Modeling the Economic Growth of Two Regions

[MATH 5131 Final Project Report]

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Chapter 1

Introduction

Modeling economic growth is crucial when it comes to grasping the development of regions over time and in understanding the factors that affect their economic bearings. The two elementary models used in the underlying work are the Solow Growth Model and the Malthusian Population Model. Using a combination of these models, one can comprehend the trends of economic growth between two regions when they are linked via capital and labor.

Developed by Robert Solow, the Solow Growth Model explains the interactions between capital accumulation, labor force growth, and technological progress in order to determine the economic output. This model is underlined by the Cobb-Douglas Function which has the form $f(K, L) = AK^{\alpha}L^{1-\alpha}$ where, Y is the output, K is the capital, L is the labor force and A is the technological progress. The growth of capital can thus be described by:

$$\frac{dK}{dt} = f(K, L) - \delta K; K(0) = K_0 \tag{1.1}$$

Proposed by Thomas Robert Malthus, the Malthusian Population Model suggested that an exponential growth in population would lead to depletion of resources using the form $L(t) = L_0 e^{rt}$. Later modifications by Pierre Francois Velhurst proved to be a better framework by making use of carrying capacity and the model took the form:

$$\frac{dL}{dt} = aL - bL^2; L(0) = L_0 \tag{1.2}$$

Where, L is the population at time t, $a > 0$ is the growth rate and $b > 0$ is the crowding effect.

The above models alone cannot accurately explain the complexity of studying modern regional economic development. Hence, this study makes use of both these classical models by combining them into a system of differential equations, to comprehend the dynamics between two regions. By setting up an environment of a large set of parameters, one can even identify the long term behaviour of this scenario.

Chapter 2

Mathematical Model

To obtain a model with a system of ODEs, we make use of a combination of the Solow Growth Model (1.1) and the Malthusian Population Model (1.2).

$$\frac{dK}{dt} = AK^{\alpha}L^{1-\alpha}; \frac{dL}{dt} = aL - bL^2 \tag{2.1}$$

This model is set in a homogeneous condition with no movement of capital and labor in any direction. But, the main interest lies in analyzing the economic trends when there lies a flow of capital and labor between the two regions. Hence, we modify this by constructing a four equation ODE system and subsequently analyze the results and the equilibrium states.
To establish the system of four ODEs, we can propose two additional factors, namely the difference of capital between the regions and the difference of labor between the regions. Further, we can assume that the rate of exchange of capital between the two regions depends on the difference of capital between them. Although for the rate of exchange of labor between the regions, we assume that it depends on both, the capital difference and the labor difference. The reason is because if labor is saturated in a particular region then people will look for jobs in other regions, and a region with a higher capital will attract a higher labor force. With these assumptions, the following model is proposed:

$$\frac{dK_1}{dt} = d_k(K_2 - K_1) + A_1K_1^{\alpha}L_1^{1-\alpha} - \delta_1 K1, \tag{2.2}$$

$$\frac{dK_2}{dt} = d_k(K_1 - K_2) + A_2K_2^{\alpha}L_2^{1-\alpha} - \delta_2 K2, \tag{2.3}$$

$$\frac{dL_1}{dt} = d_l(L_2 - L_1) + a_1L_1 - b_1L_1^2 - cH(L_1, L_2, K_1, K_2), \tag{2.4}$$

$$\frac{dL_2}{dt} = d_l(L_1 - L_2) + a_2L_2 - b_2L_2^2 + cH(L_1, L_2, K_1, K_2), \tag{2.5}$$

Where,

- $K_1(t)$ and $K_2(t)$ are the capital amounts at time t in regions 1 and 2 respectively.
- $L_1(t)$ and $L_2(t)$ are the labor amounts at time t in regions 1 and 2 respectively.
- d_k is the capital diffusion coefficient.
- d_l is the labor diffusion coefficient.
- c is the strength of capital-induced labor movement.
- $\delta_k(\delta > 0)$ is the capital depreciation rate.

The function $H(K_1, K_2, L_1, L_2)$ can be described using:

$$H(K_1, K_2, L_1, L_2) = \begin{cases} L_1(K_2 - K_1), & \text{if } K_2 - K_1 \geq 0, \\ L_2(K_2 - K_1), & \text{if } K_2 - K_1 < 0. \end{cases} \tag{2.6}$$

Here, $H(K_1, K_2, L_1, L_2)$ is a function defined piecewise, hence it is not differentiable at $K_1 = K_2$. So, we can describe it using a different form given as follows:

$$H(K_1, K_2, L_1, L_2) = (\frac{L_1 - L_2}{1 + e^{-}h(K_2 - K_1)} + L_2)(K_2 - K_1); h > 0 \tag{2.7}$$

When $K_2 - K_1 >> 0 \implies H(K_1, K_2, L_1, L_2) \approx L_1(K_2 - K_1)$;
And when $K_2 - K_1 << 0 \implies H(K_1, K_2, L_1, L_2) \approx L_2(K_2 - K_1)$.

Therefore, we have a system of four differential equations that factors in the effects of capital and labor differences between the regions as well. Now, suppose that $H(K_1, K_2, L_1, L_2)$ is defined as per (2.7). For any given initial conditions $K_1(0) = K_{1,0} \geq 0, K_2(0) = K_{2,0} \geq 0, L_1(0) = L_{1,0} \geq 0, L_2(0) = L_{2,0} \geq 0$, there exists a unique solution $(K_1(t), K_2(t), L_1(t), L_2(t)) \geq 0$ for any $t \in (0, \infty)$. At $t = 0$, either $K_1, K_2, L_1, L_2 = 0$

- If $K_1(t_0) = 0 \implies (K_1)'(t_0) = d_k K_2(t_0) \geq 0$;
- If $K_2(t_0) = 0 \implies (K_2)'(t_0) = d_k K_1(t_0) \geq 0$;
- If $L_1(t_0) = 0 \implies (L_1)'(t_0) = d_l L_2 - cH(L_1, L_2, K_1, K_2)$
  This leads to two cases:
  $$\begin{cases} (L_1)'(t_0) = d_l L_2 \geq 0; & K_2 - K_1 \geq 0 \\ (L_1)'(t_0) = d_l L_2 - L_2(K_2 - K_1) > 0; & K_2 - K_1 < 0 \end{cases}$$

- If $L_2(t_0) = 0 \implies (L_2)'(t_0) = d_l L_1 + cH(L_1, L_2, K_1, K_2)$

  This again leads to two cases:

  $$\begin{cases} (L_2)'(t_0) = d_l L_1 + L_1(K_2 - K_1) > 0; & K_2 - K_1 \geq 0 \\ (L_2)'(t_0) = d_l L_1 \geq 0; & K_2 - K_1 < 0 \end{cases}$$

Hence we can conclude that there exists a unique solution $(K_1(t), K_2(t), L_1(t), L_2(t)) \geq 0$ exists for our four ODE model.

Chapter 3

Modification of Model to show No Capital Induced Labor Movement (c = 0)

In this case, we modify the system given in (2.2) - (2.5) such that there is No Capital Induced Labor Movement i.e. c = 0. Hence,

$$\frac{dK_1}{dt} = d_k(K_2 - K_1) + A_1 K_1^{\alpha} L_1^{1-\alpha} - \delta_1 K1, \tag{3.1}$$

$$\frac{dK_2}{dt} = d_k(K_1 - K_2) + A_2 K_2^{\alpha} L_2^{1-\alpha} - \delta_2 K2, \tag{3.2}$$

$$\frac{dL_1}{dt} = d_l(L_2 - L_1) + a_1 L_1 - b_1 L_1^2, \tag{3.3}$$

$$\frac{dL_2}{dt} = d_l(L_1 - L_2) + a_2 L_2 - b_2 L_2^2 \tag{3.4}$$

For any A_1, A_2, δ_1, δ_2, a_1, a_2, b_1, $b_2 > 0$, d_k, $d_l > 0$, and $\phi \in (0, 1)$, the system (3.1)-(3.4) has a unique positive equilibrium $(K_1^*, K_2^*, L_1^*, L_2^*)$.
To prove that the system has a unique positive equilibrium, we will first prove that (3.3) and (3.4) have a unique positive equilibrium. Setting $\frac{dK_1}{dt}$ and $\frac{dK_2}{dt}$ in (3.3) and (3.4) equal to 0, we find

$$L_2 = f(L_1) = L_1(1 - \frac{a_1}{d_l}) + \frac{b_1}{d_l} L_1^2, \tag{3.5}$$

$$L_1 = g(L_2) = L_2(1 - \frac{a_2}{d_l}) + \frac{b_2}{d_l} L_2^2. \tag{3.6}$$

Solving L_2 from (3.6), we get

$$L_2 = \frac{-(1-\frac{a_2}{d_l}) \pm \sqrt{(1-\frac{a_2}{d_l})^2 + \frac{4b_2}{d_l}L_1}}{2b_2/d_l}. \tag{3.7}$$

Since $L_2 > 0$, we must have

$$L_2 = \frac{-(1-\frac{a_2}{d_l}) + \sqrt{(1-\frac{a_2}{d_l})^2 + \frac{4b_2}{d_l}L_1}}{2b_2/d_l} \equiv g^{-1}(L_1). \tag{3.8}$$

Hence,

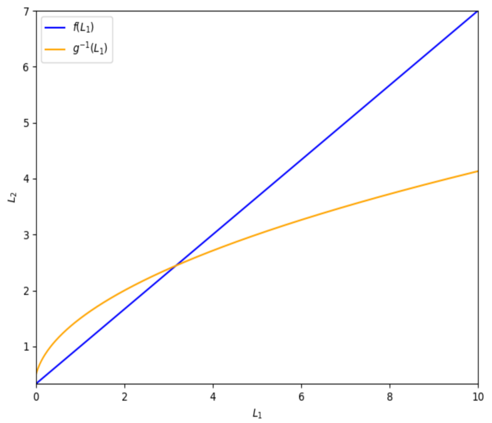
$$f'(0) = \frac{d_l - a_1}{d_l} \tag{3.9}$$

$$(g^-1)'(0) = \frac{d_l}{d_l - a_2} \tag{3.10}$$

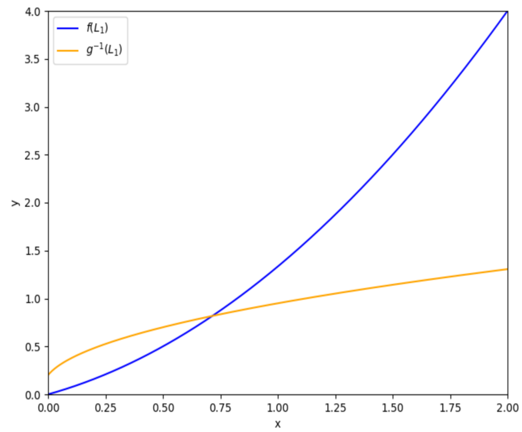
Now we can divide it into 4 cases:

- Case 1: $f'(0) > 0, (g^-1)'(0) > 0$
- Case 2: $f'(0) > 0, (g^-1)'(0) < 0$
- Case 3: $f'(0) < 0, (g^-1)'(0) > 0$
- Case 4: $f'(0) < 0, (g^-1)'(0) < 0$

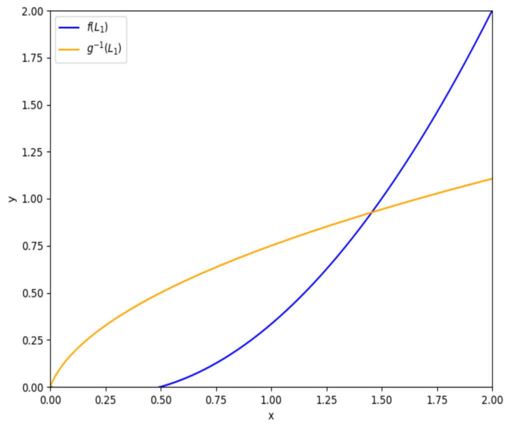
And, we obtain a positive equilibrium (L_1^*, L_2^*) for each case as shown in figure (3.1)



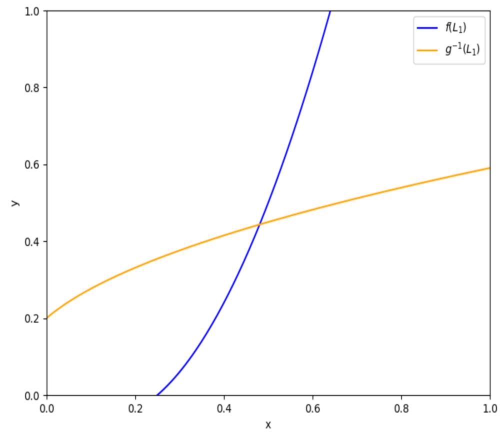
(a) Case 1: $a_1 = 1, a_2 = 2, d_l = 1.5, b_1 = 1, b_2 = 2$



(b) Case 2: $a_1 = 1, a_2 = 2, d_l = 1.5, b_1 = 1, b_2 = 2$



(c) Case 3: $a_1 = 2, a_2 = 1, d_l = 1.5, b_1 = 1, b_2 = 2$



(d) Case 4: $a_1 = 1, a_2 = 2, d_l = 0.5, b_1 = 1, b_2 = 2$

Figure 3.1: Graphs for $f(L_1)$ and $g^{-1}(L_1)$

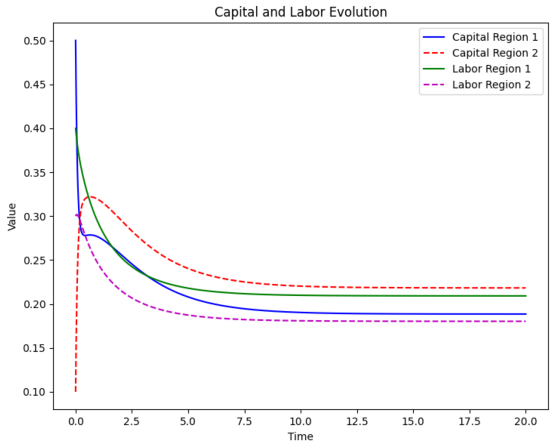
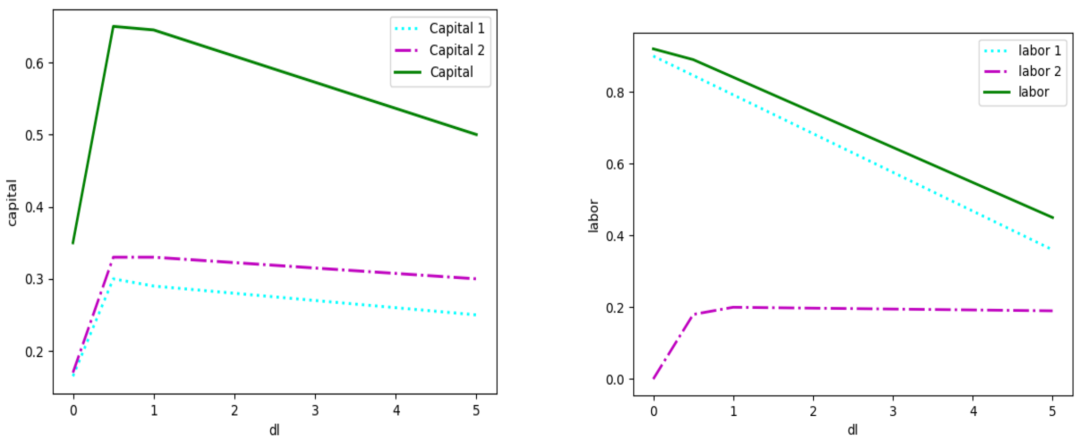


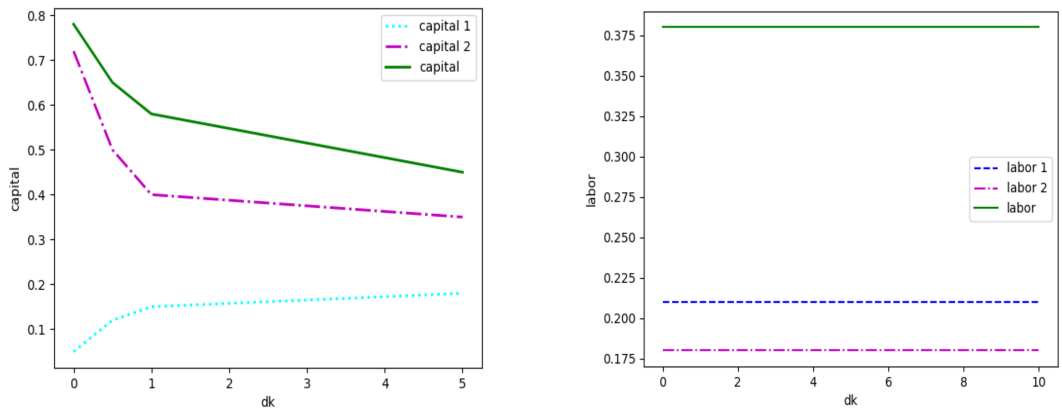
Figure 3.2: Convergence to the unique equilibrium point $(K_1^*, K_2^*, K_3^*, K_4^*) = (0.1904, 0.2203, 0.2099, 0.1809)$. Parameters used: $d_k = 6, d_l = 5, A_1 = 1, A_2 = 2, \phi = 0.5, \delta_1 = 2, \delta_2 = 1, a_1 = 0.9, a_2 = 0.1, b_1 = 1, b2_= 5$. Initial value: $(K_1(0), K_2(0), L_1(0), L_2(0)) = (0.5, 0.1, 0.4, 0.3)$



(a) Equilibrium Points for (3.1) and (3.2)

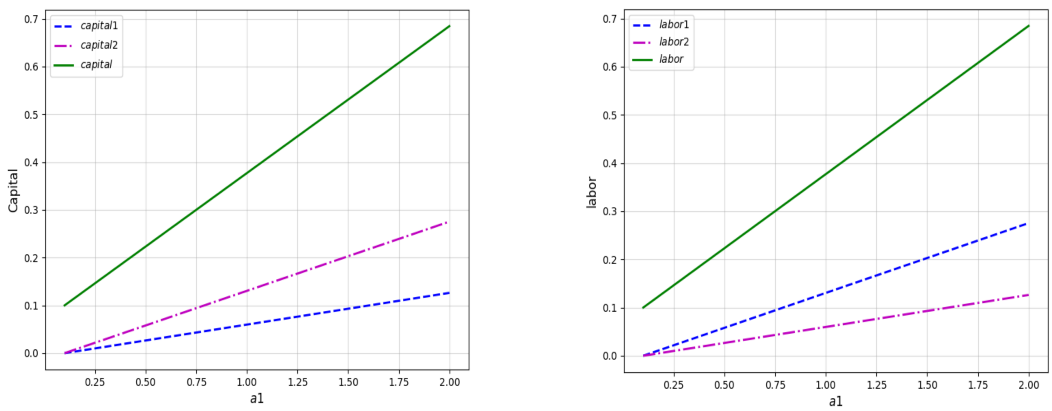
(b) Equilibrium Points for (3.3) and (3.4)

Figure 3.3: Unique Equilibrium Points for different d_l values. Parameters used: $d_k = 6, A_1 = 1, A_2 = 2, \alpha = 0.5, \delta_1 = 2, \delta_2 = 1, a_1 = 0.9, a_2 = 0.1, b_1 = 1, b_2 = 5$ and Initial Value:$(K_{1,0}, K_{2,0}, L_{1,0}, L_{2,0}) = (1, 0.1, 2, 0.3)$



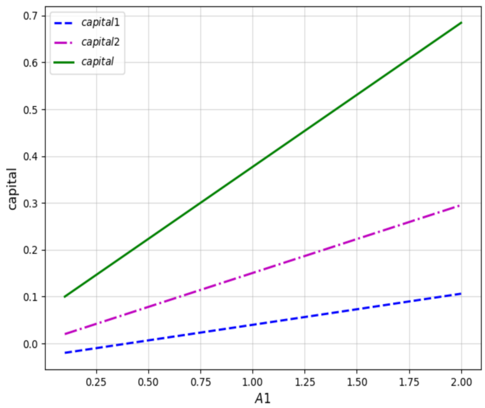
(a) Equilibrium Points for (3.1) and (3.2) (b) Equilibrium Points for (3.3) and (3.4)

Figure 3.4: Unique Equilibrium Points for different d_k values. Parameters used: $d_l = 5, A_1 = 1, A_2 = 2, \alpha = 0.5, \delta_1 = 2, \delta_2 = 1, a_1 = 0.9, a_2 = 0.1, b_1 = 1, b_2 = 5$ and Initial Value:$(K_{1,0}, K_{2,0}, L_{1,0}, L_{2,0}) = (1, 0.1, 2, 0.3)$

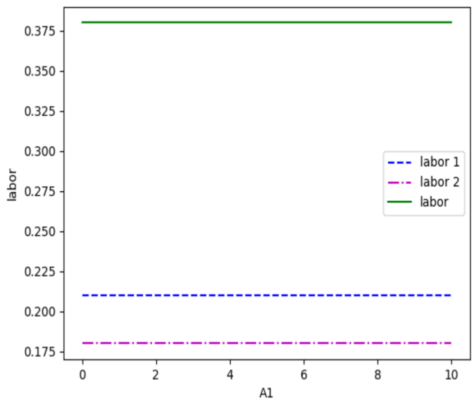


(a) Equilibrium Points for (3.1) and (3.2) (b) Equilibrium Points for (3.3) and (3.4)

Figure 3.5: Unique Equilibrium Points for different a_1 values. Parameters used: $d_k = 6, d_l = 5, A_1 = 1, A_2 = 2, \alpha = 0.5, \delta_1 = 2, \delta_2 = 1, a_2 = 0.1, b_1 = 1, b_2 = 5$ and Initial Value:$(K_{1,0}, K_{2,0}, L_{1,0}, L_{2,0}) = (1, 0.1, 2, 0.3)$



(a) Equilibrium Points for (3.1) and (3.2)



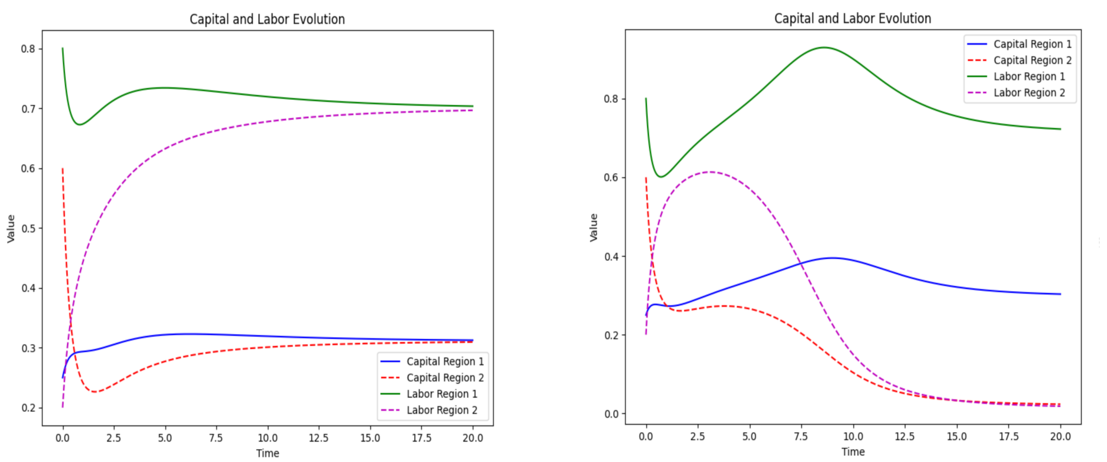
(b) Equilibrium Points for (3.3) and (3.4)

Figure 3.6: Unique Equilibrium Points for different A_1 values. Parameters used: $d_k = 6, d_l = 5, A_2 = 2, \alpha = 0.5, \delta_1 = 2, \delta_2 = 1, a_1 = 0.9, a_2 = 0.1, b_1 = 1, b_2 = 5$ and Initial Value:$(K_{1,0}, K_{2,0}, L_{1,0}, L_{2,0}) = (1, 0.1, 2, 0.3)$

Chapter 4

Modification of Model to show Capital Induced Labor Movement (c > 0)

Now, we consider a special case of the model, where $A_1 = A_2 = A, a_1 = a_2 = a, b_1 = b_2 = b, \delta_1 = \delta_2 = \delta, d_k > 0, d_l > 0$ and $c > 0$.
In this case, the symmetric equilibrium point K^*, K^*, L^*, L^* is locally asymptotically stable when $0 \leq c \leq m$ and is unstable when $c > m$, where $m = \frac{(2d_k - a_{11})(2d_l - a_{22})}{2a_{12}L^*}$ and $a_{11} = A\alpha(K^*)^{\alpha-1}(L^*)^{1-\alpha} - \delta$, $a_{12} = A(K^*)^{\alpha}(1-\alpha)(L^*)^{-\alpha}$ and $a_{22} = a - 2bL^*$.



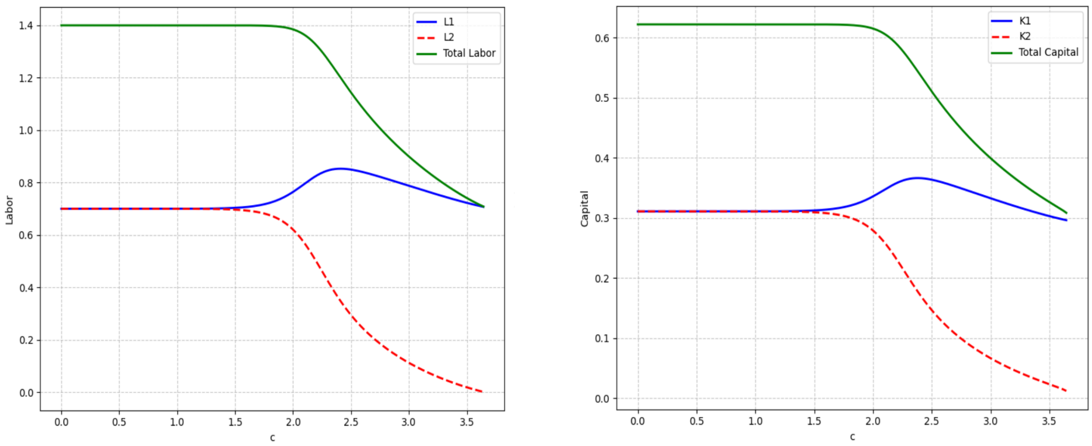
(a) Equilibrium Points for c = 1.5

(b) Equilibrium Points for c = 3.5

Figure 4.1: Parameters used: $d_k = 0.1, d_l = 0.2, A_1 = A_2 = 2, \alpha = 0.5, \delta_1 = \delta_2 = 3, a_1 = a_2 = 0.7, b_1 = b_2 = 1$ and Initial Value: $(K_{1,0}, K_{2,0}, L_{1,0}, L_{2,0}) = (0.25, 0.6, 0.8, 0.2$

From figure (4.1(a)), we can observe that eventually the labor and capital of both regions converge to the same steady state, considering that the value of c is small. Since region 2 has more capital, capital will flow into region 1, thereby increasing it for small values of t. Ultimately they converge to the same value at large values of t.

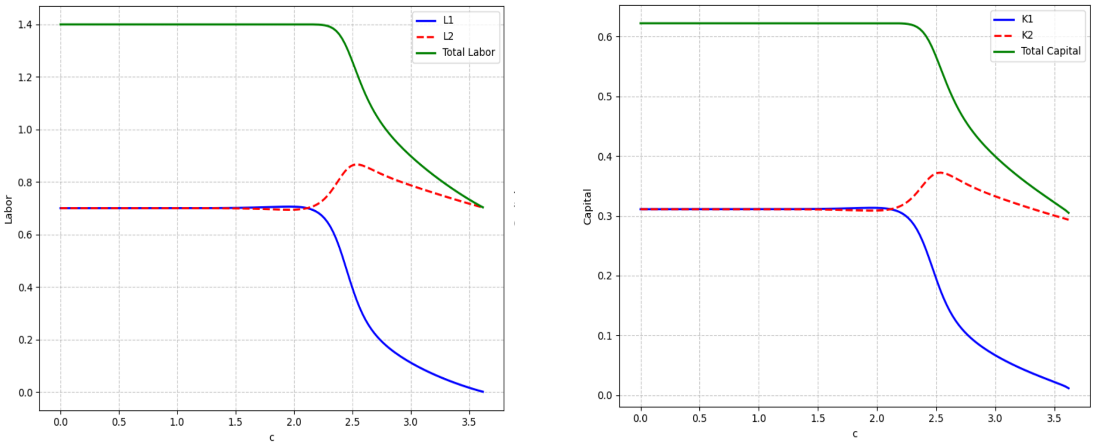
However, for large values of c as in figure (4.2(b)), the steady states of both regions are unequal. The capital of region 2 will decrease and the increase. But labor in region 1 will increase and then decrease to a steady state which is much lower than that of region 2.



(a) Equilibrium Points of Labor for Different c Values (b) Equilibrium Points of Capital for Different c Values

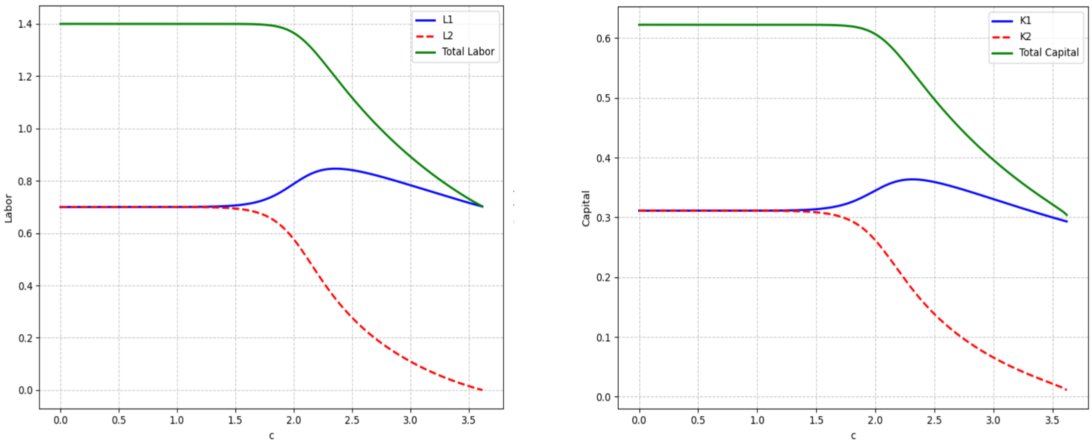
Figure 4.2: Parameters used: $d_k = 0.1, d_l = 0.2, A_1 = A_2 = 2, \alpha = 0.5, \delta_1 = \delta_2 = 3, a_1 = a_2 = 0.7, b_1 = b_2 = 1$ and Initial Value:$(K_{1,0}, K_{2,0}, L_{1,0}, L_{2,0}) = (0.25, 0.6, 0.8, 0.2)$

For figure (4.2) above and figures (4.3) and (4.4) below, we change the value of c from 0 to 5, increasing in steps of 0.5. It is evident from the graphs that until c = 2, the labor and capital in both regions is equal. For larger values of c, we observe that the equilibrium points depend upon the initial conditions assumed. One common interpretation from these graphs is that the total capital and total labor always decrease, implying that capital induced labor movement is no longer beneficial for the economies.



(a) Equilibrium Points of Labor for Different c Values (b) Equilibrium Points of Capital for Different c Values

Figure 4.3: Parameters used: $d_k = 0.1, d_l = 0.2, A_1 = A_2 = 2, \alpha = 0.5, \delta_1 = \delta_2 = 3, a_1 = a_2 = 0.7, b_1 = b_2 = 1$ and Initial Value:$(K_{1,0}, K_{2,0}, L_{1,0}, L_{2,0}) = (0.25, 0.6, 0.7, 0.3)$



(a) Equilibrium Points of Labor for Different c Values (b) Equilibrium Points of Capital for Different c Values

Figure 4.4: Parameters used: $d_k = 0.1, d_l = 0.2, A_1 = A_2 = 2, \alpha = 0.5, \delta_1 = \delta_2 = 3, a_1 = a_2 = 0.7, b_1 = b_2 = 1$ and Initial Value:$(K_{1,0}, K_{2,0}, L_{1,0}, L_{2,0}) = (0.25, 0.6, 0.9, 0.1)$

Chapter 5

Conclusion

Our mathematical model of economic growth between two geographical regions provides valuable insights into the dynamics of regional development and the factors that influence economic disparities. By combining the Solow Growth Model with population dynamics and incorporating capital and labor flows, we have demonstrated how different parameters affect the evolution of regional economies.

The numerical simulations reveal several key findings. First, when capital-induced labor movement is weak, regions tend to converge toward similar levels of economic development, regardless of initial conditions. However, when capital-induced labor movement is strong, we observe that even regions with similar initial conditions can develop asymmetrically, leading to persistent economic inequalities. This suggests that the mobility of labor in response to capital differences plays a crucial role in determining regional economic outcomes.

Our analysis also highlights the importance of policy considerations regarding economic openness between regions. The model shows that while capital and labor mobility can promote economic efficiency, they may also contribute to uneven development patterns. These results have practical implications for policymakers seeking to balance economic growth with regional equity.

Further research could extend this model to consider additional factors such as technological diffusion between regions, trade barriers, or multiple regions. Nevertheless, the current model provides a useful framework for understanding the complex interactions that shape regional economic development.