Solution - Image of a charge

Solution of Task 1

Task 1a)

As the metallic sphere is grounded, its potential vanishes, V=0.

Task1b)

Let us consider an arbitrary point B on the surface of the sphere as depicted in Fig. 1.

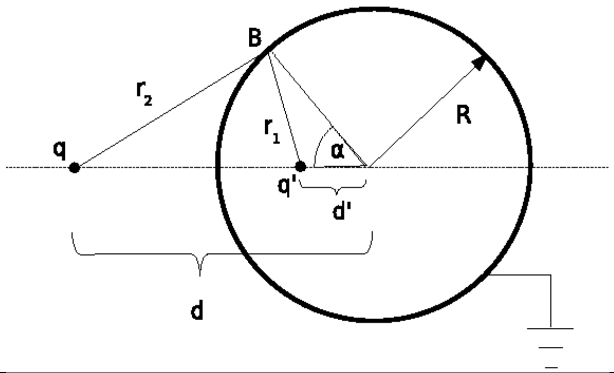

Fig 1. The potential at point *B* is zero.

The distance of point *B* from the charge q' is

$$r_1 = \sqrt{R^2 + d'^2 - 2Rd'\cos\alpha} \qquad (1)$$

whereas the distance of the point B from the charge q is given with the expression

$$r_2 = \sqrt{R^2 + d^2 - 2Rd\cos\alpha} \qquad (2)$$

The electric potential at the point B is

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_2} + \frac{q'}{r_1}\right) \qquad (3)$$

This potential must vanish,

$$\frac{q}{r_2} + \frac{q'}{r_1} = 0 \qquad (4)$$

i.e. its numerical value is 0 V.

Combining (1), (2) and (3) we obtain

$$R^2+d^2-2Rd\cos\alpha=\left(\frac{q}{q'}\right)^2(R^2+d'^2-2Rd'\cos\alpha)$$	(5)

As the surface of the sphere must be equipotential, the condition (5) must be satisfied for every angle α what leads to the following results

$$d^2+R^2=\left(\frac{q}{q'}\right)^2(R^2+d'^2)$$	(6)

and

$$dR=\left(\frac{q}{q'}\right)^2(d'R)$$	(7)

By solving of (6) and (7) we obtain the expression for the distance d' of the charge q' from the center of the sphere

$$d'=\frac{R^2}{d}$$	(8)

and the size of the charge q'

$$q'=-q\frac{R}{d}$$	(9)

Task 1c)

Finally, the magnitude of force acting on the charge q is

$$F=\frac{1}{4\pi\varepsilon_0}\frac{q^2Rd}{\left(d^2-R^2\right)^2}$$	(10)

The force is apparently **attractive.**

Solution of Task 2

Task 2a)

The electric field at the point A amounts to

$$\vec{E}_A = \left(\frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} - \frac{1}{4\pi\varepsilon_0} \frac{q\frac{R}{d}}{\left(r - d + \frac{R^2}{d}\right)^2} \right) \hat{r}$$	(11)

Task 2b)

For very large distances r we can apply approximate formula $(1+a)^{-2} \approx 1-2a$ to the expression (11) what leads us to

$$\vec{E}_A = \frac{1}{4\pi\varepsilon_0} \frac{\left(1 - \frac{R}{d}\right) q}{r^2} \hat{r} - \frac{1}{4\pi\varepsilon_0} \frac{2q\frac{R}{d}\left(d - \frac{R^2}{d}\right)}{r^3} \hat{r}$$	(12)

In general a grounded metallic sphere cannot completely screen a point charge q at a distance d (even in the sense that its electric field would decrease with distance faster than $1/r^2$) and the dominant dependence of the electric field on the distance r is as in standard Coulomb law.

Task 2c)

In the limit $d \to R$ the electric field at the point A vanishes and the grounded metallic sphere screens the point charge completely.

Solution of Task 3

Task 3a)

Let us consider a configuration as in Fig. 2.

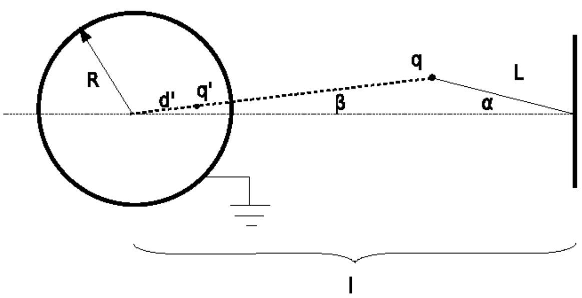

Fig 2. The pendulum formed by a charge near a grounded metallic sphere.

The distance of the charge q from the center of the sphere is

$$d = \sqrt{l^2 + L^2 - 2lL\cos\alpha}$$	(13)

The magnitude of the electric force acting on the charge q is

$$F = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{(d-d')^2} = \frac{1}{4\pi\varepsilon_0}\frac{q^2Rd}{(d^2-R^2)^2}$$	(14)

From which we have

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q^2R\sqrt{l^2+L^2-2lL\cos\alpha}}{\left(l^2+L^2-2lL\cos\alpha - R^2\right)^2}$$	(15)

Task 3b)

The direction of the vector of the electric force (17) is described in Fig. 3.

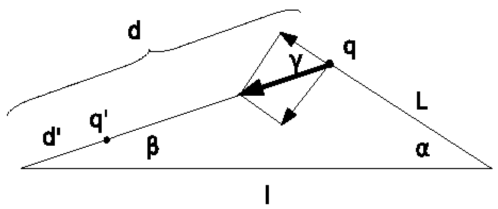

Fig 3. The direction of the force *F*.

The angles α and β are related as

$$L\sin\alpha = d\sin\beta$$	(16)

whereas for the angle γ the relation γ=α+β is valid. The component of the force perpendicular to the thread is F sin γ, that is ,

$$F_\perp = \frac{1}{4\pi\varepsilon_0}\frac{q^2R\sqrt{l^2+L^2-2lL\cos\alpha}}{\left(l^2+L^2-2lL\cos\alpha - R^2\right)^2}\sin(\alpha+\beta)$$ where $$\beta = \arcsin\left(\frac{L}{\sqrt{L^2+l^2-2Ll\cos\alpha}}\sin\alpha\right)$$	(17)

Task 3c)

The equation of motion of the mathematical pendulum is

$mL\ddot{\alpha} = -F_{\perp}$	(18)

As we are interested in small oscillations, the angle α is small, i.e. for its value in radians we have α much smaller than 1. For a small value of argument of trigonometric functions we have approximate relations sin x ≈ x and cos x ≈ 1-x^2/2. So for small oscillations of the pendulum we have $\beta \approx \alpha L/(l-L)$ and $\gamma \approx l\alpha/(l-L)$.

Combining these relations with (13) we obtain

$mL\dfrac{d^2\alpha}{dt^2} + \dfrac{1}{4\pi\varepsilon_0}\dfrac{q^2Rd}{\left(d^2-R^2\right)^2}\left(1+\dfrac{L}{d}\right)\alpha = 0$	(19)

Where $d = l - L$ what leads to

$\omega = \dfrac{q}{d^2-R^2}\sqrt{\dfrac{Rd}{4\pi\varepsilon_0}\dfrac{1}{mL}\left(1+\dfrac{L}{d}\right)} = \dfrac{q}{(l-L)^2-R^2}\sqrt{\dfrac{Rl}{4\pi\varepsilon_0}\dfrac{1}{mL}}$.	(20)

Solution of Task 4

First we present a solution based on the definition of the electrostatic energy of a collection of charges.

Task 4a)

The total energy of the system can be separated into the electrostatic energy of interaction of the external charge with the induced charges on the sphere, $E_{el,1}$, and the electrostatic energy of mutual interaction of charges on the sphere, $E_{el,2}$, i.e.

$E_{el} = E_{el,1} + E_{el,2}$	(21)

Let there be *N* charges induced on the sphere. These charges q_j are located at points $\vec{r}_j, j = 1,\ldots,N$ on the sphere. We use the definition of the image charge, i.e., the potential on the surface of the sphere from the image charge is identical to the potential arising from the induced charges:

$\frac{q'}{\left\|\vec{r}-\vec{d}'\right\|}=\sum_{j=1}^{N}\frac{q_j}{\left\|\vec{r}_j-\vec{r}\right\|}$,	(22)

where $\vec{r}$ is a vector on the sphere and $\vec{d}'$ denotes the vector position of the image charge. When $\vec{r}$ coincides with some $\vec{r}_i$, then we just have

$\frac{q'}{\left\|\vec{r}_i-\vec{d}'\right\|}=\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{q_j}{\left\|\vec{r}_j-\vec{r}_i\right\|}$.	(23)

From the requirement that the potential on the surface of the sphere vanishes we have

$\frac{q'}{\left\|\vec{r}-\vec{d}'\right\|}+\frac{q}{\left\|\vec{r}-\vec{d}\right\|}=0$,	(24)

where $\vec{d}$ denotes the vector position of the charge $\vec{q}$ ($\vec{r}$ is on the sphere).

For the interaction of the external charge with the induced charges on the sphere we have

$E_{el,1}=\frac{q}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{\left\|\vec{r}_i-\vec{d}\right\|}=\frac{1}{4\pi\varepsilon_0}\frac{qq'}{\left\|\vec{d}'-\vec{d}\right\|}=\frac{1}{4\pi\varepsilon_0}\frac{qq'}{d-d'}=-\frac{1}{4\pi\varepsilon_0}\frac{q^2R}{d^2-R^2}$	(25)

Here the first equality is the definition of this energy as the sum of interactions of the charge *q* with each of the induced charges on the surface of the sphere. The second equality follows from (21).

In fact, the interaction energy $E_{el,1}$ follows directly from the definition of an image charge.

Task 4b)

The energy of mutual interactions of induced charges on the surface of the sphere is given with

$E_{el,2}=\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\frac{q_iq_j}{\left\|\vec{r}_i-\vec{r}_j\right\|}$ $=\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}q_i\frac{q'}{\left\|\vec{r}_i-\vec{d}'\right\|}=$ $=-\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}q_i\frac{q}{\left\|\vec{r}_i-\vec{d}\right\|}=$ $=-\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{qq'}{\left\|\vec{d}'-\vec{d}\right\|}=-\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{qq'}{d-d'}=\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{q^2R}{d^2-R^2}$	(26)

Here the second line is obtained using (22). From the second line we obtain the third line applying (23), whereas from the third line we obtain the fourth using (22) again.

Task 4c)

Combining expressions (19) and (20) with the quantitative results for the image charge we finally obtain the total energy of electrostatic interaction

$E_{el}(d)=-\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{q^2R}{d^2-R^2}$	(27)

An alternative solution follows from the definition of work. By knowing the integral

$\int_d^{\infty}\frac{xdx}{\left(x^2-R^2\right)^2}=\frac{1}{2}\frac{1}{d^2-R^2}$	(28)

We can obtain the total energy in the system by calculating the work needed to bring the charge *q* from infinity to the distance *d* from the center of the sphere:

$E_{el}(d)=-\int_{\infty}^{d}F(\vec{x})d\vec{x}=\int_{d}^{\infty}F(\vec{x})d\vec{x}=$ $=\int_d^{\infty}(-)\frac{1}{4\pi\varepsilon_0}\frac{q^2Rx}{\left(x^2-R^2\right)^2}dx=$ $=-\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{q^2R}{d^2-R^2}$	(29)

This solves Task 4c).

The electrostatic energy between the charge *q* and the sphere must be equal to the energy between the charges *q* and *q'* according to the definition of the image charge:

$$E_{el,1} = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{(d-d')} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2 R}{d^2 - R^2}$$	(30)

This solves Task 4a).

From this we immediately have that the electrostatic energy among the charges on the sphere is:

$$E_{el,2} = \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{q^2 R}{d^2 - R^2}.$$	(31)

This solves Task 4b).

Solution - Chimney physics

This problem was inspired and posed by using the following two references:

- W.W. Christie, *Chimney design and theory*, D. Van Nostrand Company, New York, 1902.
- J. Schlaich, R. Bergermann, W. Schiel, G. Weinrebe, *Design of Commercial Solar Updraft Tower Systems — Utilization of Solar Induced Convective Flows for Power Generation*, Journal of Solar Energy Engineering 127, 117 (2005).

Solution of Task 1

a) What is the minimal height of the chimney needed in order that the chimney functions efficiently, so that it can release all of the produced gas in the atmosphere?

Let $p(z)$ denote the pressure of air at height *z*; then, according to one of the assumptions $p(z)=p(0)-\rho_{Air}gz$, where $p(0)$ is the atmospheric pressure at zero altitude.

Throughout the chimney the Bernoulli law applies, that is, we can write

$\frac{1}{2}\rho_{Smoke}v(z)^2+\rho_{Smoke}gz+p_{Smoke}(z)=const.,$	(1)

where $p_{Smoke}(z)$ is the pressure of smoke at height *z*, ρ_{Smoke} is its density, and $v(z)$ denotes the velocity of smoke; here we have used the assumption that the density of smoke does not vary throughout the chimney. Now we apply this equation at two points, (i) in the furnace, that is at point $z=-\varepsilon$, where ε is a negligibly small positive number, and (ii) at the top of the chimney where $z=h$ to obtain:

$\frac{1}{2}\rho_{Smoke}v(h)^2+\rho_{Smoke}gh+p_{Smoke}(h)\approx p_{Smoke}(-\varepsilon)$	(2)

On the right hand side we have used the assumption that the velocity of gases in the furnace is negligible (and also $-\rho_{Smoke}g\varepsilon\approx 0$).

We are interested in the minimal height at which the chimney will operate. The pressure of smoke at the top of the chimney has to be equal or larger than the pressure of air at altitude h; for minimal height of the chimney we have $p_{Smoke}(h)\approx p(h)$. In the furnace we can use $p_{Smoke}(-\varepsilon)\approx p(0)$. The Bernoulli law applied in the furnace and at the top of the chimney [Eq. (2)] now reads

$\frac{1}{2}\rho_{Smoke}v(h)^2+\rho_{Smoke}gh+p(h)\approx p(0).$	(3)

From this we get

$v(h)=\sqrt{2gh\left(\frac{\rho_{Air}}{\rho_{Smoke}}-1\right)}.$	(4)

The chimney will be efficient if all of its products are released in the atmosphere, i.e.,

$v(h) \geq \frac{B}{A}$,	(5)

from which we have

$h \geq \frac{B^2}{A^2} \frac{1}{2g} \frac{1}{\frac{\rho_{Air}}{\rho_{Smoke}} - 1}$.	(6)

We can treat the smoke in the furnace as an ideal gas (which is at atmospheric pressure $p(0)$ and temperature T_{Smoke}). If the air was at the same temperature and pressure it would have the same density according to our assumptions. We can use this to relate the ratio $\rho_{Air} / \rho_{Smoke}$ to T_{Smoke}/T_{Air} that is,

$\frac{\rho_{Air}}{\rho_{Smoke}} = \frac{T_{Smoke}}{T_{Air}}$, and finally	(7)
$h \geq \frac{B^2}{A^2} \frac{1}{2g} \frac{T_{Air}}{T_{Smoke} - T_{Air}} = \frac{B^2}{A^2} \frac{1}{2g} \frac{T_{Air}}{\Delta T}$.	(8)

For minimal height of the chimney we use the equality sign.

b) How high should the chimney in warm regions be?

$\frac{h(30)}{h(-30)} = \frac{\frac{T(30)}{T_{Smoke} - T(30)}}{\frac{T(-30)}{T_{Smoke} - T(-30)}}; h(30) = 145m$.	(9)

c) How does the velocity of the gases vary along the height of the chimney?

The velocity is constant,

$v = \sqrt{2gh\left(\frac{\rho_{Air}}{\rho_{Smoke}} - 1\right)} = \sqrt{2gh\left(\frac{T_{Smoke}}{T_{Air}} - 1\right)} = \sqrt{2gh\frac{\Delta T}{T_{Air}}}$.	(10)

This can be seen from the equation of continuity $Av = const.$ (ρ_{Smoke} is constant). It has a sudden jump from approximately zero velocity to this constant value when the gases enter the chimney from the furnace. In fact, since the chimney operates at minimal height this constant is equal to B, that is $v = B/A$.

d) At some height *z*, from the Bernoulli equation one gets

$p_{smoke}(z) = p(0) - (\rho_{Air} - \rho_{Smoke})gh - \rho_{Smoke}gz$.	(11)

Thus the pressure of smoke suddenly changes as it enters the chimney from the furnace and acquires velocity.

Solution of Task 2

a) The kinetic energy of the hot air released in a time interval Δt is

$E_{kin} = \frac{1}{2}(Av\Delta t\rho_{Hot})v^2 = Av\Delta t\rho_{Hot}gh\frac{\Delta T}{T_{Atm}}$,	(12)

Where the index “Hot” refer to the hot air heated by the Sun. If we denote the mass of the air that exits the chimney in unit time with $w = Av\rho_{Hot}$, then the power which corresponds to kinetic energy above is

$P_{kin} = wgh\frac{\Delta T}{T_{Air}}$.	(13)

This is the maximal power that can be obtained from the kinetic energy of the gas flow.

The Sun power used to heat the air is

$P_{Sun} = GS = wc\Delta T$.	(14)

The efficiency is evidently

$\eta = \frac{P_{kin}}{P_{Sun}} = \frac{gh}{cT_{Atm}}$.	(15)

b) The change is apparently linear.

Solution of Task 3

a) The efficiency is

$\eta = \frac{gh}{cT_{Atm}} = 0.0064 = 0.64\%$.	(16)

b) The power is

$P = GS\eta = G(D/2)^2\pi\eta = 45$ kW.	(17)

c) If there are 8 sunny hours per day we get 360kWh.

Solution of Task 4

The result can be obtained by expressing the mass flow of air w as

$w = Av\rho_{Hot} = A\sqrt{2gh\frac{\Delta T}{T_{Air}}}\rho_{Hot}$	(18)
$w = \frac{GS}{c\Delta T}$	(19)

which yields

$\Delta T = (\frac{G^2S^2T_{Atm}}{A^2c^2\rho_{Hot}^2 2gh})^{1/3} \approx 9.1$ K.	(20)

From this we get

$w = 760\,\text{kg/s}$.	(21)

Solution - model of an atomic nucleus

Solution of Task 1

a) In the SC-system, in each of 8 corners of a given cube there is one unit (atom, nucleon, etc.), but it is shared by 8 neighboring cubes – this gives a total of one nucleon per cube. If nucleons are touching, as we assume in our simplified model, then $a = 2r_N$ is the cube edge length a. The volume of one nucleon is then

$V_N = \frac{4}{3} r_N^3 \pi = \frac{4}{3}\left(\frac{a}{2}\right)^3 \pi = \frac{4a^3}{3 \cdot 8}\pi = \frac{\pi}{6}a^3$	(1)

from which we obtain

$f = \frac{V_N}{a^3} = \frac{\pi}{6} \approx 0.52$	(2)

b) The mass density of the nucleus is:

$\rho_m = f\frac{m_N}{V_N} = 0.52 \cdot \frac{1.67 \cdot 10^{-27}}{4/3 \cdot \left(0.85 \cdot 10^{-15}\right)^3 \pi} \approx 3.40 \cdot 10^{17} \frac{\mathrm{kg}}{\mathrm{m}^3}.$	(4)

c) Taking into account the approximation that the number of protons and neutrons is approximately equal, for charge density we get:

$\rho_c = \frac{f}{2}\frac{e}{V_N} = \frac{0.52}{2} \cdot \frac{1.6 \cdot 10^{-19}}{4/3 \cdot \left(0.85 \cdot 10^{-15}\right)^3 \pi} \approx 1.63 \cdot 10^{25} \frac{\mathrm{C}}{\mathrm{m}^3}$	(5)

The number of nucleons in a given nucleus is *A*. The total volume occupied by the nucleus is:

$V = \frac{AV_N}{f},$	(6)

which gives the following relation between radii of nucleus and the number of nucleons:

$R = r_N\left(\frac{A}{f}\right)^{1/3} = \frac{r_N}{f^{1/3}} A^{1/3} = \frac{0.85}{0.52^{1/3}} A^{1/3} = 1.06\,\mathrm{fm} \cdot A^{1/3}.$	(7)

The numerical constant (1.06 fm) in the equation above will be denoted as r_0 in the sequel.

Solution of Task 2

First one needs to estimate the number of surface nucleons. The surface nucleons are in a spherical shell of width $2r_N$ at the surface. The volume of this shell is

$V_{surface}=\frac{4}{3}R^3\pi-\frac{4}{3}(R-2r_N)^3\pi=$ $=\frac{4}{3}R^3\pi-\frac{4}{3}R^3\pi+\frac{4}{3}\pi 3R^2 2r_N-\frac{4}{3}\pi 3R4r_N^2+\frac{4}{3}\pi 8r_N^3$ $=8\pi Rr_N(R-2r_N)+\frac{4}{3}\pi 8r_N^3=$ $=8\pi(R^2r_N-2Rr_N^2+\frac{4}{3}r_N^3)$	(8)

The number of surface nucleons is:

$A_{surface}=f\frac{V_{surface}}{V_N}=f\frac{8\pi(R^2r_N-2Rr_N^2+\frac{4}{3}r_N^3)}{\frac{4}{3}r_N^3\pi}=$ $=f6\left(\left(\frac{R}{r_N}\right)^2-2\left(\frac{R}{r_N}\right)+\frac{4}{3}\right)=$ $=f6\left(\left(\frac{A}{f}\right)^{2/3}-2\left(\frac{A}{f}\right)^{1/3}+\frac{4}{3}\right)=$ $=6f^{1/3}A^{2/3}-12f^{2/3}A^{1/3}+8f=$ $=6^{2/3}\pi^{1/3}A^{2/3}-2\cdot 6^{1/3}\pi^{2/3}A^{1/3}+\frac{4}{3}\pi\approx$ $\approx 4.84A^{2/3}-7.80A^{1/3}+4.19.$	(9)

The binding energy is now:

$E_b=(A-A_{surface})a_V+A_{surface}\frac{a_V}{2}=$ $=Aa_V-A_{surface}\frac{a_V}{2}=$ $=Aa_V-(3f^{1/3}A^{2/3}-6f^{2/3}A^{1/3}+4f)a_V=$ $=Aa_V-3f^{1/3}A^{2/3}a_V+6f^{2/3}A^{1/3}a_V-4fa_V=$ $=(15.8A-38.20A^{2/3}+61.58A^{1/3}-33.09)\mathrm{MeV}$	(10)

Solution of Task 3 - Electrostatic (Coulomb) effects on the binding energy

a) Replacing Q_0 with Ze gives the electrostatic energy of the nucleus as:

$U_c=\frac{3(Ze)^2}{20\pi\varepsilon_0 R}=\frac{3Z^2e^2}{20\pi\varepsilon_0 R}$	(12)

The fact that each proton is not acting upon itself is taken into account by replacing Z^2 with *Z(Z-1)*:

$U_c = \dfrac{3Z(Z-1)e^2}{20\pi\varepsilon_0 R}$		(13)

b) In the formula for the electrostatic energy we should replace R with $r_N f^{-1/3} A^{1/3}$ to obtain

$\Delta E_b = -\dfrac{3e^2 f^{1/3}}{20\pi\varepsilon_0 r_N}\dfrac{Z(Z-1)}{A^{1/3}} = -\dfrac{Z(Z-1)}{A^{1/3}}\cdot 1.31\times 10^{-13}\ \mathrm{J}$ $= -\dfrac{Z(Z-1)}{A^{1/3}}\cdot 0.815\ \mathrm{MeV} \approx \text{-}0.204\mathrm{A}^{5/3}\mathrm{MeV} + 0.409\mathrm{A}^{2/3}\ \mathrm{MeV}$		(14)

where *Z≈A/2* has been used. The Coulomb repulsion reduces the binding energy, hence the negative sign before the first (main) term. The complete formula for binding energy now gives:

$E_b = Aa_V - 3f^{1/3}A^{2/3}a_V + 6f^{2/3}A^{1/3}a_V - 4fa_V - \dfrac{3e^2 f^{1/3}}{20\pi\varepsilon_0 r_N}\left(\dfrac{A^{5/3}}{4} - \dfrac{A^{2/3}}{2}\right)$		(15)

Solution of Task 4 - Fission of heavy nuclei

a) The kinetic energy comes from the difference of binding energies (2 small nuclei – the original large one) and the Coulomb energy between two smaller nuclei (with *Z/2=A/4* nucleons each):

$E_{kin}(d) = 2E_b\left(\dfrac{A}{2}\right) - E_b(A) - \dfrac{1}{4\pi\varepsilon_0}\dfrac{A^2e^2}{4\cdot 4\cdot d} =$ $= -3f^{1/3}A^{2/3}a_V(2^{1/3}-1) + 6f^{2/3}A^{1/3}a_V(2^{2/3}-1)$ $-4fa_V - \dfrac{3e^2 f^{1/3}}{20\pi\varepsilon_0 r_N}\left[\dfrac{A^{5/3}}{4}\left(2^{-2/3}-1\right) - \dfrac{A^{2/3}}{2}\left(2^{1/3}-1\right)\right]$ $-\dfrac{1}{4\pi\varepsilon_0}\dfrac{A^2e^2}{16d}$		(16)

(notice that the first term, Aa_V, cancels out).

b) The kinetic energy when $d = 2R(A/2)$ is given with:

$E_{kin} = 2E_b\left(\dfrac{A}{2}\right) - E_b(A) - \dfrac{1}{4\pi\varepsilon_0}\dfrac{2^{1/3}A^2e^2}{16\cdot 2r_N A^{1/3} f^{-1/3}} =$ $= -3f^{1/3}A^{2/3}a_V(2^{1/3}-1) + 6f^{2/3}A^{1/3}a_V(2^{2/3}-1)$ $-4fa_V - \dfrac{e^2 f^{1/3}}{\pi\varepsilon_0 r_N}\left[\dfrac{3}{80}(2^{-2/3}-1) + \dfrac{2^{1/3}}{128}\right]A^{5/3} - \dfrac{e^2 f^{1/3}}{\pi\varepsilon_0 r_N}\left[\dfrac{3}{40}(2^{1/3}-1)\right]A^{2/3} =$ $= (0.02203A^{5/3} - 10.0365A^{2/3} + 36.175A^{1/3} - 33.091)\mathrm{MeV}$		(17)

Numerically one gets:

A=100 … E_{kin}= -33.95 MeV,

A=150 … E_{kin}= -30.93 MeV,

A=200 … E_{kin}= -14.10 MeV,
A=250 … E_{kin}= +15.06 MeV.

In our model, fission is possible when $E_{kin}(d=2R(A/2))\geq 0$. From the numerical evaluations given above, one sees that this happens approximately halfway between A=200 and A=250 – a rough estimate would be A≈225. Precise numerical evaluation of the equation:

$$E_{kin}=(0.02203A^{5/3}-10.0365A^{2/3}+36.175A^{1/3}-33.091)\mathrm{MeV}\geq 0 \tag{18}$$

gives that for $A\geq 227$ fission is possible.

Solution of Task 5 – Transfer reactions

Task 5a) This part can be solved by using either non-relativistic or relativistic kinematics.

Non-relativistic solution

First one has to find the amount of mass transferred to energy in the reaction (or the energy equivalent, so-called Q-value):

$$\begin{aligned}\Delta m &= (\text{total mass})_{\text{after reaction}} - (\text{total mass})_{\text{before reaction}} = \\ &= (57.93535+12.00000)\ \text{a.m.u.} - (53.93962+15.99491)\ \text{a.m.u.} = \\ &= 0.00082\ \text{a.m.u.} = \\ &= 1.3616\cdot 10^{-30}\ \text{kg.}\end{aligned} \tag{19}$$

Using the Einstein formula for equivalence of mass and energy, we get:

$$\begin{aligned}Q &= (\text{total kinetic energy})_{\text{after reaction}} - (\text{total kinetic energy})_{\text{before reaction}} = \\ &= -\Delta m\cdot c^2 = \\ &= -1.3616\cdot 10^{-30}\cdot 299792458^2 = -1.2237\cdot 10^{-13}\ \text{J}\end{aligned} \tag{20}$$

Taking into account that 1 MeV is equal to $1.602\cdot 10^{-13}$ J, we get:

$$Q=-1.2237\cdot 10^{-13}/1.602\cdot 10^{-13}=-0.761\,\mathrm{MeV} \tag{21}$$

This exercise is now solved using the laws of conservation of energy and momentum. The latter gives (we are interested only for the case when ^{12}C and ^{16}O are having the same direction so we don't need to use vectors):

$$m(^{16}\mathrm{O})v(^{16}\mathrm{O})=m(^{12}\mathrm{C})v(^{12}\mathrm{C})+m(^{58}\mathrm{Ni})v(^{58}\mathrm{Ni}) \tag{22}$$

while the conservation of energy gives:

$$E_k(^{16}\mathrm{O})+Q=E_k(^{12}\mathrm{C})+E_k(^{58}\mathrm{Ni})+E_x(^{58}\mathrm{Ni}) \tag{23}$$

where $E_x(^{58}\mathrm{Ni})$ is the excitation energy of ^{58}Ni, and Q is calculated in the first part of this task. But since ^{12}C and ^{16}O have the same velocity, conservation of momentum reduced to:

$$\left[m(^{16}\mathrm{O})-m(^{12}\mathrm{C})\right]v(^{16}\mathrm{O})=m(^{58}\mathrm{Ni})v(^{58}\mathrm{Ni}) \tag{24}$$

Now we can easily find the kinetic energy of ^{58}Ni:

$$\begin{aligned} E_k(^{58}\mathrm{Ni}) &= \frac{m(^{58}\mathrm{Ni})v^2(^{58}\mathrm{Ni})}{2}=\frac{\left[m(^{58}\mathrm{Ni})v(^{58}\mathrm{Ni})\right]^2}{2m(^{58}\mathrm{Ni})}= \\ &= \frac{\left[\left[m(^{16}\mathrm{O})-m(^{12}\mathrm{C})\right]v(^{16}\mathrm{O})\right]^2}{2m(^{58}\mathrm{Ni})}= \\ &= E_k(^{16}\mathrm{O})\frac{\left[m(^{16}\mathrm{O})-m(^{12}\mathrm{C})\right]^2}{m(^{58}\mathrm{Ni})m(^{16}\mathrm{O})} \end{aligned} \tag{25}$$

and finally the excitation energy of ^{58}Ni:

$$\begin{aligned} E_x(^{58}\mathrm{Ni}) &= E_k(^{16}\mathrm{O})+Q-E_k(^{12}\mathrm{C})-E_k(^{58}\mathrm{Ni})= \\ &= E_k(^{16}\mathrm{O})+Q-\frac{m(^{12}\mathrm{C})v^2(^{16}\mathrm{O})}{2}-E_k(^{16}\mathrm{O})\frac{\left[m(^{16}\mathrm{O})-m(^{12}\mathrm{C})\right]^2}{m(^{58}\mathrm{Ni})m(^{16}\mathrm{O})}= \\ &= Q+E_k(^{16}\mathrm{O})-E_k(^{16}\mathrm{O})\cdot\frac{m(^{12}\mathrm{C})}{m(^{16}\mathrm{O})}-E_k(^{16}\mathrm{O})\frac{\left[m(^{16}\mathrm{O})-m(^{12}\mathrm{C})\right]^2}{m(^{58}\mathrm{Ni})m(^{16}\mathrm{O})}= \\ &= Q+E_k(^{16}\mathrm{O})\left[1-\frac{m(^{12}\mathrm{C})}{m(^{16}\mathrm{O})}-\frac{\left[m(^{16}\mathrm{O})-m(^{12}\mathrm{C})\right]^2}{m(^{58}\mathrm{Ni})m(^{16}\mathrm{O})}\right]= \\ &= Q+E_k(^{16}\mathrm{O})\frac{\left[m(^{16}\mathrm{O})-m(^{12}\mathrm{C})\right]\cdot\left[m(^{58}\mathrm{Ni})-m(^{16}\mathrm{O})+m(^{12}\mathrm{C})\right]}{m(^{58}\mathrm{Ni})m(^{16}\mathrm{O})} \end{aligned} \tag{26}$$

Note that the first bracket in numerator is approximately equal to the mass of transferred particle (the ^{4}He nucleus), while the second one is approximately equal to the mass of target nucleus ^{54}Fe. Inserting the numbers we get:

$$\begin{aligned} E_x(^{58}\mathrm{Ni}) &= -0.761+50\cdot\frac{(15.99491-12.)(57.93535-15.99491+12.)}{57.93535\cdot 15.99491}= \\ &= 10.866\,\mathrm{MeV} \end{aligned} \tag{27}$$

Relativistic solution

In the relativistic version, solution is found starting from the following pair of equations (the first one is the law of conservation of energy and the second one the law of conservation of momentum):

$$m(^{54}\mathrm{Fe})\cdot c^2+\frac{m(^{16}\mathrm{O})\cdot c^2}{\sqrt{1-v^2(^{16}\mathrm{O})/c^2}}=\frac{m(^{12}\mathrm{C})\cdot c^2}{\sqrt{1-v^2(^{12}\mathrm{C})/c^2}}+\frac{m^*(^{58}\mathrm{Ni})\cdot c^2}{\sqrt{1-v^2(^{58}\mathrm{Ni})/c^2}} \tag{28}$$

$$\frac{m(^{16}\mathrm{O})\cdot v(^{16}\mathrm{O})}{\sqrt{1-v^2(^{16}\mathrm{O})/c^2}}=\frac{m(^{12}\mathrm{C})\cdot v(^{12}\mathrm{C})}{\sqrt{1-v^2(^{12}\mathrm{C})/c^2}}+\frac{m^*(^{58}\mathrm{Ni})\cdot v(^{58}\mathrm{Ni})}{\sqrt{1-v^2(^{58}\mathrm{Ni})/c^2}}$$

All the masses in the equations are the rest masses; the ^{58}Ni is NOT in its ground-state, but in one of its excited states (having the mass denoted with m^*). Since ^{12}C and ^{16}O have the same velocity, this set of equations reduces to:

$$m(^{54}\mathrm{Fe})+\frac{m(^{16}\mathrm{O})-m(^{12}\mathrm{C})}{\sqrt{1-v^2(^{16}\mathrm{O})/c^2}}=\frac{m^*(^{58}\mathrm{Ni})}{\sqrt{1-v^2(^{58}\mathrm{Ni})/c^2}}$$
$$\frac{\left(m(^{16}\mathrm{O})-m(^{12}\mathrm{C})\right)\cdot v(^{16}\mathrm{O})}{\sqrt{1-v^2(^{16}\mathrm{O})/c^2}}=\frac{m^*(^{58}Ni)\cdot v(^{58}\mathrm{Ni})}{\sqrt{1-v^2(^{58}\mathrm{Ni})/c^2}} \tag{29}$$

Dividing the second equation with the first one gives:

$$v(^{58}\mathrm{Ni})=\frac{\left(m(^{16}\mathrm{O})-m(^{12}\mathrm{C})\right)\cdot v(^{16}\mathrm{O})}{\left(m(^{16}\mathrm{O})-m(^{12}\mathrm{C})\right)+m(^{54}\mathrm{Fe})\sqrt{1-v^2(^{16}\mathrm{O})/c^2}} \tag{30}$$

The velocity of projectile can be calculated from its energy:

$$E_{kin}(^{16}\mathrm{O})=\frac{m(^{16}\mathrm{O})\cdot c^2}{\sqrt{1-v^2(^{16}\mathrm{O})/c^2}}-m(^{16}\mathrm{O})\cdot c^2$$
$$\sqrt{1-v^2(^{16}\mathrm{O})/c^2}=\frac{m(^{16}\mathrm{O})\cdot c^2}{E_{kin}(^{16}\mathrm{O})+m(^{16}\mathrm{O})\cdot c^2}$$
$$v^2(^{16}\mathrm{O})/c^2=1-\left(\frac{m(^{16}\mathrm{O})\cdot c^2}{E_{kin}(^{16}\mathrm{O})+m(^{16}\mathrm{O})\cdot c^2}\right)^2$$
$$v(^{16}\mathrm{O})=\sqrt{1-\left(\frac{m(^{16}\mathrm{O})\cdot c^2}{E_{kin}(^{16}\mathrm{O})+m(^{16}\mathrm{O})\cdot c^2}\right)^2}\cdot c \tag{31}$$

For the given numbers we get:

$$v(^{16}\mathrm{O})=\sqrt{1-\left(\frac{15.99491\cdot 1.6605\cdot 10^{-27}\cdot(2.9979\cdot 10^8)^2}{50\cdot 1.602\cdot 10^{-13}+15.99491\cdot(2.9979\cdot 10^8)^2}\right)^2}\cdot c=$$
$$=\sqrt{1-0.99666^2}\cdot c=0.08172\cdot c=2.4498\cdot 10^7\ \mathrm{km/s} \tag{32}$$

Now we can calculate:

$$v(^{58}\mathrm{Ni})=\frac{(15.99491-12.0)\cdot 2.4498\cdot 10^7\ \mathrm{km/s}}{(15.99491-12.0)+53.93962\sqrt{1-0.08172^2}}=1.6946\cdot 10^6\ \mathrm{km/s} \tag{33}$$

The mass of ^{58}Ni in its excited state is then:

$$m^*\left({}^{58}\mathrm{Ni}\right)=\left(m\left({}^{16}\mathrm{O}\right)-m\left({}^{12}\mathrm{C}\right)\right)\frac{\sqrt{1-v^2\left({}^{58}\mathrm{Ni}\right)/c^2}}{\sqrt{1-v^2\left({}^{16}\mathrm{O}\right)/c^2}}\cdot\frac{v\left({}^{16}\mathrm{O}\right)}{v\left({}^{58}\mathrm{Ni}\right)}=$$
$$=(15.99491-12.0)\frac{\sqrt{1-\left(1.6945\cdot 10^6/2.9979\cdot 10^8\right)^2}}{\sqrt{1-0.08172^2}}\cdot\frac{2.4498\cdot 10^7}{1.6945\cdot 10^6}\text{ a.m.u.}=$$
$$=57.9470\text{ a.m.u.} \qquad (34)$$

The excitation energy of ^{58}Ni is then:

$$E_x=\left[m^*\left({}^{58}\mathrm{Ni}\right)-m\left({}^{58}\mathrm{Ni}\right)\right]\cdot c^2=(57.9470-57.93535)\cdot 1.6605\cdot 10^{-27}\left(2.9979\cdot 10^8\right)^2=$$
$$=2.00722\cdot 10^{-12}/1.602\cdot 10^{-13}\text{ MeV/J}=10.8636\text{ MeV} \qquad (35)$$

The relativistic and non-relativistic results are equal within 2 keV so both can be considered as correct –we can conclude that at the given beam energy, relativistic effects are not important.

Task 5b) For gamma-emission from the static nucleus, laws of conservation of energy and momentum give:

$$E_x\left({}^{58}\mathrm{Ni}\right)=E_\gamma+E_{\mathrm{recoil}}$$
$$p_\gamma=p_{\mathrm{recoil}} \qquad (36)$$

Gamma-ray and recoiled nucleus have, of course, opposite directions. For gamma-ray (photon), energy and momentum are related as:

$$E_\gamma=p_\gamma\cdot c \qquad (37)$$

In part a) we have seen that the nucleus motion in this energy range is not relativistic, so we have:

$$E_{\mathrm{recoil}}=\frac{p_{\mathrm{recoil}}^2}{2m\left({}^{58}\mathrm{Ni}\right)}=\frac{p_\gamma^2}{2m\left({}^{58}\mathrm{Ni}\right)}=\frac{E_\gamma^2}{2m\left({}^{58}\mathrm{Ni}\right)\cdot c^2} \qquad (38)$$

Inserting this into law of energy conservation Eq. (36), we get:

$$E_x\left({}^{58}\mathrm{Ni}\right)=E_\gamma+E_{\mathrm{recoil}}=E_\gamma+\frac{E_\gamma^2}{2m\left({}^{58}\mathrm{Ni}\right)\cdot c^2} \qquad (39)$$

This reduces to the quadratic equation:

$$E_\gamma^2 + 2m\left({}^{58}\mathrm{Ni}\right)c^2 \cdot E_\gamma + 2m\left({}^{58}\mathrm{Ni}\right)c^2 E_x\left({}^{58}\mathrm{Ni}\right) = 0 \tag{40}$$

which gives the following solution:

$$E_\gamma = \frac{-2m\left({}^{58}\mathrm{Ni}\right)c^2 + \sqrt{4\left(m\left({}^{58}\mathrm{Ni}\right)c^2\right)^2 + 8m\left({}^{58}\mathrm{Ni}\right)c^2 E_x\left({}^{58}\mathrm{Ni}\right)}}{2} = \\ = \sqrt{\left(m\left({}^{58}\mathrm{Ni}\right)c^2\right)^2 + 2m\left({}^{58}\mathrm{Ni}\right)c^2 E_x\left({}^{58}\mathrm{Ni}\right)} - m\left({}^{58}\mathrm{Ni}\right)c^2 \tag{41}$$

Inserting numbers gives:

$$E_\gamma = 10.8633\,\mathrm{MeV} \tag{42}$$

The equation (37) can also be reduced to an approximate equation before inserting numbers:

$$E_\gamma = E_x\left(1 - \frac{E_x}{2m\left({}^{58}\mathrm{Ni}\right)c^2}\right) = 10.8633\,\mathrm{MeV} \tag{43}$$

The recoil energy is now easily found as:

$$E_{\mathrm{recoil}} = E_x\left({}^{58}\mathrm{Ni}\right) - E_\gamma = 1.1\,\mathrm{keV} \tag{44}$$

Due to the fact that nucleus emitting gamma-ray (^{58}Ni) is moving with the high velocity, the energy of gamma ray will be changed because of the Doppler effect. The relativistic Doppler effect (when source is moving towards observer/detector) is given with this formula:

$$f_{\mathrm{detector}} = f_{\gamma,\mathrm{emitted}}\sqrt{\frac{1+\beta}{1-\beta}} \tag{45}$$

and since there is a simple relation between photon energy and frequency *(E=hf)*, we get the similar expression for energy:

$$E_{\mathrm{detector}} = E_{\gamma,\mathrm{emitted}}\sqrt{\frac{1+\beta}{1-\beta}} \tag{46}$$

where $\beta = v/c$ and v is the velocity of emitter (the ^{58}Ni nucleus). Taking the calculated value of the ^{58}Ni velocity (equation 29) we get:

$$E_{\mathrm{detector}} = E_{\gamma,\mathrm{emitted}}\sqrt{\frac{1+\beta}{1-\beta}} = 10.863\sqrt{\frac{1+0.00565}{1-0.00565}} = 10.925\,\mathrm{MeV} \tag{47}$$