Plenum 14/09-12

3.3: 10, 12 (1, 3, 7, 8)

3.4: 1b) d), 3a), 8, 9a), 11b), 15 a), b

3.5: 1a), b), 3a), 5, 9

(4.3: 1, 3a), b), d), 4, 11, 13, 14, 15)

Google: Løsningsforslag Kalkulus

3.3:

8) $(1+i)^{804}$: $1+i$: $r=\sqrt{1+1}=\sqrt{2}$, $\theta=\frac{\pi}{4}$

$1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

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$$(1+i)^{804}=\left(\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right)^{804}$$

$$=(\sqrt{2})^{804}\left(\cos\left(804\frac{\pi}{4}\right)+i\sin\left(804\frac{\pi}{4}\right)\right)$$

De Moivres formel

$$=\left((\sqrt{2})^2\right)^{402}\left(\cos(201\pi)+i\sin(201\pi)\right)$$

$$=2^{402}\left(\cos(\underbrace{2\cdot 100\pi}_{2\pi\cdot 100}+\pi)+i\sin(2\cdot 100\pi+\pi)\right)$$

$$=2^{402}\left(\cos(\pi)+i\sin(\pi)\right)$$

$$=2^{402}(-1+i0)=-2^{402}$$

3.4: 1) b) $z=-i$: $|z|=r=1$, $\theta=\frac{3\pi}{2}$

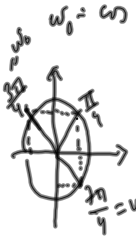
$w_0=z^{\frac{1}{2}}=\sqrt{1}e^{i\frac{3\pi}{2\cdot 2}}=e^{i\frac{3\pi}{4}}$

$w_+=e^{\frac{2\pi}{2}}=e^{i\pi}$

$w_1=w_0w_+=e^{i\frac{3\pi}{4}+i\pi}=e^{i\frac{7\pi}{4}}$

sep 14-12:23

$w_0=\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}=-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$



$w_1=\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}$

$=\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}$

$w_0=\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}=-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$

$w_1=\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}=\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}$

w_0

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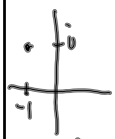
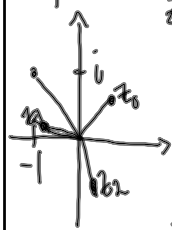
8.) a) $z^3=-1+i$: Finn alle 3dje røtter til

$w=-1+i$: $|w|=\sqrt{1+1}=\sqrt{2}$, $\theta=\frac{3\pi}{4}$,



$w=\sqrt{2}e^{i\frac{3\pi}{4}}$

$z_0=\left(\sqrt{2}e^{i\frac{3\pi}{4}}\right)^{\frac{1}{3}}=\sqrt[6]{2}e^{i\frac{\pi}{4}}$,



$w_+=e^{i\frac{2\pi}{3}}$

$z_1=z_0w_+=2^{\frac{1}{6}}e^{i\left(\frac{\pi}{4}+\frac{2\pi}{3}\right)}=2^{\frac{1}{6}}e^{i\frac{11\pi}{12}}$

$z_2=z_1w_+=2^{\frac{1}{6}}e^{i\left(\frac{11\pi}{12}+\frac{8\pi}{12}\right)}=2^{\frac{1}{6}}e^{i\frac{19\pi}{12}}$

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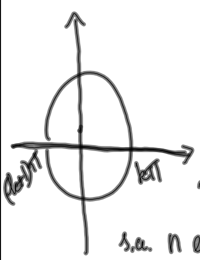
b) $w=2^{\frac{1}{6}}e^{i\frac{11\pi}{12}}$, $w^n=2^{\frac{n}{6}}e^{i\frac{11\pi n}{12}}$

For at w^n skal være reelt må $e^{i\frac{11\pi n}{12}}$ være reelt. Dvs. at $\frac{11\pi n}{12}$ må være lik $k\pi$ for et eller annet heltall k.



$\frac{11\pi n}{12}=k\pi$

$n=\frac{12k}{11}$

Vil velge k så liten som mulig s.a. n er naturlig, positivt tall. Vi må ha

$k=11 \Rightarrow n=12$

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9.) a) $x^2+2x+4=0$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-2\pm\sqrt{4-16}}{2}$$

$$=\frac{-2\pm\sqrt{-12}}{2}=\frac{-2\pm 2\sqrt{3}i}{2}=-1\pm\sqrt{3}i$$

15) a) $z^3+iz^2+z=0$

$z(z^2+iz+1)=0$

$z=0$ eller $z^2+iz+1=0$

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$$Z = \frac{-i \pm \sqrt{-1-4}}{2} = \frac{-i \pm \sqrt{-5}}{2}$$

$$= \frac{-i \pm \sqrt{5}i}{2}$$

Løsningene er $Z_0 = 0$, $Z_1 = \frac{-i + \sqrt{5}i}{2}$,

$Z_2 = \frac{-i - \sqrt{5}i}{2}$.

sep 14-12:57

b) $z^2 = 1 + \sqrt{3}i$, $w := 1 + \sqrt{3}i$,

$|w| = \sqrt{1+3} = 2$, $\begin{matrix} 2\cos\theta = 1 \\ 2\sin\theta = \sqrt{3} \end{matrix} \Rightarrow \begin{matrix} \cos\theta = \frac{1}{2} \\ \sin\theta = \frac{\sqrt{3}}{2} \end{matrix}$

$\Rightarrow \theta = \frac{\pi}{3} \Rightarrow w = 2e^{i\frac{\pi}{3}}$

$Z_0 = w^{\frac{1}{2}} = \underline{\sqrt{2}e^{i\frac{\pi}{6}}}$, $w_+ = e^{i\frac{2\pi}{2}} = e^{i\pi}$,

$Z_1 = Z_0 w_+ = \sqrt{2}e^{i\frac{\pi}{6}} \cdot e^{i\pi} = \underline{\sqrt{2}e^{i\frac{7\pi}{6}}}$

Løsningene er $Z_0 = \sqrt{2}e^{i\frac{\pi}{6}}$, $Z_1 = \sqrt{2}e^{i\frac{7\pi}{6}}$

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Merk: $Z_0 = \sqrt{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$= \sqrt{2}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$

$= \underline{\underline{\frac{\sqrt{6}}{2} + i\frac{\sqrt{2}}{2}}}$

Tilsvarende for Z_1.

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3.5: 3) a) $z^4 + 2z^2 + 1$:

$z^4 + 2z^2 + 1 = 0$

La $w := z^2$

$w^2 + 2w + 1 = 0$

Annengradsformel:

$w = \frac{-2 \pm \sqrt{4-4}}{2} = -1$ (multiplisitet 2)

$Z_1^2 = w = -1 \Rightarrow Z_1 = \pm i$

$Z_2^2 = w = -1 \Rightarrow Z_2 = \pm i$

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Kompleks faktorisering:

$z^4 + 2z^2 + 1 = (z-i)^2(z+i)^2$ Konjugat-setning

Reell faktorisering: Merk: $(z-i)(z+i) = z^2 + 1$

$z^4 + 2z^2 + 1 = (z^2+1)^2$

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5.) a) $P(z) = z^4 + 2z^3 + 4z^2 + 2z + 3$

$P(i) = i^4 + 2i^3 + 4i^2 + 2i + 3 = \cancel{1} - \cancel{2i} - 4 + \cancel{2i} + \cancel{3} = 0$

Dermed er i en rot i $P(z)$.

b) i er en rot i $P(z)$ og $P(z)$ er et reelt polynom, da gir Lemma 3.5.B at $-i$ også er en rot i $P(z)$.

Da må $P(z)$ være delelig med:

sep 14-13:31

$$(z-i)(z+i) = z^2+1$$

Gjør polynomdivisjon:

$$z^4+2z^3+4z^2+2z+3 : z^2+1 = \underline{z^2+2z+3}$$
$$-(z^4+z^2)$$
$$2z^3+3z^2+2z+3$$
$$-(2z^3+2z)$$
$$3z^2+3$$
$$-(3z^2+3)$$
$$0$$

Så:
$$P(z)=(z^2+1)(z^2+2z+3)$$

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Annengradsformel:

$$z = \frac{-2 \pm \sqrt{4-12}}{2} = \frac{-2 \pm \sqrt{-8}}{2}$$
$$= \frac{-2 \pm 2\sqrt{2}\,i}{2} = -1 \pm \sqrt{2}\,i$$

Kompleks faktorisering:

$$P(z) = (z-i)(z+i)(z-(-1+\sqrt{2}\,i))(z-(-1-\sqrt{2}\,i))$$

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Reell faktorisering:

$$P(z) = (z^2+1)(z^2+2z+3)$$

3.3:

10.) $\sin(z+w) = \sin z \cos w + \cos z \sin w$:

HS: $\sin z \cos w + \cos z \sin w$

$$= \left(\frac{e^{iz}-e^{-iz}}{2i}\right)\left(\frac{e^{iw}+e^{-iw}}{2}\right)+\left(\frac{e^{iz}+e^{-iz}}{2}\right)\left(\frac{e^{iw}-e^{-iw}}{2i}\right)$$

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$$= \frac{(e^{iz}-e^{-iz})(e^{iw}+e^{-iw})+(e^{iz}+e^{-iz})(e^{iw}-e^{-iw})}{4i}$$

$$= \frac{e^{i(z+w)} + \cancel{e^{i(z-w)}} - \cancel{e^{i(w-z)}} - e^{-i(z+w)} + e^{i(z+w)} - \cancel{e^{i(z-w)}} + \cancel{e^{i(w-z)}} - e^{-i(z+w)}}{4i}$$

$$= \frac{2e^{i(z+w)} - 2e^{-i(z+w)}}{4i} = \frac{e^{i(z+w)} - e^{-i(z+w)}}{\underline{2i}}$$

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VS:
$$\sin(z+w) = \frac{e^{i(z+w)} - e^{-i(z+w)}}{\underline{2i}}$$

Så HS = VS, og dermed er ligningen bevist.

12.) a) $\sum_{k=0}^{n} z^k = \frac{z^{n+1}-1}{z-1}$

$$\left((z-1)\sum_{k=0}^{n} z^k = z\sum_{k=0}^{n} z^k - \sum_{k=0}^{n} z^k \right)$$
$$= \dots = z^{n+1}-1$$

b) Velg $z = e^{i\theta}$ og bruk a).

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c) Hint:

Merk: Er nok å vise:

$$\underbrace{\frac{e^{i(n+1)\theta}-1}{e^{i\theta}-1}}_{\text{Fra b) er dette lik VS i det vi vil vise.}} = \underbrace{e^{i\frac{n\theta}{2}} \frac{\sin(\frac{n+1}{2}\theta)}{\sin(\frac{\theta}{2})}}$$

Regn ut m/ formler s. 121:

Siste steg -> Gang m/ $e^{i\frac{\theta}{2}}$ oppe og nede

sep 14-14:00