

2019 National Astronomy Olympiad (NAO)

1 Instructions (Please Read Carefully)

(Start the exam after reading the instructions carefully and affirming that the contestant understood the instructions given.)

The top 5 eligible scorers on the NAO will be invited to represent USA at the next IOAA. In order to qualify for the national team, you must be a high school student with US citizenship or permanent residency.

This exam consists of 3 parts: Short Questions (7 points each), Medium Questions (15 points each) and Long Questions (25 points each). The maximum score is 153 points.

The test must be completed within 2.5 hours (150 minutes). The proctor should mark the start and end time of the exam on the front page.

Please solve each problem on a blank piece of paper and mark the number of the problem at the top of the page. The contestant's full name in capital letters should appear at the top of each solution page. If the contestant uses scratch papers, those should be labeled with the contestant's name as well and marked as "scratch paper" at the top of the page. Scratch paper will not be graded. Partial credit will be available given that correct and legible work was displayed in the solution. This exam document, solution, and all the scratch paper used should be turned in to the proctor at the end of the exam.

This is a written exam. Contestants can only use a scientific calculator for this exam. A table of physical constants will be provided. Discussing the problems with other people is strictly prohibited in any way until the end of the examination period on March 8th. Receiving any external help during the exam is strictly prohibited. This means that the only allowed items are: a scientific calculator, the provided table of constants, a pencil (or pen), an eraser, blank sheets of papers, and the exam. No books, notes, laptops, mobile phones, or any other devices are allowed while taking the exam.

After reading the instructions, please make sure to sign at the bottom of this page, affirming that:

1. All work on this exam is mine.
2. I took this exam under a proctor's supervision.
3. I did not receive any external aids besides the materials provided.
4. I am not allowed to discuss the test with others throughout the period of this examination.
5. Failure to follow these rules will lead to disqualification from the exam.

To be completed by the proctor:

Full name (First, Last): _____
Position (e.g. Physics Teacher at High School x): _____
Email address: _____
Student began exam at (hh:mm): _____
Student submitted the exam at (hh:mm): _____
Signature: _____ Date (mm/dd/yy): _____

To be completed by the student:

Last name: _____ Middle name: _____
First name: _____
Date of birth (mm/dd/yy): _____ Email address: _____
Signature: _____ Date (mm/dd/yy): _____

2 Short Questions

1. (7 points) Assuming that the present density of baryonic matter is $\rho_{b0} = 4.17 \times 10^{-28} \text{ kg m}^{-3}$, what was the density of baryonic matter at the time of Big Bang nucleosynthesis (when $T \sim 10^{10} \text{ K}$)? Assume the present temperature, T_0 to be 2.7 K.
2. (7 points) On the night of January 21st, 2019, there was a total lunar eclipse during a supermoon. At the time, the moon was close to perigee, at a distance of 351837 km from the earth, which was 1.4721×10^8 km from the sun. The gamma (γ) of a lunar eclipse refers to the closest distance between the center of the moon and the center of the shadow, expressed as a fraction of the earth's radius. For this eclipse, $\gamma = 0.3684$. Given this information, find the closest estimate for the duration of totality of the eclipse.
3. (7 points) You are in the northern hemisphere and are observing rise of star A with declination $\delta = -8^\circ$, and at the same time a star B with declination $\delta = +16^\circ$ is setting. What will happen first: next setting of the star A or rising of the star B?
4. (7 points) Consider a star with mass M and radius R . The star's density varies as a function of radius r according to the equation $\rho(r) = \rho_{center}(1 - \sqrt{r/R})$, where ρ_{center} is the density at the center of the star. Derive an expression for dP/dr in terms of G , M , R , and r , where P is the pressure at a given radius r .

3 Medium Questions

5. (15 points) An alien spaceship from the planet Kepler 62f is in search of a rocky planet for a remote base. They're attracted to Earth because of a fortunate coincidence: its axis of rotation points directly at their home planet. That means they can have uninterrupted communication with home by planting fixed transmitters on The North Pole. But first, they need to find out if Earth's axis will always point in the same direction or if it undergoes precession. They can't know without years of observation, so they hope that we, its now-extinct intelligence, have left behind the answer.

While orbiting Earth, they see a few remarkable structures, including the Hoover Dam in Nevada. Zooming in on the dam, a colorful plaza with peculiar markings on its floor catches their attention. Descending on the plaza, they realize the markings are a map of the sky when the dam was built, left to indicate the date to posterity. Figure 1 is an overhead architectural map of this plaza. The center-point depicts the north ecliptic pole, and the large circle represents the path of the Earth's axis throughout its counter-clockwise procession. As they interpret the map, they're dismayed to realize that their star has not been and will not be Earth's north star for very long.

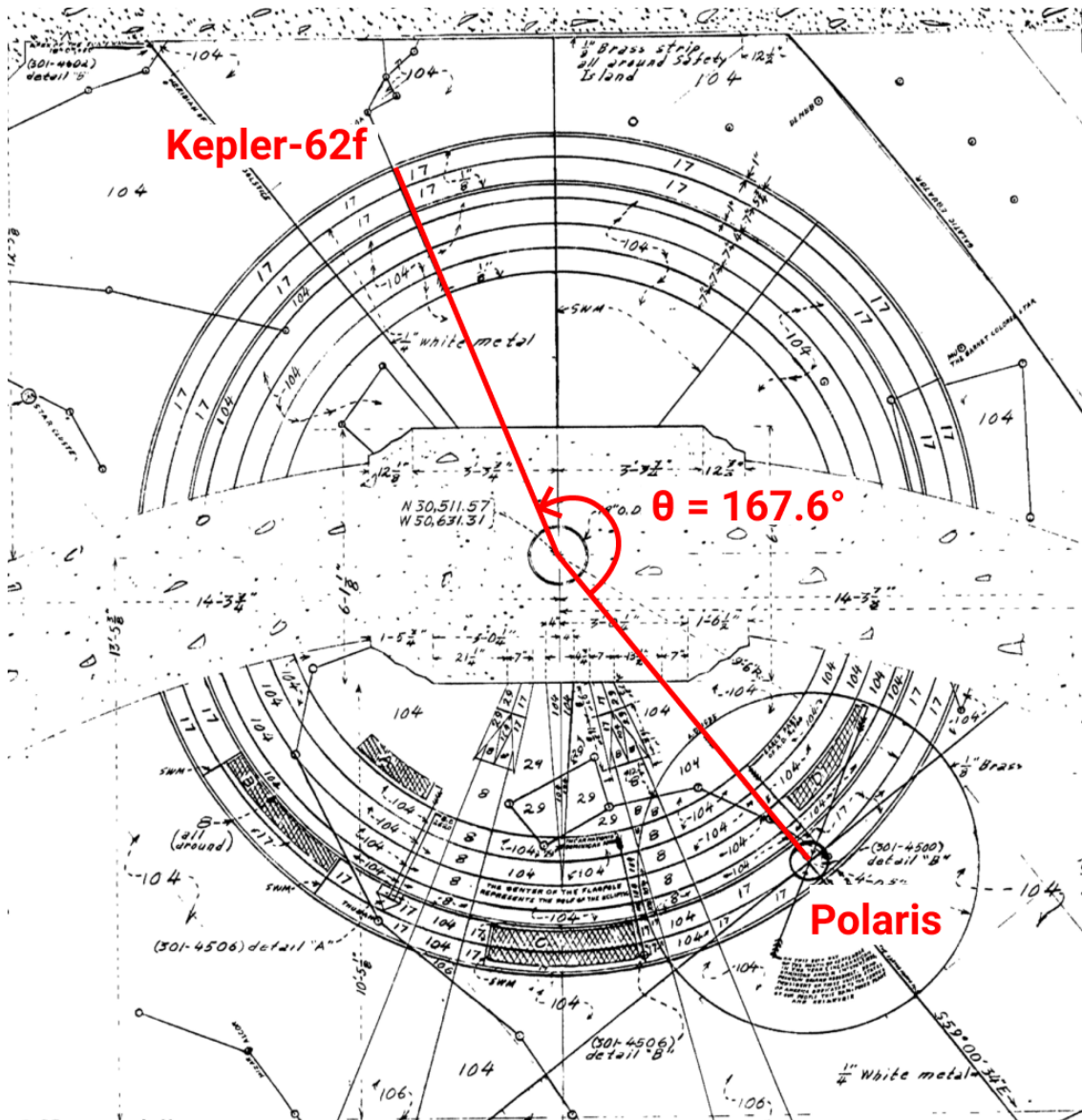


Figure 1: Overhead architectural plan of the Hoover Dam plaza depicting Polaris as north star

For the purpose of this question, assume that the Earth's axial tilt is a constant $i = 23.5^\circ$ and its axis precesses at a constant rate.

- Using the values on the map, and knowing that the aliens used carbon-aging to determine that the dam is 12,000 years old, find all possible values for the period of Earth's axial precession.
- Using the most optimistic answer (longest period) from part (a), calculate how many arcseconds the Earth's axis precesses each day. Use the period you calculate here in the next two sections.
- If they hadn't been lucky enough to come across the star map and decided to build a radio interferometer to observe the movement of the celestial pole over the course of 30 days instead, how many kilometers would the baseline of their telescope array have to be, assuming it operated at a 20cm wavelength?
- As a last resort, to keep Earth's axis fixed, the aliens decide to counter the forces that cause the Earth's precession by building giant nuclear thrusters on the Earth's surface. Assume Earth's pre-

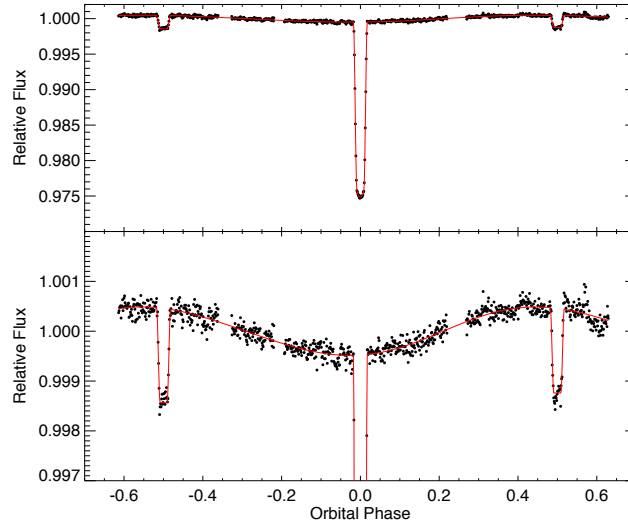


Figure 2: Full-phase light curve of HD 189733b. From Knutson et al. (2012).

cession is caused by external forces alone and calculate the average force (in kN) that a strategically positioned thruster would have to exert to counter them.

6. (15 points) Figure 2 shows a full-phase light curve (“phase curve”) of the exoplanet HD 189733b taken by the Spitzer space telescope. Use this figure to answer the following questions. The star HD 189733 has an effective temperature of 4785 K and a radius of 0.805 Solar radii.

- Use the depth of the planet’s transit to estimate the radius of HD 189733b, in Jupiter radii.
- Use the depth of the eclipse of the planet by the host star to estimate the ratio of the flux of the planet HD 189733b to that of the host star HD 189733.
- HD 189733b is so close-in to its host star that it is expected to be tidally locked. Use the phase curve to estimate the ratio of the dayside flux emitted by the planet to the nightside flux emitted by the planet.
- This phase curve also noticeably has a phase curve offset, that is, the maximum in planet and star flux does not occur exactly at secondary eclipse. What process that occurs in a planetary atmosphere could cause such a phase curve offset?

7. (15 points)

- Mass-Radius Relation** Stellar physics often involves guessing the equation of state for stars, which is typically a relation between the pressure P and the density ρ . A family of such guesses are known as polytropes and go as follows-

$$P = K \rho^\gamma \quad (1)$$

where K is a constant and the exponent γ is fixed to match a certain pressure and core temperature of a star. Given this, show that one can obtain a crude power-law scaling between the mass M of a polytropic star and its radius R of the form $M \propto R^\alpha$. Find the exponent α for polytropic stars (justify all steps in your argument). Also, indicate the exponent γ for which the mass is independent of the radius R . Bonus: Why is this case interesting?

- Black Holes as Blackbodies** The mass radius relation for ideal non-rotating, uncharged black holes is known from relativity to be

$$R = \frac{2GM}{c^2} \quad (2)$$

Moreover, Stephen Hawking showed that a black hole behaves like a blackbody, where its temperature (known as the Hawking temperature) is given by

$$T = \frac{\hbar c^3}{8\pi k_B G M} \quad (3)$$

Given this information, show that the lifetime of a black hole (justify this phrase!) t^* scales with its mass M as

$$t^* \propto M^\beta \quad (4)$$

where you should find the exponent β

- c) **Minimal Black Holes** Using the information of the previous part, and Wien's displacement law, estimate the smallest possible mass of a black hole. State any possible flaws with this estimate.

8. (15 points) In a rather weird universe, the gravitational constant G varies as a function of the scale factor $a(t)$.

$$G = G_0 f(a) \quad (5)$$

Consider the model $f(a) = e^{b(a-1)}$ where $b = 2.09$.

- a) Assuming that the universe is flat, dark energy is absent, and the only constituent is matter, estimate the present age of this weird universe according to this model. Assume that the Friedmann equation:

$$H(a)^2 = H_0^2 (\Omega_m + \Omega_r + \Omega_k + \Omega_\Lambda) \quad (6)$$

still holds in this setting.

- b) What is the behaviour of the age of the universe t as the scale factor $a(t) \rightarrow \infty$?

Note that all parameters with subscript $_0$ indicate their present value. Take the value of Hubble's constant as $H_0 = 67.8 \text{ kms}^{-1} \text{Mpc}^{-1}$

Hint: You might need the following integrals

$$\int_0^\infty x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{4} \quad \int_0^1 x^2 e^{-x^2} dx \approx 0.189471 \quad (7)$$

9. (15 points)

- a) Find the shortest distance from Boston (42.3601° N , 71.0589° W) to Beijing (39.9042° N , 116.4074° E) traveling along the Earth's surface. Assume that the Earth is a uniform sphere of radius 6371 km.
- b) What fraction of the path lies within the Arctic circle (north of 66.5608° N)?

4 Long Questions

10. (25 points) In this problem, we will try to understand the relationship between magnetic moments and angular momenta, first for charged particles and how this can be extended to planetary objects.

- (a) (5 points) Consider a charge e and mass m moving in circular orbit of radius r with constant speed v . Write down the angular momentum L of the charge and magnetic moment μ of the effective current loop. Recall that the magnetic moment of a current loop with current I and radius r is given as $\mu = IA$ where A is the area of the loop.
- (b) (3 points) Use the above results to find a relationship between the magnetic moment μ and angular momentum L in terms of intrinsic properties of the particle (charge, mass).

- (c) (2 points) The relationship from part (b) can be expressed as $\mu = \gamma L$. γ is usually referred to as the *classical* gyromagnetic ratio of a particle. Evaluate the classical gyromagnetic ratio for an electron and for a neutron in SI units.
- (d) (7 points) For extended objects such as planets, the magnetic dipole moment is not directly accessible whereas the surface magnetic field can be measured. Assuming a magnetic dipole of magnetic moment μ located at the center of a sphere of radius r , write down the expression for the surface magnetic field B_{surf} and the surface magnetic moment defined as $\mathcal{M}_{surf} = B_{surf} r^3$. You may use the value of the angular dependence at the magnetic equator for the following parts.
- (e) (3 points) Assuming a gyromagnetic relationship exists between magnetic moment μ and angular momentum L of an extended object, write down the relationship between the surface magnetic moment \mathcal{M}_{surf} and angular momentum L as $\mathcal{M}_{surf} = \kappa L$. You will observe that κ depends only on fundamental constants and intrinsic properties of the extended object.
- (f) (3 points) The surface magnetic moments for Mercury and Sun are $5 \times 10^{12} \text{ T m}^3$ and $3 \times 10^{23} \text{ T m}^3$ respectively. Assuming the bodies are perfect spheres, evaluate the constant κ for Mercury and the Sun. Comment on values obtained and if they fit into the model developed in parts (c) and (d).
- (g) (5 points) The surface magnetic moments \mathcal{M}_{surf} and angular momenta L of various solar system bodies are plotted in the figure 3. Justify that the data implies $\mathcal{M}_{surf} \sim L^\alpha$ and calculate the constant α . What is the expected value of α from the model developed in parts (c) and (d)?

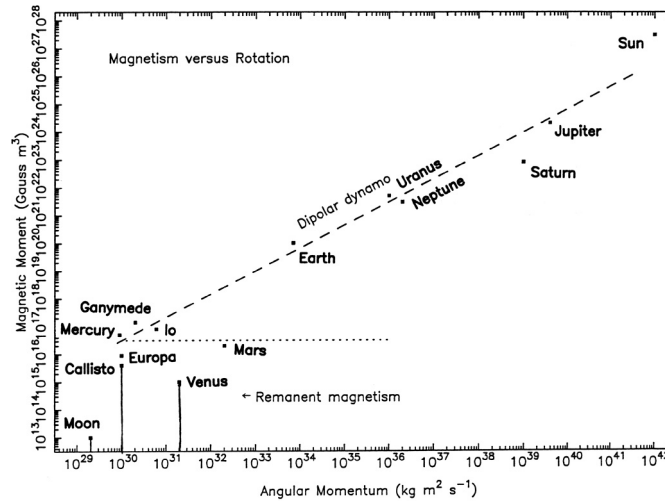


Figure 3: Surface magnetic moment vs angular momentum for solar system objects. Figure taken from Vallée, Fundamentals of Cosmic Physics, Vol. 19, pp 319-422, 1998.

- (h) (2 points) Certain bodies such as Venus, Mars and the Moon are remarkably separated from the trend observed for other bodies. What can you say about magnetism in these bodies when compared to the others?
11. (25 points)
- Cygnus X-1/HDE 226868 is a binary system consisting of a black hole Cygnus X-1 and blue supergiant HDE 226868. The mass of HDE 226868 is $30M_{\odot}$ and the period of the binary system is 5.6 days. Radial velocity data reveals that the orbital velocity of HDE 226868 is 116.68 km/s at apoapse and 123.03 km/s at periapse.
- (a) (5 points) Determine the eccentricity of the orbit of HDE 226868.
- (b) (5 points) Determine the length of the semimajor axis of the orbit of HDE 226868.

- (c) (5 points) Determine the mass of Cygnus X-1, to at least 3 significant figures.

The peak blackbody temperature of an accretion disk occurs at a distance of r_{peak} and a temperature of T_{peak} . One can determine the peak blackbody temperature by assuming that it corresponds to the peak in the x-ray spectrum. Due to relativistic effects, the actual peak blackbody temperature T_{peak} is related to the peak color temperature T_{color} derived from observed spectral data by $T_{color} = f_{GR} f_{col} T_{peak}$, where $f_{GR} \approx 0.510$ and $f_{col} \approx 1.7$. Three x-ray spectra of Cygnus X-1 are shown in Figure 4.

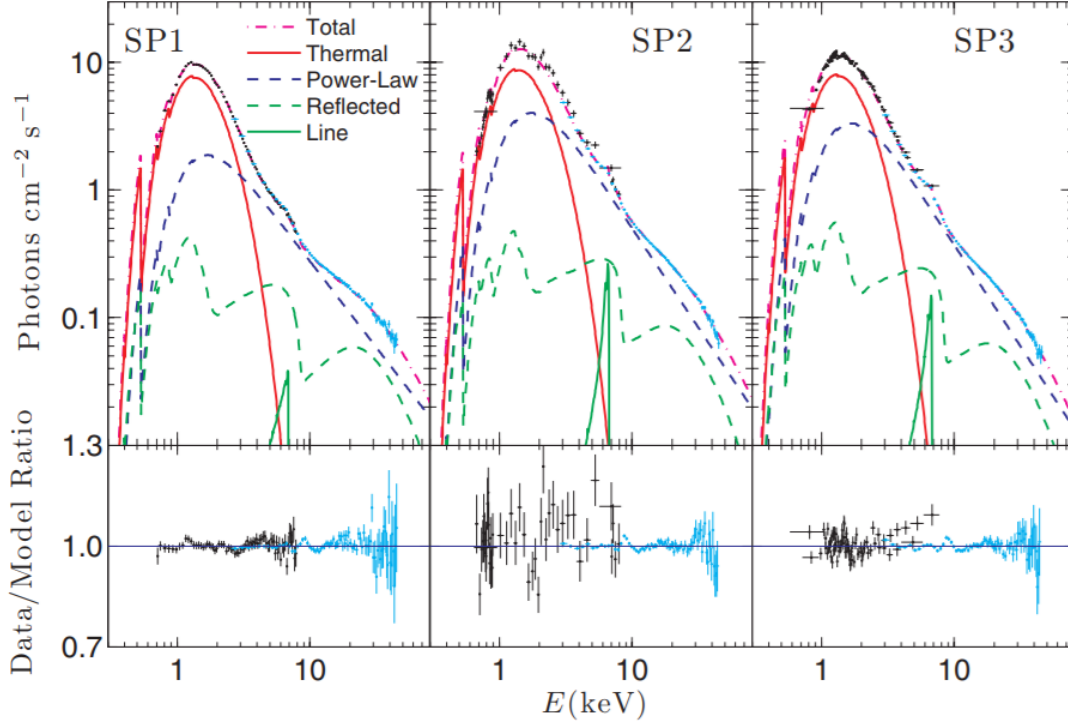


Figure 4: Three x-ray spectra from Cygnus X-1. From Gou et al. (2011).

- (d) (4 points) Using spectrum SP2, determine the peak blackbody temperature T_{peak} of the accretion disk around Cygnus X-1.

The total luminosity of the blackbody component of the accretion disk can be estimated by $L_{disk} \approx 4\pi\sigma r_{peak}^2 T_{peak}^4$ (Makishima et al. 1986). The radius r_{last} of the innermost edge of the accretion disk is related to the radius r_{peak} of the peak blackbody temperature by $r_{peak} = \eta r_{last}$, where $\eta \approx 0.63$. In 1996, the blackbody luminosity of the accretion disk around Cygnus X-1 was estimated to be 2.2×10^{37} ergs/s.

- (e) (4 points) Determine the radius r_{last} of the innermost edge of the accretion disk around Cygnus X-1.

Assume that the innermost edge of the accretion disk is located at the innermost stable circular orbit (ISCO), whose radius r_{isco} is a function of the spin of the black hole. The relationship between r_{isco} and a_* , the spin parameter of the black hole, can be estimated by:

$$r_{isco} = \frac{GM}{c^2} \left(\sqrt{8.354 \cdot [(2 - a_*)^2 - 1]} + 1 \right)$$

- (f) (2 points) Determine the spin parameter a_* of Cygnus X-1.