Assignment-3

7) There are n envelops and n letters

$E[I_K] = 1 \cdot \frac{1}{n} + 0 = \frac{1}{n}$ $\quad I_K = \begin{cases} 1 \leftarrow 1/n \\ 0 \leftarrow 1-\frac{1}{n} \end{cases}$

$Pr(I_K = 1) = \frac{1}{n}$ $\quad I_K = \begin{cases} 1 & \text{rot } K \\ 0 & \text{otherwise} \end{cases}$

X = # of correctly matched envelops. $X = f(I_K)$ K from 1 to nth

$X = f_n(I_K)$ K from 1 to n. $\quad E[X] = \sum_{K=1}^{n} E[I_K] = \sum_{K=1}^{n} 1/n = 1$

$= \sum_{K=1}^{n} I_K$

$Var(X) = \underline{E[X^2] - E(X)^2} \longrightarrow$ moments formulae

$= E\left[\left(\sum_{1}^{n} I_K\right)^2\right] - 1^2$

$= E\left[\sum_{K=1}^{n} I_K^2 + 2\left({}^nC_2\right)\left(\sum_{i \neq j}\sum I_K I_j\right)\right] - 1$

$= \sum E(I_K^2) + 2\left({}^nC_2\right)\sum\sum E(I_K I_j) - 1$

$= \sum_{1}^{n} 1 + 2\underbrace{\left({}^nC_2\right)}_{n(n-1)} \cancel{\phi}1 = \cancel{n-1+2}\, \frac{n(n-1)}{} = n$

$= \cancel{n^2 - 2 + 1}$

$E(I_K I_j)\ (K \neq j)$

$Pr(I_K = 1 \,\&\, I_j = 1) = Pr(I_j = 1 \mid I_K = 1) \cdot Pr(I_K = 1)$

$\cancel{E(X^2)}$

$= \frac{1}{n-1} \cdot \frac{1}{n}$

$E[I_K \cdot I_j] = \sum_K \sum_j I_K I_j \cdot \frac{1}{n(n-1)} =$

As $n \to \infty$ $P(\text{success}) = Pr(I_K = 1) \longrightarrow 0$

$Pr(X = r) = \frac{e^{-\lambda}\lambda^r}{r!}$

18. Sucess is a student recieiving a grade that has not been recived before.

Let X_i be the no of papers between i^{th} & $(i+1)^{th}$ sucess.

$$X = f(x_i's) = \sum_{i=1}^{5} X_i + 1 \qquad X_i = Y_{i+1} - Y_i$$

X_i is the random variable.

$$Y_K = \sum_{i=0}^{K-1} X_i \Rightarrow E[Y_K] = E\left[\sum_{i=0}^{K-1} X_i\right] = E[X_0] + E(X_1) + \text{---} + E(X_{K-1})$$

Here K=6

$$= 6\left[\frac{1}{6} + \frac{1}{5} + \frac{1}{4} + \frac{1}{3} + \frac{1}{2} + 1\right] = 14.7$$

1. $E[X|_{Y=i}] = \sum x \, P_{X|Y}(x/y)$

$$= 1\times\frac{1}{9} + 2\times\frac{1}{3} + 3\times\frac{1}{9} + 1\times\frac{1}{9} + 2\times 0 + 3\times\frac{1}{18} \times 1\times 0 + 2\times\frac{1}{6} + 3\times\frac{1}{9}$$

$$= 37/18.$$

$E[X|Y=i]$ $i=1,2,3$ $= 37/18.$

2. Given probability of element being '1' = p

probability of element being '0' = 1-p

(a) Expected value of the run start with 1 & ends with 0

$$E(X_1) = \frac{1}{1-(1-p)} = \frac{1}{p}.$$

It starts with 0 & ends with 1

$$E(X) = \frac{1}{1-p}$$

⇒ If we combine

$$E(X) = p \cdot 1/p + (1-p)(1/1-p) = 2$$

(b) If 1 run ends with 0 = 1-p

1 runs ends with 1 = p.

$$E(X_2) = p\left(\frac{1}{1-p}\right) + (1-p)\frac{1}{p} = \frac{p^2 + (1-p)^2}{(1-p)p} = \frac{2p^2 - 2p + 1}{p - p^2}$$

(8) Given $n = 10{,}000$ & $p = 10^{-6}$

poissions distribution

$\lambda = np = 10^4 \times 10^{-6} = 0.01$

$P(N=k) = \frac{\lambda^k e^{-\lambda}}{k!}$

(a) $P(N=0) = e^{-0.01}$

$P(N \le 3) = P(N=0) + P(N=1) + P(N=2) + P(N=3)$

$= e^{-0.01} + \lambda e^{-0.01} + \frac{\lambda^2}{2} e^{-0.01} + \frac{\lambda^3}{3} e^{-0.01}$

$P(N \le 3) = e^{-0.01}\left(1 + 0.01 + \frac{(0.01)^2}{2} + \frac{(0.01)^3}{3}\right).$

(b) $\lambda = np$

$P(N \ge 1) = 1 - P(N=0) = 0.99$

$P(N=0) = 0.01 = e^{-\lambda}$

$-\lambda = \ln(0.01) \Rightarrow \lambda = 4.605 \Rightarrow np = 4.605$

$\boxed{p = 4.605 \times 10^{-4}}$

(9) No. of possible passwords is 2^m.

probability of sucess in each trail is $p = \frac{1}{2^m}$.

(a) $p(x = k+j \mid x > k) \Rightarrow$ found on j^{th} trail

$= \frac{P(x = k+j \ \& \ x > k)}{P(x > k)} = \frac{(1-p)^{k+j-1} p}{(1-p)^k}$

$= (1-p)^{j-1} p$

9.

(b) $E[X|X>k]$ $E[X] = 1/p$ → ∵ Geometric distribution.

$E[X|X>k] = k + \frac{1}{p} = 2^m + k.$
↳ when correct password founded

(10) $X = r\cos\left(\frac{2\pi\theta}{8}\right) = r\cos\left(\frac{\pi\theta}{4}\right)$

$Y = r\sin\left(\frac{2\pi\theta}{8}\right) = r\sin\left(\frac{\pi\theta}{4}\right)$

(a)

	X	Y
$\theta = 0 \Rightarrow$	$X = r$	$Y = 0$
$\theta = 1$	$X = r/\sqrt{2}$	$r/\sqrt{2}$
$\theta = 2$	0	r
$\theta = 3$	$-r/\sqrt{2}$	$r/\sqrt{2}$
$\theta = 4$	$-r$	0
$\theta = 5$	$-r/\sqrt{2}$	$-r/\sqrt{2}$
$\theta = 6$	0	$-r$
$\theta = 7$	$r/\sqrt{2}$	$-r/\sqrt{2}$

(b) $P_{X,Y}(X=x, Y=y) = 1/8$

$$P(X,Y) = \begin{cases} 1/8 & \text{if } X,Y \in S_{x,y} \\ 0 & \text{otherwise.} \end{cases}$$

(11) marginal PMF of X

$P(X=r) = 1/8$, $P(X=r/\sqrt{2}) = 1/4$, $P(X=0) = 1/4$, $P(X=-r/\sqrt{2}) = 1/4$

marginal PMF of Y

$P(Y=r) = 1/8$, $P(Y=r/\sqrt{2}) = 1/4$, $P(Y=0) = 1/4$, $P(Y=-r/\sqrt{2}) = 1/4$

(b) $P(X=0) = 1/4$

$P(Y \le r/\sqrt{2}) = P(Y=r/\sqrt{2}) + P(Y=0) + P(Y=-r/\sqrt{2}) + P(Y=-r)$
$= 1/4 + 1/4 + 1/4 + 1/8 = 7/8.$

$P(X \ge r/\sqrt{2}, Y \ge r/\sqrt{2}) = P(X=r/\sqrt{2}, Y=r/\sqrt{2}) = 1/8.$

$P(X < r/\sqrt{2}) = P(X=-r) = 1/8.$

(12) $P(X=0) = 0.5$ $P(Y=1 | X=0) = 0.1$, $P(Y=0 | X=1) = 0.2$ 112

$P(Y=1 | X=0) = 1 - P(~~B~~ Y=1 | X=0) = 1 - 0.1 = 0.9$

$P(X=0 \,\&\, Y=0) = P(X=0) \cdot P(Y=1 | X=0) = 0.5 \times 0.9$
$= \underline{0.45}$

$P(X=1 \,\&\, Y=0) = P(X=1) \cdot P(Y=0 | X=1) = \underline{\underline{0.1}}$

$P(X=0 \,\&\, Y=1) = P(X=0) \cdot P(Y=1 | X=0) = 0.5 \times 0.1 = \underline{0.05}$

$P(Y=1 | X=1) = 1 - P(Y=0 | X=1) = 1 - 0.2 = 0.8$

$P(X=1, Y=1) = P(X=1) \cdot P(Y=1 | X=1) = 0.5 \times 0.8 = 0.4$

(a) joint PMF's

$$P_{X,Y}(x,y) = \begin{cases} 0.45 & x=0, y=0 \\ 0.10 & x=1, y=0 \\ 0.05 & x=0, y=1 \\ 0.4 & x=1, y=1. \end{cases}$$

(b) marginal PMF's.

$$P_X(x) = \begin{cases} 0.5 & x=1 \\ 0.5 & x=0 \end{cases} \qquad P_Y(y) = \begin{cases} 0.55 & y=0 \\ 0.45 & y=1. \end{cases}$$

(c) To prove independent of two events. $P(A) P(B) = P(A \cap B)$

$P(X=0) \cdot P(Y=0) = P(X=0 \cap Y=0) \Rightarrow 0.5 \times 0.55 \neq 0.45$

$\therefore$ Dependent

(13) $X = \frac{e^{-\lambda_1} \lambda_1^K}{K!}$ $Y = \frac{e^{-\lambda_2} \lambda_2^K}{K!}$

Let $\boxed{X+Y=Z}$

$P(Z) = \sum_{K=0}^{Z} P(X=K) P(Y=Z-K) = \sum_{K=0} \frac{e^{\lambda_1} \lambda_1^K}{K!} \frac{e^{\lambda_2} \lambda_2^{Z-K}}{(Z-K)!}$

(13)

$$p(z) = e^{\lambda_1 + \lambda_2} \sum_{k=0} \frac{\lambda_1^{k}}{k!} \frac{\lambda_2^{z-k}}{(z-k)!}$$

$$= \frac{e^{\lambda_1+\lambda_2} (\lambda_1+\lambda_2)^z}{z!}$$

(14) Let T be the time to escape

$$P_X[x] = \begin{cases} 1/3 & x=2 \\ (1/3)^2(2) & x=4. \\ 1/3(2)+1/3(2)1/3 & x=6. \\ 2^{n-1}(1/3)^n & \text{if } x=n \end{cases}$$

$$E[x] = \sum x P_X(x)$$

$$= \frac{2}{3} + 2\left[\frac{2}{3}\right]^2 + 3\left[\frac{2}{3}\right]^3 + \ldots$$

$$= 6$$

$\therefore$ 6 Hours is expected time for escaping.

(16) We know that $\sum_{m,n} P_{M,N}(m,n) = 1$

a) $\sum_{m=0}^{\alpha}\sum_{n=0}^{\alpha} c = 1 \Rightarrow (\alpha+1)^2 c = 1 \Rightarrow c = 1/(\alpha+1)^2.$

b) Marginal PMF

$$P_M(m) = \sum_{n=0} P_{M,N}(m,n) = \sum_{n=0}^{L} \frac{1}{(\alpha+1)^2} = \frac{\alpha+1}{(\alpha+1)^2} = \frac{1}{\alpha+1},$$

$$P_N(n) = \sum_{m=0}^{L} P_{M,N}(m,n) = \sum_{m=0} \frac{1}{(\alpha+1)^2} = \frac{\alpha+1}{(\alpha+1)^2} = \frac{1}{\alpha+1}.$$

c) $$P[M+N < 4/2] = \sum_{m=0}^{L} \sum_{n=0}^{\min(L, 4/2-m-1)} P_{M,N}(m,n) = \sum_{m=0}^{L} \sum_{n=0}^{\min(L, 4/2-m-1)} \frac{1}{(\alpha+1)^2}$$

$$= \frac{1}{(\alpha+1)^2} \times \sum_{m=0}^{L} \sum_{n=0}^{\min(L, 4/2-m-1)}$$

(17) The no. of transmissions [Z] within a given time interval X. a possion PMF with parameter λ.

$$P(X=K) = \frac{e^{-\lambda}\lambda^K}{K!}$$

Let Y be a R.v for no. of 1's

In K transmission there can be 1's and 0's

$$P[Y=i \mid X=K] = {}^K C_j \, p^j (1-p)^{1-j}$$

$$P(Y=1 \cap X=K) = {}^K C_j \, p^i (1-p)^{K-i} \frac{e^{-\lambda}\lambda^K}{K!}$$

$$P(Y=i) = \sum_{K=i}^{\infty} {}^K C_i \, e^{-\lambda} p^i (1-p)^{K-i} \frac{\lambda^K}{K!}$$

$$= \frac{e^{-\lambda} p^i \lambda^i}{i!} \left[\sum_{K=i}^{\infty} \frac{K!}{(K-i)!} \times \frac{\lambda^{K-i}}{K!} (1-p)^{K-i} \right]$$

$$= \frac{e^{-\lambda}(\lambda p)^K}{K!} e^{\lambda} e^{(1-p)\lambda} = \frac{(\lambda p)^K}{K!} e^{-p\lambda}$$

$\therefore$ It is possion's ratio with parameter $p\lambda$.

$\therefore$

(18) $X \to$ sucess, Y = failure, $n \to$ total trails

$X + Y = n$, $y = n - x$.

$$P(X=K) = {}^n C_K \, p^K (1-p)^{n-K}$$

$$Z = x - y = x - (n-x) = 2x - n \Rightarrow x = \frac{n+z}{2}$$

$$P(Z=z_0) = P\left(X = \frac{z_0+n}{2}\right) = \binom{n}{\frac{z_0+n}{2}} p^{\frac{n+z_0}{2}} (1-p)^{\frac{n+z_0}{2}}$$

$E(Z) = E[2X - n] = 2E[X] - n = 2np - n$

$= (2p-1)n.$

$Var(Z) = Var(2X - n)$

$= 4 Var(X) = 4np(1-p)$

(19) (a) Probability that exact 11 cars will approach the toll

$P(X=k) = \dfrac{e^{-\alpha} \alpha^k}{k!}$ Here $\alpha = 10$, $k = 11$.

$P(X=11) = \dfrac{e^{-10} \times 10^{11}}{11!}$

(b) Minimum no. of toll booths needed

traffic rate $\alpha = 30$

$N \to$ booths

Rate for each booth $\lambda = \dfrac{30}{N}$.

$P(X \leq 5) \geq 0.05$, $P(X \leq 5) = \sum_{k=0}^{5} \dfrac{e^{-\lambda} \lambda^k}{k!}$

Let $N = 5$

$\boxed{\lambda = 6}$

$P(X \leq 5) = e^{-5} \left(5^0 + \dfrac{5^1}{1!} + \dfrac{5^2}{2!} + \dfrac{5^3}{3!} + \dfrac{5^4}{4!} + \dfrac{5^5}{5!} \right)$

$\therefore N = 6$ is sufficient.

(20) $X_i = 1, 2$ $Y_i = 1, 2$

$P(1,1) = 3K$, $P(1,2) = 4K$, $P(2,1) = 5K$, $P(2,2) = 6K$.

Sum of all these $= 1$, $3K + 4K + 5K + 6K = 1 \Rightarrow 18K = 1$

$\boxed{K = 1/18}$

$P(x=1) = P(1,1) + P(1,2) = 7K = 7/18$

$P(x=2) = P(2,1) + P(2,2) = 11/18$

$P(Y=1) = P(1,1) + P(1,2) = 8/18$

$P(Y=2) = P(1,2) + P(2,1) = 10/18$

To Check wheather independent (or) not

$P(X=1)P(Y=1) = P(X=1, Y=1) \Rightarrow \frac{7}{18} \times \frac{8}{18} \neq \frac{3}{18}$

So, they are dependent.

(3) $P(X=i, Y=j) = \binom{j}{i} \frac{e^{-2\lambda} e^{j}}{j!}$

$P(Y=y) = \sum_{x} P(X=x, Y=y) = \sum_{x} \binom{j}{i} e^{-2\lambda} \frac{\lambda^{j}}{j!}$

$= \frac{e^{-2\lambda}\lambda^{j}}{j!} \sum_{x=0}^{j} \binom{j}{i}$

$= \frac{e^{-2\lambda}\, 2^{j}}{}$

$P(X=x) = \sum_{y} P(X=x, Y=y)$

$= \sum_{y} \binom{j}{i} e^{-2\lambda} \frac{\lambda^{j}}{j!} = e^{-2\lambda} \sum_{y} \binom{j}{i} \frac{\lambda^{j}}{j!}$

$= \frac{e^{-2\lambda}}{i!} \frac{\sum \lambda^{j}}{(j-i)!}$.

(4) X is uniform over (0,1)

$$E[X^n] = \int_0^1 x^n f(x)\,dx = \int_0^1 x^n\,dx = \frac{1}{n+1}.$$

$$Var(X^n) = E[X^{2n}] - (E(X^n))^2$$

$$= \int_0^1 x^{2n}\,dx - \frac{1}{(n+1)^2} = \underline{\frac{1}{2n+1} - \frac{1}{(n+1)^2}}.$$

(5) Binomial random variable

$$P(X=K) = \binom{n}{K} p^K (1-p)^{n-K}$$

~~Consider the~~ ratio of succesive probabilities is

$P(X=K+1)$, $P(X=K)$

$$\frac{P(X=K+1)}{P(X=K)} = \frac{{}^nC_{K+1}\, p^{K+1} (1-p)^{n-(K+1)}}{{}^nC_K\, p^K (1-p)^{n-K}} = \frac{(n-K)\,p}{(1-p)(K+1)}$$

Critical value $= 1$ $\qquad (n-K)p = (1-p)(K+1)$

$$np - \cancel{Kp} = 1 - p + K - \cancel{pK} \Rightarrow K = np + p - 1$$

If $K \le ((n+1)p - 1) \Rightarrow P(X=K)\uparrow$

$K \geq ((n+1)p - 1) \Rightarrow P(X=K)\downarrow$.

(a) If $(n+1)p$ is an integer $= m$

$K = (n+1)p = m$ probability is maxmised

(b) If $(n+1)p$ is not integer

maximised probability occured when $(n+1)p - 1 < K < (n+1)p$

K is closer to $(n+1)p$.

⑥ For geometric

$$P(X=k) = (1-p)^{k-1}\,p$$

$$P(X=n+k \mid X>n) = \frac{P(X=n+k \;\&\; X>n)}{P(X>n)}$$

$$= \frac{P(X=n+k)}{P(X>n)} = \frac{(1-p)^{(n+k-1)}\,p}{(1-p)^{n}}$$

$$= (1-p)^{k-1}\,p = P(X=k)$$

This is called lack of memory propery..

The process has already taken more than n ~~[illegible]~~/ trails without sucess doesnot change the probability distribution of remaining trails. is known as "lack of memory".