

# Problem 1

## Solution

### Part A

Consider the forces acting on the puck and the cylinder and depicted in the figure on the right. The puck is subject to the gravity force  $mg$  and the reaction force from the cylinder  $N$ . The cylinder is subject to the gravity force  $Mg$ , the reaction force from the plane  $N_1$ , the friction force  $F_{fr}$  and the pressure force from the puck  $N' = -N$ . The idea is to write the horizontal projections of the equations of motion. It is written for the puck as follows

$$ma_x = N \sin \alpha, \quad (A.1)$$

where  $a_x$  is the horizontal projection of the puck acceleration.

For the cylinder the equation of motion with the acceleration  $w$  is found as

$$Mw = N \sin \alpha - F_{fr}. \quad (A.2)$$

Since the cylinder moves along the plane without sliding its angular acceleration is obtained as

$$\varepsilon = w/R \quad (A.3)$$

Then the equation of rotational motion around the center of mass of the cylinder takes the form

$$I\varepsilon = F_{fr} R, \quad (A.4)$$

where the inertia moment of the hollow cylinder is given by

$$I = MR^2. \quad (A.5)$$

Solving (A.2)-(A.5) yields

$$2Mw = N \sin \alpha. \quad (A.6)$$

From equations (A.1) and (A.6) it is easily concluded that

$$ma_x = 2Mw. \quad (A.7)$$

Since the initial velocities of the puck and of the cylinder are both equal to zero, then, it follows from (A.7) after integrating that

$$mu = 2Mv. \quad (A.8)$$

It is obvious that the conservation law for the system is written as

$$mgR = \frac{mu^2}{2} + \frac{Mv^2}{2} + \frac{I\omega^2}{2}, \quad (A.9)$$

where the angular velocity of the cylinder is found to be

$$\omega = \frac{v}{R}, \quad (A.10)$$

since it does not slide over the plane.

Solving (A.8)-(A.10) results in velocities at the lowest point of the puck trajectory written as

$$u = 2\sqrt{\frac{MgR}{(2M+m)}}, \quad (A.12)$$

$$v = \frac{m}{M}\sqrt{\frac{MgR}{(2M+m)}}. \quad (A.13)$$

In the reference frame sliding progressively along with the cylinder axis, the puck moves in a circle of radius  $R$  and, at the lowest point of its trajectory, have the velocity

$$v_{rel} = u + v \quad (A.14)$$

and the acceleration

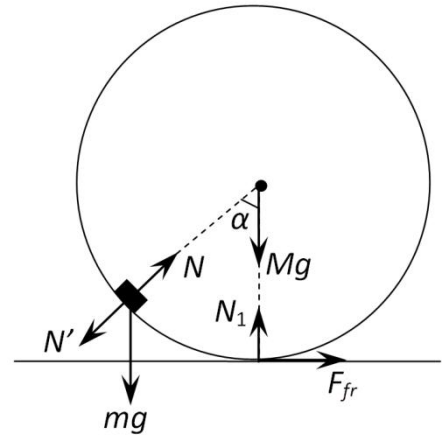
$$a_{rel} = \frac{v_{rel}^2}{R}. \quad (A.15)$$

At the lowest point of the puck trajectory the acceleration of the cylinder axis is equal to zero, therefore, the puck acceleration in the laboratory reference frame is also given by (A.15).

$$F - mg = \frac{mv_{rel}^2}{R}. \quad (A.16)$$

then the interaction force between the puck and the cylinder is finally found as

$$F = 3mg \left(1 + \frac{m}{3M}\right). \quad (A.17)$$



**Part B**

1) According to the first law of thermodynamics, the amount of heat transmitted  $\delta Q$  to the gas in the bubble is found as

$$\delta Q = \nu C_V dT + p dV, \quad (\text{B.1})$$

where the molar heat capacity at arbitrary process is as follows

$$C = \frac{1}{\nu} \frac{\delta Q}{dT} = C_V + \frac{p}{\nu} \frac{dV}{dT}. \quad (\text{B.2})$$

Here  $C_V$  stands for the molar heat capacity of the gas at constant volume,  $p$  designates its pressure,  $\nu$  is the total amount of moles of gas in the bubble,  $V$  and  $T$  denote the volume and temperature of the gas, respectively.

Evaluate the derivative standing on the right hand side of (B.2). According to the Laplace formula, the gas pressure inside the bubble is defined by

$$p = \frac{4\sigma}{r}, \quad (\text{B.3})$$

thus, the equation of any equilibrium process with the gas in the bubble is a polytrope of the form

$$p^3 V = \text{const}. \quad (\text{B.4})$$

The equation of state of an ideal gas has the form

$$pV = \nu RT, \quad (\text{B.5})$$

and hence equation (B.4) can be rewritten as

$$T^3 V^{-2} = \text{const}. \quad (\text{B.6})$$

Differentiating (B.6) the derivative with respect to temperature sought is found as

$$\frac{dV}{dT} = \frac{3V}{2T}. \quad (\text{B.7})$$

Taking into account that the molar heat capacity of a diatomic gas at constant volume is

$$C_V = \frac{5}{2} R, \quad (\text{B.8})$$

and using (B.5) it is finally obtained that

$$C = C_V + \frac{3}{2} R = 4R = 33.2 \frac{\text{J}}{\text{mole} \cdot \text{K}}. \quad (\text{B.9})$$

2) Since the heat capacity of the gas is much smaller than the heat capacity of the soap film, and there is heat exchange between them, the gas can be considered as isothermal since the soap film plays the role of thermostat. Consider the fragment of soap film, limited by the angle  $\alpha$  as shown in the figure. It's area is found as

$$S = \pi(\alpha r)^2. \quad (\text{B.10})$$

and the corresponding mass is obtained as

$$m = \rho S h. \quad (\text{B.11})$$

Let  $x$  be an increase in the radius of the bubble, then the Newton second law for the fragment of the soap film mentioned above takes the form

$$m\ddot{x} = p' S' - F_{\text{surf}}, \quad (\text{B.12})$$

where  $F_{\text{surf}}$  denotes the projection of the resultant surface tension force acting in the radial direction,  $p'$  stands for the gas pressure beneath the surface of the soap film and

$$S' = S \left(1 + 2 \frac{x}{r}\right).$$

$F_{\text{surf}}$  is easily found as

$$F_{\text{surf}} = F_{ST} \alpha = \sigma \cdot 2 \cdot 2\pi[(r+x)\alpha] \cdot \alpha. \quad (\text{B.13})$$

Since the gaseous process can be considered isothermal, it is written that

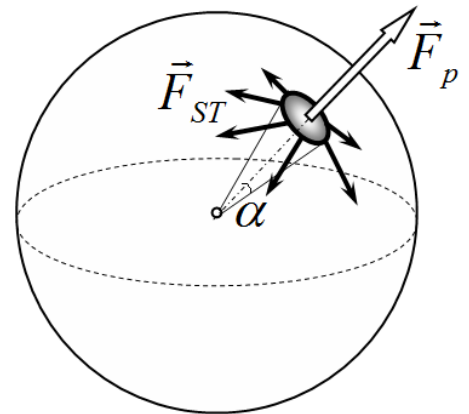
$$p' V' = pV. \quad (\text{B.14})$$

Assuming that the volume increase is quite small, (B.14) yields

$$p' = p \frac{1}{\left(1 + \frac{x}{r}\right)^3} \approx p \frac{1}{\left(1 + \frac{3x}{r}\right)} \approx p \left(1 - \frac{3x}{r}\right). \quad (\text{B.15})$$

Thus, from (B.10) - (B.16) and (B.3) the equation of small oscillations of the soap film is derived as

$$\rho h \ddot{x} = -\frac{8\sigma}{r^2} x \quad (\text{B.16})$$



with the frequency

$$\omega = \sqrt{\frac{8\sigma}{\rho h r^2}} = 108 \text{ s}^{-1}. \quad (\text{B.17})$$

### Part C

The problem can be solved in different ways. Herein several possible solutions are considered.

#### Method 1. Direct approach

At the moment when the current in the coils is a maximum, the total voltage across the coils is equal to zero, so the capacitor voltages must be equal in magnitude and opposite in polarity. Let  $U$  be a voltage on the capacitors at the time moment just mentioned and  $I_0$  be that maximum current. According to the law of charge conservation

$$q_0 = 2CU + CU, \quad (\text{C1.1})$$

thus,

$$U = \frac{q_0}{3C}. \quad (\text{C1.2})$$

Then, from the energy conservation law

$$\frac{q_0^2}{2 \cdot 2C} = \frac{LI_0^2}{2} + \frac{2LI_0^2}{2} + \frac{CU^2}{2} + \frac{2CU^2}{2} \quad (\text{C1.3})$$

the maximum current is found as

$$I_0 = \frac{q_0}{3\sqrt{2LC}}. \quad (\text{C1.4})$$

After the key  $K$  is shortened there will be independent oscillations in both circuits with the frequency

$$\omega = \frac{1}{\sqrt{2LC}}, \quad (\text{C1.5})$$

and their amplitudes are obtained from the corresponding energy conservation laws written as

$$\frac{2CU^2}{2} + \frac{LI_0^2}{2} = \frac{LJ_1^2}{2}, \quad (\text{C1.6})$$

$$\frac{CU^2}{2} + \frac{2LI_0^2}{2} = \frac{2LJ_2^2}{2}. \quad (\text{C1.7})$$

Hence, the corresponding amplitudes are found as

$$J_1 = \sqrt{5}I_0, \quad (\text{C1.8})$$

$$J_2 = \sqrt{2}I_0. \quad (\text{C1.9})$$

Choose the positive directions of the currents in the circuits as shown in the figure on the right. Then, the current flowing through the key is written as follows

$$I = I_1 - I_2. \quad (\text{C1.10})$$

The currents depend on time as

$$I_1(t) = A \cos \omega t + B \sin \omega t, \quad (\text{C1.11})$$

$$I_2(t) = D \cos \omega t + F \sin \omega t, \quad (\text{C1.12})$$

The constants  $A, B, D, F$  can be determined from the initial values of the currents and their amplitudes by putting down the following set of equations

$$I_1(0) = A = I_0, \quad (\text{C1.13})$$

$$A^2 + B^2 = J_1^2, \quad (\text{C1.14})$$

$$I_2(0) = D = I_0, \quad (\text{C1.15})$$

$$D^2 + F^2 = J_2^2. \quad (\text{C1.16})$$

Solving (C1.13)-(C1.16) it is found that

$$B = 2I_0, \quad (\text{C1.17})$$

$$F = -I_0, \quad (\text{C1.18})$$

The sign in  $F$  is chosen negative, since at the time moment of the key shortening the current in the coil  $2L$  decreases.

Thus, the dependence of the currents on time takes the following form

$$I_1(t) = I_0(\cos \omega t + 2 \sin \omega t), \quad (\text{C1.19})$$

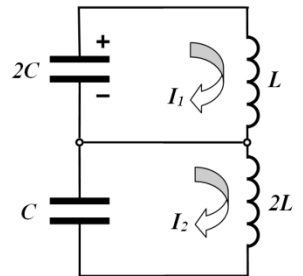
$$I_2(t) = I_0(\cos \omega t - \sin \omega t). \quad (\text{C1.20})$$

In accordance with (C.10), the current in the key is dependent on time according to

$$I(t) = I_1(t) - I_2(t) = 3I_0 \sin \omega t. \quad (\text{C1.21})$$

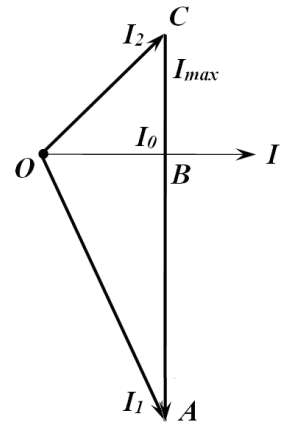
Hence, the amplitude of the current in the key is obtained as

$$I_{\max} = 3I_0 = \omega q_0 = \frac{q_0}{\sqrt{2LC}}. \quad (\text{C1.22})$$



Method 2. Vector diagram

Instead of determining the coefficients  $A, B, D, F$  the vector diagram shown in the figure on the right can be used. The segment  $AC$  represents the current sought and its projection on the current axis is zero at the time of the key shortening. The current  $I_1$  in the coil of inductance  $L$  grows at the same time moment because the capacitor  $2C$  continues to discharge, thus, this current is depicted in the figure by the segment  $OA$ . The current  $I_2$  in the coil of inductance  $2L$  decreases at the time of the key shortening since it continues to charge the capacitor  $2C$ , that is why this current is depicted in the figure by the segment  $OC$ .



It is known for above that  $OB = I_0, OA = \sqrt{5}I_0, OC = \sqrt{2}I_0$ . Hence, it is found from the Pythagorean theorem that

$$AB = \sqrt{OA^2 - OB^2} = 2I_0, \quad (C2.1)$$

$$BC = \sqrt{OC^2 - OB^2} = I_0, \quad (C2.2)$$

Thus, the current sought is found as

$$I_{\max} = AC = AB + BC = 3I_0 = \omega q_0 = \frac{q_0}{\sqrt{2LC}}. \quad (C2.3)$$

Method 3. Heuristic approach

It is clear that the current through the key performs harmonic oscillations with the frequency

$$\omega = \frac{1}{\sqrt{2LC}}. \quad (C3.1)$$

and it is equal to zero at the time of the key shortening, i.e.

$$I(t) = I_{\max} \sin \omega t. \quad (C3.2)$$

Since the current is equal to zero at the time of the key shortening, then the current amplitude is equal to the current derivative at this time moment divided by the oscillation frequency. Let us find that current derivative. Let the capacitor of capacitance  $2C$  have the charge  $q_1$ . Then the charge on the capacitor of capacitance  $C$  is found from the charge conservation law as

$$q_2 = q_0 - q_1. \quad (C3.3)$$

After shortening the key the rate of current change in the coil of inductance  $L$  is obtained as

$$\dot{I}_1 = \frac{q_1}{2LC}, \quad (C3.4)$$

whereas in the coil of inductance  $2L$  it is equal to

$$\dot{I}_2 = -\frac{q_0 - q_1}{2LC}. \quad (C3.5)$$

Since the voltage polarity on the capacitors are opposite, then the current derivative with respect to time finally takes the form

$$\dot{I} = \dot{I}_1 - \dot{I}_2 = \frac{q_0}{2LC} = \omega^2 q_0. \quad (C3.6)$$

Note that this derivative is independent of the time of the key shortening!

Hence, the maximum current is found as

$$I_{\max} = \frac{\dot{I}}{\omega} = \omega q_0 = \frac{q_0}{\sqrt{2LC}}, \quad (C3.7)$$

and it is independent of the time of the key shortening!

## Problem 2. Van der Waals equation of state

### Solution

#### Part A. Non-ideal gas equation of state

**A1.** If  $V = b$  is substituted into the equation of state, then the gas pressure turns infinite. It is obvious that this is the moment when all the molecules are tightly packed. Therefore, the parameter  $b$  is approximately equal to the volume of all molecules, i.e.

$$b = N_A d^3 \quad (\text{A1.1})$$

**A2.** In the most general case the van der Waals equation of state can be rewritten as

$$P_c V^3 - (RT_c + bP_c)V^2 + aV - ab = 0 \quad (\text{A2.1}).$$

Since at the critical values of the gas parameters the straight line disappears, then, the solution of (A2.1) must have one real triple root, i.e. it can be rewritten as follows

$$P_c(V - V_c)^3 = 0 \quad (\text{A2.2}).$$

Comparing the coefficients of expression (A2.1) and (A2.2), the following set of equations is obtained

$$\begin{cases} 3P_c V_c = RT_c + bP_c \\ 3P_c V_c^2 = a \\ P_c V_c^3 = ab \end{cases} \quad (\text{A2.3}).$$

Solution to the set (A2.3) is the following formulas for the van der Waals coefficients

$$a = \frac{27R^2 T_c^2}{64P_c} \quad (\text{A2.4}),$$

$$b = \frac{RT_c}{8P_c} \quad (\text{A2.5}).$$

#### Alternative solution

The critical parameters are achieved in the presence of an inflection point in the isotherm, at which the first and second derivatives are both zero. Therefore, they are defined by the following conditions

$$\left(\frac{dP}{dV}\right)_T = 0 \quad (\text{A2.6}),$$

and

$$\left(\frac{d^2P}{dV^2}\right)_T = 0 \quad (\text{A2.7}).$$

Thus, the following set of equations is obtained

$$\begin{cases} -\frac{RT_c}{(V_c - b)^2} + \frac{2a}{V_c^3} = 0 \\ \frac{2RT_c}{(V_c - b)^3} - \frac{6a}{V_c^4} = 0 \\ \left(P_c + \frac{a}{V_c^2}\right)(V_c - b) = RT_c \end{cases} \quad (\text{A2.8}),$$

which has the same solution (A2.4) and (A2.5).

**A3.** Numerical calculations for water produce the following result

$$a_w = 0.56 \frac{\text{m}^6 \cdot \text{Pa}}{\text{mole}^2} \quad (\text{A3.1}).$$

$$b_w = 3.1 \cdot 10^{-5} \frac{\text{m}^3}{\text{mole}} \quad (\text{A3.2}).$$

**A4.** From equations (A1.4) and (A3.2) it is found that

$$d_w = \sqrt[3]{\frac{b}{N_A}} = 3.7 \cdot 10^{-10} \text{ m} \approx 4 \cdot 10^{-10} \text{ m} \quad (\text{A4.1}).$$

#### Part B. Properties of gas and liquid

**B1.** Using the inequality  $V_G \gg b$ , the van der Waals equation of state can be written as

$$\left(p_0 + \frac{a}{V_G^2}\right)V_G = RT \quad (\text{B1.1}),$$

which has the following solutions

$$V_G = \frac{RT}{2p_0} \left(1 \pm \sqrt{1 - \frac{4ap_0}{R^2 T^2}}\right) \quad (\text{B1.2}).$$

Smaller root in (B1.2) gives the volume in an unstable state on the rising branch of the van der Waals isotherm. The volume of gas is given by the larger root, since at  $a = 0$  an expression for the volume of an ideal gas should be obtained, i.e.

$$V_G = \frac{RT}{2p_0} \left( 1 + \sqrt{1 - \frac{4ap_0}{R^2T^2}} \right) \quad (\text{B1.3}).$$

For given values of the parameters the value  $\frac{ap_0}{(RT)^2} = 5.8 \cdot 10^{-3}$ . It can therefore be assumed that  $\frac{ap_0}{(RT)^2} \ll 1$ , then (B1.3) takes the form

$$V_G \approx \frac{RT}{p_0} \left( 1 - \frac{ap_0}{R^2T^2} \right) = \frac{RT}{p_0} - \frac{a}{RT} \quad (\text{B1.4}).$$

**B2.** For an ideal gas

$$V_{G0} = \frac{RT}{p_0} \quad (\text{B2.1}),$$

hence,

$$\left( \frac{\Delta V_G}{V_{G0}} \right) = \frac{V_{G0} - V_G}{V_{G0}} = \frac{1}{2} \left( 1 - \sqrt{1 - \frac{4ap_0}{R^2T^2}} \right) \approx \frac{ap_0}{R^2T^2} = 0.58\%. \quad (\text{B2.2})$$

**B3.** Mechanical stability of a thermodynamic system is in power provided that

$$\left( \frac{dP}{dV} \right)_T < 0. \quad (\text{B3.1})$$

The minimum volume, in which the matter can still exist in the gaseous state, corresponds to a point in which

$$V_{Gmin} \rightarrow \left( \frac{dP}{dV} \right)_T = 0 \quad (\text{B3.2}).$$

Using the van der Waals equation of state (B3.2) is written as

$$\left( \frac{dP}{dV} \right)_T = -\frac{RT}{(V-b)^2} + \frac{2a}{V^3} = 0 \quad (\text{B3.3}).$$

From (B3.2) and (B3.3), and with the help of  $V_{Gmin} \gg b$ , it is found that

$$V_{Gmin} = \frac{2a}{RT} \quad (\text{B3.4}).$$

Thus,

$$\frac{V_G}{V_{Gmin}} = \frac{R^2T^2}{2ap_0} = 86 \quad (\text{B3.5}).$$

**B4.** Using the inequality  $P \ll a/V^2$ , the van der Waals equation of state is written as

$$\frac{a}{V_L^2} (V_L - b) = RT, \quad (\text{B4.1})$$

whose solution is

$$V_L = \frac{a}{2RT} \left( 1 \pm \sqrt{1 - \frac{4bRT}{a}} \right) \quad (\text{B4.2}).$$

In this case, the smaller root should be taken, since at  $T \rightarrow 0$  the liquid volume  $V_L = b$  must be obtained according to (B4.1), i.e.

$$V_L = \frac{a}{2RT} \left( 1 - \sqrt{1 - \frac{4bRT}{a}} \right) \approx b \left( 1 + \frac{bRT}{a} \right). \quad (\text{B4.3}).$$

**B5.** Since (B4.3) gives the volume of the one mole of water its mass density is easily found as

$$\rho_L = \frac{\mu}{V_L} = \frac{\mu}{b(1 + \frac{bRT}{a})} \approx \frac{\mu}{b} = 5.8 \cdot 10^2 \frac{\text{kg}}{\text{m}^3} \quad (\text{B5.1}).$$

**B6.** In accordance with (B4.3) the volume thermal expansion coefficient is derived as

$$\alpha = \frac{1}{V_L} \frac{\Delta V_L}{\Delta T} = \frac{bR}{a + bRT} \approx \frac{bR}{a} = 4.6 \cdot 10^{-4} \text{K}^{-1} \quad (\text{B6.1}).$$

**B7.** The heat, required to convert the liquid to gas, is used to overcome the intermolecular forces that create negative pressure  $a/V^2$ , therefore,

$$E = L\mu \approx \int_{V_L}^{V_G} \frac{a}{V^2} dV = a \left( \frac{1}{V_L} - \frac{1}{V_G} \right) \quad (\text{B7.1}),$$

and using  $V_G \gg V_L$ , (B7.1) yields

$$L = \frac{a}{\mu V_L} = \frac{a}{\mu b \left( 1 + \frac{bRT}{a} \right)} \approx \frac{a}{\mu b} = 1.0 \cdot 10^6 \frac{\text{J}}{\text{kg}} \quad (\text{B7.2}).$$

**B8.** Consider some water of volume  $V$ . To make a monolayer of thickness  $d$  out of it, the following work must be done

$$A = 2\sigma S \quad (\text{B8.1}).$$

Fabrication of the monomolecular layer may be interpreted as the evaporation of an equivalent volume of water which requires the following amount of heat

$$Q = Lm \quad (\text{B8.2}),$$

where the mass is given by

$$m = \rho S d \quad (\text{B8.3}).$$

Using (A4.1a), (B5.1) and (B7.2), one finally gets

$$\sigma = \frac{a}{2b^2} d_w = 0.12 \cdot 10^{-2} \frac{\text{N}}{\text{m}} \quad (\text{B8.4}).$$

### Part C. Liquid-gas systems

**C1.** At equilibrium, the pressure in the liquid and gas should be equal at all depths. The pressure  $p$  in the fluid at the depth  $h$  is related to the pressure of saturated vapor above the flat surface by

$$p = p_0 + \rho_L g h \quad (\text{C1.1}).$$

The surface tension creates additional pressure defined by the Laplace formula as

$$\Delta p_L = \frac{2\sigma}{r} \quad (\text{C1.2}).$$

The same pressure  $p$  in the fluid at the depth  $h$  depends on the vapor pressure  $p_h$  over the curved liquid surface and its radius of curvature as

$$p = p_h + \frac{2\sigma}{r} \quad (\text{C1.3}).$$

Furthermore, the vapor pressure at different heights are related by

$$p_h = p_0 + \rho_S g h \quad (\text{C1.4}).$$

Solving (C1.1)-(C1.4), it is found that

$$h = \frac{2\sigma}{(\rho_L - \rho_S) g r} \quad (\text{C1.5}).$$

Hence, the pressure difference sought is obtained as

$$\Delta p_T = p_h - p_0 = \rho_S g h = \frac{2\sigma}{r} \frac{\rho_S}{\rho_L - \rho_S} \approx \frac{2\sigma}{r} \frac{\rho_S}{\rho_L}. \quad (\text{C1.6}).$$

Note that the vapor pressure over the convex surface of the liquid is larger than the pressure above the flat surface.

**C2.** Let  $P_e$  be vapor pressure at a temperature  $T_e$ , and  $P_e - \Delta P_e$  be vapor pressure at a temperature  $T_e - \Delta T_e$ . In accordance with equation (3) from problem statement, when the ambient temperature falls by an amount  $\Delta T_e$  the saturated vapor pressure changes by an amount

$$\Delta P_e = P_e \frac{a}{b R T_e^2} \Delta T_e \quad (\text{C2.1}).$$

In accordance with the Thomson formula obtained in part C1, the pressure of saturated vapor above the droplet increases by the amount of  $\Delta p_T$ . While a droplet is small in size, the vapor above its surface remains unsaturated. When a droplet has grown up to a certain minimum size, the vapor above its surface turns saturated.

Since the pressure remains unchanged, the following condition must hold

$$P_e - \Delta P_e + \Delta p_T = P_e \quad (\text{C2.2}).$$

Assuming the vapor is almost ideal gas, its density can be found as

$$\rho_S = \frac{\mu P_e}{R T_e} \ll \rho_L \quad (\text{C2.3}).$$

From equations (C2.1)-(C2.3), (B5.1) and (C1.6) one finds

$$\frac{2\sigma}{r} \frac{\mu P_e}{R T_e} = P_e \frac{a \Delta T_e}{b R T_e^2} \quad (\text{C2.4}).$$

Thus, it is finally obtained that

$$r = \frac{2\sigma b^2 T_e}{a \Delta T_e} = 1.5 \cdot 10^{-8} \text{ m} \quad (\text{C2.5}).$$



### Problem 3. Simplest model of gas discharge

#### Solution

#### Part A. Non-self-sustained gas discharge

**A1.** Let us derive an equation describing the change of the electron number density with time. It is determined by the two processes; the generation of ion pairs by external ionizer and the recombination of electrons with ions. At ionization process electrons and ions are generated in pairs, and at recombination process they disappear in pairs as well. Thus, their concentrations are always equal at any given time, i.e.

$$n(t) = n_e(t) = n_i(t) \quad (\text{A1.1}).$$

Then the equation describing the number density evolution of electrons and ions in time can be written as

$$\frac{dn(t)}{dt} = Z_{ext} - rn(t)^2 \quad (\text{A1.2}).$$

It is easy to show that at  $t \rightarrow 0$  the function  $\tanh bt \rightarrow 0$ , therefore, by virtue of the initial condition  $n(0) = 0$ , one finds

$$n_0 = 0 \quad (\text{A1.3}).$$

Substituting  $n_e(t) = a \tanh bt$  in (A1.2) and separating it in the independent functions (hyperbolic, or 1 and  $e^x$ ), one gets

$$a = \sqrt{\frac{Z_{ext}}{r}} \quad (\text{A1.4}),$$

$$b = \sqrt{rZ_{ext}} \quad (\text{A1.5}).$$

**A2.** According to equation (A1.4) the number density of electrons at steady-state is expressed in terms of the external ionizer activity as

$$n_{e1} = \sqrt{\frac{Z_{ext1}}{r}} \quad (\text{A2.1}),$$

$$n_{e2} = \sqrt{\frac{Z_{ext2}}{r}} \quad (\text{A2.2}),$$

$$n_e = \sqrt{\frac{Z_{ext1} + Z_{ext2}}{r}} \quad (\text{A2.3}).$$

Thus, the following analogue of the Pythagorean theorem is obtained as

$$n_e = \sqrt{n_{e1}^2 + n_{e2}^2} = 20.0 \cdot 10^{10} \text{ cm}^{-3}. \quad (\text{A2.4})$$

**A3.** In the steady state, the balance equations of electrons and ions in the tube volume take the form

$$Z_{ext}SL = rn_en_iSL + \frac{I_e}{e} \quad (\text{A3.1}),$$

$$Z_{ext}SL = rn_en_iSL + \frac{I_i}{e} \quad (\text{A3.2}).$$

It follows from equations (A3.1) and (A3.2) that the ion and electron currents are equal, i.e.

$$I_e = I_i \quad (\text{A3.3}).$$

At the same time the total current in each tube section is the sum of the electron and ion currents

$$I = I_e + I_i \quad (\text{A3.4}).$$

By definition of the current density the following relations hold

$$I_e = \frac{I}{2} = en_e vS = e\beta n_e ES \quad (\text{A3.5}),$$

$$I_i = \frac{I}{2} = en_i vS = e\beta n_i ES \quad (\text{A3.6}).$$

Substituting (A3.5) and (A3.6) into (A3.1) and (A3.2), the following quadratic equation for the current is derived

$$Z_{ext}SL = rSL \left( \frac{I}{2e\beta ES} \right)^2 + \frac{I}{2e} \quad (\text{A3.7}).$$

The electric field strength in the gas is equal to

$$E = \frac{U}{L} \quad (\text{A3.8}).$$

and solution to the quadratic equation (A3.7) takes the form

$$I = \frac{e\beta^2 U^2 S}{rL^3} \left( -1 \pm \sqrt{1 + \frac{4rZ_{ext}L^4}{\beta^2 U^2}} \right) \quad (\text{A3.9}).$$



It is obvious that only positive root does make sense, i.e.

$$I = \frac{e\beta^2 U^2 S}{rL^3} \left( \sqrt{1 + \frac{4rZ_{ext}L^4}{\beta^2 U^2}} - 1 \right) \quad (\text{A3.10}).$$

**A4.** At low voltages (A3.10) simplifies and gives the following expression

$$I = 2Ue\beta \sqrt{\frac{Z_{ext}}{r}} \frac{S}{L}. \quad (\text{A4.1})$$

which is actually the Ohm law.

Using the well-known relation

$$R = \frac{U}{I} \quad (\text{A4.2})$$

together with

$$R = \rho \frac{L}{S} \quad (\text{A4.3}),$$

one gets

$$\rho = \frac{1}{2e\beta} \sqrt{\frac{r}{Z_{ext}}} \quad (\text{A4.4}).$$

### Part B. Self-sustained gas discharge

**B1.** Consider a gas layer located between  $x$  and  $x + dx$ . The rate of change in the electron number inside the layer due to the electric current is given for a small time interval  $dt$  by

$$dN_e^I = \frac{I_e(x+dx) - I_e(x)}{e} dt = \frac{1}{e} \frac{dI_e(x)}{dx} dx dt. \quad (\text{B1.1}).$$

This change is due to the effect of the external ionization and the electron avalanche formation.

The external ionizer creates the following number of electrons in the volume  $Sdx$

$$dN_e^{ext} = Z_{ext} S dx dt \quad (\text{B1.2}).$$

whereas the electron avalanche produces the number of electrons found as

$$dN_e^a = \alpha N_e dl = n_e S dx v dt = \alpha \frac{I_e(x)}{e} dx dt \quad (\text{B1.3}).$$

The balance equation for the number of electrons is written as

$$dN_e^I = dN_e^{ext} + dN_e^a \quad (\text{B1.4}),$$

which results in the following differential equation for the electron current

$$\frac{dI_e(x)}{dx} = eZ_{ext} S + \alpha I_e(x) \quad (\text{B1.5}).$$

On substituting  $I_e(x) = C_1 e^{A_1 x} + A_2$ , one derives

$$A_1 = \alpha \quad (\text{B1.6}),$$

$$A_2 = -\frac{eZ_{ext} S}{\alpha} \quad (\text{B1.7}).$$

**B2.** Given the fact that the ions flow in the direction opposite to the electron motion, the balance equation for the number of ions is written as

$$dN_i^I = dN_i^{ext} + dN_i^a \quad (\text{B2.1}),$$

where

$$dN_i^I = \frac{I_i(x) - I_i(x+dx)}{e} dt = -\frac{1}{e} \frac{dI_i(x)}{dx} dx dt \quad (\text{B2.2}).$$

$$dN_i^{ext} = Z_{ext} S dx dt \quad (\text{B2.3}).$$

$$dN_i^a = \alpha \frac{I_e(x)}{e} dx dt \quad (\text{B2.4}).$$

Hence, the following differential equation for the ion current is obtained

$$-\frac{dI_i(x)}{dx} = eZ_{ext} S + \alpha I_e(x). \quad (\text{B2.5})$$

On substituting the previously found electron current together with the ion current,  $I_i(x) = C_2 + B_1 e^{B_2 x}$ , yields

$$B_1 = -C_1 \quad (\text{B2.6}),$$

$$B_2 = \alpha \quad (\text{B2.7}).$$

**B3.** Since the ions start to move from the anode located at  $x = L$ , the following condition holds

$$I_i(L) = 0 \quad (\text{B3.1}).$$

**B4.** By definition of secondary electron emission coefficient the following condition should be imposed

$$I_e(0) = \gamma I_i(0) \quad (\text{B4.1}).$$

**B5.** Total current in each tube section is the sum of the electron and ion currents:

$$I = I_e + I_i = C_2 - \frac{eZ_{ext}S}{\alpha} \quad (B5.1).$$

After substituting the boundary conditions (B3.1) and (B4.1):

$$C_2 - C_1 e^{\alpha L} = 0 \quad (B5.2)$$

and

$$C_1 - \frac{eZ_{ext}S}{\alpha} = \gamma(C_2 - C_1) \quad (B5.3).$$

Solving (B5.2) and (B5.3) one can obtain:

$$C_2 = \frac{eZ_{ext}S}{\alpha} \left( \frac{1}{e^{-\alpha L}(1+\gamma)-\gamma} \right) \quad (B5.4).$$

So the total current:

$$I = \frac{eZ_{ext}S}{\alpha} \left( \frac{1}{e^{-\alpha L}(1+\gamma)-\gamma} - 1 \right) \quad (B5.5).$$

**B6.** When the discharge gap length is increased, the denominator in formula (B5.1) decreases. At that moment, when it turns zero, the electric current in the gas becomes self-sustaining and external ionizer can be turned off. Thus,

$$L_{cr} = \frac{1}{\alpha} \ln \left( 1 + \frac{1}{\gamma} \right) \quad (B6.1).$$