Seksjon 6.3

L'Hôpitals regel:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Gjelder også når grensen er $\pm\infty$.

når $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ eller ∞

Gjelder også for $a = \pm\infty$.

Oppgave 1

e) $\lim_{x \to 0} \frac{1-\cos(x)}{x^3} \overset{\text{l'hopital}}{=} \lim_{x \to 0} \frac{\sin(x)}{3x^2} \overset{\text{l'hopital}}{=} \lim_{x \to 0} \frac{\cos(x)}{6x}$

$= \infty.$ $= \infty.$

f) $\lim_{x \to 1} \frac{\ln(x) - x + 1}{(x-1)^2} \overset{\text{l'hopital}}{=} \lim_{x \to 1} \frac{\frac{1}{x} - 1}{2(x-1)}$

$= \lim_{x \to 1} \frac{1-x}{2x(x-1)}$

$= \lim_{x \to 1} \frac{-1}{2x} = \underline{\underline{-\frac{1}{2}}}$

3 d) $x = e^{\ln(x)}$

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln(x)} = e^{\lim_{x\to 0^+}(x\ln(x))} = e^0 = \underline{\underline{1}}$$

Regner ut $\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \dfrac{\ln(x)}{\frac{1}{x}}$

l'hopital

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0.$$

e) $\lim_{x\to\infty} \left(1+\sin\left(\frac{1}{x}\right)\right)^x$

$$= \lim_{x\to\infty} e^{x\ln(1+\sin(\frac{1}{x}))} = e^1 = \underline{\underline{e}}$$

$$\lim_{x\to\infty} x\ln\left(1+\sin\left(\tfrac{1}{x}\right)\right) = \lim_{x\to\infty} \frac{\ln\left(1+\sin\left(\frac{1}{x}\right)\right)}{\frac{1}{x}} \quad \left(\frac{0}{0}\right)$$

l'hopital

$$= \lim_{x\to\infty} \frac{\frac{\cos(\frac{1}{x})\cdot(-\frac{1}{x^2})}{1+\sin(\frac{1}{x})}}{-\frac{1}{x^2}}$$

stryke

$$= \lim_{x\to\infty} \frac{\cos(\frac{1}{x})}{1+\sin(\frac{1}{x})} = \frac{\cos(0)}{1+\sin(0)} = 1.$$

$$g)\ \lim_{x\to 1} \frac{x}{x-1} - \frac{1}{\ln(x)} = \lim_{x\to 1} \frac{x\ln(x) - (x-1)}{(x-1)\ln(x)}$$

$$= \lim_{x\to 1} \frac{x\ln(x) - x + 1}{(x-1)\ln(x)} \qquad \left(\frac{0}{0}\right)$$

l'hopital

$$= \lim_{x\to 1} \frac{\ln(x)}{\ln(x) + \frac{x-1}{x}}$$

$$= \lim_{x\to 1} \frac{1}{1 + \frac{x-1}{x\ln(x)}}$$

$$\lim_{x\to 1} \frac{x-1}{x\ln(x)} = \lim_{x\to 1} \frac{1}{\ln(x)+1} = 1$$

$$= \frac{1}{1+1} = \underline{\underline{\frac{1}{2}}}$$

9 $\lim_{x \to 0} (e^x + \sin x)^{\frac{1}{x}} = \lim_{x \to 0} e^{\frac{1}{x} \ln(e^x + \sin(x))} = e^2$

$$\lim_{x \to 0} \frac{\ln(e^x + \sin(x))}{x} \overset{\text{l'hopital}}{=} \lim_{x \to 0} \frac{\frac{e^x + \cos(x)}{e^x + \sin(x)}}{1}$$

$$= \frac{e^0 + \cos(0)}{e^0 + \sin(0)} = \underline{\underline{2}}$$

17 Finne tallet a slik at

$$\lim_{x\to\infty}\left(\frac{ax+1}{ax}\right)^x = \sqrt{e}.$$

Merk $(a \neq 0)$

$$\lim_{x\to\infty}\left(\frac{ax+1}{ax}\right)^x = \lim_{x\to\infty}\left(1+\frac{1}{ax}\right)^x$$

$$= \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{ax}\right)}$$

$(0/0)$ l'hopital

$$\lim_{x\to\infty} x\ln\left(1+\frac{1}{ax}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{ax}\right)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\frac{\frac{1}{-ax^2}}{1+\frac{1}{ax}}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{\frac{1}{a}}{1+\frac{1}{ax}} = \lim_{x\to\infty}\frac{1}{a+\frac{1}{x}} = \frac{1}{a}.$$

$$\lim_{x\to\infty}\left(\frac{ax+1}{ax}\right)^x = \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{ax}\right)} = e^{\frac{1}{a}} = \sqrt{e} = e^{\frac{1}{2}}.$$

$\Rightarrow \frac{1}{a} = \frac{1}{2}$ siden e^x er injektiv.

$\Rightarrow \underline{\underline{a = 2}}$

13. $$\lim_{x\to 0^+} x^{\sin(x)} = \lim_{x\to 0^+} e^{\sin(x)\ln(x)} = e^0 = \underline{\underline{1.}}$$

(∞/∞) l'hopital

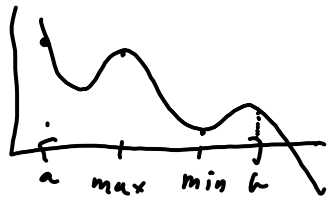
$$\lim_{x\to 0^+} \sin(x)\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{\sin(x)}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{\cos(x)}{\sin(x)^2}}$$

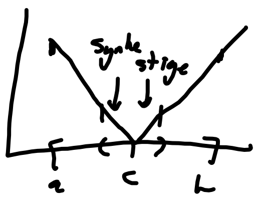
$$= \lim_{x\to 0^+} \frac{\frac{\sin(x)^2}{x}}{-\cos(x)} = \lim_{x\to 0^+} -\frac{\sin(x)^2}{x\cos(x)}.$$

$$= \lim_{x\to 0^+} \underbrace{\frac{\sin(x)}{x}}_{(\to 1)} \cdot \underbrace{\left(-\frac{\sin(x)}{\cos(x)}\right)}_{(\to 0)} = 0.$$

(Kan enkelt vises med l'hopital)

Lokale maksimum/minimum.





La $f:[a,b]\to\mathbb{R}$ ha et lokalt max/min i c.

Da er enten:

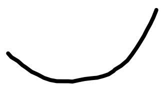
(i) $c = a$ eller b

(ii) $f'(c) = 0$

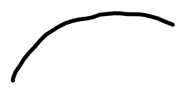
(iii) f ikke deriverbar i c.

En funksjon f er ·) Konveks på $I=[a,b]$ når $f''(x)\geq 0$ på (a,b)

Konveks:



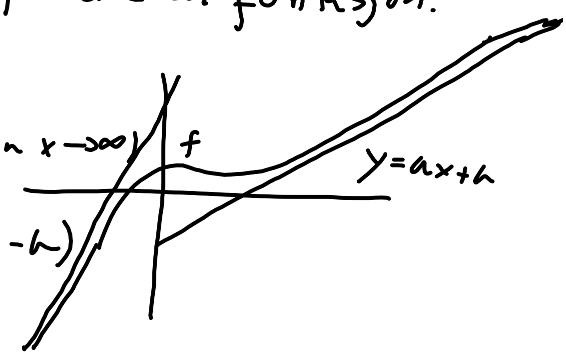
Konkav:



·) Konkav på $I=[a,b]$ når $f''(x)\leq 0$ på (a,b)

Asymptoter: La f være en funksjon.
skrå-

Da er linja $y=ax+b$ en skrå-asymptote (når $x\to\infty$) til f hvis $\lim\limits_{x\to\infty}(f(x)-ax-b) = 0$



y er en skråasymptote (når $x\to-\infty$) hvis

$\lim\limits_{x\to-\infty}(f(x)-ax-b) = 0.$

For å finne asymptoter $(x\to\infty)$

1) Beregn $\lim\limits_{x\to\infty}\frac{f(x)}{x} = a.$

2) Beregn $\lim\limits_{x\to\infty} f(x)-ax = b.$

$\Rightarrow$ $y = ax+b$ er en skrå-asymptote.

6.5 oppgave 13.

En funksjon er definert ved

$$f(x) = (3x^2 - x^3)^{\frac{1}{3}}.$$

a) Bestem nullpunkter, og finn hvor f er positiv og negativ.

b) Finn hvor f er voksende og synkende. Og lokale og globale max/min punkter.

c) Finn hvor f er konveks/konkav.

d) Finn asymptoter og skissèr.

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a) $f(x) = 0$ når $3x^2 - x^3 = 0$.

$x^2(3-x) = 0. \Rightarrow x = 0,\ x = 3.$

$$f(x) = (3x^2 - x^3)^{\frac{1}{3}}$$

$$f'(x) = \frac{1}{3}(3x^2 - x^3)^{-\frac{2}{3}}(6x - 3x^2)$$

$$= \frac{(2x - x^2)}{(3x^2 - x^3)^{\frac{2}{3}}} \leftarrow >0$$

$f'(x) = 0$ når $2x - x^2 = 0$.

$x(2-x) = 0.$ $x = 0,\ x = 2$. MEN ikke deriverbar i 0. $\Rightarrow x = 2$.

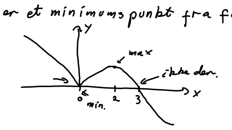
$f'(x) \geq 0$ når $x(2-x) \geq 0$.

Fortegnslinje:

$x = 0,\ x = 3$ null punkt. $x = 2$ maxpunkt.

$f(x)$ er ikke deriverbar i $x = 0,\ x = 3$.

$x = 0$ er et minimumspunkt fra fortegnslinje.



c) $$f'(x) = \frac{2x - x^2}{(3x^2 - x^3)^{\frac{2}{3}}}$$

$$f''(x) = \frac{(2-2x)(3x^2 - x^3)^{\frac{2}{3}} - \frac{2}{3}(3x^2 - x^3)^{-\frac{1}{3}}(6x - 3x^2)(2x - x^2)}{(3x^2 - x^3)^{\frac{4}{3}}}$$

$$= \frac{(2-2x)(3x^2 - x^3) - 2(2x - x^2)(2x - x^2)}{(3x^2 - x^3)^{\frac{4}{3}}}$$

$$= \frac{2(1-x)x^2(3-x) - 2x(2-x)x(2-x)}{(3x^2 - x^3)^{5/3}}$$

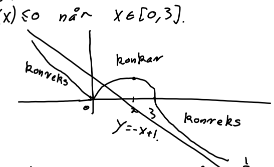
$$= \frac{2x^2\left[(1-x)(3-x) - (2-x)^2\right]}{(3x^2 - x^3)^{5/3}}$$

$$= \frac{2x^2\left[x^2 - 4x + 3 - x^2 + 4x - 4\right]}{(3x^2 - x^3)^{5/3}}$$

$$= -2x^2 / \underbrace{(3x^2 - x^3)^{5/3}}_{\text{positiv når } x \in [0,3]}.$$

$f''(x) \geq 0$ når $x \leq 0$ eller $x \geq 3$.

$f''(x) \leq 0$ når $x \in [0,3]$.



d): $$\lim_{x \to \infty} f(x)/x = \lim_{x \to \infty} \frac{(3x^2 - x^3)^{\frac{1}{3}}}{x}$$

$$= \lim_{x \to \infty} \frac{\left(x^3\left(\frac{3}{x} - 1\right)\right)^{\frac{1}{3}}}{x}$$

$$= \lim_{x \to \infty} \left(\frac{3}{x} - 1\right)^{\frac{1}{3}} = -1.$$

$a = -1.$

$$\lim_{x \to \infty} f(x) + x = \lim_{x \to \infty} x\left(\frac{3}{x} - 1\right)^{\frac{1}{3}} + x$$

$$= \lim_{x \to \infty} x\left(1 + \left(\frac{3}{x} - 1\right)^{\frac{1}{3}}\right)$$

$$= \lim_{x \to \infty} \frac{1 + \left(\frac{3}{x} - 1\right)^{\frac{1}{3}}}{\frac{1}{x}} \qquad \left(\frac{0}{0}\right)$$

l'hopital

$$\downarrow$$

$$= \lim_{x \to \infty} \frac{\frac{1}{3}\left(\frac{3}{x} - 1\right)^{-\frac{2}{3}} \cdot \left(-\frac{3}{x^2}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x \to \infty} \frac{1}{\left(\frac{3}{x} - 1\right)^{2/3}} = \frac{1}{(-1)^{2/3}} = 1.$$

$\Rightarrow y = -x + 1.$