8.2.1 La $f: [1,2] \to \mathbb{R}$ være $f(x) = \frac{1}{x}$

La $\Pi = \{ 1, \frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \frac{9}{5}, 2 \}$
$\quad x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5$

Finn Øvre og Nedre trappesum.
$Ø(\Pi) \qquad N(\Pi)$

$$Ø(\Pi) = \sum_{i=1}^{5} M_i (x_i - x_{i-1})$$

M_i er maksimumsverdien til f på $[x_{i-1}, x_i]$

Så $M_i = x_{i-1}$.

$$Ø(\Pi) = \sum_{i=1}^{5} \frac{1}{x_{i-1}} \cdot \underbrace{(x_i - x_{i-1})}_{= \frac{1}{5}}$$

$$= \sum_{i=1}^{5} \frac{1}{x_{i-1}} \cdot \frac{1}{5}$$

$$= \frac{1}{5} \left(1 + 5 \cdot \left(\frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} \right) \right)$$

Tilsvarende er

$$N(\Pi) = \sum_{i=1}^{5} m_i (x_i - x_{i-1})$$

$$= \sum_{i=1}^{5} \frac{1}{x_i} \cdot \frac{1}{5}$$

$$= \frac{1}{5}\left(5\left(\frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9}\right) + \frac{1}{2}\right)$$

$$\left(\phi(\Pi) - N(\Pi) = \frac{1}{10} \right)$$

8.2.5 La $f: [0,1] \to \mathbb{R}$, $f(x) = x$.

For $n \in \mathbb{N}$ la $\Pi_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots \frac{n-1}{n}, 1\}$
$\{x_0, x_1, x_2, \dots x_n\}$

(merk: $x_k = \frac{k}{n}$)

$$\emptyset(\Pi_n) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}), \text{ så } M_i = x_i$$

$$N(\Pi_n) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}), \text{ så } m_i = x_{i-1}$$

(merk: $x_i - x_{i-1} = \frac{1}{n}$)

$$\text{så } \emptyset(\Pi_n) = \sum_{i=1}^{n} x_i \frac{1}{n} = \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{i=1}^{n} i$$

$$= \frac{1}{n^2} \cdot \frac{(n+1) \cdot n}{2} = \frac{1}{2}\left(1 + \frac{1}{n}\right)$$

$$\text{og } N(\Pi_n) = \sum_{i=1}^{n} x_{i-1} \cdot \frac{1}{n} = \sum_{i=1}^{n} \frac{(i-1)}{n} \cdot \frac{1}{n}$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} (i-1) = \frac{1}{n^2} \sum_{i=0}^{n-1} i = \frac{1}{n^2} \cdot \frac{(n-1) \cdot n}{2}$$

$$= \frac{1}{2}\left(1 - \frac{1}{n}\right)$$

b) Beregn $\lim_{n \to \infty} \emptyset(\Pi_n) = \overline{\int_0^1} x \, dx$

$$= \lim_{n \to \infty} \frac{1}{2}\left(1 + \frac{1}{n}\right) = \frac{1}{2}, \quad \text{og}$$

$$\lim_{n \to \infty} N(\Pi_n) = \underline{\int_0^1} x \, dx = \lim_{n \to \infty} \frac{1}{2}\left(1 - \frac{1}{n}\right) = \frac{1}{2}.$$

c) Siden f er (kontinuerlig) monoton så er f integrerbar ved setning 8.2.3.

Så f har integralet

$$\int_0^1 x \, dx = \overline{\int_0^1} x \, dx = \underline{\int_0^1} x \, dx = \underline{\underline{\frac{1}{2}}}.$$