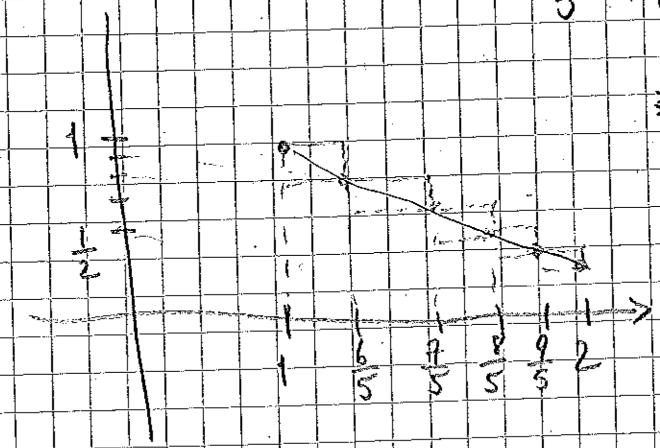
8.2: Definisjon av integralet

NB: $\Delta x = \frac{1}{5}$

1) $f: [1,2] \to \mathbb{R}$, $f(x) = \frac{1}{x}$, $\Pi = \{1, \frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \frac{9}{5}, 2\}$

øvre trappesum $= Ø(\Pi) = 1 \cdot \frac{1}{5} + \frac{\cancel{5}}{6} \cdot \frac{1}{\cancel{5}} + \frac{\cancel{5}}{7} \cdot \frac{1}{\cancel{5}} + \frac{\cancel{5}}{8} \cdot \frac{1}{\cancel{5}} + \frac{\cancel{5}}{9} \cdot \frac{1}{\cancel{5}}$

$= \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9}$



$\approx \underline{\underline{0,746}}$

nedre trappesum $= N(\Pi) = \frac{5}{6} \cdot \frac{1}{5} + \frac{5}{7} \cdot \frac{1}{5} + \frac{5}{8} \cdot \frac{1}{5} +$

$\frac{5}{9} \cdot \frac{1}{5} + \frac{1}{2} \cdot \frac{1}{5} \approx \underline{\underline{0,646}}$

5) $f: [0,1] \to \mathbb{R}$, $f(x) = x$. $\Pi_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\}$ $\to \Delta x = \frac{1}{n}$

a) $Ø(\Pi_n) = \frac{1}{n} \sum_{k=1}^{n} \frac{k}{n} = \frac{1}{n^2} \sum_{k=1}^{n} k \underset{\boxed{\text{oppg. } 1.2.1}}{=} \frac{1}{n^2} \cdot \frac{n^2+n}{2} = \underline{\frac{1}{2}\left(1 + \frac{1}{n}\right)}$

$N(\Pi_n) = \frac{1}{n} \sum_{k=0}^{n-1} \frac{k}{n} = \frac{1}{n^2} \sum_{k=0}^{n-1} k = \frac{1}{n^2} \sum_{k=1}^{n-1} k \underset{\text{oppg. } 1.2.1}{=} \frac{1}{2}\left(1 - \frac{1}{n}\right)$

b) $\int_0^1 x\,dx = \inf\{Ø(\Pi) : \Pi \text{ partisjon for } [0,1]\}$

$= \inf\{Ø(\Pi_n) : n \in \mathbb{N}\}$

$= \inf_{n \in \mathbb{N}} \left\{\frac{1}{2}\left(1 + \frac{1}{n}\right)\right\} = \frac{1}{2}$

Korollar 8.2.4: $f(x) = x$ er monoton (str. voksende) på $[0,1]$: Gjelder ikke bare for $\int$ men også $\overline{\int}$ og $\underline{\int}$ fra beregnet for korollaret

Tilsv: $\underline{\int_0^1} x\,dx = \sup_{n\in\mathbb{N}} \left\{\frac{1}{2}\left(1-\frac{1}{n}\right)\right\} = \frac{1}{2}$

c) Fra b er f integrerbar siden (evt siden f er monoton her)

$$\underline{\int_0^1} x\,dx = \int_0^1 x\,dx = \overline{\int_0^1} x\,dx = \frac{1}{2}$$

8.3: Analysens fundamentalteorem

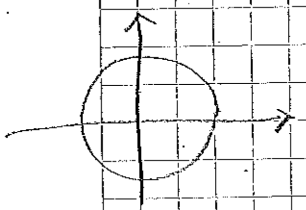
1) b) $\int_0^2 2x^3\,dx = \left[\frac{1}{2}x^4\right]_0^2 = \frac{1}{2}\left(2^4 - 0^4\right)$

$= 2^3 = \underline{\underline{8}}$

$V_{\arcsin} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

d) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx = \left[\arcsin x\right]_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{\pi}{6} - \left(-\frac{\pi}{6}\right)$

$= \frac{2\pi}{6} = \frac{\pi}{3}$



f) $\int_1^{\sqrt{3}} \frac{1}{1+x^2}\,dx = \left[\arctan x\right]_1^{\sqrt{3}}$

$= \frac{\pi}{3} - \frac{\pi}{4} = \frac{(4-3)\pi}{12} = \underline{\underline{\frac{\pi}{12}}}$

g) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin^2 x}\,dx = \left[-\cot x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = -\left(\frac{\cos\frac{\pi}{3}}{\sin\frac{\pi}{3}} - \frac{\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}}\right)$

$= -\left(\frac{\frac{1}{\cancel{2}}}{\frac{\sqrt{3}}{\cancel{2}}} - \frac{\frac{\sqrt{3}}{\cancel{2}}}{\frac{1}{\cancel{2}}}\right) = \sqrt{3} - \frac{1}{\sqrt{3}} = \underline{\underline{\frac{2\sqrt{3}}{3}}}$

3) d) $\int_0^{\frac{1}{2}} \frac{1}{1+4x^2}\,dx = \int_0^{\frac{1}{2}} \frac{1}{1+(2x)^2}\,dx$

$= \left[\frac{1}{2}\arctan 2x\right]_0^{\frac{1}{2}} = \frac{1}{2}\left(\frac{\pi}{4} - 0\right) = \underline{\underline{\frac{\pi}{8}}}$

e) $\int_0^1 \frac{1}{\sqrt{9-x^2}}\,dx = \int_0^1 \frac{1}{\sqrt{9\left(1-\frac{x^2}{9}\right)}}\,dx$

$= \int_0^1 \frac{1}{\sqrt{9}\sqrt{1-\frac{x^2}{9}}}\,dx = \int_0^1 \frac{1}{3\sqrt{1-\left(\frac{x}{3}\right)^2}}\,dx$

$= \frac{1}{3}\int_0^1 \frac{1}{\sqrt{1-\left(\frac{x}{3}\right)^2}}\,dx = \frac{1}{3}\left[3\arcsin\left(\frac{x}{3}\right)\right]_0^1$

$= \arcsin\frac{1}{3} - \arcsin 0 = \underline{\underline{\arcsin\frac{1}{3}}}$

5.) a) $f(x) = \int_0^x e^{-t^2}\,dt$

$f'(x) = \underline{\underline{e^{-x^2}}}$

Analysens fundamentalthm.

b) $f(x) = \int_1^x \frac{\sin t}{t}\,dt \Rightarrow f'(x) = \underline{\underline{\frac{\sin x}{x}}}$

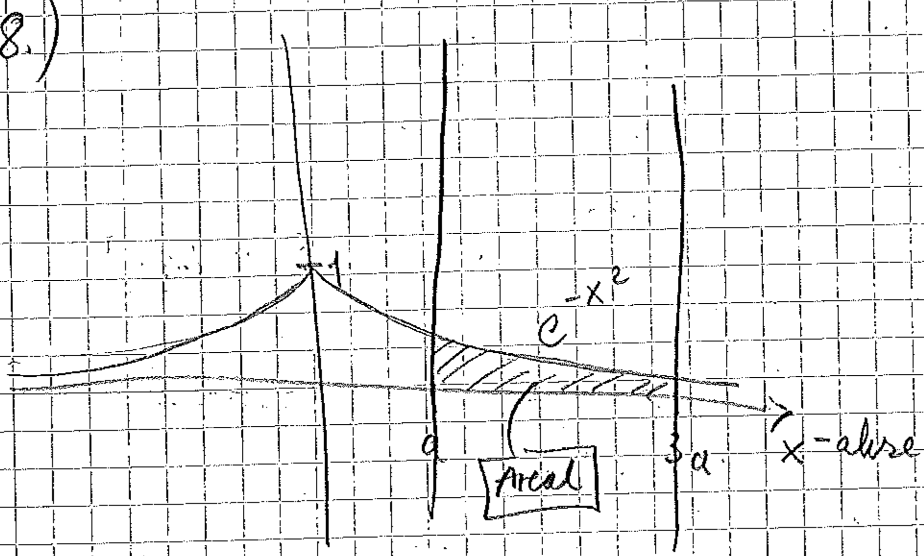
analysens fundamentalthm.

c) $f(x) = \int_1^x \arctan t^2\,dt$

$f'(x) = \underline{\underline{\arctan x^2}}$

analysens fundamentalthm

8.)



Hvorfor er $g(a) = \int_a^{3a} e^{-x^2} dx$ arealet på fig.?

$\int_a^b f(x)dx$ angir arealet mellom funksjonen f, x-aksen og linjene $x=a$, $x=b$ (fra def. av integralet som grense av Riemannsum), dermed er $\int_a^{3a} e^{-x^2} dx$ nettopp det oppgaven beskriver

$$\max_{a>0} \int_a^{3a} e^{-x^2} dx \; ?$$

$$\max_{a>0} \int_a^{3a} e^{-x^2} dx = \max_{a>0} \left(\int_0^{3a} e^{-x^2} dx - \int_0^{a} e^{-x^2} dx \right) = \max_{a>0} g$$

MERK: $F(a) := \int_0^a e^{-x^2} dx$ har derivert

$F'(a) = e^{-a^2}$ fra Analysens fundamentalthm.

$G(a) := \int_0^{3a} e^{-x^2} dx = F(3a)$. Bruker kjerneregelen til å derivere G:

$$G'(a) = F'(3a)\,3 = 3e^{-(3a)^2}$$

Dermed er

$$g'(a) = 3e^{-9a^2} - e^{-a^2}, \text{ og } g'(a) = 0 \iff$$

$$3e^{-9a^2} = e^{-a^2} \iff$$

$$e^{8a^2} = 3 \iff 8a^2 = \ln 3$$

$$a^2 = \frac{\ln 3}{8} \iff a = \sqrt{\frac{\ln 3}{8}} = \frac{\sqrt{\ln 3}}{2\sqrt{2}}$$

$\boxed{a > 0}$

$$= \frac{\sqrt{2}\,\sqrt{\ln 3}}{4}$$