Lab Session 6

MA-423: Matrix Computations — July-November 2025 — S. Bora

Important instructions:

(i) **Switch to `format long e` for all experiments.**

(ii) **Submit a <u>single livescript program</u> that contains all comments, answers and codes necessary to produce the required outputs. Ensure that the answers are correctly numbered and the file does not include any irrelevant material. The livescript program should be saved as <u>MA423YourrollnumberLab6.mlx</u>**

1. Write a MATLAB function program $[\mathtt{Q},\mathtt{R}] = \mathtt{cgs}(\mathtt{V})$ to orthonormalize the columns of an $n \times m$ matrix V, $(n \geq m)$ by the Classical Gram Schmidt procedure so that Q is an isometry satisfying

$$\begin{aligned} \operatorname{span}\{Q(:,1)\} &= \operatorname{span}\{V(:,1)\} \\ \operatorname{span}\{Q(:,1), Q(:,2)\} &= \operatorname{span}\{V(:,1), V(:,2)\}, \\ &\vdots \\ \operatorname{span}\{Q(:,1), Q(:,2), \ldots, Q(:,m)\} &= \operatorname{span}\{V(:,1), V(:,2), \ldots, V(:,m)\} \end{aligned}$$

   and R is an upper triangular matrix such that $R(i,j) = \langle V(:,j), Q(:,i)\rangle$.

2. A slight modification of the above program leads to the Modified Gram Schmidt procedure for orthonormalizing the columns of V. Perform this modification to obtain another function program $[\mathtt{Q},\mathtt{R}] = \mathtt{mgs}(\mathtt{V})$.

3. Write a function program $[\mathtt{Q},\mathtt{R}] = \mathtt{mgsrep}(\mathtt{V})$ that performs Modified Gram Schmidt with reorthogonalization. Follow the code on page 233 of *Fundamentals of Matrix Computations, 2nd Edition* for efficient execution.

   Take care to replace *for loops* by matrix-vector multiplications as far as possible in each of the above programs.

4. Given a vector $x \in \mathbb{R}^n$, write a MATLAB function program $[\mathtt{u}, \gamma, \tau] = \mathtt{reflect}(\mathtt{x})$ to compute $u \in \mathbb{R}^n$, and $\gamma \in \mathbb{R}$ so that $Qx = [-\tau, 0, \ldots, 0]^T$ where $\tau = \pm\|x\|_2$. Ensure that you choose the sign of τ as so as to avoid catastrophic cancellation.

5. Write another function program $\mathtt{B} = \mathtt{applreflect}(\mathtt{u}, \mathtt{gamma}, \mathtt{A})$ to *efficiently* perform the multiplication QA (as explained in the lecture slides) where $Q = I - \gamma uu^T$.

6. Use the programs written above to write another function program $[\mathtt{Q},\mathtt{R}] = \mathtt{reflectqr}(\mathtt{A})$ that computes the condensed QR decomposition of $A \in \mathbb{R}^{n\times m}, n \geq m$, via reflectors.

   The program should have all the features explained in class. The zeros created at each step are to be overwritten by the vectors u (apart from the leading 1 entry) required to construct the reflector used at that stage and the values of γ corresponding to each reflector are to be stored as a separate vector. Further, the Q should be assembled column-by-column as explained in the lectures.

7. Test your output $[\texttt{Q}, \texttt{R}] = \texttt{reflectqr}(\texttt{A})$ for various different randomly generated matrices A by running $[\texttt{Qhat}, \texttt{Rhat}] = \texttt{qr}(\texttt{A}, 0)$ and checking if $\texttt{norm}(\texttt{Q}*\texttt{R}-\texttt{A}), \texttt{norm}(\texttt{Q}'*\texttt{Q}-\texttt{eye}(\texttt{m}))$, $\texttt{norm}(\texttt{tril}(\texttt{R}, -1))$, $\texttt{norm}(\texttt{R} - \texttt{Rhat})$, and $\texttt{norm}(\texttt{Q} - \texttt{Qhat})$ are all $\approx \texttt{u}$.
   (If you have chosen the sign of τ correctly in the `reflect.m` program, then $\texttt{norm}(\texttt{R}-\texttt{Rhat})$ and $\texttt{norm}(\texttt{Q} - \texttt{Qhat})$ will be $\approx u$.)