Plenum 15/3

$\boxed{4.10.4}$ a) $A = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

$$\det\begin{pmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{pmatrix} = 0$$

$$-(2-\lambda)(1+\lambda) - 4 = 0$$
$$-(2+2\lambda-\lambda-\lambda^2) - 4 = 0$$
$$\lambda^2 - \lambda - 6 = 0 \longrightarrow \lambda = \frac{1 \pm \sqrt{1-(-24)}}{2} = \frac{1\pm 5}{2} = \begin{cases} -2 \\ 3 \end{cases}$$

$\underline{\lambda = -2}$: $\begin{pmatrix} 2-(-2) & 2 \\ 2 & -1-(-2) \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow v_{\lambda=-2} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$2x + y = 0$

$\underline{\lambda = 3}$: $\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \Rightarrow v_{\lambda=3} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$x - 2y = 0$

$v_{\lambda=-2} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $v_{\lambda=3} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 1 & 2 & -1 \\ -2 & 1 & 5 \end{array}\right) \overset{II+2I}{\sim} \left(\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & 5 & 3 \end{array}\right) \overset{II\cdot\frac{1}{5}}{\sim} \left(\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & 1 & 3/5 \end{array}\right)$$

$$\overset{I-2II}{\sim} \left(\begin{array}{cc|c} 1 & 0 & -11/5 \\ 0 & 1 & 3/5 \end{array}\right)$$

$$\underline{\begin{pmatrix} -1 \\ 5 \end{pmatrix} = -\frac{11}{5}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \frac{3}{5}\begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

$\boxed{4.10.6}$ $D = M^T A M$; $A = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix} = A^T$

$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $M = [v_{\lambda_1} \; v_{\lambda_2}]$

$\det \begin{pmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{pmatrix} = 0$

$(5-\lambda)(2-\lambda) - 4 = 0$

$10 - 7\lambda + \lambda^2 - 4 = 0$

$\lambda^2 - 7\lambda + 6 = 0 \longrightarrow \lambda = \begin{cases} 1 \\ 6 \end{cases}$

$\underline{\lambda = 1}$: $\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow v_{\lambda=1} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \tilde{v}$

$\underline{\lambda = 6}$: $\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \Rightarrow v_{\lambda=6} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \tilde{u}$

$\tilde{v} \cdot \tilde{u} = 1 \cdot 2 + (-2) \cdot 1 = 0 \Rightarrow \tilde{v} \perp \tilde{u}$

$v = \frac{\tilde{v}}{|\tilde{v}|} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $u = \frac{\tilde{u}}{|\tilde{u}|} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \{v, u\}$ o.n.b.

$\Rightarrow$ Korollar 4.10.13 : $D = M^T A M$, der

$D = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$, $M = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$.

$\boxed{4.10.7}$ $p_A(\lambda) = \det(A - I\lambda) = \det\left((A-\lambda I)^T\right) = \det(A^T - \lambda I) = p_{A^T}(\lambda)$

korollar 4.9.18

$\det(A) = \det(A^T)$ $\quad$ $A - I\lambda \neq A^T - I\lambda$

$\boxed{4.10.8}$ $Av = \lambda_A v$, $Bv = \lambda_B v$

$(A+B)v = Av + Bv = \lambda_A v + \lambda_B v = (\lambda_A + \lambda_B) v$

$\boxed{4.10.9}$ $A(Bv) = A(\lambda_B v) = \lambda_B (Av) = \lambda_B \lambda_A v$

$= \lambda_A \lambda_B$

v er en egenvektor for AB med egenverdi $\lambda_A \lambda_B$.

$\boxed{4.10.10}$ $B = P^{-1}AP$

Vis at A og B har samme egenverdier:

$$B v_B = \lambda_B v_B$$
$$\|$$
$$P^{-1}AP v_B$$

$$P^{-1}AP v_B = \lambda_B v_B$$
$$P(P^{-1}AP v_B) = P(\lambda_B v_B)$$
$$A\underbrace{(P v_B)}_{v_A} = \lambda_B \underbrace{P v_B}_{v_A} \quad \Rightarrow \quad A v_A = \lambda_B v_A \;, \quad v_A = P v_B$$

Den andre veien følger av at $A = PBP^{-1}$, $P^{-1} = Q$

$$A = Q^{-1} B Q$$

λ egenverdi for A $\iff$ λ egenverdi for B

$\Leftarrow$

$\Rightarrow$

$$P v_B = v_A \iff v_B = P^{-1} v_A .$$

$\boxed{4.10.14}$ a) $(Ax)\cdot x > 0 \quad \forall x \neq 0 \quad$ P.D.

A P.D. $\iff$ alle egenverdiene til A er strengt positive.

$\Rightarrow$: λ, v : $Av = \lambda v$

$$(Av)\cdot v > 0$$
$$(\lambda v)\cdot v > 0$$
$$\lambda \underbrace{|v|^2}_{>0} > 0 \quad \Rightarrow \quad \lambda > 0$$

det eksisterer

$\Leftarrow$: A symmetrisk $\Rightarrow \exists \{v_1, v_2, \dots, v_n\}$ o.n.b.

$x \neq 0,\ x \in \mathbb{R}^n$: $\quad x = c_1v_1 + c_2v_2 + \dots + c_nv_n$, minst én $c_i \neq 0$

$$\begin{aligned}(Ax)\cdot x &= (A(c_1v_1 + \dots + c_nv_n))\cdot x \\ &= (c_1\lambda_1v_1 + c_2\lambda_2v_2 + \dots + c_n\lambda_nv_n)\cdot(c_1v_1 + \dots + c_nv_n) \\ &= \lambda_1c_1^2|v_1|^2 + \lambda_2c_2^2|v_2|^2 + \dots + \lambda_nc_n^2|v_n|^2 \\ &= \lambda_1\underbrace{c_1^2}_{\geq 0} + \lambda_2\underbrace{c_2^2}_{\geq 0} + \dots + \lambda_n\underbrace{c_n^2}_{\geq 0} > 0\end{aligned}$$

b) A,B har strengt positive egenverdier $\overset{(a)}{\Rightarrow}$ A,B er P.D.

$$(A+B)^T = A^T + B^T = A + B$$

$\underline{x \neq 0:}$
$$(A+B)x\cdot x = \underbrace{(Ax+Bx)\cdot x}_{} = \underbrace{Ax\cdot x}_{>0} + \underbrace{Bx\cdot x}_{>0} > 0 \quad \Rightarrow \quad A+B \text{ er P.D.}$$

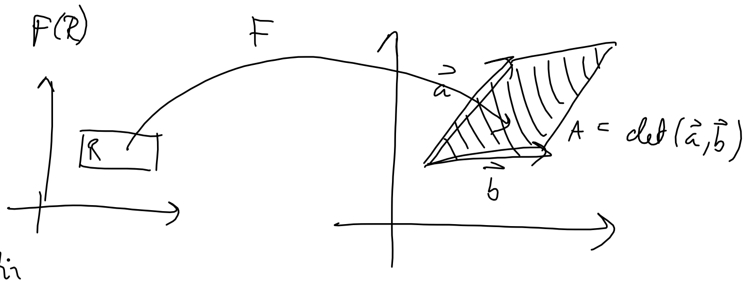
$\Rightarrow$ A+B har strengt positive egenverdier.

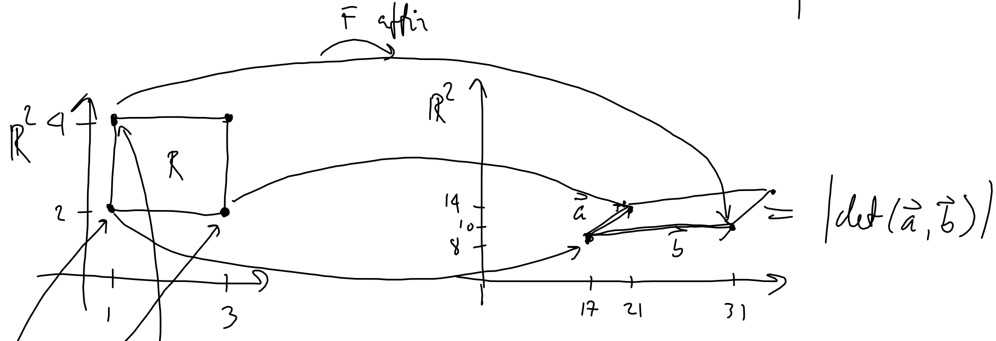
Vår 13

② $R \subset \mathbb{R}^2$ $\quad R = [1,3]\times[2,4]$

$F:\mathbb{R}^2\to\mathbb{R}^2$, $\quad F\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix} + A\begin{pmatrix}x\\y\end{pmatrix}$, der $A=\begin{pmatrix}2&7\\3&1\end{pmatrix}$

Finn arealet til bildet $F(R)$





$\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\2\end{pmatrix}, \begin{pmatrix}1\\4\end{pmatrix}$

$$F\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix} + \begin{pmatrix}2&7\\3&1\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix} + \begin{pmatrix}16\\5\end{pmatrix} = \begin{pmatrix}17\\8\end{pmatrix}$$

$$F\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix} + \begin{pmatrix}2&7\\3&1\end{pmatrix}\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix} + \begin{pmatrix}20\\11\end{pmatrix} = \begin{pmatrix}21\\14\end{pmatrix}$$

$$F\begin{pmatrix}1\\4\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix} + \begin{pmatrix}2&7\\3&1\end{pmatrix}\begin{pmatrix}1\\4\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix} + \begin{pmatrix}30\\7\end{pmatrix} = \begin{pmatrix}31\\10\end{pmatrix}$$

$$\vec{a} = \begin{pmatrix}21\\14\end{pmatrix} - \begin{pmatrix}17\\8\end{pmatrix} = \begin{pmatrix}4\\6\end{pmatrix}, \quad \vec{b} = \begin{pmatrix}31\\10\end{pmatrix} - \begin{pmatrix}17\\8\end{pmatrix} = \begin{pmatrix}14\\2\end{pmatrix}$$

$$A = \left|\det\begin{pmatrix}4&14\\6&2\end{pmatrix}\right| = |8 - 14\cdot 6| = |8-84| = \underline{\underline{76}}$$

$$A = \text{areal}(R)\cdot \det(A)$$

$\boxed{4.11.1}$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}}_{A} \begin{pmatrix} x_n \\ y_n \end{pmatrix} \quad , \quad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \end{pmatrix}$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} = A\left(A \begin{pmatrix} x_{n-2} \\ y_{n-2} \end{pmatrix}\right) = A^2 \begin{pmatrix} x_{n-2} \\ y_{n-2} \end{pmatrix} = \dots = A^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \quad , \quad A^n v = \lambda^n v$$

$$\lambda = \begin{cases} -1 \, , & v = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \\ 4 \, , & u = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{cases} \quad , \quad \begin{pmatrix} 1 & 3 & 5 \\ 1 & -2 & -5 \end{pmatrix} \overset{II-I}{\sim} \begin{pmatrix} 1 & 3 & 5 \\ 0 & -5 & -10 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 5 \\ -5 \end{pmatrix} = -\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A^n \left(2 \begin{pmatrix} 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = 2 A^n \begin{pmatrix} 3 \\ -2 \end{pmatrix} - A^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 (-1)^n \begin{pmatrix} 3 \\ -2 \end{pmatrix} - 4^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 6 \cdot (-1)^n - 4^n \\ 4 \cdot (-1)^{n+1} - 4^n \end{pmatrix}}}$$

$\boxed{4.11.2}$ $\underbrace{\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}}_{\vec{r}\,'(t)} = \underbrace{\begin{pmatrix} 1 & 8 \\ 2 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}}_{\vec{r}(t)}$, $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$

A har 2 egenverdier og egenvektorer som danner en basis i $\mathbb{R}^2$. enhver vektor i $\mathbb{R}^2$

(i) $\vec{r}(t) = c_1(t)\vec{v}_1 + c_2(t)\vec{v}_2$

(ii) $\vec{r}\,'(t) = c_1'(t)\vec{v}_1 + c_2'(t)\vec{v}_2$

$\vec{r}\,'(t) = A\vec{r}(t)$

$$\underbrace{c_1'(t)}\vec{v}_1 + c_2'(t)\vec{v}_2 = c_1(t)A\vec{v}_1 + c_2(t)A\vec{v}_2$$
$$= \underbrace{c_1(t)\lambda_1}\vec{v}_1 + c_2(t)\lambda_2\vec{v}_2$$

$c_1'(t) = c_1(t)\lambda_1$ & $c_2'(t) = c_2(t)\lambda_2$

$c_1(t) = C_1 e^{\lambda_1 t}$ $c_2(t) = C_2 e^{\lambda_2 t}$

$\vec{r}(t) = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2$

Bruk $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$ til å finne C_1 og C_2.