** $\exists x \in \mathbf{R}\ (x^2 = x)$

a) <u>Every real number has its square not equal to -1</u>. Alternatively, the square of a real number is never -1. **This is true**, since squares of real numbers are always nonnegative.
b) <u>There exists an integer whose square is 2</u>. **This is false**, since the only two numbers whose squares are 2, namely √2 and -√2, are not integers.
c) The square of every integer is positive. This is almost true, but not quite $0^2 \not> 0$
d) There is a real number equal to its own square. This is true, since x = 1 (as well as x = 0) fits the bill.

43. Find the truth set of each of these predicates where the domain is the set of integers.

a) $P(x)$: $x^2 < 3$ **b)** $Q(x)$: $x^2 > x$
c) $R(x)$: $2x + 1 = 0$

a) The truth set is $\{x \in \mathbf{Z} \mid x^2 < 3\} = \{-1, 0, 1\}$
b) All negative integers satisfy this inequality, as do all nonnegative integers other than 0 and 1. So the truth set is $\{x \in \mathbf{Z} \mid x^2 > x\} = \mathbf{Z} - \{0, 1\} = \{\ldots, -2, -1, 2, 3, 4, \ldots\}$
c) The only real number satisfying this equation is x = -1/2. Because this value is not in our domain, the truth set is empty: $\{x \in \mathbf{Z} \mid 2x + 1 = 0\} = \emptyset$

Set Operations

Union

Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

An element x belongs to the union of the sets A and B if and only if x belongs to A or x belongs to B. This tells us that

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

Example The union of the sets {1, 3, 5} and {1, 2, 3} is the set {1, 2, 3, 5}; that is, {1, 3, 5} ∪ {1, 2, 3} = {1, 2, 3, 5}.

Intersection

Let A and B be sets. The *intersection* of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.

An element x belongs to the intersection of the sets A and B if and only if x belongs to A and x belongs to B. This tells us that

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

Example The intersection of the sets {1, 3, 5} and {1, 2, 3} is the set {1, 3}; that is, {1, 3, 5} ∩ {1, 2, 3} = {1, 3}.

Disjointness

Two sets are called **disjoint** if their intersection is the **empty set**.

Let A = {1, 3, 5, 7, 9} and B = {2, 4, 6, 8, 10}. Because A ∩ B = ∅, A and B are disjoint.

Cardinality of a union of two finite sets A and B

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Difference

Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B. The difference of A and B is also called the *complement of B with respect to A*.

Note: The difference of sets A and B is sometimes denoted by A\B.

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

Complement

Let U be the universal set. The *complement* of the set A, denoted by $\overline{A}$, is the complement of A with respect to U. Therefore, the complement of the set A is $U - A$.

An element belongs to $\overline{A}$ if and only if $x \notin A$. This tells us that

$$\overline{A} = \{x \in U \mid x \notin A\}.$$

Example

Let A be the set of positive integers greater than 10 (with universal set the set of all positive integers). Then $\overline{A} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Set Identities

TABLE 1 Set Identities.	
Identity	***Name***
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Generalized Unions and Intersections

The *union* of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

We use the notation

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$$

to denote the union of the sets A1,A2, . . . , An.

The *intersection* of a collection of sets is the set that contains those elements that are members of all the sets in the collection.

We use the notation

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$

Functions

Definition

Let A and B be nonempty sets. A *function* f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \to B$.

Note **Functions** are sometimes also called mappings or **transformations**.

A function f : A → B can also be defined in terms of a relation from A to B.
A relation from A to B that contains one, and only one, **ordered pair (a, b) for every element a ∈ A**, defines a function f from A to B. This function is defined by the assignment f (a) = b, where (a, b) is the unique ordered pair in the relation that has a as its first element.

If f is a function from A to B, we say that A is the *domain* of f and B is the *codomain* of f. If $f(a) = b$, we say that b is the *image* of a and a is a *preimage* of b. The *range*, or *image*, of f is the set of all images of elements of A. Also, if f is a function from A to B, we say that f *maps* A to B.

Equal Functions
Two functions are equal when they have the **same domain**, have the **same codomain**, and **map each element of their common domain to the same element in their common codomain**.

Note that if we change either the domain or the codomain of a function, then we obtain a different function. If we change the mapping of elements, then we also obtain a different function.

Example Let **f : Z → Z** assign the square of an integer to this integer. Then, f (x) = x^2, where the domain of f is the set of all integers, the codomain of f is the set of all integers, and the range of f is the set of all integers that are perfect squares, namely, {0, 1, 4, 9, . . . }.

Let f_1 and f_2 be functions from A to $\mathbf{R}$. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\mathbf{R}$ defined for all $x \in A$ by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x),$$
$$(f_1 f_2)(x) = f_1(x) f_2(x).$$

Note that the functions **f1 + f2** and **f1f2** have been defined by specifying their values at x in terms of the values of f1 and f2 at x.

Example
Let f_1 and f_2 be functions from $\mathbf{R}$ to $\mathbf{R}$ such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. What are the functions $f_1 + f_2$ and $f_1 f_2$?

(f1 + f2)(x) = f1(x) + f2(x) = x^2 + (x – x^2) = x

And

(f1f2)(x) = x^2(x – x^2) = x^3 – x^4.

When f is a function from A to B, the **image of a subset of A can also be defined**.

Let f be a function from A to B and let S be a subset of A. The *image* of S under the function f is the subset of B that consists of the images of the elements of S. We denote the image of S by $f(S)$, so

$$f(S) = \{t \mid \exists s \in S\ (t = f(s))\}.$$

We also use the shorthand $\{f(s) \mid s \in S\}$ to denote this set.

Example

Let A = {a, b, c, d, e} and B = {1, 2, 3, 4} with f (a) = 2, f (b) = 1, f (c) = 4, f (d) = 1, and f (e) = 1. The image of the subset S = {b, c, d} is the set **f (S) = {1, 4}**.

One-to-One and Onto Functions

Some functions never assign the same value to two different domain elements. These functions are said to be **one-to-one (injective)**.

A function f is said to be *one-to-one*, or an *injunction*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be *injective* if it is one-to-one.

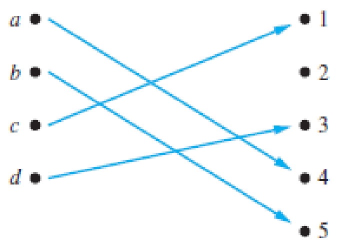


FIGURE 3 A One-to-One Function.

Note that a function **f is one-to-one if and only if f (a) != f (b)** whenever **a! = b.** This way of expressing that f is one-to-one is obtained by taking the contrapositive of the implication in the definition.

Remark: We can express that f is **one-to-one using quantifiers** as **∀a∀b(f (a) = f (b) → a = b)** or equivalently **∀a∀b(a != b → f (a) != f (b))**, where the universe of discourse is the domain of the function.

Example Determine whether the function f (x) = x^2 from the set of integers to the set of integers is one-to-one.

The function f(x) = x^2 is **not one-to-one as** f(1) = 1 and f(-1) = 1 but 1!= -1.
Note that the function f (x) = x^2 with its domain restricted to Z+ is one-to-one.

Definition

A function f whose domain and codomain are subsets of the set of real numbers is called *increasing* if $f(x) \leq f(y)$, and *strictly increasing* if $f(x) < f(y)$, whenever $x < y$ and x and y are in the domain of f. Similarly, f is called *decreasing* if $f(x) \geq f(y)$, and *strictly decreasing* if $f(x) > f(y)$, whenever $x < y$ and x and y are in the domain of f. (The word *strictly* in this definition indicates a strict inequality.)

Remark: A function f is increasing if $\forall x \forall y(x < y \rightarrow f(x) \leq f(y))$, strictly increasing if $\forall x \forall y(x < y \rightarrow f(x) < f(y))$, decreasing if $\forall x \forall y(x < y \rightarrow f(x) \geq f(y))$, and strictly decreasing if $\forall x \forall y(x < y \rightarrow f(x) > f(y))$, where the universe of discourse is the domain of f.

Note: A function that is **either strictly increasing or strictly decreasing** must be **one-to-one**. However, a function that is increasing, but **not strictly increasing**, or decreasing, but not strictly decreasing, is **not one-to-one**.

Number of one-to-one function

f: A→B

If |A| = m and |B| = n

1. m < n then **nPm**, first element from A can be mapped to n elements in B, next element in A can be mapped to (n-1), 3rd element from A can be mapped to (n-2) in B, and so on..
   nx(n-1)x(n-2)x …… x(n-(m-1) = nPm = n!/(n-m)!

2. if m = n then nPn = **n!**
3. if m > n then **not possible one-to-one** function.

ONTO Functions (Surjection)

For some functions **the range and the codomain are equal**. That is, every member of the codomain is the image of some element of the domain. Functions with this property are called **onto** functions.

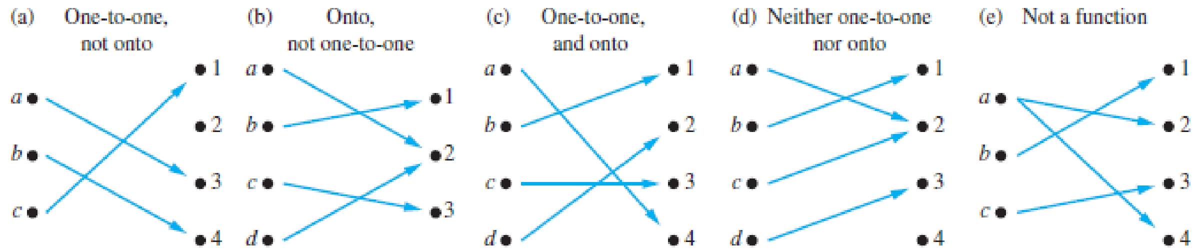
A function f from A to B is called *onto*, or a *surjection*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called *surjective* if it is onto.

Remark: A function f is onto if **∀y∃x(f (x) = y)**, where the domain for x is the domain of the function and the domain for y is the codomain of the function.

Example Let f be the function from {a, b, c, d} to {1, 2, 3} defined by f (a) = 3, f (b) = 2, f (c) = 1, and f (d) = 3. Is f an **onto** function.

Example Is the function f (x) = x^2 from the set of integers to the set of integers onto?
Not it's not ONTO function because there is no integer x with x2 = −1, for instance.

Example Is the function f (x) = x + 1 from the set of integers to the set of integers onto?
This function is onto, because for every integer y there is an integer x such that f(x) = y. To see this, note that f(x) = y **if and only if x+1 = y** which holds only if **x = y-1**.



Number of ONTO functions (f: A→B)

Suppose |A| = m= 5 and |B| =n= 4, total number of functions = n^m = 4^5.

No. of onto functions = total functions – (not onto functions)
= n^m – (nC1*(n-1)^m + nC2(n-2)^m – nC3(n-3)^m + … nCn(n-n)^m)

No. of onto functions = 4^5 – (4C1*3^4 – 4C2*2^4 + 4C3*1^4 – 4C4*0^4)

One-to-One Correspondence (Bijection) or (one-to-one and onto functions)

The function f is a *one-to-one correspondence,* or a *bijection,* if it is both one-to-one and onto. We also say that such a function is *bijective*.

Bijective functions are only possible when |A|=|B|=n hence nPn = **n! Functions**

Example Let f be the function from {a, b, c, d} to {1, 2, 3, 4} with f (a) = 4, f (b) = 2, f (c) = 1, and f(d) = 3. Is both **one-to-one and onto** hence **bijection**.

Note:-

1. Suppose that f is **a function from a set A to itself**. If A is **finite**, then f is one-to-one if and only if it is onto.
2. This is not necessarily the case if A is **infinite**.

Identity function

Let A be a set. The **identity function** on A is the function $\iota_A : A \to A$, Where
$\iota_A(x) = x$ for all x ∈ A.

In other words, the **identity function iA** is the function that assigns each element to itself. The function **iA** is **one-to-one and onto**, so it is a **bijection**.

Suppose that $f : A \to B$.
To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.
To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.
To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.
To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Inverse Functions and Compositions of Functions

Now consider a **one-to-one correspondence f** from the set A to the set B.

1. f is an **onto** function, every element of B is the image of some element in A.
2. f is also a **one-to-one** function, every element of B is the image of a unique element of A.

Hence, we **can define a new function from B to A that reverses the correspondence given by f**.

Let f be a one-to-one correspondence from the set A to the set B. The *inverse function* of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

Note: If a function f is not a one-to-one correspondence, we cannot define an inverse function of f.

A one-to-one correspondence is called **invertible** because we can define an inverse of this function. A function is not invertible if it is not a one-to-one correspondence, because the inverse of such a function does not exist.

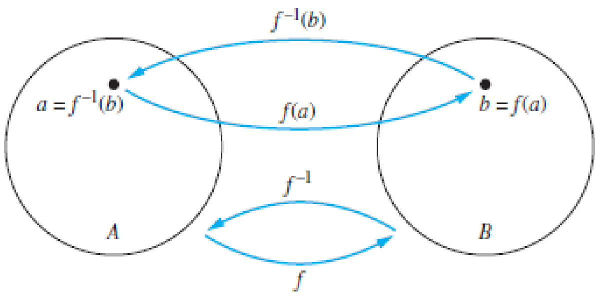


FIGURE 6 The Function f^{-1} Is the Inverse of Function f

Example Let f be the function from {a, b, c} to {1, 2, 3} such that f (a) = 2, f (b) = 3, and f (c) = 1. Is f invertible, and if it is, what is its inverse?
Solution f^-1(2) =a f^-1(3)=b f^-1(1)=c

Example Let f : Z → Z be such that f (x) = x + 1. Is f invertible, and if it is, what is its inverse?
Solution
Let's check if it's one to one? x1 +1 = x2 +1 ➔ x1=x2 (hence one to one)
Let's check if it's onto? y=x+1, x = y-1 (hence, onto)
This function is **bijective hence invertible**.

Example Show that if we restrict the function f (x) = x^2 to a function from the set of all **Non-negative real numbers** to the set of all nonnegative real numbers, then f is invertible.

One to one: every positive real number will have its unique square value.

Solution: The function $f(x) = x^2$ from the set of nonnegative real numbers to the set of nonnegative real numbers is one-to-one. To see this, note that if $f(x) = f(y)$, then $x^2 = y^2$, so $x^2 - y^2 = (x + y)(x - y) = 0$. This means that $x + y = 0$ or $x - y = 0$, so $x = -y$ or $x = y$. Because both x and y are nonnegative, we must have $x = y$. So, this function is one-to-one.

Furthermore, $f(x) = x^2$ is onto when the codomain is the set of all nonnegative real numbers, because each nonnegative real number has a square root. That is, if y is a nonnegative real number, there exists a nonnegative real number x such that $x = \sqrt{y}$, which means that $x^2 = y$. Because the function $f(x) = x^2$ from the set of nonnegative real numbers to the set of nonnegative real numbers is one-to-one and onto, it is invertible. Its inverse is given by the rule $f^{-1}(y) = \sqrt{y}$. ◀

f(x) = y = x^2, x = root(y)

<u>Composition of functions</u>

Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The *composition* of the functions f and g, denoted for all $a \in A$ by $f \circ g$, is defined by

$$(f \circ g)(a) = f(g(a)).$$

That is, to find (f ◦ g)(a) we first <u>apply the function g to a to obtain g(a)</u> and then we apply the function f to the result g(a) to obtain **(f ◦ g)(a) = f (g(a)).** Note that the composition f ◦ g <u>cannot be defined unless the range of g is a subset of the domain of f</u>.

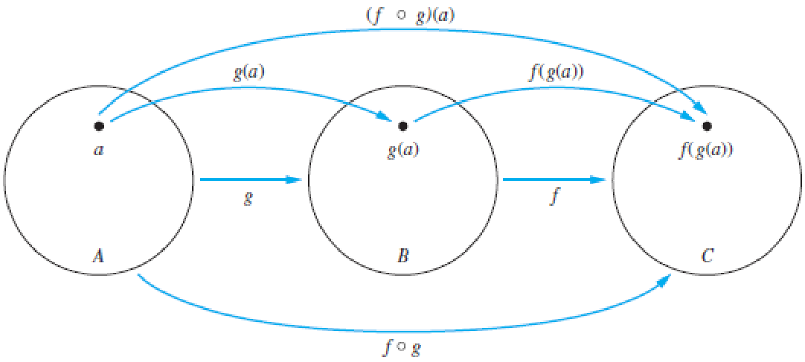


FIGURE 7 The Composition of the Functions f and g.

Example Let g be the function from the set {a, b, c} to itself such that g(a) = b, g(b) = c, and g(c) = a. Let f be the function from the set {a, b, c} to the set {1, 2, 3} such that f (a) = 3, f (b) = 2, and f (c) = 1. What is the composition of f and g, and what is the composition of g and f ?

g:
g(a) = b
g(b) = c
g(c) = a

f:
f (a) = 3
f (b) = 2
f (c) = 1

fog =(fog)(a)= f(g(a))
a->2
b->1
c->3

gof = =(gof)(a)= g(f(a)) Note that g ◦ f is not defined, because the **<u>range of f is not a subset of the domain of g</u>**.

<u>Note fog and gof are not equal</u>.

Example Let f and g be the functions from the set of integers to the set of integers defined by f(x) = 2x + 3 and g(x) = 3x + 2. What is the composition of f and g? What is the composition of g and f?

Solution

fog = f(g(x)) = f(3x+2) = 6x+4+3 = 7x+3

gof = g(f(x)) = g(2x+3) = 6x+9+2 = 6x+11

Composition of a functions and its inverse

When the composition of a function and its inverse is formed, in either order, an **identity function** is obtained.

To see this, suppose that f is a one-to-one correspondence from the set A to the set B. Then the **inverse function f ^−1 exists and is a one-to-one correspondence from B to A**. **The inverse function reverses the correspondence of the original function**.

When f^-1(b)=a then f(a)=b and f(a)=b then f^-1(b)=a. Hence,

(f^-1 o f)a = f^-1(f(a)) = f^-1(b)= a

And

(fof^-1)b = f(f^-1(b)) =f(a) = b

Consequently $f^{-1} \circ f = \iota_A$ and $f \circ f^{-1} = \iota_B$, where ι_A and ι_B are the identity functions on the sets A and B, respectively. That is, $(f^{-1})^{-1} = f$.

Excercise

1. Why is f not a function from **R** to **R** if

a) $f(x) = 1/x$?

b) $f(x) = \sqrt{x}$?

c) $f(x) = \pm\sqrt{(x^2 + 1)}$?

a) f(0) is not defined

b) f(-1) = root(-1) = i complex number but not a real number.

c) The "rule" for f is ambiguous. There are two values associated with every x

2. Determine whether f is a function from **Z** to **R** if

a) $f(n) = \pm n$.

b) $f(n) = \sqrt{n^2 + 1}$.

c) $f(n) = 1/(n^2 - 4)$.

a. Not, ambiguous definition

b. Yes, it's function

c. Not, f(2) is not defined.

3. Determine whether f is a function from the set of all bit strings to the set of integers if

a) $f(S)$ is the position of a 0 bit in S.

b) $f(S)$ is the number of 1 bits in S.

c) $f(S)$ is the smallest integer i such that the ith bit of S is 1 and $f(S) = 0$ when S is the empty string, the string with no bits.

a) S may not have any 0, or may have more than one 0. Thus there may be no value for f (S) or more than one. In either case this violates the definition of a function.

b) Yes, it's a function.

c) F(S) is not defined if S=0000.

5. Find the domain and range of these functions. Note that in each case, to find the domain, determine the set of elements assigned values by the function.

a) the function that assigns to each bit string the number of ones in the string minus the number of zeros in the string

b) the function that assigns to each bit string twice the number of zeros in that string

c) the function that assigns the number of bits left over when a bit string is split into bytes (which are blocks of 8 bits)

d) the function that assigns to each positive integer the largest perfect square not exceeding this integer

a) Clearly the domain is the set of all bit strings. The range is Z(integers);

b) Domain is set of all bit strings, the range is 0, 2, 4, ... the set of even natural numbers.

c) The domain is the set of all bit strings. Since the number of leftover bits can be any whole number from 0 to 7.

d) The domain is the set of positive integers, nly perfect squares can be function values, Therefore the range is {l, 4, 9, 16, ... }.

9. Find these values.

a) $\lceil \frac{3}{4} \rceil$ **b)** $\lfloor \frac{7}{8} \rfloor$
c) $\lceil -\frac{3}{4} \rceil$ **d)** $\lfloor -\frac{7}{8} \rfloor$
e) $\lceil 3 \rceil$ **f)** $\lfloor -1 \rfloor$
g) $\lfloor \frac{1}{2} + \lceil \frac{3}{2} \rceil \rfloor$ **h)** $\lfloor \frac{1}{2} \cdot \lfloor \frac{5}{2} \rfloor \rfloor$

a) 1 b) 0 c)0 d)-1 e)3 f)-1 g)2 h)1

12. Determine whether each of these functions from **Z** to **Z** is one-to-one.

a) $f(n) = n - 1$ **b)** $f(n) = n^2 + 1$
c) $f(n) = n^3$ **d)** $f(n) = \lceil n/2 \rceil$

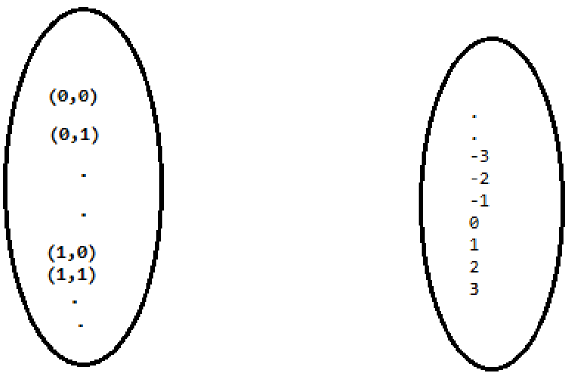
a) n1 -1 = n2 – 1 n1 = n2 hence one to one
b) n1^2 = n2^2 = n1^2 –n2^2 = (n1- n2)(n1+n2) = 0, n1=+- n2, not one to one but it's a function. f(-1)=f(1) = 2.
c) n1^3 = n2^3, one to one
d) not to one to one. f(1) = f(2) = 1

13. Which functions in Exercise 12 are onto?

a) y = n-1, n = y+1 for each n we can have n+1. f(x+1=x
b) Not Onto, y=n^2 +1, n^2=y-1, n=root(y-1). For 0 we don't have any mapping.
or, Since n^2 +1 is always positive, the range can't include any negative number.
c) Not onto, as 2 is not the cube of any number, hence 2 is not a part of the range.
d) This function is onto. If we want to obtain value x, then we simply need to start with 2x, since f(2x) = ceil(2x/2) = x for all x E Z. or y=x/2, x=2y.

14. Determine whether $f: \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ is onto if

a) $f(m, n) = 2m - n.$
b) $f(m, n) = m^2 - n^2.$
c) $f(m, n) = m + n + 1.$
d) $f(m, n) = |m| - |n|.$
e) $f(m, n) = m^2 - 4.$



f: zxz -> z

a) f: X->Y is onto if ∀y∈ Y, ∃x∈X such that f(x)=y
2m-n = y if we take m=0, and n=-y then 2*0 - (-y) = y
hence for each y we can find a pair in domain as m=0, and n=-y. f(0, -y) = y.
This function is onto.
b) f(m, n) = m^2 – n^2, it's not onto function. as if we keep n=0, and m=root(y) then m^2 – n^2 = y but root(y) is not an integer.

c) f(m,n) = m+n+1, y=m+n+1, n=-1, then y=m for each y we can have f(y,-1), hence onto.
d) Onto, keep m=0 and n=y for –ve number in co-domain, and for positive y, keep n=-y.
e) Not Onto. for example 7 is not mapped with any pair in domain.

15. Determine whether the function $f: \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ is onto if

a) $f(m, n) = m + n$.
b) $f(m, n) = m^2 + n^2$.
c) $f(m, n) = m$.
d) $f(m, n) = |n|$.
e) $f(m, n) = m - n$.

a) m+n = y, so if m=0, and n=y then 0+y = y f(0,y) = y, hence onto.
b) range contains no negative numbers, hence no onto.
c) f(m,o) = m, hence onto, for each m we have a pair of m and such that f(m,0).
d) Not onto, range contains no negative numbers.
e) Given any integer m, we have f(m,0) =m, so the function is onto.

22. Determine whether each of these functions is a bijection from **R** to **R**.

a) $f(x) = -3x + 4$
b) $f(x) = -3x^2 + 7$
c) $f(x) = (x + 1)/(x + 2)$
d) $f(x) = x^5 + 1$

a) -3x1 + 4 = -3x2 +4
x1=x2 hence one to one.
y = -3x + 4
x=(y-4)/-3 for each y we can have (y-4)/-3, hence onto, so this function is bijective.

b) f(x) = -3x^2 + 7
x1^2 = x2^2 this function is not one to one, hence not bijection.

c) This is not a function because f(-2) = -1/0 is not defined.
suppose domain is R - {-2} then
(x1 +1)/(x1+2) = (x2 +1)/(x2+2)
(x1 +1)*(x2+2)=(x2+1)*(x1+2)
x1=x2 hence one to one.

(x+1)/(x+2) = y
x+1 = xy +2y
x – xy = 2y-1
x(1-y)=2y-1
x=(2y-1)/(1-y) if y=1 then function is not defined. Function is onto if co-domain is R –{1}.

d) f(x) = x^5 + 1
x1^5 = x2^5, x1=x2 One to One
y=x^5 +1, x = (y-1)^1/5 hence onto.
this function is bijection.

23. Determine whether each of these functions is a bijection from **R** to **R**.

a) $f(x) = 2x + 1$

b) $f(x) = x^2 + 1$

c) $f(x) = x^3$

d) $f(x) = (x^2 + 1)/(x^2 + 2)$

a) This function is one-to-one.
y = 2x+1, x = (y-1)/2. To show that the function is onto, note that 2((y - 1) /2) + 1 = y, so every number is in the range.

b) Not one to one, hence not bijection. And range in this function is set to greater than or equal to 1, not all of R.

c) Yes, one to one and onto. this function is a bijection because it has inverse function namely f(y) = y^1/3.

d) X and –x have the same image, hence this function is not one-to-one (injective). And a little work shows that the range is only {y | 0.5 <= y <1} = {0.5, 1).

Note:

1. the function **f(x) = e^x** from the set of real numbers to the set of real numbers is not invertible, but if the codomain is restricted to the set of positive real numbers, the resulting function is **invertible**.
2. The function f (x) = |x| from the set of real numbers to the set of **nonnegative real numbers** is **not invertible**, but if the **domain is restricted to the set of nonnegative real numbers,** the resulting function is **invertible**.

Sequences and Summations

Sequences

A sequence is a discrete structure used to represent an ordered list. For example, 1, 2, 3, 5, 8 is a sequence with five terms and 1, 3, 9, 27, 81 , . . . , 3^n, . . . is an **infinite sequence**.

Recurrence Relations

A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer. A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation. (A recurrence relation is said to *recursively define* a sequence. We will explain this alternative terminology in Chapter 5.)

A recurrence relation is said to recursively define a sequence.

Example

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \ldots,$ and suppose that $a_0 = 2$. What are a_1, a_2, and a_3?

Solution: We see from the recurrence relation that $a_1 = a_0 + 3 = 2 + 3 = 5$. It then follows that $a_2 = 5 + 3 = 8$ and $a_3 = 8 + 3 = 11$. ◀

Example

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots,$ and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

Solution: We see from the recurrence relation that $a_2 = a_1 - a_0 = 5 - 3 = 2$ and $a_3 = a_2 - a_1 = 2 - 5 = -3$. We can find a_4, a_5, and each successive term in a similar way. ◀

The *Fibonacci sequence*, $f_0, f_1, f_2, \ldots,$ is defined by the initial conditions $f_0 = 0$, $f_1 = 1$, and the recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

for $n = 2, 3, 4, \ldots.$

Example

Suppose that $\{a_n\}$ is the sequence of integers defined by $a_n = n!$, the value of the factorial function at the integer n, where $n = 1, 2, 3, \ldots.$ Because $n! = n((n-1)(n-2)\ldots 2 \cdot 1) = n(n-1)! = na_{n-1}$, we see that the sequence of factorials satisfies the recurrence relation $a_n = na_{n-1}$, together with the initial condition $a_1 = 1$. ◀

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

Cardinality of Sets

Definition 1
The sets A and B have the same **cardinality** if and only if there is a **one-to-one correspondence** from A to B. When A and B have the same cardinality, we write |A| = |B|.

Definition 2
If there is a **one-to-one** function from A to B, the cardinality of A is less than or the same as the cardinality of B and we write |A| ≤ |B|. Moreover, when |A| ≤ |B| and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and we write |A| < |B|.

Relations

Let A and B be sets. A *binary relation from A to B* is a subset of $A \times B$.

In other words, a binary relation from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B. We use the notation aRb to denote that (a, b) ∈ R and $a \not R\, b$ to denote that $(a, b) \notin R$. Moreover,

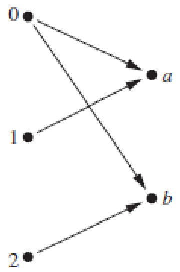
<u>When (a, b) belongs to R, a is said to be related to b by R.</u>

<u>Note: Binary relations represent relationships between the elements of two sets.</u>

<u>Example</u>

Let A = {0, 1, 2} and B = {a, b}. Then {(0, a), (0, b), (1, a), (2, b)} is a relation from A to B. This means, for instance, that 0R a, but that $1 \not R\, b$.

Relations can be represented graphically, or using a table.



R	a	b
0	×	×
1	×	
2		×

<u>Ordered Pairs in the Relation R</u>

<u>Relations on a Set</u>

Relations from a set A to itself are of special interest. In other words, a relation on a set A is a **subset of A × A**.

Example

Let A be the set {1, 2, 3, 4}. Which ordered pairs are in the relation R = {(a, b) | a divides b}?

R = {(1, 1), (2, 2), (3, 3), (4, 4),(1, 2), (1, 3),(1, 4), (2, 4) }

Example

Consider these relations on the set of integers:

$R_1 = \{(a, b) \mid a \leq b\}$,

$R_2 = \{(a, b) \mid a > b\}$,

$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$,

$R_4 = \{(a, b) \mid a = b\}$,

$R_5 = \{(a, b) \mid a = b + 1\}$,

$R_6 = \{(a, b) \mid a + b \leq 3\}$.

Which of these relations contain each of the pairs (1, 1), (1, 2), (2, 1), (1,−1), and (2, 2)?

Solution

The pair (1, 1) is in R1, R3, R4, and R6.

The pair (1, 2) is in R1, R6

The pair (2, 1) is in R2, R5 and R6

The pair (1, -1) is in R2, R3 and R6

The pair (2, 2) is in R1, R3, and R4

Example How many relations are there on a set with n elements?
Solution
A relation on a set A is a subset of A×A. Because A×A has n^2 elements when A has n elements.
Number of relations from A to itself will be 2^{n^2}

Properties of Relations

1. **Reflexive relations**

A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

Using quantifiers we see that the relation R on the set A is reflexive if ∀a((a, a) ∈ R), where the universe of discourse is the set of all elements in A.

2. **Symmetric/Antisymmetric relations**

A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$. A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.

Using quantifiers, we see that

1. The relation R on the set A is symmetric if ∀a∀b((a, b) ∈ R → (b, a) ∈ R).
2. Similarly, the relation R on the set A is antisymmetric if ∀a∀b(((a, b) ∈ R ∧ (b, a) ∈ R) → (a = b)).

Note:

1. The terms **symmetric** and **antisymmetric** are **not opposites**.
2. Because a relation **can have both** of these properties or **may lack both** of them

3. **Transitive relations**

A relation R on a set A is called *transitive* if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

Using quantifiers we see that the relation R on a set A is transitive if we have

$$\forall a \forall b \forall c(((a, b) \in R \land (b, c) \in R) \to (a, c) \in R)$$

Combining Relations

Because relations from A to B are subsets of AXB, two relations from A to B can be combined in any way two sets can be combined.

Example Let A = {1, 2, 3} and B = {1, 2, 3, 4}. The relations R1 = {(1, 1), (2, 2), (3, 3)} and R2 = {(1, 1), (1, 2), (1, 3), (1, 4)} can be combined to obtain

$$R_1 \cup R_2 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (3,3)\},$$
$$R_1 \cap R_2 = \{(1,1)\},$$
$$R_1 - R_2 = \{(2,2), (3,3)\},$$
$$R_2 - R_1 = \{(1,2), (1,3), (1,4)\}.$$

Example Let R1 be the "**less than**" relation on the set of real numbers and let R2 be the "greater than" relation on the set of real numbers, that is, **R1 = {(x, y) | x < y} and R2 = {(x, y) | x > y}**. What are **R1 ∪ R2, R1 ∩ R2, R1 − R2, R2 − R1, and R1 ⊕ R2**?

R1UR2 = {(x, y) | x != y}
R1 ∩ R2 = ∅ (empty relation)
R1 – R2 = {(x, y) | x < y}
R2 – R1 = {(x, y) | x > y}
R1 ⊕ R2 = R1 ∪ R2 − R1 ∩ R2 = {(x, y) | x != y}

Compositions of Relations

Let R be a relation from a set A to a set B and S a relation from B to a set C. The *composite* of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

Computing the composite of two relations requires that we find elements that are the second element of ordered pairs in the first relation and the first element of ordered pairs in the second relation.

Example what is the composite of the relations R and S, where R is the relation from {1, 2, 3} to {1, 2, 3, 4} with **R = {(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)}** and S is the relation from {1, 2, 3, 4} to {0, 1, 2} with **S = {(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)}**?

Solution

SoR is constructed using all ordered pairs in R and ordered pairs in S, where the **second element of the ordered pair in R agrees with the first element of the ordered pair in S**.

For example, the **ordered pairs (2, 3) in R** and **(3, 1) in S** produce the **ordered pair (2, 1) in SoR**.

[R A→B and S B→C then SoR A→C]

SoR = {(1, 0), (1, 1), (2, 1), (2, 2), (3, 1)}

Composing the Parent Relation with Itself

Let R be the relation on the set of all people such that (**a, b) ∈ R if person a is a parent of person b**. Then (a, c) ∈ RoR if and only if there is a person b such that (a, b) ∈ R and (b, c) ∈ R, that is, if and only if there is a person b such that a is a parent of b and b is a parent of c. In other words, (a, c) ∈ RoR **if and only if a is a grandparent of c**.

Power of a relation R

The powers of a relation R can **be recursively defined** from the definition of a composite of two relations.

Let R be a relation on the set A. The powers $R^n, n = 1, 2, 3, \ldots,$ are defined recursively by

$$R^1 = R \qquad \text{and} \qquad R^{n+1} = R^n \circ R.$$

The definition shows that $R^2 = R \circ R$, $R^3 = R^2 \circ R = (R \circ R) \circ R$, and so on.

Example Let R = {(1, 1), (2, 1), (3, 2), (4, 3)}. Find the powers $R^n, n = 2, 3, 4, \ldots.$

Because R^2 = RoR we find that R^2 = {(1, 1), (2, 1), (3, 1), (4, 1)}

R^3 = R^2oR = {(1, 1), (2, 1), (3, 1), (4, 1)}. Additional computation shows that R^4 is the same as R^3.

It also follows that R^n = R^3 for n = 5, 6, 7, ...

THEOREM The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots.$

Representing Relations Using Matrices

A relation between **finite sets** can be represented using a **zero–one matrix**. Suppose that R is a relation from A = {a1, a2, . . . , am} to B = {b1, b2, . . . , bn}.

The relation R can be represented by the matrix $\mathbf{M}_R = [m_{ij}]$.

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R, \\ 0 \text{ if } (a_i, b_j) \notin R. \end{cases}$$

In other words, the zero–one matrix representing R has a 1 as its (i, j) entry when ai is related to bj, and a 0 in this position if ai is not related to bj .

Example

Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R if $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$, and $b_1 = 1$ and $b_2 = 2$?

Solution: Because $R = \{(2, 1), (3, 1), (3, 2)\}$, the matrix for R is

$$\mathbf{M}_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

The 1s in $\mathbf{M}_R$ show that the pairs (2, 1), (3, 1), and (3, 2) belong to R. The 0s show that no other pairs belong to R.

Example suppose that the relation R on a set is represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Is R reflexive, symmetric, and/or antisymmetric?

Solution

Because all the diagonal elements of this matrix are equal to 1, **R is reflexive**, and R is symmetric.

Example **suppose that the relations R1 and R2 on a set A are represented by the matrices**

$$\mathbf{M}_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the matrices representing R1 ∪ R2 and R1 ∩ R2?

Solution

The matrices of these relations are

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Matrix for the composite of relations

Suppose that R is a relation from A to B and S is a relation from B to C. Suppose that A, B, and C have m, n, and p elements, respectively.

Let the zero – one matrices for S ∘R, R, and S be $\mathbf{M}_{S \circ R} = [t_{ij}]$, $\mathbf{M}_R = [r_{ij}]$, and $\mathbf{M}_S = [s_{ij}]$, respectively, (these matrices have sizes m×p, m×n, and n×p, respectively).

The ordered pair (ai, cj) belongs to $S \circ R$ if and only if there is an element b_k such that (a_i, b_k) belongs to R and (b_k, c_j) belongs to S. It follows that $t_{ij} = 1$ if and only if $r_{ik} = s_{kj} = 1$ for some k. From the definition of the Boolean product, this means that

$$\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S.$$

Example Find the matrix representing the relations S ∘R, where the matrices representing R and S are

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Solution The matrix for SoR is

$$\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Example Find the matrix representing the relation R^2, where the matrix representing R is

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Solution the matrix for R^2 is

$$\mathbf{M}_{R^2} = \mathbf{M}_R^{[2]} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Representing Relations Using Digraphs

A *directed graph*, or *digraph*, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex a is called the *initial vertex* of the edge (a, b), and the vertex b is called the *terminal vertex* of this edge.

An edge of the form (a, a) is represented using an arc from the vertex a back to itself. Such an edge is called a **loop**.

The directed graph with vertices a, b, c, and d, and edges (a, b), (a, d), (b, b), (b, d), (c, a), (c, b), and (d, b) is displayed

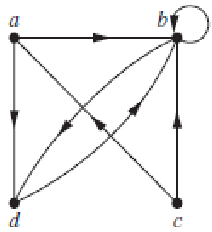


FIGURE 3
A Directed Graph.

Closures of Relations

1. **Reflexive Closure of Relation**

The relation R = {(1, 1), (1, 2), (2, 1), (3, 2)} on the set A = {1, 2, 3} is not reflexive. How can we produce a reflexive relation containing R that is **as small as possible**?

This can be done by adding (2, 2) and (3, 3) to R, because these are the only pairs of the form (a, a) that are not in R.

Given a relation R on a set A, the reflexive closure of R can be formed by adding to R all pairs of the form (a, a) with a ∈ A, **not already in R**. The addition of these pairs produces a new relation **that is reflexive**, contains R, and is contained within any reflexive relation containing R.

We see that the **reflexive closure of R** equals R∪Δ, where Δ = {(a, a) | a ∈ A} is the **diagonal relation** on A.

Example What is the reflexive closure of the relation R = {(a, b) | a < b} on the set of integers?

The reflexive closure of R is

$$R \cup \Delta = \{(a, b) \mid a < b\} \cup \{(a, a) \mid a \in \mathbf{Z}\} = \{(a, b) \mid a \leq b\}$$

2. **Symmetric Closure of Relation**

The relation {(1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3, 2)} on {1, 2, 3} is **not symmetric**. How can we produce a symmetric relation that is as small as possible and contains R? We need only add (2, 1) and (1, 3), because these are the only pairs of the form (b, a) with (a, b) ∈ R that **are not in R**.

This new relation is symmetric and contains R. Furthermore, any symmetric relation that contains R must contain this new relation, because a symmetric relation that contains R must contain (2, 1) and (1, 3). Consequently, this **new relation** is called the **symmetric closure** of R.

The symmetric closure of a relation can be constructed by taking the union of a relation with its inverse

that is, $R \cup R^{-1}$ is the symmetric closure of R, where

$$R^{-1} = \{(b, a) \mid (a, b) \in R\}.$$

Example What is the symmetric closure of the relation R = {(a, b) | a > b} on the set of positive integers?
Solution the symmetric closure of R is the relation

$$R \cup R^{-1} = \{(a, b) \mid a > b\} \cup \{(b, a) \mid a > b\} = \{(a, b) \mid a \neq b\}$$

This last equality follows because R contains all ordered pairs of positive integers where the first element is greater than the second element and R^{-1} contains all ordered pairs of positive integers where the first element is less than the second.

3. Transitive closure of relation

Consider the relation R = {(1, 3), (1, 4), (2, 1), (3, 2)} on the set {1, 2, 3, 4}. This relation is not transitive because it does not contain all pairs of the form (a, c) where (a, b) and (b, c) are in R. The pairs of this form not in R are **(1, 2), (2, 3), (2, 4), and (3, 1)**. Adding these pairs does not produce a transitive relation, because the resulting relation contains (3, 1) and (1, 4) but does not contain (3, 4).

Equivalence Relations

A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Two elements a and b that are related by an equivalence relation are called *equivalent*. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Example Let R be the relation on the set of real numbers such that **aRb if and only if a − b is an integer**. Is R an equivalence relation?
Solution
Because a – a = 0, is an integer for all real numbers a, aRa for all real numbers a. Hence, R is reflexive.
If a – b is integer then b – a is also an integer. Hence, R is symmetric.
If aRb and bRc, then a – b and b – c are integers. Therefore, a – c = (a – b)+ (b – c), a – c = int+int= int
Hence R is transitive. R is an **equivalence relation**.

Note: **One of the most widely used equivalence relations is congruence modulo m, where m is an integer greater than 1.**
Solution
Congruence Modulo m Let m be an integer with m > 1. Show that the relation.

R = {(a, b) | a ≡ b (mod m)}

is an equivalence relation on the set of integers.
Note:-

1. **a ≡ b (mod m) means if m divides a then b will be left as remainder.**
2. **a ≡ b (mod m) if and only if m divides a − b.**

a – a=0, is divisible by m. hence a ≡ a (mod m), R is **reflexive**.
Suppose a ≡ b (mod m) then (a-b) is divisible by m. so a-b=k*m, where k is an integer. It follows that b-a= (-k)m, so b ≡ a (mod m). Hence, congruence modulo m is **symmetric**.
Suppose that a ≡ b (mod m) and b ≡ c (mod m). m divides both (a-b) and (b-c). Therefore, there are integers k and l with a − b = km and b − c = lm.

Adding these 2 equations shows that **a − c = (a − b) + (b − c) = km + lm = (k + l)m**. Thus, a ≡ c (modm). Therefore, congruence modulo m is **transitive**.
It follows that congruence modulo m is an **equivalence relation**.

Example Suppose that R is the relation on the set of strings of English letters such that aRb if and only if l(a) = l(b), where l(x) is the length of the string x. Is R an equivalence relation?

Solution: Because $l(a) = l(a)$, it follows that aRa whenever a is a string, so that R is reflexive. Next, suppose that aRb, so that $l(a) = l(b)$. Then bRa, because $l(b) = l(a)$. Hence, R is symmetric. Finally, suppose that aRb and bRc. Then $l(a) = l(b)$ and $l(b) = l(c)$. Hence, $l(a) = l(c)$, so aRc. Consequently, R is transitive. Because R is reflexive, symmetric, and transitive, it is an equivalence relation. ◀

Example Let R be the relation on the set of real numbers such that xRy if and only if x and y are real numbers that differ by less than 1, that is |x − y| < 1. Show that R is not an equivalence relation.
Solution

Solution: R is reflexive because $|x - x| = 0 < 1$ whenever $x \in \mathbf{R}$. R is symmetric, for if $x R y$, where x and y are real numbers, then $|x - y| < 1$, which tells us that $|y - x| = |x - y| < 1$, so that yRx. However, R is not an equivalence relation because it is not transitive. Take $x = 2.8$, $y = 1.9$, and $z = 1.1$, so that $|x - y| = |2.8 - 1.9| = 0.9 < 1$, $|y - z| = |1.9 - 1.1| = 0.8 < 1$, but $|x - z| = |2.8 - 1.1| = 1.7 > 1$. That is, $2.8\, R\, 1.9$, $1.9\, R\, 1.1$, but $2.8\, \not R\, 1.1$. ◀
Not an **equivalence** relation.

<u>Equivalence Classes</u>

Let A be the set of all students in your school who graduated from high school. Consider the relation R on A that consists of all pairs (x, y), <u>where x and y graduated from the same high school</u>. Given a student x, <u>we can form the set of all students equivalent to x with respect to R</u>. This set consists of all students who graduated from the same high school as x did. This subset of A is called an equivalence class of the relation.

Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a. The equivalence class of a with respect to R is denoted by $[a]_R$. When only one relation is under consideration, we can delete the subscript R and write $[a]$ for this equivalence class.

In other words, if R is an equivalence relation on a set A, the equivalence class of the element a is

$$[a]_R = \{s \mid (a, s) \in R\}$$

If $b \in [a]_R$, then b is called a **representative** of this equivalence class. Any element of a class can be used as a representative of this class.

Example Let R be the relation on the set of integers such that aRb if and only if a = b or a = −b.
Solution
Because an integer is equivalent to itself and its negative in this equivalence relation, it follows that [a] = {−a, a}. This set contains two distinct integers unless a = 0. For instance
[7] = {−7, 7}, [−5] = {−5, 5}, and [0] = {0}.

Example what are the equivalence classes of 0 and 1 for congruence modulo 4?
Solution The equivalence class of 0 contains all integers a such that **a ≡ 0 (mod 4).** The integers in this class are those **divisible by 4**. Hence, the equivalence class of 0 for this relation is

$$[0] = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$$

The equivalence class of 1 contains all the integers a such that a ≡ 1 (mod 4).

$$[1] = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$$

Equivalence Classes and Partitions

Let A be the set of students at your school who are majoring in exactly one subject, and let R be the relation on A consisting of pairs **(x, y), where x and y are students with the same major**. Then R is an equivalence relation.
We can see that **R splits all students in A into a collection of disjoint subsets**, where each subset contains students with a specified major.
For instance, one subset contains all students majoring (just) in computer science, and a second subset contains all students majoring in history. Furthermore, **these subsets are equivalence classes of R**. This example illustrates how the **equivalence classes of an equivalence relation partition a set into disjoint, nonempty subsets**.

Let R be a relation on the set A. Theorem 1 shows that the equivalence classes of two elements of A are either identical or disjoint.

Partitions: **non-empty, disjoint, and finite such that union of all partitions is equal to original SET**.

Let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

(i) aRb $\quad$ (ii) $[a] = [b]$ $\quad$ (iii) $[a] \cap [b] \neq \emptyset$

1. (i) implies (ii). Assume that aRb, then [a] = [b] by because [a] ⊆ [b] and [b] ⊆ [a].
2. (ii) implies (iii). Assume that [a] = [b]. It follows that [a] ∩ [b] ——————————————= ∅ because **[a] is nonempty** (because **a ∈ [a] because R is reflexive**).

Example Suppose that S = {1, 2, 3, 4, 5, 6}. The collection of sets A1 = {1, 2, 3}, A2 = {4, 5}, and A3 = {6} forms a partition of S, because these sets are **disjoint** and their **union is S**.

1. We see that (a, a) ∈ R for every a ∈ S, because a is in the same subset as itself. Hence, R is **reflexive**.
2. If (a, b) ∈ R, then b and a are in the same subset of the partition, so that (b, a) ∈ R as well. Hence, R is **symmetric**.
3. If (a, b) ∈ R and (b, c) ∈ R, Consequently, a and c belong to the same subset of the partition, so (a, c) ∈ R. Thus, R is **transitive**.

It follows that R is an **equivalence relation**.

Example List the ordered pairs in the equivalence relation R produced by the partition A1 = {1, 2, 3}, A2 = {4, 5}, and A3 = {6} of S = {1, 2, 3, 4, 5, 6}, given in Example 12.
Solution
The pair (a, b) ∈ R if and only if a and b are in the same subset of the partition.
1. A1xA1 = the pairs (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), and (3, 3) belong to R because A1 = {1, 2, 3} is **an equivalence class**;
2. A2xA2 = the pairs (4, 4), (4, 5), (5, 4), and (5, 5) belong to R because A2 = {4, 5} is an equivalence class;
3. And finally the pair (6, 6) belongs to R because {6} is an equivalence class.

No pair other than those listed belongs to R.

EXAMPLE What are the sets in the partition of the integers arising from congruence modulo 4?
Solution
There are **four congruence classes**, corresponding to $[0]_4$, $[1]_4$, $[2]_4$, and $[3]_4$ they are sets

$$[0]_4 = \{\ldots, -8, -4, 0, 4, 8, \ldots\},$$
$$[1]_4 = \{\ldots, -7, -3, 1, 5, 9, \ldots\},$$
$$[2]_4 = \{\ldots, -6, -2, 2, 6, 10, \ldots\},$$
$$[3]_4 = \{\ldots, -5, -1, 3, 7, 11, \ldots\}.$$

These congruence classes are disjoint, and every integer is in exactly one of them. In other words, as Theorem 2 says, these **<u>congruence classes form a partition</u>**.

Partial Orderings

A relation R on a set S is called a *partial ordering* or *partial order* if it is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R). Members of S are called *elements* of the poset.

Example Show that the "**greater than or equal**" relation (≥) is a partial ordering on the set of **integers**.
Solution
Solution: Because $a \geq a$ for every integer a, $\geq$ is reflexive. If $a \geq b$ and $b \geq a$, then $a = b$. Hence, $\geq$ is antisymmetric. Finally, $\geq$ is transitive because $a \geq b$ and $b \geq c$ imply that $a \geq c$. It follows that $\geq$ is a partial ordering on the set of integers and $(\mathbf{Z}, \geq)$ is a poset. ◀

Example The divisibility relation | is a partial ordering on the set of positive integers, because it is reflexive, antisymmetric, and transitive. (Z+, |) is a poset.

Example Show that the inclusion relation ⊆ is a partial ordering on the power set of a set S.
Solution: Because $A \subseteq A$ whenever A is a subset of S, $\subseteq$ is reflexive. It is antisymmetric because $A \subseteq B$ and $B \subseteq A$ imply that $A = B$. Finally, $\subseteq$ is transitive, because $A \subseteq B$ and $B \subseteq C$ imply that $A \subseteq C$. Hence, $\subseteq$ is a partial ordering on $P(S)$, and $(P(S), \subseteq)$ is a poset. ◀

Example Let R be the relation on the set of people such that xRy if x and y are people and **x is older than y**. Show that R is not a partial ordering.
Solution
This relation is **antisymmetric** if x is older than y then y is not older than x. and **transitive** but not reflexive, **because a person is not older than himself**.

Note
1. The notation $a \preccurlyeq b$ is used to denote that (a, b) ∈ R in an **arbitrary poset** (S, R).
2. The symbol $\preccurlyeq$ is used to denote the relation in any poset, not just the "less than or equals" relation.)
3. The notation $a \prec b$ denotes that $a \preccurlyeq b$, but $a \neq b$. we say "a is less than b" or "b is greater than a" if $a \prec b$

The elements a and b of a poset $(S, \preccurlyeq)$ are called *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When a and b are elements of S such that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, a and b are called *incomparable*.

Example In the poset (Z+, |), are the integers 3 and 9 comparable? Are 5 and 7 comparable?
Solution
The integers 3 and 9 are comparable, because 3 | 9. The integers 5 and 7 are incomparable, because $5 \nmid 7$ and $7 \nmid 5$.

Totally Ordered Set

If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a *totally ordered* or *linearly ordered set*, and $\preccurlyeq$ is called a *total order* or a *linear order.* A totally ordered set is also called a *chain*.

EXAMPLE The poset (Z,≤) is **totally ordered**, because a ≤ b or b ≤ a whenever a and b are integers.
Example The poset (Z+, |) is not totally ordered because it contains elements that are incomparable, such as 5 and 7.

Well-ordered set

$(S, \preccurlyeq)$ is a *well-ordered set* if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of S has a least element.

(Z+,≤) is well-ordered, where ≤ is the usual "less than or equal to" relation.

Hasse Diagrams

Many edges in the directed graph for a **finite poset** do not have to be shown because they must be present. Consider the directed graph for the partial ordering **{(a, b) | a ≤ b}** on the set {1, 2, 3, 4}, because this relation is a partial ordering, **it is reflexive**, and its directed graph has loops at all vertices. We do not have to show these loops because they must be present; because a partial ordering is transitive, we do not have to show those edges that must be present because of transitivity.
If we assume that all edges are pointed "**upward**", we do not have to show the directions of the edges;
We can represent a finite poset $(S, \preccurlyeq)$ using this procedure.

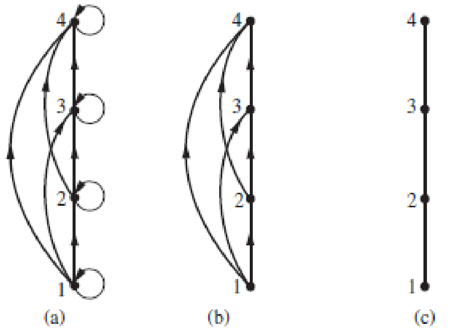
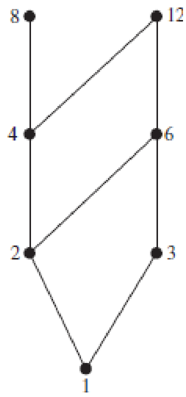


FIGURE 2 **Constructing the Hasse Diagram for ({1, 2, 3, 4}, ≤).**

Note: - **As shown in the figure, transitive edges and self-loops are removed**.

EXAMPLE Draw the **Hasse diagram** representing the partial ordering **{(a, b) |a divides b}** on {1, 2, 3, 4, 6, 8, 12}.



Note

1. **Lattice, Graph Theory part will be covered from the lecture of kiran sir.**
2. **Group Theory will be completed from notes**

Counting

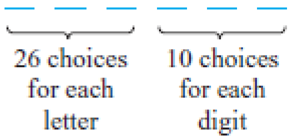
Basic Counting Principles

1. **The Product Rule** suppose that a procedure can be broken down into a sequence of two tasks. If there are n1 ways to do the first task and for each of these ways of doing the first task, there are n2 ways to do the second task, then there are **n1*n2 ways** to do the procedure.

   An extended version of the product rule is often useful. Suppose that a procedure is carried out by performing the tasks T1, T2, . . . , Tm in sequence. If each task Ti , i = 1, 2, . . . , n, can be done in ni ways, regardless of how the previous tasks were done, then there are n1 · n2 · · · · nm ways to carry out the procedure.

Example How many different bit strings of length seven are there?
Solution
Each of the seven bits can be chosen in two ways, because each bit is either 0 or 1. There the **product rule** shows there are a total of 2^7=128.

Example How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?
Solution



26*26*26*10*10*10 = 17,576,000

2. **The Sum Rule** If a task can be done **either in one of n1 ways or in one of n2 ways**, where **none** of the set of n1 ways is the same as any of the set of n2 ways, then there are **n1 + n2** ways to do the task.

Example Suppose that either a member of the mathematics faculty **or** a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are **37 members** of the mathematics faculty and **83 mathematics majors** and no one is both a faculty member and a student?
Solution
By the sum rule it follows that there are **37C1 + 83C1 = 120**

We can extend the sum rule to more than two tasks. Suppose that a task can be done in one of n_1 ways, in one of n_2 ways, . . . , or in one of n_m ways, where none of the set of n_i ways of doing the task is the same as any of the set of n_j ways, for all pairs i and j with $1 \leq i < j \leq m$. Then the number of ways to do the task is $n_1 + n_2 + \cdots + n_m$. This extended version of the

Example A student can choose a computer project from **one of three lists**. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?
Solution
The student can choose a project by selecting a project from the first list, the second list, or the third list. Because no project is on more than one list, by the sum rule there are **23 + 15 + 19 = 57** ways to choose a project.

<u>More Complex Counting Problems</u>

Example The name of a variable is a string of **<u>one or two</u>** <u>alphanumeric characters</u>, where uppercase and lowercase letters are not distinguished. An alphanumeric character is either one of the 26 English letters or one of the 10 digits. Moreover, a variable name <u>must begin with a letter</u> and <u>must be different from the **five strings of two characters** that are reserved for programming use</u>. How many different variable names are there in this version of BASIC?

Solution

Let V1 be the number of variables that are one character long = 26

Let V2 be the number of variables that are two characters long = 26*(26+10) = 26*36=936

Subtract 5 strings, hence 936-5 = 931

Total passwords possible = 26 + 931 = **957**.

Example each user on a computer system has a password, which is **six to eight** characters long, where **<u>each character is an uppercase letter or a digit</u>**. Each p**<u>assword must contain at least one digit</u>**. How many possible passwords are there?

Solution

Let P be the total number of possible passwords, and let **P6**, **P7**, and **P8** denote the number of possible passwords of length 6, 7, and 8, respectively.

By the sum rule p = p6 + p7 +p8

P6 = total 6 length passwords – (6 length passwords without any digit)

= 36^6 – 26^6 (repetitions are allowed)

= 1,867,866,560

Similarly

P7 = 36^7 – 26^7

P8 = 36^8 – 26^8

P = p6 + p7 + P8

<u>The Subtraction Rule (Inclusion–Exclusion for Two Sets)</u>

1. Suppose that a task can be done in one of **two ways**.
2. But some of the ways to do it are **<u>common to both ways</u>**.

In this situation, we **<u>cannot use the sum rule</u>** to count the number of ways to do the task. Tasks that are common to the two ways **<u>are counted twice</u>**. We must **subtract the number** of ways that are counted twice.

> THE SUBTRACTION RULE If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

The subtraction rule is also known as the principle of **inclusion–exclusion**.

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

There are **|A1∪A2|**ways to select an element in either A1 or in A2, and **|A1∩A2|** ways to select an element **common to both sets**.

Example how many bit strings of length **<u>eight either start with a 1 bit or end with the two bits 00</u>**?

Solution

|A| = Strings start with 1, 8 length strings, 2^7=128 strings possible.

|B| = Strings end with 00, 8 length strings, 2^6 = 64 strings possible.

|A1∩A2| = Strings starting with 1 and ending with 00, 8 length strings = 1_ _ _ _ _ 00 =2^5=32
|A1∪A2| = 128 + 64 – 32 = **160**

Example
A computer company receives **350 applications** from computer graduates for a job. Suppose that 220 of these applicants majored in computer science, 147 majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business?

Number of students who majored either in computer science or in business or (both)
= 220 + 147 – 51 = **316**
We conclude that 350 − 316 = 34 of the applicants majored **neither in computer science nor in business**.

Example how many different ways are there to seat **four people around a circular table**, where two seatings are considered the same when each person has the same left neighbor and the same right neighbor?
Solution

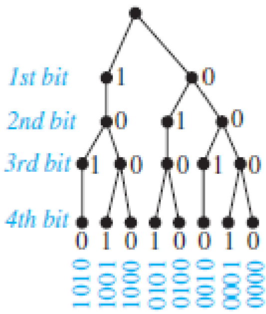
Circular permutation

1. **(N-1)! Ways**
   There is a round table, then first person can sit only in **one way** because it doesn't make any difference if he sits on any chair. It doesn't make any difference where does he sit.
   First person – 1 way
   Second person – 3 ways
   Third person – 2 ways
   Fourth person – 1 way
   (N-1)! = 3!

2. **(N-1)!/2 ways**
   If clockwise and anti-clockwise are same, then we divide by 2.

Tree Diagrams
Counting problems can be solved using **tree diagrams**. To use trees in counting, **we use a branch to represent each possible choice**.

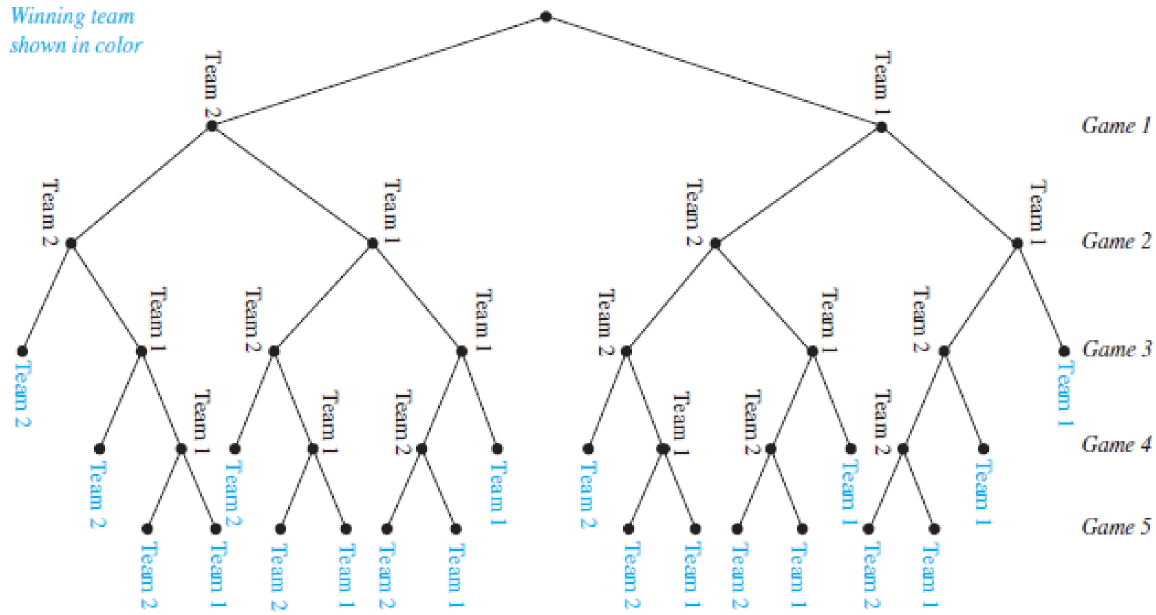
Example How many bit strings of length four do not have two consecutive 1s?



Bit Strings of Length Four without Consecutive 1s.

There are such 8 strings without two consecutive 1's.

Example A playoff between two teams consists of **at most five games**. The first team that wins **three games wins the playoff**. In how many different ways can the playoff occur?



Best Three Games Out of Five Playoffs

There are 20 different ways can the playoff occur.

Example Suppose that "I Love New Jersey" T-shirts come in five different sizes: S, M, L, XL, and XXL. Further suppose that each size comes in four colors, white, red, green, and black, except for XL, which comes only in red, green, and black, and XXL, which comes only in green and black. How many different shirts does a souvenir shop have to stock to have at least one of each available size and color of the T-shirt?

Solution

It follows that the souvenir shop owner needs to stock 17 different T-shirts.

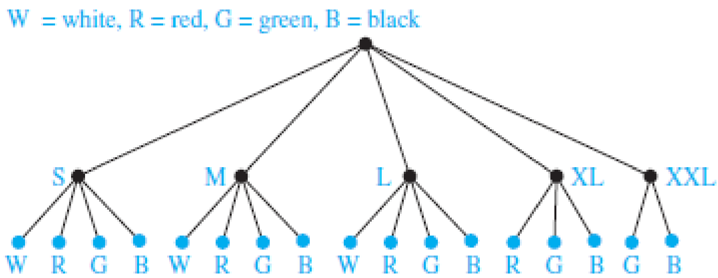


FIGURE 4 **Counting Varieties of T-Shirts.**

______________________ Exercise_________________________________

1. There are 18 mathematics majors and 325 computer science majors at a college.
   a) In how many ways can two representatives be picked so that one is a mathematics major and the other is a computer science major?
   b) In how many ways can one representative be picked who is either a mathematics major or a computer science major?

Solution

a) 18C1*325C1 = 5850

b) 18C1 + 325C1 = 343, nothing is common b/w both

2. An office building contains 27 floors and has 37 offices on each floor. How many offices are in the building?

Solution 27*37 = 999

3. A multiple-choice test contains 10 questions. There are four possible answers for each question.
   a) In how many ways can a student answer the questions on the test if the student answers every question?
   b) In how many ways can a student answer the questions on the test if the student can leave answers blank?

Solution

a) A question can be answered in 4 ways, choose any option among the given 4 options, and every question needs to be attempted hence, so option has to be chosen

4x4x4x4.... 10 times= 4^10

b) Now there are 5 ways to perform a task

5x5x5x5x5.. = 5^10

4. A particular brand of shirt comes in 12 colors, has a male version and a female version, and comes in three sizes for each sex. How many different types of this shirt are made?

Solution

12 colors*2 gender*3sizes = 72 t-shirts

5. Six different airlines fly from New York to Denver and seven fly from Denver to San Francisco. How many different pairs of airlines can you choose on which to book a trip from New York to San Francisco via Denver, when you pick an airline for the flight to Denver and an airline for the continuation flight to San Francisco?

Solution

New York - > Denver = 6 flights

Denver to San Francisco = 7 flights

New York to San Francisco = 6*7=42

7. How many different three-letter initials can people have?

Solution

26x26x26 = 26^3 = 17,576 each initial can be chosen 26 ways.

8. How many different three-letter initials with none of the letters repeated can people have?

Solution

26P3 = 26!/23!= 26*25*24= 15,600

9. How many different three-letter initials are there that begin with an A?

Solution

=1wayx26waysx26ways = 26*26 = 676

11. How many bit strings of length ten both begin and end with a 1?

Solution

1_ _ _ _ _ _ _ _ _ 1 = 2^8 = 256

12. How many bit strings are there of length six or less, not counting the empty string?

Solution

=2^6 + 2^5 + 2^4 + 2^3 + 2^2 + 2^1 = 127

13. How many bit strings with length not exceeding n, where n is a positive integer, consist entirely of 1s, not counting the empty string?

Solution

1 + 1 + 1 + .. n times = n strings

15. How many strings are there of lowercase letters of length four or less, not counting the empty string?

Solution

26^4 + 26^3 + 26^2 + 26 = 4,75,254, note that empty string is not counted

16. How many strings are there of four lowercase letters that have the letter x in them?

Solution

<u>We will solve this question indirectly</u>.

Total possible strings with length 4 – disqualified strings which doesn't have x

=26^4 – 25^4 = 66,351.

17. How many strings of five ASCII characters contain the character @ ("at" sign) at least once? [*Note:* There are 128 different ASCII characters.

Solution

"At least", solve it indirectly.

=128^5 – 127^5

19. How many 6-element RNA sequences

a) do not contain U?

b) end with GU?

c) start with C?

d) contain only A or U?

Solution

RNA sequence is a sequence of letters, each of which is **<u>one of A, C, G, or U</u>**. Thus by the product rule there are 4^6 RNA sequences of length six if we impose **no restrictions**.

a) 3^6

b) 4^4

c) 4^5

d) Contain only A or U, for each position 2 choices are only allowed, 2^6 = 64

20. How many positive integers between 5 and 31

a) are divisible by 3? Which integers are these?

b) are divisible by 4? Which integers are these?

c) are divisible by 3 and by 4? Which integers are these?

Solution

a) 31/3 = 10, 10-1 = 9 because > 5 and < 31

6, 9, 12, 15, 18, 21, 24, 27, and 30

b) 31/4 = 7, 7-1=6 because > 5 and < 31

4, 8, 12, 16, 20, 24, 28

c) Divisible by 12, 31/12 = 2. 12, and 24

21. How many positive integers between 50 and 100

a) are divisible by 7? Which integers are these?

b) are divisible by 11? Which integers are these?

c) are divisible by both 7 and 11? Which integers are these?

Solution

a) 100/7 - 50/7 = 14 – 7 = 7 ➔ 56, 63, 70, 77, 84, 91, and 98

b) 100/11 – 50/11 = 9 – 4 = 5 ➔ 55, 66, 77, 88, and 99

c) A number is divisible by both 7 and 11 if and only if it is divisible by their **least common multiple(LCM)**,

Divisible by 11 and 7 = 77, which is only one number 77.

22. How many positive integers less than 1000

a) are divisible by 7?

b) are divisible by 7 but not by 11?

c) are divisible by both 7 and 11?

d) are divisible by either 7 or 11?

e) are divisible by exactly one of 7 and 11?

f) are divisible by neither 7 nor 11?

g) have distinct digits?

h) have distinct digits and are even?

Solution

a) 999/7 = 142

b) Number should not be divisible LCM(11,7) = 77, 999/77=

Total numbers divisible by 7 = 142 – total number divisible by 77=12

142-12=130

c) 999/77=12

d) 999/7=142, 999/11=90, 999/77=12

142+90-12= 220

e) 142 + -12*2 = 208

f) 999-220=779

g) Count 1 digit number = 1, 2, 3, -- 9, 9 numbers

Count 2 digits number = 10 to 99 there are 90 numbers in which11, 22, 33, 44, 55, 66, 77, 88, 99 don't have unique digits = 90-9 = 81 numbers

Or first position can have any non-zero number hence 9 choices and second position can have any number other than first number, 9*9 = 81

Count 3 digit numbers = 9*9*8 = 648

So there are total 684+81+9 = 738 positive integers less than 1000.

h) 1 digit number = 2,4,6,8 = **4**

2 digit numbers = number ends with 0, 9x1 = 9, first digit any non-zero number and last digit will be 0

Number ends with even number. _ _, last digit has 4 choices, 2,4,6,8 and first non-zero digit except at last place = 8x4 = 32

There will be total 9+32 = **41** numbers
3 digit even numbers ending with 0, _ _ 0 = 9X8X1 = 72
3 digit even numbers ending with an even number = 8x8x4 = 256
Total positive even numbers less than 1000 = **256+72+41+4** = 338 = 373

25. How many strings of three decimal digits
- **a)** do not contain the same digit three times?
- **b)** begin with an odd digit?
- **c)** have exactly two digits that are 4s?

Solution
a) With 3 decimal digits, 1000 numbers are possible from 000 to 999. 000, 111, 222, 333, 444, 555, 666, 777, 888, 999 are 10 decimal numbers which contain same number three times.
1000 – 10 = 990
b) First digit has only 1,3,5,7,9 5 choices second digit can be 0 to 9, 10 choices and 3rd unit also has 10 choices, hence 5*10*10 = 500
c) 3 digits, 2 places will be 4, and third place can be anything except 4, means 9 choices then we can permute them in 3!/2! = 3 ways. Hence total such numbers will be 9*3 = 27.
Or
Here we need to choose the position of the digit that is not a 4 (3 ways) and choose that digit (9 ways). Therefore there are 3 · 9 = 27 such strings.

27. A committee is formed consisting of one representative from each of the 50 states in the United States, where the representative from a state is either the governor or one of the two senators from that state. How many ways are there to form this committee?

Solution
1 + 2C1 = 3 ways to choose 1 person from 1 state
There are 50 states
Hence 3^50 total committees are possible.

28. How many license plates can be made using either three digits followed by three uppercase English letters or three uppercase English letters followed by three digits?

Solution
=10^3*26^3 + 26^3*10^3

31. How many license plates can be made using either two or three uppercase English letters followed by either two or three digits?

Solution
=(26^2 + 26^3)(10^2 + 10^3) all possible 4 combinations are covered.

32. How many strings of eight uppercase English letters are there

a) if letters can be repeated?
b) if no letter can be repeated?
c) that start with X, if letters can be repeated?
d) that start with X, if no letter can be repeated?
e) that start and end with X, if letters can be repeated?
f) that start with the letters BO (in that order), if letters can be repeated?
g) that start and end with the letters BO (in that order), if letters can be repeated?
h) that start or end with the letters BO (in that order), if letters can be repeated?

Solution

a) 26^8
b) 26P8 = 26!/18! = 62990928000
c) 26^7
d)25P7 = 25!/18! // start with X, we can not permute X with other symbols.
e) 26^6
f) 26^6
g) 26^4
h) Start with BO = 26^6
   End with BO = 26^6
   Start and end with BO = 26^4
= 26^6 + 26^6 – 26^4

33. How many strings of eight English letters are there

a) that contain no vowels, if letters can be repeated?
b) that contain no vowels, if letters cannot be repeated?
c) that start with a vowel, if letters can be repeated?
d) that start with a vowel, if letters cannot be repeated?
e) that contain at least one vowel, if letters can be repeated?
f) that contain exactly one vowel, if letters can be repeated?
g) that start with X and contain at least one vowel, if letters can be repeated?
h) that start and end with X and contain at least one vowel, if letters can be repeated?

Solution Strings of eight English letters

a) 21^8
b) 21P8 = 8204716800
c) 5C1*26^7
d) 5C1*25P7
e) All possible strings with 8 length – all possible strings of length 8 without vowels
  26^8 – 21^8
f) First choose one position out of 8 to put vowels (8 choices), then choose 1 vowel (5C1 5 choices) and then fill remaining 7 places with any constants.
8C1*5C1*21^7
g) 26^7 – 21^7 = 6,230, 721,635
h) 26^6 – 21^6 = 223,149,655

34. How many different functions are there from a set with 10 elements to sets with the following numbers of elements?

a) 2 **b)** 3 **c)** 4 **d)** 5

Solution

a) 2^10 b)3^10 c)4^10 d)5^10

35. How many one-to-one functions are there from a set with five elements to sets with the following number of elements?

a) 4 **b)** 5 **c)** 6 **d)** 7

Solution

a) Not possible

b) 5P5 = 5!

c) 6P5 = 6!

d) 7P5 = 7!/2! = 2520

37. How many functions are there from the set $\{1, 2, \ldots, n\}$, where n is a positive integer, to the set $\{0, 1\}$

a) that are one-to-one?

b) that assign 0 to both 1 and n?

c) that assign 1 to exactly one of the positive integers less than n?

Solution

|n| → |2|

a) Not possible if n>=2, If n = 2, then there are again 2 such functions

b) 2^(n-2), if n>=2, 1 and n will be mapped to 0 and rest of the n-2 elements will have 2 choices for mapping. If n=2 then there will be 1 such function.

c) If n = 1, then there are no such functions, since there are **no positive integers less than n**. suppose n>1, we have to choose 1 element in domain which will be assigned to 1, there are such n-1 choice. nth element has 2 choices. So total ways are 2*(n-1).

40. How many subsets of a set with 100 elements have more than one element?

Solution

Number of subsets = 2^100

There are 100 subsets with 1 element, and 1 subset with 0 element.

Answer will be 2^100 – (100 + 1).

41. **A palindrome** is a string whose reversal is identical to the string. How many bit strings of length n are palindromes?

Solution

If n is even then n/2 place out of n places will have 2 choices each, and rest n/2 part will have 1 choice of each place.

2^(n/2) **[bit strings, made of 0 and 1]**

If n is odd then 2^(ciel(n/2)) will have 2 choices for each places and remaining places will have 1 choice for each place.

44. How many ways are there to seat four of a group of ten people around a circular table where two seatings are considered the same when everyone has the same immediate

Solution

First task: select 4 people out of 10.

Second task: place 4 people on the table

10C4*(4-1)! = 10C4*3! = 210*6 = 1,260

45. How many ways are there to seat six people around a circular table where two seatings are considered the same when everyone has the same two neighbors without regard to whether they are right or left neighbors?

Solution

Clockwise and anti-clockwise are same.

(6-1)!/2 = 120/2=60

46. In how many ways can a photographer at a wedding arrange 6 people in a row from a group of 10 people, where the bride and the groom are among these 10 people, if

a) the bride must be in the picture?
b) both the bride and groom must be in the picture?
c) exactly one of the bride and the groom is in the picture?

Solution

a) Bride is already chosen, we need to choose 5 out of 9 remaining people, and further 6 people can be arranged in 6! Ways, 9C5*6!= **90,720**

b) Both bride and groom are already selected, choose 4 out of 8 people and arrange them in 6! Ways. 8C4*6! = 50,400

c) When Bride is in picture but not groom= 8C5*6!
When groom is in picture but not bride = 8C5*6!
= 2*(8C5*6!) = 80,640

47. In how many ways can a photographer at a wedding arrange six people in a row, including the bride and groom, if

a) the bride must be next to the groom?
b) the bride is not next to the groom?
c) the bride is positioned somewhere to the left of the groom?

Solution

a) Consider bride and groom as single unit, now units can be arranged 5! ways, and groom and bride can be arranged 2 ways. Hence total possible ways 2!*5! = 240. Next to groom means immediate next to groom.

b) 6 people can be arranged in 6! Ways – when bride & groom are next to each other
6! – 2!*5! = 720-240 = 480 ways to arrange the people with the bride not standing next to the groom.

c) Note that bride need not to be immediate next to groom, it can be anywhere in the left side of groom. There are total ways to arrange 6! Ways, exactly half must have the bride somewhere to the left of the groom. Hence answer will be 720/2 = 360

48. How many bit strings of length seven either begin with two 0s or end with three 1s?

Solution

|A U B| = |A| + |B| - |A ∩ B|
=2^5 + 2^4 – 2^2 = 32+16-4 = 44

49. How many bit strings of length 10 either begin with three 0s or end with two 0s?

Solution

=2^7 + 2^8 – 2^5 = 128 + 256 – 32 = 352

***50.** How many bit strings of length 10 contain either five consecutive 0s or five consecutive 1s?

Solution

A = 5 consecutive 0's, s _ s _ s _ s _ s _s = there are 6 possible 6 places where 5 consecutive 0's can be placed, and left 5 bits can be chosen 2^5 ways.

This is wrong way of doing it, because we are counting many duplicate cases.

Correct Method

Consecutive five 0's "00000" can start from the position 1st, 2nd, 3rd, 4th, 5th or 6th

Case 1: start with 1st position, remaining 5 places can be chosen 2^5 ways 00000xxxxx

Case 2: Start with second position, x00000xxx first x has to be 1, otherwise if it is 0-00000-xxxx this case has already been counted above, hence 100000xxxx = 2^4 ways

Case 3: start with 3rd position, xx00000xxx, first x can be either 0 or 1 but second x has to be because

00-00000-xxx has already been counted in case 1

10-00000-xxx has been already counted in case 2.

Hence there will be 2^4 ways

Case 4: starting with 4th, 5th or 6th position in each case there will be 2^4 ways

Total = 32 + 5*2^4 = 32+80 = 112

The 5 consecutive 1's follow the same pattern, and have 112 possibilities.

There are two cases with both 5 consecutive 0's and 5 consecutive 1's

0000011111 or 1111100000

Hence answer to our questions will be = 112 + 112 – 2 = 222

51. How many bit strings of length eight contain either three consecutive 0s or four consecutive 1s?

Solution string length is 8.

A = three consecutive 0's

_ _ _ _ _ _ _ _

Three consecutive 0's can start either from the position 1, 2, 3, 4, 5 or 6th

Case 1: start with 1st position 000-xxxxx, any x can be chosen 2 ways hence total 2^5=32 ways

Case 2:

a. start with 2nd position, x000xxxx, 2^4 ways, 1st x will be 1

b. start with 3rd position, xx000xxx, 2nd x will be 1, 2^4 ways

c. start with 4th position, xxx000xx, 3rd x will be 1, 2^4 ways

d. start with 5th position, xxxx000x, 4th x will be 1, position from 1 to 3 can be anything except 3 0's as 0001000x has already been counted in first case, there are 7 ways we can choose first three bits, and 2 ways to choose last 1 bit, total ways = 7*2 = 14 ways

3. three consecutive 0's are kept from 6th position, xxxxx000, 5th position will be 1. First 4 bits can be chosen 2^4 ways except 0000-1-000, 0001-1-000, 1000-1-000, hence there are only such 13 ways.

Total = 32+ 3*2^4 + 14 + 13 = 107 ways.

B = four consecutive 1's

Can start with 1st position or 2nd or 3rd or 4th or 5th position

Case 1: start with 1st position 1111xxxx, 2^4 = 16 ways

Case 2: x1111xxx, first x will be 0, 2^3

Total ways = 16 + 4*2^3 = 16+32 = 48 ways

=|A & B| contain both three consecutive 0's and four consecutive 1's.

= 0-000-1111, 1-000-1111, 000-1111-0, 000-1-1111, 11110000, 11110001, 01111000, 11111000, there are such 8 cases

Answer : 107 + 48 – 8 = **147 ways**.

53. How many positive integers not exceeding 100 are divisible either by 4 or by 6?

Solution

100/4 = 25

100/6 = 16

LCM(4,6) = 12

100/12 = 8

= 25+16-8 = 23

The Pigeonhole Principle

Suppose that a flock of 20 pigeons flies into a set of 19 pigeonholes to roost. Because there are 20 pigeons but only 19 pigeonholes, a least one of these 19 pigeonholes must have at least two pigeons in it.
This illustrates a general principle called the **pigeonhole principle**, which states that if there are more pigeons than pigeonholes, then there must be **at least one** pigeonhole with at least two pigeons in it.

THE PIGEONHOLE PRINCIPLE If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

A function f from a set with $k+1$ or more elements to a set with k elements is not one-to-one.

Example Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

Example In any group of 27 English words, there must be at least two that begin with the same letter, because there are **26 letters in the English alphabet**.

Example How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?
Solution
There are 101 possible scores on the final. The pigeonhole principle shows that among any 102 students there must be at least 2 students with the same score.

THE GENERALIZED PIGEONHOLE PRINCIPLE If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Example among 100 people there are at least **Ceil**(100/12) = 9 who were born in the same month.

Example What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?
Solution
Ceil(n/5) = 6, smallest such number is 26, as ceil(26/5) = 6 **(because of ceil function)**

Example
a) How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?
b) How many must be selected to guarantee that at least three hearts are selected?
Solution
a) There are 4 decks, we need to select at least **9 cards** to guarantee that at least three cards of the same suit are chosen
Ceil(N/4) = 3, 9 is the smallest number which will give remainder 3. Note that if eight cards are selected, it is possible to have two cards of each suit, so more than eight cards are needed.

b) 39+3 = 42.
We do not use the generalized pigeonhole principle to answer this question, because we want to make sure that there are three hearts, not just three cards of one suit.

Permutations and Combinations

Permutations

A **permutation** of a **set of distinct objects** is an ordered arrangement of these objects. An **ordered arrangement of r elements of a set** is called an **r-permutation**.

The number of **r-permutations of a set with n elements** is denoted by **P(n, r)**.

Example

Let S = {a, b, c}. The **2-permutations** of S are the ordered arrangements **a, b**; **a, c**; **b, a**; **b, c**; **c, a**; and **c, b**. Consequently, there are **six 2-permutations** of this set with three elements.
First select 2 out of 3 elements, and then arrange them. 3C2 = 3 ways and then arrange them 3*2 = 6 ways.

If n is a positive integer and r is an integer with $1 \le r \le n$, then there are

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$

r-permutations of a set with n distinct elements.

If n and r are integers with $0 \le r \le n$, then $P(n, r) = \dfrac{n!}{(n-r)!}$.

Example How many ways are there to select a **first-prize winner**, a **second-prize winner**, and a **third-prize winner** from **100 different people** who have entered a contest?
Solution
100*99*98 = 100P3 = 100!/97! = 970,200.

Example Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?
Solution
The first city is **determined**, but the remaining seven can be ordered arbitrarily. Consequently, there are 7! = 7 · 6 · 5 · 4 · 3 · 2 · 1 = 5040 ways

Example How many permutations of the letters ABCDEFGH contain the string ABC ?
Solution consider ABC as one unit, and permute 'ABC'DEFGH in 6! Ways.

Combination

"Counting **unordered selections** of objects"

Example How many different committees of **three students** can be formed from a group of **four students**?
We need only find the **number of subsets with three elements** from the set containing the four students.
Choosing three students is the same as choosing one of the four students to leave out of the group. This means that there are four ways to choose the three students for the committee, where he order in which these students are chosen does not matter.

An **r-combination** of elements of a set is an unordered selection of **r elements from the set**. Thus, an **r-combination is simply a subset of the set with r elements**.

Example We see that C(4, 2) = **6**, because the **2-combinations** of {a, b, c, d} are the six subsets {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, and {c, d}.

The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \le r \le n$, equals

$$C(n, r) = \frac{n!}{r!\,(n-r)!}.$$

Example How many ways are there to **select five players** from a **10-member tennis team** to make a trip to a match at another school?

Solution

The answer is given by the number of **5-combinations** of **a set with 10 elements**.

$$C(10, 5) = \frac{10!}{5!\,5!} = 252$$

Example How many bit strings of **length n** contain **exactly r 1s**?

Solution

The positions of r 1s in a bit string of length n form an **r-combination** of the set {1, 2, 3 . . . n}. Hence, there are **C(n, r)** bit strings of length n that contain exactly r 1s.

Example Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways committee can be chosen to consist of **three** faculty members from the mathematics department and **four** from the computer science department?

Solution

9C3*11C4 = 9!/3!6! * 11!/7!4! = 84*330 = 27,720

Let n and r be nonnegative integers with $r \le n$. Then $C(n, r) = C(n, n - r)$

Exercise

7. Find the number of 5-permutations of a set with nine elements.

Solution 9P5 = 9!/4! = 9*8*7*6*5=15,120.

9. How many possibilities are there for the win, place, and show (first, second, and third) positions in a horse race with 12 horses if all orders of finish are possible?

Solution 12P3 = 12!/9! = 12*11*10 = 1320. We need to pick 3 horses from the 12 horses in the race, and we need to arrange them in order (first, second, and third).

11. How many bit strings of length 10 contain

a) exactly four 1s?
b) at most four 1s?
c) at least four 1s?
d) an equal number of 0s and 1s?

a) Select 4 places out of 10 and place 1 and in the remaining place put 0’s. 10C4 = 10!/4!6!=**210**

b) 0 1’s, one 1’s, two 1’s, three 1’s, four 1’s

10C0 + 10C1 + 10C2 + 10C3 + 10C4 = 1 + 10 + 45 + 120 + 210 = **386**

c) 10C4 + 10C5 + 10C6 + 10C7 + 10C8 + 10C9 + 10C10 = 210 + 252 + 210 + 120 + 45 + 10 + 1 = **484**

=total number of strings – less than four 1s

= 1024 – (10C0 + 10C1 + 10C2 + 10C3) = 1024 –(1+10+45+120) = 1024 – 176 = **848**

d) Five 0’s and five 1’s. Select five places out of 10 and put 1’s, 10C5 = **252**

13. A group contains n men and n women. How many ways are there to arrange these people in a row if the men and women alternate?

Solution

Suppose n = 4, either row will start with men MwMwMwMw or row will start with women wMwMwMwM there are only 2 possibilities, further we can arrange 4 men in 4! Ways and 4 women 4! Ways. Total permutations = 2*4!*4! = 2*n!*n!

14. In how many ways can a set of two positive integers less than 100 be chosen?

Solution

There are 99 **positive integers** less than 100, we can choose 2 integers out of 99 = 99C2 = 4,851

16. How many subsets with an odd number of elements does a set with 10 elements have?

Solution

=C(10, 1) + C(10, 3) + C(10, 5) + C(10, 7) + C(10, 9) = **512**

OR

There are total 1024 subsets possible, half of them will be with even element and half of them will be with odd number of elements i.e. 512.

18. A coin is flipped eight times where each flip comes up either heads or tails. How many possible outcomes

a) are there in total?
b) contain exactly three heads?
c) contain at least three heads?
d) contain the same number of heads and tails?

Solution

a) 2^8 = 256
b) 8C3 = 8!/5!3! = 56 ways, select 3 out of 8 flips.
c) Total possibilities – (less than 3 heads) = 2^8 – (8C0 + 8C1 + 8C2) = 256 – 37 = 219
d) Select 4 out of 8, 8C4 = 8!/4!4! = **70**

21. How many permutations of the letters *ABCDEFG* contain

a) the string *BCD*?
b) the string *CFGA*?
c) the strings *BA* and *GF*?
d) the strings *ABC* and *DE*?
e) the strings *ABC* and *CDE*?
f) the strings *CBA* and *BED*?

Solution

a) There are 5 letters, put BCD together A'BCD'EFG can be permutated in 5! Ways.
b) BDE'CFGA' can be permutated in 4! Ways
c) 'BA' C D E 'GF' = 5! Ways
d) 'ABC' 'DE' FG = 4! Ways
e) If both ABC and CDE are substrings, then **ABCDE has to be a substring**. So we are really just permuting three items: ABCDE, F, and G. Therefore the answer is P(3,3) = 3! = 6.
f) There are **no permutations** with both of these substrings, since B cannot be followed by both A and E at the same time.

23. How many ways are there for eight men and five women to stand in a line so that no two women stand next to each other? [*Hint:* First position the men and then consider possible positions for the women.]

Solution

Eight men, and five women. _M_M_M_M_M_M_M_M_
Task 1: Select 5 out of 9 positions = 9C5 = 126

Task2: 8 men can be permutated in 8! Ways and women can be permutated 5! Ways.
8!*5!
Total ways = 126*8!*5! = 609,638,400.

25. One hundred tickets, numbered 1, 2, 3, . . . , 100, are sold to 100 different people for a drawing. Four different prizes are awarded, including a grand prize (a trip to Tahiti). How many ways are there to award the prizes if

a) there are no restrictions?
b) the person holding ticket 47 wins the grand prize?
c) the person holding ticket 47 wins one of the prizes?
d) the person holding ticket 47 does not win a prize?
e) the people holding tickets 19 and 47 both win prizes?
f) the people holding tickets 19, 47, and 73 all win prizes?
g) the people holding tickets 19, 47, 73, and 97 all win prizes?
h) none of the people holding tickets 19, 47, 73, and 97 wins a prize?
i) the grand prize winner is a person holding ticket 19, 47, 73, or 97?
j) the people holding tickets 19 and 47 win prizes, but the people holding tickets 73 and 97 do not win prizes?

Solution
a) 100P4 = 100*99*98*97 = 94,109,400
b) 99P3 = 99*98*97 = 941,094
c)Select one out of 4 prizes and give it to person holding ticket 47, and then give remaining 3 prizes to other 3 people. 4C1*99P3 = 4*99*98*97 = 3,764,376
d) 99P4
e) There are 4 · 3 = 12 ways to determine which prizes these two lucky people will win, after which there are P(98, 2) = 9506, hence answer will be 4C1*3C1*98P2.
f) 24 ways to determine which prizes these three lucky people will win. And then choose one 97P1 = 97, answer will be 24*97
g) 4P4 = 4! = 24
h) 96P4
i) Grand prize can be given 4 ways, and the select 3 people from remaining 99 people, 4*99P3
j) 4C1*3C1*96P2

Binomial Coefficients and Identities

The Binomial Theorem

A Binomial expression is simply the sum of two terms, such as x + y.

THE BINOMIAL THEOREM Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

= nC0x^n y^(n-0) + nc1x^(n-1)y^1 + nC2 x^(n-2)y^2 + ……… nCnx^(n-n)y^n

Example What is the expansion of (x + y)^4?

Solution

= 4C0x^4y^0 + 4C1x^3y^1 + 4C2x^2y^2 + 4C3x^1y^3 + 4C4x^0y^4

$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

Note – the sum of powers of x and y is equal to n.

Example What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

Solution the power if x is 12, and the power of y is 13, it means

=25C13x^12y^13 = 5,200,300

Example What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

Solution

= 25C13(2x)^12(-3y)^13 = 25C13*2^12*(-3)^13 $= -\frac{25!}{13!\,12!} 2^{12} 3^{13}$

Let n be a nonnegative integer. Then

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

=(1+1)^n = nC0 + nC1 + nC2 + … … + nCn = 2^n

= 5C0 + 5C1 + 5C2 + 5C3 + 5C4 + 5C5 = 1 + 5 + 10 + 10 + 5 + 1 = 32 = **2^5**

PASCAL'S IDENTITY Let n and k be positive integers with $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Example 90C15 = 89C14 + 89C15

Generalized Permutations and Combinations

In many counting problems, elements may be used repeatedly. For instance, a letter or digit may be used more than once on a license plate.

Permutations with Repetition

Counting permutations when repetition of elements is allowed can easily be done using the **product rule**,

Example How many strings of length r can be formed from the uppercase letters of the English alphabet?
Solution By the product rule, every letter can be used repeatedly. 26^r strings of uppercase English letters of length r are possible.

The number of r-permutations of a set of n objects with repetition allowed is n^r.

Combinations with Repetition

Example How many ways are there to select **four pieces of fruit** from a bowl containing **apples**, **oranges**, and **pears** **if the order in which the pieces are selected does not matter**, **only the type of fruit and not the individual piece matters**, and t**here are at least four pieces of each type of fruit in the bowl**?
Solution

There are $C(n+r-1, r) = C(n+r-1, n-1)$ r-combinations from a set with n elements when repetition of elements is allowed.

X1 + x2 + x3 = 4
X1, x2, x3>=0
n = 3, as there are 3 items, and r=4
(n+r-1)Cr = 6C4 = 6C2 = **15** ways.

Example Suppose that a cookie shop has **four different kinds of cookies**. How many different ways can **six cookies be chosen**? Assume that only the type of cookie, and not the individual cookies or the order in which they are chosen, matters.
Solution
n=4, r=6
(n+r-1)Cr = (n+r-1)C(n-1) = 9C6 = 9C3 = 84

Example How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where x1, x2, and x3 are **nonnegative integers**?
Solution

$$C(3+11-1, 11) = C(13, 11) = C(13, 2) = \frac{13 \cdot 12}{1 \cdot 2} = 78$$

<u>Variables with constraints</u>

The number of solutions of this equation can also be found when the variables are subject to constraints.

$$x_1 + x_2 + x_3 = 11$$
$$x_1 \geq 1,\ x_2 \geq 2, \text{ and } x_3 \geq 3$$

There is at least <u>one item of type one</u>, <u>two items of type two</u>, <u>and three items of type three</u>. So, a solution **<u>corresponds to a choice of one item of type one, two of type two, and three of type three, together with a choice of five additional items of any type</u>**.

x1 + x2 + x3 = 11-6 = 5
x1, x2, x3 >=0

(3+5-1)C5 = 7!/(2!5!) = 21

TABLE 1 Combinations and Permutations With and Without Repetition.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

<u>Permutations with Indistinguishable Objects</u>

<u>Some elements may be indistinguishable</u> in counting problems.

<u>Example</u> How many different strings can be made by reordering the letters of the word **SUCCESS**?
Solution
<u>Because some of the letters of SUCCESS are the same</u>, the a**<u>nswer is not given by the number of permutations of seven letters</u>**.
S = 3   (their order don't matter)
U = 1
C = 2 (Order doesn't matter)
E = 1
To determine the number of different strings that can be made by reordering the letters,

1. First note that the **three S** can be placed among the seven positions in **C(7, 3)** **<u>different ways</u>** leaving four positions free.
2. Then the **two C** can be placed in **C(4, 2) ways**, **<u>leaving two free positions</u>**.
3. The **U** can be placed in **C(2, 1)** ways, leaving just one position free.
4. Hence **E** can be placed in **C(1, 1)** way.

$$C(7,3)C(4,2)C(2,1)C(1,1) = \frac{7!}{3!\,4!} \cdot \frac{4!}{2!\,2!} \cdot \frac{2!}{1!\,1!} \cdot \frac{1!}{1!\,0!}$$
$$= \frac{7!}{3!\,2!\,1!\,1!}$$
$$= 420.$$

The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, . . . , and n_k indistinguishable objects of type k, is

$$\frac{n!}{n_1!\,n_2!\cdots n_k!}.$$

Distributing Objects into Boxes

Many counting problems can be solved by enumerating the ways objects can be placed into boxes (where the order these objects are placed into the boxes does not matter).

1. The objects can be either **distinguishable**, that is, different from each other, or **indistinguishable**, that is, considered identical.
2. Distinguishable objects are sometimes said to be **labeled**, whereas indistinguishable objects are said to be **unlabeled**.
3. Similarly, **boxes can be distinguishable**, that is, different, or **indinguishable**, that is, identical.
4. Distinguishable boxes are often said to be **labeled**, while indistinguishable boxes are said to be **unlabeled**.
5. When we solve a counting problem using the **model of distributing objects into boxes**, we need to determine whether the **objects are distinguishable** and whether the **boxes are distinguishable**.

DISTINGUISHABLE OBJECTS AND DISTINGUISHABLE BOXES

We first consider the case when **distinguishable objects** are placed into **distinguishable boxes**.

Example How many ways are there to distribute hands of **5 cards to each of four players** from the standard deck of 52 cards?

Solution

We will use the **product rule** to solve this problem. To begin, note that the first player can be dealt 5 cards in **C(52, 5)** ways. The second player can be dealt 5 cards in C(47, 5) ways, because only 47 cards are left. The third player can be dealt 5 cards in C(42, 5) ways. Finally, the fourth player can be dealt 5 cards in C(37, 5) ways. Hence, the total number of ways to deal four players 5 cards each is

$$C(52,5)C(47,5)C(42,5)C(37,5) = \frac{52!}{47!\,5!} \cdot \frac{47!}{42!\,5!} \cdot \frac{42!}{37!\,5!} \cdot \frac{37!}{32!\,5!}$$
$$= \frac{52!}{5!\,5!\,5!\,5!\,32!}.$$

The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, $i = 1, 2, \ldots, k$, equals

$$\frac{n!}{n_1!\,n_2!\cdots n_k!}.$$

INDISTINGUISHABLE OBJECTS AND DISTINGUISHABLE BOXES

Counting the number of ways of placing **n indistinguishable objects into k distinguishable boxes** turns out to be the same as counting the number of n-combinations for a set with k elements when repetitions are allowed.

Example

How many ways are there to place **10 indistinguishable balls into eight distinguishable bins**?

Solution

X1 + X2 + X3 + X4 + X5 + X6 + X7 + X8 = 10

N=8, r=10 (n+r-1)Cr = C(8+10-1,10) = 17!/10!7! = 19,448.

DISTINGUISHABLE OBJECTS AND INDISTINGUISHABLE BOXES

Counting the ways to place **n distinguishable objects into k indistinguishable boxes** is more difficult.

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Ex: 5 distinct balls in 2 identical bins?

1. Keep all 5 balls in one bin and keep another bin empty = 1 way = 5C5
2. Keep 1 ball in one bin and 4 in other bin = 5 ways as we have 5 choices to keep 1 ball. = 5 ways = 5C1 = 5C4
3. Keep two balls in one bin and 3 balls in other bin = 5C2 = 5C3 10 ways

Total cases = 1 + 5 + 10 = 16

INDISTINGUISHABLE OBJECTS AND INDISTINGUISHABLE BOXES

Example How many ways are there to pack six copies of the same book into four identical boxes, where a box can contain as many as six books?

Solution

For each way to pack the books, we will list the number of books in the box with the largest number of books, followed by the numbers of books in each box containing at least one book, in order of decreasing number of books in a box. The ways we can pack the books are

6
5, 1
4, 2
4, 1, 1
3, 3
3, 2, 1
3, 1, 1, 1
2, 2, 2
2, 2, 1, 1.

For example, 4, 1, 1 indicates that one box contains four books, a second box contains a single book, and a third box contains a single book (and the fourth box is empty). We conclude that there are **nine allowable ways to pack the books**, because we have listed them all.

__________________________________Exercise__________________________________

1. In how many different ways can five elements be selected in order from a set with three elements when repetition is allowed?

Solution

5 elements can be selected from, a set with 3 elements. There are 3 ways in which the first element can be selected, 3 ways in which the second element can be selected, and so on,

3x3x3x3x3 = 3^5.

3. How many strings of six letters are there?

Solution

26^6 = for each letter there are 26 possibilities.

15. How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21,$$

where x_i, $i = 1, 2, 3, 4, 5$, is a nonnegative integer such that

a) $x_1 \geq 1$?
b) $x_i \geq 2$ for $i = 1, 2, 3, 4, 5$?
c) $0 \leq x_1 \leq 10$?
d) $0 \leq x_1 \leq 3$, $1 \leq x_2 < 4$, and $x_3 \geq 15$?

Solution

a) x1+x2+x3+x4+x5 = 20

b) x1+x2+x3+x4+x5 = 11

c) (x1+x2+x3+x4+x5 = 21) – (x1+x2+x3+x4+x5 = 21, x1>=11)

(x1+x2+x3+x4+x5 = 21) – (x1+x2+x3+x4+x5 = 10)

d) x1+x2+x3+x4+x5 = 21

0=< x1 =< 3; 1=<x2=<3; x3>=15

Case 1:

First let us impose the restrictions that x3>=15 and x2>=1. Then problem is equivalent to counting the number of solutions to x1+x2+x3+x4+x5 = 5. C(5+5-1,5) = 9C5 = 126

Case 2:

When x1>=4, and our equation is x1+x2+x3+x4+x5 = 5, hence after imposing this condition our equation will be x1+x2+x3+x4+x5 = 1, C(5+1-1,1) = 5C1 = 5

Case 3:

When x2>=3, and our equation is x1+x2+x3+x4+x5 = 5, hence after imposing this condition our equation will be x1+x2+x3+x4+x5 = 2, C(5+2-1,2) = 6C2 = 15

Using generating functions

=(1 + x + x^2 + x^3) (x+x^2+x^3)(x^15 + x^16 + …)(1 + x + x^2 + x^3 + …..)^2

= (1 – x^4/1-x)x(1+x+x^2)x^15(1 + x + x^2 + x^3 + …)(1/(1-x)^2)

= (1 – x^4/1-x)x(1-x^3/(1-x))x^15(1/(1-x))(1/(1-x)^2)

=x^16 (1-x^4)(1-x^3)(1/(1-x)^5)

= (1-x^4)(1-x^3)(1/(1-x)^5) find the coefficient of x^5

= 1 – x^3 – x^4 +x^7 , where n=5

= C(9,5) – C(6,2) – C(5,1) = 126 – 15 – 5 = 106

16. How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 29,$$

where x_i, $i = 1, 2, 3, 4, 5, 6$, is a nonnegative integer such that

a) $x_i > 1$ for $i = 1, 2, 3, 4, 5, 6$?
b) $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$, $x_4 \geq 4$, $x_5 > 5$, and $x_6 \geq 6$?
c) $x_1 \leq 5$?
d) $x_1 < 8$ and $x_2 > 8$?

Solution

a) x1 + x2 + x3 + x4 + x5 + x6 = 23

b) x1 + x2 + x3 + x4 + x5 + x6= 7

c) x1 + x2 + x3 + x4 + x5 + x6 = 29 - x1 + x2 + x3 + x4 + x5 + x6 = 23

d) case 1: x2>=9, x1 + x2 + x3 + x4 + x5 + x6= 20

Case 2: x1=<7 hence subtract those cases when x1>=8
x1 + x2 + x3 + x4 + x5 + x6= 12 (case 1 condition is already applied)
= C(6+20-1, 20) – C(6+12-1,12) = C(25,20) – C(17,12) = 46942

17. How many strings of 10 ternary digits (0, 1, or 2) are there that contain exactly two 0s, three 1s, and five 2s?

Solution
String length = 10, digits = 0, 1 or 2
Select 2 places out of 10 and put 0, then select 3 places out of remaining 8 places and put 1, and select 5 places out of remaining 5 and put 2's.
=10C2*8C3*5C5 = 45*56*1 = 2520
Or
10!/2!3!5! = 2520

18. How many strings of 20-decimal digits are there that contain two 0s, four 1s, three 2s, one 3, two 4s, three 5s, two 7s, and three 9s?

Solution
String length = 20
Digits = 0-9
20!/2!4!3!1!2!3!2!3! = 58663725120000

20. How many solutions are there to the inequality

$$x_1 + x_2 + x_3 \leq 11,$$

where x_1, x_2, and x_3 are nonnegative integers? [*Hint:* Introduce an auxiliary variable x_4 such that $x_1 + x_2 + x_3 + x_4 = 11$.]

Solution
X1 + x2 + x3 + x4 = 11, C(4+11-1,11)

21. How many ways are there to distribute six indistinguishable balls into nine distinguishable bins?

22. How many ways are there to distribute 12 indistinguishable balls into six distinguishable bins?

23. How many ways are there to distribute 12 distinguishable objects into six distinguishable boxes so that two objects are placed in each box?

24. How many ways are there to distribute 15 distinguishable objects into five distinguishable boxes so that the boxes have one, two, three, four, and five objects in them, respectively.

Solution
21. x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 = 6, C(9+6-1,6) = C(14,6)
22. x1 + x2 + x3 + x4 + x5 + x6 = 12, C(6+12-1,12) = C(17,12)
23. 12 D into 6 D, 2 objects are placed in each box.
We will use product rule to solve this problem.
Select 2 items from 12, and put them into a box, and so on.
12C2*10C2*8C2*6C2*4C2*2C2
= 12!/2!10! * 10!/2!8! * 8!/2!6! * 6!/2!4! * 4!/2!2! * 2!/2!*0!
=12!/2!2!2!2!2!2! = 7,484,400
24. 15!/1!2!3!4!5! = 37,837,800

Note: Since it used the word "respectively" there is only one way to order the boxes. If the labels of the boxes are **not** fixed and can be reassigned then we need to multiply with 5!

25. How many positive integers less than 1,000,000 have the sum of their digits equal to 19?

Solution

As the numbers are less than 1,000,000, then there will be 6 digits in the number and leading digits can be 0.

_ _ _ _ _ _ = x1+x2+x3+x4+x5+x6=19 with the condition that 0<=Xi <=9

C(6+19-1,19) = C(24,19)

We must subtract the number of solutions in which the restriction is violated. If the digits are to add up to 19 and one or more of them is to exceed 9, then exactly one of them will have to exceed 9, since 10 + 10 > 19. There are 6 ways to choose the digit that will exceed 9.

We have counted those solution also where one digit is 10, and more than one 10 as single digit is not possible as 10+10 = 20 has exceeded 19. But we counted only those solution whose sum was 19. And any 1 out of 6 digits can be 10.

x1+x2+x3+x4+x5+x6=19, x1>=10

x1+x2+x3+x4+x5+x6=9

C(6+9-1,9) = C(14,9), there are 6 choices to choose x>=10.

Hence answer will be C(24,19) – 6*C(14,9)

26. How many positive integers less than 1,000,000 have exactly one digit equal to 9 and have a sum of digits equal to 13?

Solution

x1+x2+x3+x4+x5+x6=13, such that one digit is 9, then other 5 digits can be less than 5 as 9+4 = 13

x1+x2+x3+x4+x5 = 4, C(5+4-1,4) = C(8,4) = 70

There are 6 choices for selecting 9, hence 6*70 = 420.

27. There are 10 questions on a discrete mathematics final exam. How many ways are there to assign scores to the problems if the sum of the scores is 100 and each question is worth at least 5 points?

Solution

x1+x2+x3+x4+x5+x6+x7+x8+x9+x10=100, xi>=5

We can rewrite as

x1+x2+x3+x4+x5+x6+x7+x8+x9+x10=50, 0<=x

C(10+50-1,50) = C(59,50)

31. How many different strings can be made from the letters in *ABRACADABRA*, using all the letters?

Solution

ABRACADABRA

A-5, B-2, C-1, D-1, R-2

=11!/5!2!1!1!2! = 83,160.

32. How many different strings can be made from the letters in *AARDVARK*, using all the letters, if all three *A*s must be consecutive?

Solution

AARDVARK

A-3, D-1, R-2, V-1, K-1

Consider all A's 1 unit. A-1, D-1, R-2, V-1, K-1 = 6!/1!1!2!1!1! = 360

33. How many different strings can be made from the letters in *ORONO*, using some or all of the letters?

Solution

1 length strings = O, R, N

2 length strings = OO, ON, OR, NO, RO, RN, NR

13 strings of length three = 3! using('O','R','N') + 3 using('O','O','N') + 3 using('O','O','R') + 1 using('O','O','O')

20 strings of length four = 12 using('O','O','N','R') + 4 using('O','O','O','N') + 4 using('O','O','O','R')

20 strings of length five = 20 using all characters

Total strings =63

________________________________Advanced Counting Problems__________________________

Example Find a recurrence relation and give initial conditions for the **number of bit strings of length n** that **do not have two consecutive 0s**. How many such bit strings are there of length five?

Solution

Let a_n denote the number of bit strings of length n that do not have two consecutive 0s. Assume that we are filling the string from the end.

The bit strings of length n ending with 1 that do not have two consecutive 0s are precisely the bit strings of length $n-1$ with no two consecutive 0s with a 1 added at the end. Consequently, there are a_{n-1} such bit strings.

Bit strings of length n ending with a 0 that do not have two consecutive 0s must have 1 as their $(n-1)$st bit; otherwise they would end with a pair of 0s. It follows that the bit strings of length n ending with a 0 that have no two consecutive 0s are precisely the bit strings of length $n-2$ with no two consecutive 0s with 10 added at the end. Consequently, there are a_{n-2} such bit strings.

We conclude, as illustrated in Figure 4, that

$a_n = a_{n-1} + a_{n-2}$

For n>=3.

The initial conditions are $a_1 = 2$, because both bit strings of length one, 0 and 1 do not have consecutive 0s, and $a_2 = 3$, because the valid bit strings of length two are 01, 10, and 11. To obtain a_5, we use the recurrence relation three times to find that

$$\begin{aligned} a_3 &= a_2 + a_1 = 3 + 2 = 5, \\ a_4 &= a_3 + a_2 = 5 + 3 = 8, \\ a_5 &= a_4 + a_3 = 8 + 5 = 13. \end{aligned}$$

Remark: Note that $\{a_n\}$ satisfies the same recurrence relation as the Fibonacci sequence. Because $a_1 = f_3$ and $a_2 = f_4$ it follows that $a_n = f_{n+2}$.

The Inclusion-Exclusion Principle

EXAMPLE 5 Give a formula for the number of elements in the union of four sets.

Extra Examples

Solution: The inclusion–exclusion principle shows that

$$\begin{aligned} |A_1 \cup A_2 \cup A_3 \cup A_4| = {} & |A_1| + |A_2| + |A_3| + |A_4| \\ & - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| \\ & - |A_3 \cap A_4| + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| \\ & + |A_2 \cap A_3 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_4|. \end{aligned}$$

Note that this formula contains 15 different terms, one for each nonempty subset of $\{A_1, A_2, A_3, A_4\}$.

Derangements

A derangement is a permutation of objects that leaves no object in its original position.

The permutation **21453 is a derangement of 12345** because no number is left in its original position. However, **21543 is not a derangement of 12345, because this permutation leaves 4 fixed**.

Let Dn denote the **number of derangements** of n objects. For instance, **D3 = 2**, because the derangements of 123 are 231 and 312.

The number of derangements of a set with n elements is

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right].$$

n! is total possible ways in which n objects can be permutated.

Dn = n! – [not derangement sequences]

nC1*(n-1)! Means select 1 object out of n keep it in its correct position and remaining (n-1) objects can be arranged in (n-1)! Ways.

Dn = n! – (nC1*(n-1)! – nC2(n-2)! + nC3(n-3)! - . . . . . (-1)^n nCn*(n-n)!)

Consequently, inserting these quantities into our formula for D_n gives

$$\begin{aligned} D_n &= n! - C(n,1)(n-1)! + C(n,2)(n-2)! - \cdots + (-1)^n C(n,n)(n-n)! \\ &= n! - \frac{n!}{1!(n-1)!}(n-1)! + \frac{n!}{2!(n-2)!}(n-2)! - \cdots + (-1)^n \frac{n!}{n!\,0!}0!. \end{aligned}$$

Simplifying this expression gives

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right].$$

Probability

Finite Probability

An **experiment** is a procedure that yields **one of a given set of possible outcomes**. The **sample space** of the experiment is the **set of possible outcomes**. An **event** is a subset of the sample space.

If S is a finite nonempty sample space of equally likely outcomes, and E is an event, that is, a subset of S, then the *probability* of E is $p(E) = \frac{|E|}{|S|}$.

note that if E is an event from a finite sample space S, then $0 \leq |E| \leq |S|$, because $E \subseteq S$. Thus, $0 \leq p(E) = |E|/|S| \leq 1$.

Example An urn contains four blue balls and five red balls. What is the probability that a ball chosen at random from the urn is blue?

Solution

Note that there are **nine possible outcomes**, and four of these possible outcomes produce a blue ball. Hence, the probability that a blue ball is chosen is **4/9**.

Example What is the probability that when two dice are rolled, the sum of the numbers on the two dice is 7?

Solution

There are a total of 36 equally likely possible outcomes when two dice are rolled. There are six successful outcomes, namely, (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), and (6, 1), where the values of the first and second dice are represented by an ordered pair. Hence, the probability that a seven comes up when two fair dice are rolled is 6/36 = 1/6.

Example In a lottery, players win a large prize when they pick four digits that match, in the correct order, four digits selected by a random mechanical process. A smaller prize is won if only three digits are matched. What is the probability that a player wins the large prize? What is the probability that a player wins the small prize?

Solution

P(Player wins the large prize) =

There is only one way to choose all four digits correctly.

By the product rule, there are 10^4 = 10,000 ways to choose four digits. Hence, the probability that a player wins the **large prize** is 1/10,000 = 0.0001.

P(player wins the small prize) = exactly one digit is wrong and remaining 3 digits are correct.

There are possible 10 choices for the digit, so there are 9 possible wrong digits for this first place except the correct digit. Remaining 3 digits will have only one choice. Wrong digit can be chosen 4 ways. = 9+9+9+9 = 36/10,000 = 9/2500.

Example There are many lotteries now that award enormous prizes to people who correctly choose **a "set of six numbers" (order doesn't matter) out of the first n positive integers, where n is usually between 30 and 60**. What is the probability that a person picks the **correct six numbers out of 40**?

Solution.

There is only one winning combination. The total number of ways to choose six numbers out of 40 is C(40,6) = 40!/4!36! = 38,38,380.

P = 1/38,38,380

Example What is the probability that the numbers **11, 4, 17, 39, and 23** are drawn in that order from **a bin containing 50 balls labeled with the numbers 1, 2, . . . , 50** if
(a) The ball selected is not returned to the bin before the next ball is selected and
(b) The ball selected is returned to the bin before the next ball is selected?
Solution
a) By the product rule, there are 50 · 49 · 48 · 47 · 46 = 254,251,200 ways to select the balls because each time a ball is drawn there is one fewer ball to choose from.
1/254,251,200.
b) By the product rule, there are 50^5 = 312,500,000 ways to select the balls. P = 1/312,500,000.

Probabilities of Complements and Unions of Events

Let E be an event in a sample space S. The probability of the event $\overline{E} = S - E$, the complementary event of E, is given by

$$p(\overline{E}) = 1 - p(E).$$

$$p(\overline{E}) = \frac{|S| - |E|}{|S|} = 1 - \frac{|E|}{|S|} = 1 - p(E).$$

Example A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is 0?
Solution
= 1 – P(no bit is 0)
S = each bit can be chosen 2 ways. |S| = 2^10
p = 1 – (1/2^10) = 1023/1024

Let E_1 and E_2 be events in the sample space S. Then

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2).$$

Example
What is the probability that a positive integer selected at random from the set of positive integers **not exceeding 100 is divisible by either 2 or 5**?
Solution
E1 = divisible by 2 = 100/2 = 50
E2 = divisible by 5 = 100/5 = 20
E1 Intersect E2 = divisible by 10 = 100/10=10
|S| = 100,
= 50/100 + 20/100 – 10/100 =60/100 = 3/5

Probability Theory

Assigning Probabilities

Let S be the sample space of an experiment with a finite or countable number of outcomes. We assign a probability p(s) to each outcome s.

$$(i) \quad 0 \le p(s) \le 1 \text{ for each } s \in S$$

and

$$(ii) \quad \sum_{s \in S} p(s) = 1.$$

a. Condition (i) states that the probability of each outcome is a nonnegative real number no greater than 1.
b. Condition (ii) states that the sum of the probabilities of all possible outcomes should be 1;

When there are n possible outcomes, x1, x2, . . . , xn, the two conditions to be met are

$$(i) \quad 0 \le p(x_i) \le 1 \text{ for } i = 1, 2, \ldots, n$$

and

$$(ii) \quad \sum_{i=1}^{n} p(x_i) = 1.$$

The function p from the set of all outcomes of the sample space S is called a **probability distribution**.

Example What probabilities should we assign to the outcomes H (heads) and T (tails) when a fair coin is flipped? What probabilities should be assigned to these outcomes when the coin is biased so that heads comes up twice as often as tails?

Solution

When coin is fair, P(H) = P(T) and P(H) + P(T) = 1, hence P(H) =P(T) = ½

When coin is biased
P(H) = 2*P(T)
P(H) + P(T) = 1
2*P(T) + P(T) = 1
P(T) = 1/3 and P(H)=2/3

Suppose that S is a set with n elements. The *uniform distribution* assigns the probability $1/n$ to each element of S.

Example Suppose that a **die is biased so that 3 appears twice as often as each other number** but that the other five outcomes are equally likely. What is the probability that **an odd number appears** when we roll this die?

Solution

P(1) + P(2) + P(4) +P(5) + P(6) = x
P(3) = 2x
P(1) + P(2) + p(3) + P(4) +P(5) + P(6) = 1

5x + 2x = 7x, x=1/7
p(3) = 2*x = 2/7

p(odd numbers) = P(1) + P(3) +P(5) = 1/7 + 2/7 + 1/7 = 4/7

Probabilities of Complements and Unions of Events

$$p(\overline{E}) = 1 - p(E)$$

where $\overline{E}$ is the complementary event of the event E. This equality also holds when Definition 2 is used. To see this, note that because the sum of the probabilities of the n possible outcomes is 1, and each outcome is either in E or in $\overline{E}$, but not in both, we have

$$\sum_{s \epsilon S} p(s) = 1 = p(E) + p(\overline{E}).$$

Hence, $p(\overline{E}) = 1 - p(E)$.

If E1 and E2 are disjoint Sets

Also, note that if the events E_1 and E_2 are disjoint, then $p(E_1 \cap E_2) = 0$, which implies that

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2) = p(E_1) + p(E_2).$$

Conditional Probability

Suppose that we flip a coin three times, and all eight possibilities are equally likely. Moreover, suppose **we know that the event F, that the first flip comes up tails**, occurs. Given this information, **what is the probability of the event E that an odd number of tails appears?**

Because the first flip comes up tails, there are only four possible outcomes: TTT, TTH, THT, and THH.
An odd number of tails appears only for the outcomes TTT and THH.
Hence the probability of odd number of tails will be 2/4 = ½.
This probability is called the **conditional probability** of E given F.

Let E and F be events with $p(F) > 0$. The *conditional probability* of E given F, denoted by $p(E \mid F)$, is defined as

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)}.$$

Example A bit string of **length four** is generated at random so that **each of the 16 "bit strings" of length four is equally likely**. What is the probability that it contains **at least two consecutive 0s**, given that its first bit is a 0?

Solution

Let E be the event that a bit string of length four contains at least two consecutive 0s, and let F be the event that the first bit of a bit string of length four is a 0.

The **probability that a bit string of length four has at least two consecutive 0s, given that its first bit is a 0**, equals

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)}$$

Because E ∩ F = 0001, 0100, 0000, 0010, 0011
P(F)= the first bit of a bit string of length four is 0 = 8 such strings possible.1x2x2x2 = 8
P(E/F) = (5/16)/(8/16) = 5/8

Example What is the conditional probability that a family with two children has two boys, given they have at least one boy? Assume that each of the possibilities BB, BG, GB, and GG is equally likely, where B represents a boy and G represents a girl. (Note that BG represents a family with an older boy and a younger girl while GB represents a family with an older girl and a younger boy.)
Solution
E = Family with two children has two boys. = BB, P(E) = 1/4
F = they have at least one boy.

Total possibilities = 4 = BB, GB, BG, GG
E ∩ **F** = BB, only one possibility. P(E∩F) = 1/4
F = BG, GB, BB = 3/4
$$p(E \mid F) = \frac{p(E \cap F)}{p(F)}$$
= (1/4)/(3/4)= 1/3

Independence

The events E and F are *independent* if and only if $p(E \cap F) = p(E)p(F)$.

As both the events are independent, so P(E/F) = P(E)P(F)/P(F) = P(E)

P(E/F) = P(E)

Example
Suppose **E is the event that a randomly generated bit string of length four begins with a 1** and F is the event that this **bit string contains an even number of 1s**. Are E and F independent, if the 16 "**bit strings"** of length four are equally likely?
Solution
Total possible bit strings of four length = 16
F = bit strings with even number of 1's = 8
E = bit strings of length four begins with 1 = 8

P(E) = P(F) = 8/16 = ½

E∩F = bit strings begins with 1 and contains even number of 1's = 1001, 1010, 1100, 1111
P(E∩F) = 4/16 = 1/4 = P(E)*P(F) = 1/2*1/2 = 1/4

Hence, it's proved that both events are independent.

Example What is the conditional probability that a family with two children has two boys, given they have at least one boy? Assume that each of the possibilities BB, BG, GB, and GG is equally likely, where B represents a boy and G represents a girl. Are the events E and F independent.
Solution
As shown above, P(E) = 1/4 P(F)=3/4
P(E∩F)=1/4 != P(E)*P(F) Hence, not independent.

Example Are the events E, that a family with three children has children of both sexes, and F, that this family has at most one boy, independent? Assume that the eight ways a family can have three children are equally likely.
Solution
E = a family with three children has children of both sexes. 8-2 = 6 cases, (BBB, GGG) are excluded.

P(E) = 6/8
F = this family has at most one boy. GGG, BGG, GBG, GGB, P(F) = 4/8 = 1/2

E∩F = At most one boy, and children of both sexes = BGG, GBG, GGB, P(E∩F) = 3/8
P(E)*P(F) = 6/8*1/2 = 3/8 = P(E∩F) Hence, E and F are **independent**.

Bernoulli Trials and the Binomial Distribution

Suppose that **an experiment can have only two possible outcomes**. For instance, when a bit is generated at random, the possible outcomes are **0** and **1**. When a coin is flipped, the possible outcomes are **heads** and **tails**.
Each performance of an experiment with two possible outcomes is called a **Bernoulli trial**.

In general, a possible outcome of a Bernoulli trial is called a **success** or a **failure**. If p is the probability of a **success** and q is the probability of a **failure**, it follows that p + q = 1.

If p is the probability of a success and q is the probability of a failure, it follows that p + q = 1. Bernoulli trials are mutually independent

Example A coin is biased so that the probability of heads is 2/3. What is the probability that **exactly four heads** come up when the coin is **flipped seven times**, assuming that the flips are independent?
Solution
There are 2^7 = 128 possible outcomes if a coin is tossed 128 times.
The number of ways four of the seven flips can be head is C(7,4). Because the seven flips are independent, the
probability of each of these outcomes (four heads and three tails) is $(2/3)^4(1/3)^3$

$$C(7,4)(2/3)^4(1/3)^3 = \frac{35 \cdot 16}{3^7} = \frac{560}{2187}$$

Probability of k successes in n independent Bernoulli trials.

The probability of exactly k successes in n independent Bernoulli trials, with probability of success p and probability of failure $q = 1 - p$, is

$$C(n,k)p^k q^{n-k}.$$

It is considered as a function of k, we call this function the **binomial distribution**.

$$b(k; n, p) = C(n,k)p^k q^{n-k}.$$

Example
Suppose that the probability that a 0 bit is generated is 0.9 that the probability that a 1 bit is generated is 0.1, and that bits are generated independently. What is the probability that exactly eight 0 bits are generated when 10 bits are generated?
Solution

$$b(8; 10, 0.9) = C(10,8)(0.9)^8(0.1)^2 = 0.1937102445.$$

Note that the sum of the **probabilities that there are k successes when n independent Bernoulli trials** are carried out, for k = 0, 1, 2, . . . , n, equals

$$\sum_{k=0}^{n} C(n,k)p^k q^{n-k} = (p+q)^n = 1.$$

As p + q = 1.

Random Variables

A **random variable** **is a function from the sample space of an experiment to the set of real numbers**. That is, **a random variable assigns a real number to each possible outcome**.

A *random variable* is a function from the sample space of an experiment to the set of real numbers. That is, a random variable assigns a real number to each possible outcome.

Note- A random variable is a function. It is not a variable, and it is not random!

Example 10 Suppose that a coin is flipped three times. Let **X(t) be the random variable that equals the number of heads** that appear when t is the outcome. Then X(t) takes on the following values:

Solution

$$X(HHH) = 3,$$
$$X(HHT) = X(HTH) = X(THH) = 2,$$
$$X(TTH) = X(THT) = X(HTT) = 1,$$
$$X(TTT) = 0.$$

The *distribution* of a random variable X on a sample space S is the set of pairs $(r, p(X = r))$ for all $r \in X(S)$, where $p(X = r)$ is the probability that X takes the value r. (The set of pairs in this distribution is determined by the probabilities $p(X = r)$ for $r \in X(S)$.)

Example Each of the eight possible outcomes when a **fair coin is flipped three times** has probability 1/8. So, the distribution of the random variable X(t) in **Example 10** is determined by the probabilities P(X = 3) = 1/8, P(X = 2) = 3/8, P(X = 1) = 3/8, and P(X = 0) = 1/8. Consequently, the distribution of X(t) in Example 10 is the set of **pairs (3, 1/8), (2, 3/8), (1, 3/8), and (0, 1/8)**.

Example Let **X be the sum of the numbers** that appear when a pair of dice is rolled. **What are the values of this random variable for the 36 possible outcomes (i, j),** where i and j are the numbers that appear on the first die and the second die, respectively, when these two dice are rolled?

Solution

$$X((1,1)) = 2,$$
$$X((1,2)) = X((2,1)) = 3,$$
$$X((1,3)) = X((2,2)) = X((3,1)) = 4,$$
$$X((1,4)) = X((2,3)) = X((3,2)) = X((4,1)) = 5,$$
$$X((1,5)) = X((2,4)) = X((3,3)) = X((4,2)) = X((5,1)) = 6,$$
$$X((1,6)) = X((2,5)) = X((3,4)) = X((4,3)) = X((5,2)) = X((6,1)) = 7,$$
$$X((2,6)) = X((3,5)) = X((4,4)) = X((5,3)) = X((6,2)) = 8,$$
$$X((3,6)) = X((4,5)) = X((5,4)) = X((6,3)) = 9,$$
$$X((4,6)) = X((5,5)) = X((6,4)) = 10,$$
$$X((5,6)) = X((6,5)) = 11,$$
$$X((6,6)) = 12.$$

The Birthday Problem

What is the minimum number of people who need to be in a room so that the probability that at least two of them have the same birthday is greater than 1/2?
Solution
The birthday of the first person certainly does not match the birthday of someone already in the room.
The probability that the birthday of the second person is different from that of the first person is 365/366 because the second person has a different birthday when he or she was born on one of the 365 days of the year other than the day the first person was born.
The probability that the third person has a birthday different from both the birthdays of the first and second people given that these two people have different birthdays is 364/366.

For nth person, subtract n from 367, e.g. 4^{th} person = (367-4)/366=363/366

= 1 – P(everyone is born on different date) > 1/2

= 1 – 366/366*365/366*364/366*…………*(367-n)/n >1/2

If n=23 $1 - p_n \approx 0.506.$

Probability of a Collision in Hashing Functions

To calculate this probability, we assume that the probability that a randomly selected key is mapped to a location is **1/m**, where m is the number of available locations, that is, the hashing function distributes **keys uniformly**.
Suppose that the keys are k1, k2, . . . , kn. When we add the second record, the probability that it is mapped to a **location different from the location of the first record**, that h(k2) != h(k1), is **(m − 1)/m** because there are m − 1 free locations after the first record has been placed.

The probability that the third record is mapped to a free location after the first and second records have been placed without a collision is **(m − 2)/m**.

In general, the probability that the jth record is mapped to a free location after the first j − 1 records have been mapped to locations is **(m − (j − 1))/m**.

Because the keys are independent, the **probability that all n keys are mapped to different locations** is

$$p_n = \frac{m-1}{m} \cdot \frac{m-2}{m} \cdot \cdots \cdot \frac{m-n+1}{m}.$$

Note – probability for the first element will be m/m.

The probability that there is **at least one collision**, that is, at least two keys are mapped to the same location, is

$$1 - p_n = 1 - \frac{m-1}{m} \cdot \frac{m-2}{m} \cdot \cdots \cdot \frac{m-n+1}{m}$$

Bayes' Theorem

Let sample space 'S' be partitioned into n mutually exclusive and collectively exhausted events say E1, E2, E3 …. En and A be any arbitrary event which is subset of E1 U E2 U E3 U …. U En.

P(Ei/A) = P(Ei)*P(A/Ei) / Total Probability

The formula is:

$$P(A|B) = \frac{P(A)\,P(B|A)}{P(B)}$$

It tells us how often A happens *given that B happens*, written **P(A|B)**, when we know how often B happens *given that A happens*, written **P(B|A)** , and how likely A and B are on their own.

- P(A|B) is "Probability of A given B", the probability of A given that B happens
- P(A) is Probability of A
- P(B|A) is "Probability of B given A", the probability of B given that A happens
- P(B) is Probability of B

Example We have two boxes. The first contains **two green balls and seven red balls**; the second contains **four green balls and three red balls**. Bob selects a ball by first choosing one of the two boxes at random. He then selects one of the balls in this box at random. If Bob has selected a red ball, what is the probability that he selected a ball from the first box?
Solution

P(B1/R) = the selected red ball is from the first box.
P(R) = probability of selecting a red ball.

Bob selects a ball randomly, hence selecting a ball from either box1 or box2 is ½.
P(R/B1) = Red ball from box 1 =7/9
P(R/B2) = Red ball from box 2 =3/7
P(B1) = Probability to select a ball from box 1/2.
P(B2) = Probability to select a ball from box 1/2.

P(B1/R) = P(B1)*P(R/B1)/ P(R)
= (1/2*7/9)/(1/2*7/9 + 1/2*3/7) = (7/18)/(38/63) = 49/76

----- Noted in the notebook

Expected Value and Variance

Expected Values

The *expected value*, also called the *expectation* or *mean*, of the random variable X on the sample space S is equal to

$$E(X) = \sum_{s \in S} p(s)X(s).$$

The *deviation* of X at $s \in S$ is $X(s) - E(X)$, the difference between the value of X and the mean of X.

Example Expected value of a die let X be the number that comes up when a fair die is rolled. What is the expected value of X?

Solution

The random variable X takes the values 1, 2, 3, 4, 5, or 6, each with probability **1/6**.

It follows that

E(X) =

$$\frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 = \frac{21}{6} = \frac{7}{2}$$

Example A fair coin is flipped three times. Let S be the sample space of the eight possible outcomes, and let X be the random variable that assigns to an outcome the number of heads in this outcome. What is the expected value of X?

Solution

$$\begin{aligned} E(X) = \frac{1}{8}[&X(HHH) + X(HHT) + X(HTH) + X(THH) + X(TTH) \\ &+ X(THT) + X(HTT) + X(TTT)] \\ = \frac{1}{8}&(3 + 2 + 2 + 2 + 1 + 1 + 1 + 0) = \frac{12}{8} \\ = \frac{3}{2}&. \end{aligned}$$

Consequently, the **expected number of heads that come up when a fair coin is flipped three times is 3/2**.

Example

What is the expected value of the sum of the numbers that appear when a **pair of fair dice** is rolled?

Solution

Possible sums = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12. Their respective probabilities are:

$$p(X = 2) = p(X = 12) = 1/36,$$

$$p(X = 3) = p(X = 11) = 2/36 = 1/18,$$

$$p(X = 4) = p(X = 10) = 3/36 = 1/12,$$

$$p(X = 5) = p(X = 9) = 4/36 = 1/9,$$

$$p(X = 6) = p(X = 8) = 5/36,$$

$$p(X = 7) = 6/36 = 1/6.$$

Substituting these values in the formula, we have

$$E(X) = 2 \cdot \frac{1}{36} + 3 \cdot \frac{1}{18} + 4 \cdot \frac{1}{12} + 5 \cdot \frac{1}{9} + 6 \cdot \frac{5}{36} + 7 \cdot \frac{1}{6} + 8 \cdot \frac{5}{36} + 9 \cdot \frac{1}{9} + 10 \cdot \frac{1}{12} + 11 \cdot \frac{1}{18} + 12 \cdot \frac{1}{36} = 7.$$

Advanced Counting Techniques

Applications of Recurrence Relations

A **sequence** is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

Generating functions:-

Generating functions are used to represent sequences efficiently by coding the terms of a sequence as **coefficients of powers of a variable x** in a formal power series.

Generating functions can be used to **solve recurrence relations** by translating a recurrence relation for the terms of a sequence **into an equation involving a generating function**.

This equation can then be solved to find a closed form for the generating function. From this closed form, the coefficients of the power series for the generating function can be found, solving the original recurrence relation.

The *generating function for the sequence* $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k.$$

EXAMPLE 1 Extra Examples The generating functions for the sequences $\{a_k\}$ with $a_k = 3, a_k = k + 1$, and $a_k = 2^k$ are $\sum_{k=0}^{\infty} 3x^k$, $\sum_{k=0}^{\infty} (k+1)x^k$, and $\sum_{k=0}^{\infty} 2^k x^k$, respectively. ◀

EXAMPLE 3 Let m be a positive integer. Let $a_k = C(m, k)$, for $k = 0, 1, 2, \ldots, m$. What is the generating function for the sequence $a_0, a_1, \ldots, a_m$?

Solution: The generating function for this sequence is

$$G(x) = C(m, 0) + C(m, 1)x + C(m, 2)x^2 + \cdots + C(m, m)x^m.$$

The binomial theorem shows that $G(x) = (1 + x)^m$. ◀

EXAMPLE 4 The function $f(x) = 1/(1 - x)$ is the generating function of the sequence $1, 1, 1, 1, \ldots$, because

$$1/(1 - x) = 1 + x + x^2 + \cdots$$

for $|x| < 1$. ◀

EXAMPLE 5 The function $f(x) = 1/(1 - ax)$ is the generating function of the sequence $1, a, a^2, a^3, \ldots$, because

$$1/(1 - ax) = 1 + ax + a^2x^2 + \cdots$$

when $|ax| < 1$, or equivalently, for $|x| < 1/|a|$ for $a \neq 0$. ◀

We also will need some results on how to add and how to multiply two generating functions. Proofs of these results can be found in calculus texts.

Counting Problems and Generating Functions

TABLE 1 Useful Generating Functions.

$G(x)$	a_k
$(1 + x)^n = \sum_{k=0}^{n} C(n, k)x^k = 1 + C(n, 1)x + C(n, 2)x^2 + \cdots + x^n$	$C(n, k)$
$(1 + ax)^n = \sum_{k=0}^{n} C(n, k)a^k x^k = 1 + C(n, 1)ax + C(n, 2)a^2x^2 + \cdots + a^n x^n$	$C(n, k)a^k$
$(1 + x^r)^n = \sum_{k=0}^{n} C(n, k)x^{rk} = 1 + C(n, 1)x^r + C(n, 2)x^{2r} + \cdots + x^{rn}$	$C(n, k/r)$ if $r \mid k$; 0 otherwise
$\frac{1 - x^{n+1}}{1 - x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$	1 if $k \leq n$; 0 otherwise
$\frac{1}{1 - x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$	1
$\frac{1}{1 - ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$	a^k

$\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots$	1 if $r \mid k$; 0 otherwise
$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + \cdots$	$k+1$
$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k$ $= 1 + C(n,1)x + C(n+1,2)x^2 + \cdots$	$C(n+k-1,k) = C(n+k-1,n-1)$
$\frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)(-1)^k x^k$ $= 1 - C(n,1)x + C(n+1,2)x^2 - \cdots$	$(-1)^k C(n+k-1,k) = (-1)^k C(n+k-1,n-1)$
$\frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)a^k x^k$ $= 1 + C(n,1)ax + C(n+1,2)a^2x^2 + \cdots$	$C(n+k-1,k)a^k = C(n+k-1,n-1)a^k$
$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$1/k!$
$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$(-1)^{k+1}/k$

Using Generating Functions to Solve Recurrence Relations