Plenum 12/3-15

6.3: (1 d f g)

6.4: 1(d) f, 4, 7, 10, 17

6.5: 10

oo

6.4: Anvendelser av dobbeltintegraler

1) f) E: over xy-plan, under $z = 4 - (x-2)^2 - (y+1)^2$.

I xy-planet: Dvs. $z = 0$;

$$0 = 4 - (x-2)^2 - (y+1)^2$$

$$(x-2)^2 + (y+1)^2 = 4 = 2^2$$

Sirkel med sentrum i $(2,-1)$ og radius 2; Kaller for S.

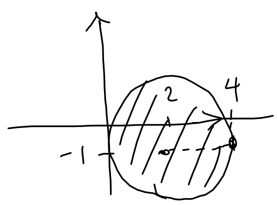
$$V = \iint_S f(x,y)\,dx\,dy$$

sirkel over

Beskrive S: $x = 2 + r\cos\theta$

$y = -1 + r\sin\theta$

$r \in [0,2]$, $\theta \in [0, 2\pi]$.



$$V = \int_0^{2\pi}\int_0^{2} f(2 + r\cos\theta, -1 + r\sin\theta)\, r \, dr\, d\theta$$

polar-koordinater

Mellomregning:

$$z = f(2 + r\cos\theta, -1 + r\sin\theta)$$

$$= 4 - r^2\cos^2\theta - r^2\sin^2\theta$$

$$= 4 - r^2(\underbrace{\cos^2\theta + \sin^2\theta}_{=1}) = 4 - r^2$$

$$V = \int_0^{2\pi}\int_0^{2} (4 - r^2) r \, dr\, d\theta = \int_0^{2\pi} \left[2r^2 - \frac{1}{4}r^4\right]_{r=0}^{2} d\theta$$

$$= \int_0^{2\pi} 4 \, d\theta = 2\pi \cdot 4 = \underline{\underline{8\pi}}$$

4) Parametrisering:

kulekoord.

$$\vec{r}(u, v) = (R\sin u\cos v, R\sin u\sin v, R\cos u)$$

$$v \in [0, 2\pi], \quad u = [0, \pi].$$

$$A = \iint_K \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| du\, dv$$

$$\frac{\partial \vec{r}}{\partial u} = (R\cos u\cos v, R\cos u\sin v, -R\sin u)$$

$$\frac{\partial \vec{r}}{\partial v} = (-R\sin u\sin v, R\sin u\cos v, 0)$$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \Big(-R^2 \sin^2 u \cos v,\ R^2 \sin^2 u \sin v,$$

$$R^2 \underbrace{(\cos u \sin u \cos^2 v + \cos u \sin u \sin^2 v)}_{\cos u \sin u}\Big)$$

Så:

$$\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| = \left(R^4 \sin^4 u \cos^2 v + R^4 \sin^4 u \sin^2 v + R^4 \cos^2 u \sin^2 u\right)^{\frac{1}{2}}$$

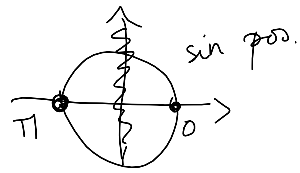
$\cos^2 + \sin^2 = 1$

$$= R^2 \left(\sin^4 u + \cos^2 u \sin^2 u\right)^{\frac{1}{2}}$$

$$= R^2 \Big(\sin^2 u \,\big(\underbrace{\sin^2 u + \cos^2 u}_{1}\big)\Big)^{\frac{1}{2}}$$

$$= R^2 (\sin^2 u)^{\frac{1}{2}} = R^2 |\sin u|$$

OBS: $\sqrt{}$ er positiv!



$$A = \iint_K \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| du\,dv = \int_0^{2\pi}\!\!\int_0^{\pi} R^2 |\sin u|\, du\, dv$$

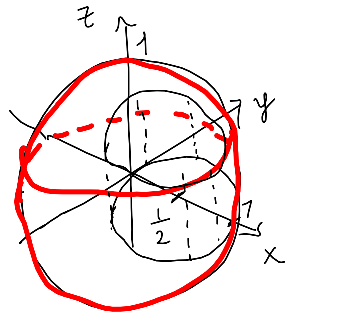
$$= R^2\, 2\pi \int_0^{\pi} |\sin u|\, du = 2\pi R^2 \int_0^{\pi} \sin u\, du$$

$$= 2\pi R^2 \left[-\cos u\right]_{u=0}^{\pi}$$

$$= 2\pi R^2 \underbrace{(-(-1)-(-1))}_{2} = \underline{\underline{4\pi R^2}}$$

7.) Areal av del av kuleflate $x^2+y^2+z^2=1=1^2$
som ligger over sirkelen $(x-\frac{1}{2})^2+y^2 \leq \frac{1}{4} = (\frac{1}{2})^2$

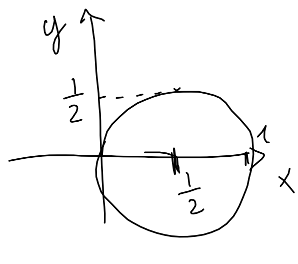
sirkel med sentrum $(\frac{1}{2},0)$ og radius $\frac{1}{2}$



Kuleflaten er beskrevet ved:

$$z = f(x,y) = \sqrt{1-x^2-y^2} = \sqrt{1-r^2}$$

Polarkoord:
$x = r\cos\theta$
$y = r\sin\theta$



$$\frac{\partial f(x,y)}{\partial x} = \frac{-x}{\sqrt{1-x^2-y^2}} \qquad \frac{\partial f(x,y)}{\partial y} = \frac{-y}{\sqrt{1-x^2-y^2}}$$

$$\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2} = \sqrt{1+\frac{x^2}{1-x^2-y^2}+\frac{y^2}{1-x^2-y^2}}$$

$$= \sqrt{\frac{1}{1-x^2-y^2}} = \frac{1}{\sqrt{1-x^2-y^2}} = \frac{1}{\sqrt{1-r^2}}$$

polarkoord

Integrasjonsgrenser i polarkoordinater

$$\left(x-\frac{1}{2}\right)^2+y^2 \le \frac{1}{4}$$

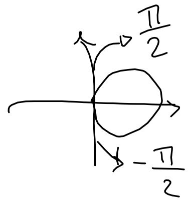
Polarkoord: $\left(r\cos\theta-\frac{1}{2}\right)^2+(r\sin\theta)^2 \le \frac{1}{4}$

$$r^2\cos^2\theta + r^2\sin^2\theta - r\cos\theta + \cancel{\tfrac{1}{4}} \le \cancel{\tfrac{1}{4}}$$

$$r^2 - r\cos\theta \le 0$$

$$r^2 \le r\cos\theta \qquad r>0$$

$$r \le \cos\theta$$



Fra figuren er sirkelen inneholdt i kvadrant 1 og 4, dvs.
$\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. I tillegg: $r \in [0, \cos\theta]$.

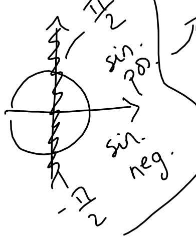
$$A = \iint_R \frac{1}{\sqrt{1-x^2-y^2}}\,dx\,dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\cos\theta} \frac{1}{\sqrt{1-r^2}}\, r\, dr\, d\theta$$

$(1-r^2)^{-\frac{1}{2}}$

substitusjon
$u = 1 - r^2$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[-\sqrt{1-r^2} \right]_{r=0}^{\cos\theta} d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 - \sqrt{\underbrace{1-\cos^2\theta}_{\sin^2\theta}}\right) d\theta$$



$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - |\sin\theta|)\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\, d\theta - \int_{-\frac{\pi}{2}}^{0} |\sin\theta|\, d\theta$$

$$- \int_{0}^{\frac{\pi}{2}} |\sin\theta|\, d\theta$$

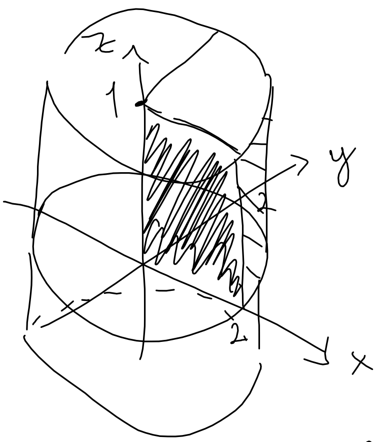
$$= \pi + \int_{-\frac{\pi}{2}}^{0} \sin\theta\, d\theta - \int_{0}^{\frac{\pi}{2}} \sin\theta\, d\theta$$

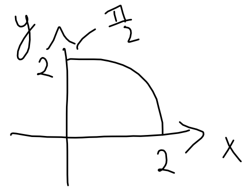
$$= \pi + \left[-\cos\theta\right]_{\theta=-\frac{\pi}{2}}^{0} - \left[-\cos\theta\right]_{\theta=0}^{\frac{\pi}{2}}$$

$$= \pi - 1 - 1 = \underline{\underline{\pi - 2}}$$

10.) $\iint_T xyz^2\, dS$, T; del av sylinder $x^2+y^2=4$ $= 2^2$

$x \geq 0$, $y \geq 0$ og $0 \leq z \leq 1$.





Sylinderen kan parametriseres:

$\vec{r}(\theta, z) = (2\cos\theta, 2\sin\theta, z)$

der $\theta \in [0, 2\pi]$ og $z \in [0, 1]$

Da er: $\dfrac{\partial \vec{r}}{\partial \theta} = (-2\sin\theta, 2\cos\theta, 0)$

$\dfrac{\partial \vec{r}}{\partial z} = (0, 0, 1)$

$$\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial z} = (2\cos\theta, 2\sin\theta, 0)$$

$$\left|\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial z}\right| = \sqrt{4\cos^2\theta + 4\sin^2\theta} = \underline{2}$$

$\Downarrow$

Flateint. er:

$$\iint_T xyz^2\, dS = \iint_T r\cos\theta\, r\sin\theta\, z^2 \left|\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial z}\right| dz\, d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 8z^2 \sin\theta\cos\theta\, dz\, d\theta = \int_0^{\frac{\pi}{2}}\int_0^1 4z^2 \sin(2\theta)\, dz\, d\theta$$

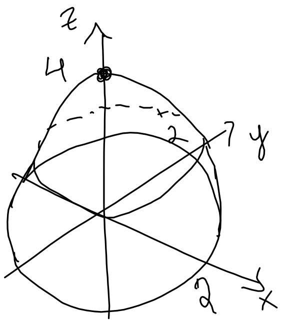
Trig. formel

$$= \int_0^{\frac{\pi}{2}} \left[\frac{4}{3} z^3 \right]_{z=0}^{1} \sin(2\theta)\, d\theta$$

$$= \frac{4}{3} \int_0^{\frac{\pi}{2}} \sin(2\theta)\, d\theta = \left[-\frac{2}{3} \cos(2\theta) \right]_{\theta=0}^{\frac{\pi}{2}}$$

$$= \frac{2}{3} + \frac{2}{3} = \underline{\underline{\frac{4}{3}}}$$

17.) $D \subseteq \mathbb{R}^3$; over xy-plan, inni $z = 4 - x^2 - y^2$ og $x^2 + y^2 = 1$



Figur:

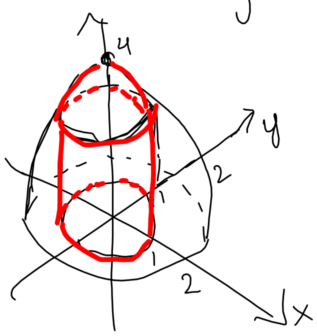
Paraboloide: $x = y = 0 \Rightarrow z = 4$

$z = 4 - x^2 - y^2 = 4 - r^2$ (i polar-koord)

Når r øker, så synker z!

Skjæring xy-plan; dvs. $z = 0$:

$4 - x^2 - y^2 = 0 \Rightarrow x^2 + y^2 = 4 = 2^2$



Hva er skjæringen mellom paraboloiden & sylinderen?

Skjær når $r = 1$. Nok å se på paraboloiden for $r \in [0,1]$. Da får vi sylinder i bunn & paraboloiden på toppen.

$$V = \iint_A \underbrace{f(x,y)}_{z}\, dx\, dy = \int_0^{2\pi}\int_0^{1} (4 - r^2)\, r\, dr\, d\theta$$

$$= 2\pi \left[2r^2 - \frac{1}{4} r^4 \right]_{r=0}^{1} = 2\pi \left(2 - \frac{1}{4}\right)$$

$$= \underline{\underline{\frac{7\pi}{2}}}$$

b) $z = f(x,y) = 4 - x^2 - y^2$

$$\frac{\partial f}{\partial x} = -2x \qquad \frac{\partial f}{\partial y} = -2y$$

$$\text{Areal} = \iint_A \sqrt{1 + 4x^2 + 4y^2}\, dx\,dy$$

$$= \int_0^1 \int_0^{2\pi} \sqrt{1+4r^2}\; r\, d\theta\, dr$$

polar-koord: a)

$$= \ldots = \frac{\pi}{4}\left[\frac{2}{3} u^{\frac{3}{2}}\right]_{u=1}^{5} = \ldots =$$

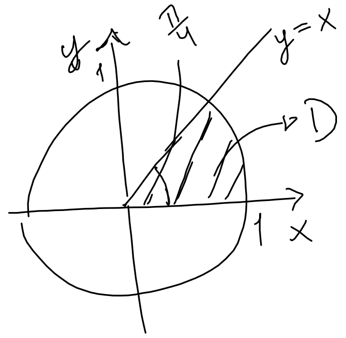
$u = 1+4r^2$
$du = 8r\,dr$
$r=0 \Rightarrow u=1$
$r=1 \Rightarrow u=5$

$$= \frac{\pi}{6}\left(5\sqrt{5} - 1\right)$$

6.5:

10.) $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1,\ x \geq 0,\ y \geq 0,\ 0 \leq y \leq x\}$

a)



Polarkoordinater:

$x = r\cos\theta$
$y = r\sin\theta$ $\Rightarrow$ $r \in [0,1]$, $\theta \in [0, \frac{\pi}{4}]$

$$\iint_D (x + y^2)\,dx\,dy = \int_0^{\frac{\pi}{4}} \int_0^1 (r\cos\theta + r^2\sin^2\theta)\, r\, dr\, d\theta$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

Trig formel