Løsningsforslag prøveeksamen Mat1100 5. des 2015

Oppgave 1

$$(1, 2i, 4+i, 9)\cdot(2, -i, 4+i, 1)$$
$$= 1\cdot\bar{2} + 2i\,\overline{(-i)} + (4+i)\overline{(4+i)} + 9\cdot\bar{1}$$
$$= 2 + 2i\cdot i + (4+i)(4-i) + 9$$
$$= 11 - 2 + 16 - i^2 = 9 + 16 + 1 = 26$$

$\boxed{D}$

Oppgave 2

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$
$$= \left(1 + \frac{1}{1+(xyz)^2}\cdot yz,\ 2y + \frac{1}{1+(xyz)^2}\cdot xz,\ \frac{1}{1+(xyz)^2}\cdot xy\right)$$

$$\overset{(1,1,1)}{=} \left(\frac{3}{2}, \frac{5}{2}, \frac{1}{2}\right)$$

$$f'(\vec{a};\vec{r}) = \nabla f(\vec{a})\cdot\vec{r} = \left(\frac{3}{2}, \frac{5}{2}, \frac{1}{2}\right)\cdot(1, -1, 2)$$
$$= \frac{3}{2} - \frac{5}{2} + 1 = 0$$

$\boxed{A}$

Oppgave 3

$$f(x,y,z) = e^{\cos(x^2 y)} \quad \text{gir}$$

$$\frac{\partial f}{\partial x} = e^{\cos(x^2 y)} \cdot \left(-\sin(x^2 y)\right) \cdot 2xy$$

$\boxed{B}$

Oppgave 4

$$\frac{1}{6}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 17 & 19 & 18 \end{vmatrix} = \frac{1}{6} \cdot 1 \cdot \begin{vmatrix} 1 & 0 \\ 19 & 18 \end{vmatrix} - 0 + 0$$

$$= \frac{1}{6} \cdot (1 \cdot 18 - 0) = 3$$

$\boxed{B}$

Oppgave 5

$$\int_0^{\pi/2} \cos^n x \overbrace{\sin x \, dx}^{-du} = -\int_1^0 u^n \, du = \int_0^1 u^n \, du$$

$u = \cos x$ gir $\frac{du}{dx} = -\sin x$

$du = -\sin x \, dx$

$$= \left[\frac{1}{n+1} u^{n+1}\right]_0^1$$

$$= \frac{1}{n+1}$$

$\boxed{C}$

Oppgave 6

$$f(x) = x \cdot 7^{1/x} = x \cdot \left(e^{\ln 7}\right)^{\frac{1}{x}} = x \cdot e^{\frac{\ln 7}{x}}$$

$\frac{\ln 7}{x} \to 0$

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} e^{\frac{\ln 7}{x}} = e^0 = 1$$

$$b = \lim_{x\to\infty} \left[f(x) - ax\right] = \lim_{x\to\infty} \left[f(x) - x\right]$$

$$= \lim_{x\to\infty} \left[x e^{\frac{\ln 7}{x}} - x\right]$$

$[\infty - \infty]$

$$= \lim_{x\to\infty} x\left[e^{\frac{\ln 7}{x}} - 1\right]$$

$[\infty \cdot 0]$

$$= \lim_{x\to\infty} \frac{e^{\frac{\ln 7}{x}} - 1}{\frac{1}{x}}$$

$\left[\frac{0}{0}\right]$

$$= \lim_{x\to\infty} \frac{e^{(\ln 7)/x} \cdot \left(-\frac{\ln 7}{x^2}\right)}{-\frac{1}{x^2}}$$

$$= \ln 7 \cdot \lim_{x\to\infty} e^{\frac{\ln 7}{x}} = \ln 7$$

Skråasymp: $\underline{y = x + \ln 7}$

$\boxed{C}$

Oppgave

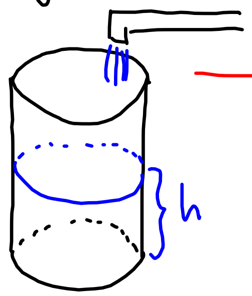
$$M^2 = \begin{array}{cc|cc} & & 2 & 1 \\ & & 5 & 1 \\ \hline 2 & 1 & 9 & 3 \\ 5 & 1 & 15 & 6 \end{array} \qquad M^3 = \begin{array}{cc|cc} & & 9 & 3 \\ & & 15 & 6 \\ \hline 2 & 1 & 33 & 12 \\ 5 & 1 & 60 & 21 \end{array}$$

$$3M + M^3 = \begin{bmatrix} 6 & 3 \\ 15 & 3 \end{bmatrix} + \begin{bmatrix} 33 & 12 \\ 60 & 21 \end{bmatrix} = \begin{bmatrix} 39 & 15 \\ 75 & 24 \end{bmatrix}$$

$$\det(3M + M^3) = 39 \cdot 24 - 15 \cdot 75 = \underline{\underline{-189}}$$

$\boxed{D}$

Oppgave 8



0,8 liter/sek

$$V = \pi r^2 \cdot h$$

$$V = \pi \cdot 20^2 \cdot h = 400\pi \cdot h$$

Tenker: $V(t) = 400\pi \cdot h(t)$

$$V'(t) = 400\pi \cdot h'(t)$$

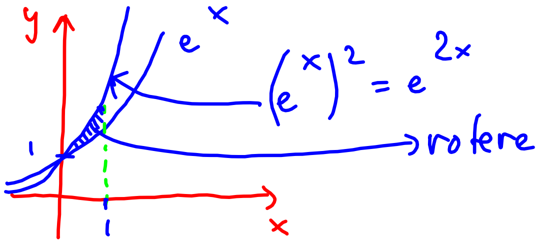
0,8 ℓ $= 800\ \text{cm}^3$

$$800 = 400\pi \cdot h'(t)$$

$$h'(t) = \frac{2}{\pi} \quad (\text{cm/sek}).$$

$\boxed{B}$

Oppgave 9



$$V_1 = \int_0^1 \pi \left[e^x\right]^2 dx$$

$$= \pi \int_0^1 e^{2x}\, dx \overset{\text{regn}}{=} \frac{\pi}{2}(e^2 - 1)$$

$$V_2 = \int_0^1 \pi \left[e^{2x}\right]^2 dx \overset{\text{regn}}{=} \frac{\pi}{4}\left[e^4 - 1\right]$$

Så $V = V_2 - V_1 \overset{\text{regn mye}}{=} \frac{\pi}{4}(e^2 - 1)^2$

$\boxed{A}$

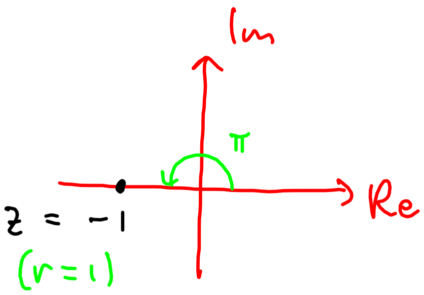
Oppgave 10

$$\lim_{x\to\infty}\frac{f(x)}{x^3} = \lim_{x\to\infty}\frac{\int_0^x \sqrt{1+t^4}\,dt}{x^3} \qquad \text{(ved fund. teo.)}$$

$$\left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty}\frac{\sqrt{1+x^4}}{3x^2} = \frac{1}{3}\lim_{x\to\infty}\sqrt{\frac{1+x^4}{x^4}}$$

$$= \frac{1}{3}\cdot\lim_{x\to\infty}\sqrt{\underbrace{\frac{1}{x^4}}_{\to 0}+1} = \frac{1}{3} \qquad \boxed{B}$$

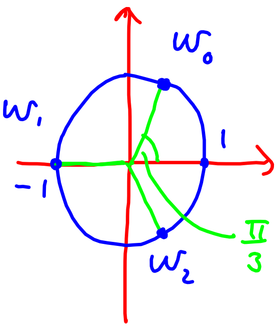
Oppgave 11



$z = -1$

$$w_0 = \sqrt[3]{1}\, e^{i(\pi/3)}$$

$$= 1\cdot e^{i(\pi/3)} = e^{i(\pi/3)}$$



$$w_+ = e^{i(2\pi/3)}$$

$$w_1 = w_+ w_0 = e^{i(2\pi/3)}\cdot e^{i(\pi/3)} = e^{i\pi}$$

$$w_2 = w_+ w_1 = e^{i(2\pi/3)}\cdot e^{i\pi} = e^{i(5\pi/3)}$$

Oppgave 12

$$f(x) = \begin{cases} x \ln|x| & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

a) $$\lim_{x \to 0} f(x) = \lim_{x \to 0} x \ln|x| \overset{[0\cdot\infty]}{=} \lim_{x \to 0} \frac{\ln|x|}{\frac{1}{x}} \overset{[\frac{\infty}{\infty}]}{=} \lim_{x \to 0} \frac{\frac{1}{x} \cdot x^2}{-\frac{1}{x^2} \cdot x^2}$$

$$= \lim_{x \to 0} (-x) = 0 = f(0)$$

Altså kontinuerlig i $x=0$

b) $$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h} \cdot \ln|h| - 0}{\cancel{h}} = -\infty$$

Altså ikke deriverbar i $x=0$

Oppgave 13

$$\int_1^\infty \frac{1}{1+x^2+\ln(1+x^2)}\,dx$$

Vet at $\int_1^\infty \frac{1}{x^2}\,dx$ konvergerer (p-integral med $p=2$)

Har $\frac{1}{1+x^2+\ln(1+x^2)} < \frac{1}{x^2}$ for $x > 0$

så integralet konvergerer ved sammenlikningstesten.

Oppgave 14

a) $$\int \frac{1}{x^2-x+1}\,dx = \frac{4}{3}\int \frac{1}{1+\left[\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right]^2}\,dx$$

Mellomregning:

$$\begin{aligned} x^2-x+1 &= x^2-x+\tfrac{1}{4}+\tfrac{3}{4} \\ &= \left(x-\tfrac{1}{2}\right)^2+\tfrac{3}{4} \\ &= \tfrac{3}{4}\left\{\tfrac{4}{3}\left(x-\tfrac{1}{2}\right)^2+1\right\} \\ &= \tfrac{3}{4}\left\{\left[\tfrac{2}{\sqrt{3}}\left(x-\tfrac{1}{2}\right)\right]^2+1\right\} \end{aligned}$$

$u = \frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)$

$du = \frac{2}{\sqrt{3}}\,dx$

$dx = \frac{\sqrt{3}}{2}\,du$

$$= \frac{4}{3}\int \frac{1}{1+u^2}\,\frac{\sqrt{3}}{2}\,du = \frac{4\cancel{\sqrt{3}}}{\cancel{\sqrt{3}}\cdot\sqrt{3}\cdot 2}\int\frac{1}{1+u^2}\,du$$

$$= \frac{2}{\sqrt{3}}\arctan u + C$$

$$= \frac{2}{\sqrt{3}}\arctan\left[\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right] + C$$

b) $$\int \frac{x-2}{x^2-x+1}\,dx = \int \frac{\frac{1}{2}(2x-1)+\frac{1}{2}-2}{x^2-x+1}\,dx$$

$$= \frac{1}{2}\int \frac{2x-1}{x^2-x+1}\,dx - \frac{3}{2}\int \frac{1}{x^2-x+1}\,dx$$

a)

$u = x^2 - x + 1$

$$= \frac{1}{2}\ln|x^2-x+1| - \frac{3}{2}\left(\text{svaret på a)}\right) + C$$

c) $\int \frac{1}{1+x^3}\,dx$ $\qquad x^3+1=0$ gir $x^3=-1$

løsn. $x=-1$

$(x^3+1):(x+1) \overset{\text{regn}}{=} x^2-x+1$

$x^2-x+1=0$ gir ingen løsn.

Delbrøksoppspalting: Vi har altså $x^3+1=(x+1)(x^2-x+1)$:

$$\frac{1}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

$$\boxed{1 = A(x^2-x+1) + (Bx+C)(x+1)}$$

$x=-1$ gir $1 = A(1+1+1)$, $A=\frac{1}{3}$.

x^2-ledd gir $B=-\frac{1}{3}$

konst.-ledd gir $1 = A + C$, dvs. $C = \frac{2}{3}$

Ergo

$$\int \frac{1}{1+x^3}\,dx = \frac{1}{3}\int \frac{1}{x+1}\,dx + \int \frac{-\frac{1}{3}x + \frac{2}{3}}{x^2-x+1}$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{3}\int \frac{x-2}{x^2-x+1}\,dx$$

$$= \underline{\underline{\frac{1}{3}\ln|x+1| - \frac{1}{3}\cdot\left(\text{svaret på b)}\right) + C}}$$