Matlab Work 2 - Convex regularisation and contour aware image restoration

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Exercise 1.

$$\varphi_Q(\delta) = \delta^2$$
$$\varphi'_Q(\delta) = 2\delta$$
$$\varphi''_Q(\delta) = 2$$

$$\varphi_H(\delta) = \begin{cases} \delta^2, & \text{if } |\delta| \leq T \\ 2T|\delta| - T^2, & \text{otherwise} \end{cases}$$

$$\varphi'_H(\delta) = \begin{cases} 2\delta, & \text{if } |\delta| \leq T \\ 2T, & \text{if } \delta \geq T \\ -2T, & \text{if } \delta \leq -T \end{cases}$$

$$\varphi''_H(\delta) = \begin{cases} 2, & \text{if } |\delta| \leq T \\ 0, & \text{otherwise} \end{cases}$$

Exercise 2.

2a.

We want to make smoother the homogeneous areas (δ is small) and preserve the edges (δ is big) at the same time. φ_Q penalizes both cases in a quadratic manner unlike φ_H, which penalizes bigger δ (above a threshold T) linearly. With such a penalization edges will have fewer weights and the blurring effect will be applied more to the homogeneous areas.

2b.

T is a threshold of δ, where the linear part of φ_H begins. The smaller the T, the sharper edges are preserved, but the homogeneous parts will not be blurred so much.

Exercise 4.

$$J_H(\mathbf{x}, \mathbf{a}) = ||\mathbf{y} - H\mathbf{x}||^2 + \mu' \left[\sum_p \sum_q \frac{1}{2}((x_p - x_q) - a_{pq})^2 - \hat{\xi}_\alpha(apq) \right]$$

$$J_H(\mathbf{x}, \mathbf{a}) = ||\mathbf{y} - H\mathbf{x}||^2 + \mu' \left[\sum_p \sum_q \frac{1}{2}((x_p - x_q) - a_{pq})^2 \right] - \mu' \left[\sum_p \sum_q \hat{\xi}_\alpha(a_{pq}) \right]$$

$$J_H(\mathbf{x}, \mathbf{a}) = ||\mathbf{y} - H\mathbf{x}||^2 + \mu' ||D\mathbf{x} - \mathbf{a}||^2 - \mu' \left[\sum_p \sum_q \hat{\xi}_\alpha(a_{pq}) \right]$$

$$\nabla J_H(\hat{\mathbf{x}}) = 0$$
$$\nabla J_H(\hat{\mathbf{x}}) = (2H^tH + \mu' D^tD)\hat{\mathbf{x}} - 2H^t\mathbf{y} - \mu' D^t\mathbf{a}$$
$$(2H^tH + \mu' D^tD)\hat{\mathbf{x}} - 2H^t\mathbf{y} - \mu' D^t\mathbf{a} = 0$$
$$\hat{\mathbf{x}} = (2H^tH + \mu' D^tD)^{-1}(2H^t\mathbf{y} + \mu' D^t\mathbf{a})$$

$$\mathring{\hat{\mathbf{x}}} = (2\Lambda_h^\dagger\Lambda_h) - \mu'\Lambda_d^\dagger\Lambda_d)^{-1}(2\Lambda_h^\dagger\mathring{\mathbf{y}} + \mu'\Lambda_d^\dagger\mathring{\mathbf{a}})$$

If $\mathbf{a} = \mathbf{0}$, we get back the Wiener-Hunt solution.

Exercise 5.

5a.

The set of a_{qp}'s can be updated in an independent and parallel manner since we can compute that without a loop. This expression has no inner iteration, so it is explicit.

Exercise 6.

6a.

See Matlab code for comments.

6b.

In the Huber Potential Function, threshold T is defining the point where the function starts to behave linearly instead of the quadratic part. With a higher T value, we will get back the Wiener-Hunt method, since the whole function will be a quadratic one. Furthermore, with higher T, we smooth more the edges of the images. The parameter μ controls the effect of the Huber penalty.

6c.

In the frequency domain, we can clearly see the blurring effect, ie. the loss of information in the higher frequencies (sharp edges). Figure 4 shows a state where the convergence is not yet reached, hence higher frequencies are absent. In comparison with a converged solution (1), we can see that the edges of the frequency image contain fewer details.

6d.

Constant parameters:

- Threshold = 0.2
- Number of iterations = 3000, early stopping if $d_2(I_k, I_{k+1}) < 10^{-14}$ for two consequent iterations
- $\mu = 0.1$

The theoretical range of α value is $(0.0, 0.5)$, above 0.5 the solution is not guaranteed to converge. We can also note that the value of α does not effect the reconstructed image, only the speed of the convergence. The higher α is, the shorter time it takes to converge. In figure 1, 2, 3, and 4 we show the algorithm output for $\alpha = 0.5, 0.05, 0.005, 10^{-5}$, respectively. We can observe that with higher α (1), the solution is found in around 220 iterations, while with an α close to 0 (4), we can't get a solution in 3000 iterations.

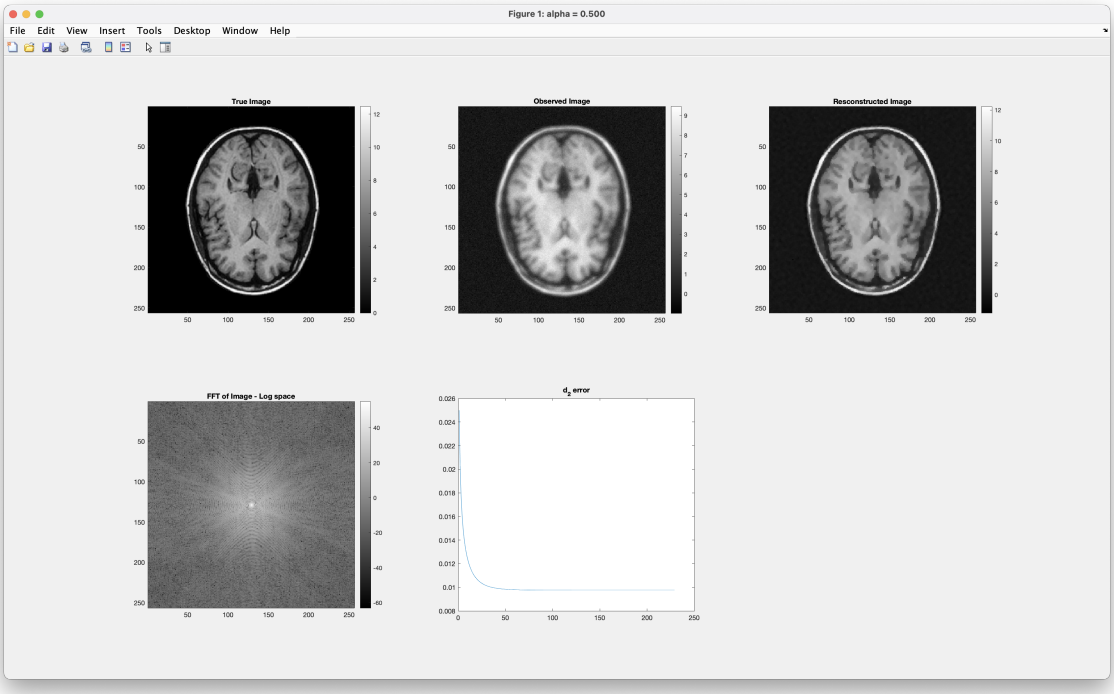

Figure 1: $\alpha = 0.5$

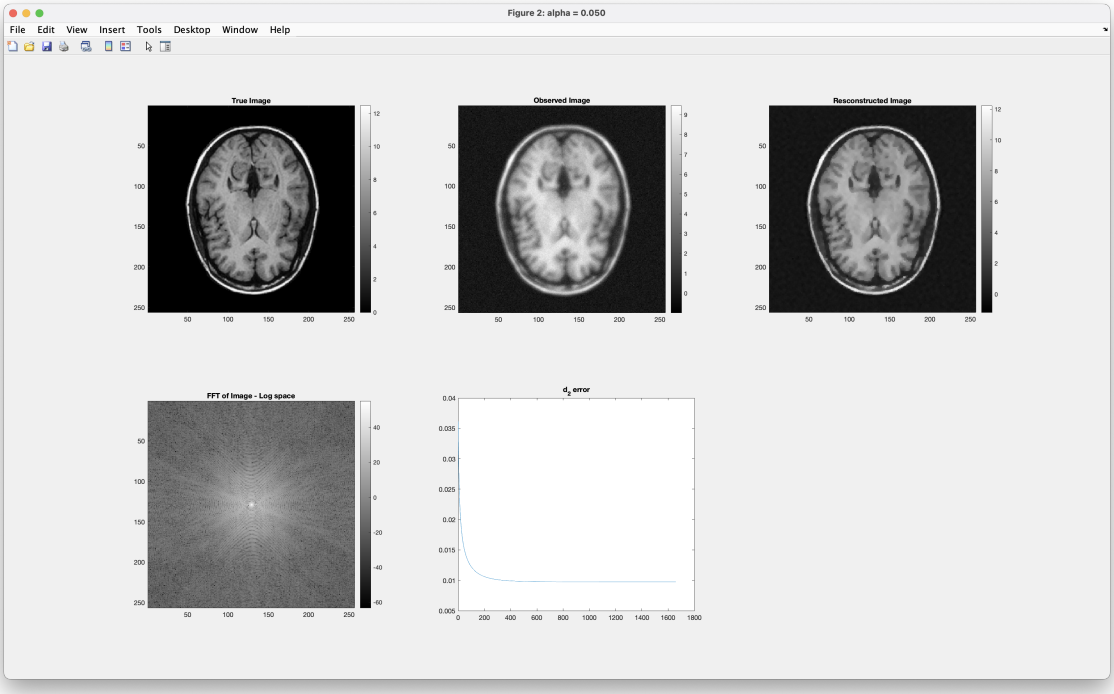

Figure 2: $\alpha = 0.05$

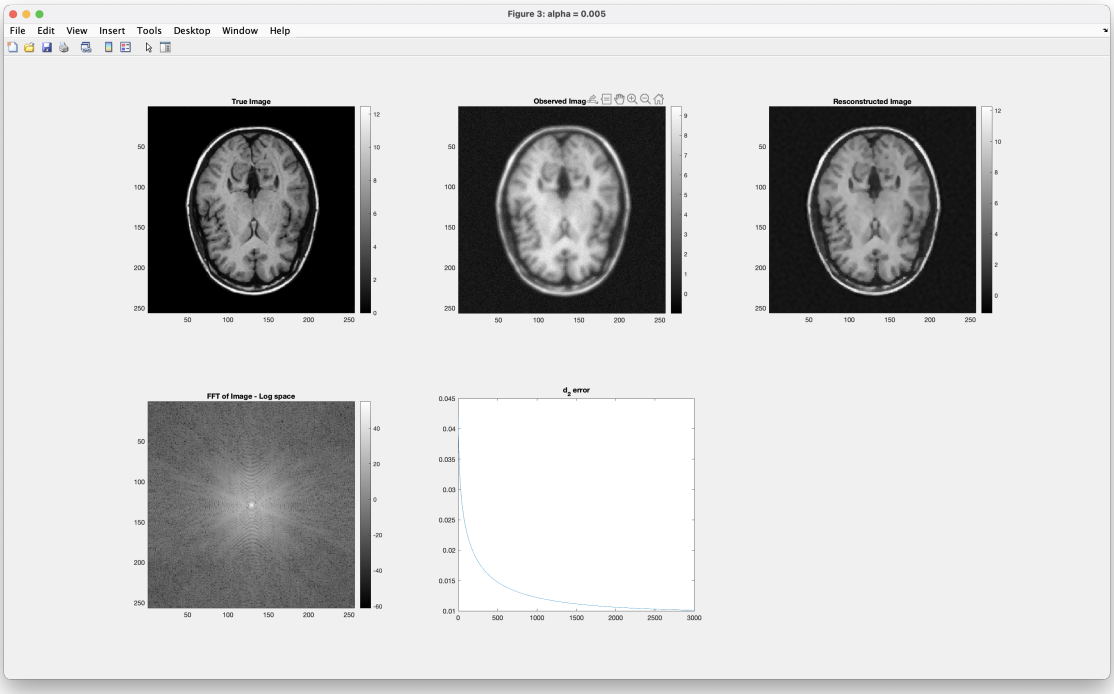

Figure 3: $\alpha = 0.005$

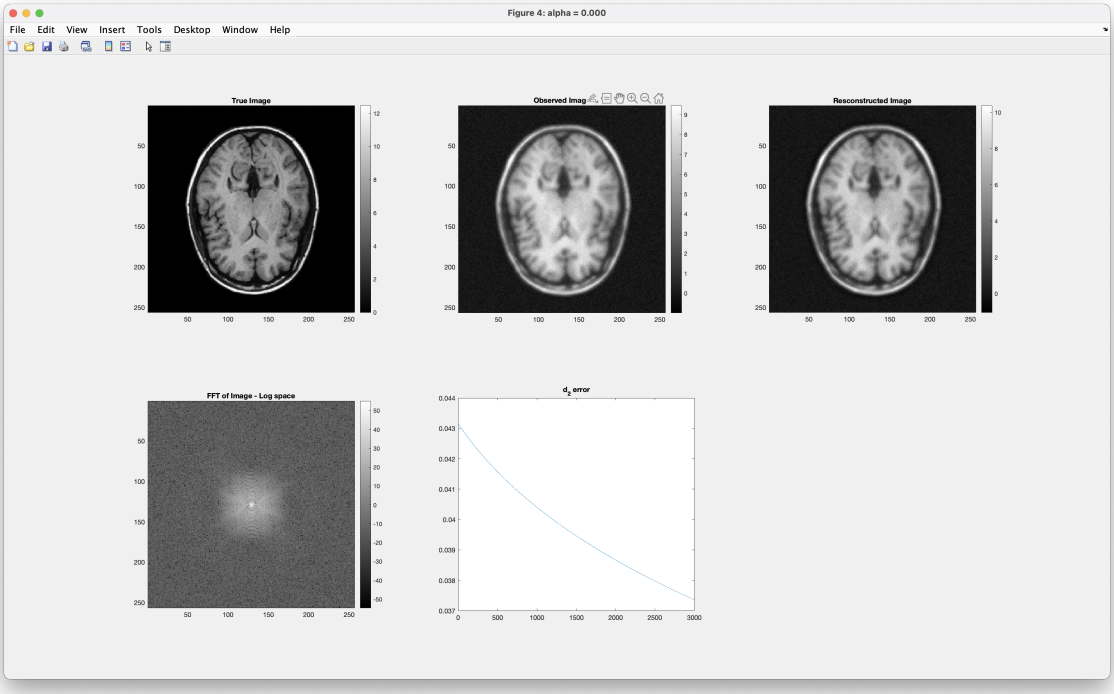

Figure 4: $\alpha = 10^{-5}$