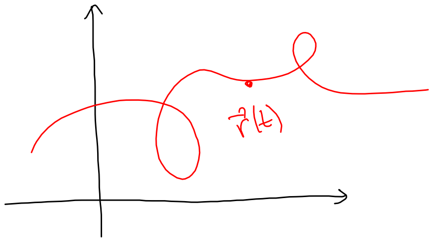
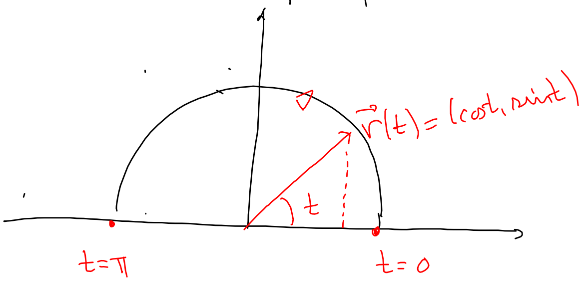
Parametriserte kurver



Definisjon: En parametrisert kurve i $\mathbb{R}^n$ er en kontinuerlig funksjon $\vec r: I \to \mathbb{R}^n$ der $I \subseteq \mathbb{R}$ er et intervall. Parametriserte kurver kalles også vektorvaluerte funksjoner.

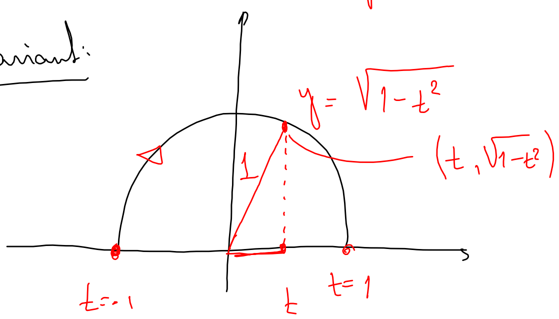
F.eks: $I = (a,b)$
$I = [a,b]$
$I = [a,\infty)$
$I = (-\infty,\infty)$

Eks:



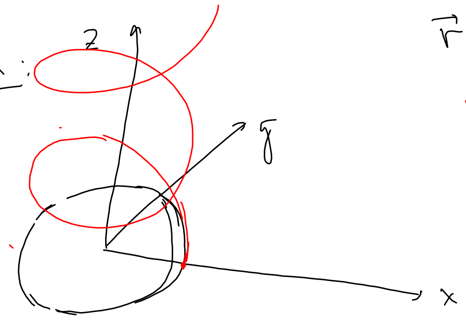
$$\vec r(t) = \cos t\,\vec i + \sin t\,\vec j = (\cos t, \sin t), \quad I = [0,\pi]$$

Variant:



$$\vec r(t) = t\,\vec i + \sqrt{1-t^2}\,\vec j = (t, \sqrt{1-t^2}), \quad I = [-1,1]$$

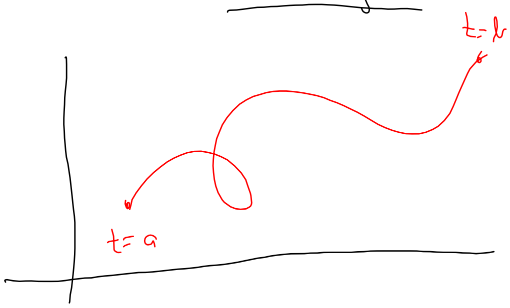
Eksempel:



$$\vec r(t) = \cos t\,\vec i + \sin t\,\vec j + t\,\vec k$$

Spiral:

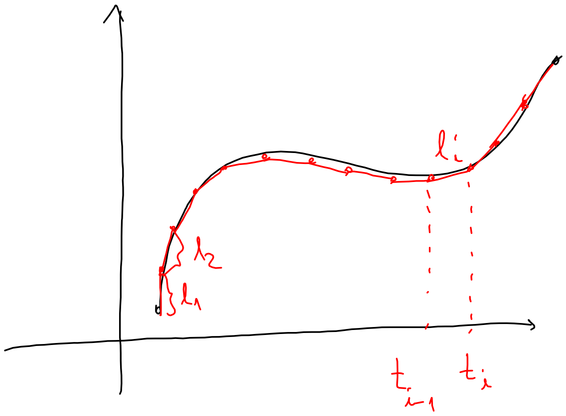
Buelengde



$\vec r(t) = (x_1(t), x_2(t), \dots, x_n(t))$

Hvor lang er kurven?

$$L(a,b) = \int_a^b \sqrt{x_1'(t)^2 + x_2'(t)^2 + \dots + x_n'(t)^2}\,dt$$



Lengden til den Ovadue kurven:

$$l_1+l_2+\dots+l_n=\sum_{i=1}^{n} l_i$$

$$l_i=|\vec r(t_i)-\vec r(t_{i-1})|$$

$$=\sqrt{(x_1(t_i)-x_1(t_{i-1}))^2+\dots+(x_n(t_i)-x_n(t_{i-1}))^2}$$

$$=\sqrt{\Big(\underbrace{\frac{x_1(t_i)-x_1(t_{i-1})}{t_i-t_{i-1}}}_{\approx x_1'(t_{i-1})}\Big)^2+\dots+\Big(\underbrace{\frac{x_n(t_i)-x_n(t_{i-1})}{t_i-t_{i-1}}}_{\approx x_n'(t_{i-1})}\Big)^2}\,(t_i-t_{i-1})$$

$$\approx\sqrt{x_1'(t_{i-1})^2+x_2'(t_{i-1})^2+\dots+x_n'(t_{i-1})^2}\,(t_i-t_{i-1})$$

Hele lengden: $L(a,b)\approx\sum_{i=1}^{n} l_i=\underbrace{\sum_{i=1}^{n}\sqrt{x_1'(t_{i-1})^2+x_2'(t_{i-1})^2+\dots+x_n'(t_{i-1})^2}\,(t_i-t_{i-1})}$

(Riemannsum til funksjonen $\sqrt{x_1'(t)^2+x_2'(t)^2+\dots+x_n'(t)^2}$)

$$\longrightarrow\int_a^b\sqrt{x_1'(t)^2+x_2'(t)^2+\dots+x_n'(t)^2}\,dt$$

Definisjon: Anta at $\vec r(t)=(x_1(t),\dots,x_n(t))$, $t\in[a,b]$, er en parametrisert kurve der $x_1',\dots,x_n'$ er kontinuerlige. Da er buelengden av kurven definert ved

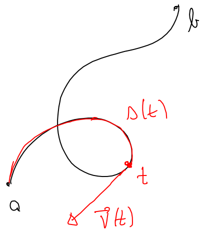
$$L(a,b)=\int_a^b\sqrt{x_1'(t)^2+x_2'(t)^2+\dots+x_n'(t)^2}\,dt.$$

Eksempel: Finn buelengden til

$$\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\vec{k} = (\cos t, \sin t, t),\ t \in [0, 2\pi]$$

$$L(0,2\pi) = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt = \int_0^{2\pi} \sqrt{\underbrace{(-\sin t)^2 + (\cos t)^2}_{1} + 1^2}\,dt$$

$$= \int_0^{2\pi} \sqrt{2}\,dt = 2\pi\sqrt{2} = \underline{\underline{2\sqrt{2}\,\pi}}$$



Fart: Anta at t står for tid og la

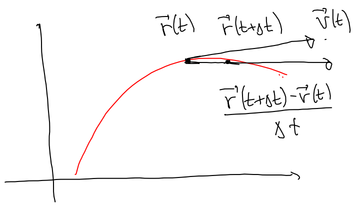
$$s(t) = \int_a^t \sqrt{x_1'(s)^2 + x_2'(s)^2 + \dots + x_n'(s)^2}\,ds$$

(strekningen ved tiden t).

Farten $v(t)$ er den deriverte av $s(t)$:

$$v(t) = s'(t) = \sqrt{x_1'(t)^2 + x_2'(t)^2 + \dots + x_n'(t)^2}$$

Hastighet er fart med retning:



$$\lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \underbrace{\vec{v}(t)}_{\text{hastighet}}$$

Definisjonen: Anta at $\vec{r}(t) = (x_1(t), x_2(t), \dots, x_n(t))$ er en parametrisert kurve der $x_1, x_2, \dots, x_n$ er deriverbar. Da er den deriverte $\vec{r}\,'(t)$ definert ved

$$\vec{r}\,'(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \lim_{\Delta t \to 0} \left(\frac{x_1(t+\Delta t) - x_1(t)}{\Delta t}, \dots, \frac{x_n(t+\Delta t) - x_n(t)}{\Delta t} \right)$$

$$= (x_1'(t), x_2'(t), \dots, x_n'(t))$$

Dersom t står for tid og $\vec{r}(t)$ står for posisjon kaller vi $\vec{r}\,'(t)$ for hastighet og skriver også $\vec{v}(t) = \vec{r}\,'(t)$

velocity

Sammenheng mellom fart og hastigheten:

$$\vec{v}(t) = (x_1'(t), x_2'(t), \dots, x_n'(t))$$

$$|\vec{v}(t)| = \sqrt{x_1'(t)^2 + x_2'(t)^2 + \dots + x_n'(t)^2} = v(t)$$

Regneregler for derivasjon: $\vec{r}(t)$ og $\vec{s}(t)$ er to deriverbare funksjoner:

(i) $(\vec{r}(t) + \vec{s}(t))' = \vec{r}\,'(t) + \vec{s}\,'(t)$

(ii) $(\vec{r}(t) - \vec{s}(t))' = \vec{r}\,'(t) - \vec{s}\,'(t)$

(iii) $(c(t)\vec{r}(t))' = c'(t)\vec{r}(t) + c(t)\vec{r}\,'(t)$

(iv) $(\vec{r}(t) \cdot \vec{s}(t))' = \vec{r}\,'(t) \cdot \vec{s}(t) + \vec{r}(t) \cdot \vec{s}\,'(t)$

(v) $(\vec{r}(t) \times \vec{s}(t))' = \vec{r}\,'(t) \times \vec{s}(t) + \vec{r}(t) \times \vec{s}\,'(t)$ i $\mathbb{R}^3$.

Onsdag: Forelesn. 3.1 – 3.2 – 3.3 + MATLAB