Lab Session 5

MA-423: Matrix Computations July-November 2025 S. Bora

Important instructions:

(i) **Switch to `format long e` for all experiments.**

(ii) **Submit a <u>single livescript program</u> that contains all comments, answers and codes necessary to produce the required outputs. Ensure that the answers are correctly numbered and the file does not include any irrelevant material. The livescript program should be saved as <u>MA423YourrollnumberLab5.mlx</u>**

1. The purpose of this exercise is to solve the Least-Squares Problem (in short, LSP) $Ax = b$ by different methods and compare the solutions. Here $A \in \mathbb{R}^{n \times m}$ and $b \in \mathbb{R}^n$, and usually n is much bigger than m.

   Origin: Suppose that we have a data set (t_i, b_i), for $i = 1 : m$, that have been obtained from some experiment. These data are governed by some unknown laws. So, the task is to come up with a model that best fits these data. A model is generated by a few functions, called model functions, $\phi_1, \ldots, \phi_n$. Therefore once a model is chosen, the task is to find a function p from the span of the model functions that best fits the data.

   Suppose that the model functions $\phi_1, \ldots, \phi_n$ are given. For $p \in \text{span}(\phi_1, \ldots, \phi_n)$, we have $p = x_1\phi_1 + \cdots + x_n\phi_n$ for some $x_j \in \mathbb{R}$. Now, forcing p to pass through the data (t_i, b_i) for $i = 1 : m$, we have $p(t_i) = b_i + r_i$, where r_i is the error. We want to choose that p for which the sum of the squares of the errors r_i is the smallest, that is, $\sum_{i=1}^m |r_i|^2$ is minimized.

   Now $p(t_i) = b_i + r_i$ gives $x_1\phi_1(t_i) + \cdots + x_n\phi_n(t_i) = b_i + r_i$. Thus in matrix notation,

$$\begin{bmatrix} \phi_1(t_1) & \cdots & \phi_n(t_1) \\ \phi_1(t_2) & \cdots & \phi_n(t_2) \\ \vdots & \cdots & \vdots \\ \phi_1(t_m) & \cdots & \phi_n(t_m) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix}.$$

   This is of the form $Ax = b + r$ and we have to choose $x \in \mathbb{R}^n$ for which the 2-norm of the residual vector $\|r\|_2$, is minimized. We write this as LSP $Ax = b$.

   Your task is to find the polynomial of degree 17 that best fits the function $f(t) = \sin(\frac{\pi}{5}t) + \frac{t}{5}$ for $t_1 = -5, t_2 = -4.5, \ldots, t_{23} = 6$.

   Set up the LSP $Ax = b$ by using the standard basis $\{1, x, \ldots, x^{17}\}$ of the space of polynoials of degree at most 17 as the model functions and determine the best fitting polynomial p whose coefficients are determined by x in three different ways:

   (a) By using the Matlab command
   `>> A \ b`
   This uses QR factorization to solve the LSP $Ax = b$. Call this polynomial p_1.

(b) By setting up the normal equation $A^T Ax = A^T b$ and solving them for x. You will need to use the Cholesky method for this. Call this polynomial p_2.

(c) By solving the argumented system $\begin{bmatrix} I_n & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} -r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$. Call this polynomial p_3.

2. Set the formatting to `format long e` and compute the condition number of the coefficient matrix associated with each of the systems that you are solving. Which one is the most ill conditioned?

   Compute $\|r\|_2/\|b\|_2$ which gives the value of $\sqrt{\Sigma_{i=1}^{23}|p_j(t_i) - f(t_i)|^2}/\sqrt{\Sigma_{i=1}^{23}|f(t_i)|^2}$, $j = 1, 2, 3$ for each of these methods (again in `format long e`). This is a measure of the goodness of the fit in each case. Which of the methods provide the best fit?

3. Repeat the above steps by using the basis $\{1, \frac{x-2}{6}, \ldots, \left(\frac{x-2}{6}\right)^{17}\}$.

   What is the effect of the change of basis on `cond(A)`, cond$(A' * A)$ and the condition number of the coefficient matrix of the argumented system? Does this have any effect on $\|r\|_2/\|b\|_2$?