Solutions to Jaan Kalda's Problems in Mechanics

With detailed diagrams and walkthroughs

Edition 1.2.0

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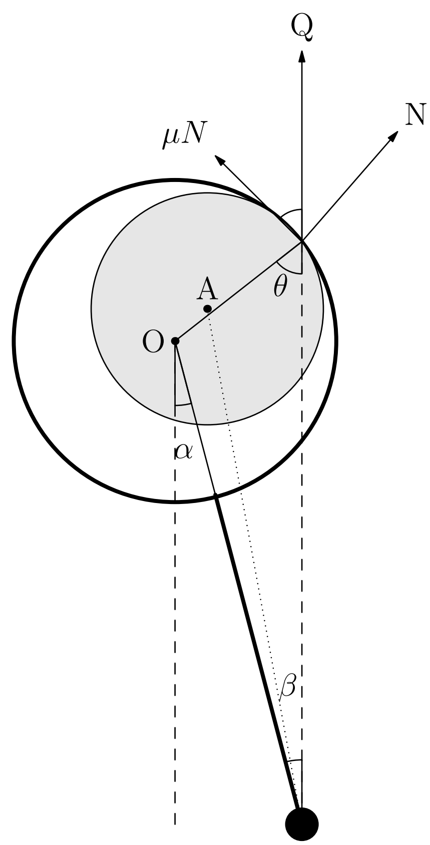

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Preface

Jaan Kalda's handouts are beloved by physics students both in for a quick challenge, to students preparing for international Olympiads. As of writing, the current mechanics handout (ver 1.2) has 86 unique problems and 74 main 'ideas'.

This solutions manual came as a pilot project from the online community at artofproblemsolving.com. Although there were detailed hints provided, full solutions have never been written. The majority of the solutions seen here were written on a private forum given to those who wanted to participate in making solutions. In an amazing show of an online collaboration, students from around the world came together to discuss ideas and methods and created what we see today.

This project would not have been possible without the countless contributions from members of the community. Online usernames were used for those who did not wish to be named:

Hermab Podar, Ameya Deshmukh, Viraj Jayam, Rakshit, dbs27, Anant Lunia, Jai, Sean Chen, Killer_Instinct, Joshua S, Tarun Agarwal, c_deng

Structure of The Solutions Manual

Each chapter in this solutions manual will be directed towards a section given in Kalda's mechanics handout. There are three major chapters: statics, dynamics, and revision problems. If you are stuck on a problem, cannot make progress even with the hint, and come here for reference, look at only the start of the solution, then try again. Looking at the entire solution wastes the problem for you and ruins an opportunity for yourself to improve.

Contact Us

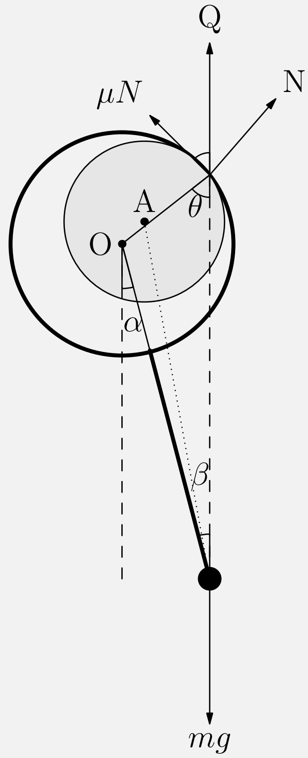
Despite editing, there is almost zero probability that there are *no* mistakes inside this book. If there are any mistakes, you want to add a remark, have a unique solution, or know the source of a specific problem, then please contact us at hello@physoly.tech. The most current and updated version can be found on our website physoly.tech

Please feel free to contact us at the same email if you are confused on a solution. Chances are that many others will have the same question as you.

1 Solutions to Statics Problems

This section will consist of the solutions to problems from problem 1-23 of the handout. Statics is typically the analysis of objects not in motion. However, objects travelling at constant velocity or with a uniform acceleration can be treated as a statics problem with a frame of reference change. This usually involves balancing forces, torques, and more to achieve equilibrium.

pr 1. The hardest thing about this problem, as Kalda noted, was drawing a diagram. Here we provide a diagram for us to work with. Let O be the center of the hoop and A the center of the revolving shaft.



Let Q be the vector sum of the friction and normal forces[a],

$$Q = \sqrt{\mu^2 N^2 + N^2} = N\sqrt{\mu^2+1}$$

because the system is in equilibrium, then the frictional force, μN, must be equal to $mg\sin\theta$. We also know by simple trigonometry that $\mu N = Q\sin\theta$. Therefore, because the sum of forces are zero we have,

$$\mu N = mg\sin\theta = N\sqrt{\mu^2+1}\sin\theta.$$

We must now establish this relation in terms of β. One may look towards a torque analysis, however a more elegant mathematical approach is by the law of sines. We know by law of sines that

$$\frac{\sin\beta}{r} = \frac{\sin\theta}{r+\ell} \implies \sin\theta = \frac{(r+\ell)\sin\beta}{r}$$

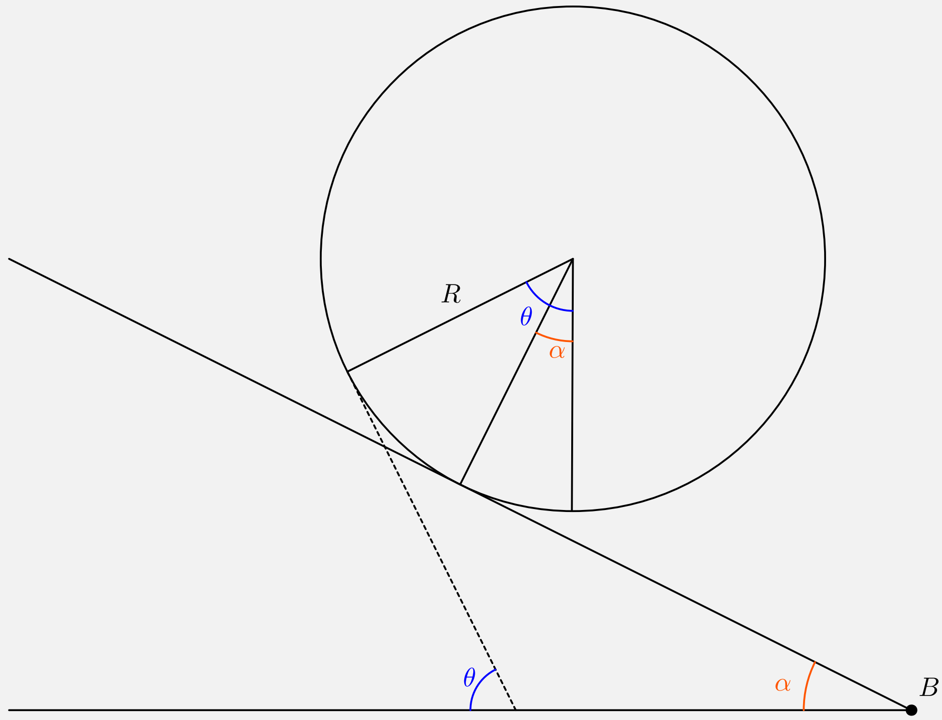
Substituting this in for $\sin\theta$ we find

$$\mu N = N\sqrt{\mu^2+1}\frac{(r+\ell)\sin\beta}{r}$$

$$\sin\beta = \frac{r\mu}{(r+\ell)\sqrt{\mu^2+1}} \implies \boxed{\beta = \sin^{-1}\left(\frac{r\mu}{(r+\ell)\sqrt{\mu^2+1}}\right)}$$

[a] The frictional force is not constant throughout the entire process of slipping however it is maximum (or μN) when the shaft is at equilibrium angle.

pr 2.



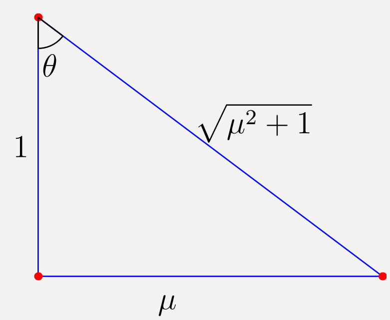
Let the angle formed from the mass, the center of the cylinder O, and the vertical be θ. By summing forces on the mass, we get

$$mg\sin\theta - \mu mg\cos\theta = 0 \implies \mu = \tan\theta.$$

This is unsurprising, as it is the typical condition for an object to not slip. You can verify yourself that the effective angle of the incline is equal to the angle the normal force makes with the vertical, θ. Next, we sum up the torques with respect to the contact point between the ramp and the cylinder. The moment arm for the cylinder is $R\sin\alpha$ and the moment arm for the block is $R\sin\theta - R\sin\alpha$. Therefore, we can write the torque balance equation as:

$$(M+m)g\sin\alpha = mg\sin\theta$$

Because $\tan\theta = \mu$, we have a right triangle that can be constructed:



Therefore, $\sin\theta = \frac{\mu}{\sqrt{\mu^2+1}}$. Substituting this result into our equation of sum of torques at point P gives us

$$(M+m)g\sin\alpha = mg\frac{\mu}{\sqrt{\mu^2+1}}$$

which implies the answer is

$$\boxed{\alpha = \arcsin\left(\frac{m}{M+m}\frac{\mu}{\sqrt{\mu^2+1}}\right)}$$

.

Solution 2: First, consider just the block in the cylinder. If the surface the block is on makes an angle $\mu = \tan\theta$ with the horizontal, then it will be on the verge of slipping. During static equilibrium, a tiny disturbance will not affect the total energy. We can represent the rotation in two steps. First, we purely translate the center of the cylinder by a distance $Rd\theta$. Then, we purely rotate the cylinder about the center by an angle $d\theta$. We can sum up the change in potential energy in these two steps and sum it to zero.

By translating the cylinder a downwards distance of $Rd\theta$ along the ramp, we are changing the potential energy by:

$$dU_{\text{cylinder}} = -(M+m)g(Rd\theta)(\sin\alpha)$$

Next up, we rotate the cylinder counterclockwise by an angle $d\theta$. This will cause the block to rise and increase its potential energy by:

$$dU_{\text{block}} = mgR\sin\theta d\theta$$

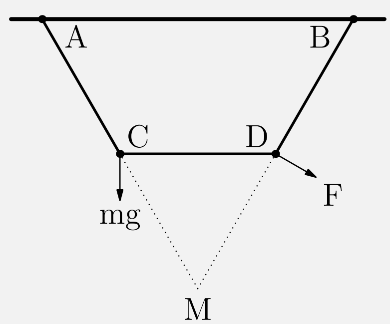
We know that the total energy change will be zero (since $\frac{dU}{d\theta} = 0$ at a local minimum) so we have:

$$0 = -(M+m)gR\sin\alpha d\theta + mgR\sin\theta d\theta$$
$$(M+m)\sin\alpha = m\sin\theta$$
$$\sin\alpha = \frac{m\sin\theta}{M+m}$$

We can determine $\sin\theta$ by using the fact that $\tan\theta = \mu$ which gives us:

$$\boxed{\alpha = \sin^{-1}\left(\frac{m\mu}{(M+m)\sqrt{\mu^2+1}}\right)}$$

pr 3.



Due to symmetry of the system and by the fact that $AB - CD = CD = AC = BD$, we see that

$$\angle CAB = \angle DBA = 60^\circ$$

such that $AC\cos\angle CAB = \frac{AC}{2}$. Now consider the net torque acting on rod CD about the meeting point of extended AC and DB, which we call M. The tension forces due to rods AC and DB pass through M and exert no torque. The only torques are due to mg and F. The torque due to mg is

$$mg\ell\sin 30^\circ = \frac{mg\ell}{2}$$

where $\ell \equiv MC = MD = CD$. For the minimum value of F, it must point perpendicular to MB and its value must be

$$\frac{mg\ell}{2\ell} = \boxed{\frac{mg}{2}}$$

By using symmetry, we can determine angle $\angle C = \angle D = 120°$ of the isosceles trapezoid. We will use fact 20 cited in the handout which in brief states that if a mass-less rod is freely hinged at both ends, the force at the hinge must point along the rod. This is the only way for the torque to be zero.
Therefore, the rod AC must provide an upwards force of mg. By using the given angle above and breaking the force up into its components, we can see that the horizontal force is $mg\tan 30°$. This must also be the horizontal force the rod exerts on hinge D due to Newton's third law. Therefore, we know that the vertical forces BD and F exerts on D has to sum up to zero and their horizontal forces have to sum up to $mg/2$. Therefore, we have:

$$F\cos\theta + T\sin 30° = mg\tan 30°$$

and

$$F\sin\theta = T\cos 30°$$

Combining them together to remove T gives:

$$F\left(\cos\theta + \frac{\sin\theta}{\sqrt{3}}\right) = mg/2$$

You can minimize F by taking the derivative or you can recognize that the second part of the left hand side reaches a maximum of

$$\sqrt{1^2 + \left(\frac{1}{\sqrt{3}}\right)^2} = \frac{2}{\sqrt{3}}$$

so the minimum F is achieved when this value is reached or:

$$F\left(\frac{2}{\sqrt{3}}\right) = \frac{mg}{\sqrt{3}}$$
$$F = \frac{mg}{2}$$

The same problem was presented in the book *Problems in Physics* by SS Krotov.

pr 4. Let's just tackle part B straight away. Part A follows the same reasoning and can be derived through part B. The normal force is given by:

$$N = mg\cos\alpha - F\sin\theta$$

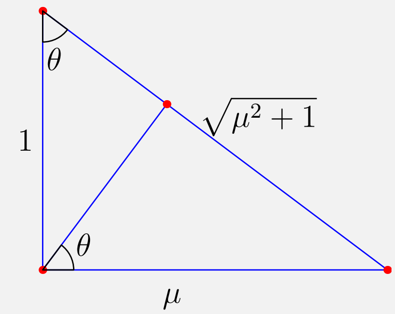
While adding forces in the direction along the plane gives (assuming maximum friction):

$$F\cos\theta + mg\sin\alpha - \mu N = 0$$

Substituting our expression for N we can write F in terms of all our variables:

$$F = \frac{mg(\mu\cos\alpha - \sin\alpha)}{\cos\theta + \mu\sin\theta}$$

We can determine when F is at a minimum when the denominator is at a maximum. Taking the derivative of $\cos\theta + \mu\sin\theta$ and setting it to zero gives the minimum F when $\tan\theta = \mu$.



Using the large triangle, we can see that in this particular set-up, we have $\tan\theta = \mu$. Using the two smaller triangles, we can express the hypotenuse as $\cos\theta + \mu\sin\theta$, the hypotenuse of the expression above which happens to be $\sqrt{\mu^2+1}$ as well. Therefore, for the minimum force we can re-write it as:

$$\boxed{F_{\text{min}} = \frac{mg(\mu\cos\alpha - \sin\alpha)}{\sqrt{\mu^2+1}}}$$

which is equivalent (though not obviously) the same as the given answer. If we replace $\alpha = 0$ we get:

$$F_{\text{min}} = mg \cdot \frac{\mu}{\sqrt{\mu^2+1}}$$

In the solution given above, we tried to find the maximum value of $\cos\theta + \mu\sin\theta$ by taking the derivative, then substituting it back in with a clever triangle. However, we can trivialize this step if we know that given:

$$f(x) = A\cos x + B\sin x$$

the maximum value for $f(x)$ is:

$$\sqrt{A^2+B^2}$$

which will give the intended result of

$$\sqrt{1+\mu^2}$$

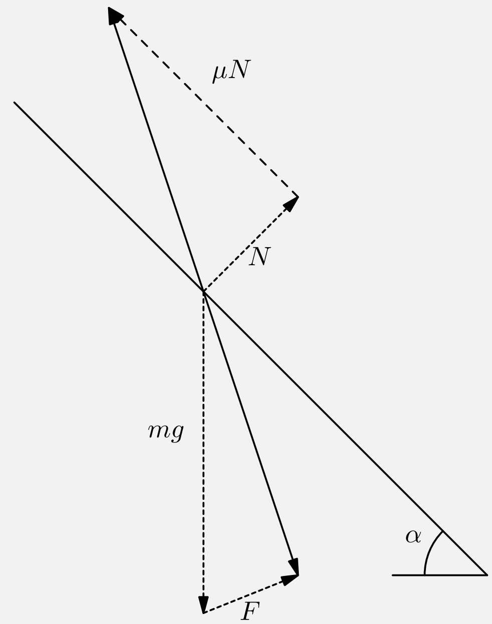
Solution 2: There are four forces: The gravitational force, the normal force, the applied force, and the friction force. We can break this up into two forces. Let

$$\vec{F_1} = \vec{N} + \vec{f}$$

and let

$$\vec{F_2} = \vec{F} + m\vec{g}$$

The angle F_1 makes with the perpendicular is $\theta = \tan^{-1}(\mu)$



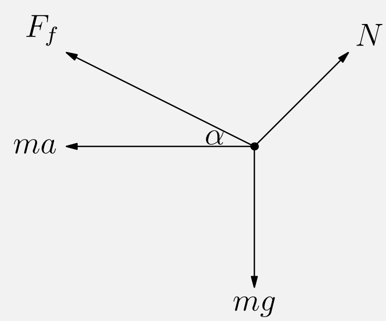
Secondly, F_2 must also fall upon the same line as F_1. In order for F to be drawn such that it meets this requirement *and* F is at a minimum, we want F to be perpendicular to the ray F_1 makes. Doing some angle tracing gives us $\phi = \theta - \alpha$ as the angle between mg and the ray formed by F_1.

Separating $\vec{F}_2$ into its components gives:

$$\boxed{F = mg\sin\phi = mg\sin\left(\tan^{-1}(\mu) - \alpha\right)}$$

This is equivalent to the answer given above.

pr 5. In the frame of the plane, the free body diagram of the block is



Analyzing the forces involved, we see that for the block to remain still, we must have

$$F_f = mg\sin\alpha - ma\cos\alpha$$
$$N = mg\cos\alpha + ma\sin\alpha$$

Because the normal force must be greater than zero, we have that

$$N > 0 \implies g\cos\alpha + a\sin\alpha > 0$$
$$g + a\tan\alpha > 0$$

We also have that, since the frictional force must be less than or equal to μN that

$$f \leq \mu N \implies \frac{g\sin\alpha - a\cos\alpha}{g\cos\alpha + a\sin\alpha} \leq \mu$$

If friction acts in the opposite direction, we then have

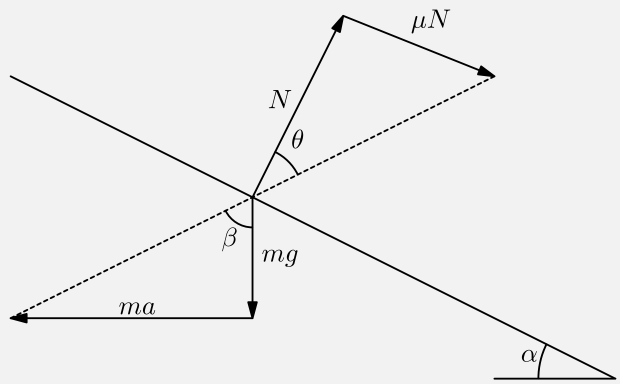
$$-\frac{g\sin\alpha - a\cos\alpha}{g\cos\alpha + a\sin\alpha} \leq \mu$$

Therefore:

$$\boxed{\frac{|g\sin\alpha - a\cos\alpha|}{g\cos\alpha + a\sin\alpha} \leq \mu}$$

but only if $g + a\tan\alpha > 0$.

Solution 2: Here are two extreme scenarios that can happen. First, the plane can have a large acceleration and the block is just about to slip upwards. Second, the plane can have an acceleration just low enough such that it prevents the block from slipping downwards. Let us first focus on the first scenario.



Similar to problem 4, let us consider the four forces geometrically as a whole instead of via components. Moving into an accelerated reference frame, we introduce our fictitious force $f = ma$. The mg and ma vectors combine to give a single "effective" force. The normal and maximum static friction force combine to give us our second "effective" force. Now the problem becomes a static equilibrium problem when there are only two forces. This is a trivial case - they have to point in opposite directions. Geometrically, this occurs when:

$$\beta = \theta + \alpha$$

The angle θ is given by:

$$\tan\theta = \frac{N\mu}{N} = \mu$$

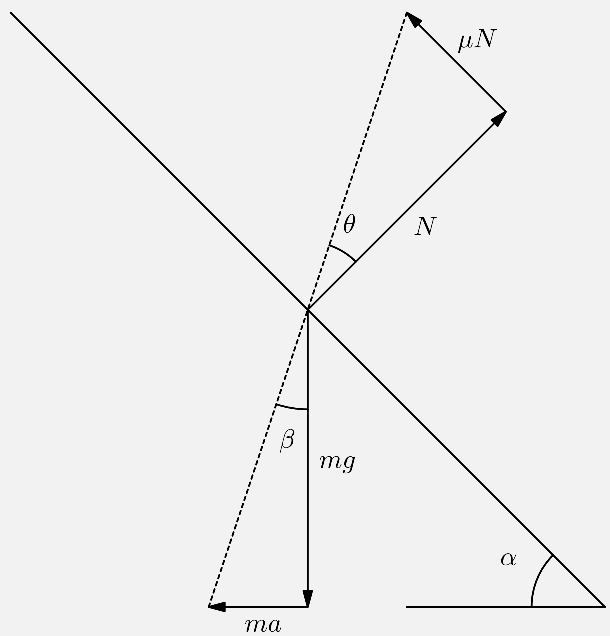
and the angle β relates ma and mg through:

$$\tan\beta = \frac{ma}{mg} \implies \tan(\tan^{-1}\mu + \alpha) = \frac{a}{g}$$

Solving for μ gives:

$$\mu_{\text{max}} = \tan\left(\tan^{-1}\left(\frac{a}{g}\right) - \alpha\right)$$

Now let's consider the case where friction is at a minimum and it is at a verge of slipping downwards.



Again, we pair the forces up as before. This time, the condition for equilibrium is:

$$\beta = \alpha - \theta$$

The value for θ remains constant so we can solve for μ to be:

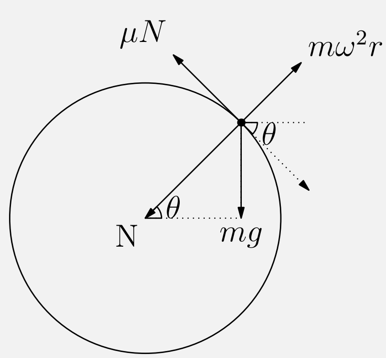
$$\mu_{\min} = \tan\left(\alpha - \tan^{-1}\left(\frac{a}{g}\right)\right)$$

Therefore, we have:

$$\boxed{\tan\left(\alpha - \tan^{-1}\left(\frac{a}{g}\right)\right) < \mu < \tan\left(\tan^{-1}\left(\frac{a}{g}\right) - \alpha\right)}$$

While the answers for these two solutions are very different, they are actually equivalent!

pr 6. a) We examine the forces involved in a cross-section of the cylinder. Assuming the block behaves like a point mass, and noting there is a centrifugal force, we create following diagram



Because the system is in equilibrium we must set the resultant force to be zero in both directions. We assume a tilted coordinate of θ to perform our calculations on. In the vertical direction we have

$$0 = N + mg\sin\theta - m\omega^2 r$$

this in turn implies that the normal force is

$$N = m\omega^2 r - mg\sin\theta.$$

Looking in the horizontal direction we note that

$$\mu N - mg\cos\theta = 0$$

$$mg\cos\theta = \mu N$$

However, we remember that μN is the maximum amount of friction obtained from slipping, thus we have to put a less than or equal sign to obtain

$$mg\cos\theta \leq \mu N$$

substituting in N from our previous calculation we have

$$mg\cos\theta \leq \mu(m\omega^2 r - mg\sin\theta)$$

moving variables to the other side and canceling out m gives

$$\omega^2 r \geq g(\mu^{-1}\cos\theta + \sin\theta)$$

Our goal is to now to find a maximal value of $\mu^{-1}\cos\theta + \sin\theta$ on the interval $[0, 2\pi]$. It is known that a sinusoid $A\cos\theta + B\sin\theta$, can be represented as a single trigonometric function:

$$A\cos\theta + B\sin\theta = \sqrt{A^2 + B^2} \cdot \cos(\theta + \phi)$$

From these expressions of 1 sinusoid, it is clear the maximum value is $\sqrt{A^2 + B^2}$, giving the maximum of $\mu^{-1}\cos\theta + \sin\theta$ as $\sqrt{1 + \mu^{-2}}$. Thus replacing this value in for our final expression gives us

$$\boxed{\omega^2 r \geq g(\sqrt{\mu^{-2} + 1})}$$

b) In this part we work with cylindrical coordinates. We decompose gravity upon two axes. If we rotate the cylinder by α we have

$$g_z = g\sin\alpha$$
$$g_{r,\theta} = g\cos\alpha$$

All we do now is plug in g_{eff} for our two equations. For our radial equation we had

$$N = m\omega^2 r - mg\sin\theta$$

Since the normal force is radial we use $g_{r,\theta} = g\cos\alpha$ we plug in for gravity to get

$$N = m\omega^2 r - mg\cos\alpha\sin\theta$$

In our second equation who have two components of gravity, F_θ and F_z, who's combined modulus must be less than friction or μN.

$$\sqrt{F_\theta^2 + F_r^2} \leq \mu N$$

$$\sqrt{(mg\cos\alpha\cos\theta)^2 + (mg\sin\alpha)^2} \leq \mu(m\omega^2 r - mg\cos\alpha\sin\theta)$$

Taking out m and factoring we have

$$\boxed{\omega^2 r \geq g\cos\alpha(\sqrt{\cos^2\theta + \tan^2\alpha} + \mu\sin\theta)}$$

Again we must maximize our right hand equation. Inevitably, there is no neat trick to maximize this apart from differentiating and setting the result to zero.

pr 7. The center of mass of the object can be calculated by treating the wheel as a superposition of two objects, one with positive density ρ and one with negative mass density $-\rho$. Taking $r = 0$ to be the center, the center of mass is:

$$\begin{aligned} r_{\text{cm}} &= \frac{\rho\pi R^2(0) - \rho\pi\left(\frac{R}{2}\right)^2\left(\frac{R}{3}\right)}{\rho\pi R^2 - \rho\pi\left(\frac{R}{2}\right)^2} \\ &= -\frac{\frac{1}{4}\left(\frac{R}{3}\right)}{1 - 1/4} \\ &= -R/9 \end{aligned}$$

Therefore, when the normal force is zero, we have:

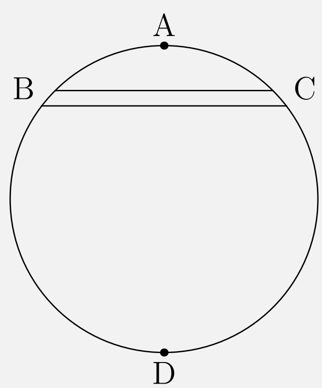
$$\begin{aligned} m\omega^2(R/9) &= mg \\ \omega &= 3\sqrt{g/R} \end{aligned}$$

And therefore the speed would be

$$\boxed{v = 3\sqrt{gR}}$$

pr 8. **a)** Let the point where the rope meets the cylinder be A, and the two points where friction band meets the cylinder be B and C. Let D be the point diagonally opposite A.

Claim. D is the instantaneous centre of rotation (ICOR).



Proof. Let us assume a contradictory case. Let D^* be the ICOR. Since the velocity of point A is perpendicular to AD, D^* must lie somewhere on AD. The velocities of B and C are perpendicular to DB and DC (due to definition of ICOR), and the friction forces are anti-parallel to these. The only forces acting on the cylinder is the tension T due to the rope, and the two friction forces. As the cylinder is in equilibrium, by setting torque to be 0 about the point where the two friction vectors intersect, we see that the tension vector must also pass through it. However, due to symmetry, the point of intersection must lie on AD and thus it must be A itself. Thus, $\angle ABD^* = \angle ACD^* = 90°$. Therefore this means that $ABCD^*$ is cyclic, which implies $D^* \equiv D$.

Now let the angular velocity about D be ω. The velocity of A is

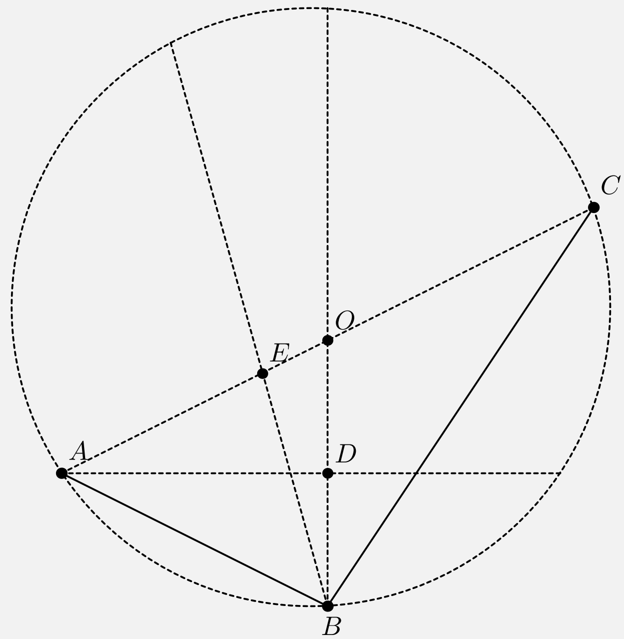
$$v = \omega \times 2R$$

and the velocity at the centre is:

$$\boxed{v_{\text{center}} = \omega \times R = \frac{v}{2}}$$

b) Dividing the floor into various infinitesimally thin strips like in a), we can conclude that the ICOR is still D and the answer remains the same.

pr 9. We shall use a property in geometry. Thales's theorem states that if A, B, and C are distinct points on a circle where the line AC is a diameter, then the angle $\angle ABC$ is a right angle.



Therefore if we draw a circle where the corners of the two pillars form the ends of the diameter AC, the outline of the circle gives the possible locations the mass can be located as. Let the location of the mass be B. We wish to minimize the height of B which so happens to be at the very bottom of the circle. Let $\angle EBD = \alpha$ such that $\angle ABE = 45°$. Doing some angle tracing, we can verify that

$$\angle BAD = 45° - \alpha$$

Now since OA and OB are both the radius, that means OAB is an isosceles triangle where:

$$\angle OAB = \angle ABO \implies 45° - \alpha + \angle OAD = 45° + \alpha \implies \angle OAD = 2\alpha$$

This angle relates the horizontal distance of the two pillars and the vertical distance of the two pillars through:

$$\tan OAD = \boxed{\tan(2\alpha) = \frac{h}{a}}$$

Solution 2: Let y be the vertical distance between the mass and the top of the left pillar. Then let b and c be the horizontal distances between the mass and the left and right pillars, respectively, such that $a = b + c$. Doing basic angle tracing, we can see that:

$$b = \frac{y}{\tan(45 - \alpha)}$$

and

$$c = (h + y)\tan(45 - \alpha)$$

Adding them together and letting $\beta \equiv 45 - \alpha$ yields:

$$\begin{aligned}
a &= b + c \\
a &= \frac{y}{\tan(\beta)} + (h + y)\tan(\beta) \\
a\tan(\beta) &= y + (h + y)\tan^2(\beta) \\
a\tan(\beta) - h\tan^2(\beta) &= y + y\tan^2(\beta) \\
\frac{\tan(\beta)(a - h\tan(\beta))}{1 + \tan^2(\beta)} &= y
\end{aligned}$$

Doing a quick sanity check, this yields the correct answer of $y = a/2$ when $\beta = 45°$ and $h = 0$
We can simplify this further with a few trig identities. You can verify that the above expression is equivalent to

$$y = \frac{1}{2}a\sin(90 - 2\alpha) - \frac{h}{2}\tan(45 - \alpha)$$

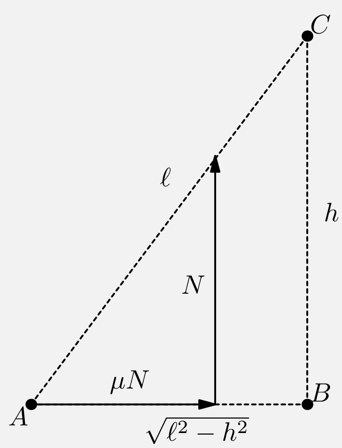
From the energy approach, the system will be in static equilibrium if no work is needed to rotate the system by a differential amount (change in potential energy is zero). This occurs when the gravitational potential energy is at a minimum or y is minimized. Taking the derivative with respect to α we get:

$$\begin{aligned}
\frac{dy}{d\alpha} &= \frac{1}{2}a\cos(90 - 2\alpha)(-2) - (2h\sin(45 - \alpha))(\cos(45 - \alpha)(-1) \\
0 &= -a\cos(90 - 2\alpha) + h\sin(90 - 2\alpha) \\
\frac{a}{h} &= \tan(90 - 2\alpha)
\end{aligned}$$

But since $\tan(90 - 2\alpha) = \cot(2\alpha)$, we can rewrite this to get:

$$\tan(2\alpha) = \frac{h}{a}$$

pr 10.



Consider what happens when the applied force approaches infinity. To maintain equilibrium, the friction force between the rod and the board must also increase. This friction force will also approach infinity. When dealing with large forces, we can ignore constant forces such as the weight of both the board and the rod.

As a result, since the weight of the rod is negligible we can pretend it's a mass-less rod. We also know that the forces at the ends of a massless rod will always point along the rod. For example, the force exerted on the rod by the board must point along the rod as well. The angle of this force is solely dependent on the friction coefficient μ_1. Therefore:

$$\tan\alpha < \frac{\mu_1 N}{N} \implies \boxed{\mu_1 > \frac{\sqrt{\ell^2 - h^2}}{h}}$$

Solution 2: We want that when the board is on the verge of slipping then the rod should exert a larger force on the board (the rod should be pulled towards the board and not away from it). Consider the torque on the rod about the hinge point. We want that it should be clockwise when the block is on the verge of slipping.

Let the sum of normal reaction and friction force on the rod be f (the normal points upwards and the friction points to the right). When the block is on the verge of slipping, the resultant makes an angle $\tan^{-1}\mu$ from the normal. We have:

$$\tau = mg\sin\alpha\frac{l}{2} + f\sin(\tan^{-1}\mu - \alpha)$$

considering clockwise torque to be positive. As the applied force on the block increases, f also increases without bounds and because we want the torque to be clockwise no matter how much force we apply, the mg term can be neglected. So

$$f\sin(\tan^{-1}\mu - \alpha) \geq 0$$

Since both $\tan^{-1}\mu$ and α are less than 90°, we can conclude that

$$\boxed{\tan^{-1}\mu \geq \alpha \implies \mu \geq \frac{\sqrt{l^2 - h^2}}{h}}$$

pr 11. We will use a virtual work approach.[a] In a static situation, the net force will be zero and as a result the potential energy will be at a minimum. Any slight displacement will create no change to the potential energy in first order.

Consider what happens when the mass is lowered by a distance dh. The potential energy would drop by $-mgdh$. The distance between hinges would each increase by $dh/3$ to compensate for the length increase. This means the string gets stretched by $dh/3$. The energy stored thus is:

$$Tdh/3$$

Setting these changes to zero gives:

$$-mgdh + Tdh/3 = 0 \implies \boxed{T = 3mg}$$

[a] If you are unfamiliar with virtual work, refer to the explanation in Kalda's handout, or to this pdf: http://www.ce.siue.edu/examples/Worked_examples_Internet_text-only/Data_files-Worked_Exs-Word_&_pdf/Virtual_work.pdf

pr 12. First, we'll look at the behavior of the tension at the bottom. The vertical component of the tension has to support the weight of the block so we have:

$$2T_{\text{bottom,y}} = 2T_{\text{bottom}} \cos(\beta/2) = Mg$$

The horizontal component is thus:

$$T_x = T_{\text{bottom}} \sin(\beta/2) = \frac{Mg}{2} \tan(\beta/2)$$

Notice that this horizontal tension force will be constant in a massive rope. If we look at a differential area of the string, the only other force other than tension is the gravitational force downwards. To balance horizontal forces, the horizontal components of tension have to be constant. At the top of the rope, the vertical component of the tension has to support the weight of the block and the string. We have:

$$2T_{\text{top,y}} = 2T_{\text{top}} \sin\alpha = (M+m)g$$

The horizontal component will thus be:

$$T_x = T_{\text{top}} \cos\alpha = \frac{(M+m)g}{2} \cot\alpha$$

Setting these two expressions for the horizontal tension equal gives:

$$M\tan(\beta/2) = (M+m)\cot\alpha \implies \boxed{\beta = 2\tan^{-1}\left(\left(1+\frac{m}{M}\right)\cot\alpha\right)}$$

pr 13. As Kalda notes, if the weight of a hanging part of a rope is much less than its tension then the curvature of the rope is small and its horizontal mass distribution can quite accurately be regarded as constant.

First, we use torques to find the tension in the y-direction on the rope. We sum torques from the hand that's pulling the rope.

$$\lambda g\ell\frac{l}{2} = T_y\ell \implies T_y = \frac{\lambda g\ell}{2}$$

We can sum forces to find the tension in the x-direction. Using the note we gave at the beginning of the solution, the curvature is small so we can approximate the rope that is above the ground as a straight line. Call the angle this makes with the ground θ. Therefore, we have

$$T_x = \mu g(L-\ell)\lambda\cos\theta$$

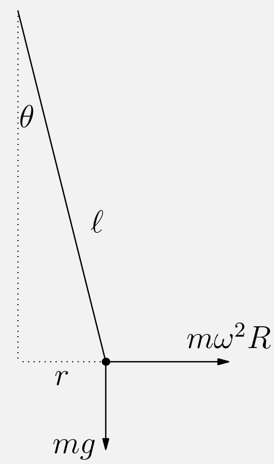
The relation $T_y/T_x = \tan\theta$ gives us that

$$\tan\theta = \frac{\lambda g\ell}{2\mu g(L-\ell)\lambda\cos\theta} \implies 2\sin\theta = \frac{\ell}{\mu(L-\ell)}$$

Recall that the rope above the ground is a straight line. Therefore, $\sin\theta = H/\ell$. Thus, we have after some simplification

$$2H\mu L - 2\mu H\ell = \ell^2 \implies \boxed{\ell = \sqrt{\mu^2H^2 + 2\mu HL} - \mu H \approx 7.169 \text{ m}}$$

pr 14. **a)** Let us draw the free body diagram:



We note by analysis of torques that in order for there to be a restoring torque, we must have:

$$mgr > m\omega^2 r\ell\cos\theta$$

We use the small angle approximation $\cos\theta = 1$ to then yield

$$g > \omega^2\ell$$

Thus

$$\boxed{\omega^2 < \frac{g}{\ell}}$$

b) Let the angles produced by the rods be φ_1 and φ_2 respectively. We then have the potential energy to be

$$V = -mgl\cos\varphi_1 - mgl(\cos\varphi_1 + \cos\varphi_2)$$

Using the small angle approximation $\cos\varphi = 1 - \frac{\varphi^2}{2}$ gives us

$$V = -mgl\left(1 - \frac{\varphi_1^2}{2}\right) - mgl\left(1 - \frac{\varphi_1^2}{2} + 1 - \frac{\varphi_2^2}{2}\right).$$

According to idea 19, we can remove all constants or non-quadratic terms

$$V_g = mgl\left(\varphi_1^2 + \frac{\varphi_2^2}{2}\right)$$

Finding the potential energy produced by the centrigal force will have a similar approach.

$$V = \frac{1}{2}m(l\sin\varphi_1\omega^2)^2 + \frac{1}{2}m(l(\sin\varphi_1 + \sin\varphi_2)\omega)^2$$

Using the small angle approximation $\sin\theta = \theta$ we have

$$V = \frac{1}{2}m(l\varphi_1\omega^2)^2 + \frac{1}{2}m(l(\varphi_1 + \varphi_2)\omega)^2$$

$$V_c = \frac{1}{2}m\omega^2 l^2(2\varphi_1^2 + 2\varphi_1\varphi_2 + \varphi_2^2)$$

The total potential energy is then

$$V = V_g + V_c = mgl\left(\varphi_1^2 + \frac{\varphi_2^2}{2}\right) + \frac{1}{2}m\omega^2 l^2(2\varphi_1^2 + 2\varphi_1\varphi_2 + \varphi_2^2)$$

Rearranging this, gives us the quadratic

$$V = (mgl - m\omega^2 l^2)\varphi_1^2 + (m\omega^2 l^2)\varphi_1\varphi_2 + \frac{1}{2}(mgl - m\omega^2 l^2)\varphi_2^2$$

The equilibrium $\varphi_1 = \varphi_2 = 0$ is stable if it corresponds to the potential energy minimum, i.e, if the polynomial yields positive values for any departure from the equilibrium point; this condition leads to two inequalities. First, upon considering $\varphi_2 = 0$ (with $\varphi_2 \neq 0$) we conclude that the multiplier of φ_1^2 (or $mgl - m\omega^2 l^2$) has to be positive. Second, for any $\varphi_2 \neq 0$, the polynomial should be strictly positive, i.e. if we equate this expression to zero and consider it as a quadratic equation for φ_1, there should be no real-valued roots, which means that the discriminant should be negative. Thus, by looking at our discriminant we find that

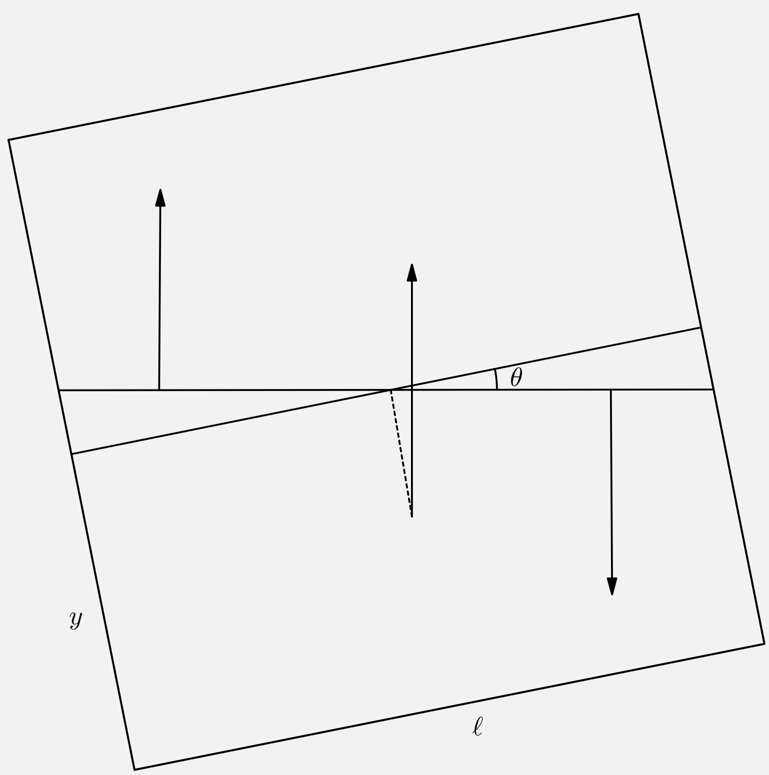
$$(m\omega^2 l^2)^2 - 4(mgl - m\omega^2 l^2)\cdot\frac{1}{2}(mgl - m\omega^2 l^2) < 0$$

$$m\omega^2 l^2 < \sqrt{2}(mgl - m\omega^2 l^2)$$

$$\left(\frac{1}{\sqrt{2}} + 1\right) m\omega^2 l^2 < mgl$$

$$\boxed{\omega^2 < \frac{(2-\sqrt{2})g}{l}}$$

pr 15. Conceptually what would happen is that if the beam is extremely light and you give the square cross section a tiny push, there will be a restoring torque causing it to be in stable equilibrium. However, at a certain density, the equilibrium position will not be when the sides of the square are parallel to the water. At some critical density ρ the new equilibrium position will be rotated a tiny angle $d\theta$.



We can represent the submerged portion as three separate masses. The submerged part represents a trapezoid. This can be represented as a rectangle that has the same area as the trapezoid. However, if we try to balance torques with this setup, we will fail because there are certain geometries that are not covered. Therefore, we need to add a triangle of density ρ an identical triangle with density $-\rho_o$.

Let the width of the square be ℓ and the height of the rectangle be y. Balancing forces we have:

$$\rho_o \ell^2 g = \rho_w \ell y g \implies \frac{y}{\ell} = \frac{\rho_o}{\rho_w}$$

Let us now balance torques around the center of mass. In an equilibrium position, the torques will sum to zero. The torque from the buoyant force from the rectangle is:

$$\tau_1 = \rho_w g\,(\ell y)\left(\frac{1}{2}(\ell - y)\right)\sin\theta$$

where θ is the angle the bottom of the beam makes with the horizontal. The triangular parts will also provide a torque from the buoyant force. Note that the buoyant force caused by the negative mass triangle will be negative and point in the other direction. The torque of each is:

$$\tau_2 = \rho_w g\left(\frac{1}{2}(\ell/2)^2\sin\theta\right)\left(\frac{2}{3}(\ell/2)\right)$$

where $\frac{2}{3}(\ell/2)$ is the perpendicular distance from the center of mass of the triangle to the center of mass of the square. Notice that since $\theta \ll 1$ we can sum torques and set it to zero:

$$\begin{aligned} 0 &= \tau_1 - 2\tau_2 \\ \frac{\ell^3}{12} &= \frac{\ell y(\ell - y)}{2} \\ \frac{\ell^2}{6} &= y(\ell - y) \end{aligned}$$

From earlier, let's substitute $\frac{y}{\ell} = \frac{\rho_o}{\rho_w} \equiv f$ and we'll get:

$$\frac{\ell^2}{6} = \ell^2 f(1-f) \implies f^2 - f + 1/6 = 0$$

Using the quadratic formula we get:

$$f = \frac{\rho_o}{\rho_w} = \frac{1}{2}\left(1 - 3^{-1/2}\right)$$

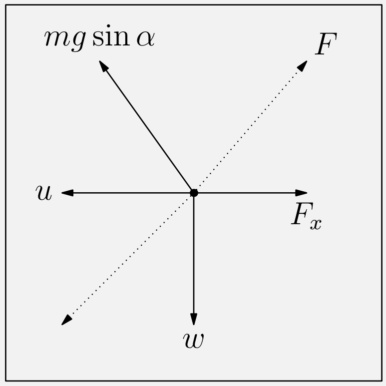
pr 16. Initially the force on the container is given by the total force of gravity that is acting on the hemispherical container, or in other words

$$F_{\text{bef}} = Mg - \rho g V = Mg - \frac{2}{3}\pi\rho g R^2$$

When the water leaks out, the containers weight no longer acts on the surface. This means the new pressure is equal to pressure of the water times the area of the base of the hemispherical container. The water pressure is the same on all places of the container so $P = \rho g R \cdot \pi R^2$. From force balancing, we have

$$(\rho g R)\pi R^2 = Mg + \frac{2}{3}\pi\rho g R^3 \implies \boxed{M = \frac{1}{3}\pi\rho R^3}$$

pr 17. The main idea is that the block slowly moves down the slope. This is because since the block moves back and forth in a very short period, it is never able to gain significant horizontal velocity. With this information let us create a freebody diagram of the block where we treat the block as a point mass



Define w as the average velocity and F_w to be the component of friction that opposes it or $F_w = mg\sin\alpha$. The component of friction pointing up the gradient must be $F_w = mg\sin\alpha$ to prevent the block from accelerating. The friction force points opposite the velocity, so $\frac{F_x}{F_w} = \frac{v}{w}$. Thus,

$$\frac{F_x}{F_w} = \frac{v}{w} \implies \frac{F_x}{mg\sin\alpha} = \frac{v}{w} \implies F_x = \frac{v}{w}mg\sin\alpha$$

We have that $F^2 = F_x^2 + F_w^2$, and since the frictional force $F = \mu N = \mu mg\cos\alpha$ we have

$$(mg\cos\alpha)^2 = \left(\frac{v}{w}mg\sin\alpha\right)^2 + (mg\sin\alpha)^2$$

$$\mu^2\cos^2\alpha = \frac{v+w}{w}\sin^2\alpha$$

Simplifying gives

$$w = \frac{v}{\sqrt{\mu^2\cot^2\alpha - 1}}$$

pr 18. We will use the property where the water is always at an equipotential surface. Consider the water surface at two points: one directly above the iron deposit and one infinitely far away. The potential infinitely far away is just:

$$U_\infty = mgh$$

while the potential on top of the iron deposit will be:

$$U_0 = mgH - \frac{G\left(\frac{4}{3}\pi r^3\Delta\rho\right)m}{(h+r)^2}$$

Setting these equal and solving for H will give:

$$H = \frac{\frac{4}{3}\pi G r^3 \Delta p}{g(r+h)}$$

pr 19. In a rotating reference frame, we have that

$$\vec{\omega}_3 = \vec{\omega}_1 + \vec{\omega}_2$$

where $\vec{\omega}_1$ is the angular velocity in the reference frame, $\vec{\omega}_2$ is the angular velocity of the body in the rotating reference, and $\vec{\omega}_3$ is that in the stationary frame. If you consider the reference point to be at infinity, then you find that the rotational motion of the disk becomes negligible. Therefore, we have that

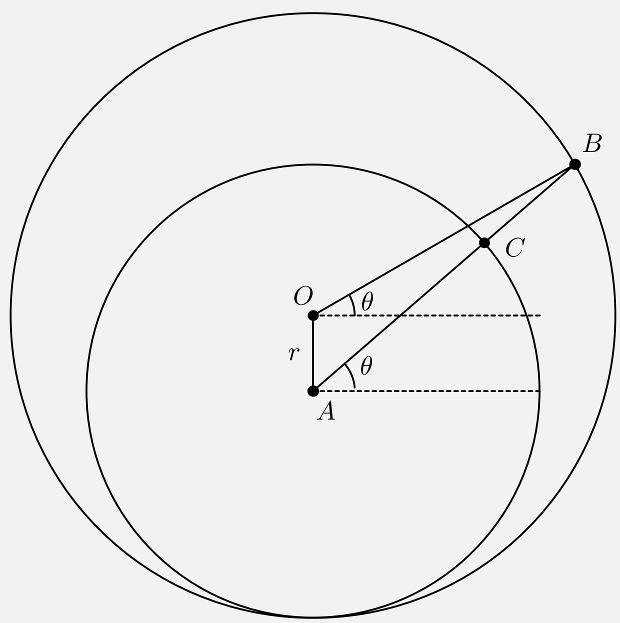
$$0 = \vec{\omega}_1 + \vec{\omega}_2$$

$$\vec{\omega}_1 = -\vec{\omega}$$

This problem was found in the book 'Aptitude Test Problems in Physics' by S.S. Krotov.

pr 20. When the waxing machine is stationary, no force is needed in order to maintain equilibrium because all the force vectors cancel out. However, when we are moving the disk at a constant velocity v, we are adding an extra component to the rotation. Let us set the rotation in the clockwise direction and velocity v to the right.

Then, if we move into a frame moving v to the right, there is an instantaneous axis of rotation on the disk at a position A located $r = v/\omega$ below the center. We can trace out a center with radius $R' = R - r$ centered around this point where the net force of this circle sums to zero. We now concern ourselves with the crescent-like shaped border.



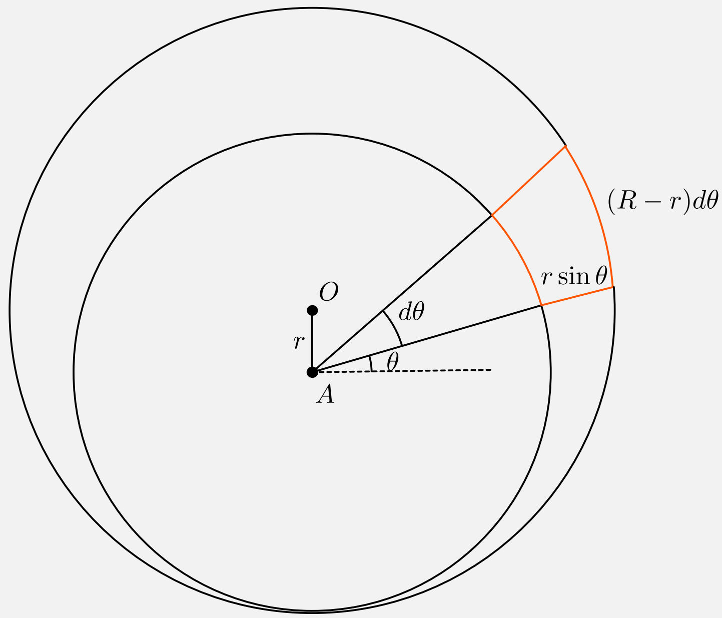
The distance from A to the rim of the disk at an angle θ can be calculated using the cosine law to be:

$$AB = \sqrt{r^2 + R^2 - 2Rr\cos(90^\circ + \theta)} = R\sqrt{1 + (r/R)^2 + 2(r/R)\sin\theta}$$

Since $r/R \ll 1$, we can perform a first order power expansion to get:

$$AB = R + r\sin\theta$$

This means the distance from the inner to outer circle is $CD = r\sin\theta$. We can determine the net (horizontal) force by considering the force of friction acting on a section as shown below.



The mass of this section is

$$dM = \sigma(r\sin\theta)(R - r\sin\theta)d\theta \approx \sigma Rr\sin\theta d\theta$$

The magnitude of the friction force acting on this segment is:

$$dF = \sigma\mu gRr\sin\theta d\theta$$

and is pointed tangent to the circumference of the circle. The horizontal component is thus:

$$dF_x = \sigma\mu gRr\sin\theta d\theta(\sin\theta)$$

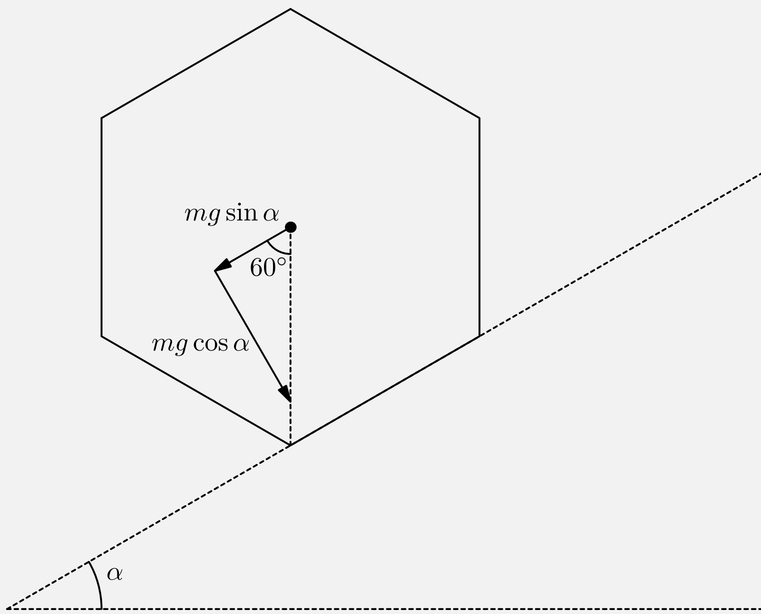
Integrating this from $\theta = 0$ to $\theta = 2\pi$ gives:

$$F_x = \int_0^{2\pi} \sigma\mu gRr\sin^2\theta d\theta = \sigma\mu gRr\pi$$

Letting $M = \sigma\pi R^2$ and $r = v/\omega$ gives us:

$$F = \boxed{\frac{\mu mgv}{\omega R}}$$

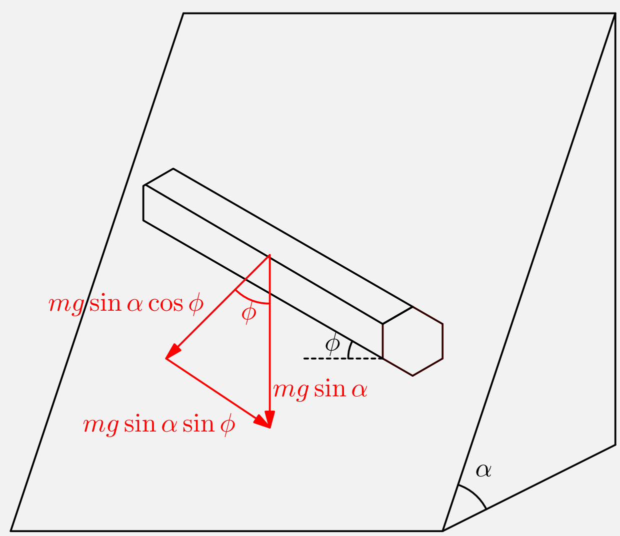
pr 21. We assume the condition $\mu > \tan\alpha$ is met so for any angle ϕ, the pencil will not slide down. However, it is able to roll. For an object to just start rolling, the force of gravity needs to form a vertical line. Let us first look at the simple case where $\phi = 0$



We break up the gravitational force into two components. One component is perpendicular to the plane $mg\cos\alpha$, and the other is along the plane and perpendicular to the pencil $mg\sin\alpha$. For rolling to begin, the sum of these two components need to lie on top of an edge, which is satisfied when:

$$\tan 30° = \frac{mg_\perp}{mg_{\text{normal}}} = \frac{mg\sin\alpha}{mg\cos\alpha}$$

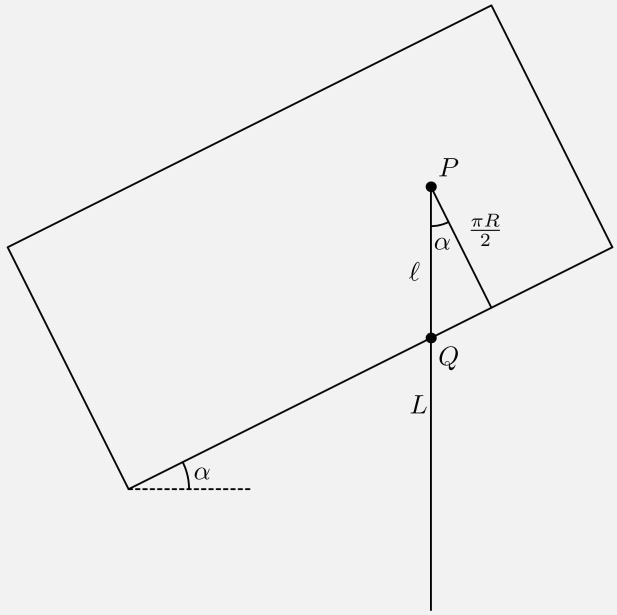
When $\phi \neq 0$, we perform a similar task however the pencil will no longer be rolling directly down the ramp but rather at an angle.



We break up the gravitational force into three components. One component is perpendicular to the plane $mg\cos\alpha$, the second is along the plane and perpendicular to the pencil $mg\sin\alpha\cos\phi$, and the third is along the plane and parallel to the pencil $mg\sin\alpha\sin\phi$. We note that the parallel component does not contribute to whether or not the pencil will roll down. Again, the sum of the first two components need to lie on top of an edge, which is satisfied when:

$$\tan 30° = \frac{mg_\perp}{mg_{\text{normal}}} = \frac{mg\sin\alpha\cos\phi}{mg\cos\alpha} \implies \boxed{\tan\alpha\cos\phi < \tan 30°}$$

pr 22. Consider a vertical plane parallel to the free hanging portion of the string.



Move this plane until it contacts the point in which the cylinder and the string first meet. Call this point Q, which is where we unfold half of the cylinder into a rectangle where the width is πR. The angle between PQ and the width of the rectangle is α so we have:

$$PQ = \frac{\pi R}{2\cos\alpha}$$

and thus:

$$\ell = L - \frac{\pi R}{2\cos\alpha}$$

When the weight oscillates, the trace of the string still stays straight on the unfolded cylinder. Therefore the length of the hanging string (and thus the weight's potential energy) do not depend in any oscillatory state on whether the surface of the cylinder is truly cylindrical or is unfolded into a planar vertical surface. Therefore the period of oscillations is still

$$T = 2\pi\sqrt{L/g}$$

pr 23. Label the strings from left to right as 1, 2, 3, 4. If string 4 is cut then in equilibrium state:

$$T_1 + T_2 + T_3 = mg$$

Let the rod be inclined at an angle θ with the horizontal in equilibrium position. As extensions in the strings will be small θ will be very small. Balancing torques about 1, we get:

$$T_2\left(\frac{\ell}{3}\right) + T_3\left(\frac{2\ell}{3}\cos\theta\right) = mg\frac{\ell}{2}\cos\theta \implies T_2 + 2T_3 = \frac{3mg}{2}$$

As the rod is rigid, we can write our third equation as a conservation law:

$$\frac{\Delta x_2 - \Delta x_1}{\ell/3} = \frac{\Delta x_3 - \Delta x_1}{2\ell/3} \implies \frac{T_2 - T_1}{k\ell/3} = \frac{T_3 - T_2}{k\ell/3}$$

As strings are identical:

$$2T_2 = T_3 + T_1$$

We have three equations and three unknowns so solving them yields:

$$T_1 = \frac{1}{12}mg$$
$$T_2 = \frac{1}{3}mg$$
$$T_3 = \frac{7}{12}mg$$

2 Solutions to Dynamics Problems

This chapter will focus on problems 24-55 of the handout. Dynamics, unlike statics, analyzes the motion of objects in motion. Lots of dynamics problems require analyzing the accelerations of the system or finding the forces acting in a system. In this section, Prof Kalda introduces a technique called Lagrangian formalism that is used to find the acceleration of a object in a system using the generalized coordinate ξ. We are also introduced to the majority of mechanics problems in this section of the handout.

pr 24. Let the acceleration of small block on top be a_1, the small block on the side be a_2, and the big block be a_3. We create three $F = ma$ equations

$$mg - T = ma_2 \tag{1}$$
$$T = ma_1 \tag{2}$$
$$-T = Ma_3 \tag{3}$$

If the block on the side moves a small distance d_1, then the block on top will move a small distance d_2 to the right which makes the whole big block move a distance d_3 to the left. Therefore as a result,

$$d_2 = d_1 + d_3$$

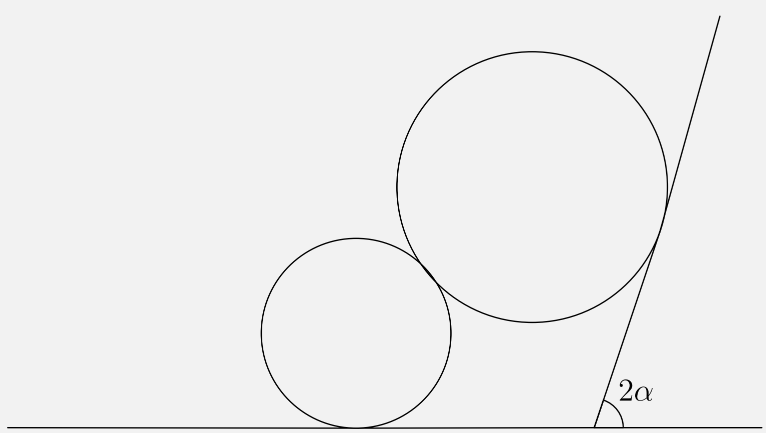
and the acceleration

$$a_2 = a_1 + a_3$$

From equation (1) we have

$$\begin{aligned} mg - T &= m(a_1 + a_3) \\ mg - T &= T + ma_3 \\ mg - ma_3 &= 2Ma_3 \\ mg &= (2M + m)a_3 \\ &\boxed{a_3 = \frac{mg}{2M + m}} \end{aligned}$$

pr 25.



We tilt the plane by an angle 2α. This makes the effective gravity in this scenario become

$$g_{\text{eff}} = mg \sin\alpha \cos\alpha$$

Since the wedge is weightless, the normal force between the wedge of both blocks have to be equal otherwise, the wedge will experience an infinite acceleration. Setting these two forces equal to each other in the horizontal direction gives us

$$F_g \sin\alpha \cos(2\alpha) = F_g \sin\alpha$$

$$\cos 2\alpha = \frac{m}{M}$$

The lower ball will then 'climb up' if

$$m < M\cos 2\alpha$$

Solution 2: Since the wedge is weightless, the normal force between the wedge of both blocks have to be equal otherwise, the wedge will experience an infinite acceleration. Therefore, setting the forces of inertia and weight at the point when both balls make contact, produces the equation

$$mg\cos\alpha + ma\sin\alpha = Mg\cos\alpha + Ma\sin\alpha$$

We also note, that by trigonometry, after contact the smaller mass must have the ratio of the translational fictitious force to the weight of the ball must be greater than $\tan\alpha$ for the ball to slide up the ramp

$$\frac{ma}{mg} > \tan\alpha \implies a > g\tan\alpha.$$

We now go to the first equation and solve for acceleration. Moving variables to the same side results in

$$a\sin\alpha(m+M) = g\cos\alpha(M-m) \implies a = \frac{g\cos\alpha(M-m)}{\sin\alpha(m+M)}$$

Substituting our minimum value of acceleration yields

$$\frac{g\cos\alpha(M-m)}{\sin\alpha(m+M)} > g\tan\alpha$$

Solving this inequality yields

$$\boxed{m < M\cos 2\alpha}$$

This problem was found in the book 'Aptitude Test Problems in Physics' by S.S. Krotov.

pr 26. Let's take the displacement ξ of the wedge as coordinate describing the system's position.[a] If the wedge moves by ξ, then the block moves the same amount with respect to the wedge because the rope is unstretchable. The kinetic energy changes by

$$\Pi = mg\xi\sin\alpha.$$

To find the velocity of the wedge and that of the block, let us add the two vectors $\dot\xi$ separated by an angle α. The vertical components of the vector $(\dot\xi \sin\frac{\alpha}{2})$ cancel out due to symmetry. The horizontal components add up together to get $2\dot\xi\sin\frac{\alpha}{2}$. Thus, the velocity of the wedge is $\dot\xi$ and that of the block is $2\dot\xi\sin\frac{\alpha}{2}$, therefore the net kinetic energy is

$$K = \frac{1}{2}\dot\xi^2\left(M + 4M\sin^2\frac{\alpha}{2}\right).$$

Then we find $\Pi'(\xi) = mg\sin\alpha$ and $\mathcal{M} = M + 4m\sin^2\frac{\alpha}{2}$; their ratio gives the answer of

$$\boxed{a = \frac{mg\sin\alpha}{M + 4m\sin^2\frac{\alpha}{2}}}$$

This problem was found in the book 'Aptitude Test Problems in Physics by S.S. Krotov.

[a]This is a solution that is based off the one given in hints, and is mainly expanding on some of the points that the hint did not give.

pr 27. Let us call ξ a generalised coordinate if the entire state of a system can be described by this single number. Say we need to find the acceleration $\ddot{\xi}$ of coordinate ξ. If we can express the potential energy Π of the system as a function $\Pi(\xi)$ of ξ and the kinetic energy in the form $K = \mathcal{M}\dot{\xi}^2/2$ where coefficient $\mathcal{M}$ is a combination of masses of the bodies (and perhaps of moments of inertia), then

$$\ddot{\xi} = -\Pi'(\xi)/\mathcal{M}.$$

Let us take the wedge's displacement as the coordinate ξ; if the displacement of the block along the surface of the wedge is η, then the center of mass from rest is

$$\eta(m_1\cos\alpha_1 + m_2\cos\alpha_2) = (M + m_1 + m_2)\xi.$$

We then find that

$$\eta = \frac{(M + m_1 + m_2)\xi}{(m_1\cos\alpha_1 + m_2\cos\alpha_2)}$$

We note that if we substitute this expression everywhere, we will get an extremely contrived answer. Thus, let us substitute this expressions with more sightful variable. Let

$$\varrho \equiv \frac{M + m_1 + m_2}{m_1\cos\alpha_1 + m_2\cos\alpha_2}.$$

The potential energy as a function of ξ is given by

$$\Pi(\xi) = m_1 g\eta\sin\alpha_1 - m_2 g\eta\sin\alpha_2$$

It is given that $\ddot{\xi} = \frac{\Pi'(\xi)}{\mathcal{M}}$. Thus by differentiating $\Pi(\xi)$ we get

$$\Pi(\xi) = \varrho(m_1\sin\alpha_1 - m_2\sin\alpha_2).$$

Finding $\mathcal{M}$ will be a bit harder. The kinetic energy of the block is given as the sum of the horizontal and vertical energies or

$$\begin{aligned}
K &= \frac{1}{2}M\dot{\xi}^2 + \frac{1}{2}m_1(\dot{\xi} - \dot{\eta}\cos\alpha_1)^2 + \frac{1}{2}m_1(\dot{\eta}\sin\alpha_1)^2 + \frac{1}{2}m_2(\dot{\xi} - \dot{\eta}\cos\alpha_2)^2 + \frac{1}{2}m_2(\dot{\eta}\sin\alpha_2)^2 \\
&= \frac{1}{2}M\dot{\xi}^2 + \frac{1}{2}m_2(\dot{\xi}^2 - 2\dot{\eta}\dot{\xi}\cos\alpha_2 + \dot{\eta}^2) + \frac{1}{2}m_1(\dot{xi}^2 - 2\dot{\eta}\dot{\xi}\cos\alpha_1 + \dot{\eta}^2)
\end{aligned}$$

We have that $\eta = \varrho\xi \implies \dot{\eta} = \varrho\dot{\xi}$. Thus, by substituting this into our expression for kinetic energy we have

$$\begin{aligned}
K &= \frac{1}{2}M\dot{\xi}^2 + \frac{1}{2}m_2(\dot{\xi}^2 - 2\dot{\eta}\dot{\xi}\cos\alpha_2 + \varrho\dot{\eta}^2) + \frac{1}{2}m_1(\dot{xi}^2 - 2\dot{\eta}\dot{\xi}\cos\alpha_1 + \dot{\eta}^2) \\
&= \frac{1}{2}M\dot{\xi}^2 + \frac{1}{2}m_2(\dot{\xi}^2 - 2\varrho\dot{\xi}^2\cos\alpha_2 + \varrho\dot{\xi}^2) + \frac{1}{2}m_1(\dot{\xi}^2 - 2\varrho\dot{\xi}^2\cos\alpha_1 + \varrho\dot{\xi}^2) \\
\mathcal{M} &= M + m_2(1 - 2\varrho\cos\alpha_2 + \varrho^2) + m_1(1 - 2\varrho\cos\alpha_1 + \varrho^2)
\end{aligned}$$

Now we apply our Lagrangian Formalism Identity,

$$\ddot{\xi} = \frac{\Pi'(\xi)}{\mathcal{M}} = \frac{\varrho(m_1\sin\alpha_1 - m_2\sin\alpha_2)}{M + m_2(1-2\varrho\cos\alpha_2+\varrho^2) + m_1(1-2\varrho\cos\alpha_1+\varrho^2)}$$

$$= \frac{\varrho(m_1\sin\alpha_1 - m_2\sin\alpha_2)}{(M+m_1+m_2) - 2\varrho(m_1\cos\alpha_1 + m_2\cos\alpha_2) + \varrho^2(m_1+m_2)}$$

$$= \frac{\varrho(m_1\sin\alpha_1 - m_2\sin\alpha_2)}{(M+m_1+m_2) - 2\frac{M+m_1+m_2}{m_1\cos\alpha_1+m_2\cos\alpha_2}(m_1\cos\alpha_1 + m_2\cos\alpha_2) + \varrho^2(m_1+m_2)}$$

$$= \frac{\varrho(m_1\sin\alpha_1 - m_2\sin\alpha_2)}{\varrho^2(m_1+m_2) - (M+m_1+m_2)} = \frac{\frac{M+m_1+m_2}{m_1\cos\alpha_1+m_2\cos\alpha_2}(m_1\sin\alpha_1 - m_2\sin\alpha_2)}{\left(\frac{M+m_1+m_2}{m_1\cos\alpha_1+m_2\cos\alpha_2}\right)^2(m_1+m_2) - (M+m_1+m_2)}$$

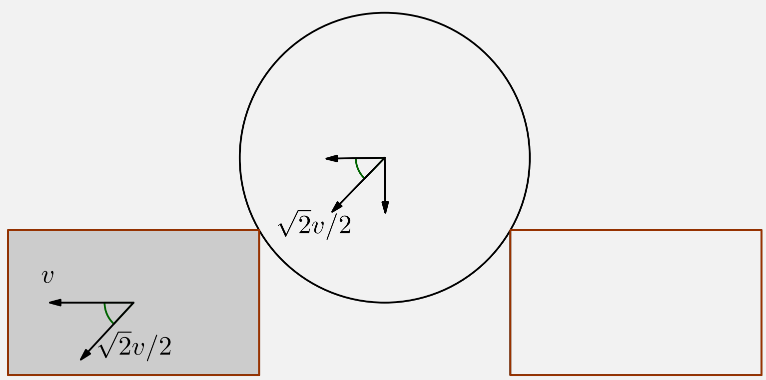
$$= \frac{\left(\frac{M+m_1+m_2}{m_1\cos\alpha_1+m_2\cos\alpha_2}(m_1\sin\alpha_1 - m_2\sin\alpha_2)\right)}{\left(\frac{(M+m_1+m_2)^2(m_1+m_2) - (M+m_1+m_2)(m_1\cos\alpha_1+m_2\cos\alpha_2)}{(m_1\cos\alpha_1+m_2\cos\alpha_2)^2}\right)}$$

$$\boxed{a_0 = \frac{(m_1\cos\alpha_1+m_2\cos\alpha_2)(m_1\sin\alpha_1 - m_2\sin\alpha_2)}{(m_1+m_2+M)(m_1+m_2) - (m_1\cos\alpha_1+m_2\cos\alpha_2)^2}}$$

This is extremely long, yes, but to do well at the International Olympiad, you must be familiar and not scared to bash it all out.

This problem is from the 1997 IPhO Problem 1. Refer to https://www.jyu.fi/tdk/kastdk/olympiads/problems.html#71prob for a solution without lagrangian formalism.

pr 28.



Let us denote the horizontal velocity of the block as v. When the distance between the block and the step is $\sqrt{2}r$, the cylinder pushes on the block at an angle of 45°. By trigonometry, we see that the cylinder would have to push on the block with a velocity of $\sqrt{2}v/2$ for the block to move horizontally with a velocity v. Now it is easy to see that velocity of cylinder is just

$$\vec{v_c} = -\frac{v_b}{2}\hat{i} - \frac{v_b}{2}\hat{j}$$

where v_b is the speed of the block (directed towards the negative x-axis). By energy conservation

$$mg\left(r - \frac{r}{\sqrt{2}}\right) = \frac{1}{2}m{v_b}^2 + \frac{1}{2}m{v_c}^2$$

Also project Newton's 2nd law onto the axis that passes through the top corner of the step and the cylinder's centre: this axis is perpendicular both to the normal force between the block and the cylinder and to the cylinder's tangential acceleration.

$$\frac{mg}{\sqrt{2}} = N + \frac{m{v_c}^2}{r} \implies mg\frac{\sqrt{2}}{2} - N = m\frac{(\sqrt{2}v/2)^2}{r}$$

where N is the normal force by the wall. Now, we solve these systems of equations for N. In our first equation, we have

$$\begin{aligned} mgr\left(\frac{2-\sqrt{2}}{2}\right) &= \frac{1}{2}mv^2 + \frac{1}{2}m\left(\frac{\sqrt{2}v}{2}\right)^2 \\ &= \frac{1}{2}m\left(\frac{3(\sqrt{2}v)^2}{2}\right) \end{aligned}$$

Taking out common factors from both sides gives us

$$gr(\sqrt{2}-2) = \left(\frac{3(\sqrt{2}v)^2}{2}\right) \implies \frac{g(2-\sqrt{2})}{3} = \frac{(\sqrt{2}v/2)^2}{r}.$$

Substituting this result into our conservation of energy equation gives us

$$mg\frac{\sqrt{2}}{2} - N = m\frac{g(2-\sqrt{2})}{3}$$

Solving for N gives us the result

$$N = \left(\frac{\sqrt{2}}{2} - \frac{2\sqrt{2}}{3}\right)mg = \boxed{\left(\frac{5\sqrt{2}-4}{6}\right)mg}.$$

Let the normal force from the other block be Q. From here we can project Newton's Second Law onto the cylinder and block on the horizontal direction (and noting that the aceleration of the cylinder is half that of the block because it's horizontal velocity is half that of the blocks velocity) gives us

$$\begin{aligned} m(a/2) &= N\sin\theta - Q\sin\theta \\ ma &= Q\sin\theta \end{aligned}$$

where θ is the angle the normal force makes with respect to the vertical. Substituting the second equation into the first gives us

$$\frac{1}{2}Q\sin\theta = N\sin\theta - Q\sin\theta \implies \frac{3}{2}Q = N$$

Therefore the ratio between the two normal forces are $\frac{Q}{N} = \frac{2}{3}$. As mentioned in the hint, the ratios of the normals is fixed, hence they blow up at the same instant. (only differing by a constant factor) which means that they would give 0 at the same values.

This problem was found in the book 'Aptitude Test Problems in Physics by S.S. Krotov.

pr 29. Consider the time when the angle between the line joining the edge to the rod makes an angle θ, with the vertical. We want that the normal should always be positive (outwards). The velocity of the rod at this time, ν, can be calculated using energy conservation $\frac{1}{2}m\nu^2 = \frac{1}{2}mv^2 + mgR(1-\cos\theta)$. Setting total forces along the line joining the rod to the edge to 0, we get that

$$\begin{aligned}
\frac{m\nu^2}{R} + N &= mg\cos\theta \\
N &= mg\cos\theta - \frac{2\cdot\frac{1}{2}m\nu^2}{R} \\
&= mg\cos\theta - \frac{mv^2 + 2mgR(1-\cos\theta)}{R} \\
0 &\leq 3mg\cos\theta - \frac{mv^2 + 2mgR}{R} \\
\cos\theta &\geq \frac{1}{3}\left(2 + \frac{v^2}{gR}\right)
\end{aligned}$$

for all values of θ that can be achieved. The maximum value of θ will be alpha, so

$$\cos\alpha \geq \frac{1}{3}\left(2 + \frac{v^2}{gR}\right)$$

is a necessary and sufficient condition.

pr 30. First, we choose a frame that we will work from in this problem. To cancel out as many variables as possible, we should work in the frame of the large block when it is set into motion.

From reference of the bottom of the circular cavity, the initial potential energy of the small block at the top is mgr. When it gets to the bottom of the circular cavity, it gains a kinetic energy of $\frac{1}{2}mv^2$. By conservation of energy we get

$$\frac{1}{2}mv^2 = mgr \implies v = \sqrt{2gr}.$$

When the small block is at the bottom of the cavity, it will move backwards with a velocity v_1 in the reference frame of the big block, while the big block itslef moves with a velocity v_2 forwards. Thus, conservation of momentum and energy gives us

$$\begin{aligned}
Mv_1 - mv_2 &= m\sqrt{2gr} \\
\frac{1}{2}Mv_1^2 + \frac{1}{2}mv_2^2 &= mgr
\end{aligned}$$

From our conservation of momentum equation we have

$$v_2 = \frac{M}{m}v_1 - \sqrt{2gr}$$

Thus by substituting v_2 back into our conservation of momentum equation we result in

$$\begin{aligned}
\frac{1}{2}Mv_1^2 + \frac{1}{2}m\left(\frac{M}{m}v_1 - \sqrt{2gr}\right)^2 &= mgr \\
\frac{1}{2}Mv_1^2 + \frac{1}{2}m\left(\frac{M^2}{m^2}v_1^2 + 2gr - \frac{2M}{m}v_1\sqrt{2gr}\right) &= mgr
\end{aligned}$$

Expanding $\frac{1}{2}m$ inside gives

$$\frac{1}{2}Mv_1^2 + \frac{1}{2}\frac{M^2}{m}v_1^2 + mgr - Mv_1\sqrt{2gr} = mgr$$

Taking away mgr from both sides and dividing both sides by Mv_1 gives us

$$\frac{1}{2}v_1 + \frac{M}{2m}v_1 - \sqrt{2gr} = 0$$

Factoring and taking $\sqrt{2gr}$ to the other side gives us

$$v_1\left(\frac{M+m}{2m}\right) = \sqrt{2gr}$$

$$\boxed{v_1 = 2\frac{m}{M+m}\sqrt{2gr}}$$

This problem was found in the book 'Aptitude Test Problems in Physics by S.S. Krotov.

pr 31. Let the centre of mass of the system be C and the point where the right string and the rod meet be A. Let the required tension be T.
Claim. A must have no acceleration.

Proof: Acceleration of A cannot be downwards as the string is inextensible. If the acceleration of A is upwards, then the string will slack and T will be 0, so the acceleration of the centre of mass will be downwards and it there would be no torque and hence no rotation implying that A has acceleration downwards. This results in a contradiction. So the only case left is that A has zero acceleration.

Due to this, $a = \alpha \times AC$, where a and α are the linear acceleration of C and the angular acceleration of the rod about A respectively. Also, the distance $AC = l + \frac{Ml}{M+m} = \frac{(m+2M)l}{M+m}$. The acceleration of the centre of the mass can be calculated from Newton's second law,

$$(M+m)g - T = (M+m)a \implies a = g - \frac{T}{M+m}$$

, where a is positive downwards. The torque on the rod about A is,

$$\tau = (M+m)g \times AC = I\alpha = (m+4M)l^2\frac{a}{AC} \implies a = \frac{(m+2M)^2}{(M+m)(m+4M)}g$$

Substituting this in the previous equation, we get that $\boxed{T = \frac{Mmg}{m+4M}}$

Solution 2: Let the acceleration of the mass m right after the second string is cut be a, it then follows the acceleration of the second mass M right after is given by $2a$. If the normal force produced from the first mass is N_1 and the normal force produced from the second mass is N_2 then our two $F = ma$ equations are

$$mg - N_1 = ma$$
$$Mg - N_2 = M(2a)$$

We also have our equation of torque to be

$$mg\ell + Mg(2\ell) = I\alpha = (m\ell^2 + 4M\ell^2)\alpha$$

Lastly, by Newton's third law the tension is given by

$$T = N_1 + N_2$$

We now can solve this problem given four equations and four unknowns. We first manipulate the torque equation.

$$\begin{aligned} mg\ell + 2Mg\ell^2 &= (m\ell^2 + 4M\ell^2)\frac{a}{\ell} \\ mg + 2Mg &= (m + 4M)a \\ a &= \frac{m+2M}{m+4M}g \end{aligned}$$

We now go back to our first two $F = ma$ equations. We substitute our first equation to get

$$mg - N_1 = ma \implies N_1 = mg - m\left(\frac{m+2M}{m+4M}g\right)$$

our second equation gives us

$$Mg - N_2 = M(2a) \implies N_2 = Mg - 2M\left(\frac{m+2M}{m+4M}g\right).$$

Our equation for Newton's third law then gives us

$$\begin{aligned} T &= mg - m\left(\frac{m+2M}{m+4M}g\right) + Mg - 2M\left(\frac{m+2M}{m+4M}g\right) \\ &= (m+M)g - (m+2M)\frac{m+2M}{m+4M}g \\ &= \frac{(m+M)(m+4M) - (m+2M)^2}{m+4M} \\ &= \frac{m^2 + 4Mm + Mm - m^2 - 2Mm - 2Mm - 4M^2 + 4M^2}{m+4M} \\ T &= \boxed{\frac{Mm}{m+4M}g} \end{aligned}$$

pr 32. When the pulley is let go, one side of the rope will go up a distance ξ while the other side will go down a distance ξ. The change in potential energy of this is the categorized as

$$\Pi(\xi) = -\rho g(L - 2l - \pi R)\xi \implies -\Pi'(\xi) = \rho g(L - 2l - \pi R)$$

The kinetic energy of the system will then be

$$K = \frac{1}{2}m\dot{\xi}^2 + \frac{1}{2}\rho L\dot{\xi}^2$$

This implies that the effective mass, $\mathcal{M}$, is $\mathcal{M} = m + \rho L$. We then get the acceleration of the system to be

$$a \equiv \frac{\rho g(L - 2l - \pi R)}{m + \rho L}$$

Now, we write for the displacement of parts of the system times their mass divided by the total effective mass of the system $\mathcal{M}$. Differentiating that with respect to ξ will give us our accelerations in the x and

y direction. In the x direction, we have

$$x = \frac{2R\xi\rho}{m+\rho L}$$
$$a_x = \frac{2R\rho a}{m+\rho L}$$

In the y direction, we have

$$y = \frac{(L-2l-\pi R)\rho\xi}{m+\rho L}$$
$$a_y = \frac{(L-2l-\pi R)\rho a}{m+\rho L}$$

By $F = ma$, we have the direction of force in the x and y direction to then be

$$\boxed{F_x = 2R\rho a}$$

$$F_y - (m+\rho L)g = -\rho a(L-\pi R-2l)$$
$$\boxed{\therefore F_y = -\rho a(L-\pi R-2l) + (m+\rho L)g}$$

pr 33. Since the second block is being pushed rightwards with some velocity it is in turn pulling more string outwards. By conservation of string, the block that isn't pushed will be pushed upwards because less string is there to sustain it's mass. Thus the answer is that the block that isn't pushed will reach higher after subsequent motion.

Let the tension in the string be T. Then at a certain instant, when the angle between the right mass and the vertical is α, we have the component of vertical force to be

$$F_{y_1} = mg - T\cos\alpha.$$

At the other end, we have the component of vertical force to be

$$F_{y_2} = mg - T.$$

Comparing the two accelerations at both ends gives us

$$a_2 - a_1 = \left(g - \frac{T}{m}\right) - \left(g - \frac{T}{m}\cos\alpha\right) = \frac{T}{m}(1-\cos\alpha)$$

which is always a non negative number. This implies that at any instant, the right load is lower than the left load.

pr 34. Let the friction force directed on the block to the right be f_1, the friction force directed on to the block on the left be f_2, and let the tension directed from the string be T. Drawing a freebody diagram results in 4 equations. (The two small blocks have the same accelerations because they are connected by the same string).

$$F - f_1 = Ma_1 \tag{4}$$
$$f_1 - T = ma_2 \tag{5}$$
$$f_2 = Ma_3 \tag{6}$$
$$T - f_2 = ma_2 \tag{7}$$

We consider four options: all the bodies move together, the rightmost block moves separately, all three components move separately, or the left block moves separately.

Case 1: (all the bodies move together, $a_1 = a_2 = a_3$)

Since all the bodies move together, then they move at the same acceleration. This means that our equations are now

$$F - f_1 = Ma \tag{8}$$
$$f_1 - T = ma \tag{9}$$
$$f_2 = Ma \tag{10}$$
$$T - f_2 = ma \tag{11}$$

Substituting equation (3) into equation (4) gives us

$$T - Ma = ma$$
$$T = (m + M)a$$

Substituting our result for tension into equation (2) gives us

$$f_1 - (m + M)a = ma$$
$$f_1 = (2m + M)a$$

Taking this result and now substituting into equation (1) gives us

$$F - (2m + M)a = Ma$$
$$F = 2(m + M)a$$
$$\boxed{a = \frac{F}{2(m + M)}}$$

From equation (1), we note that

$$F - Ma = f_1 \leq \mu mg$$

Plugging in our equation for acceleration gives us

$$F - M\frac{F}{2(m + M)} \leq \mu mg$$
$$F\left(1 - \frac{1}{2(m + M)}\right) \leq \mu mg$$
$$F\left(\frac{M + 2m}{2(m + M)}\right) \leq \mu mg$$
$$\boxed{F \leq 2\mu mg\frac{m + M}{m + 2M}}$$

Case 2: (The rightmost block moves separately, $a_2 = a_3$, $f_2 = \mu mg$)

In this case, we have our equations to be

$$F - \mu mg = Ma_1 \tag{12}$$
$$\mu mg - T = ma_2 \tag{13}$$
$$f_2 = Ma_2 \tag{14}$$
$$T - f_2 = ma_2 \tag{15}$$

From this, we find that

$$\boxed{a_1 = \frac{F - \mu mg}{M}}$$
$$\boxed{a_2 = \frac{\mu mg}{M + 2m}}$$

For the block to move, we must have

$$Ma_2 \leq \mu mg$$
$$\frac{\mu mMg}{M + 2m} \leq \mu mg$$
$$\frac{M}{M + 2m} \leq 1$$

This works, since both the denominator is greater than the numerator. Thus, we can continue with our calculations and find that from case 1, we have that if the force is

$$F \leq 2\mu mg\frac{m + M}{m + 2M}$$

then our acceleration would be

$$\boxed{a = \frac{F}{2(m + M)}}$$

if the force does not satisfy that constraint, then we have the accelerations to be the results we found before.

Case 3: (all three components move separately, $a_1 \neq a_2 \neq a_3$)

If $a_1 \neq a_2 \neq a_3$, then that implies that the $f_1 = \mu mg$ and $f_2 = \mu mg$. Looking at our systems of equations

$$F - f_1 = Ma_1 \tag{16}$$
$$f_1 - T = ma_2 \tag{17}$$
$$f_2 = Ma_3 \tag{18}$$
$$T - f_2 = ma_2 \tag{19}$$

We find that $a_2 = 0$ which is impossible.

Case 4: (the left block moves separately, $a_1 = a_2$, $f_2 = \mu mg$) Our systems of equation would then be

$$F - f_1 = Ma_1 \tag{20}$$
$$f_1 - T = ma_1 \tag{21}$$
$$\mu mg = Ma_3 \tag{22}$$
$$T - \mu mg = ma_1 \tag{23}$$

Solving these equations gives us

$$a_1 = \frac{F - \mu mg}{M + 2m}$$

Substituting back into equation (1) gives us

$$F - f_1 = M\frac{F - \mu mg}{M + 2m}$$
$$F - M\frac{F - \mu mg}{M + 2m} \le \mu mg$$

Multiplying across gives us

$$(M + 2m)F - M(F - \mu mg) \le \mu mg(M + 2m)$$

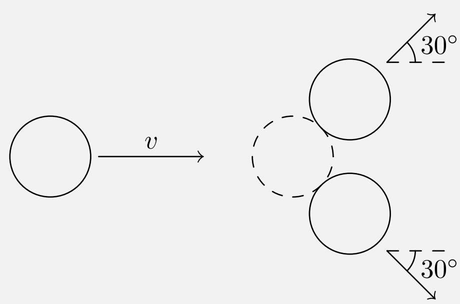
Solving this inequality gives us

$$F \le \mu mg$$

which is impossible because that would then imply that $a_1 \le 0$.

pr 35. The angle two equal masses make after an elastic collision will be a right angle, and thus if the stationary ball is placed on a semi-circle where the two holes form the diameter, then the description is possible according to Thales' Theorem.

pr 36. **a)** Denote the first ball with a final speed v_1 and the other two balls with final speed v_2.



In this problem, we're given an elastic collision, so we know that both momentum and energy are conserved. Conservation of momentum gives

$$v_1 + \frac{\sqrt{3}}{2}v_2 + \frac{\sqrt{3}}{2}v_2 = v \implies v_1 + \sqrt{3}v_2 = v,$$

and conservation of energy gives

$$2\left(\frac{1}{2}mv_2^2\right) + \frac{1}{2}mv_1^2 = \frac{1}{2}mv^2 \implies 2v_2^2 + v_1^2 = v^2.$$

This gives us two equations. Our goal is to find v_1, thus we first rearrange our conservation of momentum equation to get v_2 in terms of v_1 and then substitute back in to get v_1.

$$v_1 = v - \sqrt{3}v_2$$

putting this in to our conservation of energy equation gives us

$$2(v - \sqrt{3}v_2)^2 + v_1^2 = v^2.$$

Expanding this equation out gives us

$$2v_2^2 + 3v_2^2 - 2\sqrt{3}v_2 v + v^2 = v^2$$

Taking out v^2 from both sides, and then dividing by v_2 on both sides gets us the equation

$$2v_2 + 3v_2 - 2\sqrt{3}v = 0 \implies 5v_2 = 2\sqrt{3}v \implies v_2 = \frac{2\sqrt{3}}{5}v$$

Substituting v_2 back into our conservation of momentum equation gives us

$$v_1 = v - \sqrt{3}\frac{2\sqrt{3}}{5}v \implies \boxed{|v_1| = \frac{1}{5}v.}$$

b) Suppose the moving ball first strikes the lower ball. Let the x-direction point in the line joining their centers. Therefore, $v_{1,x} = v\cos 30°$ and the perpendicular component of velocity is $v_{1,y} = v\sin 30°$. Note that the impulse acts along line joining their center, therefore the perpendicular component of its velocity is unchanged. Conservation of momentum gives:

$$v_{1,x} = v_1' + v_2'$$

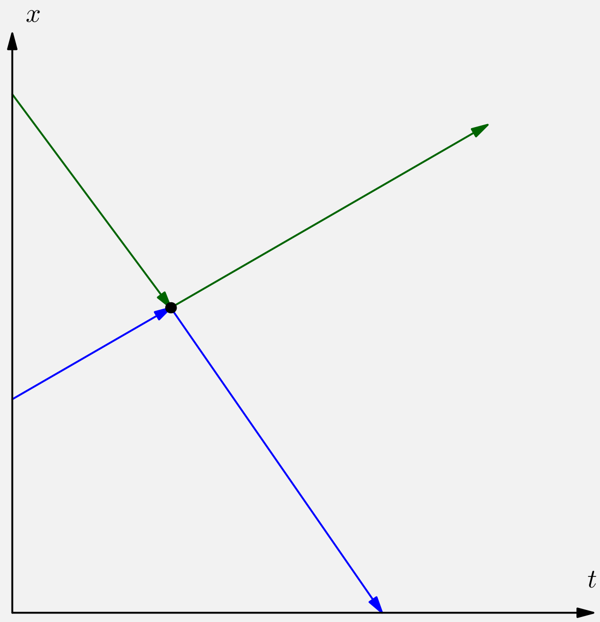
Conservation of energy:

$$v_{1,x}^2 + v_{1,y}^2 = v_1'^2 + v_{1,y}^2 + v_2'^2 \implies v_{1,x}^2 = v_1'^2 + v_2'^2$$

Notice that in the x-direction, this gives the same behavior as a head-on collision between two identical balls. Therefore, the velocity of the moving ball becomes zero along the x-direction. Now the moving ball will strike the upper ball with a speed $v\cos 30°$.
This second collision is identical to the first. The component of velocity along the line joining their centers is $(v\cos 30°)\sin 30°$ and the component of velocity perpendicular to this line is $(v\cos 30°)\sin 30°$. Again, only the component of velocity perpendicular to this line will survive at the end so the final answer is:

$$\boxed{v_f = v\sin^2 30° = \frac{v}{4}}$$

pr 37. We use idea 55, and visualise the motion using a graph, particularly let us plot the $x-t$ graph of each bead on the same plane. For simplifications, let us consider the collision of just two beads. Since they have the same mass, the elastic collision will cause them to swap velocities. The below graph shows the interaction between a blue and green particle. Even though each individual bead exhibits a zig-zag behaviour, together it appears as if it is two straight lines intersecting, with the intersection point representing the point of collision.



Since we have all the n lines intersecting with another line at exactly one point, and no three lines intersect at a point (the probability that more than two beads will collide at the same time is negligibly small), we have the number of intersections (collisions) in the graph as

$$\binom{n}{2} = \boxed{\frac{n(n-1)}{2}}$$

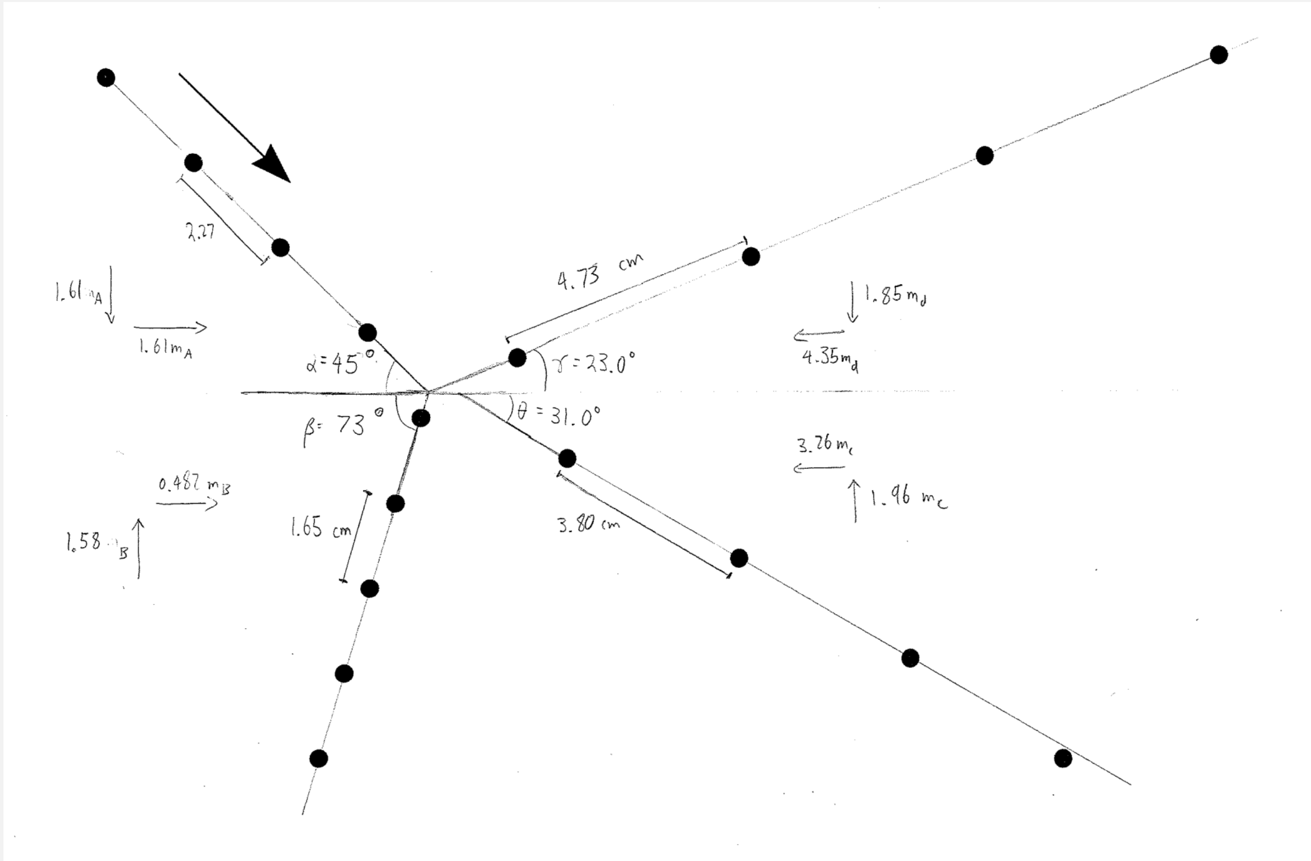
pr 38. The velocity of the small block in the center of mass frame is $\frac{Mv}{m+M}$ while the velocity of the large block in the center of mass frame is $\frac{mv}{m+M}$. The work done by friction is μmgL, thus we can create a conservation of energy equation

$$\frac{1}{2}m\left(\frac{Mv}{m+M}\right)^2 + \frac{1}{2}M\left(\frac{mv}{m+M}\right)^2 = \mu mgL$$

$$v^2\left(\frac{mM^2+Mm^2}{(m+m)^2}\right) = 2\mu mgL \implies v^2\left(\frac{mM(m+M)}{(m+m)^2}\right) = 2\mu mgL$$

$$\boxed{v = \sqrt{2\mu gL\left(1+\frac{m}{M}\right)}}$$

pr 39. Between successive images, the time difference Δt is constant. Therefore, the velocity and consequently momentum is directly proportional to the distance between successive images. Let us assume the time differences is $\Delta t = 0.01$ s.



Therefore, we can list the following linear momenta (starting from top left going counterclockwise)

$$\begin{aligned}
\text{a: } p_x &= 1.61m_a \\
p_y &= 1.61m_a \\
\text{b: } p_x &= 0.482m_b \\
p_y &= 1.58m_b \\
\text{c: } p_x &= 3.26m_c \\
p_y &= 1.96m_c \\
\text{d: } p_x &= 4.35m_d \\
p_y &= 1.85m_d
\end{aligned}$$

These results were measured by averaging the distance between consecutive points. It is clear that the second ball has to come from the right. If it came from the bottom left, it is impossible to increase the system's horizontal momentum. Therefore, we really only have two options.

1) Second ball comes from top right. Again, there are two options. Let us select $m_a = m_c$ and $m_b = m_d$. In this case, we have:

$$1.61m_a - 4.35m_d = 3.26m_a - 0.482m_d \implies m_a = -0.43m_d$$

Clearly this doesn't work. Let us now select $m_a = m_b$ and $m_c = m_d$

$$1.61m_a - 4.35m_d = -0.482m_a + 3.26m_d \implies m_a = 0.275m_d$$

$$1.61m_a + 1.85m_d = 1.58m_a + 1.96m_d \implies m_a = 0.273m_d$$

This could work, though let us look at the second case before deciding.

2) Second ball comes from bottom right. Again, there are two options. Let us select $m_a = m_d$ and $m_b = m_c$. In this case, we have:

$$1.61m_a - 3.26m_c = 4.35m_a - 0.482m_c \implies m_a = -0.99m_c$$

Clearly this doesn't work. Finally, we must have $m_a = m_b$. In this case, we have:

$$1.61m_a - 3.26m_c = -0.482m_a + 4.35m_c \implies m_a = 0.275m_c$$

$$1.61m_a - 1.96m_c = 1.58m_a - 1.85m_c \implies m_a = 0.273m_c$$

This gives a mass ratio of $\boxed{m_c/m_a = 3.6}$.

Now, notice that this mass ratio makes both coming from the top right and coming from the bottom right possible. However, notice that coming from the bottom right to the top right, the ball will pick up momentum. Due to Newton's third law, the ball coming from the top left must lose momentum, which is indeed the case. If the ball instead came from the top right, both balls would be losing momentum, violating Newton's third law. Therefore, the ball came from $\boxed{\text{the bottom right}}$

pr 40. The key difference between the barrels is that the walls in barrel provide a non-zero momentum to every small elemental mass that exits through the tap, while the other does not. Some non-zero work is done by the force exerted by these walls on the water molecules, which is not true for the other . So we first write the conservation of energy equation for the barrel: Let the small dm mass of water element exit at a velocity v_1 from the tap of the barrel,

$$\frac{1}{2}dm{v_1}^2 = dmgH$$

and by impulse momentum theorem,

$$F_{\text{walls}} = dmv_2 \implies (\rho g A_0 H)dt = (\rho A_0 v_2 dt)v_2$$

From these two equations, we have the answers $\boxed{\sqrt{2gH}}$ and $\boxed{\sqrt{gH}}$.

pr 41. At a small incremental time dt, the mass dm that is poured onto the conveyor belt is given by

$$dm = \mu dt$$

This implies that the change in momentum dp is given by (v is the velocity of the conveyor belt that is pulling the sand up)

$$dp = dmv = \mu v dt.$$

By Newton's second law, the amount of force that is directed upwards on the crane is given by

$$F_{\text{up}} = \frac{dp}{dt} = \frac{\mu v dt}{dt} = \mu v.$$

We note that there is a force of gravity that is directed downwards. The total mass of all the sand grains on the conveyor belt with length ℓ is given by $m = \frac{\mu}{v}\ell$[a] thus, the force that is directed downwards is given by

$$F_{\text{down}} = mg\sin\alpha = \frac{\mu}{v}\ell\sin\alpha$$

This gives us the total force to bring sand grains up on a conveyor belt as

$$F = \mu v + \frac{\mu}{v}\ell \sin\alpha.$$

Minimizing this function by differentiating with respect to v gives us

$$\mu - \frac{\mu}{v^2}\ell \sin\alpha \implies v = \sqrt{g\ell \sin\alpha}.$$

Substituting this expression back into our expression for force gives us

$$F_{\text{min}} = \mu\sqrt{g\ell\sin\alpha} + \frac{\mu}{\sqrt{g\ell\sin\alpha}}\ell\sin\alpha = 2\mu\sqrt{g\ell\sin\alpha}$$

The minimum torque is then given by

$$\tau = F_{\text{min}} \cdot R = \boxed{2\mu R\sqrt{g\ell\sin\alpha}}$$

[a] $m = \sigma L$ where $\sigma v = \mu$.

pr 42. The velocity of the blob just when the blob is about to hit the surface is found by conservation of mechanical energy

$$\frac{1}{2}mv^2 = mgh \implies v = \sqrt{2gh}$$

The Impulse (change in momentum) imparted perpendicular to the blob is clearly

$$\Delta p_\perp = m\sqrt{2gh}$$

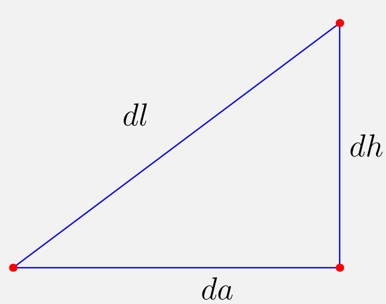
From idea 60,

$$\Delta p_\perp = \int \mu N dt = \mu \Delta p_\parallel \implies \Delta p_\perp = \mu m\sqrt{2gh}$$

Hence

$$\boxed{v = u - \mu\sqrt{2gh}}$$

pr 43. Let us observe what happens to the work done at small changes of da and dh.



The work due to friction will be

$$W_f = \mu mg\cos\phi dl$$

Since $dl\cos\phi = da$ then

$$W_f = \mu mgda$$

Integrating all of these small work variables from 0 to a gives us the work produced by friction as

$$W_f = \mu mga$$

The work produced by gravity is mgh thus the total work W_{tot} is

$$W_{\text{tot}} = mgh + \mu mga = \boxed{mg(h + \mu a)}$$

pr 44. First, let us use a little bit of lagrangian formalism to make the problem slightly easier. Let ξ be the displacement along the slanted surface. The kinetic energy is given by

$$K = \frac{1}{2}I\omega^2 + \frac{1}{2}(M+m)\dot{\xi}^2$$

We know by the basic formula $v = \omega r$, that $\omega = \frac{\dot{\xi}}{R}$, thus

$$K = \frac{1}{2}MR^2\left(\frac{\dot{\xi}}{R}\right)^2 + \frac{1}{2}(M+m)\dot{\xi}^2$$

Moving around variables gives us

$$K = M\dot{\xi}^2 + \frac{1}{2}m\dot{\xi}^2$$

From here we find that $\mathcal{M}$ is given by

$$K = \frac{1}{2}(2M)\dot{\xi}^2 + \frac{1}{2}m\dot{\xi}^2 \implies \mathcal{M} = 2M + m$$

The potential energy at a small displacement ξ is given by

$$\Pi(\xi) = (M+m)g\sin\alpha\xi \implies \Pi'(\xi) = (M+m)g\sin\alpha$$

Therefore, we get that the acceleration is given by

$$a = \frac{\Pi'(\xi)}{\mathcal{M}} = \frac{(M+m)g\sin\alpha}{2M+m}$$

Substituting in $m = M/2$ and $\alpha = 45^\circ$ given in the problem gives us

$$a = \frac{3g}{5\sqrt{2}}$$

Having found the acceleration a, we now move into the non-inertial frame. With a geometrical argument, we see that by law of sines

$$\frac{\sin(\alpha-\beta)}{ma} = \frac{\sin(90+\beta)}{mg}$$

Rearranging gives us

$$\frac{a}{g} = \frac{\sin(\alpha-\beta)}{\cos\beta} = \sin\alpha - \cos\alpha\tan\beta$$

Substituting our value for a gives us

$$\frac{3}{5\sqrt{2}} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\tan\beta$$

$$\tan\beta = \frac{2}{5} \implies \boxed{\beta = \arctan\frac{2}{5}}$$

pr 45. **a)** The angular momentum of the rod about the end of it's axis before the collision is defined by

$$L_0 = Mvl - \frac{1}{3}Ml^2\omega.$$

After the rod collides with the post it's angular momentum right after impact is

$$L_a = Mv'l - \frac{1}{3}Ml^2\omega'.$$

Since angular momentum is conserved in the entire process we have

$$L_0 = L_a \implies Mvl - \frac{1}{3}Ml^2\omega = Mv'l - \frac{1}{3}Ml^2\omega' \implies v - \frac{1}{3}\omega l = v' - \frac{1}{3}\omega' l.$$

We know that the condition for the rod being at the end, is the relation

$$v' + l\omega' = 0 \implies \omega' = -\frac{v'}{l}$$

Substituting our relation of ω' and v' into our simplified angular momentum equation gives us

$$v - \frac{1}{3}\omega l = \frac{4}{3}v'$$

$$\boxed{v' = \frac{3v - \omega l}{4}}$$

b) From part a) we have the equation

$$v - \frac{1}{3}\omega l = v' - \frac{1}{3}\omega' l.$$

The kinetic energy before is

$$K = \frac{1}{2}Mv^2 + \frac{1}{2}\left(\frac{1}{3}Ml^2\right)\omega^2 = \frac{1}{2}Mv^2 + \frac{1}{6}Ml^2\omega^2.$$

Therefore we have to equations of conservation of kinetic energy and angular momentum

$$3v - \omega l = 3v' - \omega' l$$
$$3v^2 + \omega^2 l^2 = 3v'^2 + \omega'^2 l^2$$

rearranging both of these equations and factoring gives us two new equations of

$$3(v - v') = l(\omega - \omega')$$
$$3(v^2 - v'^2) = l^2(\omega'^2 - \omega^2)$$

Solving these equations gives us $\boxed{v' = \frac{v - \omega l}{2}}$.

pr 46. We use the idea that if a body collides with something, then its angular momentum is conserved with respect to the point of impact. Upon impact with the ball, the rotation is reversed. When the ball hits the sweet spot of the bat, the hand-held end of the bat should come to halt without receiving any impulse from the hand. Let us use this information to try to solve this problem. Let x be the distance from the center of rotation to the center of percussion. The angular momentum with respect to the impact point before collision will then be

$$L_i = mv\left(x - \frac{\ell}{2}\right) - I_0\omega$$

where $v = \omega\frac{\ell}{2}$ and $I_0 = \frac{1}{12}m\ell^2$. After the impact, the bat turns backwards with an angular velocity ω', thus the angular momentum after is

$$L_a = mv'\left(x - \frac{\ell}{2}\right) - I_0\omega'$$

where $v' = \omega'\frac{\ell}{2}$. We also remember that the bat should come to a halt without recieving any impulse from the hand which means that the angular momentum with respect to the center of rotation after is actually 0. This means that

$$L_a = mv'\left(x - \frac{\ell}{2}\right) - I_0\omega' = 0.$$

This intuitively makes sense because ω' will have to be zero after collision. Setting up our angular momentum equations $L_i = L_a$ gives us

$$L_i = mv\left(x - \frac{\ell}{2}\right) - I_0\omega = 0$$

$$m\left(\frac{\omega\ell}{2}\right)\left(x - \frac{\ell}{2}\right) = \frac{1}{12}m\ell^2\omega$$

$$x - \frac{\ell}{2} = \frac{\ell}{6} \implies \boxed{x = \frac{2\ell}{3}}$$

pr 47. Let f be the frictional force created by the floor. We then have two equations

$$F - f = m\ddot{x}$$
$$fR - F(R-a) = I\ddot{\theta}$$

We first find the force created by friction. Noting that the YoYo rolls without slipping. We use the relation $\ddot{x} = r\ddot{\theta}$. Substituting this into the second equation gives us

$$fR - F(R-a) = \frac{1}{2}mR^2\frac{\ddot{x}}{R} \implies fR - F(R-a) = \frac{1}{2}R(m\ddot{x})$$

Substituting in our first equation gives us

$$fR - F(R-a) = \frac{1}{2}R(F - f)$$

Rearranging and simplifying gives us

$$\frac{3}{2}fR = \frac{3}{2}FR + Fa$$

This tells us

$$f = F + \frac{2}{3}\frac{Fa}{R}.$$

Now substituting our relation for friction into our first equation gives us

$$F - \left(F + \frac{2}{3}\frac{Fa}{R}\right) = m\ddot{x}$$

$$\boxed{|a| = \frac{2}{3}\frac{Fa}{mR}}$$

By Parallel axis theorem we see that

$$I' = I_0 + m\ell^2 \implies I' = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$$

The torque produced by the string is given by

$$\tau = Fa = I\alpha \implies Fa = \frac{3}{2}MR^2\left(\frac{a_r}{R}\right)$$

Simplifying gives

$$Fa = \frac{3}{2}MRa_r$$

$$a_r = \boxed{\frac{2}{3}\frac{Fa}{MR}}$$

pr 48. Let us direct the z axis upward (this will fix the signs of the angular momenta). We first attempt to find the initial angular momentum. We note that

$$L = MvR + I\omega$$

Substituting in $I = \frac{2}{5}MR^2$ and $v = \omega/R$ gives us $L = \frac{7}{5}MvR$. In the x-axis, the sign of angular momentum is negative because of the right hand rule, and in the y-axis the sign of angular momentum is postive. This gives us,

$$L_x = -\frac{7}{5}Mv_{y_0}R$$

$$L_y = \frac{7}{5}Mv_{x_0}R$$

The ball will continue to move in the same velocity in the y-direction as no non-conservative forces are acting in the horizontal direction. In the x-axis, the ball will have a final velocity of u, which implies that the final angular momentum is

$$L_{x_f} = -\frac{7}{5}Mv_yR - MuR$$

Setting this equal to the initial angular momentum because of conservation of angular momentum, we get

$$-\frac{7}{5}Mv_yR = -\frac{7}{5}Mv_yR - MuR$$

$$\frac{7}{5}v_{y_0} = \frac{7}{5}v_y + u$$

$$v_y = v_{y_0} - \frac{5}{7}u$$

This gives the final velocity to be $\boxed{(v_{x0}, v_{y0} - \frac{5}{7}u)}$.

pr 49. Immediately after the first collision, the center of mass of both dumbbells are at rest. Then, the velocities of the colliding balls reverse direction and the non-colliding balls' velocities don't change. Both dumbbells act like pendula and complete half an oscillation period, after which the second collision occurs – analogous to the first one where the dumbells expand outwards and hit each other. After that they separate and move a distance L to create SHM.

Thus, let us create three times t_1, t_2, and t_3 summing all the individual time components results in the total time t for SHM.

Calculating t_1: t_1 is the time when the dumbells' first hit each other when they are first initially separated a distance L. Both dumbells move at an initial velocity v_0, thus the time when both of them hit at the same time is equivalent to when one of the dumbbells travels a distance $L/2$. Therefore, using $v = d/t$, we get

$$t_1 = \frac{L}{2v_0}$$

Calculating t_2: After the collision, the velocity of the colliding balls reverse direction and the non-colliding ball's velocities don't change. This results in the spring to fully compress, the time for the spring to do so and then expand again to the second collision is t_2. Both dumbbells will act like an oscillator and complete half an oscillation period during time t_2. Both dumbells will move towards each other and compress the dumbell to half its length making the spring constant two times larger before recoil. Therefore, the oscillation period is given by

$$\omega = \sqrt{\frac{2k}{m}} \implies t_2 = \pi\sqrt{\frac{m}{2k}}$$

Calculating t_3: The last time, t_3 is simply the time for both dumbbells to move outwards a distance L. This is the same as t_1 or $\frac{L}{2v_0}$.

Thus, the total time is

$$t = t_1 + t_2 + t_3 = \frac{L}{2v_0} + \pi\sqrt{\frac{m}{2k}} + \frac{L}{2v_0} = \boxed{\frac{L}{v_0} + \pi\sqrt{\frac{m}{2k}}}$$

pr 50. We use the fact that effective gravity is given as $g_{\text{eff}} = g\cos\alpha$. This directly means that

$$T = 2\pi\sqrt{\frac{R}{g\cos\alpha}}.$$

The particle will exit at B if the time to cross the trough along its axis is an integer multiple of the oscillation's half-period. Thus, the length will be given as [a]

$$L = \left(n + \frac{1}{2}\right)\frac{T}{2}.$$

Thus,

$$\begin{aligned}L &= \frac{1}{2} g \sin\alpha \left(\left(n+\frac{1}{2}\right)\frac{T}{2}\right)^2 \\ &= \frac{1}{2} g \sin\alpha \left(n+\frac{1}{2}\right)^2 \frac{\pi^2 r}{g\cos\alpha} \\ &= \boxed{\frac{\pi^2}{2}\tan\alpha\left(n+\frac{1}{2}\right)^2}\end{aligned}$$

[a] There is a factor of one half added to the statement because not all the particles exit at the bottom of the gutter

pr 51. Label the three scenarios from left to right as A, B, and C.
The period is proportional to $T^2 \propto \frac{I}{\ell_{\rm cm}}$. Since all the hangers have the center period, this ratio must be the same for all three situations. If the moment of inertia about the center is I_0 then we have:

$$\frac{I_0 + M\ell_a^2}{\ell_a} = \frac{I_0 + M\ell_b^2}{\ell_b} = \frac{I_0 + M\ell_c^2}{\ell_c}$$

Due to symmetry, the center of mass must lie on the vertical line passing through the position of the pin at A and B. This gives us:

$$\ell_a + \ell_b = 0.1$$

Looking at the first pair of this three-way equality, we see that it is a quadratic. We do not have to invoke the quadratic formula here! From inspection we see that the trivial solution is $\ell_a = \ell_b$. However, there is another solution if this condition isn't satisfied! However, if $\ell_a \neq \ell_b$, then we can see that either $\ell_a = \ell_c$ or $\ell_b = \ell_c$ per the same reasoning. Or in short, at least two of ℓ_a, ℓ_b, or ℓ_c are the same.
However, since the center of mass can't lie outside the clothes hanger, we know that $\ell_a \neq \ell_c$ and $\ell_b \neq \ell_c$ so we must have $\boxed{\ell_a = \ell_b = 0.05 \text{ m}}$
We can now determine the third length to be $\ell_c = \sqrt{0.21^2 + 0.05^2} = 0.216$ m Using the first and third pair of the three-way equality, we can solve for I_0 to be:

$$I_0 = M(\ell_c + \ell_a)\ell_a\ell_c$$

thus, plugging it into the formula for period gives:

$$T = 2\pi\sqrt{\frac{M(\ell_c+\ell_a)\ell_a\ell_c + M\ell_a^2}{Mg\ell_a}} = \boxed{1.03 \text{ s}}$$

pr 52. Using method 6, we find that the kinetic energy of the system is

$$K = \frac{1}{2}(m + \alpha\rho_0 V)$$

where the constant α is a number that characterizes the geometry of the body that correspond to the extent of the region of the liquid that will move (compared to the volume of the body itself). This expression is obtained by noticing that the characteristic speed of the liquid around the body is v, and the characteristic size of the region where the liquid moves (the speed is not much smaller than v) is estimated as the size of the body itself. If a body is acted on by a force F, then the power produced by this force is

$$P = Fv = \frac{dK}{dt} = va(m + \alpha\rho_0 V).$$

Thus

$$F = a_0(m + \alpha\rho_0 V).$$

We also know by Archimedes principle that,

$$F = \rho V g - \rho_0 V g$$

Thus, by equating these two forces to each other, we get

$$a_0(m + \alpha\rho_0 V) = \rho V g - \rho_0 V g.$$

We know that $m = \rho V$ so,

$$a_0(\rho V + \alpha\rho_0 V) = \rho V g - \rho_0 V g \implies a_0(\rho + \alpha\rho_0) = \rho g - \rho_0 g$$

Dividing by a_0 on both sides, and subtracting ρ gives us

$$\alpha\rho_0 = \frac{\rho g - \rho_0 g}{a_0} - \rho \implies \alpha = \frac{1}{\rho_0}\left(\frac{\rho g - \rho_0 g}{a_0} - \rho\right)$$

Substituting known relations gives us $\alpha = 0.5$. For the rising bubble, the effective mass is exactly the same, however this time, the mass dm of the bubble is negligibly small. Thus, we have the equation

$$F = a_0(dm + \alpha\rho_0 V) = \rho V g - dmg$$

Taking $dm \approx 0$ gives us

$$a_0\alpha\rho_0 = \rho_0 g \implies a = \frac{g}{\alpha} = \boxed{2.0g}$$

pr 53. Let the incoming mass flow rate be labelled μ_i and it is divided into the flow rates μ_1 and mu_2. By the equation of continuity, we have

$$\mu_i = \mu_1 + \mu_2$$

By Bernoulli's law (or by idea 71 and fact 30), the velocity of both the left and the right compartment is same and equal to v. Now, by conservation of momentum in the horizontal direction (idea 72), we get

$$\mu_i\rho(v\cos\alpha) = \mu_1\rho(v) - \mu_2\rho(v)$$

which simplifies to

$$\mu_i \cos\alpha = mu_1 - \mu_2$$

From this equation and our initial equation of continuity, we have

$$\mu_1 = \mu_i \cos^2\frac{\alpha}{2}$$

and

$$\mu_2 = \mu_i \sin^2\frac{\alpha}{2}$$

Hence,

$$\frac{\mu_1}{\mu_2} = \frac{\mu_i\cos^2\frac{\alpha}{2}}{\mu_i\sin^2\frac{\alpha}{2}} = \boxed{\cot^2\frac{\alpha}{2}}$$

pr 54. Let $h(x)$ represent the height of the water above H at a point x and $v(x)$ be the velocity of the water. We then have by equation of continuity that

$$(H+h)(u+v) = Hu$$

Expanding this equation gives us

$$\begin{aligned} Hu + hu + Hv + hv &= Hu \\ hu + Hv &= 0 \end{aligned}$$

By Bernoulli's equation we have that

$$\begin{aligned} \frac{1}{2}p(u+v)^2 + pg(H+h) &= \frac{1}{2}pu^2 + pgH \\ uv + gh = 0 &\implies v = -\frac{gh}{v} \end{aligned}$$

Substituting this back into the equation we found for continuity gives us

$$hu = g\frac{Hh}{u} \implies \boxed{u = \sqrt{gH}}$$

A Generalization: For any given height h, the dispersion relationship is:

$$\omega^2 = gk\tanh(kh)$$

where ω is the angular frequency and k is the wavenumber (number of wavelengths per unit distance). For small values, we have $\tanh(kh) = kh$ or:

$$\omega^2 = gk^2h \implies \omega = k\sqrt{gh}$$

The speed the waves will be travelling at, or the phase velocity, will be:

$$\boxed{v = \frac{\omega}{k} = \sqrt{gh}}$$

For large values of h, we have $\tanh(kh) = 1$ or:

$$\omega^2 = gk \implies v = \sqrt{\frac{g}{k}}$$

This tells us that waves with higher wavenumbers (e.g. tsunamis) travel faster than waves with lower wavenumbers.

pr 55. **a)** We can draw an analogy with thermodynamics. Since the process is slow, and there are no external work being done, we can say that the process is adiabatic, that is:

$$TV^{\gamma-1} = \text{constant}$$

Here,

$$\gamma = \frac{1+2}{1} = 3$$

since there is only one degree of freedom. Therefore, when the distance doubles, the volume doubles and $V^{\gamma-1}$ will quadruple. As a result, T will decrease by a factor of four. However, since $T \propto v^2$, the speed must decrease by a factor of two. Therefore,

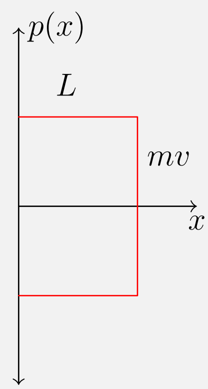
$$v = \boxed{5 \text{ m/s}}$$

b) The average force corresponds with the pressure. We have:

$$PV^\gamma = \text{constant}$$

Since the V^γ term increases by a factor of eight, pressure, and thus the average force will decrease by a factor of eight.

Solution 2: We can analyze the adiabatic invariant of the system. Let us draw a phase diagram of the entire system.



From here, we see that the adiabatic invariant of the system is given by

$$I = 2mvL$$

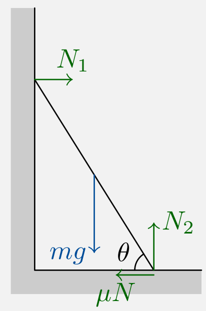
where v and L can change. Thus, the initial adiabatic invariant, I_0, is given by $2mv_0L$, and the final adiabatic invariant, I_f, is given by $2mv(2L) = 4mvL$. This means that

$$I_0 = I_f \implies 2mv_0L = 4mvL$$
$$v = \frac{1}{2}v_0 = \boxed{5 \text{ m/s}}$$

3 Solutions to Revision Problems

This section will contain problem 55-86 of the handout. Revision problems take concepts and ideas from earlier problems and places them in a new context. As a result, many of the problems in this section will seem familiar. This however, does not mean that all the problems in this section are easy. Some of the hardest problems originate in this section.

pr 56. **a)** Let the normal force from the floor on the ladder be N. Then, at the cutoff case, the friction force takes on it's maximum, so the friction from the floor is μN.



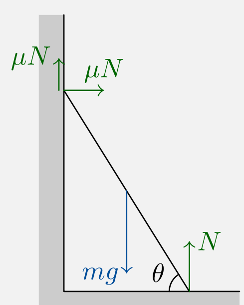
Since the ladder is in equilibrium, we have three equations. These is the equation of equilibrium of force in the horizontal and vertical direction and as well as torques. Looking quickly at the vertical forces, we can see easily that $N_1 = mg$. Then by looking at the horizontal forces, we see that $N_2 = \mu N$. Therefore, there is only one equation of torque remaining.
We first have to find the pivot point of the ladder. Generally, the pivot point of the system is located where there are more forces. Thus, by looking at the ladder, we see that the pivot point of the system is the bottom of the ladder. Balancing the torques due to gravity and N_2, we have

$$N_2 \ell \sin\theta = mg(\ell/2)\cos\theta \implies N_2 = \frac{mg}{2\tan\theta}$$

This is also the value of the frictional force F as we have found before. Thus, by using $F \geq \mu mg$ we find

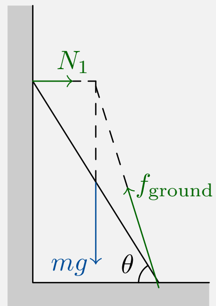
$$\frac{mg}{2\tan\theta} \leq \mu mg \implies \boxed{\tan\theta \geq \frac{1}{2\mu}}$$

b) Drawing a freebody diagram gives us the following diagram



We see that from force balance that there is not an opposite force to oppose the force of μN from the wall. This means that it is impossible for the ladder to stay still in this case.

There is an easier way to solve part a. Let us project the gravitational force vector mg and the normal force from the wall N_1 such that they meet at a point above the middle of the ladder. At this location, the torque caused by these two forces is zero. In order to be in static equilibrium, the force from the ground must also intersect this point. The slope the force from the ground makes with the horizontal is $2\tan\theta$.



Since the force from the ground is consisted of both the normal force N_2 and the friction force f_s, we have:

$$2\tan\theta = \frac{N_2}{f_s}$$

Combining this with $f_s \leq \mu N_2$ gives:

$$\boxed{\tan\theta \geq \frac{1}{2\mu}}$$

pr 57. Let us consider the situation when the bug has travelled a distance x from the upper end and is moving with a speed v_b. The stick always remains at rest, that is, making an angle α with the floor. When at a distance $\ell - x$ from the lower end of the rod, the torque on the bug about the bottom-most point (call this point O) is due to gravitational force on the bug and is equal to

$$\vec{\tau}_O = \vec{r} \times \vec{F} = mg(\ell - x)\cos\alpha\ \hat{k}$$

At this moment, the angular momentum of the bug can be written as

$$\vec{L}_O = mv\ell\sin\alpha\cos\alpha\ \hat{k}$$

So, we have

$$\vec{\tau}_O = \frac{d\vec{L}_O}{dt} \implies m\ell\sin\alpha\cos\alpha\frac{dv}{dt} = mg(\ell - x)\cos\alpha$$

$$\implies \frac{dv}{dt} = \frac{d^2x}{dt^2} = \frac{g}{\sin\alpha}(1 - x/\ell) = -\frac{g}{\sin\alpha}(x - \ell)$$

Notice that the second derivative of position of the bug (with respect to the point O) is proportional to the negative of its distance from point O. But this resembles the equation of a simple harmonic motion being executed about the mean position O! So, as found above, we have

$$a_{\text{bug}} = \boxed{\frac{g}{\sin\alpha}(1 - x/\ell)}$$

where $\sqrt{\frac{g}{l\sin\alpha}}$ can be considered to be the angular velocity of this hypothetical simple harmonic motion. The time taken by the bug to reach the bottom-most point is simply one-fourth the time period of the simple harmonic motion (T_0), so

$$T_{\text{bug}} = \frac{T_0}{4} = \boxed{\frac{\pi}{2}\sqrt{\frac{\ell\sin\alpha}{g}}}$$

Solution 2: The stick provides an acceleration a to the bug, so the bug exerts a force ma to the rod, pointing along the rod. In order to be stationary, we must have the normal force from the ground be $N_1 = mg$. To balance torques about the bug, we have:

$$N_1 \cos\alpha(\ell - x) = N_2 \sin\alpha\ell \implies N_2 = mg \cot\alpha\,(1 - x/\ell)$$

The net force has to be zero, so we can add the vectors N_1, N_2, and ma (which forms a right angle triangle). The horizontal component of the force the bug exerts on the rod $ma\cos\alpha$ has to balance out N_2 or:

$$ma\cos\alpha = mg \cdot \frac{\cos\alpha}{\sin\alpha}(1 - x/\ell) \implies \boxed{a = \frac{g(1 - x/\ell)}{\sin\alpha}}$$

This can also be written as:

$$\ddot{x} = -\frac{g}{\ell\sin\alpha}x + \frac{g}{\sin\alpha}$$

This gives the equation for simple harmonic motion with a period of:

$$T = 2\pi\sqrt{\frac{\ell\sin\alpha}{g}}$$

Travelling from the top to the bottom corresponds with one quarter of the period (maximum potential energy to maximum kinetic energy), so:

$$\boxed{t = \frac{\pi}{2}\sqrt{\frac{\ell\sin\alpha}{g}}}$$

pr 58. The key insight is noting if the net vertical forces of the normal and friction forces were directed downwards then the stopper would be blocked. Let us then try to calculate the vertical components of forces that are involved. Let the normal force directed on the wedge be N. We then know that the vertical component of the normal force is clearly either $N\cos\alpha$ or $N\sin\alpha$. We can figure the component by chasing angles around, but an easier way is to imagine $\alpha \to 0$. In this case, the horizontal component of the normal force also goes to zero, which is the behavior of a sine function, so the horizontal component is $N\sin\alpha$. This in turn means that the vertical component of force involved is $N\cos\alpha$.

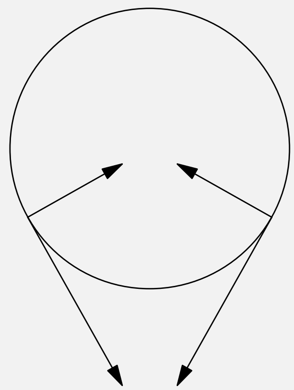
We now try to find the vertical component of friction involved. The friction force directed downwards on the direction of the wedge is μN (because N is already perpendicular, you do not have to manipulate it with trigonometric functions). This means that the vertical component of friction is $\mu N\sin\alpha$.

We now equate these, with an inequality where the vertical component of the normal force greater than the friction force for the wedge to pass through.

$$N\cos\alpha > \mu N\sin\alpha$$
$$\boxed{\mu < \cot\alpha}$$

pr 59. Two forces act on the rod in the vertical direction, it's weight and the force of friction, where at its maximum is μN_1. As the weight increases, we must have N_1 increase as well. Let us look at the limiting case where $W_{\text{rod}} \to \infty$. The normal and friction forces acting on the cylinder will be so large that the mass of the cylinder will be negligible, thus we can ignore the force mg. This allows us to effectively turn gravity off.

Let us now rotate the setup by an angle $\alpha/2$ such that it is completely symmetrical along its vertical axis. It is clear that the horizontal forces will cancel each other out.



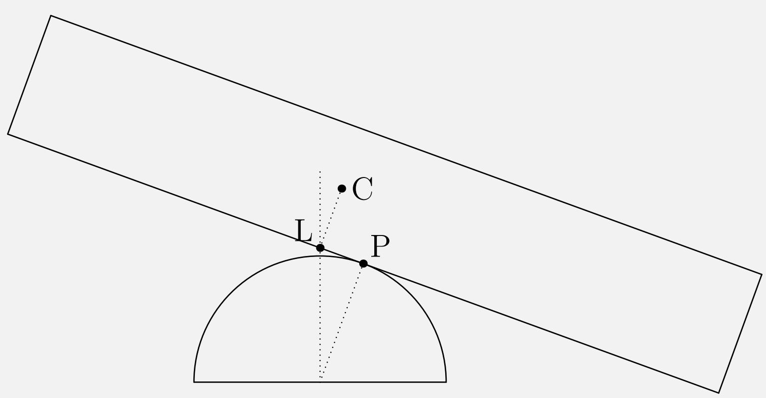
Now we just have to balance out vertical forces. Due to symmetry, the y-component of each friction force cancels out with the y-component of each normal force. For the left side, we have:

$$N_1 \sin(\alpha/2) = \mu_1 N_1 \cos(\alpha/2) \implies \boxed{\mu_1 > \tan(\alpha/2)}$$

We have the inequality since the force balance equation gives the maximum friction. Similarly for the other side:

$$\boxed{\mu_1 > \tan(\alpha/2)}$$

pr 60.



Here P is the point of contact of the plank with the hemisphere after turning through θ and L is the original contact point. C is the centre of mass of the plank.

To solve the problem, let us turn the plank through an angle of θ and consider the torques at that moment. First of all, we see that

$$\angle POL = 90^\circ - \angle OLP = \theta$$

We want C to generate a clockwise torque about P, so the distance between C and OL must be greater than the distance between P and OL. This means

$$\frac{h}{2}\sin\theta < R\sin\theta \implies \boxed{R > \frac{h}{2}}$$

If the initial position was stable, then slight deviations would cause the center of mass to be higher. Therefore, we want:

$$c\cos\theta + R\theta\sin\theta + \frac{h}{2}\cos\theta > R + \frac{h}{2}$$

Using $\cos\theta \approx 1 - \frac{\theta^2}{2}$ and $\sin\theta \approx \theta$, we can rewrite the above inequality as:

$$R\left(1 - \frac{\theta^2}{2}\right) + R\theta(\theta) + \frac{h}{2}\left(1 - \frac{\theta^2}{2}\right) > R + \frac{h}{2}$$

Simplifying, we see that the angle θ cancels out and we are left with:

$$\boxed{R > \frac{h}{2}}$$

pr 61. Similar to problem 16, we want:

$$\rho g h(\pi R^2) = mg + V\rho g$$

except this time:

$$\begin{aligned}
V &= \frac{2}{3}\pi R^3 - \pi H^2\left(R - \frac{H}{3}\right) \\
&= \frac{2}{3}\pi R^3 - \pi(R-h)^2\left(\frac{2R+h}{3}\right) \\
&= \frac{2}{3}\pi R^3 - \frac{\pi}{3}(2R^3 + R^2h - 4R^2h - \cancel{2Rh^2} + \cancel{2Rh^2} + h^3) \\
&= \pi R^2 h - \frac{\pi}{3}h^3
\end{aligned}$$

Plugging this in gives:

$$\cancel{\rho g h(\pi R^2)} = mg + \cancel{\pi R^2 h\rho g} - \frac{\pi}{3}h^3\rho g$$

or:

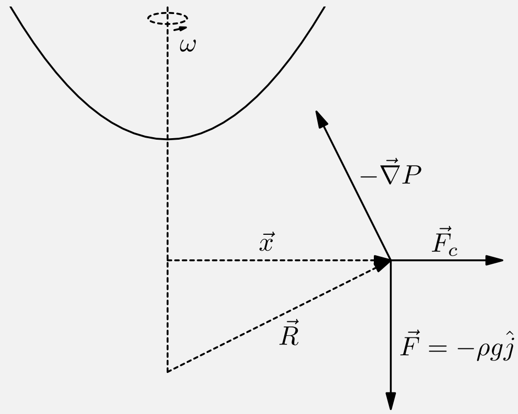
$$mg = \frac{\pi}{3}h^3\rho g \implies \boxed{h = \sqrt[3]{\frac{3m}{\pi\rho}}}$$

Verifying, if we plug $m = \frac{\pi}{3}\rho R^3$, we do indeed get $h = R$.

pr 62. Assume the water surface pressure uniform. In the rotating frame of the water, every water element is at rest. So in this rotating frame, the hydrostatics equation is

$$\vec{F} + \vec{F}_{\text{centrifugal}} - \vec{\nabla}P = 0$$

where $\vec{F} = -\rho g\hat{j}$ is the force on the body per unit volume, $\vec{F}_{\text{centrifugal}}$ the centrifugal force, and $-\vec{\nabla}P$ is the force due to the pressure gradient.



Clearly, by definition we have $\vec{F}_{\text{centrifugal}} = -\rho\vec{\omega} \times (\vec{\omega} \times \vec{R})$, hence the hydrostatic equation becomes

$$\vec{\nabla}P = -\rho g\hat{j} + \rho\omega^2 x\hat{i}$$

$$\Rightarrow \frac{\partial P}{\partial x} = \rho\omega^2 x \quad ; \quad \frac{\partial P}{\partial y} = -\rho g$$

Integrating, we have

$$P = \frac{\rho\omega^2 x^2}{2} - \rho g y$$

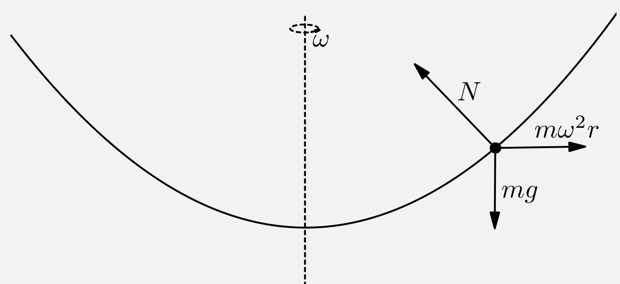
And assuming the surface pressure constant, this yields

$$\frac{\rho\omega^2 x^2}{2} = \rho g y \Rightarrow y = \frac{\omega^2}{2g}x^2$$

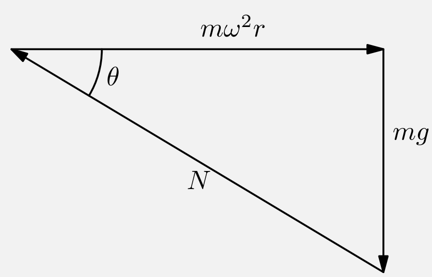
Thus the cross sectional surface of the rotating water is a parabola with this equation Hence substituting $x = R$ gives the relative height near the edges of the vessel, which is just

$$\Delta h = \boxed{\frac{\omega^2}{2g}R^2}$$

Solution 2: Consider a water particle on the surface. In the rotating frame, it experiences three forces, a gravitational force downwards, a centrifugal force outwards, and a normal force perpendicular to the surface.



The three forces must sum up to zero. If we add them geometrically, they form a closed right angle triangle



where $\tan\theta = \frac{g}{\omega^2 r}$ Since the normal force is perpendicular to the surface, the slope of the surface at this point is:

$$\frac{dh}{dr} = \cot\theta = \frac{\omega^2 r}{g}$$

Integrating, from 0 to R, we get:

$$\boxed{\Delta h = \frac{\omega^2 R^2}{2g}}$$

pr 63. First, let us move into an accelerated reference frame such that M is stationary. The acceleration of M is:

$$Ma = T - T\sin\alpha \implies a_M = T\left(\frac{1-\sin\alpha}{M}\right)$$

Thus, m will have a fictitious force acting towards the right. The actual gravitational force and the fictitious force combine together to give us the effective gravity.
Now keep in mind that even in this accelerated reference frame, m is not stationary. It is actually moving in the direction parallel to the rope holding it and due to conservation of rope, the acceleration of m in this frame is $a_m = a_M$. Balancing forces, we get:

$$mg_{\text{eff}} - T = ma_M$$

Substituting in $g_{\text{eff}} = \frac{mg}{\cos\alpha}$ and a_M, we get:

$$\frac{mg}{\cos\alpha} - T = T\left(\frac{m}{M}\right)(1-\sin\alpha)$$

We can solve for T to be:

$$T = \frac{mg}{\cos\alpha}\cdot\frac{1}{1+(m/M)(1-\sin\alpha)}$$

The ratio of the fictitious force and the gravitational force form a right angle, such that:

$$\tan\alpha = \frac{a_M}{g} = \frac{T}{mg}(1-\sin\alpha)$$

Substituting in T, cancelling out $\cos\alpha$ on both sides, and solving for m/M (the algebra takes some time), we get:

$$\boxed{\frac{m}{M}} = \frac{\sin\alpha}{(1-\sin\alpha)^2}$$

pr 64. Let the normal force on the cylinder be N_1, the normal force on the wedge be N_2, and the normal force between the cylinder and wedge be N. If the cylinder moves downwards with an acceleration a_1 and the wedge moves upwards with an acceleration a_2, we find from geometry that we have a constraint equation of

$$Na_1\cos(180-\alpha) + Na_2\cos\alpha = 0$$
$$-a_1 + a_2 = 0 \implies a_1 = a_2$$

To find the acceleration, we use lagrangian formalism. Let the cylinder move down by a small amount ξ, the wedge will then move upwards ξ meaning that the change in potential energy will become

$$\Pi(\xi) = (m-M)g\sin\alpha\xi$$

Differentiating this with respect to ξ gives us

$$\Pi'(\xi) = (m - M)g \sin\alpha.$$

The kinetic energy of the system in this case will then be given by

$$K = \frac{1}{2}(m + M)\dot{\xi}^2$$

which implies that $\mathcal{M} = m + M$. This lets the acceleration become

$$a = \frac{m - M}{m + M} g \sin\alpha.$$

Projecting Newton's laws onto the cylinder gives us the equation

$$mg \sin\alpha - N \cos\alpha = ma$$

Substituting our value of accceleration gives us

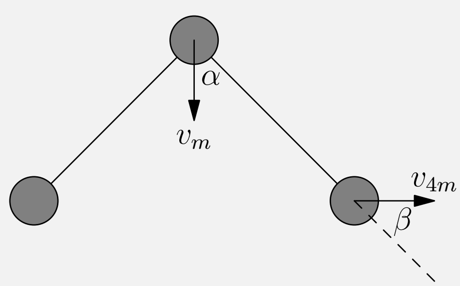
$$mg \sin\alpha - N \cos\alpha = m\frac{m - M}{m + M} g \sin\alpha$$

rearranging variables gives us

$$mg \sin\alpha \left(1 - \frac{m - M}{m + M}\right) = N \cos\alpha$$

$$\boxed{N = 2\frac{Mm}{m + M} g \tan\alpha}$$

pr 65.



Let us analyze the velocities of the masses. Because the rod is in-compressible, their relative velocities radial to the rod must sum up to zero, or:

$$v_m \cos\alpha = v_{4m} \cos\beta$$

Differentiating, we get:

$$a_2 \cos\alpha + (-v_m \sin\alpha)\frac{d\alpha}{dt} = a_1 \cos\beta + (-v_{4m} \sin\alpha)\frac{d\beta}{dt}$$

Since the initial velocities are 0, we have:

$$a_2 \cos\alpha = a_1 \cos\beta$$

Since $\cos\alpha = \cos\beta$, this implies that their accelerations along the rod are the same. For the top mass, we have the net force along the right rod as:

$$\frac{mg}{\sqrt{2}} - F_1 = ma_2 \cos\alpha$$

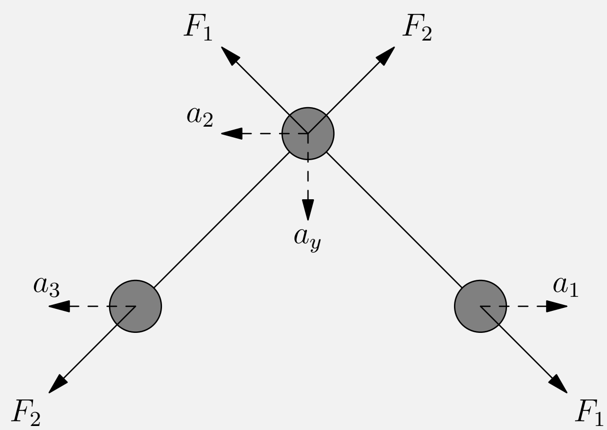
where F_1 is the force exerted by the rod. Balancing forces in the horizontal direction for the rightmost mass, we get:

$$\frac{F}{\sqrt{2}} = 4ma_1$$

Combining, we have:

$$\boxed{a_1 = \frac{g}{4}}$$

Solution 2: Denote the horizontal accelerations of the three masses from right to left as a_1, a_2, and a_3 as shown in the diagram below. Let the vertical acceleration of the top mass be a_y.



There is no external force in the horizontal direction, therefore:

$$ma_2 + ma_3 = 4ma_1 \implies a_2 + a_3 = 4a_1$$

It can be shown as in the first solution that the component of acceleration of the top mass along the left rod must be the same as the component of acceleration of the left mass along the left rod. This means:

$$a_2 \cos 45^\circ + a_y \sin 45^\circ = a_3 \cos 45^\circ \implies a_y + a_y = a_3$$

The same is true along the right rod, giving:

$$a_y - a_2 = a_1$$

Solving these three equations, we get:

$$a_y = 2a_1$$
$$a_2 = a_1$$
$$a_3 = 3a_1$$

Using Newton's Second Law on the rightmost mass, we have:

$$F_1 \cos 45^\circ = (4m)a_1$$

For the left mass,

$$F_2 \cos 45^\circ = ma_3 = m(3a_1)$$

For the top mass in the vertical direction. We get:

$$mg - (F_1 \cos 45^\circ + F_2 \cos 45^\circ) = ma_y = m(2a_1)$$

Substituting in F_1 and F_2 from above gives:

$$mg - 7ma_1 = 2ma_1 \implies \boxed{a_1 = \frac{g}{9}}$$

Solution 3: The top mass has 3 forces acting on it, a force F_2 exerted by the rod on left (compressive), F_1 exerted by the rod on right (compressive), and the force of gravity mg. The $4m$ mass has only horizontal acceleration. Therefore:

$$4mg + \frac{F_1}{\sqrt{2}} = N$$

Now, in the non inertial reference frame of the top mass, the $4m$ mass has only a tangential acceleration since the rod that separates them is fixed in length. As a result, we get the force balance:

$$\sum F_{\text{radial}} = \sum F_{\text{inertial}} + \sum F_{\text{pseudo}}$$

The inertial force acting on $4m$ in the radial direction is simply:

$$\frac{N - 4mg}{\sqrt{2}} - F_1$$

By switching into an accelerated reference frame where the top mass is at rest, we need to apply a pseudo-force which acts in the opposite direction of the top mass's net force. The acceleration of the top mass in the radial direction is:

$$a = \frac{F}{m} = \frac{1}{m}\left(\frac{mg}{\sqrt{2}} + F_1\right)$$

Thus, the radial component of the pseudo-force must point in the opposite direction. The pseudo-force the rightmost mass experiences is thus:

$$\sum F_{\text{pseudo}} = \frac{-\frac{mg}{m} \cdot 4m}{\sqrt{2}} - \frac{F_1}{m} \cdot 4m = \frac{-4mg}{\sqrt{2}} - 4F_1$$

Thus, we have:

$$0 = \left(\frac{N - 4mg}{\sqrt{2}} - F_1\right) + \left(\frac{-4mg}{\sqrt{2}} - 4F_1\right)$$

Substituting in N gives

$$F_1 = \frac{4\sqrt{2}mg}{9}$$

Therefore,

$$a_1 = \frac{F_1}{\sqrt{2} \cdot 4m} = \boxed{\frac{g}{9}}$$

pr 66. Let the acceleration of mass m along the incline be a and acceleration of mass M in downward direction be a_1. Since length of string remains constant therefore we have

$$a \sin \alpha = a_1$$

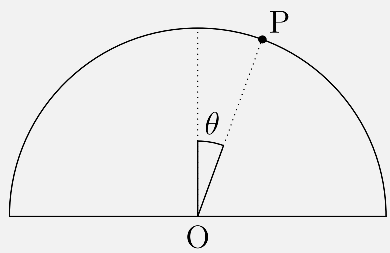
Writing the force equations for both masses gives us

$$T \sin\alpha + mg \sin\alpha = ma$$

$$mg - T = ma_1$$

Solving the three equations we get $\boxed{a_1 = g \sin^2\alpha \frac{M+m}{m + M sin^2\alpha}}$

pr 67.



Let the point at where the object slips of the hemisphere be P. We then have by energy conservation that

$$\frac{1}{2}mv^2 = mgR(1-\cos\theta)$$

This implies that v^2 is

$$v^2 = 2gR(1-\cos\theta)$$

At any point on the circle we have the Newton's third law pair of

$$F_g = F_c + F_N$$

However at the point where the object loses contact, the normal force becomes zero. This implies that

$$mg\cos\theta = \frac{mv^2}{R}$$

Taking out m from both sides and substituting v^2 gives us

$$g\cos\theta = \frac{2gR(1-\cos\theta)}{R}$$
$$2 - 2\cos\theta = \cos\theta$$
$$\cos\theta = \frac{2}{3}$$

We know that $\cos\theta = \frac{h}{R}$, thus the height at which the object loses contact is $\boxed{h = \frac{2}{3}R}$

pr 68. Consider the reference frame moving with velocity v to the right. This frame is easier to work with because here the end of the rod on the ground is at rest. Let's call this end A. Let $t = 0$ correspond to the time when the rod is vertical. At time t, the distance between A and the vertical wall is $d = vt$.

Let the angle between the rod and horizontal wall be

$$\theta = \frac{\pi}{2} - \alpha$$

such that

$$\cos\theta = \frac{vt}{2l}$$

Differentiating with respect to time gives

$$\dot{\theta}\sin\theta = \frac{v}{2l} \implies \dot{\theta} = \frac{v}{2l\sin\theta}$$

The velocity of the sphere is

$$\dot{\theta}(2l - x) = \frac{v(2l - x)}{2l\sin\theta}$$

perpendicular to the rod. Therefore, the $x-$ component of the velocity

$$v_x = \frac{v(2l - x)\sin\theta}{2l\sin\theta} = \frac{v(2l - x)}{2l}$$

is constant, implying that $a_x = 0$. Now the centripetal acceleration of the sphere

$$a_c = \dot{\theta}^2(2l - x) = \frac{v^2(2l - x)}{4l^2\sin^2\theta}$$

which is pointed towards A. We know that the acceleration is entirely in y, as $a_x = 0$. As a result:

$$\begin{aligned}
a\sin\theta &= \frac{v^2(2l - x)}{4l^2\sin^2\theta} \\
a &= \frac{v^2(2l - x)}{4l^2\sin^3\theta} \\
&= \frac{v^2(2l - x)}{4l^2\cos^3\alpha} \\
&= \frac{v^2}{2l\cos^3\alpha}(\frac{x}{2l} - 1)
\end{aligned}$$

which is pointed in negative y direction. By Newton's second law, we have:

$$N = mg - ma \implies \boxed{N = m(g - \frac{v^2(2l - x)}{\sqrt{2}l^2})}$$

Solution 2: We solve it for the general case. Let the place where both walls meet be the origin then we can write the coordinates of sphere as

$$X = x_1 - x\sin\alpha$$

$$Y = y_1 - x\cos\alpha$$

Now differentiating Y with respect to time we get

$$v_y = v\tan\alpha - v\tan\alpha\left(\frac{x}{2\ell}\right)$$

Now again differentiating it with respect to time we get

$$a_y = \omega v\sec^2\alpha\left(\frac{x}{2\ell} - 1\right)$$

Also we have

$$\omega = \frac{v}{2\ell\cos\alpha}$$

Substituting it we get

$$a_y = \frac{v^2}{2\ell}\sec^3\alpha\left(\frac{x}{2\ell} - 1\right)$$

Now using Newton's law in y direction we get

$$4mg - N = ma_y$$

Solving we get

$$N = m\left(g - v^2\frac{(2\ell - x)}{l^2\sqrt{2}}\right)$$

Note that $a_x = 0$. Since $v_x = v\left(1 - \frac{x}{2l}\right)$ is constant, therefore the rod/sphere will apply no force on each other in the horizontal direction.

pr 69. Let the velocity of the stick be v_1, the velocity of the box be v_2, m be the mass of the mass on the stick, M be the mass of the box, and α be the angle v_1 makes with the weight of the mass mg. Because v_2 is purely horizontal, we can easily see that[a]

$$v_1\sin\alpha = v_2$$

Next, we use conservation of energy. Comparing initial to final, we get

$$mgL = \frac{1}{2}mv_1^2 + \frac{1}{2}Mv_2^2 + mgL\sin\alpha$$
$$mgL(1-\sin\alpha) = v_1^2\left(\frac{m}{2} + \frac{M}{2}\sin^2\alpha\right)$$
$$v_1^2 = = \frac{2mgL(1-\sin\alpha)}{m + M\sin^2\alpha}$$

We now use idea 40. v_1 and v_2 are at maximum, thus $\mathbf{N}$ and $\mathbf{F} = 0$. From our first and third relation, we also have that

$$v_2 = 0 = \sqrt{\frac{2mgL(1-\sin\alpha)\sin^2\alpha}{m + M\sin^2\alpha}}.$$

Using Newton's Second Law on the block, we have the following $F = ma$ equation,

$$Ma_0\sin\alpha - \mathbf{N} = \frac{Mv_1^2}{L}\cos\alpha.$$

Using idea 40, $\mathbf{N} = 0$ at the moment of leaving contact and thus,

$$a_0\sin\alpha = \frac{v_1^2}{L}\cos\alpha$$

Substituting in v_1 and simplifying gives

$$a_0 = \frac{2mgL(1-\sin\alpha)\cos\alpha}{\sin\alpha(m + M\sin^2\alpha)}$$

which is equal to $g\cos\alpha$ as gravity is the only force causing the acceleration and $N = F = 0$). This means that

$$\begin{aligned} g\cos\alpha &= \frac{2mgL(1-\sin\alpha)\cos\alpha}{\sin\alpha(m+M\sin^2\alpha)} \\ 2m(1-\sin\alpha) &= m\sin\alpha + M\sin^3\alpha \\ m(2-3\sin\alpha) &= M\sin^3\alpha \\ \frac{M}{m} &= \frac{2-3\sin\alpha}{\sin^3\alpha} \end{aligned}$$

For $\alpha = \frac{\pi}{6}$, this means that $\boxed{\frac{M}{m} = 4}$. Using this information from what we found, simplifies our expression for v_2 into

$$v_2 = \sqrt{\frac{2mgL(1-\sin\alpha)\sin^2\alpha}{m+4m\sin^2\alpha}} = \sqrt{gL \times \frac{1}{2} \times \frac{1}{4}} = \boxed{\sqrt{\frac{gL}{8}}}.$$

[a]This problem came in the book 'Aptitude Test Problems in Physics' by S.S. Krotov.

pr 70. The centre of mass of the system is initially on the pulley. In the horizontal direction, the net external force is provided by the tension forces (pulley), and it is always rightwards. (For the block to have covered an angle α, the net horizontal force is $\vec{F_{\text{ext}}} = T(1-\cos\alpha)\hat{i}$ which is always in positive $\hat{i}$ direction. Hence the centre of mass moves right to the pulley throughout the motion. At the time of collision, it is thus right of the pulley, which is only possible if the **left block** reaches the pulley first.[a]

[a]This problem came in the Moscow physics olympics in the 1970s, and is also in the book 'Aptitude Test Problems in Physics by S.S. Krotov. It is quite a famous problem.

pr 71. See problem 20, the hockey puck will travel in a straight line. Consider a differential piece located at (x, y) and has a vertical component of velocity v_y. At a location $(-x, y)$, there will be a differential piece with a vertical component of velocity $-v_y$. Their horizontal components of velocity will be the same. Thus, if we change their horizontal components of velocity, say by pushing the puck rightwards, their vertical components will change in the same way such that they still cancel out to zero. As a result, since there is always going to be another point which cancels out the perpendicular component of velocity, the net force caused by friction in the perpendicular direction will be zero. The puck will travel in a straight line.

The puck will travel further if $\omega \neq 0$. This is simply because it starts off with a higher energy so it'll take a longer distance for the energy to be completely dissipated.

pr 72. Note that due to the fact that the thread is extremely long, there are no horizontal forces exerted by the string and as a result, the horizontal momentum is conserved. We have:

$$mv = (M+m)u$$

Conservation of energy then gives:

$$\frac{1}{2}mv^2 = \frac{1}{2}(M+m)u^2 = mg\ell \implies \frac{1}{2}mv^2 = \frac{1}{2}\frac{m^2v^2}{M+m} + mg\ell$$

Solving for v, we get:[a]

$$v = \sqrt{2g\ell\,(1 + m/M)}$$

[a]It is good to check for limiting cases. If $m/M \to 0$, we can ignore the horizontal components of velocity and just conserve energy. We get the standard result $v = \sqrt{2g\ell}$. If $M \to \infty$, then we need to provide an infinite speed to raise the height as we stated the string is extremely long.

pr 73. We use idea 51 and list out all the possible combinations of objects moving together. Let us label the surface between the top two masses as S_1 and the surface between the bottom two as S_2. We have a few options:

- S_1 static, S_2 static
- S_1 kinetic, S_2 static
- S_1 static, S_2 kinetic
- S_1 kinetic, S_2 kinetic

(1) Let us look at the first case. If this is the case, then we have:

$$F = 3ma \implies a = \frac{F}{3m}$$

To analyze the conditions for this to occur, we must look at the friction forces. The second block experiences two friction forces from two surfaces. Both forces f_1 and f_2 have to satisfy $f_1 < mg\mu$ and $f_2 < 2mg\mu$. Since the first condition is harder to meet (and if met, the second one is also met), we will look at the first surface. The top-most block experiences two forces, a force of tension with magnitude $F/2$ and a friction force $-f_1$. Combined together, these forces give the bottom block the acceleration calculated above. We have:

$$m\left(\frac{F}{3m}\right) = \frac{F}{2} - f_1 \implies f_1 = \frac{F}{6}$$

Using the inequality $f_1 < mg\mu$ we get the condition:

$$\frac{F}{6} < mg\mu \implies \frac{F}{mg\mu} < 6$$

(2) Now let's look at the second case where the bottom two blocks stay together but the top two blocks slide against each other. We wish to balance forces on the bottom two blocks together, but we need to be careful of which direction the friction force points. The top block and the bottom two blocks are both being pulled by a string but since the top block is lighter, it will be pulled faster. As a result, the kinetic friction the top two blocks experiences points directly to the right. Balancing forces, we have:

$$(2m)a = \frac{F}{2} + mg\mu \implies a = \frac{F}{4m} + \frac{1}{2}g\mu$$

The conditions that has to be met in order for this to take place is: $f_1 = mg\mu$ and $f_2 < 2mg\mu$. For the second to be satisfied, we can balance forces for the bottommost block. Balancing forces, we have:

$$m\left(\frac{F}{4m} + \frac{1}{2}g\mu\right) = \frac{F}{2} - f_2 \implies f_2 = \frac{F}{4} - \frac{1}{2}mg\mu$$

Using the condition $f_2 < mg\mu$, we get:

$$\frac{F}{mg\mu} < 10$$

For the first condition to be met, we must have $\frac{F}{mg\mu} > 6$, if this was not the case, then all three blocks would start sliding together. We can prove this by balancing forces on the middle block. We have:

$$m\left(\frac{F}{4} + \frac{mg\mu}{2}\right) = f_1 + f_2 \implies \frac{F}{4} - f_1 + \frac{mg\mu}{2} < mg\mu$$

Setting $f_1 = mg\mu$ and isolating for F does indeed give us:

$$\frac{F}{mg\mu} > 6$$

(3) This is impossible. If the friction is strong enough such that the top two blocks can move together, then it must be so that the friction is strong enough the bottom two blocks can move together. This is because the normal force between the bottom two blocks will always be stronger than the normal force between the upper two blocks.

(4) This is an easy case. Balancing forces directly on the second block, we get:

$$ma = f_1 + f_2 = mg\mu + 2mg\mu \implies a = 3g\mu$$

This is the case when the applied force crosses the upper boundary set above, which was $\frac{F}{mg\mu} < 10$. Therefore, complete slipping occurs when

$$\frac{F}{mg\mu} > 10$$

Finally, we can summarize our results:

$$\boxed{a = \begin{cases} \frac{F}{3m} & \frac{F}{mg\mu} < 6 \\ \frac{F}{4m} + \frac{g\mu}{2} & 6 < \frac{F}{mg\mu} < 10 \\ 3g\mu & \frac{F}{mg\mu} > 10 \end{cases}}$$

pr 74. During the collision, we can conserve momentum. We have a perfectly inelastic collision where:

$$mv = (M+m)u \implies u = \frac{mv}{M+m}$$

However, since the boy continuously pushes against the other boy, the applied force will be $mg\mu$. Balancing forces, we get:

$$(m+M)a = \mu mg - \mu Mg$$

Dividing through gives the magnitude of the deceleration to be

$$a = \frac{\mu g(M-m)}{M+m}$$

Thus:

$$d = \frac{v^2}{2a} = \frac{\left(\frac{mv}{M+m}\right)^2}{2\frac{(M-m)\mu g}{M+m}} = \boxed{\frac{m^2v^2}{2(M^2-m^2)\mu g}}$$

pr 75. Energy conservation gives:

$$\frac{1}{2}v^2 = \frac{1}{2}g\ell(1 - \sin\theta)$$

where θ is the angle the rod makes with the ground at the point of maximum extension of the string. We are restricted to a total vertical length of 2ℓ so we have:

$$H = 2\ell\sin\theta \implies \sin\theta = \frac{H}{2\ell}$$

Applying this to our energy conservation expression gives:

$$v^2 = g\ell(1 - H/2\ell) \implies \boxed{v = \sqrt{g(\ell - H/2)}}$$

Now, we use idea 44 and notice that horizontal acceleration of the centre must be zero; this follows from the Newton's 2nd law for the horizontal motion (there are no horizontal forces at that moment). Further, notice that the vertical coordinate of the centre of mass is arithmetic average of the coordinates of the endpoints,

$$x_O = \frac{1}{2}(x_A + x_B)$$

Noting that x_B must be constant, taking the time derivatives gives us

$$\dot{x_O} = \frac{1}{2}\dot{x}_A$$
$$\ddot{x_O} = \frac{1}{2}\ddot{x}_A$$

Hence, the acceleration of O can be found as half of the vertical acceleration of the rod's upper end A; this is the radial, i.e. centripetal component of the acceleration of point A on its circular motion around the hanging point. From here, we know from the common formula, that

$$a = \frac{v^2}{\ell}$$

substituting our expression for v^2 from part a) gives us

$$a = \frac{g\ell(1 - H/2\ell)}{\ell} \implies a = g(1 - H/2\ell).$$

At point x_O, the acceleration is then given by $\boxed{\frac{g}{2}\left(1 - \frac{H}{2\ell}\right)}$.

Solution 2: First, we make the following claim:

Claim. At any position the potential energy lost is converted into $E_{\text{rotational}} + E_{\text{translational}}$. i.e.

$$\Delta U = \frac{1}{2}I_{CM}\omega^2 + \frac{1}{2}mv_{CM}^2$$

Coincidentally for this system ΔU reaches its maxima and ω becomes 0 at the same time. When the thread becomes vertical, the angle made by the rod with the ground, β is minimum $\implies \omega = 0$.

Proof. If α is the angle made by the thread with the vertical,

$$l\cos\alpha + l\sin\beta = H$$
$$\sin\beta = \frac{H - l\cos\alpha}{l}$$

$|\alpha|$ is always acute here so $\cos\alpha$ reaches its maxima and β reaches its minimum at $\alpha = 0$. At the same instant, $y_{CM} = \frac{l\sin\beta}{2}$ reaches its minima.

When the thread is vertical:

$$y_{CM} = \frac{H-l}{2}$$

Initially:

$$y_{CM} = \frac{H}{4}$$

$$\therefore v_{max} = \sqrt{2g\left(\frac{H}{4} - \frac{(H-l)}{2}\right)} = \sqrt{g\left(l - \frac{H}{2}\right)}$$

At this instant, let the angular acceleration of the rod be α (into the plane) and COM's acceleration a (upwards)

$$\frac{ml^2}{12}\alpha = (T-N)\frac{l}{2}\cos\beta$$
$$T + N - mg = ma$$

The bottom most point must have 0 vertical acceleration:

$$\alpha\frac{l}{2}\cos\beta = a$$

The point connected to thread must have vertical acceleration $= \frac{v^2}{l}$

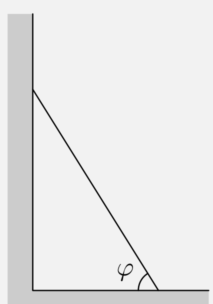
$$\alpha\frac{l}{2}\cos\beta + a = \frac{v^2}{l} = g\left(1 - \frac{H}{2l}\right)$$

Also, $\sin\beta = \frac{H-l}{l}$ Solving gives:

$$a = \frac{g}{2}\left(1 - \frac{H}{2l}\right)$$

$$T = \frac{mg}{4}\left(3 + \frac{l}{6H} - \frac{H}{2l}\right)$$

pr 76. First, we make the following claim:



Claim. The horizontal component of acceleration in the rod becomes zero at $\sin\varphi_c = \frac{2}{3}$.

Proof 1: Define a coordinate system with origin at the initial position of the lowest point of the rod. At any instant of time, the coordinates of the centre of mass of the system is $\mathrm{P}_{\mathrm{C},\varphi}(\frac{L}{2}\cos\varphi, \frac{L}{2}\sin\varphi)$ when the rod makes an angle φ with the horizontal. This means that the locus of the centre of mass is a circle of radius $\frac{L}{2}$[a]. Since we have

$$\omega = \frac{v_x}{L\sin\varphi} = \frac{\sqrt{gL\sin^2\varphi(1-\sin\varphi)}}{L\sin\varphi} = \sqrt{\frac{g}{L}(1-\sin\varphi)}$$

Differentiating gives

$$\alpha = \ddot{\varphi} = \sqrt{\frac{g}{L}} \times \frac{1}{2\sqrt{1-\sin\varphi}} \times -\cos\varphi \times \dot{\varphi} = -\frac{g}{2L}\cos\varphi$$

This is the angular acceleration of the centre of mass on its circular orbit. Now, for the top-most point, we have

$$a_x = \omega^2\frac{L}{2}\cos\varphi - \alpha\frac{L}{2}\sin\varphi$$

Substituting the values of ω and α found above, we have

$$a_x = g\cos\varphi(1-\frac{3}{2}\sin\varphi)$$

which is clearly zero at $\sin\varphi_c = \frac{2}{3}$, and we are done. $\square$

Proof 2: Let φ be the angle made by the rod with the horizontal. From conserving energy between $t=0$ and the moment the top-most point leaves contact, we get

$$mg\frac{L}{2} + 0 = \frac{1}{2}m{v_x}^2 + \frac{1}{2}m{v_y}^2 + mg\frac{L}{2}\sin\varphi$$

$$\Rightarrow v_x = \sqrt{gL\sin^2\varphi(1-\sin\varphi)}$$

Also by definition,

$$\dot{\varphi} = \omega = \frac{v_x\sin\varphi + v_y\cos\varphi}{L}$$

and by constraint relation, $v_y = v_x\cot\varphi$. Substituting constraint relation in ω, we get

$$\omega = \frac{v_x}{L\sin\varphi}$$

Now, for a_x, we differentiate v_x with time:

$$\begin{aligned} a_x &= \sqrt{gL}\cdot\frac{\sin\varphi\cos\varphi(2-3\sin\varphi)}{2\sqrt{\sin^2\varphi(1-\sin\varphi)}}\cdot\dot{\varphi} \\ &= \sqrt{gL}\cdot\frac{\sin\varphi\cos\varphi(2-3\sin\varphi)}{2\sqrt{\sin^2\varphi(1-\sin\varphi)}}\cdot\frac{\sqrt{gL\sin^2\varphi(1-\sin\varphi)}}{L\sin\varphi} \\ &= \frac{g\cos\varphi}{2}(2-3\sin\varphi) \end{aligned}$$

Clearly, $a_x = 0$ at $\sin\varphi_c = \frac{2}{3}$ and we are done.

Now, from the claim, we have that at $\sin\varphi_c = \frac{2}{3}$, $a_x = 0$. At this moment, the horizontal component of the system's acceleration is zero (or the horizontal velocity of the system is maximised). Thus there is no horizontal force on the rod at this moment. Then, if tension exists, every point on the rod would be accelerating towards each other, and their x-distance would decrease. But the y-distance is also decreasing, which is a contradiction. Hence, the tension in the rod must be zero at this moment.

Solution 2: Let φ be the angle between the rod and the horizontal surface. y is the vertical position of the upper mass, and y the horizontal position of the lower mass.

$$\begin{aligned}
x &= r\cos\varphi \\
\dot{x} &= -r\sin\varphi\dot{\varphi} \\
\ddot{x} &= -r\sin\varphi\ddot{\varphi} - r\cos\varphi\dot{\varphi}^2 \\
y &= r\sin\varphi \\
\dot{y} &= r\cos\varphi\dot{\varphi} \\
\ddot{y} &= r\cos\varphi\ddot{\varphi} - r\sin\varphi\dot{\varphi}^2
\end{aligned}$$

By conservation of energy,

$$\begin{aligned}
\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + mg\frac{r}{2}\sin\varphi &= mg\frac{r}{2} \\
\frac{1}{2}mr^2\dot{\varphi}^2 + mg\frac{r}{2}\sin\varphi &= mg\frac{r}{2}
\end{aligned}$$

Taking the derivative with respect to time,

$$\frac{d}{dt}\left[\frac{1}{2}mr^2\dot{\varphi}^2 + mg\frac{r}{2}\sin\varphi\right] = \frac{d}{dt}\left[mg\frac{r}{2}\right]$$

$$\ddot{\varphi} = -\frac{g}{2r}\cos\varphi$$

We may also find with the energy equation that

$$\dot{\varphi}^2 = \frac{g}{r}(1-\sin\varphi)$$

When the top-most point loses contact with the wall, there is no horizontal force acting on the rod, so the horizontal acceleration of the rod must be 0. We solve for φ_c such that this happens.

$$\begin{aligned}
0 &= \ddot{x} \\
&= -r\sin\varphi_c\ddot{\varphi}_c - r\cos\varphi_c{\dot{\varphi}_c}^2 \\
&= -r\sin\varphi_c\left(-\frac{g}{2r}\cos\varphi_c\right) - r\cos\varphi_c\left(\frac{g}{r}(1-\sin\varphi_c)\right) \\
&= \frac{g\cos\varphi_c}{2}(3\sin\varphi_c - 2)
\end{aligned}$$

Hence, we have

$$\boxed{\sin\varphi_c = \frac{2}{3}}$$

Fun Fact: Any arbitrary point on the rod undergoes an elliptical motion.

Proof: At any moment, consider a point at a distance r along the rod from its bottom-most point. The coordinates of this point are simply

$$\mathrm{P}_{\mathrm{r},\varphi}((L-r)\cos\varphi, r\sin\varphi)$$

or

$$\cos\varphi = \frac{x}{L-r}$$
$$\sin\varphi = \frac{y}{r}$$

Thus the locus of this point is

$$\left(\frac{x}{L-r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

which is an ellipse that degenerates to a circle at $L-r=r \Rightarrow r=\frac{L}{2}$, or the centre of mass of the rod.

[a] In fact, it is the quarter of this circle, since the motion is constrained in the first quadrant.

pr 77. Clearly the angular momentum is conserved about any point lying on the line passing through the rod. For convenience, let us choose the point where the collision occurs:

$$Mv\frac{\ell}{2}\,\hat{k} - \frac{M\ell^2}{12}\omega\,\hat{k} = 0 \Rightarrow \vec{\omega} = -\frac{6v}{\ell}\hat{k}$$

By momentum conservation, we have

$$Mv = mv_f \Rightarrow v_f = \frac{M}{m}v$$

where v_f is the final velocity of the puck. Since the collision is elastic,

$$e = 1 = \frac{(v_f) - 0}{(v+\frac{\omega\ell}{2}) - 0} \Rightarrow v_f = v + \frac{\omega\ell}{2}$$

From these three equations, we obtain

$$\frac{M}{m} = 4$$

Instead of using the equation of the restitution coefficient, we use energy conservation.

$$\frac{1}{2}Mv^2 + \frac{1}{2}(\frac{M\ell^2}{12})\omega^2 = \frac{1}{2}m{v_f}^2$$

Solving this equation with the equation for linear and angular momentum conservation yields the same answer. □

pr 78. **a)** The main difference between the two parts is that in the first part the friction does not act for the whole time during which the ball is in contact. Once pure rolling is achieved friction becomes zero. Also because this is an elastic collision and the floor's mass is much greater than the mass of the mall, v_y is reversed after the collision. So, $v_y = \sqrt{2gh}$. This is the same in the two cases. The impulse due to the normal force is

$$\int N\,dt = m(\sqrt{2gh} - (-\sqrt{2gh})) = 2m\sqrt{2gh}$$

Let the friction force (as a function of time) be $f\,(-\hat{\imath})$, final velocity in the $x-$ direction be v_x and the angular velocity be ω. Because the point of contact is at rest, $v_x = \omega R$. The impulse due to friction is then,

$$J = \int f\,dt = m(v_0 - v_x)$$

The angular impulse due to friction is

$$\int fR\,dt = \frac{2}{5}MR^2\omega \implies \int f\,dt = \frac{2}{5}Mv_x$$

Solving the above two equations gives

$$\boxed{v_x = \frac{5}{7}v_0}\ \boxed{\omega = \frac{5v_0}{7R}}$$

b) Here the friction acts for the entire time while the ball is in contact with the floor. Also $f = \mu N$ for the entire time. The impulse due to friction

$$J = \int \mu N\,dt = m(v_0 - v_x) \implies \boxed{v_x = v_0 - 2\mu v_y}$$

Finally, the angular impulse due to friction is

$$\int \mu NR\,dt = \frac{2}{5}MR^2\omega \implies \boxed{\omega = \frac{5\mu\sqrt{2gh}}{R}}$$

pr 79. If the coefficient of friction μ surpasses a certain value then the block will start rolling without slipping, and in turn, will roll with a different total acceleration. This means that before finding the acceleration of the ball, we must find this coefficient of friction and break the acceleration into two cases.

The coefficient of friction can be found from considering the boundary case of static friction. From

$$ma = mg\sin\theta - F_s$$

and

$$I\alpha = F_s r$$

we get

$$a = g\sin\theta - \frac{F_s}{m}$$

and

$$\alpha = \frac{F_s r}{I}$$

With static friction there is no slipping thus we combine using $a = \alpha r$ to get

$$F_s = \frac{mI\sin\theta}{I + mr^2}$$

Since $F_s \leq F_{s,max} = \mu mg\cos\theta$, the angle where the "rimless wheel" stops rolling without slipping can be found as

$$\frac{mI\sin\theta}{I + mr^2} = \mu mg\cos\theta \implies \tan\theta = \mu\frac{I + mr^2}{I}$$

The moment of inertia of a ball about it's central axis is $\frac{2}{5}mr^2$, so by substituting this we find

$$\mu = \frac{I}{I + mr^2}\tan\theta \implies \mu = \frac{\frac{2}{5}mr^2}{\frac{2}{5}mr^2 + mr^2}\tan\theta = \frac{2}{7}\tan\theta.$$

Now, we have two cases:
Case 1 $\mu > \frac{2}{7}\tan\theta$. The ball will start rolling without slipping down the ramp. We know that

$$mg\sin\theta - f = ma$$

Newton's second law of rotation gives

$$-fr = I_{\mathrm{cm}}\alpha \implies f = \frac{-I_{\mathrm{cm}}\alpha}{R}$$

Substituting $I = \frac{2}{5}mr^2$ into this result for our equation gives us

$$f = \frac{-\left(\frac{2}{5}mr^2\right)(-a_{\mathrm{cm}}/r)}{r} = \frac{2}{5}ma$$

Taking this result back to our first equation

$$mg\sin\theta - \frac{2}{5}ma = ma$$

$$a = \frac{5}{7}g\sin\theta$$

Case 2 $\mu < \frac{2}{7}\tan\theta$. The ball will simply slide down the ramp in this case, so we have

$$mg\sin\theta - \mu mg\cos\theta = ma$$

$$a = g\sin\theta - \mu g\cos\theta.$$

pr 80. The centre of mass of the entire system is initially at rest. The walls of the hoop are frictionless, which means that there is no net impulse throughout the motion in the horizontal direction. The impulse due to gravity only pulls the centre of mass downwards after the motion has started, and not in the horizontal direction. In fact, the hoop first moves rightwards, and after some time, leftwards. When the block has made an angle of φ as in the problem, let it have a velocity of

$$\vec{v_b} = v_x\hat{\imath} - v_y\hat{\jmath}$$

in the ground frame. Since the momentum is conserved in the horizontal direction, we have

$$Mv_0 = mv_x$$

where v_0 is the speed of the hoop's central point (directed leftward). Now, since the mechanical energy of the system is conserved, we have

$$2mgr = mgr(1-\cos\varphi) + \frac{1}{2}M{v_0}^2 + \frac{1}{2}m{v_b}^2$$

$$\implies mgr(1+\cos\varphi) = \frac{1}{2}M{v_0}^2 + \frac{1}{2}m{v_b}^2$$

Also, one can notice in the hoop's frame of reference that

$$\tan\varphi = \frac{v_y}{v_x + v_0}$$

Now we solve these three equations:

$$\frac{1}{2}M{v_1}^2 + \frac{1}{2}m\left(\frac{M}{m}\sqrt{1+\tan^2\varphi(1+\frac{m}{M})^2}\right)^2 = mgR(1+\cos\varphi)$$

Isolating for v_1:

$$v_1 = \sqrt{\frac{2mgR(1+\cos\varphi)}{M(1+\frac{M}{m}\tan^2\varphi(1+\frac{m}{M})^2)}}$$
$$= \sqrt{\frac{2m^2gR(1+\cos\varphi)\cos^2\varphi}{Mm\cos^2\varphi + M^2 + m^2\sin^2\varphi + 2Mm\sin^2\varphi}}$$
$$= \sqrt{\frac{2m^2\cos^2\varphi(1+\cos\varphi)gR}{(M+m)(M+m\sin^2\varphi)}}$$
$$= \boxed{m\cos\varphi\sqrt{\frac{2gR(1+\cos\varphi)}{(M+m)(M+m\sin^2\varphi)}}}$$

Now, for the acceleration of the hoop, we use Newton's second law of motion in the non-inertial frame of the hoop. For this, let the acceleration of the hoop's centre at the moment be a_0, directed leftwards. At this moment, suppose the acceleration of the block in the ground frame $\vec{a}_b = a_c\hat{r} + a_t\hat{\theta}$. In the frame of the hoop in the $\hat{r}$ direction, the block's radial acceleration is simply

$$a_{b,h_r} = a_c - a_0\sin\varphi$$

By Newton's second law on the hoop in the horizontal direction, we have

$$N\sin\varphi = Ma_0$$

where N is the normal force exerted by the block on the hoop. In the frame of the hoop, force balancing on the block in the radial direction yields the equation

$$N - mg\cos\varphi = ma_{b,h_r} = m\frac{(v_b + v_0\cos\varphi)^2}{R}$$

From these equations, we have

$$a_0 = \frac{\sin\varphi}{M}\left(\frac{2m^3\ \cos^2\varphi(1+\cos\varphi)}{(M+m)(M+m\sin^2\varphi)}\left(\frac{M^2}{m^2}(\tan^2\varphi\left(1+\frac{m}{M}\right)^2\right.\right.$$
$$\left.\left. + \cos^2\varphi + 2\frac{M}{m}\cos\varphi\sqrt{1+\tan^2\varphi\left(1+\frac{m}{M}\right)^2}\right) + mg\cos\varphi\right)$$

which on simplifying gives

$$\boxed{a = \frac{mg\sin 2\phi}{M+m\sin^2\phi}\left(\frac{1}{2} + \frac{(M+m)(1+\cos\phi)}{\cos\phi(M+m\sin^2\phi)}\right)}$$

Let v_1 be the velocity of of the block relative to the hoop, and it will be directed tangent to the hoop. Let v_2 represent the velocity of the hoop relative to the ground. Conservation of linear horizontal momentum gives us:

$$m(v_1\cos\phi - v_2) = Mv_2$$

because the net force on the system is zero. Conservation of energy gives us:

$$\frac{1}{2}Mv_2^2 + \frac{1}{2}m(v_1^2 + v_2^2 + 2v_1v_2\cos(\pi-\phi)) = mgR(1+\cos\phi)$$

Solving, we have:

$$v_1 = \sqrt{\frac{2(M+m)gR(1+\cos\phi)}{M+m\sin^2\phi}}$$

$$\boxed{v_2 = m\cos\varphi\sqrt{\frac{2gR(1+\cos\varphi)}{(M+m)(M+m\sin^2\varphi)}}}$$

Now, in the hoop's frame the block does circular motion and thus the only radial component of acceleration is

$$a_r = \frac{v_1^2}{R}$$

Let the hoop's acceleration be a_2 (directed horizontally backwards). Therefore in ground frame the net radial acceleration of the block is:

$$= a_r - a_2\sin\phi$$

Applying Newton's second law on both gives:

$$N\sin\phi = Ma_2$$

where N is exerted by hoop on block towards the centre. Finally,

$$N - mg\cos\phi = m(a_r - a_2\sin\phi) = m(\frac{v_1^2}{R} - a_2\sin\phi)$$

Solving gives[a]

$$\boxed{a_2 = \frac{mg\sin 2\phi}{M+m\sin^2\phi}\left(\frac{1}{2} + \frac{(M+m)(1+\cos\phi)}{\cos\phi(M+m\sin^2\phi)}\right)}$$

[a]Note that the two solutions provided in the text essentially use the same idea. Both the solutions have been provided to show that working in the relative frame of the hoop is a much more convenient way to obtain the answers.

pr 81. First, let us notice that the period of oscillations $T = 0.01$ s is extremely small, so any deviation in the velocity caused by friction can be ignored if we only focus on the average velocity. We assume that the block travels at a constant velocity u rightwards in the positive direction. Let us examine the movement qualitatively.

As the board starts moving rightwards, it is important to note that the velocity of the block relative to the board is rightwards, so the friction force $mg\mu_1$ points leftwards. This goes on for a time t_1 until the velocity of the board matches the velocity of the block and overtakes it. This goes on for a time t_2 where the board reaches a maximum and starts to slow down all the way until it has a velocity of u again. During this time period, the friction force points to the right with a magnitude $mg\mu_2$. Finally, for a time $t_3 = t_1$, the board is still moving towards the right but the friction force points towards the left. The total duration is $t_1 + t_2 + t_3 = T/2$.

Finally, the board starts travelling in the leftwards direction. The friction force here is a constant $mg\mu_1$ directed towards the left and lasts for a time $t_4 = T/2$

Now let's do the math. Let's work with the assumption we made that the block has a roughly constant average velocity. If this was not the case, then friction forces would either speed it up or slow it down until the motion is roughly constant. As a result, the total change in momentum, or impulse is zero. We have:

$$(-mg\mu_1)t_1 + (mg\mu_2)t_2 + (-mg\mu_1)t_3 + (-mg\mu_1)t_4 = 0$$

Letting $t_1 = t_3$ we get:

$$\mu_2 t_2 = \mu_1 t_4 + 2\mu_1 t_1$$

Having $2t_1 + t_2 = t_4$ then we have:

$$\mu_2(t_4 - 2t_1) = \mu_1 t_4 + 2\mu_2 t_1 \implies (\mu_2 - \mu_1)t_4 = 4\mu_1 t_1$$

or:

$$t_1 = \frac{(\mu_2 - \mu_1)t_4}{2(\mu_2 + \mu_1)}$$

Since $t_4 = 0.005$ s we get:

$$t_1 = \frac{t_4}{8}$$

this is an eighth of half a period and corresponds to the time where the board has the same velocity as the block. I put this through a visual program and determined this corresponds to 0.64 m/s. To one significant digit, , the average velocity of the board is $\boxed{v = 0.6 \text{ m/s}}$

pr 82. As the water moving with a speed v collides completely inelastically with the paddles moving with a speed u, a momentum of $dp = dm(v - u)$ is imparted to the paddles so the force on the paddle is $F = \mu(v - u)$ where in the frame of the paddle,

$$\mu = \rho S(v - u)$$

The power is therefore:

$$P = Fu = \rho S(v - u)^2 u$$

We can maximize this by taking the derivative and set it to zero, or when:

$$3u^2 - 4vu + v^2 = 0$$

This gives $u = v/3$ or $u = v$. Obviously, $u = v$ would give the minimum power so $u = v/3$ gives the maximum power to be:

$$P_{\max} = \frac{4}{27}\rho S v^3$$

Moving back to the lab frame, we have

$$\mu = \rho S v$$

or:

$$\boxed{P_{\max} = \frac{4}{27}\mu v^2}$$

pr 83. Note that the velocity vector in the picture is drawn wrong, it should be normal to the board.

Since the board is flat, we wish to move into a reference frame where the velocity of the board is parallel to its surface. We also want the stream lines of the water to be nice, so we don't want a vertical component of the water's velocity in the new reference frame. Thus, a reference frame moving with velocity $v/\cos\alpha$ to the left will used, where the board's velocity is parallel to its surface (thus can be ignored) and the water is flowing horizontally.

The setup now is very similar to that of problem 53. The water is flowing to a wall with velocity $v/\cos\alpha$ to the right, and Bernoulli's equation along the stream line near the surface (constant pressure, no significant change in height) tells us that the water will be directed with the same speed of $v/\cos\alpha$ and along the surface of the board.

Now that we have the velocity of the water in the frame of the board, we shift back to the lab frame. Adding the velocity vectors of equal magnitudes of $v/\cos\alpha$ with one horizontally to the left and one at an angle α from the vertical gives

$$\vec{u} = -\left(\frac{v}{\cos\alpha} + v\tan\alpha\right)\hat{i} + v\,\hat{j} \Rightarrow u = v\sqrt{1 + \left(\frac{1+\sin\alpha}{\cos\alpha}\right)}$$

$$\Rightarrow u = \frac{v}{\cos\alpha}\sqrt{2(1+\sin\alpha)} = \boxed{\frac{2v}{\cos\alpha}\cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)}$$

pr 84. Before we solve the problem, let us build some intuition for what is happening. As the wagon accelerates, the new equilibrium position will start oscillating around a new equilibrium angle θ. This angle is very small and therefore we can assume it undergoes small angle oscillations with an amplitude of θ. Every time it gets back to its vertical position, it will be stationary. We want it such that after it travels a distance L, the load is in this position.

We propose that at the moment the wagon starts decelerating, the load must also be in the vertical direction and motionless. This is because as soon as it starts decelerating, there will be a new equilibrium angle of $-\theta$. If we want the magnitude of the amplitudes to stay the same, we need the position of the load to be vertical as it starts decelerating.

With this intuition, it is not difficult to solve the problem as we only need to focus on one half of the journey. In the time t it takes to travel a distance $L/2$, the load must have undergone N full oscillations. This can be rewritten as:

$$t = NT$$

where $T = 2\pi\sqrt{\frac{\ell}{\sqrt{g^2+a^2}}}$ is the period of oscillations. However, since $a \ll g$, we can ignore the second order term and rewrite the period with the standard equation: $T = 2\pi\sqrt{\frac{\ell}{g}}$

The time it takes to travel half the length can be determined using simple kinematics. We have:

$$\frac{L}{2} = \frac{1}{2}at^2 \implies t = \sqrt{\frac{L}{a}}$$

Using the condition we stated, we can link the two times together:

$$\sqrt{\frac{L}{a}} = 2\pi N\sqrt{\frac{\ell}{g}} \implies \boxed{a = \frac{Lg}{N^2 4\pi^2 \ell}}$$

for all positive integers N provided that $a \ll g$.

pr 85. Let us first consider the case of the triangular prism. We shall do this by looking at the force caused by the shockwave at a particular moment. The force exerted at any particular point is:

$$F = S(p_1 - p_0)$$

where S is the cross sectional area at that point. We shall assume that the shock-wave passes through the object extremely quickly so throughout the entire process the object is stationary and only moves with the momentum imparted after it has passed. The impulse is thus:

$$J = \int S(p_1 - p_0)dt$$

where $dt = dx/c_s$ and the area of the cross section varies with x as $S = c(\frac{-b}{a}x + b)$. Plugging this in, we can determine the change in momentum as:

$$J = m\Delta v = \frac{c(p_1 - p_0)}{c_s}\int_0^a \left(\frac{-b}{a}x + b\right)dx \implies \Delta v = \frac{p_1 - p_0}{mc_s}\left(\frac{abc}{2}\right)$$

Note however that the volume of the triangular prism is also $V = \frac{1}{2}abc$. We can therefore generalize this result to any arbitrary shape. As before, the impulse is given by:

$$m\Delta v = \frac{p_1 - p_0}{c_s}\int S(x)dx$$

where $\int S(x)dx$ gives the volume. Therefore, the change in velocity for any shape, including the triangular prism above after a shock-wave passes through is:

$$\boxed{\Delta v = \frac{V(p_1 - p_0)}{mc_s}}$$

pr 86. The rod will act like a spring (since the rod is thin and made out of steel, while steel is elastic). After the left sphere has collided with the stationary sphere, the latter will acquire velocity v_0 and the former will stay at rest. Using momentum conservation when considering the entire system at impact, we find that,

$$(2m)v_0 = mv_0 + (2m)v_f \implies v_f = \frac{1}{2}v_0.$$

We can then compare the initial and final kinetic energies

$$K_i = \frac{1}{2}(2m)v_0^2 = mv_0^2$$
$$K_f = \frac{1}{2}(2m)v_f^2 + \frac{1}{2}mv_0^2 = \frac{3}{4}mv_0^2$$

Here, we can see that kinetic energy is not conserved. However, we still do not know where this energy goes to. The dumbbell, as a system of spheres and springs, will begin oscillating around its centre of mass. After a half period the dumbbell will be too far away to expand outwards and hit the mass again. Thus, we can say that the rest of the energy is stored in the oscillating dumbbell after.