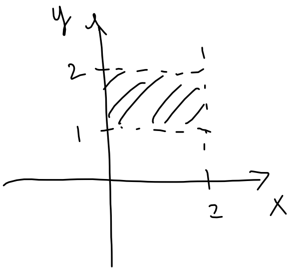
6.1: Dobbeltintegraler over rektangler

1) e) $\iint_R xy e^{x^2 y}\,dx\,dy = \int_1^2 \int_0^2 xy e^{x^2 y}\,dx\,dy$



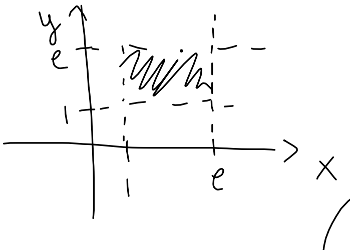
$$= \int_1^2 \left[\frac{1}{2} e^{x^2 y} \right]_{x=0}^{2} dy$$

$$= \int_1^2 \left(\frac{1}{2} e^{4y} - \frac{1}{2} \right) dy$$

$$= \left[\frac{1}{8} e^{4y} - \frac{1}{2} y \right]_{y=1}^{2} = \frac{1}{8} e^8 - 1 - \frac{1}{8} e^4 + \frac{1}{2}$$

$$= \underline{\underline{\frac{1}{8} e^8 - \frac{1}{8} e^4 - \frac{1}{2}}}$$

f) $\int_1^e \int_1^e \ln(xy)\,dx\,dy = \int_1^e \int_1^e (\ln(x) + \ln(y))\,dx\,dy$



$$= \int_1^e \left\{ [x\ln(x)]_{x=1}^{e} - \int_1^e 1\,dx + [x\ln(y)]_{x=1}^{e} \right\} dy$$

Delvis int.
$u' = 1$
$v = \ln x$
$u = x$
$v' = \frac{1}{x}$

$$= \int_1^e \{ \cancel{e} - \cancel{e} + 1 + (e-1)\ln(y) \}\,dy$$

$$= \int_1^e \{ 1 + (e-1)\ln(y) \}\,dy$$

$$= [y]_{y=1}^{e} + (e-1)[y\ln(y)]_{y=1}^{e} - (e-1)\int_1^e 1\,dy$$

$$= e-1+e(e-1)-(e-1)^2$$

$$= 2e-2 = \underline{\underline{2(e-1)}}$$

g) $$\int_0^1\int_1^{\sqrt{3}} \frac{1}{1+x^2y}\,dx\,dy =: I$$

$\underline{M}$: $$\int \frac{1}{1+x^2y}\,dx \underset{\downarrow}{=} \int \frac{1}{\sqrt{y}\,(1+u^2)}\,du$$

$\underline{\text{Substitusjon:}}$
$u = x\sqrt{y}$
$du = \sqrt{y}\,dx$

$$= \frac{1}{\sqrt{y}}\arctan(u) + C = y^{-\frac{1}{2}}\arctan(x\sqrt{y}) + C$$

$$I = \int_0^1 \left[y^{-\frac{1}{2}}\arctan(x\sqrt{y})\right]_{x=1}^{\sqrt{3}}\,dy$$

$$= \int_0^1 y^{-\frac{1}{2}}\left(\arctan(\sqrt{3}\sqrt{y}) - \arctan(\sqrt{y})\right)dy$$

$$= 2\int_0^1 \left(\arctan(\sqrt{3}u) - \arctan(u)\right)du$$

Substitusjon:

$u = \sqrt{y} = y^{\frac{1}{2}}$

$du = \frac{1}{2}y^{-\frac{1}{2}}\,dy$

$y = 0 \Rightarrow u = 0$

$y = 1 \Rightarrow u = 1$

$$= 2\left[u\left(\arctan(\sqrt{3}u) - \arctan(u)\right)\right]_{u=0}^{1} - 2\int_0^1 \left(\frac{\sqrt{3}u}{1+3u^2} - \frac{u}{1+u^2}\right)du$$

Delvis int:

$v' = 1$

$w = \arctan(\sqrt{3}u) - \arctan(u)$

$$= 2\left(\arctan(\sqrt{3}) - \arctan(1)\right) - 2\left[\frac{\sqrt{3}}{6}\ln(1+3u^2) - \frac{1}{2}\ln(1+u^2)\right]_{u=0}^{1}$$

$$= 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right) - 2\frac{\sqrt{3}}{6}\ln(4) + 2\frac{1}{2}\ln(2)$$

$$= \frac{\pi}{6} + \left(1 - \frac{2\sqrt{3}}{3}\right)\ln(2)$$

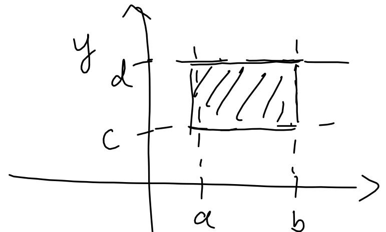
	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	ikke def.

tan $= \frac{\sin}{\cos}$

7.) Middelverdisetningen for dobbeltintegraler

Anta $f : R \to \mathbb{R}$, kont. funk. $R = [a, b] \times [c, d]$

rektangel



Ekstremalverdiset.

Vis: Fins pkt $(\bar{x}, \bar{y}) \in R$ s.a.

$$\frac{\iint_R f(x, y)\, dx\, dy}{|R|} = f(\bar{x}, \bar{y})$$

arealet til R

Disse eksisterer fordi R er lukket og begrenset og f kontinuerlig

Bevis: La $m := \min_{(x,y) \in R} f(x, y)$ og $M = \max_{(x,y) \in R} f(x, y)$

Da er:

f er mindre enn max

$$\iint_R f(x, y)\, dx\, dy \leq \iint_R M\, dx\, dy$$

$$= M \iint_R 1\, dx\, dy = M\, \underline{|R|}$$

def. av dobbeltint.

f er større enn min

og $$\iint_R f(x, y)\, dx\, dy \geq \iint_R m\, dx\, dy$$

$$= m \iint_R 1\, dx\, dy = \underline{m\, |R|}$$

Så:

$$m\,|R| \le \iint_R f(x,y)\,dx\,dy \le M\,|R|$$

$$\Downarrow \quad (|R|>0;\ \text{hvis ikke er det ingenting å vise})$$

$$m \le \frac{\iint_R f(x,y)\,dx\,dy}{|R|} \le M \qquad (\star)$$

Fra skjæringssetningen vet vi at den kont. funk. $f(x,y)$ tar alle verdier mellom minimumet & maksimumet sitt.

m M

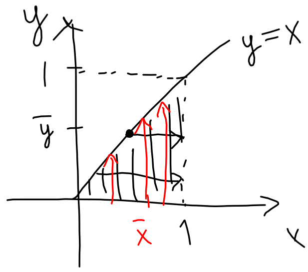
Siden $(\star)$ gir at $\dfrac{\iint_R f(x,y)\,dx\,dy}{|R|}$ er en slik verdi mellom min & maks for f, så må det finnes et punkt $(\bar{x},\bar{y}) \in R$ s.a.

$$f(\bar{x},\bar{y}) = \frac{\iint_R f(x,y)\,dx\,dy}{|R|}$$

$\blacksquare$

6.2: Dobbeltint. over begrensede områder

3) a) $\int_0^1 \int_y^1 e^{x^2}\,dx\,dy =: I$



• = opprinnelig tenkemåte

• = ny tenkemåte

Kan skrives:

$x \in [0,1]$

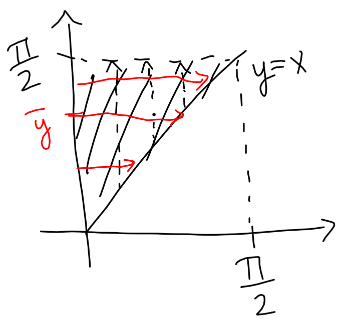
$y \in [0,x]$

$$I = \int_0^1 \int_0^x e^{x^2}\,dy\,dx = \int_0^1 x e^{x^2}\,dx = \left[\frac{1}{2}e^{x^2}\right]_{x=0}^{1}$$

$$\int_0^1 \left[y e^{x^2}\right]_{y=0}^{x} dx$$

$$= \frac{1}{2}e - \frac{1}{2} = \underline{\underline{\frac{1}{2}(e-1)}}$$

b) $\int_0^{\frac{\pi}{2}} \int_x^{\frac{\pi}{2}} \frac{\sin y}{y}\,dy\,dx =: I$



• = oppr. tenkemåte

• = ny — " —

$$I = \int_0^{\frac{\pi}{2}} \int_0^y \frac{\sin y}{y}\,dx\,dy$$

Evt:

$0 \le x \le \frac{\pi}{2}$

$x \le y \le \frac{\pi}{2}$

$0 \le x \le y \le \frac{\pi}{2}$

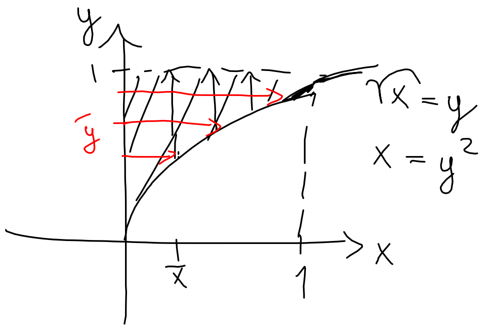
$0 \le y \le \frac{\pi}{2}$

$0 \le x \le y$

$$= \int_0^{\frac{\pi}{2}} \left[x \, \frac{\sin y}{y} \right]_{x=0}^{y} dy = \int_0^{\frac{\pi}{2}} \sin y \, dy$$

$$= \left[-\cos y \right]_{y=0}^{\frac{\pi}{2}} = \underline{\underline{1}}$$

c) $$\int_0^1 \int_{\sqrt{x}}^1 e^{\frac{x}{y^2}} \, dy \, dx =: I$$



$$\begin{aligned} 0 &\le x \le 1 \\ \sqrt{x} &\le y \le 1 \end{aligned} \Rightarrow 0 \le \sqrt{x} \le y \le 1$$

$$\Downarrow$$

$$\begin{aligned} 0 &\le y \le 1 \\ 0 &\le x \le y^2 \end{aligned}$$

$$I = \int_0^1 \int_0^{y^2} e^{\frac{x}{y^2}} \, dx \, dy = \dots = \int_0^1 (y^2 e - y^2) dy$$

$$= \dots = \underline{\underline{\frac{e-1}{3}}}$$