PDE - TP3

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From the existence and uniqueness of the proximal operator we know,

$$prox_h^F(x) = argmin_u \left(\frac{||u-x||^2}{2h} + F(u) \right) \tag{1}$$

1 Exercise 1

Given,

$$F(u) = \frac{1}{2}||u||_2^2 \tag{2}$$

In order to show that,

$$prox_h^F(x) = \frac{x}{h+1}$$

differentiating equation 2, we get

$$F'(u) = u$$

from equation 1, $\implies$

$$\frac{1}{h}(\tilde{u} - x) + F'(\tilde{u}) = 0$$

$\because F'(u) = u$

$$\frac{1}{h}(\tilde{u} - x) + \tilde{u} = 0$$
$$\tilde{u} - x + h\tilde{u} = 0$$
$$(h+1)\tilde{u} = x$$

Hence,

$$prox_h^F(x) = \tilde{u} = \frac{x}{h+1}$$

2 Exercise 2

Given,

$$F(u) = \frac{1}{2}||u-f||_2^2 \tag{3}$$

In order to show that,

$$prox_h^F(x) = \frac{x+hf}{1+h}$$

differentiating equation 3, we get

$$F'(u) = u - f$$

from equation 1, $\implies$

$$\frac{1}{h}(\tilde{u} - x) + F'(\tilde{u}) = 0$$

$\because F'(u) = u - f$

$$\frac{1}{h}(\tilde{u} - x) + (\tilde{u} - f) = 0$$
$$\tilde{u} - x + h\tilde{u} - hf = 0$$
$$prox_h^F(x) = \tilde{u} = \frac{x+hf}{1+h}$$

3 Exercise 3

Given,

$$F(u) = \frac{1}{2}||Ku - f||_2^2 \tag{4}$$

In order to show that,

$$prox_h^F(x) = (I + hK^*K)^{-1}(x + hK^*f)$$

differentiating equation 4, we get

$$F'(u) = K^*(Ku - f)$$

from equation 1, $\implies$

$$\frac{1}{h}(\tilde{u} - x) + F'(\tilde{u}) = 0$$

$\because F'(u) = K^*(Ku - f)$

$$\frac{1}{h}(\tilde{u} - x) + K^*(K\tilde{u} - f) = 0$$

$$\tilde{u} - x + hK^*K\tilde{u} - hK^*f = 0$$

$$(I + hK^*K)\tilde{u} = x + hK^*f$$

$$prox_h^F(x) = \tilde{u} = (I + hK^*K)^{-1}(x + hK^*f)$$

4 Exercise 4

Given,

$$F(u) = ||u||_1 \tag{5}$$

In order to show that,

$$prox_h^F(x) = ST(x, h)$$

with ST(x, h) the Soft-thresholding of x with parameter h, i.e. ST(x, h) = x - h if x > h , ST(x, h) = x + h if x < -h , and ST(x, h) = 0 if $|x| \leq h$. differentiating equation 5, we get

$$F'(u) = \begin{cases} 1 & \text{if } u > 0 \\ -1 & \text{if } u < 0 \\ [-1, 1] & \text{if } u = 0 \end{cases}$$

$$prox_h^F(x) = ST(x, h) \tag{6}$$

$$\frac{1}{h}(\tilde{u} - x) + \begin{cases} 1 & \text{if } \tilde{u} > 0 \\ -1 & \text{if } \tilde{u} < 0 = 0 \\ [-1, 1] & \text{if } \tilde{u} = 0 \end{cases}$$

$$\tilde{u} = x - h \begin{cases} 1 & \text{if } \tilde{u} > 0 \\ -1 & \text{if } \tilde{u} < 0 \\ [-1, 1] & \text{if } \tilde{u} = 0 \end{cases}$$

$$\tilde{u} = \begin{cases} x - h & \text{if } \tilde{u} > 0 \\ x + h & \text{if } \tilde{u} < 0 \\ [x - h, x + h] & \text{if } \tilde{u} = 0 \end{cases}$$

$$prox_h^F(x) = \tilde{u} = \begin{cases} x - h & \text{if } x > h \\ x + h & \text{if } x < -h \\ 0 & \text{if } |x| < h \end{cases}$$

σ	K	λ	h	**PSNR**
10	200	0.1	0.5	32.812021
20	200	0.05	0.5	28.516784
30	200	0.05	0.5	26.129980

Table 1: PSNR of the Forward-Backward algorithm for different σ values

5 Exercise 5

Given,

$$F(u) = ||\nabla u||_1 \tag{7}$$

In order to show that,

$$prox_h^F(x) = x + h \cdot div(z)$$

differentiating equation 7, we get

$$F'(u) = \begin{cases} 1 & \text{if } u > 0 \\ -1 & \text{if } u < 0 \\ [-1, 1] & \text{if } u = 0 \end{cases}$$

From equation 1, $\Longrightarrow$

$$\frac{1}{h}(\tilde{u} - x) + F'(\tilde{u}) = 0$$

$$\frac{1}{h}(\tilde{u} - x) - \nabla^* \nabla(\tilde{u}) = 0$$

$$(\nabla \tilde{u} = z)$$

$$prox_h^F(x) = x + h \cdot div(z)$$

6 Exercise 6

The functions to compute proximal operators can be found in the submitted notebook.

7 Image restoration

The functions to perform the forward-backward algorithm can be found in the submitted notebook. Running the algorithm for an image with different gaussian noise added ($\sigma = 10, 20, 30$), we searched for the best parameters K, h, and λ. Table 1 shows the best values for the different amounts of noise.

After implementing the FISTA algorithm, we can see that we achieve the same results with a significantly lower number of iterations (around 50). Figure 1 shows the function of the reconstructed PSNR value with respect to the number of iterations for different amounts of noise. As of the other parameters, we used the best results from our previous experiments.

8 Image restoration (salt and pepper noise)

Similarly to the previous task, we run a grid search on the parameters h, K, and λ and compared the PSNR results. In this case, the different noise variance σ values were [0.05, 0.1, 0.2]. Table 2 show the optimal parameter values for different σ-s.

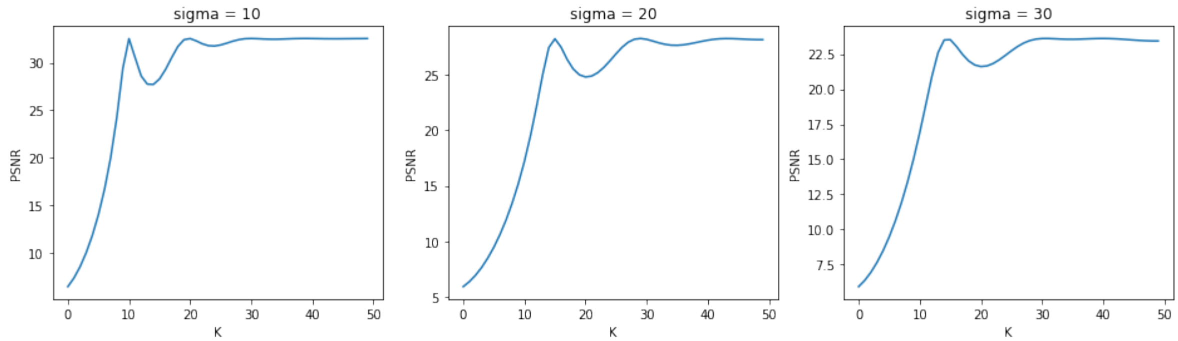

Figure 1: PSNR values in the function of the number of iterations

σ	K	λ	h	**PSNR**
0.05	200	2.0	1.0	29.049547
0.1	200	2.0	1.0	27.517936
0.15	200	2.0	2.0	25.837116

Table 2: Image restoration for different σ values with salt and pepper noise

9 Image deconvolution (gaussian noise)

Our grid search included the following parameters for $\sigma = [10, 20, 30]$:

- $\lambda = [0.5, 1]$
- $h = [0.1, 0.5]$
- $K = [100, 200]$

Table 3 illustrates the best parameters and the corresponding PSNR values for each σ.

After speeding up the algorithm with FISTA we can observe a significantly faster convergence, see Figure 2.

10 Image inpainting

Figure 3 illustrates the output of our inpainting implementation using the forward-backward algorithm with and without FISTA speedup. We can observe that a lower number of iterations (40 instead of 200) is enough to achieve similar results. Since we cannot measure the PSNR, we can only determine the reconstruction quality visually.

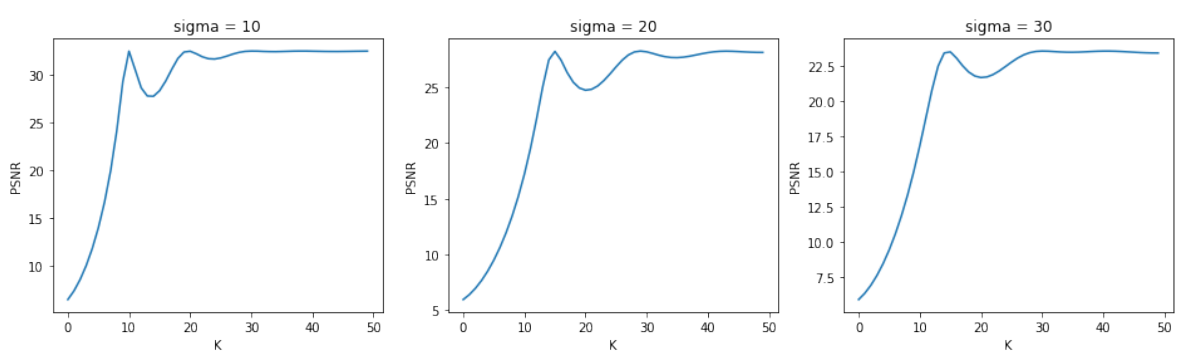

Figure 2: FISTA for deconvolution algorithm

σ	K	λ	h	**PSNR**
10	100	0.1	0.5	32.706888
20	200	0.05	0.5	28.801524
30	200	0.05	0.5	25.164115

Table 3: Image deconvolution for different σ values

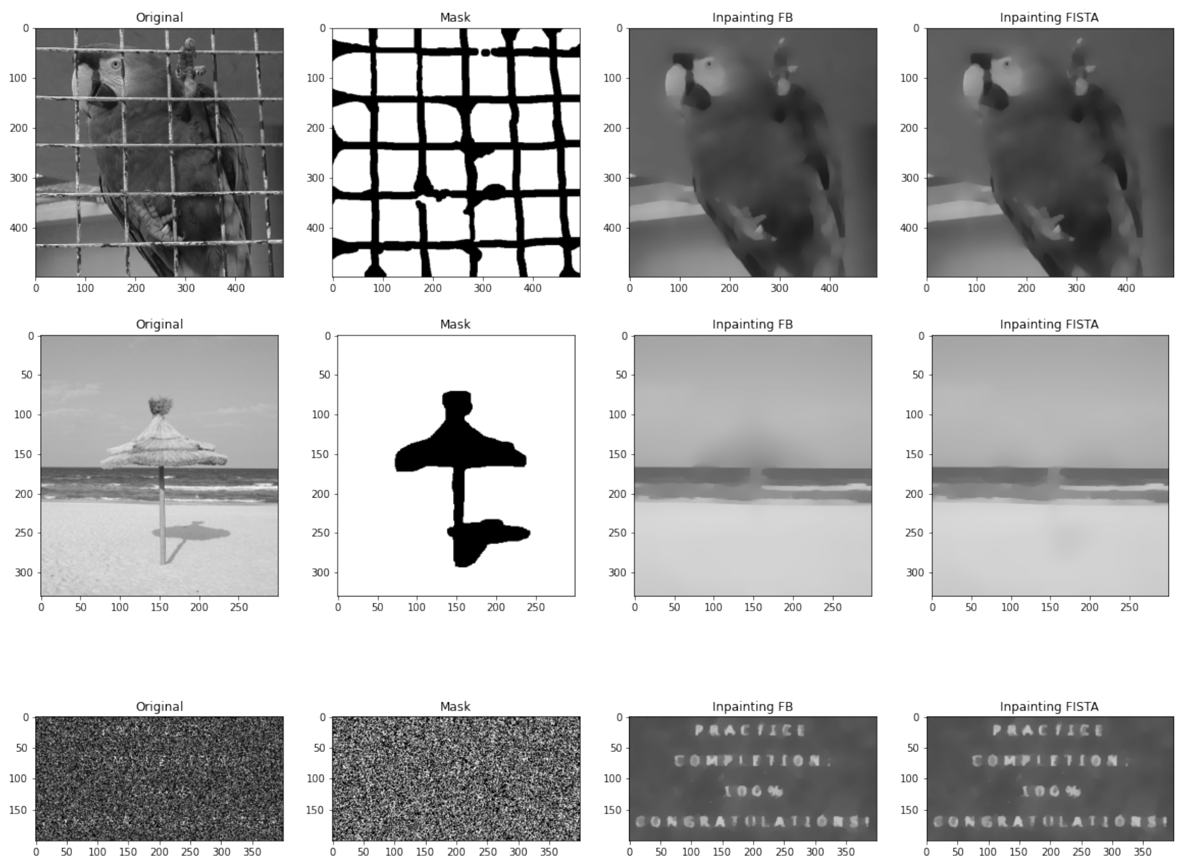

Figure 3: Inpainting