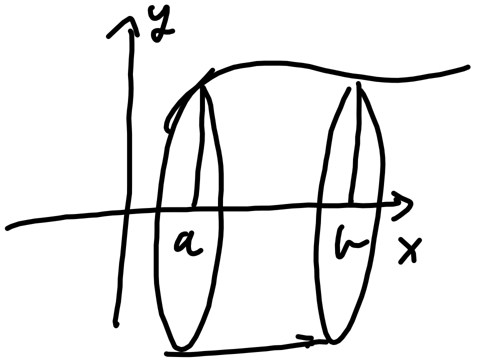
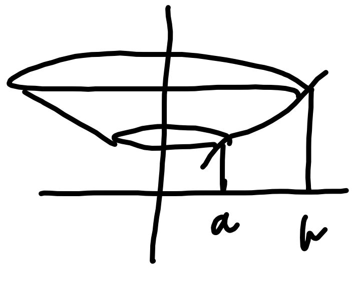
Seksjon 8.6/9.1/9.2



$$V = \int_a^b \pi f(x)^2 \, dx.$$

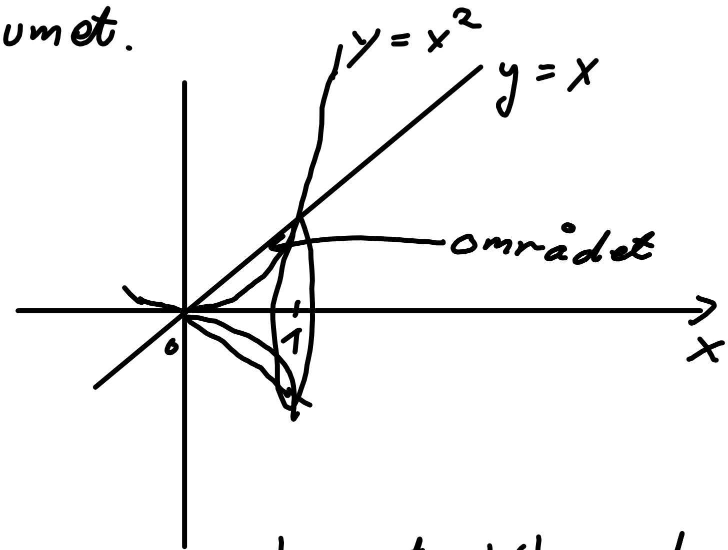
om x-aksen.



$$V = \int_a^b 2\pi x f(x) \, dx$$

om y-aksen.

9 a) Området avgrenset av $y=x$ og $y=x^2$ dreies om x-aksen. Hva er volumet.



Volumet blir volumet til omdreiningslegemet til $y=x$ minus —— " —— $y=x^2$.

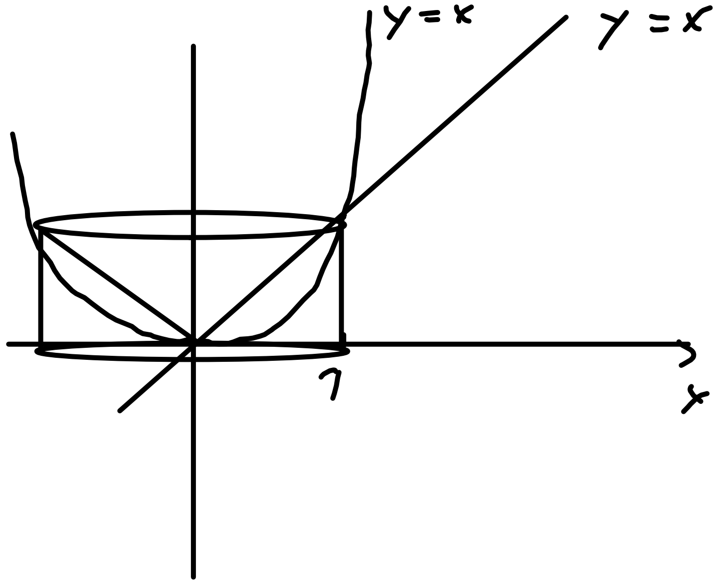
Volum til $y=x$: $V = \int_0^1 \pi x^2 dx = \pi\left[\frac{x^3}{3}\right]_0^1 = \pi\left[\frac{1^3}{3} - \frac{0^3}{3}\right] = \frac{\pi}{3}.$

$$\left[\int_a^b x^n dx = \left[\frac{x^{n+1}}{n+1}\right]_a^b \leftarrow\right]$$

Volum til $y=x^2$: $V = \int_0^1 \pi x^4 dx = \pi\left[\frac{x^5}{5}\right]_0^1 = \frac{\pi}{5}.$

Volumet til omdreiningslegemet til området blir $\frac{\pi}{3} - \frac{\pi}{5} = \underline{\underline{\frac{2\pi}{15}}}.$

b)



Bruker samme metode som i stad:

volum til $y=x$:

$$V = \int_0^1 2\pi x^2 dx = 2\int_0^1 \pi x^2 dx = \frac{2\pi}{3}.$$

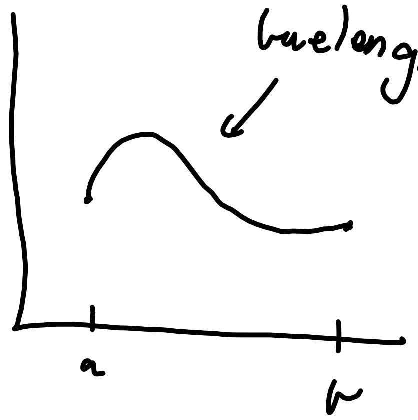
Volum til $y=x^2$:

$$V = \int_0^1 2\pi x^3 dx = 2\pi\left[\frac{x^4}{4}\right]_0^1 = \frac{\pi}{2}.$$

Det resulterende volumet blir: $\frac{2\pi}{3} - \frac{\pi}{2} = \underline{\underline{\frac{\pi}{6}}}.$

Buelengde til en graf av en funksjon $f(x)$

er $L = \int_a^b \sqrt{1 + f'(x)^2}\, dx$.



11c) Finn buelengden til

$f(x) = y = \frac{x^2}{2} - \frac{1}{4}\ln(x)$ fra $x=1$ til $x=e$.

Løsning: $f'(x) = x - \frac{1}{4x}$.

$$L = \int_1^e \sqrt{1 + \left(x - \frac{1}{4x}\right)^2}\, dx = \int_1^e x + \frac{1}{4x}\, dx$$

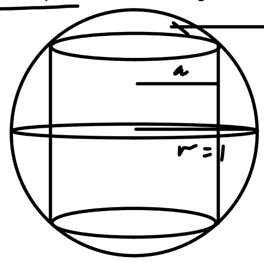
$$\left(x - \frac{1}{4x}\right)^2 = x^2 - \frac{1}{2} + \frac{1}{16x^2}.$$

$$1 + \left(x - \frac{1}{4x}\right)^2 = x^2 + \frac{1}{2} + \frac{1}{16x^2} = \left(x + \frac{1}{4x}\right)^2$$

$$\left[\frac{x^2}{2} + \frac{1}{4}\ln(x)\right]_1^e = \left(\frac{e^2}{2} + \frac{1}{4}\right) - \left(\frac{1}{2} + \frac{1}{4}\cdot 0\right)$$

$$= \frac{e^2}{2} - \frac{1}{4}$$

15. Gjennom en kule med radius 1 bores et sylindrisk hull med radius $a<1$ og akse gjennom kulens sentrum. Finn volumet til den gjenværende delen.



Finner grafen til en sirkelbue:

$y=\sqrt{1-x^2}$. $\left(y^2+x^2=(1-x^2)+x^2=1\right)$

Dreier om y-aksen:

$$V=\int_a^1 2\pi x\sqrt{1-x^2}\,dx=2\pi\int_a^1 x\sqrt{1-y^2}\,dx.$$

Setter $u=1-x^2$: $du=-2x\,dx \Rightarrow -\frac{1}{2x}du=dx$.

$$V=2\pi\int_{1-a^2}^{0} x\sqrt{u}\cdot\left(-\frac{1}{2x}\right)du$$

$$=2\pi\int_{1-a^2}^{0}-\frac{1}{2}\sqrt{u}\,du=2\pi\int_0^{1-a^2}\frac{1}{2}\sqrt{u}\,du$$

$$=\pi\int_0^{1-a^2}\sqrt{u}\,du=\pi\left[\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^{1-a^2}$$

$$=\pi\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{1-a^2}=\pi\left(\frac{(1-a^2)^{\frac{3}{2}}}{\frac{3}{2}}\right)=\frac{2}{3}\pi(1-a^2)^{\frac{3}{2}}$$

Må gange dette med 2: og får

$$V=\frac{4}{3}\pi(1-a^2)^{\frac{3}{2}}.$$

Substitusjon av grenser: (9.2.7)

$$\int_a^b f(g(x))\,dx = \int_{g(a)}^{g(b)} f(u)\,h'(u)\,du.$$

$u = g(x)$. h er den omvendte funksjonen til g.

I praksis: $du = g'(x)dx.$

$$\Rightarrow \frac{1}{g'(x)}du = dx.$$

$$\int_a^b f(g(x))dx = \int_{g(a)}^{g(b)} \frac{f(u)}{g'(x)}du.$$

9.1

a) Finn $\int \sin(\ln(x))\,dx$.

Prøver: $u = \ln x$. $du = \frac{1}{x}dx$

$\Rightarrow x\,du = dx$. $\Rightarrow e^u du = dx$.

$\rightsquigarrow \int e^u \sin(u)\,du$.

Delvis integrasjon:

$$\underline{\int e^u \sin(u)\,du} = e^u \sin(u) - \underline{\int e^u \cos(u)\,du}.$$

$$\int e^u \cos(u)\,du = e^u \cos(u) - \int e^u(-\sin(u))\,du$$

$$= e^u \cos(u) + \int e^u \sin(u)\,du.$$

Setter inn:

$$\underline{\int e^u \sin(u)\,du} = e^u \sin(u) - \left(e^u \cos(u) + \int e^u \sin(u)\,du\right)$$

$$= e^u \sin(u) - e^u \cos(u) - \underline{\int e^u \sin(u)\,du}$$

$$\Rightarrow 2\int e^u \sin(u)\,du = e^u(\sin(u) - \cos(u))$$

$$\Rightarrow \int e^u \sin(u)\,du = \frac{e^u(\sin(u) - \cos(u))}{2} + C$$

$u = \ln(x)$. Setter inn:

$$\int \sin(\ln(x))\,dx = \frac{x(\sin(\ln(x)) - \cos(\ln(x)))}{2} + C$$

11.
$$I = \int \frac{x^2 \arctan(x)}{1+x^2}\,dx$$

Setter: $u = \arctan(x)$. $du = \frac{1}{1+x^2}\,dx$.

$\Rightarrow (1+x^2)\,du = dx$.

Så: $I = \int \frac{x^2 u}{1+x^2} \cdot (1+x^2)\,du$

$= \int x^2 u\,du. = \int u \tan^2 u\,du$

(Har $x = \tan(u)$.)

Husk: $(\tan u)' = \tan^2 u + 1$.

Skriver om:

$$\int u \tan^2 u\,du = \int u(\tan^2 u + 1 - 1)\,du$$
$$= \int u(\tan^2 u + 1) - u\,du.$$
$$= \int u(\tan^2 u + 1)\,du - \int u\,du.$$

Bruker delvis integrasjon på

$I_2 = \int u(\tan^2 u + 1)\,du = u \tan u - \int \tan u\,du.$

$I = u\tan(u) - \int u\,du - \int \tan(u)\,du.$

$I_3 = \int \tan(u)\,du = \int \frac{\sin(u)}{\cos(u)}\,du$

Ny substitusjonsvariabel t:

$t = \cos(u)$. $dt = -\sin(u)\,du$.

$\Rightarrow -\frac{1}{\sin(u)}\,dt = du$.

$$I_3 = \int \frac{\sin(u)}{t} \cdot \left(-\frac{1}{\sin(u)}\right) dt$$
$$= \int -\frac{1}{t}\,dt = -\ln|t| + C$$
$$= -\ln|\cos(u)| + C.$$

Gjenstår:

$$\int u\,du = \frac{u^2}{2} + C.$$

Får da:

$$I = u\tan(u) - \int u\,du - \int \tan(u)\,du$$
$$= u\tan(u) - \frac{u^2}{2} + \ln|\cos(u)| + C.$$

$u = \arctan(x)$. $\left(\cos(\arctan(x)) \stackrel{?}{=} 1 + x^2\right)$

$\Rightarrow$

$$\underline{\underline{I = \arctan(x)\,x - \frac{\arctan(x)^2}{2} + \ln|1+x^2| + C}}$$