9.1.3a $\int e^{-x}\cos x\,dx = (-e^{-x})\cdot\cos x - \int(-e^{-x})(-\sin x)\,dx$

$\begin{pmatrix} u' = e^{-x} \\ v = \sin x \end{pmatrix}$

$= -e^{-x}\cos x - \left[(-e^{-x})\cdot\sin x - \int(-e^{-x})\cdot\cos x\,dx\right]$

$\begin{pmatrix} u' = e^{-x} \\ v = \sin x \end{pmatrix}$

$= e^{-x}(\sin x - \cos x) - \int e^{-x}\cos x\,dx$

$\Rightarrow 2\int e^{-x}\cos x\,dx = e^{-x}(\sin x - \cos x) + C$

$\Rightarrow \int e^{-x}\cos x\,dx = \underline{\underline{e^{-x}\frac{(\sin x - \cos x)}{2} + C}}$

9.1.3b $\begin{pmatrix} u' = \sin x \\ v = \sin x \end{pmatrix}$

$\int \sin^2 x = (-\cos x)\cdot\sin x - \int(-\cos x)\cdot\cos x\,dx$

$= -\cos x\sin x + \int\cos^2 x\,dx = -\cos x\sin x$
$+ \int(1 - \sin^2 x)\,dx$

$\Rightarrow 2\int\sin^2 x\,dx = -\cos x\sin x + x + C$

$\Rightarrow \int\sin^2 x\,dx = \underline{\underline{\frac{x - \cos x\sin x}{2} + C}}$

(Eller bruk $\sin^2 x = \frac{1 - \cos 2x}{2}$
& $\sin 2x = 2\sin x\cos x$)

9.1.5 $\begin{pmatrix} u' = \frac{1}{x^2} \\ v = \ln x^2 \end{pmatrix}$

$\int\frac{\ln(x^2)}{x^2}\,dx = \left(-\frac{1}{x}\right)\cdot\ln x^2 - \int\left(-\frac{1}{x}\right)\cdot\frac{2}{x}\,dx$

$= -\frac{\ln x^2}{x} - \frac{2}{x} + C = -\frac{2}{x}(\ln x + 1) + C$

9.1.19 La $I_n = \int (\ln x)^n dx$

Vis at $I_n = x(\ln x)^n - n \cdot I_{n-1}$

Bevis: $\int 1 \cdot (\ln x)^n \begin{pmatrix} u' = 1 \\ v = (\ln x)^n \end{pmatrix} = x(\ln x)^n$

$- \int \cancel{x} \cdot n(\ln x)^{n-1} \cdot \frac{1}{\cancel{x}} dx = x(\ln x)^n - n \int (\ln x)^{n-1} dx$

$= x(\ln x)^n - n \cdot I_{n-1}$

Finn I_3: $I_0 = x(+C)$

$I_1 = x \cdot \ln x - I_0 = x \cdot \ln x - x$ $\quad \int (\ln x)^2 dx$

$\boxed{I_2 = x(\ln x)^2 - 2I_1 = x(\ln x)^2 - 2 \cdot x \ln x + 2x}$

$I_3 = x(\ln x)^3 - 3 \cdot I_2 = x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C$

9.1.15. Finn volumet til omdreiningslegeme til $y = \ln x$ for $1 \leq x \leq 2$

Volum: $\pi \int_1^2 y^2 dx = \pi \int_1^2 (\ln x)^2 dx$

$= \pi \Big[x(\ln x)^2 - 2x\ln x + 2x \Big]_1^2$

$= \pi \Big[2(\ln 2)^2 - 2 \cdot 2 \cdot \ln 2 + 2 \cdot 2 - 0 + 0 - 2 \Big]$

$= 2\pi \left((\ln 2)^2 - 2\ln 2 + 1 \right)$

$= 2\pi (\ln 2 - 1)^2$

9.2.3 a)

$$\int_0^{\sqrt{2}} x e^{x^2}\,dx \underset{\begin{pmatrix} u = x^2 \\ u' = 2x \end{pmatrix}}{=} \int_0^{\sqrt{2}} \frac{1}{2} u' e^{u}\,dx$$

$$= \int_0^{2} \frac{1}{2} e^{u}\,du = \left[\frac{1}{2} e^{u}\right]_0^2 = \underline{\underline{\frac{1}{2}\left(e^2 - 1\right)}}$$

b)

$$\int_1^{e} \frac{\ln x}{x}\,dx \overset{\begin{pmatrix} u = \ln x \\ u' = \frac{1}{x} \end{pmatrix}}{=} \int_1^{e} u(x) \cdot u'(x) \cdot dx$$

$$= \int_0^{1} u\,du = \left[\frac{1}{2} u^2\right]_0^1 = \underline{\underline{\frac{1}{2}}}$$

9.2.11 $\displaystyle\int \frac{x^3}{\sqrt{x^2+1}}\,dx \quad \begin{pmatrix} u = x^2+1 \Rightarrow x^2 = u-1 \\ u' = 2x \Rightarrow x^3 = \frac{1}{2}u'\cdot(u-1) \end{pmatrix} = \int \frac{\frac{1}{2}u'(u-1)}{\sqrt{u}}\,dx$

$$= \frac{1}{2}\int \frac{u-1}{\sqrt{u}}\,du = \frac{1}{2}\int\left(\sqrt{u} - \frac{1}{\sqrt{u}}\right)du$$

$$= \frac{1}{2}\left(\frac{2}{3}u^{3/2} - \frac{2}{1}u^{1/2}\right) = \frac{1}{3}u^{3/2} - u^{1/2} + C$$

$$= \frac{1}{3}(x^2+1)^{3/2} - (x^2+1)^{1/2} + C$$

9.2.23 grafen til $y = \arcsin x$ dreies om x-aksen. $(0 \leq x \leq 1)$

Finn volumet til omdreiningslegemet

$$V = \pi \int_0^1 (\arcsin x)^2 \, dx, \quad \int_0^1 (\arcsin x)^2 \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} \, dx$$

$$\left(\begin{matrix} u = \arcsin x \\ u' = \frac{1}{\sqrt{1-x^2}} \end{matrix} \Rightarrow \sin u = x \Rightarrow \sqrt{1-x^2} = \cos u \right)$$

$$= \int_0^1 u^2 \cdot \cos u \cdot u' \, dx = \int_0^{\pi/2} u^2 \cos u \, du$$

$v' = \cos u$
$w = u^2$

$$= \left[u^2 \cdot \sin u \right]_0^{\pi/2} - \int_0^{\pi/2} 2u \cdot \sin u \, du$$

$v' = \sin u$
$w = u$

$$= \left[u^2 \sin u \right]_0^{\pi/2} - 2 \left(\left[-\cos u \cdot u \right]_0^{\pi/2} - \int_0^{\pi/2} (-\cos u) \, du \right)$$

$$= \left(\frac{\pi}{2} \right)^2 \cdot 1 - 2 \left(0 - \left[-\sin u \right]_0^{\pi/2} \right)$$

$$= \left(\frac{\pi}{2} \right)^2 - 2 \cdot 1 = \frac{\pi^2}{4} - 2$$

$$\Rightarrow V = \pi \left(\frac{\pi^2}{4} - 2 \right) = \underline{\underline{\frac{\pi^3}{4} - 2\pi}}$$

9.2.28

La f være en kontinuerlig og strengt monoton på $[a,b]$, la g være den omvendte funksjonen til f. og anta at f er deriverbar på (a,b)

Vis at $\int_{f(a)}^{f(b)} g(t)\,dt = b f(b) - a f(a) - \int_a^b f(x)\,dx$

Bevis: $\int_{f(a)}^{f(b)} 1\cdot g(t)\,dt \overset{\begin{pmatrix} u'=1 \\ v=g(t)\end{pmatrix}}{=} \left[t\, g(t)\right]_{f(a)}^{f(b)}$

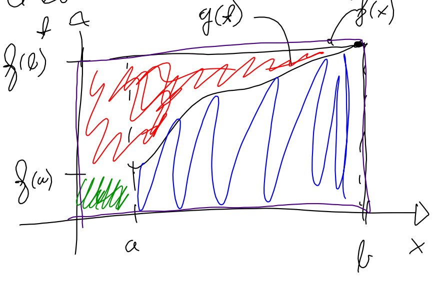
$$- \int_{f(a)}^{f(b)} t\, g'(t)\,dt \overset{\begin{pmatrix} u=g(t) \\ u'=g'(t)\end{pmatrix} \Rightarrow f(u)=t}{=} - \int_{f(a)}^{f(b)} f(u)\cdot u'\,dt + \left[t\, g(t)\right]_{f(a)}^{f(b)}$$

$$= \left[t\, g(t)\right]_{f(a)}^{f(b)} - \int_{g(f(a))=a}^{g(f(b))=b} f(u)\,du = f(b)\cdot g(f(b)) - f(a)\cdot g(f(a)) - \int_a^b f(u)\,du$$

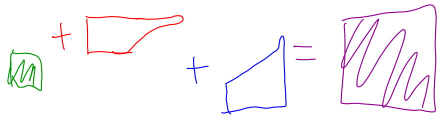
$$= f(b)\cdot b - f(a)\cdot a - \int_a^b f(u)\,du$$

Setning 9.2.7.

Geometrisk tolkning:



$= \int_{f(a)}^{f(b)} g(t)\,dt + \int_a^b f(x)\,dx + a\cdot f(a)$



$= b\cdot f(b)$

9.3.3a $\int \frac{2}{x^2+6x+10}\,dx = \int \frac{2}{(x+3)^2+1}\,dx$

$\begin{pmatrix} u=x+3 \\ u'=1 \end{pmatrix} \int \frac{2}{u^2+1}\,du = 2\cdot\arctan u + C$

$= \underline{\underline{2\arctan(x+3)+C}}$

b) $\int \frac{2x-2}{x^2+4x+8}\,dx = \int \frac{2(x+2-3)\,dx}{(x+2)^2+4}$

$\begin{pmatrix} u=x+2 \\ u'=1 \end{pmatrix} = \int \frac{2\cdot(u-3)}{u^2+4}\,du$

$= \int \frac{2u}{u^2+4}\,du - 2\cdot\frac{3}{4}\int \frac{1}{\left(\frac{u}{2}\right)^2+1}\,du$

$= \ln(u^2+4) - 2\cdot\frac{3}{4}\cdot 2\arctan\left(\frac{u}{2}\right) + C$

$= \ln(x^2+4x+8) - 3\cdot\arctan\left(\frac{x+2}{2}\right) + C$

9.3.5a $\int \frac{x^2+2x-3}{x+1}\,dx = \int \frac{x^2+2x+1-1-3}{x+1}\,dx$

$= \int \frac{(x+1)^2-4}{x+1}\,dx = \int \left((x+1) - \frac{4}{x+1}\right) dx$

$= \frac{x^2}{2} + x - 4 \cdot \ln|x+1| + C$

d) $\int \frac{3x^2+x}{(x-1)(x+1)^2}\,dx$, $u = (x-1)(x+1)^2$

$u' = (x+1)^2 + 2(x-1)\cdot(x+1)$

$= (x+1)(x+1+2x-2)$

$= (x+1)(3x-1)$

$= 3x^2+2x-1$

$= 3x^2+x+x-1$

$\Rightarrow u' - (x-1) = 3x^2+x$

$\int \frac{u' - (x-1)}{(x-1)(x+1)^2}\,dx$

$\int \frac{u'}{u} \cdot dx - \int \frac{(x-1)}{(x-1)(x+1)^2}\,dx$

$\int \frac{1}{u}\,du - \int \frac{1}{(x+1)^2}\,dx = \ln|u| + \frac{1}{x+1}$

$= \ln|(x-1)(x+1)^2| + \frac{1}{x+1} + C$

$= \ln|x-1| + 2 \cdot \ln|x+1| + \frac{1}{x+1} + C$

(Alternativt: gjør delbrøksoppspalting)

9.3.5.g) $\int \frac{-x^2+2x-1}{(x+1)(x^2+1)}\,dx$

Delbrøksoppspalting:

$\frac{-x^2+2x-1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$

$= \frac{Ax^2+A+Bx^2+Cx+Bx+C}{(x+1)(x^2+1)}$

$\Rightarrow$ $A+B=-1$, $B+C=2$, $A+C=-1$ $\Rightarrow B-C=0 \Rightarrow B=C$

$\Rightarrow B+B=2 \Rightarrow B=1=C$

$\Rightarrow A=-2$

$\int \frac{-x^2+2x-1}{(x+1)(x^2+1)}\,dx = \int \left(\frac{-2}{x+1} + \frac{x+1}{x^2+1}\right) dx$

$= -2\ln|x+1| + \frac{1}{2}\int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$

$= -2 \cdot \ln|x+1| + \frac{1}{2} \cdot \ln(x^2+1) + \arctan x + C$