Plan: 7.5.3, 7.6.3fg, 7.6.5, 7.6.8, 8.2.1, 8.2.5, deler av 7.6.3, 7.6.2, 7.5.2

7.5.3 a) $\lim_{x\to 0} x\cot x = ?$

$$= \lim_{x\to 0} \frac{x\cos x}{\sin x} = \lim_{x\to 0}\cos x \lim_{x\to 0}\frac{x}{\sin x}$$

$$\overset{L'H}{=} \cos 0 \cdot \lim_{x\to 0}\frac{1}{\cos x} = 1\cdot 1 = 1$$

b) $$\lim_{x\to \frac{\pi}{2}} \frac{\cot x}{\pi/2 - x} = \lim_{u\to 0}\frac{\cot(u-\frac{\pi}{2})}{u}$$

$$\left(u = \tfrac{\pi}{2} - x\right)$$

$$= \lim_{u\to 0}\frac{\left(\frac{\cos(u-\frac{\pi}{2})}{\sin(u-\frac{\pi}{2})}\right)}{u} \,\underline{\underline{=}}\, \lim_{u\to 0}\frac{\left(\frac{\sin u}{\cos u}\right)}{u}$$

$$= \lim_{u\to 0}\frac{1}{u\cdot\cot u} = \left(\lim_{u\to 0} u\cot u\right)^{-1}$$

$$= 1$$

c) $$\lim_{x\to 0^+}(\cot x)^x \overset{\lim_{x\to 0}}{=} e^{x\ln\cot x}$$

$$= \lim_{x\to 0^+} e^{(x\ln\cos x - x\ln\sin x)}$$

$$= \underbrace{\lim_{x\to 0^+} e^{x\ln\cos x}}_{\substack{\parallel \\ e^{0\cdot\ln\cos 0 = 0} \quad e = 1}} \cdot \lim_{x\to 0^+} e^{-x\ln\sin x}$$

$$\lim_{x\to 0^+} x\ln\sin x = \lim_{x\to 0^+}\frac{\ln\sin x}{1/x}$$

$$\overset{L'H}{=} \lim_{x\to 0^+}\frac{\frac{\cos x}{\sin x}}{\left(-1/x^2\right)} = \lim_{x\to 0^+} -\cos x\cdot\frac{x^2}{\sin x}$$

$$= -\underbrace{\lim_{x\to 0^+}\cos x}_{\substack{\parallel\\ \cos 0}} \cdot \underbrace{\lim_{x\to 0^+}\frac{x}{\sin x}}_{\substack{\parallel\\ 1}} \cdot \underbrace{\lim_{x\to 0^+} x}_{\substack{\parallel\\ 0}} = 0$$

$$\lim_{x\to 0^+} e^{-x\ln\sin x} = e^0 = \underline{\underline{1}}$$

d) $$\lim_{x\to\infty}\left(\cot\left(\tfrac{1}{x}\right) - x\right) = \lim_{x\to\infty} x\left(\frac{\cot(\frac{1}{x})}{x} - 1\right)$$

$$= \lim_{x\to\infty}\frac{\left(\frac{\cot(\frac{1}{x})}{x} - 1\right)}{1/x} = \lim_{u\to 0}\left(\frac{u\cot u - 1}{u}\right)$$

$$\left(u = \tfrac{1}{x}\right)$$

$$\overset{L'H}{=} \lim_{u\to 0}\frac{\cot u - \frac{u}{\sin^2 u}}{1} = \lim_{u\to 0}\frac{\cos u}{\sin u} - \frac{u}{\sin^2 u}$$

(siden $u\cdot\cot u \to 1$ når $u\to 0$)

$$= \lim_{u\to 0}\frac{\cos u\cdot\sin u - u}{\sin^2 u}$$

(Husk: $2\cdot\cos x\cdot\sin x = \sin 2x$)

$$= \lim_{u\to 0}\frac{\frac{1}{2}\sin 2u - u}{\sin^2 u}$$

$$\overset{L'H}{=} \lim_{u\to 0}\frac{\cos 2u - 1}{2\sin u\cdot\cos u} =$$

$$\lim_{u\to 0}\frac{\cos 2u - 1}{\sin 2u} \overset{L'H}{=} \lim_{u\to 0}\frac{-2\sin 2u}{2\cdot\cos 2u}$$

$$= \frac{-2\cdot\sin(2\cdot 0)}{2\cdot\cos(2\cdot 0)} = \frac{0}{1} = \underline{\underline{0}}$$

7.6.3 d) $\lim_{x\to 1^-} \frac{\arcsin(x) - \frac{\pi}{2}}{\sqrt{1-x}}$

$$\overset{L'H}{=} \lim_{x\to 1^-} \frac{\left(\frac{1}{\sqrt{1-x^2}}\right)}{\left(\frac{-1}{2\sqrt{1-x}}\right)} = \lim_{x\to 1^-} -2\frac{\sqrt{1-x}}{\sqrt{1-x^2}}$$

$$= \lim_{x\to 1^-} -2\sqrt{\frac{1-x}{(1-x)(1+x)}} = \lim_{x\to 1^-} -2\sqrt{\frac{1}{1+x}}$$

$$= -2\cdot\frac{1}{\sqrt{2}} = \underline{\underline{-\sqrt{2}}}$$

f) $\lim_{x\to\infty} x^{(\pi/2 - \arctan x)} = \lim_{x\to\infty} e^{\ln x\left(\frac{\pi}{2} - \arctan x\right)}$

$$\lim_{x\to\infty} \ln x\left(\frac{\pi}{2} - \arctan x\right)$$

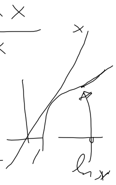
$$= \lim_{x\to\infty} \frac{\frac{\pi}{2} - \arctan x}{\left(\frac{1}{\ln x}\right)}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{-\frac{1}{1+x^2}}{\left(-\frac{(1/x)}{(\ln x)^2}\right)} = \lim_{x\to\infty} \frac{x(\ln x)^2}{1+x^2}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{(\ln x)^2 + 2\ln x\cdot x\cdot\frac{1}{x}}{2x}$$

$$= \lim_{x\to\infty} \frac{(\ln x)^2}{2x} + \lim_{x\to\infty} \frac{\ln x}{x}$$

$$\overset{L'H}{=} \underbrace{\lim_{x\to\infty} \frac{2\ln x\cdot\frac{1}{x}}{2}}_{0} + \underbrace{\lim_{x\to\infty} \frac{\frac{1}{x}}{1}}_{0} = 0$$



$$\Rightarrow \lim_{x\to\infty} e^{\ln x(\pi/2 - \arctan x)} = e^0 = \underline{\underline{1}}$$

g) $\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\arctan x}\right)$

$$= \lim_{x\to 0} \frac{\arctan x - x}{x\arctan x} \overset{L'H}{=} \lim_{x\to 0}$$

$$\frac{\frac{1}{1+x^2} - 1}{\arctan x + \frac{x}{1+x^2}} = \lim_{x\to 0} \frac{1-(1+x^2)}{(1+x^2)\arctan x + x}$$

$$= \lim_{x\to 0} \frac{-x^2}{(1+x^2)\arctan x + x} \overset{L'H}{=} \lim_{x\to 0} \frac{-2x}{2x\arctan x + \left[(1+x^2)\cdot\frac{1}{(1+x^2)}\right] + 1}$$

$$= \lim_{x\to 0} \frac{-2x}{2x\arctan x + 1 + 1} = \frac{-2\cdot 0}{2\cdot 0\cdot\arctan 0 + 2} = \underline{\underline{0}}$$

7.6.5 La $f(x) = x \cdot \arctan x$

a) Hvor er f voksende og avtagende?

$$f'(x) = \arctan x + \frac{x}{1+x^2}$$

Da $\arctan x > 0$ for $x \in (0, \infty)$

& $\frac{x}{1+x^2} > 0$ —— " ——

og $\arctan x < 0$ for $x \in (-\infty, 0)$

& $\frac{x}{1+x^2} < 0$ —— " ——

og $f'(0) = 0$

Så f er voksende på $[0, \infty)$

og f er avtagende på $(-\infty, 0]$

og $x = 0$ er et globalt minimum.

b) Avgjør hvor f er konkav/konveks?

$$f'(x) = \arctan x + \frac{x}{1+x^2}$$

$$f''(x) = \frac{1}{1+x^2} + \frac{1}{1+x^2} + \frac{-(2x)\cdot x}{(1+x^2)^2}$$

$$= \frac{2}{(1+x^2)}\left(1 - \frac{x^2}{1+x^2}\right) = \frac{2}{1+x^2}\left(\frac{1+x^2-x^2}{1+x^2}\right)$$

$$= \frac{2}{(1+x^2)^2} > 0 \text{ for alle } x$$

Så f er konveks på $\mathbb{R}$

c) Finn asymptotene til f

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{x \arctan x}{x} = \lim_{x \to \pm\infty} \arctan x$$

$$= \pm \frac{\pi}{2}$$

$$\lim_{x \to \pm\infty} f(x) - \left(\pm\frac{\pi}{2} x\right) = \lim_{x \to \pm\infty} \left(x \arctan x \mp \frac{\pi}{2} x\right)$$

$$= \lim_{x \to \pm\infty} x\left(\arctan x \mp \frac{\pi}{2}\right) = \lim_{x \to \pm\infty} \frac{\arctan x - \frac{\pi}{2}}{1/x}$$

$$\overset{L'H}{=} \lim_{x \to \pm\infty} \frac{\frac{1}{1+x^2}}{-1/x^2} = \lim_{x \to \pm\infty} \frac{-x^2}{1+x^2}$$

$$= \lim_{x \to \pm\infty} \frac{-1}{1/x^2 + 1} = \frac{-1}{0+1} = -1$$

Så f har asymptotene $\underline{\underline{y = \pm\frac{\pi}{2} x - 1}}$ når x går mot $\pm\infty$.

7.6.8a) La $f(x) = \begin{cases} \dfrac{\arctan x}{1+x^2} & x \geq 0 \\ Ae^x + B & x < 0 \end{cases}$

a) Finn A og B slik at f er kontinuerlig og deriverbar.

Det er tilstrekkelig å finne A og B slik at f er kontinuerlig og deriverbar i 0.

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\arctan x}{1+x^2} = \frac{\arctan 0}{1+0} = 0$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} Ae^x + B = Ae^0 + B = A + B$$

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x} \overset{\lim_{x\to 0^+}}{=} \frac{\frac{\arctan x}{1+x^2} - 0}{x}$$

$$= \lim_{x\to 0^+} \frac{\arctan x}{x(1+x^2)} \overset{L'H}{=} \lim_{x\to 0^+} \frac{\frac{1}{1+x^2}}{1+3x^2} = \frac{1}{1} = 1$$

$$\lim_{x\to 0^-} \frac{Ae^x + B - 0}{x} \overset{L'H}{=} \lim_{x\to 0} \frac{Ae^x}{1} = A \cdot e^0 = A$$

Så $\underline{\underline{A = 1}}$ og dermed er $\underline{\underline{B = -1}}$

b) Vis at f har et maksimumspunkt mellom 0 og 1.

$$f'(x) = D\left[\frac{\arctan x}{1+x^2}\right] = \frac{\left(\frac{1}{1+x^2}\right)(1+x^2) - 2x \cdot \arctan x}{(1+x^2)^2}$$

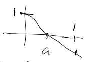
$$= \frac{1 - 2x \arctan x}{(1+x^2)^2}$$

Da er $f'(0) = 1 > 0$, $f'(1) = \dfrac{1 - 2\cdot 1 \cdot \arctan 1}{(1+1)^2}$

$$= \frac{1 - 2\cdot \frac{\pi}{4}}{4} = \frac{1 - \frac{\pi}{2}}{4} < 0$$

Så skjæringssetningen sier oss at det finnes en $a \in (0,1)$ slik at $f'(a) = 0$

Videre så er f' monoton, siden $1 - 2x \arctan x$ er monoton.



Så f har et maksimumspunkt for $x = a$.

Siden $f'(x) = e^x$ for $x < 0$, så er $f'(x) > 0$ for $x \in (-\infty, 0)$. Så f har ingen ekstremalpunkt i $(-\infty, 0)$. og $x = a$ er eneste ekstremalpunkt for $x > 0$, så vi har funnet alle ekstremalpunktene til f. og $x = a$ er et globalt maksimum.

c) Bestem asymptotene til f.

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\arctan x}{x(1+x^2)} = 0,$$

$=$ siden $\arctan x \to \frac{\pi}{2}$ og $x(1+x^2) \to \infty$.

$$\lim_{x\to -\infty} \frac{f(x)}{x} = \lim_{x\to -\infty} \frac{(e^x - 1)}{x} = 0$$

$$\lim_{x\to\infty} (f(x) - 0\cdot x) = \lim_{x\to\infty} \frac{\arctan x}{1+x^2} = 0$$

og $\lim_{x\to -\infty} (f(x) - 0\cdot x) = \lim_{x\to\infty} (e^x - 1) = -1$

Så f har asymptotene $\underset{\text{når } x\to\infty}{y = 0}$ og $\underset{\text{når } x\to -\infty}{y = -1}$.

Graf: