Konjugasjon

Hvis $z = a+ib$, så er det konjugerte tallet

$$\bar{z} = a - ib.$$

Eksempel: $z = 3+4i$, $\bar{z} = 3-4i$

$z = 7-2i$, $\bar{z} = 7+2i$

Regneregler for konjugasjon:

(i) $\overline{z+w} = \bar{z} + \bar{w}$

(ii) $\overline{z-w} = \bar{z} - \bar{w}$

(iii) $\overline{zw} = \bar{z}\,\bar{w}$

(iv) $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$

Bevis (i) Anta $z = a+ib$ $\bar{z} = a-ib$

$w = c+id$ $\bar{w} = c-id$

$$\overline{z+w} = \overline{(a+ib)+(c+id)} = \overline{(a+c)+i(b+d)}$$

$$= (a+c) - i(b+d)$$

$$\bar{z}+\bar{w} = \overline{a+ib} + \overline{c+id} = a-ib + c-id$$

$$= (a+c) - i(b+d)$$

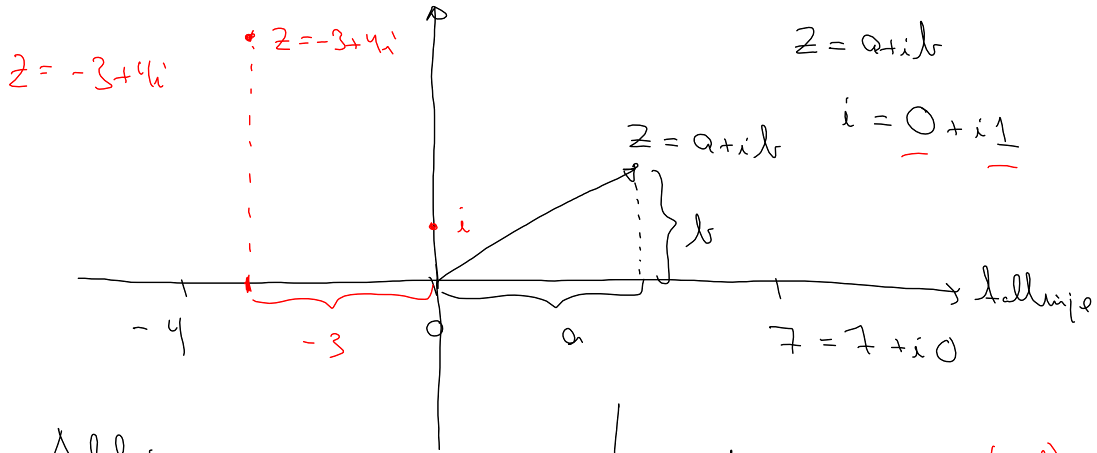
Geometrisk tolkning av komplekse tall

Forrige gang: formelle regneregler, fare for selvmotsigelser.

Eksempel: Husk $\sqrt{ab} = \sqrt{a}\sqrt{b}$ $\quad a, b > 0$

$$-1 = i^2 = \sqrt{(-1)}\sqrt{(-1)} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$$

Caspar Wessels komplekse plan

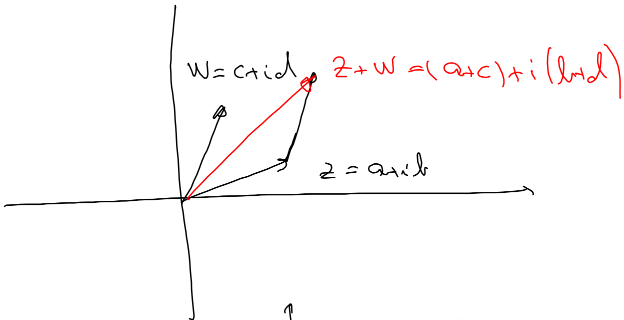


Addisjon

$z = a + ib$

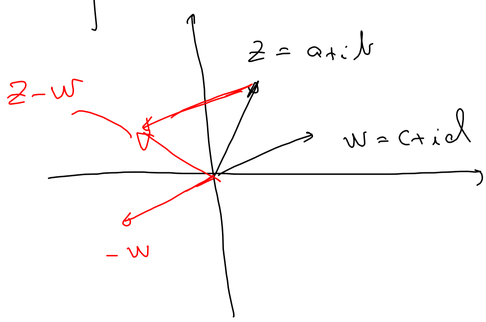
$w = c + id$

$z + w = (a+c) + i(b+d)$



Subtraksjon

$z - w = (a-c) + i(b-d)$

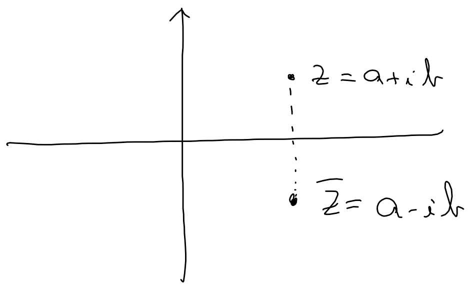


Konjugasjon

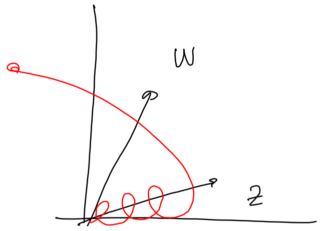
$z = a + ib$

$\bar{z} = a - ib$

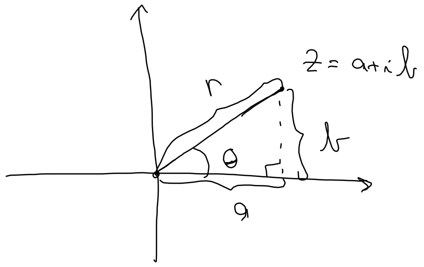
Speiling om x-aksen.



Men hva er den geometriske tolkningen av multiplikasjon og divisjon.



Polarform



$\Theta = \vartheta$ = theta : vinkel, argumentet til z

r – avstand

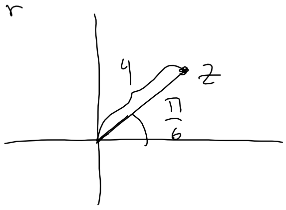
↳ modulus til z.

Hva er sammenhengen mellom a, b og r, ϑ?

$a = r\cos\vartheta$, $b = r\sin\vartheta$

$r = \sqrt{a^2+b^2}$, $\cos\vartheta = \frac{a}{r}$, $\sin\vartheta = \frac{b}{r}$

Eksempel: $r = 4$, $\vartheta = \frac{\pi}{6}$



$a = 4\cos\frac{\pi}{6} = 4\cdot\frac{\sqrt{3}}{2} = 2\sqrt{3}$

$b = 4\sin\frac{\pi}{6} = 4\cdot\frac{1}{2} = 2$

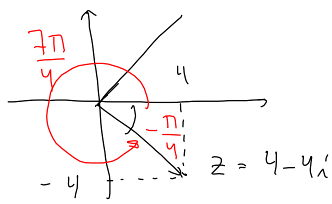
$z = 2\sqrt{3} + 2i$

Eksempel: $z = 4 - 4i$, vil finne r og ϑ

$r = \sqrt{a^2+b^2} = \sqrt{4^2+(-4)^2}$

$= \sqrt{16+16} = \sqrt{2\cdot 16} = \sqrt{2}\underbrace{\sqrt{16}}_{4}$

$= 4\sqrt{2}$



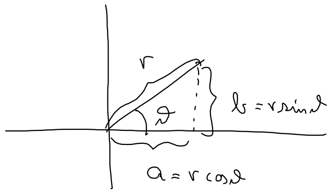
$\sin\vartheta = \frac{b}{r} = \frac{-4}{4\sqrt{2}} = -\frac{1}{\sqrt{2}} = -\frac{1\cdot\sqrt{2}}{\sqrt{2}\sqrt{2}} = -\frac{\sqrt{2}}{2}$, $\vartheta = -\frac{\pi}{4}$ siden z er i fjerde kvadrant

ϑ	$\sin\vartheta$	$\cos\vartheta$
0	0 $\frac{\sqrt{0}}{2}$	1
$\frac{\pi}{6}$	$1/2$ $\frac{\sqrt{1}}{2}$	$\sqrt{3}/2$
$\frac{\pi}{4}$	$\sqrt{2}/2$ $\frac{\sqrt{2}}{2}$	$\sqrt{2}/2$
$\frac{\pi}{3}$	$\sqrt{3}/2$ $\frac{\sqrt{3}}{2}$	$1/2$
$\frac{\pi}{2}$	1 $\frac{\sqrt{4}}{2}$	0

Oppsummering pluss litt til:

$z = a + ib = r\cos\vartheta + i\,r\sin\vartheta$

$= r(\cos\vartheta + i\sin\vartheta)$



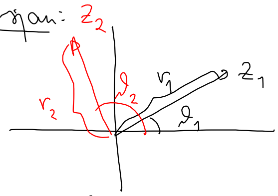
Husk: $\cos(u+v) = \cos u\cos v - \sin u \sin v$

$\sin(u+v) = \sin u \cos v + \cos u \sin v$

Geometrisk tolkning av multiplikasjon:

$z_1 = r_1\cos\vartheta_1 + i\,r_1\sin\vartheta_1$

$z_2 = r_2\cos\vartheta_2 + i\,r_2\sin\vartheta_2$



$$z_1 z_2 = (r_1\cos\vartheta_1 + i\,r_1\sin\vartheta_1)(r_2\cos\vartheta_2 + i\,r_2\sin\vartheta_2)$$

$$= r_1r_2\cos\vartheta_1\cos\vartheta_2 + i\,r_1r_2\cos\vartheta_1\sin\vartheta_2$$
$$+ i\,r_1r_2\sin\vartheta_1\cos\vartheta_2 - r_1r_2\sin\vartheta_1\sin\vartheta_2$$

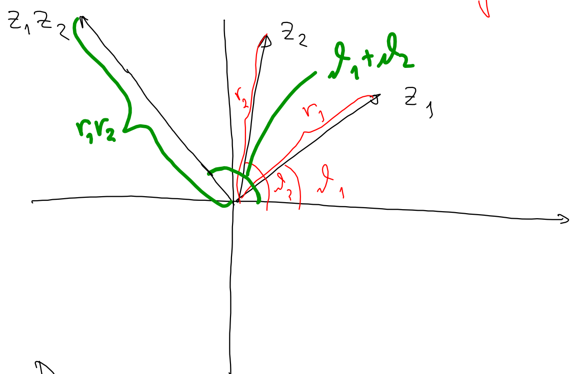
$$= r_1r_2(\overbrace{\cos\vartheta_1\cos\vartheta_2 - \sin\vartheta_1\sin\vartheta_2}^{\cos(\vartheta_1+\vartheta_2)})$$
$$+ i\,r_1r_2(\overbrace{\cos\vartheta_1\sin\vartheta_2 + \sin\vartheta_1\cos\vartheta_2}^{\sin(\vartheta_1+\vartheta_2)})$$

$$= \underbrace{r_1r_2}_{r}\cos(\underbrace{\vartheta_1+\vartheta_2}_{\vartheta}) + i\,\underbrace{r_1r_2}_{r}\sin(\underbrace{\vartheta_1+\vartheta_2}_{\vartheta})$$

$z = r\cos\vartheta$
$+ i\sin\vartheta$

Dette er et komplekst tall med modulus r_1r_2 og argument $\vartheta_1 + \vartheta_2$

Regel: Når vi ganger sammen to komplekse tall, multipliserer vi modulusene og adderer vinklene.



Divisjon: z_1 med polarkoordinater ϑ_1 og r_1

z_2 —"— ϑ_2 og r_2

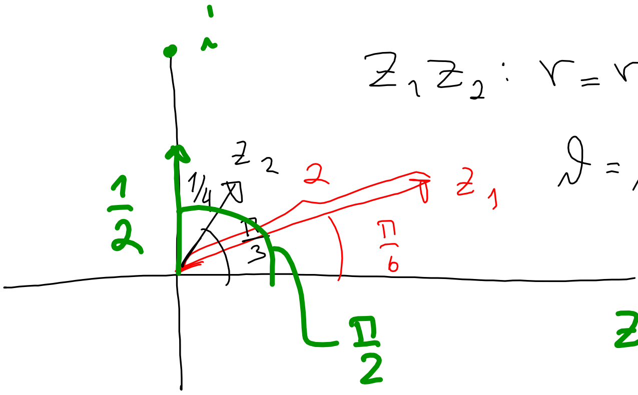
$\frac{z_1}{z_2}$ med polarkoordinater $\frac{r_1}{r_2}$ og $\vartheta_1 - \vartheta_2$

Eksempel: $z_1 : \; r_1 = 2 \,, \; \vartheta_1 = \frac{\pi}{6}$

$z_2 : \; r_2 = \frac{1}{4} \,, \; \vartheta_2 = \frac{\pi}{3}$

$z_1 z_2 : \; r = r_1 r_2 = 2 \cdot \frac{1}{4} = \frac{1}{2}$

$\vartheta = \vartheta_1 + \vartheta_2 = \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{6} + \frac{2\pi}{6} = \frac{3\pi}{6} = \frac{\pi}{2}$



$z_1 z_2 = \frac{1}{2} i$

Sjekk: $z_1 = r_1 \cos\vartheta_1 + i\, r_1 \sin\vartheta_1 = 2 \underbrace{\cos\frac{\pi}{6}}_{\frac{\sqrt{3}}{2}} + i\, 2 \underbrace{\sin\frac{\pi}{6}}_{\frac{1}{2}} = \sqrt{3} + i$

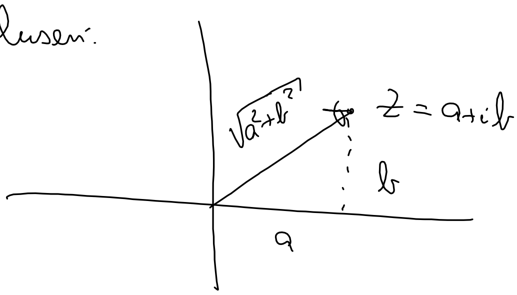
$z_2 = r_2 \cos\vartheta_2 + i\, r_2 \sin\vartheta_2$

$= \frac{1}{4} \underbrace{\cos\frac{\pi}{3}}_{\frac{1}{2}} + i\, \frac{1}{4} \underbrace{\sin\frac{\pi}{3}}_{\frac{\sqrt{3}}{2}} = \frac{1}{8} + i\, \frac{\sqrt{3}}{8}$

$z_1 z_2 = (\sqrt{3} + i)\left(\frac{1}{8} + i\frac{\sqrt{3}}{8}\right) = \cancel{\frac{\sqrt{3}}{8}} + i\,\frac{3}{8} + i\,\frac{1}{8} - \cancel{\frac{\sqrt{3}}{8}} = \frac{1}{2} i$

Absoluttverdien = tallverdien = modulusen.

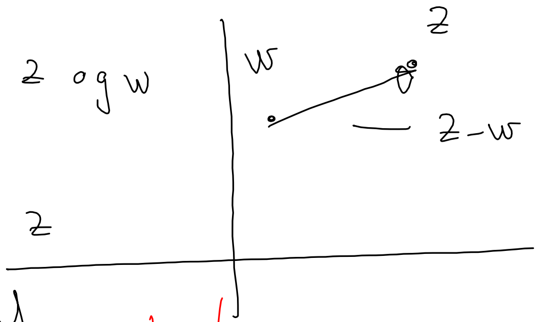
$$|z| = \sqrt{a^2+b^2}$$



$$|z|^2 = a^2+b^2 = z\bar{z}$$

$$z\cdot\bar{z} = (a+ib)(a-ib) = a^2 - i^2 b^2 = a^2+b^2$$

$|z-w|$ = avstanden mellom z og w



Eksempel: Finn de punktene z i det komplekse planet slik at

$$\underbrace{|z-i|}_{\text{avstanden fra } z \text{ til } i} < \underbrace{|z+1|}_{|z-(-1)|}$$

$\underbrace{|z-(-1)|}_{\text{avstanden fra } z \text{ til } -1}$