Oppgaver: 3.8: 1, 2, 3.9: 1, 2, 3, 5, 7, 8, 10, 11, 14
6.1: 1a) d) e) f) g), 2 b) c), 3, 4, 6, 7

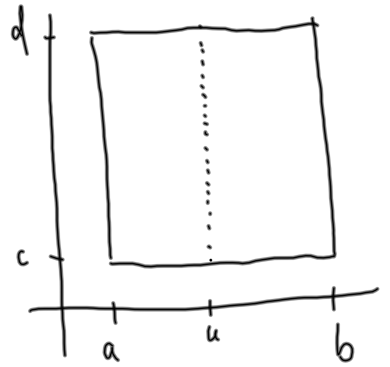
OBS: Nytt kontor: B533 i 5 et. NHAs Hus

6.1.3:

Vi skriver $\Pi \subset \Pi'$, dersom alle punkter i partisjonen Π er med i partisjonen Π'.

Da er

$$N(\Pi) \le N(\Pi') \le \varnothing(\Pi') \le \varnothing(\Pi)$$



Π

$a = x_0 < x_1 = b$

$c = y_0 < y_1 = d$

Π'

$a = x_0 < u < x_1 = b$

$c = y_0 < y_1 = d$

$\Pi \subset \Pi'$

$$N(\Pi) \le N(\Pi') \le \varnothing(\Pi') \le \varnothing(\Pi)$$

$m = \min\{f(x,y) : (x,y) \in [a,b] \times [c,d]\}$, $M = \max\{f(x,y) : (x,y) \in [a,b] \times [c,d]\}$

$$N(\Pi) = m(b-a)(d-c), \quad \varnothing(\Pi) = M(b-a)(d-c)$$

$m_1 = \min\{f(x,y) : (x,y) \in [a,u] \times [c,d]\}$, $m_2 = \min\{f(x,y) : (x,y) \in [u,b] \times [c,d]\}$

$$N(\Pi') = m_1(u-a)(d-c) + m_2(b-u)(d-c)$$

$M_1 = \max\{f(x,y) : (x,y) \in [a,u] \times [c,d]\}$, $M_2 = \max\{f(x,y) : (x,y) \in [u,b] \times [c,d]\}$

$$\varnothing(\Pi') = M_1(u-a)(d-c) + M_2(b-u)(d-c)$$

$$m \le m_1, \quad m \le m_2, \quad M_1 \le M, \quad M_2 \le M$$

$$N(\Pi') = m_1(u-a)(d-c) + m_2(b-u)(d-c) \ge m(u-a)(d-c) + m(b-u)(d-c)$$
$$= m(b-a)(d-c) = N(\Pi)$$

$$\varnothing(\Pi') = M_1(u-a)(d-c) + M_2(b-u)(d-c)$$
$$\le M(u-a)(d-c) + M(b-u)(d-c) = M(b-a)(d-c) = \varnothing(\Pi)$$

$$N(\Pi) \le N(\Pi') \le \varnothing(\Pi') \le \varnothing(\Pi)$$

6.1.3. La Π_1, Π_2 være to partisjoner. Da er

$$N(\Pi_1) \leq \emptyset(\Pi_2)$$

Bevis: La Π være partisjonen som inneholder alle punkter fra Π_1 og Π_2

Da er $\Pi_1 \subset \Pi$, og $\Pi_2 \subset \Pi$

Da blir

$$N(\Pi_1) \leq N(\Pi) \leq \emptyset(\Pi) \leq \emptyset(\Pi_2) \blacksquare$$

6.1.4: f integrerbar $\Leftrightarrow$ for alle $\varepsilon > 0$, fins Π slik at

$$\emptyset(\Pi) - N(\Pi) < \varepsilon$$

Bevis:

$\Leftarrow$: Anta at for alle $\varepsilon > 0$, fins Π slik at $\emptyset(\Pi) - N(\Pi) < \varepsilon$

Da er f integrerbar.

La $\varepsilon > 0$. Fra definisjon av $\overline{\iint_R} f := \overline{\iint_R} f(x,y)\,dx\,dy$, velg en partisjon

Π_1 slik at $\overline{\iint_R} f - \emptyset(\Pi_1) < \frac{\varepsilon}{3}$

Tilsvarende for $\underline{\iint_R} f$, velg Π_2 slik at $N(\Pi_2) - \underline{\iint_R} f < \frac{\varepsilon}{3}$

Velg Π_3 slik at $\emptyset(\Pi_3) - N(\Pi_3) < \frac{\varepsilon}{3}$

La nå Π bestå av alle punkter fra Π_1, Π_2 og Π_3. Da er

$$\overline{\iint_R} f - \emptyset(\Pi) \leq \overline{\iint_R} f - \emptyset(\Pi_1) < \frac{\varepsilon}{3}$$

$$N(\Pi) - \underline{\iint_R} f \leq N(\Pi_2) - \underline{\iint_R} f < \frac{\varepsilon}{3}$$

og

$$\emptyset(\Pi) - N(\Pi) \leq \emptyset(\Pi_3) - N(\Pi_3) < \frac{\varepsilon}{3}$$

$$\left|\overline{\iint_R} f - \underline{\iint_R} f\right| = \left|\overline{\iint_R} f - \emptyset(\Pi) + \emptyset(\Pi) - N(\Pi) + N(\Pi) - \underline{\iint_R} f\right|$$

$$\leq \left|\overline{\iint_R} f - \emptyset(\Pi)\right| + \left|\emptyset(\Pi) - N(\Pi)\right| + \left|N(\Pi) - \underline{\iint_R} f\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Da må $\overline{\iint} f = \iint f$

Anta at f er integrerbar, $\overline{\iint_R} f = \underline{\iint_R} f$

La $\varepsilon > 0$. Velg Π_1 slik at

$$\left| \overline{\iint_R} f - \emptyset(\Pi_1) \right| < \frac{\varepsilon}{2}, \quad \left(\emptyset(\Pi_1) - \overline{\iint_R} f < \frac{\varepsilon}{2} \right)$$

velg Π_2 slik at

$$\left| N(\Pi_2) - \underline{\iint_R} f \right| < \frac{\varepsilon}{2}$$

La Π være partisjonen som består av alle punkter fra Π_1 og Π_2.

- $\left| \overline{\iint_R} f - \emptyset(\Pi) \right| \leq \left| \overline{\iint_R} f - \emptyset(\Pi_1) \right| < \frac{\varepsilon}{2}$
- $\left| N(\Pi) - \underline{\iint_R} f \right| \leq \left| N(\Pi_2) - \underline{\iint_R} f \right| < \frac{\varepsilon}{2}$

Da blir $\left| \emptyset(\Pi) - N(\Pi) \right| = \left| \emptyset(\Pi) - \iint f + \iint f - N(\Pi) \right|$

$$= \left| \emptyset(\Pi) - \overline{\iint_R} f + \underline{\iint_R} f - N(\Pi) \right| \leq \left| \emptyset(\Pi) - \overline{\iint} f \right| + \left| \underline{\iint} f - N(\Pi) \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

<u>6.1.5</u>:

Vi skriver

$$\emptyset(\Pi, f) = \sum_i \sum_j M_{ij} |R_{ij}|, \quad M_{ij} = \max\{ f(x,y) : (x,y) \in R_{ij} \}$$

tilsvarende for $N(\Pi, f)$

i) Dersom f er integrerbar, så kf integrerbar,

$$\iint_R k f(x,y)\,dx\,dy = k \iint_R f(x,y)\,dxy$$

<u>Bevis</u>: Må vise at for alle $\varepsilon > 0$, fins Π slik at

$$\emptyset(\Pi, kf) - N(\Pi, kf) < \varepsilon$$

Velg Π slik at

$$\emptyset(\Pi, f) - N(\Pi, f) < \frac{\varepsilon}{|k|}$$

Observer at $\emptyset(\Pi, kf) = k\,\emptyset(\Pi, f)$, $N(\Pi, kf) = k\,N(\Pi, f)$

Da blir

$$\emptyset(\Pi, kf) - N(\Pi, kf) = k\left(\emptyset(\Pi, f) - N(\Pi, f) \right) < k \frac{\varepsilon}{|k|} = \varepsilon$$

ii) f, g integrerbare $\Rightarrow$ $f+g$ integrerbar og

$$\iint_R (f+g)(x,y)\,dx\,dy = \iint_R f(x,y)\,dx\,dy + \iint_R g(x,y)\,dx\,dy$$

Bevis: La $\varepsilon > 0$, vil vise at det fins Π slik at

$$\emptyset(\Pi, f+g) - N(\Pi, f+g) < \varepsilon$$

Velg Π_1 slik at

$$\emptyset(\Pi_1, f) - N(\Pi_1, f) < \frac{\varepsilon}{2}$$

Velg Π_2 slik at

$$\emptyset(\Pi_2, g) - N(\Pi_2, g) < \frac{\varepsilon}{2}$$

La Π være partisjonen som består av alle punkter fra Π_1 og Π_2

$$\emptyset(\Pi, f) - N(\Pi, f) \leq \emptyset(\Pi_1, f) - N(\Pi_1, f) < \frac{\varepsilon}{2}$$

$$\emptyset(\Pi, g) - N(\Pi, g) \leq \emptyset(\Pi_2, g) - N(\Pi_2, g) < \frac{\varepsilon}{2}$$

Merk at

$$\emptyset(\Pi, f+g) = \emptyset(\Pi, f) + \emptyset(\Pi, g)$$

$$N(\Pi, f+g) = N(\Pi, f) + N(\Pi, g)$$

Da blir

$$\emptyset(\Pi, f+g) - N(\Pi, f+g) = \emptyset(\Pi, f) + \emptyset(\Pi, g) - N(\Pi, f) - N(\Pi, g)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \blacksquare$$

iii) Hvis $f(x,y) \leq g(x,y)$ så er $\iint_R f(x,y)\,dx\,dy \leq \iint_R g(x,y)\,dx\,dy$

Bevis: La $h(x,y) \geq 0$, skal vise at

$$\iint_R h(x,y)\,dx\,dy \geq 0$$

Men siden $N(\Pi, h) = \sum_i \sum_j m_{ij} |R_{ij}|$, $m_{ij} = \min\{h(x,y) : (x,y) \in R_{ij}\}$

$m_{ij} \geq 0$.

Da må $\iint h(x,y)\,dx\,dy \geq 0$, som vi skulle vise.

Anta $g(x,y) \geq f(x,y)$. Definér $h(x,y) = g(x,y) - f(x,y)$. Da er

$h(x,y) \geq 0$, $\iint_R h(x,y)\,dx\,dy \geq 0$

Men

$$0 \leq \iint_R h(x,y)\,dx\,dy = \iint_R g(x,y) - f(x,y)\,dx\,dy = \iint_R g(x,y)\,dx\,dy - \iint_R f(x,y)\,dx\,dy$$

så

$$\iint_R f(x,y)\,dx\,dy \leq \iint_R g(x,y)\,dx\,dy \quad \blacksquare$$

<u>6.1.7.</u> $f : R \to \mathbb{R}$ kont

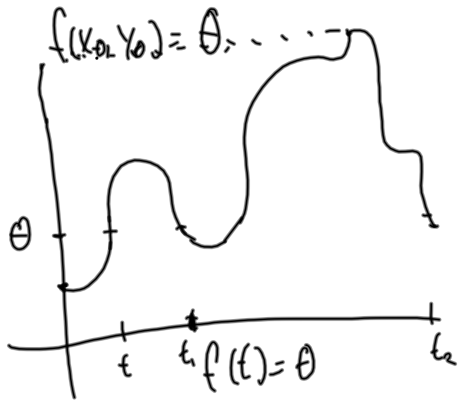
Vis at det finnes et punkt $(\bar{x},\bar{y}) \in R$ slik at

$$\frac{\iint_R f(x,y)\,dx\,dy}{|R|} = f(\bar{x},\bar{y})$$

<u>Bevis:</u> $m = \min\{f(x,y) : (x,y) \in R\}$, $M = \max\{f(x,y) : (x,y) \in R\}$

$m \le f(x,y) \le M$ for alle $(x,y) \in R$.

Men fra skjæringssetningen, siden f er kont., for alle tall $\theta \in [m,M]$, fins et punkt (x_θ, y_θ) slik at



$m \le f(x,y) \le M$

Da blir $\iint_R m\,dx\,dy \le \iint_R f(x,y)\,dx\,dy \le \iint_R M\,dx\,dy$

$$|R|\,m \le \iint_R f(x,y)\,dx\,dy \le |R|\,M$$

så

$$m \le \frac{\iint_R f(x,y)\,dx\,dy}{|R|} \le M$$

så $\dfrac{\iint_R f(x,y)\,dx\,dy}{|R|} \in [m,M]$, så fra skjæringssetningen fins $(\bar{x},\bar{y}) \in R$ slik at

$$f(\bar{x},\bar{y}) = \frac{\iint_R f(x,y)\,dx\,dy}{|R|}$$