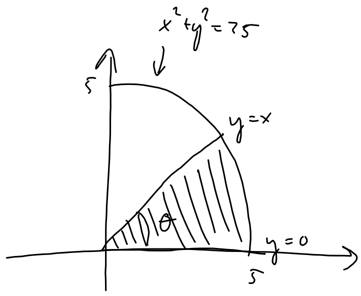
Plenum 03/05

$\boxed{6.3.1}$ b) $\iint_R (x^2+y^2)\,dxdy$



$$\left[\begin{array}{ll} x = r\cos\theta & \theta\in[0,\frac{\pi}{4}] \\ y = r\sin\theta & r\in[0,5]\end{array}\right]$$

$$= \int_0^5\int_0^{\frac{\pi}{4}} r^2\cdot r\,d\theta\,dr = \frac{\pi}{4}\int_0^5 r^3\,dr = \underline{\underline{\frac{5^4\pi}{4^2}}}$$

g) $\iint_R \underbrace{(x^2+y^2)}^{\frac{3}{2}}\,dxdy$ $\qquad R=\{(x,y)\in\mathbb{R}^2 \mid (x-1)^2+y^2\le 1\}$

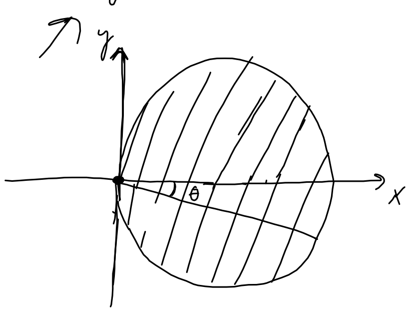
$\cancel{x-1 = r\cos\theta}$

$\cancel{y = r\sin\theta}$

$\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$

$x = r\cos\theta$

$y = r\sin\theta$



$$(x-1)^2+y^2 = (r\cos\theta-1)^2 + r^2\sin^2\theta$$
$$= \underline{r^2\cos^2\theta} - 2r\cos\theta + 1 + \underline{r^2\sin^2\theta}$$
$$= \underbrace{r^2 - 2r\cos\theta + 1 \le 1}$$
$$r^{\cancel{2}} \le 2\cancel{r}\cos\theta$$
$$r \le 2\cos\theta$$

$$\iint_R (x^2+y^2)^{\frac{3}{2}}\,dxdy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} \underbrace{r^{2\cdot\frac{3}{2}}}_{r^4} r\,dr\,d\theta = \frac{1}{5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[r^5\right]_0^{2\cos\theta} d\theta$$

$$= \frac{32}{5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^5\theta\,d\theta$$

$\cos^2\theta = 1-\sin^2\theta$

$\cos^4\theta = (1-\sin^2\theta)^2$

$\cos^5\theta = (1-\sin^2\theta)^2\cos\theta$

$$= \frac{32}{5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\sin^2\theta)^2\cos\theta\,d\theta$$

$u = \sin\theta$

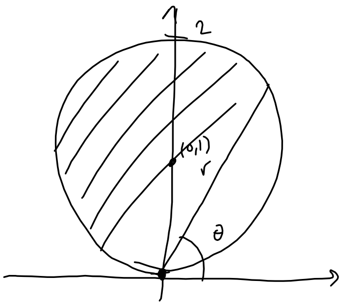
$du = \cos\theta\,d\theta$

$\theta = -\frac{\pi}{2} \Rightarrow u = \sin(-\frac{\pi}{2}) = -1$

$\theta = \frac{\pi}{2} \Rightarrow u = \sin(\frac{\pi}{2}) = 1$

$$= \frac{32}{5}\int_{-1}^{1} (1-u^2)^2\,du = \ldots = \underline{\underline{\frac{512}{75}}}$$

$\boxed{6.3.3}$ a)



$\theta \in [0, \pi]$

$r \in [0, 2\sin\theta]$

$(x-1)^2 + y^2 \le 1)$

$\boxed{x^2 + (y-1)^2 \le 1}$

$x = r\cos\theta$

$y = r\sin\theta$

$$r^2\cos^2\theta + (r\sin\theta - 1)^2 \le 1$$

$$\underbrace{r^2\cos^2\theta + r^2\sin^2\theta}_{r^2} - 2r\sin\theta + \cancel{1} \le \cancel{1}$$

$$r^2 \le 2r\sin\theta \qquad r \ge 0$$

$$r \le 2\sin\theta$$

$$\iint_R f(x,y)\,dx\,dy = \int_0^{\pi}\int_0^{2\sin\theta} f(r\cos\theta, r\sin\theta)\cdot r\,dr\,d\theta$$

b) $f(x,y) = (x^2+y^2)^{\frac{1}{2}}$

$$\iint_R (x^2+y^2)^{\frac{1}{2}}\,dx\,dy = \int_0^{\pi}\int_0^{2\sin\theta} \underbrace{r^{2\cdot\frac{1}{2}}\cdot r}_{r^2}\,dr\,d\theta$$

$$= \frac{1}{3}\int_0^{\pi} 8\sin^3\theta\,d\theta$$

$\sin^2\theta = 1 - \cos^2\theta$

$\sin^3\theta = (1-\cos^2\theta)\sin\theta$

$$= -\frac{8}{3}\int_1^{-1}(1-u^2)\,du$$

$u = \cos\theta$

$du = -\sin\theta\,d\theta$

$$= \frac{8}{3}\int_{-1}^{1}(1-u^2)\,du = \ldots$$

$\theta = 0 \Rightarrow u = \cos 0 = 1$

$\theta = \pi \Rightarrow u = \cos\pi = -1$

$$= \underline{\underline{\frac{32}{9}}}$$

$\boxed{6.4.1}$ d) $\iiint_E 1 \, dx\,dy\,dz$ $\qquad \iint_A 1 \, dx\,dy$

$$\left[\begin{array}{l} z = \sqrt{32 - 2x^2 - 2y^2} \\ \\ x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{array}\right] \qquad \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = \left|\begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}\right| = r$$

$$z = \sqrt{32 - 2\underbrace{(x^2+y^2)}_{r^2}} = \underbrace{\sqrt{32 - 2r^2}}$$

$32 - 2r^2 = 0$

$16 = r^2$

$r = 4$

$$\iiint_E 1 \, dx\,dy\,dz = \iint_A \int_0^{\sqrt{32-2(x^2+y^2)}} 1 \, dx\,dy\,dz$$

$$= \int_0^4 \int_0^{2\pi} \int_0^{\sqrt{32-2r^2}} r \, dz\,d\theta\,dr$$

$$= \int_0^4 \int_0^{2\pi} \sqrt{32-2r^2}\, r \, d\theta\,dr$$

$$= 2\pi \int_0^4 \sqrt{32-2r^2}\, r \, dr$$

$u = 32 - 2r^2$

$du = -4r\,dr \rightarrow r\,dr = -\frac{1}{4}du$

$$= -\frac{\pi}{2}\int_{32}^{0} \sqrt{u}\, du$$

$r = 0 \Rightarrow u = 32 - 2\cdot 0^2 = 32$

$r = 4 \Rightarrow u = 32 - 2\cdot 4^2 = 0$

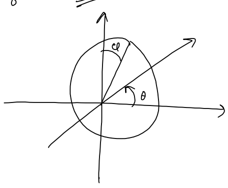
$$= \frac{\pi}{2}\int_0^{32} \sqrt{u}\, du = \frac{\pi}{2}\cdot\frac{2}{3}\left[u^{\frac{3}{2}}\right]_0^{32} = \underline{\underline{\frac{\pi}{3} 32^{\frac{3}{2}}}}$$

$\boxed{6.4.4}$

$x = \rho\cos\theta\sin\varphi$

$y = \rho\sin\varphi\sin\theta$

$z = \rho\cos\varphi$

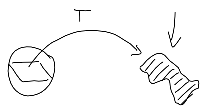


$\theta \in [0, 2\pi]$

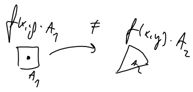
$\varphi = [0, \pi]$

$\rho = R$

$$\vec{r}(\theta,\varphi) = (R\cos\theta\sin\varphi, R\sin\theta\sin\varphi, R\cos\varphi)$$



$$\left|\frac{\partial \vec{r}}{\partial\theta} \times \frac{\partial \vec{r}}{\partial\varphi}\right|$$



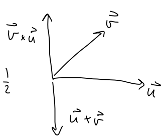
$$\frac{\partial\vec{r}}{\partial\theta} = (-R\sin\theta\sin\varphi, R\cos\theta\sin\varphi, 0)$$

$$\frac{\partial\vec{r}}{\partial\varphi} = (R\cos\theta\cos\varphi, R\sin\theta\cos\varphi, -R\sin\varphi)$$

$$\frac{\partial r}{\partial\theta}\times\frac{\partial r}{\partial\varphi} = \begin{vmatrix} i & j & k \\ -R\sin\theta\sin\varphi & R\cos\theta\sin\varphi & 0 \\ R\cos\theta\cos\varphi & R\sin\theta\cos\varphi & -R\sin\varphi \end{vmatrix}$$

$$= -R^2\cos\theta\sin^2\varphi\,\vec{i} - R^2\sin\theta\sin^2\varphi\,\vec{j} - \underbrace{(R^2\sin^2\theta\sin\varphi\cos\varphi + R^2\cos^2\theta\cos\varphi\sin\varphi)}_{R^2\sin\varphi\cos\varphi(\underbrace{\sin^2\theta + \cos^2\theta}_{1})}\vec{k}$$

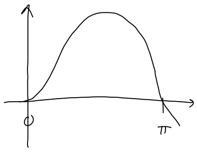
$$= (-R^2\cos\theta\sin^2\varphi, -R^2\sin\theta\sin^2\varphi, -R^2\sin\varphi\cos\varphi)$$



$$\left|\frac{\partial r}{\partial\theta}\times\frac{\partial r}{\partial\varphi}\right| = \Big(\underbrace{R^4\cos^2\theta\sin^4\varphi + R^4\sin^2\theta\sin^4\varphi}_{R^4\sin^4\varphi(\underbrace{\cos^2\theta + \sin^2\theta}_{1})} + \underline{R^4\sin^2\varphi}\cos^2\varphi\Big)^{\frac{1}{2}}$$

$$= R^2|\sin\varphi|\Big(\underbrace{\sin^2\varphi + \cos^2\varphi}_{1}\Big)^{\frac{1}{2}} = R^2|\sin\varphi|$$

$$\iint_R \underbrace{\left|\frac{\partial r}{\partial\theta}\times\frac{\partial r}{\partial\varphi}\right|}_{R^2|\sin\varphi|} d\theta\,d\varphi = \int_0^{\pi}\int_0^{2\pi} R^2|\sin\varphi|\, d\theta\,d\varphi$$



$$= 2\pi R^2 \int_0^{\pi} |\sin\varphi|\, d\varphi$$

$$= 2\pi R^2 \int_0^{\pi} \sin\varphi\, d\varphi$$

Eks. 2010

4 Avstand: $f(x,y,z) = \sqrt{(x-p_1)^2 + (y-p_2)^2 + (z-p_3)^2}$

(p_1, p_2, p_3)

$\sqrt{u} \leq \sqrt{v} \qquad u, v \geq 0$

$\Updownarrow$

$u \leq v$

$f(x,y,z) = (x-p_1)^2 + (y-p_2)^2 + (z-p_3)^2$

$f(x,y,z) = (x-1)^2 + (y-2)^2 + z^2$

$\begin{cases} \min & f(x,y,z) \\ \text{gitt} & \underbrace{z^2 = x^2+y^2+1} \end{cases} \qquad g(x,y,z) = z^2 - x^2 - y^2 = 1$

$\begin{cases} \min & f(x,y,z) \\ \text{gitt} & g(x,y,z) = 1 \end{cases} \qquad z^2 = x^2+y^2+1$

$\nabla f = \lambda \nabla g \qquad$ dersom $\nabla g \neq 0$

$\nabla g = (-2x, -2y, 2z) \Rightarrow \nabla g = 0 \iff x = y = z = 0$

$g(0,0,0) = 0^2 - 0^2 - 0^2 = 0 \neq 1$

$\nabla f = (2(x-1), 2(y-2), 2z)$

I $\quad 2(x-1) = -2\lambda x$

II $\quad 2(y-2) = -2\lambda y$

III $\quad 2z = 2\lambda z$

IV $\quad z^2 - x^2 - y^2 = 1 \Rightarrow z^2 - \underbrace{(x^2+y^2)}_{\geq 0} = 1$

$0 - \underbrace{(x^2+y^2)}_{\geq 0} = 1$

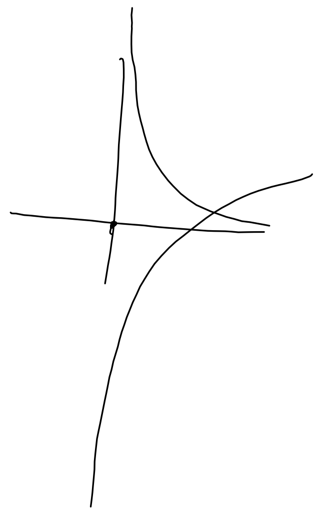
III $\Rightarrow \dfrac{2z}{2z} = \dfrac{2\lambda z}{2z} \iff 1 = \lambda$

$x = \frac{1}{2}, \quad y = 1, \quad z = \frac{3}{2}$

$f(\frac{1}{2}, 1, \frac{3}{2}) = (\frac{1}{2} - 1)^2 + (1-2)^2 + (\frac{3}{2})^2 = \frac{7}{2}$

avstand $= \sqrt{\frac{7}{2}}$

2011 ② $W = \{(x,y) \in \mathbb{R}^2 \mid x, y > 0\}$



$f(x,y) = x \ln y - y \ln x$

a) $\frac{\partial f}{\partial x} = \ln y - \frac{y}{x}$, $\frac{\partial f}{\partial y} = \frac{x}{y} - \ln x$

Ja, kont.

b) $\frac{\partial f}{\partial y}(\bar{x}, \bar{y}) \neq 0$

$$\boxed{g'(x) = - \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}}$$

c) $\frac{\partial f}{\partial y}(4,2) = \frac{4}{2} - \ln 4 = 2 - 2\underbrace{\ln 2}_{<1} > 0 \Rightarrow \frac{\partial f}{\partial y}(4,2) \neq 0$

$$g'(4) = - \frac{\frac{\partial f}{\partial x}(4,2)}{\frac{\partial f}{\partial y}(4,2)} = - \frac{\ln 2 - \frac{2}{4}}{2 - 2\ln 2} = \frac{\frac{1}{2} - \ln 2}{2 - 2\ln 2}$$