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# Probability :

sol 1)

$p(1,1) = 1/9$ ; $p(2,1) = 1/3$ ; $p(3,1) = 1/9$

$p(1,2) = 1/9$ ; $p(2,2) = 0$ ; $p(3,2) = 1/18$

$p(1,3) = 0$ ; $p(2,3) = 1/6$ ; $p(3,3) = 1/9$

$$E[X|Y=i] \quad i = 1,2,3$$

we know that,

$$E[X|Y=y] = \sum_x x\, p_{X/Y}(x/y)$$

$$p_{X/Y}(x/y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

$\rightarrow$ $p_Y(y) = \sum_x p_{X,Y}(x,y)$

$p_Y(1) = \sum_x p_{X,Y}(x,1) = 1/9 + 1/9 + 1/3 = 5/9$

$p_Y(2) = \sum p_{X,Y}(x,2) = 1/9 + 0 + 1/18 = 3/18 = 1/6$

$p_Y(3) = \sum_x p_{X,Y}(x,3) = 0 + 1/6 + 1/9 = 5/18$

$\rightarrow$ $E[X/Y=1] = \sum_x x\, p_{X/Y}(x/1)$

$$= \sum_x x \frac{p_{X,Y}(x,1)}{p_Y(1)}$$

$$= \frac{1(1/9) + 2(1/3) + 3(1/9)}{5/9}$$

$$= 10/8$$

$$= 1$$

P.T.O $\rightarrow$

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$\rightarrow$ $$E[X|Y=2] = \sum_x x\, p_{X/Y}(x/2)$$

$$= \sum_x x \frac{p_{X,Y}(x,2)}{p_Y(2)}$$

$$= \frac{1(1/9) + 2(0) + 3(1/18)}{1/6}$$

$$= 5/3$$

$\rightarrow$ $$E[X|Y=3] = \sum_x x\, p_{X/Y}(x/3)$$

$$= \sum_x x \frac{p_{X,Y}(x|3)}{p_Y(3)}$$

$$= \frac{1(0) + 2(1/6) + 3(1/9)}{5/18}$$

$$= (1/3 + 1/3)/5/18$$

$$= (12/5)$$

Sol 2. let X be a RV that represents the length of first run

(a) Case 1: First element is 1
$\hookrightarrow$ run continues till first 0 appears

$\rightarrow$ the length of this run follows a geometric distribution with success probability $\rightarrow 1-p$

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→ $E(X \mid \text{first run starts with } 1) = 1/(1-p)$

Case-2: first element is 0
↳ run continues till first 1 appears
→ the length of this run follows a geometric distribution with success = probability $p$

~~→ E (X / first run~~

→ $E(X \mid \text{first run starts with } 0) = 1/p$

→ the total expected length $E(X)$ is given by

$$E(X) = p(1/1-p) + \frac{(1-p)}{p}$$

$$= \frac{2p^2 - 2p + 1}{p(1-p)}$$

(b) let $Y$ be a RV that represents the length of the 2nd run

→ similar to the first time's run, this run will too follow a geometric distribution

Case 1: first run was of 1's

$E(Y \mid \text{second run is } 0\text{'s}) = 1/p$

Case 2: first run was of 0's

$E(Y \mid \text{second run is } 1\text{'s}) = 1/(1-p)$

∴ total expected length of the second run $E(L_2)$ is given by → $E(L_2) = p \cdot 1/p + (1-p) \, 1/(1-p)$
$= 2$ → Ans

P.T.O →

Sol3. $P(X=i, Y=j) = {}^{j}C_i \, e^{-2\lambda} \lambda^j/(j)!$

$(0 \le i \le j)$

(a) $p_Y(j) = \sum_{i=0}^{j} {}^{j}C_i \, e^{-2\lambda} \frac{\lambda^j}{j!}$

$= e^{-2\lambda} \frac{\lambda^j}{j!} \sum_{i=i}^{j} {}^{j}C_i$

$= e^{-2\lambda} \frac{\lambda^j}{j!} (2)^j$

(b) $p_X(i) = \sum_{j=0}^{\infty} {}^{j}C_i \, e^{-2\lambda} \frac{\lambda^j}{j!}$

$= e^{-2\lambda} \sum_{j=i}^{\infty} \left(j!/(i!)(i-j)!\right) \frac{\lambda^j}{j!}$

$= \frac{e^{-\lambda}}{i!} \sum_{j=i}^{\infty} \frac{e^{-\lambda}\lambda^j}{(i-j)!}$

$= \frac{e^{-\lambda}(\lambda^i)}{i!}$

(c) let $z = y - x = k$

$p_z(k) = ???$

$\rightarrow X = i \;, Y = i + k$

P.T.O.:-

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$$p_Z(k) = P(X=i, Y=i+k) = {}^{i+k}C_i \, e^{-2\lambda} \left(\frac{\lambda^{(i+k)}}{(i+k)!}\right)$$

$$= \frac{(e^{-2\lambda})\,\lambda^{i+k}}{i!\,k!}$$

$$= \left(\frac{e^{-\lambda}(\lambda^{i})}{(i!)}\right)\left(\frac{e^{-\lambda}\cdot\lambda^{k}}{k!}\right)$$

$$= P(X=i) \cdot P(Y-i=k)$$

$$\rightarrow P(Y-X=k) = \frac{e^{-\lambda}\cdot\lambda^{k}}{k!}$$

sol 4. let $f_X(x) = k$ for $x \in (0,1)$

using total probability rule

$$\int_0^1 k \cdot dx = 1 \rightarrow (k=1)$$

$$E[x^n] = \int_{-\infty}^{0} x^n \cancel{f_X(x)}^{\,0} \cdot dx + \int_0^1 x^n \cancel{f_X(x)}^{\,1} \cdot dx + \int_1^{\infty} x^n \cancel{f_X(x)}^{\,0} \cdot dx$$

$$\text{var}(X^n) = \left[\int_{-\infty}^{0} x^{2n} \cancel{f_X(x)}^{\,0} \cdot dx + \int_0^1 x^{2n} \cdot \cancel{f_X(x)}^{\,1} \cdot dx + \int_1^{\infty} x^{2n} \cancel{f_X(x)}^{\,0} \cdot dx\right] - \left(1/(n+1)\right)^2$$

$$= 1/(2n+1) - 1/(n+1)^2 \;\} \text{Ans}$$

P.T.O →

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Sol5. $P(X=k) = {}^nC_k\, p^k (1-p)^{n-k}$

let $r = \dfrac{P(X=k)}{P(X=k-1)} = \dfrac{{}^nC_k\, p^k (1-p)^{n-k}}{{}^nC_{k-1}\, (1-p)^{n-k+1} (p^{k-1})}$

$$= \dfrac{\dfrac{n!}{(n-k)!\,k!}\,(p)}{\dfrac{n!}{(k-1)!\,(n-k+1)!}\,(1-p)}$$

$$= \dfrac{p(n-k+1)}{(1-p)\cdot k}$$

→ if $r > 1 \rightarrow P(X=k) > P(X=k-1) \rightarrow$ probability ↑

$r < 1 \rightarrow P(X=k) < P(X=k-1) \rightarrow$ probability ↓

probability is reaching max. at $r = 1$

$\rightarrow r = 1$

$\rightarrow \dfrac{p(n-k+1)}{(1-p)k} = 1$

$\therefore pn + \cancel{pk} + p = k + \cancel{pk}$

So, $p(n+1) = k$

$\therefore$ probability increase upto $k = p(n+1)$

and will decrease after that

(a) If $(n+1)p$ is an integer, possibilities of $P(X=k)$ will be max. at - $k = (n+1)p - 1$
or $k = (n+1)p$

(b) If $(n+1)p$ isn't an integer → possibilities of $P(X=k)$ will be max. at some value of $k$ such that

$$(1+n)(p)-1 < k < (1+n)p$$

Sol 6. Given,
$X$ is geometric
∴ let '$p$' be the probability of success,
∴ probability of failure will be $1-p$

→ $p_X(X=k) = (1-p)^{k-1}(p)$

$$P = (X = n+k \,/\, X > n)$$

$$= \frac{P[(X=n+k) \cap (X>n)]}{P(X>n)}$$

will just be $P(X = n+k)$

$$= (1-p)^{n+k-1} \cdot p \quad \text{—①}$$

$$P(X>n) = 1 - P(X \le n)$$

$$= 1 - \left[ \sum_{i=1}^{n} p(1-p)^{i-1} \right]$$

$$= 1 - \left[ p \left[ 1 + (1-p) \ldots (1-p)^{n-1} \right] \right]$$

P.T.O →

$$= 1 - p\left(\frac{1 - (1-p)^n (1)}{1 - 1 + p}\right)$$

$$= 1 - 1 + (1-p)^n$$

$$\rightarrow P(X > n) = (1-p)^n \quad - (2)$$

→ Putting together (1) and (2) to get

$$P(X = n+k \mid X > n) = \frac{(1-p)^{n+k-1} \cdot p}{(1-p)^n}$$

$$= (1-p)^{k-1} p$$

$$= P(X = k)$$

→ The result is referred to as the "lack of memory" property, as it shows no change in probability of getting success after k extra trials, irrespective of the number of trials already occurred.

→ we can see this property in exponential distribution over a continuous interval as well

$$P(X > k) = e^{-\lambda k}$$

$$[P(X > x) = 1 - P(X \leq x)$$

$$= 1 - F_X(x)]$$

$$\to P(X > n+k \mid X > n) = \frac{P(X > n+k)}{P(X > n)}$$

$$= \frac{e^{-\lambda(n+k)}}{e^{-\lambda n}}$$

$$= e^{-\lambda k} = P(X = k)$$

where, $n, k \geq 0$

Sol 7. let $I_{A_i} = \begin{cases} 1 & ; A_i \text{ occurs} \\ 0 & ; A_i \text{ doesn't occur} \end{cases}$

where $A_i$ is the event that the $i^{th}$ letter from the pile gets into its corresponding envelope

we define $X$ as the number of correctly [redacted] matched pairs, [redacted] it can be given as

$$X = I_{A_1} + I_{A_2} \ldots I_{A_n}$$

$$E[X] = E[I_{A_1} + I_{A_2} \ldots I_{A_n}]$$

$$= \sum_{i=1}^{n} E[I_{A_i}]$$

$$= \sum_{i=1}^{n} P(A_i)$$

$$= \sum_{i=1}^{n} 1/n = 1$$

$$var(X) = var(I_{A_1} + I_{A_2} \ldots I_{A_n})$$

$$= \sum_{i=1}^{n} var(I_{A_i}) + 2 \sum_{\substack{i,j=1 \\ i<j}}^{n} cov(I_{A_i}, I_{A_j})$$

$$var(I_{A_i}) = P(A_i)(1 - P(A_i))$$

$$= 1/n \, (1 - 1/n)$$ [redacted] P.T.O.

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$\text{cov}(I_{A_i}, I_{A_j}) = E[I_{A_i} \cdot I_{A_j}] - E[I_{A_i}] \cdot E[I_{A_j}]$

$= \frac{1}{n(n-1)} - 1/n^2$

$= 1/n^2 (n-1)$

$\rightarrow \text{var}[X] = \sum_{i=1}^{n} 1/n (1 - 1/n)$

$+ 2 \sum_{\substack{i,j=1 \\ i<j}} 1/n^2(n-1)$

$= n \left( 1/n (1 - 1/n) \right)$

$+ 2 \frac{n(n-1)}{2} \frac{1}{n^2(n-1)}$

$= 1 - 1/n + 1/n$

$= 1$

As $n \rightarrow \infty$, $p$ gets smaller and smaller and the probability can be approx. as a Poisson distribution

$\rightarrow P_X[k] = \frac{e^{-\lambda} \cdot \lambda^k}{k!}$

$\lambda = E[X] = 1$

$\therefore P_X[k] = \frac{e^{-1} (1)^k}{k!}$

$= e^{-1}/k!$

P.T.O. $\rightarrow$

Sol 8. $P_r(\text{error}) = 10^{-6}$ $(p)$

Block length $= 10^4$ $(n)$

$N$ - RV representing number of error in block

$\lambda = np = 10^{-2}$

$P_N(N=k) = e^{-\lambda}\dfrac{\lambda^k}{k!}$

(a) $P(N=0) = e^{-1/100}\dfrac{(1/100)^0}{0!} = e^{-1/100}$

$P(N \leq 3) = P(N=0) + P(N=1) + P(N=2) + P(N=3)$

$= e^{-1/100} + \dfrac{e^{-1/100}}{1!}(1/10^2)$

$+ \dfrac{e^{-1/100}}{2!}(1/10^4)$

$+ \dfrac{e^{-1/100}}{3!}(1/10^6)$

(b) $P(N \geq 1) = 0.99$

$1 - e^{-\lambda} = 0.99$

$e^{-\lambda} = 0.99$

$e^{-\lambda} = 0.01$

$\lambda = \ln(100)$

$\lambda = 2\ln(10)$

$\therefore \lambda = np \rightarrow p = 4.606\,(10)^{-4}$

P.T.O

$X$ = number of patterns tested before success
For an $m$-bit password, total patterns $2^m = n$

After $k$ unsuccessful trials, $n-k$ patterns remain which contain the correct password with each pattern having the probability of $(1/n-k)$

$$\to P(X = i \mid X > k) = 1/n-k$$

$$k < i \le n$$

$$\to P(X = i \mid X > k) = \frac{1}{2^m k}$$

$$\text{for } k < i \le 2^m$$

$$E(X \mid X > k) = \sum_{i=1}^{m} i \cdot P_X(X = i \mid X > k)$$

$$= \sum_{i=1}^{2^m} i \cdot 1/2^m - k$$

$$= 1/2^m - k \left[ (k+1) + (k+2) \ldots 2^m \right]$$

$$\Rightarrow 1/(\cancel{2^m - k}) \left( 1/2 \, (\cancel{2^m - k}) (k + 2^m + 1) \right)$$

$$= \frac{k + 2^m + 1}{2}$$

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Sol 1a. $X = r\cos(2\pi\theta/8)$

$Y = r\sin(2\pi\theta/8)$

where $\theta \sim U\{0, 1 \dots 7\}$

$(0, \pi/4, \pi/2, \dots 7\pi/4)$

(a) $X = \{r, r/\sqrt{2}, 0, -r/\sqrt{2}, -r, -r/\sqrt{2}, 0, r/\sqrt{2}\}$

$Y = \{0, r/\sqrt{2}, r, r/\sqrt{2}, 0, -r/\sqrt{2}, -r, -r/\sqrt{2}\}$

$S_{XY} = \{(r,0), (r/\sqrt{2}, r/\sqrt{2}), (0,r), (-r/\sqrt{2}, r/\sqrt{2}), (-r,0), (-r/\sqrt{2}, -r/\sqrt{2}), (0,-r), (r/\sqrt{2}, -r/\sqrt{2})\}$

$\rightarrow \theta = 0 \rightarrow (r,0)$

$= \pi/4 \rightarrow (r/\sqrt{2}, r/\sqrt{2})$

$= \pi/2 \rightarrow (0,r)$

$= 3\pi/4 \rightarrow (-r/\sqrt{2}, r/\sqrt{2})$

$= 3\pi/2 \rightarrow (-r,0)$

$= 5\pi/4 \rightarrow (-r/\sqrt{2}, r/\sqrt{2})$

$= 3\pi/2 \rightarrow (0,-r)$

$= 7\pi/4 \rightarrow (r/\sqrt{2}, -r/\sqrt{2})$

(b) Joint PMF of X and Y

| X \ Y | $r$ | $r/\sqrt{2}$ | $0$ | $-r/\sqrt{2}$ | $-r$ |
|---|---|---|---|---|---|
| $r$ | 0 | 0 | 1/8 | 0 | 0 |
| $r/\sqrt{2}$ | 0 | 1/8 | 0 | 1/8 | 0 |
| $0$ | 1/8 | 0 | 0 | 0 | 1/8 |
| $-r/\sqrt{2}$ | 0 | 1/8 | 0 | 1/8 | 0 |
| $-r$ | 0 | 0 | 1/8 | 0 | 0 |

1.5.0 →

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Sol 11. (a) $p_X(x) = \sum_y p_{X,Y}(x,y)$

$p_X(1) = 1/8$

$p_X(1/\sqrt{2}) = 1/4$

$p_X(0) = 1/4$

$p_X(-1/\sqrt{2}) = 1/4$

$p_X(-1) = 1/8$

$p_Y(y) = \sum_x p_{X,Y}(x,y)$

$\rightarrow p_Y(1) = 1/8$

$p_Y(1/\sqrt{2}) = 1/4$

$p_Y(0) = 1/4$

$p_Y(-1/\sqrt{2}) = 1/4$

$p_Y(-1) = 1/8$

(b) $P_1(A) = p_X(0) = 1/4$

$P_1(B) = p_Y(1/\sqrt{2}) + p(0) + p(-1/\sqrt{2}) + p(-1) = 7/8$

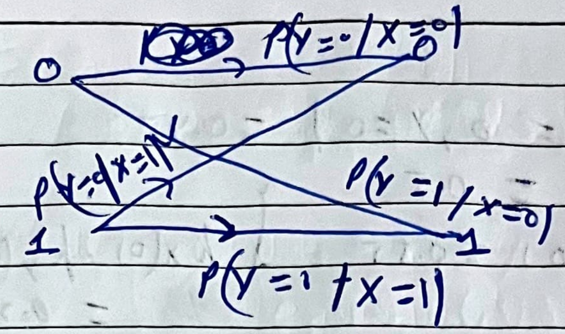
$P_1(C) = p_{X,Y}(1/\sqrt{2}, 1/\sqrt{2}) + p_{X,Y}(-1/\sqrt{2}, 1) + p_{X,Y}(1, 1/\sqrt{2}) + p_{X,Y}(a,1)$

$= 1/2$

$P_1(D) = p_X(-1) = 1/8$

P.T.O →

Sol 12.




$P(X=0) = 0.5 = P(X=1)$

$P(Y=1/X=1) = 0.8$

$P(Y=1/X=0) = 0.1$

$P(Y=0/X=1) = 0.2$

$P(Y=0/X=0) = 0.9$

(a) $P(Y=i / X=j) = \dfrac{P(Y=i, X=j)}{P(X=j)}$

$i, j \in \{0,1\}$

$\rightarrow P(Y=i, X=j) = P(Y=i/X=j) \cdot P(X=j)$

$$\rightarrow p_{X,Y}(x,y) = \begin{cases} 0.45 & ; (x,y) = (0,0) \\ 0.05 & ; (x,y) = (0,1) \\ 0.10 & ; (x,y) = (1,0) \\ 0.40 & ; (x,y) = (1,1) \\ 0 & ; \text{otherwise} \end{cases}$$

(b) $p_X(x) = \sum_y p_{X,Y}(x,y)$

$p_X(0) = 0.5$ $\qquad$ $p_Y(1) = 0.45$

$p_Y(0) = 0.55$ $\qquad$ $p_X(1) = 0.5$

(c) If X and Y are independent,

$p_{X,Y}(X=x, Y=y) = p_X(X=x) \cdot p_Y(Y=y) \ \forall_{x,y}$

checking for $x=0, y=0$ →

$15 \cdot 0$ →

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$\rightarrow$ $p_{X,Y}(X=0, Y=0) = 0.45$

$p_X(0) = 0.5$

$p_Y(0) = 0.55$ $\Big\}$ $p_X(0) \cdot p_Y(0)$

$= 0.275 \neq 0.45$

$\therefore$ X and Y aren't independent RV's

Sol 13. $p_X(X=k) = e^{-\lambda_1} \dfrac{(\lambda_1)^k}{k!}$, $p_Y(Y=k)$

$= \dfrac{e^{-\lambda_2}(\lambda_2^k)}{k!}$

$Z = X + Y$

$p_Z(z) = P(X+Y=z) = \sum_{k=0}^{z} P(X=k) \cdot P(Y=z-k)$

$= \sum_{k=0}^{z} \left( \dfrac{\lambda_1^k e^{-\lambda_1}}{k!} \right) \cdot$

$\left( \dfrac{\lambda_2^{z-k}(e^{-\lambda_2})}{(z-k)!} \right)$

$= e^{-(\lambda_1+\lambda_2)} \sum_{k=0}^{z} \dfrac{\lambda_1^k (\lambda_2^{z-k})}{k!(z-k)!}$

$= e^{-(\lambda_1+\lambda_2)} \dfrac{(\lambda_1+\lambda_2)^z}{z!}$ $\Big\}$ Ans

P.T.O. $\rightarrow$

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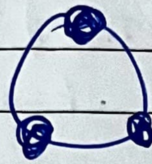
Sol 14. Let $M$ be the RV that represents total time taken for trips, 2 hours of each

$$\therefore M = (2 \text{ hours}) \times T$$



$p_T(T=t) = p(1-p)^{t-1}$ → is the geometric RV with success → $p$

$E[T] = 1/p$

$P(\text{door } 1) = 1/3$

→ $E[T] = 3$

$E[M] = 2 \times 3$

$= 6$ hours

Sol 15. Success → getting a grade that you already haven't gotten

Let $X_i$ be a RV that counts the number of papers to be submitted between $i^{th}$ and $(i+1)^{th}$ successes

Let $X$ be the RV that calculates the total number of papers submitted before grades obtained

→ $X = (X_1 + X_2 \dots X_5 + 1)$ → we can start only if one paper is submitted

(I-0 →

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$\therefore E[X] = ????$

$\rightarrow E[X] = E[X_1 + X_2 \cdots X_5 + 1]$
$= E(X_1) + E(X_2) \cdots E(X_5) + 1$

$\therefore p_{X_1}(k) = (1/6)^{k-1}(5/6) \rightarrow$ success probability $= 5/6$ $(E(X_1)) = 6/5$

similarly

$p_{X_2}(k) = (2/6)^{k-1} 4/6 = 4/6$ $(E(X_2)) = 6/4$
$p_{X_3}(k) = (3/6)^{k-1} 3/6 = 3/6$ $(E(X_3)) = 6/3$
$p_{X_4}(k) = (4/6)^{k-1} 2/6 = 2/6$ $(E(X_4)) = 6/2$
$p_{X_5}(k) = (1/6)^{k-1}(1/6) = (1/6)$ $(E(X_5)) = 6/1$

$\therefore E(X) = 6(1/6 + 1/5 + 1/4 + 1/3 + 1/2 + 1)$

$= 14.7$

So, student $\rightarrow$ to get all possible grades — must submit all 16 papers

Sol 17. $p_1 = p$ , $p_0 = 1-p$ , $X$ = RV of number of transmissions in a given time interval

$p_X(X=k) = e^{-\lambda} \frac{\lambda^k}{k!}$ , $Y$ = RV of number of transmission in same time interval

$p_X(X=k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$ , $Y$ = RV of number of transmission in same time interval of 1s

∴ To show : $p_Y(Y=k) = e^{-p\lambda} \frac{(p\lambda)^k}{k!}$

$\rightarrow p_X(X=k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$ ; $k = 0, 1, 2 \ldots$

k transmission have occurred consisting of both 1's and 0's $Y$ is defined as RV that counts number of 1's transmitted

we can say that

$P_r(Y=m / X=k) = {}^kC_m \, p^m (1-p)^m$

$m = 0, 1, \ldots, k-1, k$

$P_r(Y=m) = \sum_{k=m}^{\infty} P(Y=m / X \geq k) \cdot P(X=k)$

P.T.O →

$$= \sum_{k=m}^{\infty} {}^{k}C_{m}\, p^{m} (1-p)^{k-m} \frac{\lambda^{k} e^{-\lambda}}{k!}$$

$$= \frac{p^{m} \lambda^{m}}{m!} \sum_{k=m}^{\infty} \frac{(1-p)^{k-m}}{(k-m)!} (\lambda^{k-m})(e^{-\lambda})$$

$$= \frac{p^{m}(\lambda^{m})}{m!} \sum_{k=m}^{\infty} \frac{(\lambda(1-p))^{k-m}}{(k-m)!} \cdot e^{-\lambda}$$

let, $k-m = n$

$$= p^{m} \left(\frac{\lambda^{m}}{m!}\right) \sum_{n=0}^{\infty} \frac{[(1-p)\cdot\lambda]^{n}}{n!} (e^{-\lambda})$$

$$= p^{m} \left(\frac{\lambda^{m}}{m!}\right) \cdot e^{(1-p)\cdot\lambda} \cdot e^{-\lambda}$$

$$= e^{-p\lambda} \frac{(p\lambda)^{m}}{m!}$$

$$\therefore p_{Y}(Y=m) = \frac{e^{-p\lambda} (p\lambda)^{m}}{m!}$$

↳ poisson distribution with poisson ratio $- p\lambda$

Sol18. $X$ = number of success | $P$ (success) $= p$
$Y$ = number of failure |

$$p_X(x) = {}^nC_x\, p^x (1-p)^{n-x}$$

$$p_Y(y) = {}^nC_y (1-p)^y (p)^{n-y}$$

$$Z = X - Y$$
$$= X - (n - X)$$
$$= 2X - n$$

$$\rightarrow p_Z(z) = p_X\left(\frac{z+n}{2}\right)$$
$$= {}^nC_{\frac{z+n}{2}}\, p^{\frac{z+n}{2}} (1-p)^{(z-n)/2}$$

$z = -n, -n+2 \ldots n$

$$E[Z] = E[2X - n] = 2E[X] - n$$
$$= 2(np) - n$$
$$= n(2p-1)$$

$$\mathrm{var}(Z) = \mathrm{var}(2X - n) = 4\,\mathrm{var}(X)$$
$$= 4np(1-p)$$

Sol. Let $X$ be a R.V. measuring number of card per minute approaching toll booth

$$\rightarrow p_X(k) = e^{-\alpha} \frac{(\alpha)^k}{k!}$$

(a) $p_X(11) = ????$ $\alpha = 10$

$$\rightarrow p_X(11) = e^{-11} (20)^{11} / 11!$$

1) N toll booth, $\lambda = 30$, let $x_i$ be no. of cars approaching toll booth $i$ per minute

→ Traffic approaching each booth is independent

→ Traffic at each booth follows poisson ~~PMF~~ PMF with $\lambda = 30/N$ $\min(N) = ?$ such that

$$P(x_i < 5) \geq 0.05 \quad \text{where } i = 1, 2, \ldots N$$

$$\sum_{j=1}^{5} P(x_i = j) \geq 0.05$$

$$\sum_{j=1}^{5} e^{-30/N} \frac{(30/N)^j}{j!} \geq 0.05$$

we can solve this equation to find N.

2) $$p_{x,y}(x_i, y_i) = \begin{cases} k(2x_i + y_i) & x_i = 1,2 \; ; y_i = 1,2 \\ 0 & o/w \end{cases}$$

$$p_{x,y}(1,1) = k(2+1) = 3k$$

$$p_{x,y}(1,2) = k(2+2) = 4k$$

$$p_{x,y}(2,1) = k(4+1) = 5k$$

$$p_{x,y}(2,2) = k(4+2) = 6k.$$

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(a) Using TPR $\sum_x \sum_y p_{x,y}(x,y) = 1$

$\Rightarrow p_{x,y}(1,1) + p_{x,y}(1,2) + p_{x,y}(2,1) + p_{x,y}(2,2)$

$3k + 4k + 5k + 6k = 1$

$\Rightarrow k = 1/18$

(b) $p_x(x) = \sum_y p_{x,y}(x,y)$

$$p_x(x) = \begin{cases} 7/18 & 1 \\ 11/18 & 2 \\ 0 & o/w \end{cases}$$

$p_y(y) = \sum_x p_{x,y}(x,y)$

$$p_y(y) = \begin{cases} 8/18 & 1 \\ 10/18 & 2 \\ 0 & o/w \end{cases}$$

(c) for X and Y to be independent

$p_{x,y}(x,y) = p_x(x) \cdot p_y(y) \ \forall x, y$

$p_{x,y}(1,1) = 3/18$

$p_x(1)\, p_y(1) = \frac{7 \times 8^{4}}{18 \times 18_{9}} = \frac{28}{162}$

Clearly $p_{x,y}(1,1) \neq p_x(1) \cdot p_y(1)$

$\therefore$ X and Y are not independent

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1c) $\sum\sum P_{m,n}(m,n) = 1$ $\quad m+n < L$

$= \sum_{m=0}^{L-1} \sum_{n=0}^{L-m-1} c$

$= \sum_{m=0}^{L-1} (L-m-1) = L(L-1) - \frac{(L-1)(L-2)}{2} - (L-1)$

$= \frac{L(L-1)}{2}$

d) $p_m(m) = \sum_{n=0}^{L-m-1} P_{m,n}(m,n)$

$= \sum_{n=0}^{L-m-1} c = c(L-m)$

$= \frac{2}{L(L-1)}(L-m)$

$p_m(m) = \frac{2(L-m)}{L(L-1)}$ $\quad$ ~~$p_n(n) = \sum_{m=0}$~~ $\quad p_n(n) = \sum_{m=0}^{L-n-1} p_{m,n}(m,n)$

$= (L-n)c$

$= \boxed{\frac{2\cdot(L-n)}{L(L-1)}}$

e) $P(M+N < L/2)$ $\quad$ Let $K = L/2$

$\sum_{m=0}^{K-1} \sum_{n=0}^{K-m-1} P_{m,n}(m,n)$

$P(M+N < L/2) = c \sum_{m=0}^{K-1} (K-m)$

$= \frac{cK(K+1)}{2}$

Substituting $K = L/2$

$\sum_{m=0}^{K-1}(K-m) = \frac{\frac{L}{2}(\frac{L}{2}+1)}{2} = \frac{L(L+2)}{8}$

$= \frac{2}{L(L-1)} \cdot \frac{L(L+2)}{8} = \frac{1}{4}\frac{(L+2)}{(L-1)}$

$P(M+N < \frac{L}{2}) = \boxed{\frac{L+2}{4(L-1)}}$