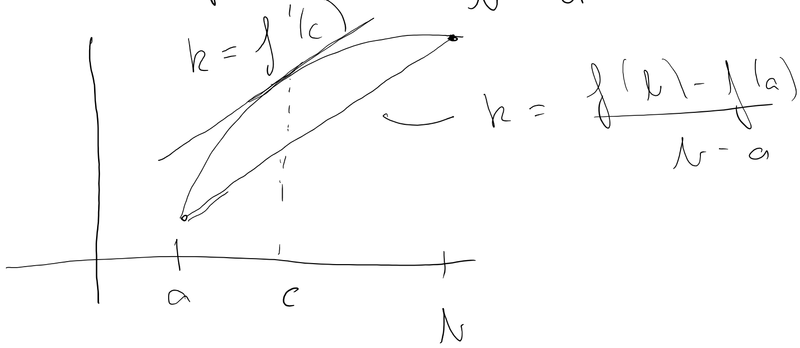
Middelverdisetningen

MVS: Hvis $f : [a,b] \to \mathbb{R}$ er kontinuerlig og deriverbar i alle indre punkter $x \in (a,b)$, så finnes det en $c \in (a,b)$ slik at

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Spørsmål: Hvis f er konstant, så er $f' = 0$. Hva med den andre veien? Hvis $f' = 0$ i alle punkter, er da funksjonen konstant.

Setning: Hvis $f'(x) = 0$ i alle punkter i et intervall I, så er f konstant på I.

Bevis: Velg et punkt $a \in I$. Det er nok å vise at for enhver $b \in I$, så er $f(b) = f(a)$. Ifølge MVS finnes det et punkt c mellom a og b slik at

$$\frac{f(b)-f(a)}{b-a} = f'(c) = 0$$

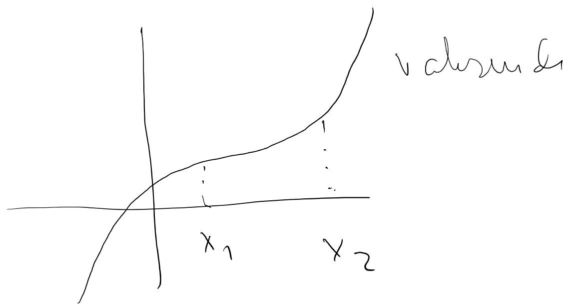
Altså er $f(b)-f(a)=0$, dvs $f(b)=f(a)$.

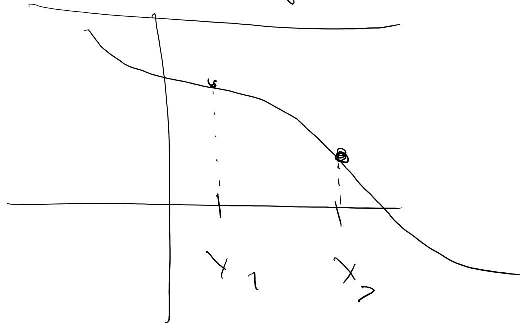
Definisjon: En funksjon $f: A \to \mathbb{R}$ kalles **voksende** dersom

$$x_1 \le x_2 \implies f(x_1) \le f(x_2)$$

f er **avtagende** dersom

$$x_1 \le x_2 \implies f(x_1) \ge f(x_2)$$





Setning: Anta at $f'(x) \ge 0$ for alle x i et intervall I. Da er f voksende på I. Tilsvarende, hvis $f'(x) \le 0$ for alle $x \in I$, så er f avtagende på I.

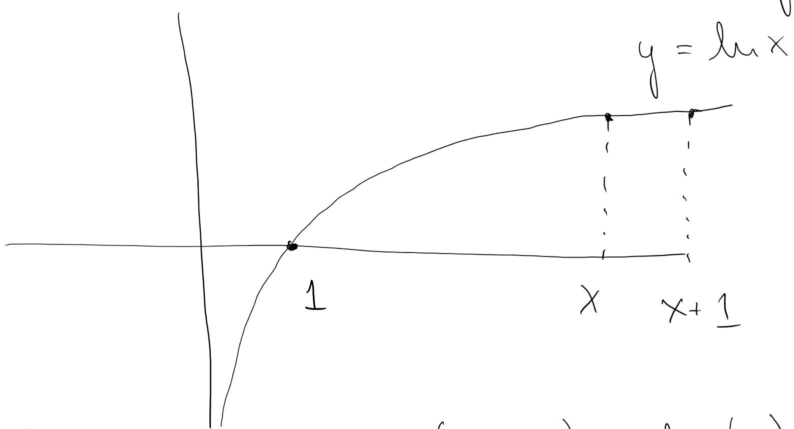
Bevis for voksende funksjon: Anta at $f'(x) \ge 0$ for alle $x \in I$. Anta videre at $x_1 \leq x_2$. Da sier MVS

$$\frac{f(x_2)-f(x_1)}{\underbrace{x_2-x_1}_{\geq 0}} = f'(c) \geq 0$$

≥ 0

Dette betyr at $f(x_2)-f(x_1) \geq 0$, dvs $f(x_2) \geq f(x_1)$.

Eksempel på ulikheter: $f(x) = \ln x$



$\ln(x+1) - \ln(x)$

Påstand: $\ln(x+1) - \ln(x) < \frac{1}{x}$

Bevis: Bruker MVS på $f(x) = \ln x$ over intervallet fra x til $x+1$.

$$\frac{f(x+1)-f(x)}{(x+1)-x} = f'(c), \quad c \in (x, x+1)$$

$$f'(c) = \frac{1}{c}$$

$$\frac{\ln(x+1)-\ln(x)}{1} = \frac{1}{c} < \frac{1}{x}$$

$$\ln(x+1) - \ln(x) < \frac{1}{x}$$

Hva er styrken i MVS?

$$\underline{f(x+h) - f(x) \approx f'(\underline{x})\, h} \longleftarrow$$

Bruker MVS på intervallet $[x, x+h]$:

$$\frac{f(x+h) - f(x)}{\underbrace{(x+h) - x}_{h}} = f'(c) \qquad \underline{c \in (x, x+h)}$$

$$\underline{f(x+h) - f(x) = f'(\underline{c})\, h}$$

6.3 L'Hôpitals regel

Hvis $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, så vet vi i utgangspunktet ingen ting om

$$\left.\lim_{x \to a} \frac{f(x)}{g(x)} \right\} \text{"}\frac{0}{0}\text{"}$$

Ubestemte uttrykk: "$\frac{0}{0}$", "$\frac{\infty}{\infty}$", "$0 \cdot \infty$"

"$\infty - \infty$", "1^{∞}", "0^{0}"

Idé: $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

$$\boxed{\begin{array}{c} f(x), g(x) \\ \downarrow \\ 0 \\ f(x), g(x) \\ \downarrow \\ \infty \end{array}}$$

L'Hôpitals regel: Anta at enten

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$$

eller

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty$$

Anta videre at $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ eksisterer

(OK at grensen er $+\infty$ eller $-\infty$). Da er

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

<u>ADVARSEL</u>: Det kan tenkes at

$\lim_{x \to a} \frac{f(x)}{g(x)}$ eksisterer selv om

$\lim_{x \to a} \frac{f'(x)}{g'(x)}$ <u>ikke</u> gjør det.

<u>Eksempel</u>: $\lim_{x \to 0} \frac{e^x - 1}{\sin x}$ "$\frac{0}{0}$"

$$\overset{L'H}{=} \lim_{x \to 0} \frac{e^x}{\cos x} = \frac{e^0}{\cos 0} = \frac{1}{1} = \underline{\underline{1}}$$

<u>Eksempel</u>: $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$ "$\frac{0}{0}$"

$$\overset{L'H}{=} \lim_{x \to 0} \frac{e^x - 1}{2x} \overset{L'H}{=} \lim_{x \to 0} \frac{e^x}{2} = \frac{e^0}{2} = \underline{\underline{\frac{1}{2}}}$$

<u>Eksempel</u>: $\lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}}$ "$\frac{-\infty}{\infty}$"

$$\overset{L'H}{=} \lim_{x \to 0^+} \frac{\frac{1}{x} \cdot x^2}{-\frac{1}{x^2} \cdot x^2} = \lim_{x \to 0^+} \frac{x}{-1} = \underline{\underline{0}}$$

Kan skrive $\dfrac{\ln x}{\frac{1}{x}} = \underbrace{x}_{\downarrow\, 0}\ \underbrace{\ln x}_{\downarrow\, -\infty}$

<u>Eksempel:</u> "$0 \cdot \infty$" $\displaystyle\lim_{x \to 0^+} \underbrace{\sin x}_{\downarrow\, 0}\ \underbrace{\ln x}_{\downarrow\, -\infty}$

$$= \lim_{x\to 0^+} \frac{\overset{\nearrow -\infty}{\ln x}}{\underset{\nwarrow \infty}{\frac{1}{\sin x}}} \overset{L'H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{\sin^2 x}\cos x}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{\cos x}{\sin^2 x}} = -\lim_{x\to 0^+} \frac{1}{x}\,\frac{\sin^2 x}{\cos x}$$

$$\frac{\sin x}{x} \to 1$$

$$= -\lim_{x\to 0^+} \frac{\sin^2 x}{x\cos x} = -\lim_{x\to 0} \underbrace{\frac{\sin x}{x}}_{\downarrow\, 1}\ \underbrace{\sin x}_{\downarrow\, 0} \cdot \underbrace{\frac{1}{\cos x}}_{\downarrow\, \frac{1}{1}=1}$$

$$= -1 \cdot 0 \cdot 1 = \underline{\underline{0}}$$

<u>Eksempel</u> "$\infty - \infty$"

$$\lim_{x\to\infty} \left[\underbrace{x\ln(x+1)}_{\downarrow\, \infty} - \underbrace{x\ln x}_{\downarrow\, \infty}\right]$$

$\ln(x+1) - \ln(x) < \frac{1}{x}$

$$= \lim_{x\to\infty} \underbrace{x}_{\downarrow\, \infty}\left[\underbrace{\ln(x+1) - \ln(x)}_{\downarrow\, 0}\right]$$

$$= \lim_{x\to\infty} \frac{\ln(x+1) - \ln(x)}{\frac{1}{x}} \qquad \text{"}\frac{0}{0}\text{"}$$

$$x \to \infty \qquad \frac{1}{x} \qquad 0$$

$$= \lim_{x \to \infty} \frac{\frac{1}{x+1} - \frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to \infty} \frac{\frac{x}{x(x+1)} - \frac{x+1}{x(x+1)}}{-\frac{1}{x^2}}$$

$$= \lim_{x \to \infty} \frac{\frac{-1}{x(x+1)}}{-\frac{1}{x^2}} = \lim_{x \to \infty} \frac{1}{\cancel{x}(x+1)} \cdot \frac{x^{\cancel{2}}}{1}$$

$$= \lim_{x \to \infty} \frac{x}{x+1} \overset{L'H}{=} \lim_{x \to \infty} \frac{1}{1} = \underline{\underline{1}}$$

<u>Eksempel</u>: "1^∞" $\lim_{n \to \infty} \underline{\left(1 + \frac{x}{n}\right)^n}$

$$= \lim_{n \to \infty} \left[e^{\ln\left(1+\frac{x}{n}\right)}\right]^n = \lim_{n \to \infty} e^{\underbrace{n \ln\left(1+\frac{x}{n}\right)}_{x}}$$

$\infty \cdot 0$

<u>Mellomregning</u>: $\lim_{n \to \infty} n \ln\left(1+\frac{x}{n}\right)$

$$= \lim_{n \to \infty} \frac{\ln\left(1+\frac{x}{n}\right)}{\frac{1}{n}} \overset{L'H}{=} \lim_{n \to \infty} \frac{\frac{1}{1+\frac{x}{n}}\left(-\frac{x}{n^2}\right)}{-\frac{1}{n^2}}$$

$$= \lim_{n \to \infty} \frac{1}{1+\frac{x}{n}} x = \underline{\underline{x}}$$

$\to 1$

Tilbake til utgangspunktet:

$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = \lim_{n \to \infty} e^{n \ln\left(1+\frac{x}{n}\right)} = \underline{\underline{e^x}}$$

$\to x$

<u>Konklusjon</u>:

$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

Eksempel: "0^0" $\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} \left(e^{\ln x}\right)^x$

$$= \lim_{x \to 0^+} e^{x \ln x}$$

$0 \cdot (-\infty)$

Mellomregning: $\lim_{x \to 0^+} x \ln x$

$$= \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} \overset{L'H}{=} \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \cdot \frac{x^2}{x^2}$$

$$= \lim_{x \to 0^+} \frac{x}{-1} = \underline{\underline{0}}$$

Tilbake til utgangspunktet:

$$\lim_{x \to 0^+} x^x = \lim_{x \to 0} e^{x \ln x} = e^0 = \underline{\underline{1}}$$

$x \ln x \to 0$