1.1 N = negative test result     S = student is stressed

$P(N|S^c) = .9$     $P(N^c|S) = .8$     $P(S) = .6$

$$P(S^c|N) = ? = \frac{P(N|S^c)\cdot P(S^c)}{P(N)} = \frac{.9(.4)}{P(S^c)P(N|S^c) + P(S)P(N|S)}$$

$$= \frac{.36}{.4(.9) + .6(.2)} = \frac{.36}{.48} = \boxed{.75}$$

1.2 $E[(X_1 + \ldots + X_n)^2] = cE[X_1^2] + dE[X_1]^2$     $Var(x) = E[x^2] - (E[x])^2$

$$\Downarrow$$

$E[x^2] = Var(x) + (E[x])^2$

$E[(X_1 + \ldots + X_N)] = E[X_1]$

$W = X_1 + \ldots + X_n$

$$E[(W)^2] = Var(W) + (E[W])^2$$

$$= nVar(X_1) + (nE[X_1])^2$$

$$= n(E[X_1^2] - E[X_1]^2) + n^2E[X_1]^2$$

$$= nE[X_1^2] + (n^2 - n)E[X_1]^2$$

$$\boxed{c = n \quad d = n^2 - n}$$

2.1 ⓐ $F_X(x) = P(X \le x) = P(F^{-1}(U) \le x)$

$$= P(F(F^{-1}(U)) \le F(x)) = P(U \le F(x))$$

$$= F(x)$$

CDF of X

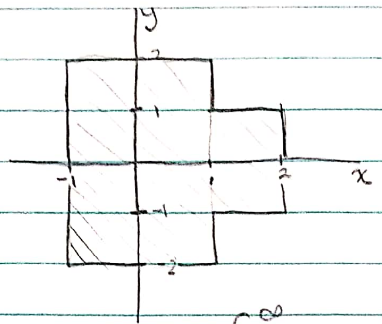
$$\boxed{F_X(x) = F(x)}$$

ⓑ CDF of exponential distribution: $F(x) = y = 1 - e^{-\lambda x}$

$$F^{-1}(y) = -\frac{1}{\lambda}\ln(1-y)$$

$$\boxed{X = F^{-1}(U) = -\frac{1}{\lambda}\ln(1-U)}$$

2.2 ⓐ



$f_{X,Y}(x,y) = \begin{cases} .1 & \text{in shaded area} \\ 0 & \text{otherwise} \end{cases}$

ⓑ $E[X|Y=y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, dx$

$f_{X|y}(x| -2 \le y \le -1 \text{ and } 1 \le y \le 2) = \begin{cases} \frac{1}{2} & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

$f_{X|Y}(x| -1 \le y \le -1) = \begin{cases} \frac{1}{3} & -1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$

$$\boxed{\begin{array}{l} E[x| -2 \le Y \le -1] = \int_{-1}^{1} x\left(\frac{1}{2}\right)dx = \frac{1}{4}x^2\Big|_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = \underline{0} \\ E[X| -1 \le y \le 1] = \int_{-1}^{2} \frac{1}{3}x\, dx = \underline{.5} \\ E[X | 1 \le Y \le 2] = \underline{0} \\ \\ Var(X| -2 \le Y \le -1) = E(X^2|Y) - (E(X|Y))^2 = .333 - 0 = \underline{\frac{1}{3}} \\ var(X| -1 \le Y \le 1) = \int_{-2}^{2} \frac{1}{3}x^2 dx - .5^2 = \underline{.75} \\ var(X| 1 \le y \le 2) = \underline{\frac{1}{3}} \end{array}}$$

ⓒ $E[X] = P(-2 \le Y \le -1)\, E(X| -2 \le Y \le -1) + P(-1 \le Y \le 1)\, E[X| -1 \le Y \le 1] + P(1 \le Y \le 2)\, E[X| 1 \le Y \le 2]$

$= 0 + \frac{3}{5}(.5) + 0 = \boxed{\frac{3}{10}}$

ⓓ $Var(X) = var(E[X|Y]) + E[var(X|Y)]$

$= \dfrac{(0-\frac{3}{10})^2 + (.5-\frac{3}{10})^2 + (0-\frac{3}{10})^2}{3} + \left(\frac{1}{5}\right)\frac{1}{3} + \frac{3}{5}(.75) + \frac{1}{5}\left(\frac{1}{3}\right)$

$\boxed{= .65667}$

3.1 $X = N(\mu, \Sigma)$

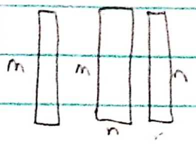
$$\boxed{f_X(x) = \frac{e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)}}{\sqrt{(2\pi)^n}\ |\Sigma|}}$$

3.2 $Y = 5X$

mean of $Y$: $\mu_y = 5\mu_x$

Covariance of $Y$: $\Sigma_y = 5 \cdot \Sigma_x \cdot 5 = 25\,\Sigma_x$

3.3 $Z = AX$



$\mu_z = A\mu_x$

$\text{cov}(z) = A\,\Sigma_x A^T$

3.4 $R = X + N$

$$\boxed{f_R(r) = \frac{e^{-\frac{1}{2}(x-(\mu_x+\mu))\,(\Sigma_1+\Sigma_x)^{-1}(x-\mu_x-\mu_1)}}{\sqrt{(2\pi)^n\ |\Sigma_1+\Sigma_x|}}}$$

$\mu_R = \mu_x + \mu_1 \qquad \Sigma_R = \Sigma_1 + \Sigma_x$

4.1 $\Sigma(y_i - \beta x^i)^2 = 0$

$$\frac{\partial\beta}{\partial\beta} = 2\Sigma(y_i - \beta x_i)x_i = 0$$

$$\Downarrow$$

$\Sigma(y_i - \beta x_i)x_i = 0$

$\Sigma\, x_i y_i - x_i^2 \beta = 0$

$\beta\,\Sigma x_i^2 = \Sigma x_i y_i$

$$\boxed{\beta = \frac{\sum_i x_i y_i}{\sum_i x_i^2}}$$

4.2 $\min_\beta \|y - X\beta\|^2 \Rightarrow y = X\beta$

if $n \geq d$:

$X^T y = X^T X \beta$

$$\boxed{\beta = (X^T X)^{-1} X^T y}$$

if $n < d$:

$\beta^{-1} y = X$

$\beta^{-1} y X^T = X X^T$

$\beta^{-1} = X^{T^{-1}}(XX^T)\,y^{-1}$

$$\boxed{\beta = X^T (XX^T)^{-1} y}$$

5

$$\begin{array}{rl} & \qquad n_1 \qquad\quad \overset{X}{\downarrow} \\ A = \begin{array}{r} \\ \\ \\ n_2 \\ \\ a_i \rightarrow \end{array} & \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & & \vdots \\ \vdots & & \vdots \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \end{array} \qquad \begin{array}{rl} & \qquad n_3 \qquad \overset{Y}{\downarrow} \\ B = \begin{array}{r} \\ \\ n_2 \\ \\ b_i \Rightarrow \end{array} & \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} \end{array}$$

Let colums of $A = X$
" " of $B = Y$

$A'B : n_1 \times n_3$

$$\sum_1^{n_2} a_i b_i' = \sum_1^{n_2} \begin{bmatrix} a_{i1} \\ \vdots \\ a_{in_1} \end{bmatrix} \begin{bmatrix} b_{i1} & \cdots & b_{in_3} \end{bmatrix} = \sum_1^{n_2} \begin{bmatrix} a_{i1} \cdot b_{i1}, & \cdots & a_{i1} \cdot b_{in_3} \\ \vdots & & \vdots \\ a_{in_2} \cdot b_{in_3} & & a_{in_1} \cdot b_{in_3} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_1^{n_2} a_{i1} \cdot b_{i1} & \cdots & \sum_1^{n_2} a_{i1} \cdot b_{in_3} \\ \vdots & & \vdots \\ \sum_1^{n_2} a_{in_2} b_{in_3} & \cdots & \sum_1^{n_2} a_{in_1} b_{in_3} \end{bmatrix} = \begin{bmatrix} X_1 Y_1 & \cdots & X_1 Y_{n_3} \\ \vdots & & \vdots \\ X_{n_1} Y_1 & \cdots & X_{n_1} Y_{n_3} \end{bmatrix}$$

$$A'B = \begin{bmatrix} X_1 Y_1 & \cdots & X_1 Y_{n_3} \\ \vdots & & \\ X_{n_1} Y_1 & \cdots & X_{n_1} Y_{n_3} \end{bmatrix} = \sum_1^{n_2} a_i b_i'$$

6 $v_1' \cdot v_2 = 0$

$Av_1 = \lambda_1 v_1 \qquad Av_2 = \lambda_2 v_2$

$$\begin{aligned} \lambda_1 (v_1 \cdot v_2) &= (\lambda v_1) \cdot v_2 \\ &= Av_1 \cdot v_2 \\ &= (Av_1)^T v_2 \\ &= v_1^T A^T v_2 \\ &= v_1^T A v_2 \\ &= v_1^T \lambda_2 v_2 \\ &= \lambda_2 (v_1 \cdot v_2) \end{aligned}$$

so,

$\lambda_1 (v_1 \cdot v_2) = \lambda_2 (v_1 \cdot v_2)$

$(\lambda_1 - \lambda_2)(v_1 \cdot v_2) = 0$

but $\lambda_1 \neq \lambda_2$ so $(v_1' v_2) = 0$

and they are orthoganal