LAB3: INTERPOLATION

1. Chebyshev polynomials

Chebyshev polynomials have many important applications in (numerical) analysis. The Chebyshev polynomial T_n of degree n on $[-1,1]$ is defined by

$$T_n(x) = \cos(n\cos^{-1}(x)), \quad x \in [-1,1],\ n = 0,1,2,\dots.$$ [1]

At first glance it is not straightforward that T_n is a polynomial. To see this, observe, that we have $T_0(x) = 1$ and $T_1(x) = x$. Using the trigonometric identity[2]

$$\cos[(n+1)y] + \cos[(n-1)y] = 2\cos(y)\cos(ny)$$

and then the change of variables $y = \cos^{-1}(x)$ we obtain

$$T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x), \quad n = 1,2,\dots$$

and thus

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n = 1,2,\dots \tag{1}$$

As T_0 and T_1 are polynomials of degree 0 and 1, respectively, it follows by induction and (1) that T_n is a polynomial of degree n with leading term $2^{n-1}x^n$. It can also be observed that T_n maps $[-1,1]$ to itself and that T_n is even for n even and odd for n odd. It is also straightforward to see that for $n \geq 1$ the zeros of T_n are given by

$$x_j = \cos\left[\frac{(2j-1)\pi}{2n}\right], \quad j = 1,\dots,n.$$

We summarize some important features of Chebyshev polynomials:

(a) The polynomial p_n of degree n defined by

$$p_n(x) = x^{n+1} - 2^{-n}T_{n+1}(x), \quad x \in [-1,1],$$

is the minmax approximation of degree n to the function $f(x) = x^{n+1}$ on $[-1,1]$.

(b) Let $n \geq 0$. Among all the polynomials of degree $n+1$ with leading coefficient 1 the polynomials $\pm 2^{-n}T_{n+1}$ have the smallest ∞-norm on $[-1,1]$.

(c) If p_n denotes the Lagrange interpolation polynomial of degree n of a function $f \in C^1[-1,1]$ with interpolation points given by the zeros of T_{n+1}, then

$$\|p_n - f\|_\infty \to 0 \text{ as } n \to \infty.$$

That is, if one chooses the interpolation points cleverly for Lagrange interpolation, then Runge's phenomenon does not occur. This is of course true for any interval $[a,b]$, where we must apply the linear mapping $t \to \frac{1}{2}(b-a)t + \frac{1}{2}(b+a)$ to the zeros of T_{n+1}.

[1] Some students would be more familiar with the notation $\arccos(x)$ instead of $\cos^{-1}(x)$.
[2] Prove it!

Exercise 1. In this exercise we investigate Runge's phenomenon.

(a) For the function $f(x) = \frac{1}{1+x^2}$ on $[-5, 5]$, consider the Lagrange interpolation polynomial p_n of degree n with $n+1$ equidistant interpolation points (always choose the first and the last interpolation points to be the endpoints of $[-5, 5]$). Calculate the values of $\|f - p_n\|_\infty$ for $n = 2, 4, 6, \ldots, 24$.

(b) For the function $f(x) = \frac{1}{1+x^2}$ on $[-5, 5]$, consider the Lagrange interpolation polynomial p_n of degree n but with interpolation points given by the zeros of the appropriate Chebyshev polynomial. Calculate the values of $\|f - p_n\|_\infty$ for $n = 2, 4, 6, \ldots, 24$. Compare the results with the equidistant case. What do you observe?

2. Spline interpolation

Unlike Lagrange and Hermite interpolation, spline interpolation is local in nature. Given an interval $[a, b]$, a continuous function $f : [a, b] \to \mathbb{R}$ and a set of knots $K = \{a = x_0 < x_1, \cdots < x_m = b\}$ we look for a function s that interpolates f at the knots, where p is a polynomial (of typically low degree) on each interval $[x_{i-1}, x_i]$ and has a certain number of continuous derivatives. Thus, in general, the polynomial s is not a global polynomial on $[a, b]$.

2.1. Linear interpolating splines.

Given an interval $[a, b]$, a continuous function $f : [a, b] \to \mathbb{R}$ and set of knots $K = \{a = x_0 < x_1, \cdots < x_m = b\}$ the *linear interpolating spline* s_L of f at the knots x_i, $i = 0, 1, \ldots, m$ is given by

$$s_L(x) = \frac{x_i - x}{x_i - x_{i-1}} f(x_{i-1}) + \frac{x - x_{i-1}}{x_i - x_{i-1}} f(x_1), \quad x \in [x_{i-1}, x_i],\ i = 1, \ldots, m.$$

A linear spline can be represented as

$$s_L(x) = \sum_{k=0}^{m} \phi_k(x) f(x_k),$$

where ϕ_k, $k = 0, 1, \ldots, m$ are basis functions. The function ϕ_k is defined as the unique continuous function, that is linear (that is, affine) on each interval $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, m$, and $\phi_k(x_i) = \delta_{ik}$[3], $i, k = 0, 1, \ldots, m$.

2.2. Cubic interpolating splines.

Given an interval $[a, b]$, a continuous function $f : [a, b] \to \mathbb{R}$ and set of knots $K = \{a = x_0 < x_1, \cdots < x_m = b\}$ we consider the set $\mathcal{S}$ of all functions $s : [a, b] \to \mathbb{R}$ such that

- $s \in C^2[a, b]$,
- $s(x_i) = f(x_i)$, $i = 0, 1, \ldots, m$,
- s is a cubic polynomial on each interval $[x_{i-1}, x_i]$, $i = 1, 2, .., m$.

An element of $\mathcal{S}$ is called an *interpolating cubic spline* for f. Clearly there are more that one interpolating cubic spline for f, but there are many special ones conditions ensuring uniqueness. The interpolating cubic spline $s_2 \in \mathcal{S}$ for f which, in addition, satisfies that $s_2''(x_0) = s_2''(x_m) = 0$ is called the *natural cubic spline*. There is no explicit formula for s_2, however, one may write a system of linear equations for the

[3] The symbol δ_{ik} denotes the Kronecker delta.

coefficients of s_2 which has exactly one solution. Let $h_i = x_i - x_{i-1}$ for $i = 1, 2, \dots, m$. The spline s_2 has the form

$$s_2(x) = \frac{(x_i - x)^3}{6h_i}\sigma_{i-1} + \frac{(x - x_{i-1})^3}{6h_i}\sigma_i + \alpha_i(x - x_{i-1}) + \beta_i(x_i - x),$$
$$x \in [x_{i-1}, x_i],\ i = 1, 2, \dots, m.$$

Here

$$\alpha_i = \frac{1}{h_i} f(x_i) - \frac{1}{6}\sigma_i h_i,\ i = 1, 2, \dots, m$$
$$\beta_i = \frac{1}{h_i} f(x_{i-1}) - \frac{1}{6}\sigma_{i-1} h_i,\ i = 1, 2, \dots, m,$$

while σ_i it the solution of the linear system of equations

$$h_i\sigma_{i-1} + 2(h_{i+1} + h_i)\sigma_i + h_{i+1}\sigma_{i+1} = 6\left(\frac{f(x_{i+1}) - f(x_i)}{h_{i+1}} - \frac{f(x_i) - f(x_{i-1})}{h_i}\right)$$
$$i = 1, 2, \dots, m-1, \quad (2)$$

together with $\sigma_0 = \sigma_m = 0$.

2.3. **Hermite cubic splines.** Given an interval $[a, b]$, a function $f \in C^1[a, b]$ and set of knots $K = \{a = x_0 < x_1, \dots < x_m = b\}$ the function $s \in C^1[a, b]$ which is the Hermite interpolation polynomial of degree 3 of f on each interval $[x_{i-1}, x_i]$ is called the *Hermite cubic spline*.

Exercise 2. Consider the function

$$f(x, t) = \sin(5\pi x)\cos(10\pi t) + 2\sin(7\pi x)\cos(14\pi t)$$

for $(x, t) \in [0, 1] \times [0, 1]$ and the set of equidistant knots $x_i = i/50$, $i = 0, 1, \dots, 50$, and plot the (a) linear, (b) natural cubic, and (c) Hermite cubic spline interpolation polynomials for the function $x \to f(x, t)$ for each $t = t_j = j/50$, $j = 0, 1, \dots, 50$, as an animation in t. For determining the natural cubic spline interpolation, solve the linear system (2) by standard Gaussian elimination.