Sammenlikningstester forts.

Eks. $\sum \frac{3n^2+4n}{8n^4-2} = a_n$

sammenlikner med $\sum \frac{1}{n^2} = b_n$

$$\frac{a_n}{b_n} = \frac{3n^2+4n}{8n^4-2}\cdot\frac{n^2}{1}$$

$$= \frac{3n^4+4n^3}{8n^4-2}$$

$$= \frac{3+\frac{4}{n}}{8-\frac{2}{n^4}} \longrightarrow \frac{3}{8} < \infty$$

Forenklede tester:

$\sum a_n$ konvergerer $\quad \lim_{n\to\infty} \frac{b_n}{a_n} < \infty \Rightarrow \sum b_n$ konvergerer

$\sum a_n$ divergerer $\quad \lim_{n\to\infty} \frac{b_n}{a_n} > 0 \Rightarrow \sum b_n$ divergere

Vet at $\sum \frac{1}{n^2}$ konvergerer $\Longrightarrow$ $\sum a_n$ konvergerer

Eks $\sum e^{\frac{1}{n}} - 1$

S.l. med $\sum \frac{1}{n}$

$$\left\{ \lim_{n\to\infty} \frac{e^{\frac{1}{n}}-1}{\frac{1}{n}} = \lim_{x\to 0} \frac{e^x-1}{x} = \lim_{x\to 0}\frac{e^x}{1} = 1 \right.$$

$$= \lim_{x\to 0} \underbrace{\frac{\cancel{1}+x+\frac{1}{2}x^2+\dots-\cancel{1}}{x}}_{=1+\frac{1}{2}x+} = 1$$

Forholdstest

$\sum a_n$ positiv.

$a = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ eksisterer

1) $a < 1$ Konvergens
2) $a > 1$ Divergens
3) $a = 1$??

Bevis

$$\frac{a_{n+1}}{a_n} < r < 1 \quad \text{når } n \geq N$$

$$\Rightarrow a_{n+1} < a_n \cdot r \;, \; a_{n+2} < a_{n+1} \cdot r < a_n \cdot r^2$$

$$\Rightarrow \sum_{n=N}^{\infty} a_n < \sum_{n=N}^{\infty} a_N \cdot r^{n-N} \leftarrow$$ Konvergerer (geometrisk rekke med $|r| < 1$.)

Eks

1) $\sum \frac{n}{2^n}$

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{2^{n+1}} \frac{2^n}{n} = \frac{n+1}{n} \frac{1}{2} \longrightarrow \frac{1}{2} < 1$$

Konvergens

2) $\sum \frac{n!}{n^n}$

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{(n+1)!}^{\,n+1}}{(n+1)^{n+1}} \frac{n^n}{\cancel{n!}} = \frac{\cancel{n+1}}{(n+1)^{n\cancel{+1}}} n^n$$

$$= \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(\frac{n+1}{n}\right)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \xrightarrow[n\to\infty]{} \frac{1}{e} <$$

Konvergerer.

Rottest

$\sum a_n$ positiv

$a = \lim_{n\to\infty} \sqrt[n]{a_n}$ eksisterer

1) $a < 1$ konvergere
2) $a > 1$ divergere
3) $a = 1$??

Bevis $a < r < 1$ For $n \geq N$ $\sqrt[n]{a_n} < r \iff a_n < r^n$

$$\sum_{n=N}^{\infty} a_n < \sum_{n=N}^{\infty} r^n \text{ Konvergent } (r<1)$$

Eks. $\sum \left(2+\frac{1}{n}\right)^{-n}$ $a_n^{\frac{1}{n}} = \left(\left(2+\frac{1}{n}\right)^{-n}\right)^{\frac{1}{n}} = \left(2+\frac{1}{n}\right)^{-1} \xrightarrow[n\to\infty]{} \frac{1}{2} < 1$

Konvergens

Alternerende rekker

En rekke hvor annet hvert ledd er positivt og negativt.

Eks. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots = \sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{1}{n}$

Setning

$\sum a_n$ alternerende rekke, $|a_n| \xrightarrow[n\to\infty]{} 0$, avtagende $\Rightarrow$ $\sum_n a_n$ er konvergent med sum S og $S_n = \sum_{j=1}^{n} a_j$ gir

$$|S - S_n| \leq |a_{n+1}|$$

Bevis. Anta $a_1 > 0$

$S_1, S_3, S_5, \ldots$ avtagende følge

$$S_{2j+1} = S_{2j-1} - |a_{2j}| + |a_{2j+1}| < S_{2j-1}$$

$$\underbrace{+\cdots+a_{2n-1}}_{S_{2n-1}}\underbrace{-|a_{2n}|+a_{2n+1}}_{S_{2n+1}}$$

$S_2, S_4, S_6, \ldots$ voksende følge

$a_1 + a_2 = S_2$, $a_1+a_2+a_3+a_4 = S_4$, $a_1+a_2+a_3 = S_3$, $a_1 = S_1$

Begge følgene konverger, med grenser $S_{odd} \to S$, $S_{even} \to T$

$$|S-T| = |S - S_{2n+1} + S_{2n+1} - S_{2n} + S_{2n} - T|$$

$$\leq |S - S_{2n+1}| + \underbrace{|S_{2n+1} - S_{2n}|}_{|a_{2n+1}|} + |S_{2n} - T|$$

$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Har valgt $\varepsilon > 0$ og funnet en N

$$\Rightarrow \underline{S = T}$$

Noen begreper

$\sum a_n$ absolutt konvergent dersom $\sum |a_n|$ konvergent

Motsatt begrep: Betinget konvergens

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ konvergent $\sum a_n$ konv. men

men $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ divergent $\sum |a_n|$ divergere

Viktig: I en absolutt konvergent rekke kan vi bytte om på ledd og den vil fortsatt konvergere.

Riemanns teorem

Anta at $\sum a_n$ er betingent konvergent. For ethvert reelt tall a så finnes en ombytting av leddene i rekka slik at den nye rekka konverger mot a.

Eks.

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots=\ln 2$$ Alternerende harmonisk

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\cdots=\frac{3}{2}\ln 2$$

$$\Big(\overbrace{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{4N}}^{S_{4N}}\Big)-\frac{1}{2}\Big(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2N}\Big)-\frac{1}{2}\Big(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{N}\Big)$$

$$=S_{4N}-\frac{1}{2}S_{2N}-\frac{1}{2}S_N=(S_{4N}-S_{2N})+\frac{1}{2}(S_{2N}-S_N)\longrightarrow \ln 2+\frac{1}{2}\ln 2$$

$$=\frac{3}{2}\ln 2$$

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}\cdots$$

$$\Big(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2N}\Big)-2\Big(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2N}\Big)=S_{2N}-\Big(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{N}\Big)$$

$$=S_{2N}-S_N\longrightarrow \ln 2 \qquad N\to\infty$$

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\cdots=\frac{1}{1-(-\frac{1}{2})}$$ Geometrisk rekke $a_0=1$, $r=-\frac{1}{2}$

$$=\frac{2}{3}$$