PDE - TP2

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1 Exercise 1

Since we're given,

$$J_H(u) = \frac{1}{2}\sum_{x\epsilon\Omega} \|\nabla u(x)\|^2 = \frac{1}{2}\|\nabla u\|_Y^2 \tag{1}$$

$$J_{PM} = \sum_{x\epsilon\Omega} \sqrt{\|\nabla u(x)\|^2 + 1} \tag{2}$$

- Convexity: From the 1 we can observe that the J_H function is a parabola, Hence, it is convex. The second order derivative of J_{PM} (2) $J''_{PM}(x) = \frac{1}{(x^2+1)^{\frac{3}{2}}}$ is positive. Hence, it is also a convex function.

- Gradient computation: We know that,

$$\Delta^T = (-\nabla^T\nabla) = \Delta \tag{3}$$

  To obtain the

$$\nabla J_H(u)$$

  differentiating the equation 1, w.r.t u we get

$$\nabla\nabla u(x)$$

  from (3) we can say that,

$$\nabla J_H(u) = -div(\nabla J(u)) = -\Delta u\epsilon X$$

  Similarly, to obtain the ∇J_{PM} differentiate equation (2) w.r.t u we get

$$\frac{1}{\sqrt{\|\nabla u(x)\|^2 + 1}}\nabla\nabla u(x)$$

  from equation(3) we can say that, $\nabla J_{PM} = -div(\frac{\nabla u}{\sqrt{\|\nabla u\|^2+1}})$

2 Exercise 2

Given,

$$J_D(u) = \frac{1}{2}\|u - f\|_X^2 \tag{4}$$

- Convexity: From the figure 3 (on the plot, $f = 0$) we can observe that equation 4 is a parabola. Hence, it is convex, furthermore, it is strictly convex.

- Gradient: to compute the gradient of equation 4, differentiate it w.r.t u we get,

$$\nabla J_D(u) = 2 \cdot \frac{1}{2}(u - f) = (u - f)\epsilon X \tag{5}$$

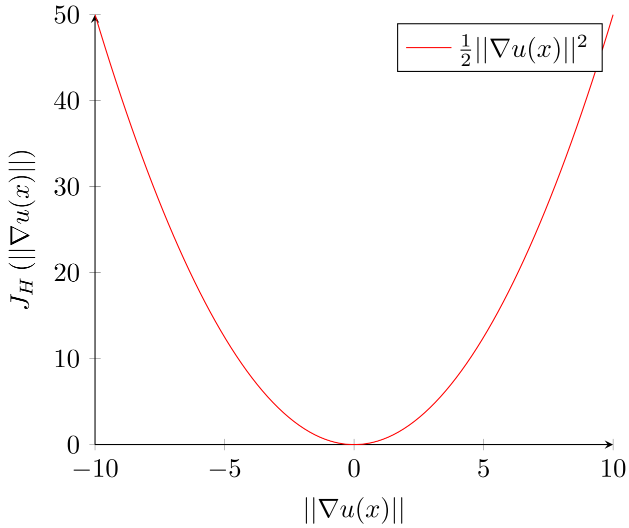


Figure 1: J_H function

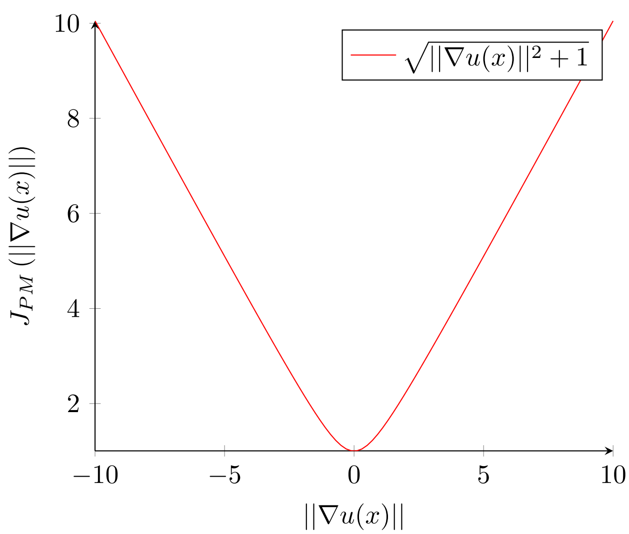


Figure 2: J_{PM} function

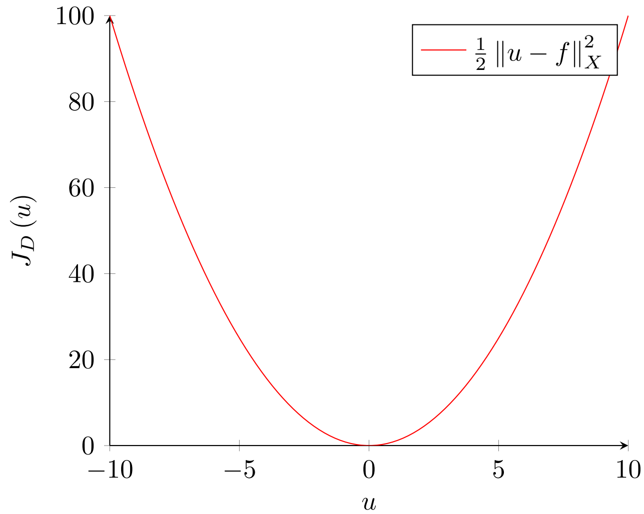
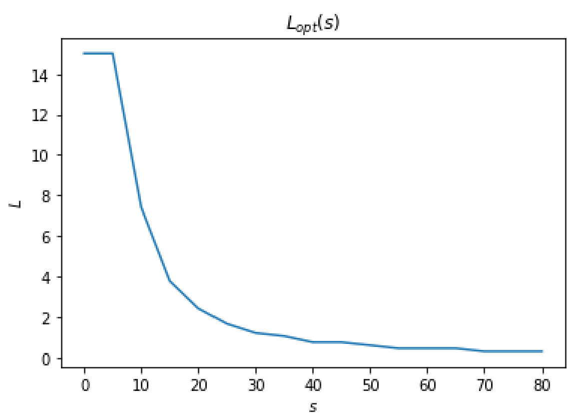


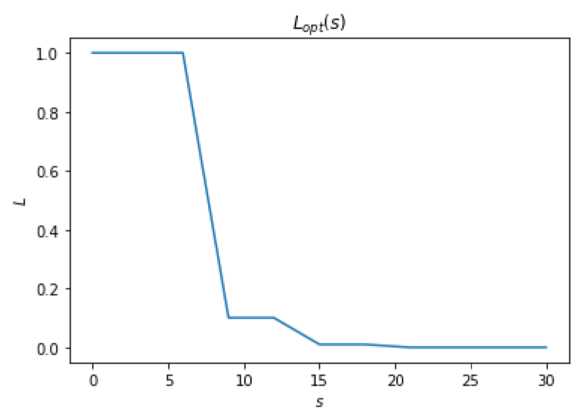
Figure 3: J_D function

3 Exercise 5

Figure 4a and 4b show the evolution of the optimal λ value as the function of the noise variance σ for Denoise_Tikhonov and Denoise_TV functions. We can conclude that the optimal λ value is inversely proportional to the amount of noise, ie. for smaller noise, we need higher λ. This is because, with a higher λ value, we are considering more the data part, but with large noise, we need to do a more intensive smoothing.



(a) Denoise Tikonov function



(b) Denoise TV function

Figure 4: λ_{opt} vs amount of noise in the image

4 Exercise 6

For lower λ values, the Tikhonov method produces better results, but for higher λ the results are approximately the same. Below are the PSNR values for different λ values.

```
l: 0.01
   FFT    16.853099943101434
   Tikh   20.817611838172787
l: 0.1
   FFT    20.713523652909238
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```
   Tikh   21.614436516743755
l: 1
   FFT    24.277167237811636
   Tikh   24.302238109326105
l: 10
   FFT    21.358861918042532
   Tikh   21.011991956097972
```

5 Exercise 7

Given equations,

$$\mathcal{F}(\Delta f)(p,q) = -4F(f)(p,q)(\sin^2(\frac{\pi p}{m}) + \sin^2(\frac{\pi q}{n})) \tag{6}$$

$$\mathcal{F}(u)(p,q) = \frac{\lambda F(f)(p,q)}{\lambda + 4(\sin^2(\frac{\pi p}{m}) + \sin^2(\frac{\pi q}{n}))} \tag{7}$$

We know that,

$$\lambda(u - f) - \Delta u = 0$$
$$\lambda u - \Delta u = \lambda f \tag{8}$$

Replacing u by $\mathcal{F}(u)$ in equation 8 and from equation 6, we get

$$\lambda\mathcal{F}(u)(p,q) + 4\mathcal{F}(u)(p,q)[\sin^2(\frac{\pi p}{m}) + \sin^2(\frac{\pi q}{n})] = \lambda\mathcal{F}(f)(p,q)$$

$$\mathcal{F}(u)(p,q)[\lambda + 4[\sin^2(\frac{\pi p}{m}) + \sin^2(\frac{\pi q}{n})]] = \lambda\mathcal{F}(f)(p,q)$$

$$\mathcal{F}(u)(p,q) = \frac{\lambda F(f)(p,q)}{\lambda + 4[\sin^2(\frac{\pi p}{m}) + \sin^2(\frac{\pi q}{n})]}$$

which is equal to equation 7.

6 Exercise 8

Computation of

$$\nabla J_A(u)$$

We know,

$$J_A(u) = \frac{1}{2}\sum_{x\epsilon\Omega} \|(Au)(x) - f(x)\|^2 = \frac{1}{2}\|Au - f\|_X^2 \tag{9}$$

differentiating the equation 9 w.r.t u we get,

$$\nabla J_A(u) = 2 \cdot \frac{1}{2}A^*(A(u - f)) = A^T(A(u - f))$$

G^T: as we know that the G is an isotropic Gaussian kernel. Hence $G^T = G$. We can compute M^T, but since M is a mask, most of the time we do not consider its transpose, flipping the mask will spoil the recovery process.

7 Exercise 13

- The Tikhonov and TV methods are different in their nature, hence they will have different optimal λ values. The Tikhonov regularization is a quadratic function, and the Total Variation is piece-wise linear. This can also be observed in Figure 4.
- As seen in the previous cases, the optimal λ values for the Deconvolution_TV function as the function of σ also follows a hyperbolical curve, see Figure 5

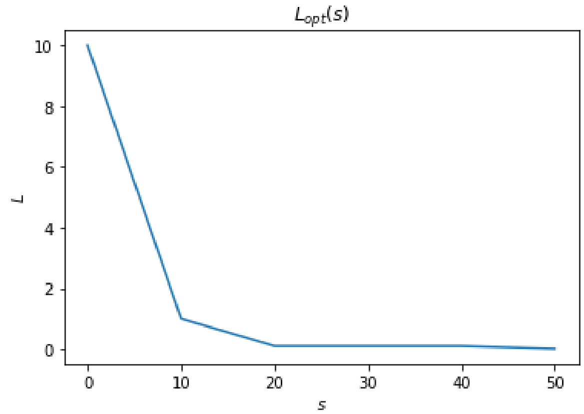

Figure 5: λ_{opt} values for the Deconvolution_TV function

- We can also note that the algorithm can be stopped if a convergence criterion is met, even before reaching, K, the maximum number of iterations. We can measure the normalized root-mean-square error (RMSE) between successive iterations to check convergence.
- We can see the inpainting results for Tikhonov and TV methods in Figure 6 and 7, respectively. Both solutions work better with a larger value of λ (0.01), and a large number of iterations was required for the solution to converge.

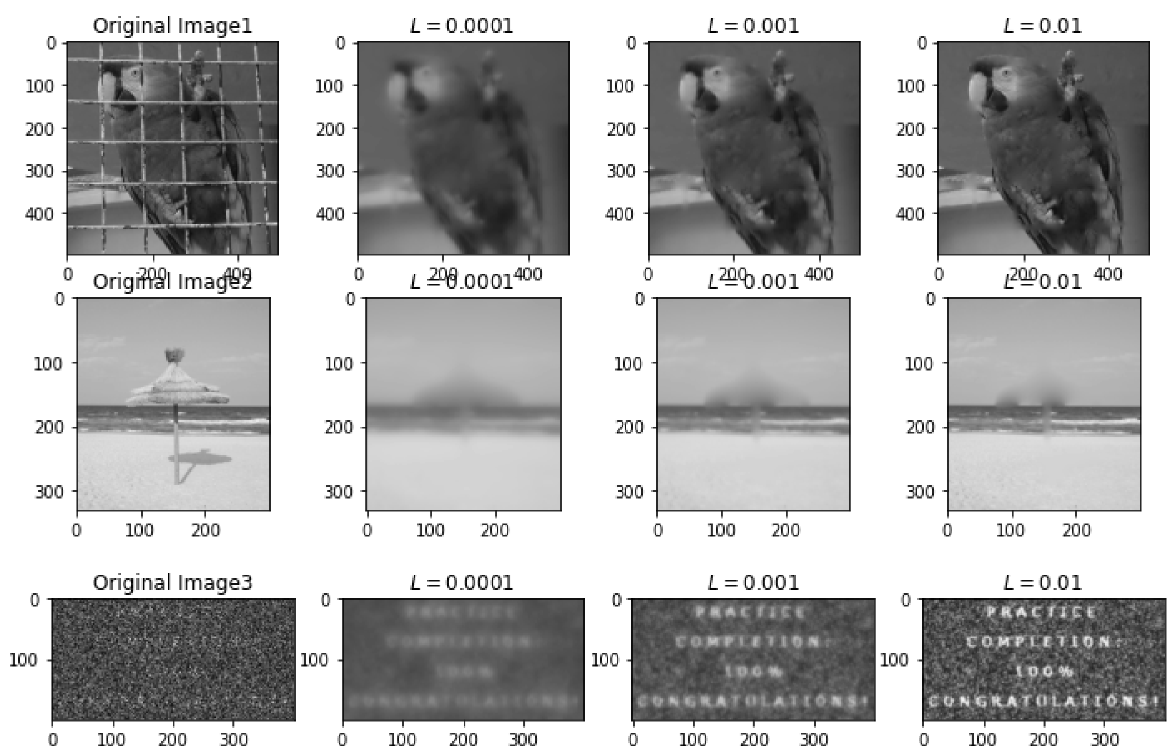

Figure 6: In painting using Tikhonov method

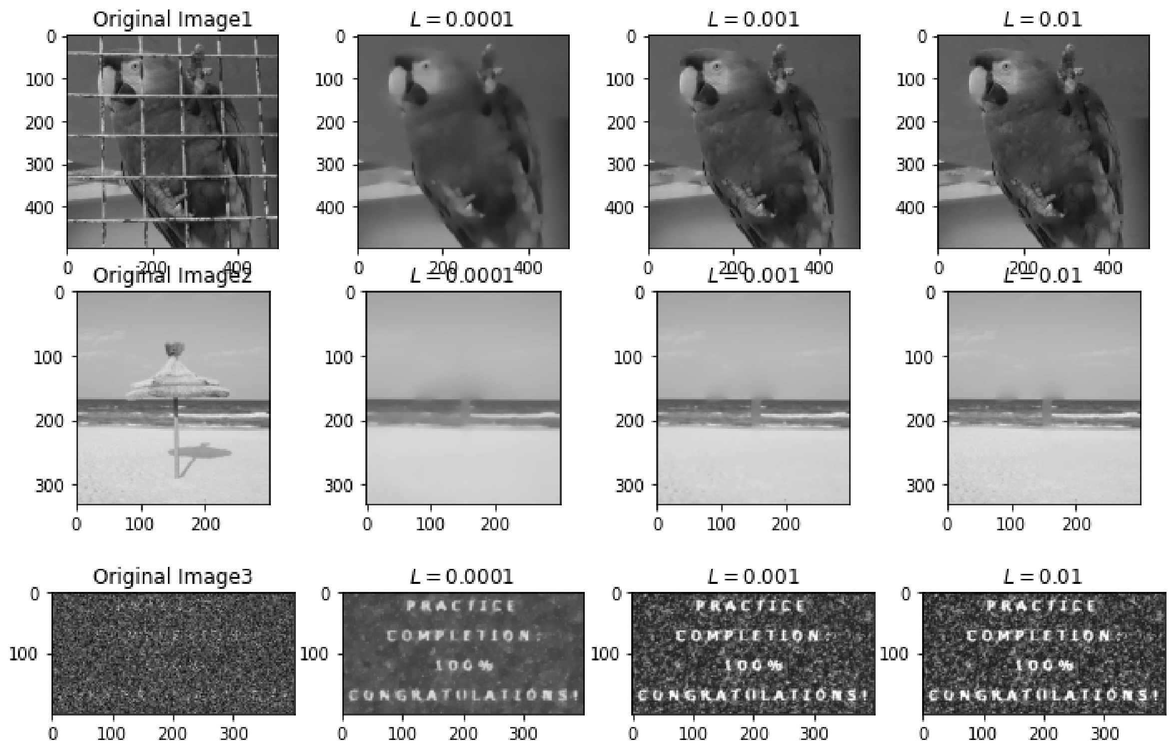

Figure 7: In painting using TV method

8 Exercise 14

For the algorithm, we need to find the minimizer for Equation 10.

$$A^*\left(Au - f\right) - \lambda \mathrm{div}\left(\frac{\psi'\left(\nabla u\right)}{2|\nabla u|}\nabla u\right) = 0 \tag{10}$$

Which will later be used as the step of the algorithm:

$$u^{k+1} = u^k + \tau\left(\lambda\left(f - u^k\right) + \mathrm{div}\left(\frac{\psi'\left(\nabla u^k\right)}{2|\nabla u^k|}\nabla u^k\right)\right) \tag{11}$$

To get the optimality condition, we need to replace ψ with ϕ_1 and ϕ_2, thus we need the derivative of ϕ_1 and ϕ_2. We know that

$$\phi_1(\xi) = \frac{\xi^2}{\xi^2 + 1}$$

and

$$\phi_2(\xi) = \log(\xi^2 + 1)$$

Hence,

$$\phi_1{}'(\xi) = \frac{2\xi}{(1 + \xi^2)^2}$$

$$\phi_2{}'(\xi) = \frac{2\xi}{1 + \xi^2}$$

Figure 8 illustrates the results of g_1 and g_2, in comparison with the Denoise_TV method. We can see that for a given λ value, the PSNR value for the Denoise_TV method is the highest, hence that is the best solution.

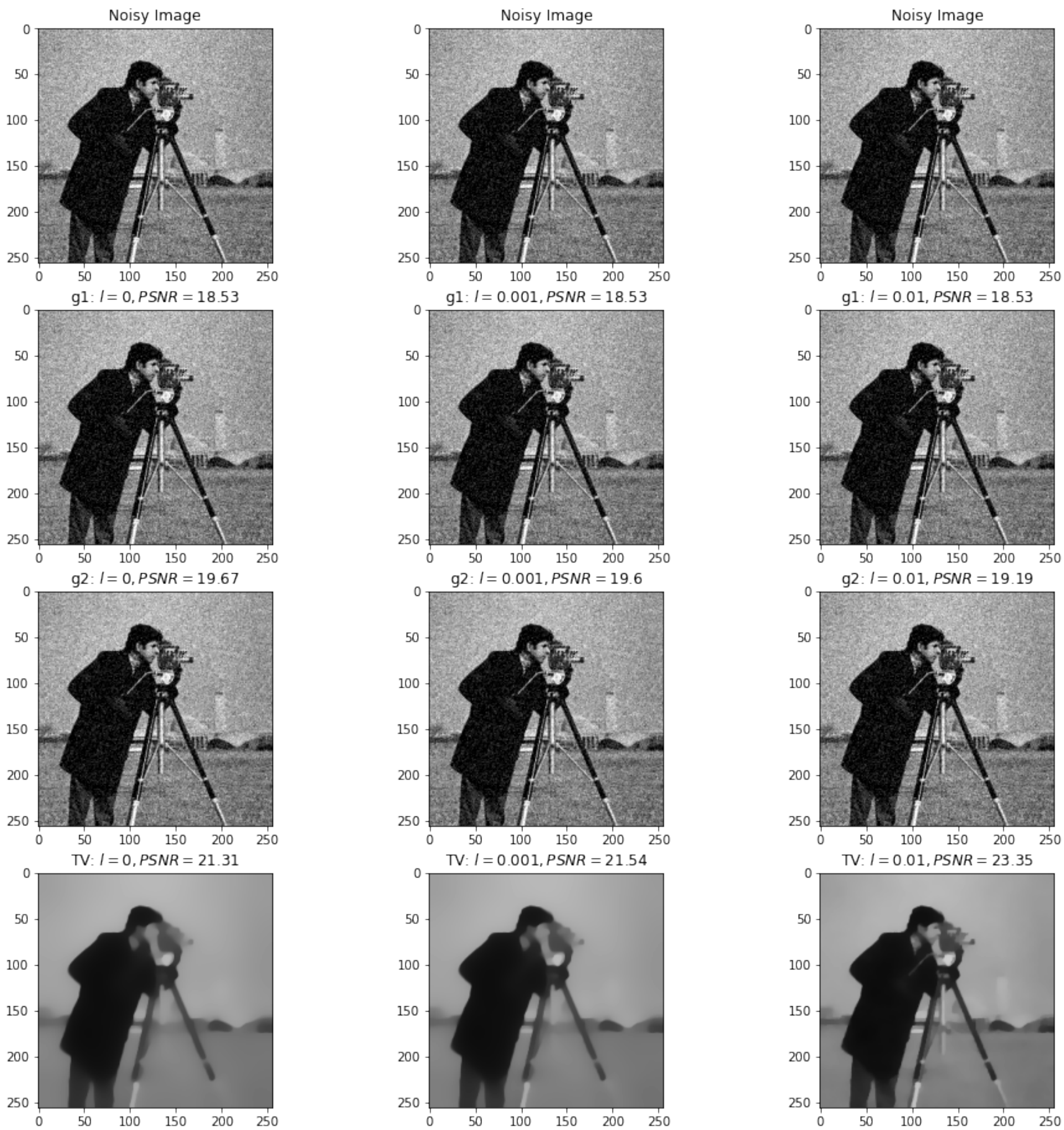

Figure 8: Comparing the results for different $\phi(\xi)$