LAB1: SOLUTION OF EQUATIONS BY ITERATION
DUE: 3RD OF OCTOBER, MIDNIGHT

In this lab we will (mostly) solve equations of the form $f(x) = 0$ by iteration, where $f : [a, b] \to \mathbb{R}$ is a continuous function. A value $\xi \in [a, b]$ such that $f(\xi) = 0$ is called a solution to the equation $f(x) = 0$.

For finding a solution of $f(x) = 0$ we considered the following three methods.

- *Newton's method*: this method is given by the recursion
$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, ...,$$
where $x_0 \in [a, b]$ is given.
- *Secant method*: this method is given by the recursion
$$x_{k+1} = x_k - f(x_k)\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}, \quad k = 1, ...,$$
where $x_0, x_1 \in [a, b]$ are given.
- *Bisection method*: let $k \geq 0$ and suppose that $f(a_k)$ and $f(b_k)$ have opposite signs. By the Intermediate Value Theorem there exists $\xi \in (a_k, b_k)$ such that $f(\xi) = 0$, Let $c_k = \frac{a_k+b_k}{2}$. If $f(c_k) = 0$, then we are done. If not, then define
$$(a_{k+1}, b_{k+1}) = \begin{cases} (a_k, c_k) & \text{if } f(c_k)f(b_k) > 0, \\ (c_k, b_k) & \text{if } f(c_k)f(b_k) < 0 \end{cases}$$
and repeat the procedure.

In certain situations it is more advantageous to consider an equation of the form $x = g(x)$ which is equivalent to $f(x) = 0$ with a suitabe g in the sense that that for $\xi \in [a, b]$ we have $f(\xi) = 0$ if and only if $\xi = g(\xi)$, in which case ξ is called a fixed point of the function g on $[a, b]$.

A simple iteration is a recursion of the form
$$x_{k+1} = g(x_k), \quad k = 0, 1, 2, ..., \tag{1}$$
where $x_0 \in [a, b]$ is given. If $g : [a, b] \to \mathbb{R}$ is a contraction whose range is contained in $[a, b]$, then the sequence (x_k) given by the simple iteration converges to the unique fixed point of g.

Exercise 1. Implement the above mentioned methods, namely Newton's method, the secant method and the bisection method. The three methods should be implemented in three different functions which return the result and the number of iterations. The function arguments are the following:

- Newton's method: function, derivative function, staring point (x_0), precision;

- Secant method: function, x_0, x_1, precision;
- Bisection method: function, left staring point, right starting point, precision.

Of course other helper functions and classes can be used. The design of the code is up to you as long as it is reasonable. The derivative function has to be calculated and hard-coded.

Exercise 2. In 1225 Leonardo Pisano (also known as Fibonacci) calculated one of the solutions of

$$f(x) = x^3 + 2x^2 + 10x - 20 = 0, \tag{2}$$

which is $x \approx 1.3688$. We do not know how he obtained this result.

(a) Show that there is a solution of (2) in $[1, 2]$. Show that it is the only real solution of (2) on $\mathbb{R}$.
(b) Solve the equation using Newton's method starting from 1.
(c) Solve the equation using the secant method with $x_0 = 1$ and $x_1 = 2$.
(d) Solve the equation using the bisection method in the interval of $[1, 2]$.

Set the precision to 10^{-4}; that is, iterate until $|f(x_k)| < 10^{-4}$ holds.

Exercise 3. Consider the equation

$$f(x) = \tanh(x) = 0.$$

This equation is not ideal for some of the above methods. Find out why and handle it in some way in the algorithms.

(a) Solve the equation using Newton's method starting from $x_0 = -5$.
(b) Solve the equation using the secant method with $x_0 = -5$ and $x_1 = -4$.
(c) Solve the equation using the bisection method in the interval $[5, 10]$.

Set the precision to 10^{-4}; that is, iterate until $|f(x_k)| < 10^{-4}$ holds.

Exercise 4. Rewrite (2) as $x = g(x)$ with

$$g(x) = \frac{20}{x^2 + 2x + 10}$$

and show that the Contraction Mapping Theorem is applicable on $[1, 2]$. Implement the simple iteration (1) and use the theoretical stopping criterion from class to estimate the solution to 4 decimal places. How pessimistic is the theoretical criterion?