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Boolean Algrebra

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Digital Design: With an Introduction to the Verilog HDL (M. M. Mano and M. D. Ciletti, Prentice Hall, 2012)

Introduction

- In order to reduce the cost of a digital system, simpler and cheaper (but equivalent) realizations of digital circuits are required to be derived.
- The mathematical methods employed to simplify digital circuits mostly rely on Boolean algebra.
- The algebraic system, now called Boolean algebra, is developed by George Boole in 1854.
- In 1938, Claude E. Shannon introduced a two-valued Boolean algebra called switching algebra in order to represent the properties of *bistable* electrical switching circuits.

Definition of Boolean Algebra

- Boolean algebra can be formally defined, by employing the postulates formulated by E. V. Huntington in 1904, as

  an algebraic structure defined by a set of elements, $\mathcal{B}$, together with two binary operators, $+$ and $\cdot$, provided that the following postulates are satisfied:

  1. (a) The structure is closed with respect to the operator $+$.
     (b) The structure is closed with respect to the operator $\cdot$.
  2. (a) The element 0 is an identity element with respect to $+$; that is, $x + 0 = 0 + x = x$.
     (b) The element 1 is an identity element with respect to $\cdot$; that is, $x \cdot 1 = 1 \cdot x = x$.
  3. (a) The structure is commutative with respect to $+$; that is, $x + y = y + x$.
     (b) The structure is commutative with respect to $\cdot$; that is, $x \cdot y = y \cdot x$.

4. (a) The operator $\cdot$ is distributive over $+$; that is, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.
   (b) The operator $+$ is distributive over $\cdot$; that is, $x + (y \cdot z) = (x + y) \cdot (x + z)$.
5. For every element $x \in \mathcal{B}$, there exists an element $\bar{x} \in \mathcal{B}$ (called the complement of x) such that (a) $x + \bar{x} = 1$ and (b) $x \cdot \bar{x} = 0$.
6. There exist at least two elements $x, y \in \mathcal{B}$ such that $x \neq y$.

- Notes:
  - A binary operator defined on a set $\mathcal{S}$ of elements is a *rule* that assigns, to each pair of elements from $\mathcal{S}$, a *unique* element from $\mathcal{S}$.
  - A set $\mathcal{S}$ is said to be closed with respect to a *binary operator* if, for every pair of elements of $\mathcal{S}$, the binary operator specifies a rule for obtaining a unique element of $\mathcal{S}$.
  - A set $\mathcal{S}$ is said to have an identity element with respect to a binary operation $*$ on $\mathcal{S}$ if there exists an element $e \in \mathcal{S}$ with the property that $e * x = x * e = x$ for every $x \in \mathcal{S}$.
  - The complement employed in Boolean algebra and the inverse employed in ordinary algebra are *not* the same.

Two-Valued Boolean Algebra

- A two-valued Boolean algebra (henceforth referred to as Boolean algebra for brevity) is defined on a set of two elements, $\mathcal{B} = \{0, 1\}$, with rules for the two binary operators $+$ and $\cdot$ as shown in the following operator tables:

x	y	$x \cdot y$
0	0	0
0	1	0
1	0	0
1	1	1

x	y	$x + y$
0	0	0
0	1	1
1	0	1
1	1	1

- The rule for the complement operator is defined for verification of the postulate 5 as:

x	$\bar{x}$
0	1
1	0

Basic Theorems and Properties of Boolean Algebra

- Duality principle: every algebraic expression deducible from the postulates of Boolean algebra remains valid if the *binary operators* and the *identity elements* are interchanged.
- In order to obtain the dual of an algebraic expression, we need to simply interchange $\cdot$ and $+$ binary operators and replace 1's by 0's and 0's by 1's.
- Four postulates and six basic theorems:

Table 2.1
Postulates and Theorems of Boolean Algebra

Postulate 2	(a)	$x + 0 = x$	(b)	$x \cdot 1 = x$
Postulate 5	(a)	$x + x' = 1$	(b)	$x \cdot x' = 0$
Theorem 1	(a)	$x + x = x$	(b)	$x \cdot x = x$
Theorem 2	(a)	$x + 1 = 1$	(b)	$x \cdot 0 = 0$
Theorem 3, involution		$(x')' = x$		
Postulate 3, commutative	(a)	$x + y = y + x$	(b)	$xy = yx$
Theorem 4, associative	(a)	$x + (y + z) = (x + y) + z$	(b)	$x(yz) = (xy)z$
Postulate 4, distributive	(a)	$x(y + z) = xy + xz$	(b)	$x + yz = (x + y)(x + z)$
Theorem 5, DeMorgan	(a)	$(x + y)' = x'y'$	(b)	$(xy)' = x' + y'$
Theorem 6, absorption	(a)	$x + xy = x$	(b)	$x(x + y) = x$

Basic Theorems and Properties of Boolean Algebra *cont'd*

- Similar to postulates, a theorem has a dual.
- The postulates are the basic axioms of Boolean algebra and need no proof. The theorems must be proven from the postulates.
- Examples:
  - Prove the theorem 1(a) and 1(b).
  - Prove the theorem 2(a) and 2(b).
- The theorems involving two or three variables may be proven algebraically from the postulates and the theorems that have already been proven.
- The theorems of Boolean algebra can be proven by means of *truth tables*.
- Examples:
  - Prove the theorem 5(a) (DeMorgan's theorem) using truth tables.
  - Prove the theorem 4(b) (associative law) using truth tables.

Boolean Functions

- A Boolean function
  - expresses the logical relationship between binary variables
  - is evaluated by determining the binary value of the expression for all possible values of the variables.
- A Boolean function can be represented in a truth table.
- A Boolean function can be transformed from an algebraic expression into a circuit diagram composed of logic gates connected in a particular structure.
- Examples:
  - Draw the logic-circuit diagram for the Boolean function $F = x + \bar{y}z$.
  - Draw the logic-circuit diagram for the Boolean function $F = x\bar{y}z + x\bar{z}$.
- By manipulating a Boolean expression according to the rules of Boolean algebra, it is sometimes possible to obtain a simpler expression for the same function, which reduces
  - the number of gates in the circuit (because of the reduced number of terms)
  - the number of inputs to the gates (because of the reduced number of literals).

Boolean Functions *cont'd*

- Examples:
  - Simplify the Boolean functions F_1 and F_2 to a minimum number of literals;
    (a) $F_1(x, y, z) = \bar{x}\bar{y}z + \bar{x}yz + x\bar{y}$ (b) $F_2(x, y, z) = xy + \bar{x}z + yz$.
- Karnaugh maps can be employed to manually simplify Boolean functions of up to *five* variables.
- Computer minimization programs that are capable of producing optimal circuits with millions of logic gates are used to simplify complex Boolean functions.
- The complement of a function F, denoted as $\bar{F}$, is obtained from an interchange of 0's for 1's and 1's for 0's in the value of F.
- The complement of a function may be algebraically derived through DeMorgan's theorems, i.e.,

$$\overline{x_1 + x_2 + \cdots + x_n} = \bar{x}_1\bar{x}_2\cdots\bar{x}_n$$

$$\overline{x_1x_2\cdots x_n} = \bar{x}_1 + \bar{x}_2 + \cdots + \bar{x}_n.$$

Boolean Functions *cont'd*

- Examples:
  - Find the complements of the Boolean functions F_1 and F_2;
    (a) $F_1(x, y, z) = \bar{x}y\bar{z} + \bar{x}\bar{y}z$ (b) $F_2(x, y, z) = x(\bar{y}\bar{z} + yz)$.
- Note that the complement of a function can be derived by taking the dual of the function and complementing each literal.

Canonical and Standard Terms

- A minterm (or a standard product) having n literals is obtained from an AND term of n binary variables.
- The 2^n minterms can be obtained from the binary numbers between 0 and $2^n - 1$, with each binary variable being complemented if the corresponding bit of the binary number is 0 and uncomplemented if 1.
- A maxterm (or a standard sum) having n literals is obtained from an OR term of n binary variables.
- The 2^n maxterms can be obtained from the binary numbers between 0 and $2^n - 1$, with each binary variable being complemented if the corresponding bit of the binary number is 1 and uncomplemented if 0.
- Example:
  - Obtain the minterms and the maxterms for three binary variables.

Canonical and Standard Terms *cont'd*

Table 2.3
Minterms and Maxterms for Three Binary Variables

			Minterms		Maxterms	
x	**y**	**z**	**Term**	**Designation**	**Term**	**Designation**
0	0	0	$x'y'z'$	m_0	$x + y + z$	M_0
0	0	1	$x'y'z$	m_1	$x + y + z'$	M_1
0	1	0	$x'yz'$	m_2	$x + y' + z$	M_2
0	1	1	$x'yz$	m_3	$x + y' + z'$	M_3
1	0	0	$xy'z'$	m_4	$x' + y + z$	M_4
1	0	1	$xy'z$	m_5	$x' + y + z'$	M_5
1	1	0	xyz'	m_6	$x' + y' + z$	M_6
1	1	1	xyz	m_7	$x' + y' + z'$	M_7

- A Boolean function can be expressed algebraically from a given truth table by forming a minterm (maxterm) for each combination of the variables that produces a 1 (0) in the function and then taking the OR (AND) of all those terms.
- Boolean functions expressed as a sum of minterms or product of maxterms are said to be in canonical form.

Canonical and Standard Terms *cont'd*

- Notation for a Boolean function $F(x, y, z)$ expressed in sum-of-minterms form:

$$F(x,y,z) = \underbrace{\Sigma}_{OR} \underbrace{(1,4,5,6,7)}_{\text{indices of minterms}}$$

- Notation for a Boolean function $F(x, y, z)$ expressed in product-of-maxterms form:

$$F(x,y,z) = \underbrace{\Pi}_{AND} \underbrace{(0,2,4,5)}_{\text{indices of maxterms}}$$

- The conversion from one canonical form to another can be done by interchanging the symbols Σ and Π and listing those numbers missing from the original form.

Canonical and Standard Terms *cont'd*

- The two canonical forms very rarely express a Boolean function with the least number of literals.
- Another way to express Boolean functions is in standard form, where a term may contain one, two, or any number of literals.
- There are two types of standard forms: sum of products and products of sums.
- The logic diagram of a sum-of-products expression consists of a group of AND gates followed by a single OR gate.
- The logic diagram of a products-of-sums expression consists of a group of OR gates followed by a single AND gate.