Plenum 29/3

$\boxed{4.11.8}$ a) A

- 10% nye kjøpere (målt i antall tidligere abr.)
- 10% $A \to B$
- 10% $B \to A$

$$x_{n+1} = 0.1 x_n + 0.9 x_n + 0.1 y_n$$

B

- 20% nye
- 10% $B \to A$
- 10% $B \to C$
- 10% $A \to B$
- 10% $C \to B$

$\left.\begin{matrix} 10\%\ B\to A \\ 10\%\ B \to C\end{matrix}\right\} 20\%$

antall abr. i C

$$y_{n+1} = 0.2 y_n + 0.8 y_n + 0.1(x_n + z_n)$$

C

- 10% nye
- 10% $B \to C$
- 10% $C \to B$

$$z_{n+1} = 0.1 z_n + 0.1 y_n + 0.9 z_n$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} x_n + 0.1 y_n + 0 \\ 0.1 x_n + y_n + 0.1 z_n \\ 0 + 0.1 y_n + z_n \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0.1 & 0 \\ 0.1 & 1 & 0.1 \\ 0 & 0.1 & 1 \end{pmatrix}}_{M} \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$$

$$\left[\begin{matrix} \det(M - I\lambda) = 0 \\ \det(I\lambda - M) = 0 \end{matrix}\right]$$

b)
$$p(\lambda) = \det \begin{pmatrix} 1-\lambda & 0.1 & 0 \\ 0.1 & 1-\lambda & 0.1 \\ 0 & 0.1 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 1-\lambda & 0.1 \\ 0.1 & 1-\lambda \end{vmatrix} - 0.1 \begin{vmatrix} 0.1 & 1-\lambda \\ 0 & 0.1 \end{vmatrix}$$

$$= \underbrace{(1-\lambda)}\underbrace{(\lambda^2 - 2\lambda + 0.98)} = (1-\lambda)\left(\lambda - \left(1 + \tfrac{\sqrt{2}}{10}\right)\right)\left(\lambda - \left(1 - \tfrac{\sqrt{2}}{10}\right)\right) = 0$$

$$\Rightarrow \lambda = 1,\ 1 + \tfrac{\sqrt{2}}{10},\ 1 - \tfrac{\sqrt{2}}{10}$$

$\underline{\lambda_1 = 1}$: $M - 1 I_3 = \begin{pmatrix} 0 & 0.1 & 0 \\ 0.1 & 0 & 0.1 \\ 0 & 0.1 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Samme for λ_2 og λ_3:

$\lambda_2: v_2 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$, $\lambda_3: v_3 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$

c)
$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \vec{x}_0 = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3$$

$$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = \vec{x}_n = M\vec{x}_{n-1} = MM\vec{x}_{n-2} = M^2 \vec{x}_{n-2} = \dots = M^n \vec{x}_0 = M^n\left(c_1\vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3\right)$$

$$= c_1 M^n \vec{v}_1 + c_2 M^n \vec{v}_2 + c_3 M^n \vec{v}_3$$

$$= c_1 1^n \vec{v}_1 + c_2\left(1 + \tfrac{\sqrt{2}}{10}\right)^n \vec{v}_2 + c_3\left(1 - \tfrac{\sqrt{2}}{10}\right)^n \vec{v}_3$$

$$= \begin{pmatrix} c_1 + c_2\left(1+\frac{\sqrt{2}}{10}\right)^n + c_3\left(1-\frac{\sqrt{2}}{10}\right)^n \\ 0 + c_2\left(1+\frac{\sqrt{2}}{10}\right)^n \cdot \sqrt{2} + c_3\left(1-\frac{\sqrt{2}}{10}\right)^n \cdot (-\sqrt{2}) \\ -c_1 + c_2\left(1+\frac{\sqrt{2}}{10}\right)^n + c_3\left(1-\frac{\sqrt{2}}{10}\right)^n \end{pmatrix}$$

$$\lim_{n\to\infty} \frac{c_2\cancel{\left(1+\frac{\sqrt{2}}{10}\right)^n}}{\sqrt{2}c_2\cancel{\left(1+\frac{\sqrt{2}}{10}\right)^n}} = \frac{1}{\sqrt{2}}$$

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{\cancel{c_1} + c_2\cancel{\left(1+\frac{\sqrt{2}}{10}\right)^n} + \cancel{c_3\left(1-\frac{\sqrt{2}}{10}\right)^n}}{c_2\cancel{\left(1+\frac{\sqrt{2}}{10}\right)^n}\cdot\sqrt{2} + \cancel{c_3\left(1-\frac{\sqrt{2}}{10}\right)^n \cdot(-\sqrt{2})}} = \underline{\underline{\frac{1}{\sqrt{2}}}}$$

<1, $0 < \cdot < 1$

$$\lim_{n\to\infty} \frac{y_n}{z_n} = \frac{\sqrt{2}}{1} = \underline{\underline{\sqrt{2}}}$$

Eks. 2006

1) a)

$$C = \left(\begin{array}{ccc|c} \multicolumn{3}{c|}{\overbrace{\hphantom{1 \quad 0 \quad a^2-a}}^{A}} & \overbrace{\hphantom{1}}^{\vec{b}} \\ 1 & 0 & 1 & 1 \\ 2 & 1 & a^2-a & 3 \\ -1 & 1 & -3 & a \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & a^2-a-2 & 1 \\ 0 & 0 & -a^2+a & a \end{array}\right)$$

1) $II - 2I$ 2) $III + I - II$

b) $A\vec{x} = \vec{b}$

én: $-a^2+a \neq 0 \iff -a(a-1) \neq 0 \iff a \neq 0,\ a \neq 1$

ingen: $\underbrace{-a^2+a=0}$ & $\underbrace{a \neq 0}$ $\iff a = 1$

uendelig mange: $-a^2+a=0$ & $a=0 \iff a=0$

Eks. 2007

2) $A = \begin{pmatrix} 0.9 & 0.4 \\ 0.2 & 1.1 \end{pmatrix}$

a) $\det(A - \lambda I) = \begin{vmatrix} 0.9-\lambda & 0.4 \\ 0.2 & 1.1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 0.91$

$\lambda = 0.7,\ 1.3$ $\qquad x + 2y = 0 \quad x = -2y$

$\lambda = 0.7: \begin{pmatrix} 0.2 & 0.4 \\ 0.2 & 0.4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\lambda = 1.3: \begin{pmatrix} -0.4 & 0.4 \\ 0.2 & -0.2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

b) $\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \vec{X}_{n+1} = A\vec{X}_n \quad , \quad \vec{X}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$

$\left(\begin{array}{cc|c} 1 & 2 & 6 \\ 1 & -1 & 3 \end{array}\right) \sim \begin{pmatrix} 1 & 2 & 6 \\ 0 & -3 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 6 \\ 0 & 1 & 1 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \end{pmatrix}$

$\begin{pmatrix} 6 \\ 3 \end{pmatrix} = v_1 + 4v_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + 4\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2+4 \\ -1+4 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$

$\vec{X}_n = A\vec{X}_{n-1} = \ldots = A^n\vec{X}_0 = A^n(v_1 + 4v_2) = \underbrace{A^n v_1}_{\lambda_1^n v_1} + 4\underbrace{A^n v_2}_{\lambda_2^n v_2}$

$= 0.7^n\begin{pmatrix} 2 \\ -1 \end{pmatrix} + 4 \cdot 1.3^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 2 \cdot 0.7^n + 4 \cdot 1.3^n \\ -0.7^n + 4 \cdot 1.3^n \end{pmatrix}$

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(6)

$$A = \begin{pmatrix} 0.2 & 0.4 & 0.4 \\ \cdot & \cdot & \cdot \\ \cdot & - & - \end{pmatrix} \quad \Sigma = 1$$

a) $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ } n lang , $A\vec{v} = \lambda \vec{v}$

$$A\vec{v} = \begin{pmatrix} a_{11}v_1 + a_{12}v_2 + \dots + a_{1n} \cdot v_n \\ \vdots \\ a_{n1}v_1 + a_{n2}v_2 + \dots + a_{nn}v_n \end{pmatrix} = \begin{pmatrix} a_{11} + a_{12} + \dots + a_{1n} \\ \vdots \\ a_{n1} + a_{n2} + \dots + a_{nn} \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \underset{\lambda}{1} \cdot \underset{\vec{v}}{\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}}$$

b) $\boxed{|\lambda| \leq 1}$

λ egenverdi m/egenvektor $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_i \\ \vdots \\ v_n \end{pmatrix} \leftarrow |v_i| \geq |v_j| \quad \forall\ j \in \{1, 2, \dots, n\}$

$$|\lambda||v_i| = |\lambda v_i| = \left| \sum_{j=1}^{n} a_{ij} v_j \right| \leq \sum_{j=1}^{n} a_{ij} |v_j| \leq \sum_{j=1}^{n} a_{ij} |v_i| = |v_i|$$

$|v_i| \cdot \underbrace{\sum_{j=1}^{n} a_{ij}}_{=1}$

trekantulik.

$|v_i| \geq |v_j| \quad \forall j$

$A\vec{v} = \lambda \vec{v}$

$$(A\vec{v})_i = \boxed{\sum_{j=1}^{n} a_{ij} v_j = \lambda v_i}$$

$$\frac{|\lambda| \cdot |v_i|}{|v_i|} \leq \frac{|v_i|}{|v_i|} \Rightarrow \underline{\underline{|\lambda| \leq 1}}$$