PHY250: Formulas and Notes

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This document is for reviewing references and is based on NG's lecture notes and Griffiths E & M Textbook. Use in the course only.

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1 Vector Calculus

1.1 Dot product, Cross product, Triple Product

The **dot product** of two vectors $\overrightarrow{a}$, $\overrightarrow{b}$ is defined as:

$$\overrightarrow{a} \cdot \overrightarrow{b} = ||\overrightarrow{a}||||\overrightarrow{b}|| \cos\theta \tag{1}$$

Algebraically, if $\overrightarrow{a} =< a_1, a_2, ..., a_n >$, $\overrightarrow{b} =< b_1, b_2, ..., b_n >$, then:

$$\overrightarrow{a} \cdot \overrightarrow{b} = \sum_{i=1}^{n} a_i b_i$$

The **cross product** of two vectors (usually in $\mathbb{R}^3$ is defined as:

$$\overrightarrow{a} \times \overrightarrow{b} = ||\overrightarrow{a}||||\overrightarrow{b}|| \sin\theta\ \hat{n} \tag{2}$$

where $\hat{n}$ is the unit vector pointing to direction of cross product vector. And algebraically:

$$\overrightarrow{a} \times \overrightarrow{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Notice that, $\overrightarrow{a} \times \overrightarrow{b} = -\overrightarrow{b} \times \overrightarrow{a}$.
Some properties of **Triple products** (Cross-Cross, Cross-Dot, Dot-Cross):

- $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$
- $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$
- $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$, *BAC-CAB Rule*
- From BAC-CAB we can derive that $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = \mathbf{0}$

1.2 Linear transformation

1.3 Differential Calculus

1.3.1 Gradient vector

In the 3D coordinate system, we first define a point $\overrightarrow{r} = (x, y, z)$. Then its infinitesimal displacement vector from (x, y, z) to $(x + dx, y + dy, z + dz)$ will be:

$$d\overrightarrow{r} = dx\hat{x} + dy\hat{y} + dz\hat{z} \tag{3}$$

Suppose we have a function of temperature with respect to position in 3D:

$$T(\overrightarrow{r}) = T(x, y, z)$$

When we move from $\overrightarrow{r}$ to $\overrightarrow{r} + d\overrightarrow{r}$, we get the infinitesimal of temperature dT:

$$dT = (\frac{\partial T}{\partial x})dx + (\frac{\partial T}{\partial y})dy + (\frac{\partial T}{\partial z})dz$$

We relate the above equation to the vector, given that the dot product of two identical unit vector equals to 1:

$$\begin{aligned} dT &= \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right) \cdot (dx\hat{x} + dy\hat{y} + dz\hat{z}) \\ &= \vec{\nabla} T \cdot d\vec{r} \end{aligned}$$

The vector ∇T is a vector quantity, which is called **gradient vector** of function T. Geometrically, the gradient vector ∇T points to the direction of **maximum increase** in temperature.

Meaning of maximum increase?
Consider a contour map showing an inclined plane, where these lines are curves of function $z = h(x, y)$. Suppose we stand on 60m contour line, and we want to move to one direction with highest increase of altitude. Then what will be this direction? In other

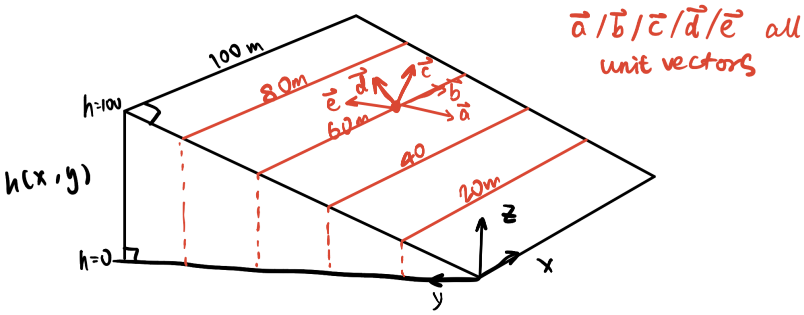

Figure 1: Meaning of gradient vector

words, we need a vector such that it can move from 60m to 80m height with shortest distance. Among these unit vectors, $\vec{d}$ is the desired one for approaching this, because it's perpendicular to all contour lines, indicating "shortest distance between contour lines". This vector $\vec{d}$ is the gradient vector $\vec{\nabla}$ we want. Notice that vector $\vec{b}$ is parallel to the contour lines, and actually in the reality this will be the directional derivative of this contour. Therefore, we have concluded a very important property of gradient:

Gradient vector at one point is always perpendicular to directional derivative vector.

1.3.2 Divergence and Curl

Consider a black hole and sun: while the sun emits EMR (such as visible light, and UV radiation), the black hole absorbs all particles, including the EMR like visible light, from its surrounding environments.

We therefore define the term **divergence**: geometrically speaking, the divergence is a measure of how much the vector $\vec{v}$ spreads out/diverge from a given point. We further define the vectors which converge into one point have negative divergence. So based on the above example, radiation vectors near a black hole have *negative* divergence, and those near stars have *positive* divergence.

Algebraically, the **divergence** is defined as dot product of Del and vector:

$$\mathrm{Div}(\mathbf{v}) = \nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \tag{4}$$

Notice that divergence is a scalar value. Fig 1.18 of Griffith textbook show three vector fields: the (a) has a source point emitting vectors, therefore the divergence is positive. In part (b) there is no difference in each of the vector and therefore has zero divergence. In part (c), instead of a point source, we now have a line source emitting the vectors with increasing lengths, therefore this vector function has positive divergence.

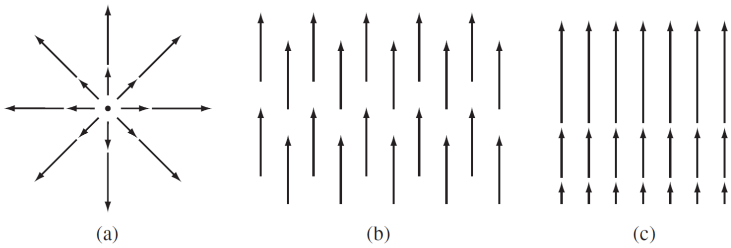

FIGURE 1.18

Figure 2: Fig 1.18 from Griffith

The **curl** of vectors is defined as cross product of gradient and vectors:

$$\mathrm{Curl}(\mathbf{v}) = \nabla \times \mathbf{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix} \tag{5}$$

Expanding this equation we have:

$$\nabla \times \mathbf{v} = \hat{x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

Notice that, the curl of vector function is still a vector quantity. Geometrically, the curl of vector shows how this vector swirl around one point. For example, all three vector fields in Griffith Fig 2 have zero curls since all these vectors don't swirl around any points. However, in Griffith Fig 1.19, both vector functions have nonzero curls, and they both point to the $\hat{z}$ direction.

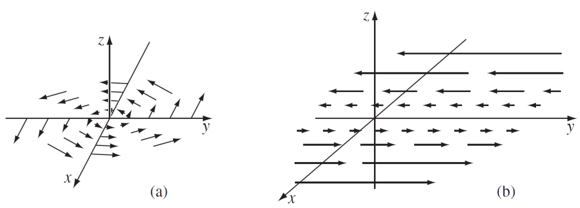

FIGURE 1.19

Figure 3: Fig 1.19 from Griffith: notice that both vector fields have non-zero curls

1.3.3 Product rules for gradient

- $\nabla(f+g) = \nabla f + \nabla g$
- $\nabla \cdot (\mathbf{A}+\mathbf{B}) = (\nabla \cdot \mathbf{A}) + (\nabla \cdot \mathbf{B})$ (similar for cross product)
- $\nabla \cdot (k\mathbf{A}) = k(\nabla \cdot \mathbf{A})$ (similar for cross product)
- Two rules for gradient:

  $\nabla(fg) = f\nabla g + g\nabla f$

  $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$
- Two rules for divergence:

  $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

  $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$
- Two rules for curls:

  $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times\nabla f$

  $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$ (*Proof: see PHY250 TUT1/TODO: transfer to this doc*)

1.4 ~~Double Del~~ Second derivatives

- Gradient derivatives (∇f)':

  a) Divergence of gradient of a scalar function:

$$\nabla\cdot(\nabla T) = \nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

  where $\nabla^2 T$ is called **Laplacian** of T

  b) The Laplacian of a vector function $\mathbf{v}$:

$$\nabla^2\mathbf{v} = (\nabla^2 v_x)\hat{x} + (\nabla^2 v_y)\hat{y} + (\nabla^2 v_z)\hat{z}$$

  which means we need to evaluate the Laplacian three times, each time for each component of vector function $\mathbf{v}$.

  c) Curl of gradient: $\nabla\times(\nabla T)$, always **zero**.

- Divergence derivatives $(\nabla\cdot\mathbf{v})$':

  a) Gradient of divergence: $\nabla(\nabla\cdot\mathbf{v})$, this seldom appears in physical applications. Differentiate from the Laplacian of vector:

$$\nabla^2\mathbf{v} = (\nabla\cdot\nabla\mathbf{v}) \neq \nabla(\nabla\cdot\mathbf{v})$$

  b) Divergence of divergence: **Undefined** because divergence is scalar instead of vector.

  c) Curl of divergence: **Undefined** because divergence is scalar instead of vector.

- Curl derivatives ($\nabla \times \mathbf{v}$)':

  a) Divergence of curl: $\nabla \cdot (\nabla \times \mathbf{v})$, always **zero**.

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0$$

  b) Curl of curl: $\nabla \times (\nabla \times \mathbf{v})$

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$$

1.5 Vector Integrals

Recall from year 1 calculus class:

- We denote $\frac{df}{dx}$ to show the derivative of function f w.r.t. x, in geometry this means how fast function f varies.
- We do anti-derivatives of function's derivative to get its original function expression, which is called "indefinite integral", and by Fundamental Theorem of Calculus, we define the definite integral to the difference of functions evaluated at two bounds:

$$\int \left(\frac{df}{dx}\right) dx = f(x) + C$$

$$\int_a^b \left(\frac{df}{dx}\right) dx = f(b) - f(a)$$

- From now on, we will focus on integral calculus on vector functions.

There are three different types of vector integrals: line integral, surface integral and volume integral.

1.5.1 Line integral

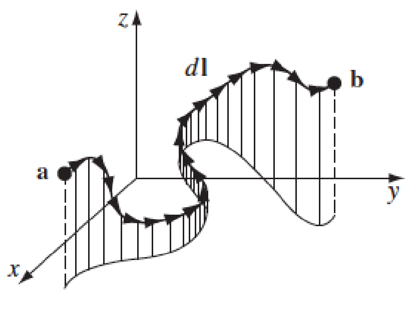

FIGURE 1.20

This is very similar to the definite integral, but we use vectors instead of scalars:

$$\int_{\vec{a}}^{\vec{b}} \vec{v} \cdot d\vec{l} = \int_{\vec{a}}^{\vec{b}} v_x dx + v_y dy + v_z dz \tag{6}$$

where $d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$

If $\vec{a} = \vec{b}$, the path will be closed and we will put a circle on the integral notation, where $\mathcal{P}$ is perimeter:

$$\oint_{\mathcal{P}} \vec{v} \cdot d\vec{l}$$

To calculate closed line integral, divide the whole perimeter into some pieces and integrate w.r.t them, and then add these integral values up.

1.5.2 Surface integral

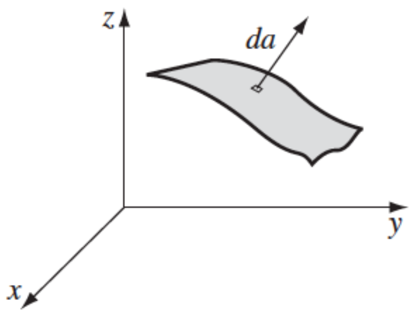

FIGURE 1.22

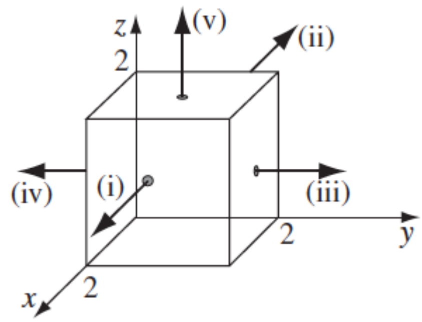

FIGURE 1.23

The surface integral is defined as:

$$\iint_{\mathcal{S}} \vec{v} \cdot d\vec{a} \tag{7}$$

where $d\vec{a}$ is the area element, **and by convention, it always points outward of surface, which means "flux" of this infinitesimal surface**. And surface integral evaluates the total flux of vector function passing through the surface.

Therefore, if $\vec{v} \parallel d\vec{a}$, which means vector $\vec{v}$ the flux of surface would reach maximum, then:

$$\vec{v} \cdot d\vec{a} = vda$$

If $\vec{v} \perp d\vec{a}$, which means vector $\vec{v}$ will be parallel to the surface, hence there will be no flux on the surface:

$$\vec{v} \cdot d\vec{a} = 0$$

If the surface $\mathcal{S}$ is closed, then we put the circle on the double integral sign to represent closed surface integral:

$$\oiint_{\mathcal{S}} \vec{v} \cdot d\vec{a}$$

To evaluate the closed surface integral, we also need to separate the surface into some pieces, evaluating the surface integrals individually and add them up. (Fig 1.23, and see Ex 1.7 on Griffiths)

1.5.3 Volume integral

Similar to the surface integral, volume integral is defined as:

$$\iiint_{\mathcal{V}} T d\tau \tag{8}$$

where $d\tau$ is the infinitesimal solid volume, i.e. $d\tau = dxdydz$. For a vector function $\mathbf{v}$, the equation becomes:

$$\iiint_{\mathcal{V}} \mathbf{v} d\tau = \hat{x} \iiint_{\mathcal{V}} v_x d\tau + \iiint_{\mathcal{V}} v_y d\tau + \iiint_{\mathcal{V}} v_z d\tau$$

1.6 Appendix I: Proofs of Some Important Properties of Unit 1

1. *Claim*: The divergence of curl is always zero.
   Proof

$$
\begin{aligned}
\nabla \cdot (\nabla \times \mathbf{v}) &= \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \\ \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \\ \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \end{bmatrix} \\
&= \frac{\partial^2 v_z}{\partial x \partial y} - \frac{\partial^2 v_y}{\partial x \partial z} + \frac{\partial^2 v_x}{\partial y \partial z} - \frac{\partial^2 v_z}{\partial y \partial x} + \frac{\partial^2 v_y}{\partial z \partial x} - \frac{\partial^2 v_x}{\partial z \partial y} \\
&= 0
\end{aligned}
$$

$\blacksquare$

2. *Claim*: $\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$ (Curl of Curl expression, HW1-2)

   NOTE: to prove this, we need first evaluate the LHS, then evaluate RHS, and compare whether they are equal. If not equal, then there is some algebra mistakes.

   Proof

   - LHS:

$$
\begin{aligned}
\nabla \times (\nabla \times \mathbf{v}) &= \nabla \times \begin{bmatrix} \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \\ \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \\ \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \end{bmatrix} \\
&= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} & \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} & \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \end{vmatrix} \\
&= \hat{x}\left(\frac{\partial}{\partial y}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\right) + \\
&\quad \hat{y}\left(\frac{\partial}{\partial z}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\right) + \\
&\quad \hat{z}\left(\frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\right) \\
&= \hat{x}\left(\frac{\partial^2 v_y}{\partial y \partial x} - \frac{\partial^2 v_x}{\partial y^2} - \frac{\partial^2 v_x}{\partial z^2} + \frac{\partial^2 v_z}{\partial z \partial x}\right) + \\
&\quad \hat{y}\left(\frac{\partial^2 v_z}{\partial z \partial y} - \frac{\partial^2 v_y}{\partial z^2} - \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_x}{\partial x \partial y}\right) + \\
&\quad \hat{z}\left(\frac{\partial^2 v_x}{\partial x \partial z} - \frac{\partial^2 v_z}{\partial x^2} - \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_y}{\partial y \partial z}\right)
\end{aligned}
$$

   - RHS:

$$
\begin{aligned}
\nabla(\nabla\cdot\mathbf{v}) &= \nabla(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}) \\
&= \hat{x}\frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)+ \\
&\quad \hat{y}\frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)+ \\
&\quad \hat{z}\frac{\partial}{\partial z}\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)
\end{aligned}
$$

$$
\begin{aligned}
\nabla^2\mathbf{v} &= \hat{x}\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)+ \\
&\quad \hat{y}\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)+ \\
&\quad \hat{z}\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
\nabla(\nabla\cdot\mathbf{v})-\nabla^2\mathbf{v} &= \hat{x}\left(\cancel{\frac{\partial^2 v_x}{\partial x^2}}+\frac{\partial^2 v_y}{\partial x\partial y}+\frac{\partial^2 v_z}{\partial x\partial z}-\cancel{\frac{\partial^2 v_x}{\partial x^2}}-\frac{\partial^2 v_x}{\partial y^2}-\frac{\partial^2 v_x}{\partial z^2}\right)+ \\
&\quad \hat{y}\left(\frac{\partial^2 v_x}{\partial y\partial x}+\cancel{\frac{\partial^2 v_y}{\partial y^2}}+\frac{\partial^2 v_z}{\partial y\partial z}-\frac{\partial^2 v_y}{\partial x^2}-\cancel{\frac{\partial^2 v_y}{\partial y^2}}-\frac{\partial^2 v_y}{\partial z^2}\right)+ \\
&\quad \hat{z}\left(\frac{\partial^2 v_x}{\partial z\partial x}+\frac{\partial^2 v_y}{\partial z\partial y}+\cancel{\frac{\partial^2 v_z}{\partial z^2}}-\frac{\partial^2 v_z}{\partial x^2}-\frac{\partial^2 v_z}{\partial y^2}-\cancel{\frac{\partial^2 v_z}{\partial z^2}}\right)
\end{aligned}
$$

- Comparing the items, we found LHS = RHS, and that completes the proof.

■

2 Electrostatics