Plenum 9/11-12

9.1: 1a, b, e, f, 5, 9, 11

9.2: 1 b, d, g, h, 3, 7, 9, 15, 23, 25

9.3: 1d, 3a, b, e(*), 5a, f, g, 9, 17, 21, 25, 27, 31

9.5: 1a, b, 3a, c, 6, 10

9.1: 5.) $\int \frac{\ln(x^2)}{x^2}\,dx = -\frac{\ln(x^2)}{x} - \int \frac{2}{x}\left(-\frac{1}{x}\right)dx$

$u(x) = \ln(x^2)$

$v'(x) = \frac{1}{x^2} = x^{-2}$

$u'(x) = \frac{1}{x^2}\,2x = \frac{2}{x}$

$v(x) = -x^{-1} = -\frac{1}{x}$

$$= \frac{-\ln(x^2)}{x} + 2\int x^{-2}dx = -\frac{\ln(x^2)}{x} + 2(-x^{-1}) + C$$

$$= -\frac{\ln(x^2)}{x} - \frac{2}{x} + C = -\frac{2}{x}(\ln(x) + 1) + C$$

$\ln(x^2)$
$= 2\ln x$

9.1) 9.) $\int \sin(\ln x)dx = \int 1 \cdot \sin(\ln x)dx$

$= x\sin(\ln x) - \int x\cos(\ln x)\frac{1}{x}dx$

$u(x) = \sin(\ln x)$
$v'(x) = 1$
$u'(x) = \cos(\ln x)\frac{1}{x}$
$v(x) = x$

$= x\sin(\ln x) - \int \cos(\ln x)dx$

M: $\int \cos(\ln x)dx = \int 1 \cdot \cos(\ln x)dx$

$= x\cos(\ln x) + \int x\sin(\ln x)\frac{1}{x}dx$

$= x\cos(\ln x) + \int \sin(\ln x)dx$

$$\int \sin(\ln x)dx = x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)dx$$

$$2\int \sin(\ln x)dx = x\sin(\ln x) - x\cos(\ln x) + C$$

$$\int \sin(\ln x)dx = \frac{1}{2}x\left(\sin(\ln x) - \cos(\ln x)\right) + C$$

11.) $\int \frac{x^2 \arctan x}{1+x^2}\,dx =$

$u(x) = \arctan x$

$v'(x) = \frac{x^2}{1+x^2}$

$\Downarrow$

$u'(x) = \frac{1}{1+x^2}$

$v(x) = x - \arctan x$

M: $\int \frac{x^2}{1+x^2}dx = \int \frac{(1+x^2)-1}{1+x^2}dx = \int 1\,dx - \int \frac{1}{1+x^2}dx$

$= x - \arctan x + D$

$= (x - \arctan x)\arctan x - \int \frac{x - \arctan x}{1+x^2}dx$

$= x\arctan x - \arctan^2 x - \int \frac{x}{1+x^2}dx + \int \frac{\arctan x}{1+x^2}dx$

$= x\arctan x - \arctan^2 x - \frac{1}{2}\ln(1+x^2)$

$+ \frac{1}{2}\arctan^2 x + C$

$= x\arctan x - \frac{1}{2}\underline{\underline{\arctan^2}} x - \frac{1}{2}\ln(1+x^2) + C$

9.2: 1) h) $\int \arcsin\sqrt{x}\,dx =$

$u = \sqrt{x}$
$du = \frac{1}{2\sqrt{x}}dx$
$2u\,du = dx$

$= \int \arcsin(u)\, 2u\,du$

$= 2\int u \arcsin u\,du$

$= 2\left(\frac{1}{2}u^2 \arcsin u\right.$

$v'(u) = u$
$w(u) = \arcsin u$
$\Downarrow$
$v(u) = \frac{1}{2}u^2$
$w'(u) = \frac{1}{\sqrt{1-u^2}}$

$\left. - \int \frac{1}{2}u^2 \frac{1}{\sqrt{1-u^2}}du\right)$

M: $\int \frac{-u^2}{\sqrt{1-u^2}}du$

$= \int \frac{1-u^2-1}{\sqrt{1-u^2}}du = \int \frac{1-u^2}{\sqrt{1-u^2}}du$

$- \int \frac{1}{\sqrt{1-u^2}}du = \int \sqrt{1-u^2}\,du - \arcsin u$

$= \int \sqrt{1-\sin^2 v}\,\cos v\,dv - \arcsin u$

$u = \sin v$
$du = \cos v\,dv$

$= \int \sqrt{\cos^2 v}\,\cos v\,dv - \arcsin u$

$= \int \cos^2 v\,dv - \arcsin u$

M2: $\int \cos^2 v\,dv = \int \frac{\cos 2v + 1}{2}dv$

$= \frac{1}{2}\left[\frac{1}{2}\sin 2v + v\right] + C$

$= \frac{1}{4}\sin 2v + \frac{1}{2}v + C$

M: $\int \frac{-u^2}{\sqrt{1-u^2}}du = \frac{1}{4}\sin(2\arcsin u)$

$+ \frac{1}{2}\arcsin u - \arcsin u + C$

$\int \arcsin(\sqrt{x})\,dx = u^2 \arcsin u +$

$u = \sqrt{x}$

$+ \frac{1}{4}\sin(2\arcsin u) - \frac{1}{2}\arcsin u + C$

$= x \arcsin\sqrt{x} + \frac{1}{4}\sin(2\arcsin\sqrt{x})$

$- \frac{1}{2}\arcsin\sqrt{x} + C$

3.) C) $\displaystyle\int_4^9 \frac{\sqrt{x}+1}{1-\sqrt{x}}\,dx = \int_2^3 \frac{(u+1)2u}{1-u}\,du$

$u=\sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2u\,du = dx$

$x=4 \Rightarrow u=2$

$x=9 \Rightarrow u=3$

$\displaystyle = -2\int_2^3 \frac{u^2+u}{u-1}\,du$

M: Polynom-div.

$\displaystyle = -2\int_2^3 \left(u+2+\frac{2}{u-1}\right)du$

$\displaystyle = -2\left[\frac{1}{2}u^2 + 2u + 2\ln|u-1|\right]_{u=2}^{3}$

$= \dots = -(9+4\ln 2)$

<u>M:</u> $u^2+u : u-1 = \underline{\underline{u+2+\frac{2}{u-1}}}$

$\underline{-(u^2-u)}$

$2u$

$\underline{-(2u-2)}$

2

15.) $\int_0^{\sqrt{3}} \frac{1+x}{\sqrt{4-x^2}}\,dx = \int_0^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx$

$+ \int_0^{\sqrt{3}} \frac{x}{\sqrt{4-x^2}}\,dx = \int_0^{\sqrt{3}} \frac{1}{\sqrt{4\left(1-\left(\frac{x}{2}\right)^2\right)}}\,dx$

$x=0 \Rightarrow u=4$
$x=\sqrt{3} \Rightarrow u=1$

$u = 4-x^2$
$du = -2x\,dx$
K: $\int \frac{x}{\sqrt{4-x^2}}\,dx = \int \frac{x}{\sqrt{u}}\left(-\frac{1}{2x}\right)du$
$= -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$

$+ \int_4^1 \left(-\frac{1}{2}\right)\frac{1}{\sqrt{u}}\,du$

$= \frac{1}{2}\int_0^{\sqrt{3}} \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\,dx + \frac{1}{2}\int_1^4 u^{-\frac{1}{2}}\,du$

$= \frac{1}{2}\left[\arcsin\left(\frac{x}{2}\right)2\right]_{x=0}^{\sqrt{3}} + \frac{1}{2}\left[2u^{\frac{1}{2}}\right]_{u=1}^{4}$

$= \arcsin\frac{\sqrt{3}}{2} - \arcsin 0 + 2 - 1$

$= \frac{\pi}{3} + 1$

9.3 : 5.) g) $\int \frac{-x^2+2x-1}{(x+1)(x^2+1)}\,dx$

Dbos: $\frac{-x^2+2x-1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$

$$-x^2+2x-1 = A(x^2+1)+(Bx+C)(x+1)$$
$$= x^2(A+B)+x(B+C) + (A+C)$$

$A+C=-1$, $B+C=2$, $A+B=-1$

$\Downarrow$ $\Downarrow$ $\Downarrow$

$C=-1-A \Rightarrow -1-A-1-A=2 \Leftarrow B=-1-A$

$$-2A-2=2$$
$$-2A=4 \Rightarrow A=-2$$

$B=C=-1-(-2)=\underline{1}$

$$\int\frac{-x^2+2x-1}{(x+1)(x^2+1)}\,dx = -2\int\frac{1}{x+1}\,dx + \int\frac{x+1}{x^2+1}\,dx$$

$$= -2\ln|x+1| + \int\frac{x}{x^2+1}\,dx + \int\frac{1}{x^2+1}\,dx$$

$$= -2\ln|x+1| + \frac{1}{2}\int\frac{2x}{x^2+1}\,dx + \arctan x$$

$$= -2\ln|x+1| + \frac{1}{2}\int\frac{1}{u}\,du + \arctan x$$

$u=x^2+1$
$du=2x\,dx$

$$= -2\ln|x+1| + \frac{1}{2}\ln(x^2+1) + \arctan x + C$$

9.) $\int \frac{x+1}{(x-1)(x^2+x+1)}\,dx$

$\to x = \frac{-1 \pm \sqrt{1-4}}{2}$ $\to$ Kan ikke faktoriseres mer!

Dbos: $\frac{x+1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$

$$x+1 = A(x^2+x+1) + (Bx+C)(x-1)$$
$$= x^2(A+B) + x(A-B+C) + (A-C)$$

$A+B=0$, $A-B+C=1$, $A-C=1$

$A=-B \Rightarrow -2B+C=1$

$C = 1+2B \Rightarrow -B-1-2B=1$

$-3B=2$

$A=\frac{2}{3}$, $C=-\frac{1}{3} \Leftarrow B=-\frac{2}{3}$

$$\int \frac{x+1}{(x-1)(x^2+x+1)}\,dx = \frac{2}{3}\int \frac{1}{x-1}\,dx$$

$$-\frac{1}{3}\int \frac{2x+1}{x^2+x+1}\,dx = \frac{2}{3}\ln|x-1|$$

$u=x^2+x+1$
$du=(2x+1)dx$

$$-\frac{1}{3}\int \frac{1}{u}\,du = \frac{2}{3}\ln|x-1| - \frac{1}{3}\ln(x^2+x+1) + C$$

$$= \frac{1}{3}\ln\left(\frac{(x-1)^2}{x^2+x+1}\right) + C$$

$2\ln|x-1|$
$= \ln((x-1)^2)$
... $\ln y - \ln z$

21.) a) $\int \frac{u+2}{u^2+2u+5}\,du = \frac{1}{2}\int \frac{2u+4}{u^2+2u+5}\,du$

$= \frac{1}{2}\int \frac{2u+2}{u^2+2u+5}\,du + \frac{2}{2}\int \frac{1}{u^2+2u+5}\,du$

$= \frac{1}{2}\int \frac{1}{v}\,dv + \int \frac{1}{(u+1)^2+4}\,du$

$v = u^2+2u+5$
$dv = (2u+2)\,du$

$= \frac{1}{2}\ln(u^2+2u+5) + \frac{1}{4}\int \frac{1}{\left(\frac{u+1}{2}\right)^2+1}\,du$

$= \frac{1}{2}\ln(u^2+2u+5) + \frac{1}{2}\arctan\left(\frac{u+1}{2}\right) + C$

b) $$\frac{1}{u(u^2+2u+5)} = \frac{A}{u} + \frac{Bu+C}{u^2+2u+5}$$

$$1 = A(u^2+2u+5) + (Bu+C)u$$
$$= u^2(A+B) + u(2A+C) + 5A$$

$A+B=0$, $2A+C=0$, $5A=1$

$\Downarrow$ $\Downarrow$ $\Downarrow$

$B = -\frac{1}{5}$ $C = -\frac{2}{5}$ $A = \frac{1}{5}$

c) $$\int \frac{\tan x}{\cos^2 x + 2\cos x + 5}\,dx = \int \frac{\sin x}{\cos x(\cos^2 x + 2\cos x + 5)}\,dx$$

$$\underset{\downarrow}{=} -\int \frac{1}{u(u^2+2u+5)}\,du \underset{\text{(b)}}{=} -\frac{1}{5}\int \frac{1}{u}\,du + \frac{1}{5}\int \frac{u+2}{u^2+2u+5}\,du$$

$u = \cos x$
$du = -\sin x\,dx$

$$\underset{\text{(a)}}{=} -\frac{1}{5}\ln|u| + \frac{1}{10}\ln(u^2+2u+5) + \frac{1}{10}\arctan\left(\frac{u+1}{2}\right) + C$$

$$= \frac{1}{10}\ln(\cos^2 x + 2\cos x + 5) + \frac{1}{10}\arctan\left(\frac{\cos x+1}{2}\right) - \frac{1}{5}\ln|\cos x| + C$$

9.5: 10.) For hvilke p konv.

$$\int_0^{\frac{1}{2}} \frac{1}{x|\ln x|^p}\,dx \text{ og } \int_2^{\infty} \frac{1}{x|\ln x|^p}\,dx \,?$$

$$\int_0^{\frac{1}{2}} \frac{1}{x|\ln x|^p}\,dx \underset{|}{=} \overbrace{\int_{-\infty}^{-\ln 2} \frac{1}{|u|^p}\,du}$$

$u = \ln x$
$du = \frac{1}{x}dx$
$x\,du = dx$

$x = 0 \Rightarrow u = -\infty$
$x = \frac{1}{2} \Rightarrow u = \ln\frac{1}{2} = -\ln 2$

$$\underset{\text{symmetri}}{=} \int_{\ln 2}^{\infty} \frac{1}{u^p}\,du$$

$$\underset{\ln 2<1}{=} \underbrace{\underbrace{\int_{\ln 2}^{1} \frac{1}{u^p}\,du}_{\text{Konv. for alle } p.} + \underbrace{\int_1^{\infty} \frac{1}{u^p}\,du}_{\substack{\text{Set. 9.5.4:} \\ \text{Konv. for } p>1, \\ \text{div. for } p\le 1}}}_{\substack{\text{Konv. for } p>1, \text{ og} \\ \text{div. for } p\le 1.}}$$

Så: $\int_0^{\frac{1}{2}} \frac{1}{x|\ln x|^p}\,dx$ konv. for $p>1$, og div. for $p \le 1$.

Andre integral: Samme metode og svar.

9.3: 31.) $\int \ln(x^2+2x+10)\,dx$

$$= x\ln(x^2+2x+10) - \int \frac{2x^2+2x}{x^2+2x+10}\,dx$$

Delvis int:

$u(x) = \ln(x^2+2x+10)$

$v'(x) = 1$

$u'(x) = \frac{1}{x^2+2x+10}(2x+2)$

$v(x) = x$

M: $\int \frac{2x^2+2x}{x^2+2x+10}\,dx$

Polynomdiv:

$$2x^2+2x : x^2+2x+10 = 2 - \frac{2x+20}{x^2+2x+10}$$

$$\int \frac{2x^2+2x}{x^2+2x+10}\,dx = \int 2\,dx - \int \frac{2x+20}{x^2+2x+10}\,dx$$

$$= 2x - \int \frac{2x+2}{x^2+2x+10}\,dx - 18\int \frac{1}{x^2+2x+10}\,dx$$

$$= 2x + \int \frac{1}{u}\,du - \frac{18}{9}\int \frac{1}{\left(\frac{x+1}{3}\right)^2+1}\,dx$$

$u = x^2+2x+10$
$du = (2x+2)\,dx$

$$= 2x + \ln|u| - 2\arctan\left(\frac{x+1}{3}\right)3 + C$$

$$= 2x + \ln(x^2+2x+10) - 6\arctan\left(\frac{x+1}{3}\right. + C$$

$$I = x\ln(x^2+2x+10) + 6\arctan\left(\frac{x+1}{3}\right) - 2x - \ln(x^2+2x+10) + C$$