PROBABILITY ASSIGNMENT 3

1. $p(Y=1) = \sum_{x} p(x,y)$

$= \frac{1}{9} + \frac{1}{3} + \frac{1}{9} = \frac{5}{9}$

$p(X \mid Y=1) \Rightarrow$

$p(X=1 \mid Y=1) = \frac{p(1,1)}{p(Y=1)} = \frac{1/9}{5/9} = \frac{1}{5}$

$p(X=2 \mid Y=1) = \frac{1/3}{5/9} = \frac{3}{5}$

$p(X=3 \mid Y=1) = \frac{1/9}{5/9} = \frac{1}{5}$

$E[X \mid Y=1] = \sum_{\forall x} x \cdot p(x \mid Y=1)$

$= \frac{1}{5} + \frac{6}{5} + \frac{3}{5} = 2$

IIly, $p(Y=2) = \frac{1}{9} + 0 + \frac{1}{18}$

$= \frac{1}{6}$

$E[X \mid Y=2] = 1 \times \frac{2}{3} + 2 \times 0 + 3 \times \frac{1}{3}$

$= 5/3$

$p(Y=3) = 0 + \frac{1}{6} + \frac{1}{9} = \frac{5}{18}$

$E[X \mid Y=3] = 1 \times 0 + 2 \times \frac{3}{5} + 3 \times \frac{2}{5} = \frac{12}{5}$

2. $P(1) = p$, $P(0) = 1-p$

(a) $P(R_1 = 1) = p^{k}(1-p)$ $\qquad$ $K_1 =$ length of run 1

$P(R_1 = 0) = (1-p)^{k}\, p$

$E[K_1 \mid R_1 = 1] = \frac{1}{1-p}$

$E[K_1 \mid R_1 = 0] = \frac{1}{p}$

$E[K]$ = P with which run is 11... (expected length)
+ (P with which run is 00 ...)(expected length)

$= p \times \frac{1}{1-p} + (1-p) \times \frac{1}{p}$

$= \frac{p}{1-p} + \frac{1-p}{p}$

$$\boxed{E[K] = \frac{p^2 + (1-p)^2}{p(1-p)}}$$

(b) For 2^{nd} run $\rightarrow$ if run 1 was 111...,
failure will be p
else if run was 00...,
failure will be $1-p$

$\therefore E[K_2 | R_2 = 1] = \frac{1}{p}$, $E[K_2 | R_2 = 0] = \frac{1}{1-p}$

$\therefore E[K_2] = p \times \frac{1}{p} + (1-p) \times \frac{1}{1-p}$

$\boxed{E[K_2] = 2}$

3. $P(X=i, Y=j) = {}^{j}C_{i}\, e^{-2\lambda} \dfrac{\lambda^{j}}{j!} \qquad 0 \leq i \leq j$

(a) $P(Y=j) = \displaystyle\sum_{i=0}^{j} {}^{j}C_{i}\, \underbrace{e^{-2\lambda} \dfrac{\lambda^{j}}{j!}}_{\text{const}}$

$$\boxed{P(Y=j) = \dfrac{e^{-2\lambda} \lambda^{j}}{j!} \times 2^{j}}$$

(b) $p(X=i) = \sum_{\forall j} {}^{j}C_i \, \dfrac{e^{-2\lambda}\lambda^j}{j!}$

$= e^{-2\lambda} \sum_{j=i}^{\infty} \dfrac{j!}{(j-i)!\,i!} \, \dfrac{\lambda^j}{j!}$

Let $j-i=k$ $\therefore j=k+i$

If $j=i$, $k=0$

$\therefore p(X=i) = \dfrac{e^{-2\lambda}}{i!} \sum_{k=0}^{\infty} \dfrac{\lambda^{i+k}}{k!\,i!}$

$= \dfrac{e^{-2\lambda}\lambda^i}{i!} \underbrace{\sum_{k=0}^{\infty} \dfrac{\lambda^k}{k!}}_{e^{\lambda}}$

$\therefore p(X=i) = \dfrac{e^{-\lambda}\lambda^i}{i!}$

(c) $p(Y-X)$

Take $Y-X=k$ $\quad Y=k+X = i+k$

$\therefore p(X=i, Y=i+k) = {}^{i+k}C_i \, \dfrac{e^{-2\lambda}\lambda^{i+k}}{(i+k)!}$

$= \dfrac{\cancel{(i+k)!}}{i!\,k!} \, \dfrac{e^{-2\lambda}\lambda^i\lambda^k}{\cancel{(i+k)!}}$

$= \dfrac{e^{-\lambda}\lambda^i}{i!} \times \dfrac{e^{-\lambda}\lambda^k}{k!}$

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$\Rightarrow p(X-X) = p(X=i) \cdot p(Y-X=k) \quad (5)$

4. $f_X(x) = \begin{cases} 1 & x \in (0,1) \\ 0 & o/w \end{cases}$

$$E[X^n] = \int_0^1 x^n f_X(x)\, dx$$

$$= \int_0^1 x^n\, dx$$

$$= \left[\frac{x^{n+1}}{n+1}\right]_0^1 = \boxed{\frac{1}{n+1}}$$

$$Var[X^n] = E[(X^n)^2] - (E[X^n])^2$$

$$= \int_0^1 x^{2n}\, dx - \frac{1}{(n+1)^2}$$

$$= \left[\frac{x^{2n+1}}{2n+1}\right]_0^1 - \frac{1}{(n+1)^2}$$

$$= \frac{1}{2n+1} - \frac{1}{(n+1)^2}$$

$$= \frac{n^2+2n+1-2n-1}{(2n+1)(n+1)^2}$$

$$\boxed{Var[X^n] = \frac{n^2}{(2n+1)(n+1)^2}}$$

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5. $P(X=K) = {}^nC_K \, p^K (1-p)^{n-K}$

If $P(X=K)$ increases, $P(X=K+1) > P(X=K)$

$\therefore \; {}^nC_{K+1} \, p^{K+1} (1-p)^{n-K-1} > {}^nC_K \, p^K (1-p)^{n-K}$

$$\frac{p \cdot \cancel{n!} \, \cancel{p^K} \, \cancel{(1-p)^{n-K}}}{(K+1)! \, \cancel{(n-K-1)!} \, (1-p)} > \frac{\cancel{n!} \, \cancel{p^K} \, \cancel{(1-p)^{n-K}}}{\cancel{K!} \, (n-K)!}$$

$$\frac{p}{(K+1)(1-p)} > \frac{1}{n-K}$$

$$\boxed{\frac{p(n-K)}{(K+1)(1-p)} > 1}$$

$P(X=K)$ decreases if $\dfrac{p(n-K)}{(K+1)(1-p)} < 1$

$pn - \cancel{pK} > K + 1 - \cancel{pK} - p$

$pn + p > K + 1$

$\underline{p(n+1) > K+1}$

$K < p(n+1) - 1 \longrightarrow$ increases

$K > p(n+1) - 1 \longrightarrow$ decreases

max at $= 1$

(a) $P(X = K)$ increases while $K < p(n+1) - 1$, maxes out at $K = p(n+1) - 1$ then decreases for $K > p(n+1) - 1$

(b) If $(n+1)p$ is not an integer, K is max for the integral part of $(n+1)p - 1$.

6. T.P $\rightarrow P(X = n+k \mid X > n) = P(X = k)$

$$= \frac{P(X = n+k \wedge X > n)}{P(X > n)}$$

~~P(~~ $X > n$ is a subset of $X = n+k$,

$\therefore P(X = n+k \wedge X > n) = P(X = n+k)$

$$\therefore \frac{P(X = n+k)}{P(X > n)} = P(X = k)$$

$P(X > n)$ → implies that all trials before $(n+1)^{th}$ are failures.

$$\frac{(1-p)^{n+k-1} \cdot p}{(1-p)^n}$$

$= (1-p)^{k-1} \cdot p = P(X = k)$ Hence Proved.

It is called lack of memory property because probability of the trials that have already been done does not affect the probability of rest of the trials.

7. $I_K = \begin{cases} 1 & \text{if } K^{th} \text{ ~~envelope~~ letter is in correct envelope} \\ 0 & \text{o/w} \end{cases}$

$\therefore$ Total correct envelopes $= \sum_{K=1}^{n} I_K = X$

$$E[X] = \sum_{i=1}^{n} E[I_i] = \sum_{i=1}^{n} 1/n$$

$$E[I_i] = 1/n$$

$$\therefore E[X] = \sum_{i=1}^{n} 1/n = n \times \frac{1}{n} = 1$$

$$Var[X] = E[X^2] - (E[X])^2$$

$\because$ These are dependent events.

$$Var\left[\sum_{i=1}^{n} I_i\right] = \sum_{i=1}^{n} Var[I_i] + 2 \sum_{1 \le i < j \le n} Cov[I_i I_j]$$

$$E[I_i I_j] - E[I_i]E[I_j]$$

$$Var[I_i] = E[(I_i)^2] - (E[I_i])^2$$

$$= \frac{1}{n} - \frac{1}{n^2}$$

$$Var[I_i] = \frac{n-1}{n^2}$$

$$Cov[I_i I_j] = \frac{1}{n} \times \frac{1}{n-1} - \frac{1}{n} \times \frac{1}{n}$$

$$= \frac{1}{n^2-n} - \frac{1}{n^2} = \frac{n^2 - n^2 + n}{n^3(n-1)}$$

$$= \frac{1}{n^2(n-1)}.$$

$$\therefore Var[x] = \underbrace{\sum_{i=1}^{n} \frac{n-1}{n^2}}_{n \text{ terms}} + \underbrace{2 \sum_{1 \le i < j \le n} \frac{1}{n^2(n-1)}}_{^nC_2 \text{ terms}}$$

$$Var[x] = \frac{n-1}{n} + \frac{\cancel{2}\, \cancel{n}\cancel{(n-1)}}{\cancel{2}} \times \frac{1}{n^{\cancel{2}}\cancel{(n-1)}}$$

$$= \frac{n-1}{n} + \frac{1}{n}$$

$$Var[x] = \frac{n}{n} = \underline{\underline{1}}$$

$\therefore$ x has mean and variance 1.

As $n \to \infty$, probability of success becomes smaller and the events become almost independent. This makes the distribution Poisson with the parameter 1 which is the expected value of x for all n.

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8. $P(\text{bit error}) = 10^{-6}$ $\underline{10{,}000 \text{ bits}}$

For 1 block, $p = 10^{-6}$ & $n = 10^{4}$

$$np = 10^{-2}$$

(a) $P(N=0) = \dfrac{e^{-\lambda}\,\lambda^{0}}{0!}$, $\lambda = np = 10^{-2}$

$$= \underline{\underline{e^{-\frac{1}{100}}}}$$

$$P(N \le 3) = \sum_{K=0}^{3} \lambda^{K}\,\frac{e^{-\lambda}}{K!}, \quad \lambda = 10^{-2}$$

$$= e^{-1/100} + \frac{e^{-1/100}}{100} + \frac{e^{-1/100}}{20000} + \frac{e^{-1/100}}{6000000}$$

(b) $1 - P(N=0) = 0.99$ $\lambda \neq 10^{-2}$ here.

$1 - e^{-\lambda} = 0.99$

$$e^{-\lambda} = 0.01.$$

$$\lambda = \ln 100$$

$$\lambda = 4.6 = np \qquad n = 10^{4}$$

$\therefore$ $\underline{p = 4.6 \times 10^{-4}}$

9. Total possible patterns $= 2^m$.

$p(\text{Success}) = \frac{1}{2^m}$

(a) $P(X=n \mid X>K) = \frac{P(X=n \cap X>K)}{P(X>K)}$

As in Q6, $P(X=n \cap X>K) = P(X=n)$ $\forall n>K$.

$$\therefore \frac{P(X=n)}{P(X>K)} = \frac{(1-p)^{n-1} p}{(1-p)^{K}}$$

$$= \underline{(1-p)^{n-K-1} \cdot p}.$$

(b) $E[X \mid X>K] = ?$

As $n > K$, we can say

$$E[X \mid X>K] = E[X] = \frac{1}{p} = 2^m.$$

2^m is the expectde no of trials after K trials have already been done

$\therefore$ Total expected trials $= 2^m + K$.

10. $X = r\cos\left(2\pi\theta/8\right)$ $\quad\theta \in \{1,\dots,7\}$

$Y = r\sin\left(2\pi\theta/8\right)$

$\theta=0$	$(r, 0)$
1	$(r/\sqrt{2}, r/\sqrt{2})$
2	$(0, r)$
3	$(-r/\sqrt{2}, r/\sqrt{2})$
4	$(-r, 0)$
$\theta=5$	$(-r/\sqrt{2}, -r/\sqrt{2})$
6	$(0, -r)$
7	$(r/\sqrt{2}, -r/\sqrt{2})$

(b) For Joint Probability →

X \ Y	r	$r/\sqrt{2}$	0	$-r$	$-r/\sqrt{2}$
r	0	0	1/8	0	0
$r/\sqrt{2}$	0	1/8	0	0	1/8
0	1/8	0	0	1/8	0
$-r$	0	0	1/8	0	0
$-r/\sqrt{2}$	0	1/8	0	0	1/8

11. (a) $\cancel{P(X=i)}$ $P_X(x) = \sum_{\forall j} P(X=i, Y=j)$

$P_X(r) = 1/8$ $\quad$ $P_X(-r) = 1/8$

$P_X(r/\sqrt{2}) = 1/4$ $\quad$ $P_X(-r/\sqrt{2}) = 1/4$.

$P_X(0) = 1/4$

$$P_Y(y) = \sum_{\forall i} P_{XY}(X=i, Y=j)$$

$P_Y(r) = 1/8$ $\quad$ $P_Y(-r) = 1/8$

$P_Y(r/\sqrt{2}) = 1/4$ $\quad$ $P_Y(-r/\sqrt{2}) = 1/4$.

$P_Y(0) = 1/4$

12. $P(X=0) = 0.5$

$\therefore P(X=1) = 0.5$

$P(Y=1 \mid X=0) = 0.1$

$\therefore P(Y=0 \mid X=0) = 0.9$

$P(Y=0 \mid X=1) = 0.2$

$\therefore P(Y=1 \mid X=1) = 0.8$

$$P(X=0, Y=0) = P(Y=0 \mid X=0) \times P(X=0)$$
$$= 0.9 \times 0.5$$
$$= \underline{0.45}$$

$$P(X=1, Y=0) = P(Y=0 \mid X=1) \times P(X=1)$$
$$= \underline{0.1}$$

$$P(X=0, Y=1) = P(Y=1 \mid X=0) \times P(X=0)$$
$$= \underline{0.05}$$

$$P(X=1, Y=1) = P(Y=1 \mid X=1) \times P(X=1)$$
$$= \underline{0.4}$$

(b) $P(X=1) = 0.1 + 0.4 = \underline{0.5}$

$P(X=0) = \underline{0.5}$

$P(Y=1) = \underline{0.45}$

$P(Y=0) = \underline{0.55}$

(c) For Independence, $P(X=i, Y=j) = P(X=i) \times P(Y=j)$
$\forall i \,\&\, j$.

$P(1, 0) = \underline{0.1}$

$P(X=1) \cdot P(Y=0) = 0.5 \times 0.55$

$= \underline{0.275}$ $\therefore$ not independent

13. $P(X=i) = e^{-\lambda_1} \dfrac{\lambda_1^i}{i!}$ $\qquad$ $P(Y=j) = e^{-\lambda_2} \dfrac{\lambda_2^j}{j!}$

X & Y are independent

$Z = X + Y$

$\therefore \; P(Z=n) = P(X+Y=n)$

$$p(Z=n) = \sum_{i=0}^{n} P(X=i)\, P(Y=n-i)$$

$$= \sum \frac{e^{-\lambda_1} \lambda_1^i}{i!} \cdot \frac{e^{-\lambda_2} \lambda_2^{n-i}}{(n-i)!}$$

$$\boxed{p(Z=n) = e^{-(\lambda_1+\lambda_2)} \sum \frac{\lambda_1^i \, \lambda_2^{n-i}}{i!\,(n-i)!}}$$

14. $P(\text{escape}) = 1/3$

$t = 2$ Hrs.

Assume R.V. N for total time spent in the maze.

Avg. time of escape = $E[N]$

If the wrong door is used, we can say that we start the 2 hour trip again in a loop.

$$E[N] = \frac{1}{3} \times 2 + \frac{2}{3} \times (2 + E[N])$$

$$E[N] = \frac{2}{3} + \frac{4}{3} + \frac{2E[N]}{3}$$

$$E[N] = 2 + \frac{2E[N]}{3}$$

$$E[N] = 6$$

$\therefore$ Average time spent in the maze = 6 hrs.

The answer can be guessed ahead of time.

15. Grades - $\{A, A-, B+, B, B-, C+\}$

$P(\text{unique grade at } \#1) = 1.$

$P(\text{unique grade at } \#2) = \frac{5}{6}$

$\#3 = \frac{4}{6}$

$\#4 = 3/6$

$\#5 = 2/6$

$\#6 = 1/6$

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$P(G_1) = \begin{cases} \text{Yes} & p = 1 \\ \text{No} & p = 0 \end{cases}$ $\quad E[G_1] = 1$ paper

$P(G_2) = \begin{cases} \text{Yes} & p = 5/6 \\ \text{No} & p = 1/6 \end{cases}$ $\quad$ Geometric Distribution

$\therefore E[G_2] = \frac{1}{5/6} = \frac{6}{5}$ ~~trials~~ papers.

||ly, $E[G_3] = \frac{6}{4}$ papers $\quad E[G_5] = \frac{6}{2}$ papers

$E[G_4] = \frac{6}{3}$ papers $\quad E[G_6] = 6$ papers

$\Rightarrow$ Total expected no. of papers $= 6 + \frac{6}{2} + \frac{6}{3} + \frac{6}{4} + \frac{6}{5} + 1$

$= 6 + 3 + 2 + 1.5 + 1.2 + 1$

$= \underline{\underline{14.7 \text{ papers}}}$

16. $P_{M,N}(m,n) = \begin{cases} C & m \geq 0,\ n \geq 0\ ;\ m+n < L. \\ 0 & \text{o/w} \end{cases}$

$$\sum_m \sum_n P_{M,N}(m,n) = 1.$$

$$m+n < L$$

$$\underline{n < L-m}$$

max. of m (when n=0) = L-1

As m changes, max. of n changes.

$$\therefore \sum_{m=0}^{L-1} \sum_{n=0}^{L-1-m} C = 1$$

$$C \cdot \sum_{m=0}^{L-1} \left(\underbrace{\sum_{n=0}^{L-1-m} 1}_{\text{no. of terms} = L-m} \right) = 1$$

$$C \cdot \sum_{m=0}^{L-1} (L-m) = 1$$

$$C \cdot \left[L + (L-1) + (L-2) + \dots + 1 \right] = 1$$

$$C \cdot \left[\frac{L(L+1)}{2} \right] = 1$$

(a) $$\boxed{C = \frac{2}{L(L+1)}}$$

(b) $$P_M(m) = \sum_{n=0}^{L-1-m} \frac{2}{L(L+1)}$$

$$\boxed{P_M(m) = \frac{2(L-m)}{L(L+1)}}$$

$$P_N(n) = \sum_{m=0}^{L-1-n} \frac{2}{L(L+1)}$$

$$P_N(n) = \frac{2(L-n)}{L(L+1)}$$

(c) $$\cancel{C = \frac{2}{L/2(L/2+1)}} = \cancel{\frac{4}{L(L+2)}}$$

(c) $$\sum_m \sum_n P_{M,N}(m,n) = 1$$

$m+n < L/2$

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$$\therefore \quad c \times \frac{\frac{L}{2}\left(\frac{L}{2}-1\right)}{2} = \cancel{2}\, Pr\left(m+n<\frac{L}{2}\right)$$

$$\frac{\cancel{2}}{\cancel{L}(L-1)} \times \frac{\cancel{L}(L-2)}{\cancel{8}\,4}$$

$$Pr\left(m+n<\frac{L}{2}\right) = \frac{L-2}{4(L-1)}$$

17. For K transmissions,

$$P(N=K) = e^{-\lambda} \frac{\lambda^K}{K!}$$

no. of 1's $= i$

$$P(i \mid N=K) = {}^K C_i \, p^i (1-p)^{K-i}$$

MARGINAL Pr.

$$P(i) = \sum_{K=i}^{\infty} P(i \mid N=K) \times P(N=K)$$

$$= \sum_{K=i}^{\infty} {}^K C_i \, p^i (1-p)^{K-i} \cdot e^{-\lambda} \frac{\lambda^K}{K!}$$

$$= \frac{e^{-\lambda} p^i \lambda^i}{i!} \sum \frac{\cancel{K!}}{(K-i)!} \frac{[(1-p)\lambda]^{K-i}}{\cancel{K!}}$$

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$$\sum_{k=i}^{\infty} \frac{[(1-p)\lambda]^{k-i}}{(k-i)!} = e^{(1-p)\lambda}$$

$$\therefore \quad \frac{e^{-\lambda} p^i \lambda^i}{i!} \times e^{\lambda - \lambda p}$$

$$P(i) = \frac{e^{-\lambda p} (\lambda p)^i}{i!}$$

$\Rightarrow$ No. of 1s transmitted in that time internal has a Poisson PMF with parameter $p\lambda$.

18. Let total number of trials be n.

$$x+y=n$$
$$z=x-y$$
$$\therefore \underline{z=2x-n}$$

$$z=\begin{cases} -n & 0 \text{ success} \\ -n+2 & 1 \quad " \\ -n+4 & 2 \quad " \\ \vdots & \\ n & 0 \text{ failure} \end{cases}$$

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$$P(x=k) = {}^nC_k\, p^k (1-p)^{n-k}$$

$$z = 2k+n$$

$$\frac{z-n}{2} = k$$

$$\therefore P(x=k) = {}^nC_{\frac{z-n}{2}}\, p^{\frac{z-n}{2}} (1-p)^{\frac{3n-z}{2}}$$

$$E[z] = E[2x-n]$$

$$= 2E[x] - n$$

$$= 2np - n$$

$$\underline{E[z] = n(2p-1)}$$

$$Var[z] = E[z^2] - (E[z])^2$$

$$= E[4x^2 - 4xn + n^2] - n^2(2p-1)^2$$

$$= 4E[x^2] - 4n^2(2p-1) + n^2 - n^2(2p-1)^2$$

$$Var[z] = Var[2x-n]$$

$$= 4\,Var(x)$$

$$= 4np(1-p)$$

19. (a) $P(11) = \dfrac{e^{-10}\lambda^{11}}{11!}$

$\lambda = 10$, $\therefore$ $P(11) = \dfrac{e^{-10} \times 10^{11}}{11!}$.

(b) not more than 5 cars $\therefore$ ≤ 5 cars.

$\therefore$ $\underline{P(X \leq 5)} = \sum_{K=0}^{5} \dfrac{e^{-\frac{30}{N}} \times (30/N)^{K}}{K!}$

solving this, we can find N.

20. $p_{XY}(x_i, y_j) = \begin{cases} K(2x_i + y_j), & x_i = 1, 2;\ y_j = 1, 2 \\ 0 & \text{otherwise.} \end{cases}$

$p(1,1) = 3K$ $\qquad$ $p(2,1) = 5K$

$p(1,2) = 4K$ $\qquad$ $p(2,2) = 6K$

$18K = 1$

(a) $\underline{K = 1/18}$.

$p(X=i) = \sum_{\forall j} p_{X,Y}(i, j)$

(b) $p(X=1) = 7/18$, $p(X=2) = 11/18$.

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llly, $p(Y=1) = 8/18$, $p(y=2) = 10/18$.