Plan: 9.5.3a, 9.5.3b, 9.5.3e, 9.5.5, 9.5.14, 9.3.17, 9.3.25, 9.3.31, 9.4.8, 8.4.10, 9.5.1c, 9.5.1e, FVA.1.2.17, FVA.1.2.21, FVA.1.2.25

9.5.3a

Avgjør om $\int_0^\infty \frac{x+4}{x^2+2x+1}dx$ divergerer eller konvergerer?

$\approx \frac{1}{x}$

$$\int_0^\infty \frac{x+4}{x^2+2x+1}dx = \int_0^\infty \frac{x+1+3}{(x+1)^2}dx$$

$$= \int_0^\infty \frac{(x+1)}{(x+1)^2}dx + \int_0^\infty \frac{3}{(x+1)^2}dx$$

$$\int_0^\infty \frac{1}{x+1}dx = \int_1^\infty \frac{1}{x}dx \quad \text{som divergerer}$$

3b $\int_1^\infty \frac{x+2}{\sqrt{x^3+x^5}}dx$ konvergerer eller divergerer

$$\left(\frac{\text{1. grads polynom}}{\text{5/2. grads polynom}}\right)$$

" $-3/2$ polynom

$\Rightarrow$ konvergerer

$$= \int_1^\infty \frac{x+2}{\sqrt{x^2(x+x^3)}}$$

$$= \int_1^\infty \frac{x+2}{x\sqrt{x+x^3}} = \int_1^\infty \frac{1+2/x}{\sqrt{x+x^3}}$$

$$\leq \int_1^\infty \frac{1+2}{\sqrt{x+x^3}} \leq \int_1^\infty \frac{3}{\sqrt{x^3}} = \int_1^\infty 3x^{-\frac{3}{2}}$$

Som konvergerer.

(siden $\int_1^\infty \frac{1}{x^p}$ konvergerer for $p > 1$)

$u = e^x - 1$

$u' = e^x = u+1$

3e
$$\int_0^1 \frac{dx}{e^x-1} = \int_0^1 \frac{u'\,dx}{u\,u'} = \int_0^1 \frac{u'\cdot dx}{u\cdot(u+1)}$$

$\boxed{\begin{matrix} u(0)=0 \\ u(1)=e-1 \end{matrix}}$

$$= \int_0^{e-1} \frac{du}{u(u+1)} \geq \int_0^{e-1} \frac{du}{u\cdot(e-1+1)} = \int_0^{e-1} \frac{du}{u\cdot e}$$

$\left(\frac{du}{dx} = u'\right)$

$du \Downarrow = u'\cdot dx$

Som divergerer.

$$= \frac{1}{e}\left[\ln u\right]_0^{e-1}$$

9.5.5 For hvilke p konvergerer

$$\int_1^\infty \frac{\ln x}{x^p}\,dx\,?$$

$$\left(\begin{array}{l} u = \ln x \\ u' = \frac{1}{x} \end{array}\right)$$

$$= \int_1^\infty \frac{\ln x}{x^{p-1}} \cdot \frac{1}{x}\,dx$$

(Anta $p \neq 1$)

(når $p = 1$: $\int_1^\infty \frac{\ln x}{x} = \int_1^e \frac{\ln x}{x} + \int_e^\infty \frac{\ln x}{x} \geq \int_1^e \frac{\ln x}{x} + \int_e^\infty \frac{1}{x} \to \infty$, så divergerer)

$$= \int_{\ln 1 = 0}^{\infty = \ln\infty} \frac{u}{(e^u)^{p-1}}\,du = \int_0^\infty u \cdot e^{u(1-p)}\,du$$

$$\left(\begin{array}{l} w = u \\ w' = e^{u(1-p)} \\ w = \frac{1}{1-p}e^{u(1-p)} \end{array}\right)$$

$$= \left[u \cdot \frac{1}{1-p} e^{u(1-p)}\right]_0^\infty - \int_0^\infty \frac{1 \cdot e^{u(1-p)}}{1-p}\,du$$

$$= \left[u \cdot \frac{1}{1-p} e^{u(1-p)}\right]_0^\infty - \left[\frac{1}{(1-p)^2} e^{u(1-p)}\right]_0^\infty$$

$$= \left[\left(u\left(\frac{1}{1-p}\right) - \frac{1}{(1-p)^2}\right) e^{u(1-p)}\right]_0^\infty$$

$$= \lim_{u \to \infty} \left(u\left(\frac{1}{1-p}\right) - \frac{1}{(1-p)^2}\right) \cdot e^{u(1-p)} + \frac{1}{(1-p)^2} e^0$$

$$= \begin{cases} \infty & \text{hvis } 1-p > 0 \text{ divergerer} \\ < \infty & \text{hvis } 1-p < 0 \text{ konvergerer} \end{cases}$$

$$L = \lim_{u \to \infty} \left(\left(\frac{1}{1-p}\right) \cdot u - \frac{1}{(1-p)^2}\right) \cdot e^{u(1-p)}$$

hvis $(1-p) > 0$ så går $e^{u(1-p)} \to \infty$

og $\to \infty$, så grensen går mot ∞.

Om $(1-p) < 0$ så går $e^{u(1-p)} \to 0$

$$L = \lim_{u \to \infty} \frac{\left(u\left(\frac{1}{1-p}\right) - \left(\frac{1}{1-p}\right)^2\right)}{\frac{1}{e^{u(1-p)}}}$$

$$\overset{L'H}{=} \lim_{u \to \infty} \frac{\frac{1}{(1-p)}}{\left(\frac{-(1-p)}{e^{u(1-p)}}\right)} \to 0$$

Så grensen er 0 og integralet konvergerer.

9.5.14 La $f(x) = \frac{1}{x}$, $1 \leq x < \infty$

og drei $f(x)$ om x-aksen.

a) Vis at Gabriels trompet har endelig volum.

$$V = \int_1^\infty \pi \cdot f(x)^2 \, dx = \pi \int_1^\infty \frac{1}{x^2} \, dx$$

$$= \pi \left[-\frac{1}{x} \right]_1^\infty = \underline{\underline{\pi}}$$ er volumet til trompeten.

b) Vis at trompeten har uendelig overflate.

Vi har oppgitt:

$$A = 2\pi \int_1^\infty f(x) \sqrt{1 + f'(x)^2} \, dx = 2\pi \int_1^\infty \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} \, dx$$

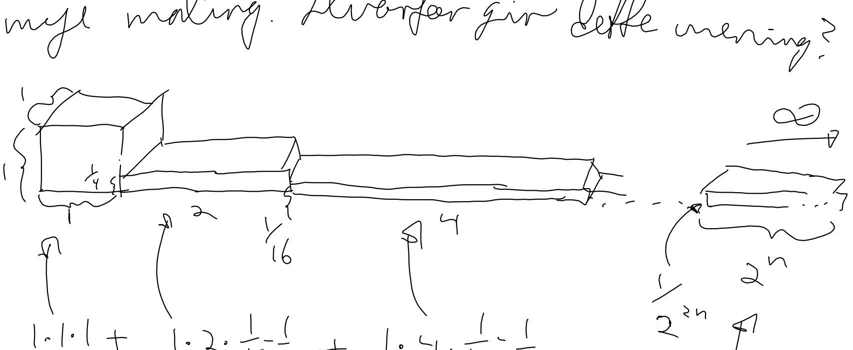
$$\geq 2\pi \int_1^\infty \frac{1}{x} \quad \left(\text{siden } 1 \leq \sqrt{1 + \frac{1}{x^4}} \text{ for alle } x \right)$$

$\downarrow$

∞

Så overflaten til trompeten er ikke endelig.

c) Vi maler den uendelige insiden av trompeten med å fylle trompeten med endelig mye maling. Hvorfor gir dette mening?



$$1 \cdot 1 \cdot 1 + 1 \cdot 2 \cdot \frac{1}{4} = \frac{1}{2} + 1 \cdot 4 \cdot \frac{1}{16} = \frac{1}{4} + \quad 1 \cdot 2^n \cdot 2^{-2n} = \frac{1}{2^n}$$

$$= 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots = 2$$

9.3, 17

Beregn $\int \frac{dx}{x^3+8}$

Geometrisk rekke: $x^3+8 = x^3-(-2)^3 = (x-(-2))\cdot$
$(x^2+(-2)\cdot x+(-2)^2) = (x+2)(x^2-2x+4)$

$$\frac{1}{x^3+8} = \frac{1}{(x+2)(x^2-2x+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2-2x+4}$$

$$\Rightarrow \frac{Ax^2-2Ax+4A+Bx^2+Cx+2Bx+2C}{x^3+8}$$

$(A+B)x^2 = 0 \quad \Rightarrow A = -B$

$(-2A+C+2B)x = 0 \quad \Rightarrow -2A+C-2A = 0$
$\Rightarrow C = 4A$

$4A+2C = 1 \quad \Rightarrow 4A+8A = 1$
$\Rightarrow A = \frac{1}{12} \Rightarrow B = -\frac{1}{12}$
$C = \frac{1}{3}$

$$\Rightarrow \frac{1}{x^3+8} = \frac{1}{12}\left(\frac{1}{x+2} + \frac{-x+4}{x^2-2x+4}\right)$$

$$\Rightarrow \int \frac{dx}{x^3+8} = \frac{1}{12}\int\frac{dx}{x+2} + \frac{1}{12}\int\frac{4-x}{x^2-2x+4}$$

$\frac{1}{12}\ln|x+2|$

$$\int\frac{4-x}{x^2-2x+4} = -\frac{1}{2}\int\frac{2x-8}{x^2-2x+4} = -\frac{1}{2}\int\frac{2x-2-6}{x^2-2x+4}$$

$$\begin{pmatrix} u = (x^2-2x+4) \\ u' = 2x-2 \end{pmatrix}$$

$$= -\frac{1}{2}\int\frac{2x-2}{x^2-2x+4}dx + \frac{6}{2}\int\frac{dx}{x^2-2x+1+3}$$

$u = x^2-2x+4$

$$= -\frac{1}{2}\int\frac{u'}{u}\cdot dx + \frac{3}{3}\int\frac{dx}{\left(\frac{x-1}{\sqrt{3}}\right)^2+1}$$

$$= -\frac{1}{2}\ln|u| + \int\frac{\sqrt{3}\,dv}{v^2+1} \qquad v = \frac{x-1}{\sqrt{3}},\quad v' = \frac{1}{\sqrt{3}}$$

$$= -\frac{1}{2}\ln|x^2-2x+4| + \sqrt{3}\arctan v$$

$$= -\frac{1}{2}\ln|x^2-2x+4| + \sqrt{3}\cdot\arctan\left(\frac{x-1}{\sqrt{3}}\right)$$

$$\int\frac{1}{x^3+8} = \frac{1}{12}\left(\ln|x+2| - \frac{1}{2}\ln|x^2-2x+4| + \sqrt{3}\arctan\left(\frac{x-1}{\sqrt{3}}\right)\right) + C$$

9.3.25a) Vis at $2+i$ er en kompleks rot for $f(z) = z^3 - 11z + 20$

$$f(2+i) = (2+i)^3 - 11\cdot(2+i) + 20$$

$$= 2^3 + 3\cdot 2^2\cdot i + 3\cdot 2\cdot i^2 + i^3 \quad - 22 - 11i + 20$$

$$8 + 12i - 6 - i - 22 - 11i + 20 = 0$$

$= 2$, $= 22$, $= 0$

Finn de andre røttene til f.

Merk at f er et reelt polynom, så $\overline{2+i} = 2-i$ er en rot.

Videre vet vi at $f(z) = (z-r_1)(z-r_2)(z-r_3)$

$$= z^3 + \cdots \underbrace{- r_1\cdot r_2\cdot r_3}_{=20}, \quad r_1 = 2+i, \quad r_2 = 2-i, \quad r_1\cdot r_2 = 4+1 = 5$$

og $-r_1\cdot r_2\cdot r_3 = 20$

$$-5\cdot r_3 \Rightarrow r_3 = -4$$

Så den siste roten er -4.

og $f(z) = (z-(2+i))(z-(2-i))(z+4)$

$$= (z^2 - 4z + 5)(z+4)$$

Beregn $\displaystyle\int \frac{10x+3}{(x^3 - 11x + 20)} = \int \frac{10x+3}{(x^2-4x+5)(x+4)}$

$$= \int\left(\frac{A}{x+4} + \frac{Bx+C}{x^2-4x+5}\right) = I$$

$$\Rightarrow Ax^2 - 4Ax + 5A$$
$$Bx^2 + Cx + 4Bx + 4C = 10x + 3$$

$\Rightarrow A + B = 0 \quad \Rightarrow A = -B$

$-4A + C + 4B = 10 \quad \Rightarrow -4A + C - 4A = 10 \Rightarrow C = 8A + 10$

$5A + 4C = 3 \quad \Rightarrow 5A + 32A + 40 = 3 \Rightarrow 37A = -37 \Rightarrow A = -1 \Rightarrow B = 1 \Rightarrow C = 2$

$$I = \int\left(\frac{-1}{x+4} + \frac{x+2}{x^2-4x+5}\right)dx$$

$$= -\ln|x+4| + \frac{1}{2}\int\frac{2x-4+8}{x^2-4x+5}$$

$$= -\ln|x+4| + \frac{1}{2}\int\frac{(2x-4)\,dx}{x^2-4x+5} + \int\frac{4\,dx}{x^2-4x+4+1}$$

$u = x^2 - 4x + 5$, $u' = 2x - 4$

$$\frac{1}{2}\int\frac{1}{u}\,du + 4\int\frac{dx}{(x-2)^2+1}$$

$$= \frac{1}{2}\ln|u| + 4\cdot\arctan(x-2) + C$$

$$= \frac{1}{2}\ln|x^2-4x+5|$$

$$\Rightarrow I = -\ln|x+4| + \frac{1}{2}\ln|x^2-4x+5| + 4\arctan(x-2) + C$$

9.3.31 $\int 1 \cdot \ln(x^2+2x+10)\,dx$

$u' = 1 \Rightarrow u = x$
$v = \ln(x^2+2x+10)$

$= x \cdot \ln(x^2+2x+10) - \int x \cdot \frac{(2x+2)}{x^2+2x+10}$

$\int \frac{x^2+x}{x^2+2x+10} = \int \frac{x^2+2x+10-x-10}{x^2+2x+10}$

$= \int \left(1 + \frac{-x-10}{x^2+2x+10}\right) = x - \int \frac{x+10}{x^2+2x+10}$

$\int \frac{x+10}{x^2+2x+10} = \frac{1}{2}\int \frac{2x+2+18}{x^2+2x+10}$

"(sider
$2x+2+18$
$\parallel$
$2x+20 = 2(x+10)$
$\frac{2}{2} = x+10$)"

$= \frac{1}{2}\int \frac{2x+2}{x^2+2x+10}\,dx + \frac{18}{2}\int \frac{1}{x^2+2x+1+9}\,dx$

$\left(\begin{array}{l} u = x^2+2x+10 \\ u' = 2x+2 \end{array}\right)$

$= \frac{1}{2}\int \frac{1}{u}\,du + 9\int \frac{1}{(x+1)^2+3^2}\,dx$

$= \frac{1}{2}\ln|u| + \frac{9}{9} \cdot \int \frac{dx}{\left(\frac{x+1}{3}\right)^2+1}$

$= \frac{1}{2}\ln|x^2+2x+10| + 1 \cdot 3 \cdot \arctan\left(\frac{x+1}{3}\right) + C$

$$\int \ln(x^2+2x+10) =$$

$$2\left(-x + \frac{1}{2}\ln|x^2+2x+10) + 3\cdot\arctan\left(\frac{x+1}{3}\right)\right) + C$$

$$+ x\cdot\ln(x^2+2x+10)$$

$$= -2x + \ln|x^2+2x+10)$$

$$+ 6\arctan\left(\frac{x+1}{3}\right)$$

$$\underline{\underline{+ x\ln(x^2+2x+10)}}$$