Seksjon 9.2

1e) $\int e^{\sqrt{x}}dx = \int e^{u}\cdot 2u\,du = \int 2ue^{u}du.$

Substituerer $u=\sqrt{x}$. $du = \frac{1}{2\sqrt{x}}dx$

$\Rightarrow 2u\,du = dx.$

Delvis integrasjon:

$2\int ue^{u}du = 2\left(ue^{u} - \int 1\cdot e^{u}du\right) = 2(ue^{u}-e^{u})+C.$

$$= \underline{\underline{2(\sqrt{x}e^{\sqrt{x}} - e^{\sqrt{x}})+C}}$$

f) $I = \int \sin\sqrt[3]{x}\,dx.$

Substituerer $u = \sqrt[3]{x} = x^{\frac{1}{3}}$.

$du = \frac{1}{3}x^{\frac{1}{3}-1}dx = \frac{x^{-\frac{2}{3}}}{3}dx$

$3x^{\frac{2}{3}}du = dx.$

$\left(x^{\frac{2}{3}} = (x^{\frac{1}{3}})^{2} = u^{2}\right)$ $3u^{2}du = dx$

$I = \int \sin(u)\cdot 3u^{2}du = \int 3u^{2}\sin(u)du.$

$I_1 = \int u^{2}\sin(u)du \overset{\text{delvis integrasjon}}{=} -u^{2}\cos(u) - \int 2u(-\cos(u))du$

$= -u^{2}\cos(u) + 2\underbrace{\int u\cos(u)\,du}_{I_2}$

$I_2 = \int u\cos(u)du$

Delvis integrasjon:

$I_2 = u\sin(u) - \int \sin(u)du = u\sin(u)+\cos(u)+C$

$\Rightarrow I_1 = -u^{2}\cos(u) + 2(u\sin(u)+\cos(u)+C)$

$= -u^{2}\cos(u) + 2u\sin(u) + 2\cos(u) + 2C$

Husk: $I = 3I_1$:

$I = -3u^{2}\cos(u) + 6u\sin(u) + 6\cos(u) + \underset{}{\overset{\text{ny konstant}}{C'}}$

3 d) $I = \int_0^3 \arctan\sqrt{x}\, dx.$

Substituerer $u = \sqrt{x}$. $du = \frac{1}{2\sqrt{x}} dx$

$\Rightarrow 2u\,du = dx.$

Nye grenser: $u(x) = \sqrt{x}$. $u(0) = 0,\ u(3) = \sqrt{3}$.

$I = \int_0^{\sqrt{3}} 2u \arctan u\, du.$ Delvis integrasjon

$= \left[u^2 \arctan u\right]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{u^2}{1+u^2} du.$

$= 3\arctan(\sqrt{3}) - \int_0^{\sqrt{3}} \frac{u^2}{1+u^2} du.$

$I_1 = \int_0^{\sqrt{3}} \frac{u^2}{1+u^2} du$ $\left(\frac{u^2}{1+u^2} = \frac{(u^2+1)-1}{u^2+1} = 1 - \frac{1}{u^2+1} \right)$

$= \int_0^{\sqrt{3}} 1 - \frac{1}{u^2+1} du$

$= \left[u - \arctan(u)\right]_0^{\sqrt{3}} = \sqrt{3} - \arctan(\sqrt{3})$

$I = 3\arctan(\sqrt{3}) - (\sqrt{3} - \arctan(\sqrt{3}))$
$= 4\arctan(\sqrt{3}) - \sqrt{3}.$

9 $I = \int_0^1 e^{\arcsin(x)}\,dx.$

Substituerer:

$u = \arcsin(x).\qquad du = \frac{1}{\sqrt{1-x^2}}\,dx$

$\sin u = x$

$$\sqrt{1-x^2}\,du = dx$$

$$\sqrt{1-\sin^2 u}\,du = dx$$

$$\sqrt{\cos^2 u}\,du = dx$$

$$\cos(u)\,du = dx$$

$I = \int_0^{\frac{\pi}{2}} e^u \cos(u)\,du.$ Delvis integrerer 2 ganger.

$$I_1 = \int e^u \cos(u)\,du.$$

$$= e^u\cos(u) + \underbrace{\int e^u \sin(u)\,du}_{I_2}$$

Delvis int. igjen:

$$I_2 = e^u\sin(u) - \underbrace{\int e^u \cos(u)\,du}_{I_1} = e^u\sin(u) - I_1$$

$$I_1 = e^u\cos(u) + I_2 = e^u\cos(u) + e^u\sin(u) - I_1$$

$$\Rightarrow \quad 2I_1 = e^u\left(\cos(u) + \sin(u)\right)$$

$$\Rightarrow \quad I_1 = \frac{e^u}{2}\left(\cos(u) + \sin(u)\right)$$

$$I = \int_0^{\frac{\pi}{2}} e^u\cos(u)\,du = \left[\frac{e^u}{2}\left(\cos(u) + \sin(u)\right)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{e^{\frac{\pi}{2}}}{2}(0+1) - \frac{e^0}{2}(1+0)$$

$$= \frac{e^{\frac{\pi}{2}}}{2} - \frac{1}{2} = \frac{e^{\frac{\pi}{2}}-1}{2}.$$

9.5 Husk sammenligningskriteriene:

•) $f(x) \geq g(x) \overset{\geq 0}{\Rightarrow}$ Hvis $\int_a^\infty f(x)dx$ konvergerer, gjør $\int_a^\infty g(x)dx$ det og.

•) Hvis $\int_a^\infty g(x)dx$ divergerer, gjør $\int_a^\infty f(x)dx$ det og.

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i) Anta at $\int_a^\infty f(x)dx$ konvergerer og at $\lim_{x\to\infty} \frac{g(x)}{f(x)} < \infty$, $\Rightarrow \int_a^\infty g(x)$ konvergerer.

ii) Bytt ut "konvergerer" med "divergerer" i i).

9.5 3) c) Bestem om disse konvergerer eller divergerer:

c) $\int_0^1 \frac{1}{\sqrt{x+x^3}}\,dx$.

9.5.8) $\int_0^1 \frac{1}{x^p}\,dx$ konvergerer for $p<1$ og diverenerer for $p \geq 1$.

Prøver å vise konvergens:
Prøver å finne en øvre grense på formen $\frac{1}{x^p}$. Vil finne en $p \in \mathbb{R}$ s.a.

$$\frac{1}{\sqrt{x+x^3}} \leq \frac{1}{x^p} \iff x^p \leq \sqrt{x+x^3}$$

$$\iff x^{2p} \leq x + x^3.$$

Hvis $2p = 1$: $\iff x \leq x + x^3$ som stemmer.

$$\iff 0 \leq x^3$$

$p = \frac{1}{2}$ fungerer. Dvs: $\frac{1}{\sqrt{x+x^3}} \leq \frac{1}{x^{\frac{1}{2}}}$.

Men $\int_0^1 \frac{1}{x^{\frac{1}{2}}}\,dx$ konvergerer

$\Rightarrow \int_0^1 \frac{1}{\sqrt{x+x^3}}\,dx$ konvergerer.

6) Konvergerer eller divergerer

$$I = \int_0^1 \ln(x^3+x^2)\,dx\,?$$

$$\ln(x^3+x^2) = \ln\left(x^2(x+1)\right)$$
$$= \ln(x^2) + \ln(x+1)$$
$$= 2\ln(x) + \ln(x+1)$$

$$\Rightarrow I = \int_0^1 2\underline{\ln(x)} + \ln(x+1)\,dx$$

$\int_0^1 \ln(x+1)\,dx$ eksisterer fordi $\ln(x+1)$ er kontinuerlig på $[0,1]$.
Gjenstår å sje kke konvergens til $I_0 = \int_0^1 \ln(x)\,dx$.

La

$I_1 = \int \ln(x)\,dx$ være det ubestemte integralet. Bruker delvis int.

$$\ln(x) = 1 \cdot \ln(x)$$

$$I_1 = x\ln(x) - \int x \cdot \frac{1}{x}\,dx = x\ln(x) - \int 1\,dx$$
$$= x\ln(x) - x + C$$

Per definisjon:

$$I_0 = \int_0^1 \ln(x)\,dx = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \ln(x)\,dx$$
$$= \lim_{\varepsilon \to 0^+} \Big[x\ln(x) - x\Big]_\varepsilon^1$$
$$= \lim_{\varepsilon \to 0^+} \left(-1 - \varepsilon\ln(\varepsilon) + \varepsilon\right)$$

$\lim_{\varepsilon \to 0^+} \varepsilon = 0$

$\lim_{\varepsilon \to 0^+} -1 = -1.$

$$\lim_{\varepsilon \to 0^+} -\varepsilon\ln(\varepsilon) = -\lim_{\varepsilon \to 0^+} \frac{\ln(\varepsilon)}{\frac{1}{\varepsilon}}$$

l'hôpital $\left(\frac{\infty}{\infty} - \text{uttrykk}\right)$

$$\to -\lim_{\varepsilon \to 0^+} \frac{\frac{1}{\varepsilon}}{-\frac{1}{\varepsilon^2}} = \lim_{\varepsilon \to 0^+} \varepsilon = 0.$$

$$\Rightarrow \int_0^1 \ln(x)\,dx = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \ln(x)\,dx = \underline{\underline{-1.}}$$

$\Rightarrow$ integralet konvergerer.

Seksjon 9.3:

31) Beregn $I = \int 1 \cdot \ln(x^2+2x+10)\,dx$

Kan bruke delvis int.:

$$I = x\ln(x^2+2x+10) - \underbrace{\int x \cdot \frac{2x+2}{x^2+2x+10}\,dx}_{I_1}$$

$$I_1 = \int \frac{2x^2+2x}{x^2+2x+10}\,dx$$

Polynomdividerer:

$$\begin{array}{l} 2x^2+2x : x^2+2x+10 = 2 - \dfrac{2x+20}{x^2+2x+10}. \\ -(2x^2+4x+20) \\ \quad -2x-20 \end{array}$$

$$I_1 = \int 2 - \frac{2x+20}{x^2+2x+10}\,dx = 2x - \underbrace{\int \frac{2x+20}{x^2+2x+10}\,dx}_{I_2}$$

$I_2 = \int \frac{2x+20}{x^2+2x+10}\,dx$. Fullfører kvadratet

$$x^2+2x+10 = (x+1)^2+9$$

Skriver om: $9\left(\frac{1}{9}(x+1)^2+1\right)$

$$= 9\left(\left(\frac{x+1}{3}\right)^2+1\right)$$

$I_2 = \int \frac{2x+20}{9\left\{\left(\frac{x+1}{3}\right)^2+1\right\}}\,dx$. Substituerer $u = \frac{x+1}{3}$.

$du = \frac{1}{3}dx \Rightarrow 3du = dx$. $3u = x+1$

$\Rightarrow x = 3u-1$.

$$I_2 = \int \frac{2(3u-1)+20}{9(u^2+1)} \cdot 3du$$

$$= \int \frac{6u+18}{3(u^2+1)}\,du = \int \frac{2u+6}{u^2+1}\,du$$

$$= \int \frac{2u}{u^2+1} + \frac{6}{u^2+1}\,du = \ln(u^2+1) + 6\arctan(u) + C$$

$$I = x\ln(x^2+2x+10) - (2x - I_2)$$

$$= x\ln(x^2+2x+10) - 2x + \ln\left(\left(\frac{x+1}{3}\right)^2+1\right) + 6\arctan\left(\frac{x+1}{3}\right) + C$$

$$\left(\ln\left(\left(\frac{x+1}{3}\right)^2+1\right) = \ln\left(\frac{x^2+2x+1}{9}+1\right)\right.$$

$$= \ln\left(\frac{1}{9}\left((x^2+2x+1)+9\right)\right)$$

$$= \ln\left(\frac{1}{9}(x^2+2x+10)\right)$$

$$= \ln\left(\frac{1}{9}\right) + \ln(x^2+2x+10)$$

$$= -2\ln(3) + \ln(x^2+2x+10)$$

$\left(\ln\left(\frac{1}{9}\right) = \ln(3^{-2}) = -2\ln(3)\right)$

$C' = C - 2\ln(3)$

$$I = x\ln(x^2+2x+10) - 2x + (-2\ln(3) + \ln(x^2+2x+10)) + 6\arctan\left(\frac{x+1}{3}\right) + C$$

$$= (x+1)\ln(x^2+2x+10) - 2x + 6\arctan\left(\frac{x+1}{3}\right) + C'$$

9.2

1) $I = \int \cos(\ln(x))\,dx$

$u = \ln(x)$. $\quad du = \dfrac{dx}{x} \Rightarrow x\,du = dx$

$x = e^u \quad \Rightarrow e^u\,du = dx.$

$\Rightarrow I = \int e^u \cos(u)\,du = \dfrac{e^u}{2}(\cos(u) + \sin(u)) + C.$

$= \dfrac{x}{2}(\cos(\ln(x)) + \sin(\ln(x)))$

$+ C.$