6.9.2 d)

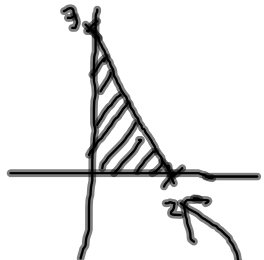
$$\iiint_A (3y^2-3z)\,dx\,dy\,dz$$

$3x+2y-z=6$

Skjæring med xy-planet:

$3x+2y=6$

$y=-\frac{3}{2}x+3$



$x=y=0 \Rightarrow z=-6$

$\Rightarrow 3x+2y-z=6$ ligger under xy-planet i området vi ser på

$$=\int_0^2\int_0^{-\frac{3}{2}x+3}\int_{3x+2y-6}^{0}(3y^2-3z)\,dz\,dy\,dx$$

$$=\int_0^2\int_0^{-\frac{3}{2}x+3}\left[3y^2z-\frac{3}{2}z^2\right]_{3x+2y-6}^{0}dy\,dx$$

$$=\int_0^2\int_0^{-\frac{3}{2}x+3}\left(\underline{-3y^2(3x+2y-6)}+\frac{3}{2}(3x+2y-6)^2\right)dy\,dx$$

$$=\int_0^2\left(\left[-\frac{3}{4}y^2(3x+2y-6)^2\right]_0^{-\frac{3}{2}x+3}+\int_0^{-\frac{3}{2}x+3}\underline{\frac{3}{2}y}\,\underline{(3x+2y-6)^2}\,dy\right)dx+\int_0^2\left[\frac{1}{4}(3x+2y-6)^3\right]_0^{-\frac{3}{2}x+3}$$

$$=\int_0^2\left(\underline{\left[\frac{1}{4}y(3x+2y-6)^3\right]_0^{-\frac{3}{2}x+3}}-\int_0^{-\frac{3}{2}x+3}\frac{1}{4}(3x+2y-6)^3\,dy\right)dx-\int_0^2\frac{1}{4}(3x-6)^3\,dx$$

$$=\int_0^2\left(\left[-\frac{1}{32}(3x+2y-6)^4\right]_0^{-\frac{3}{2}x+3}-\frac{1}{4}(3x-6)^3\right)dx$$

$$=\int_0^2\frac{1}{32}(3x-6)^4-\frac{1}{4}(3x-6)^3\,dx=\left[\frac{1}{480}(3x-6)^5-\frac{1}{48}(3x-6)^4\right]_0^2$$

$$=-\frac{1}{480}(-6)^5+\frac{1}{48}(-6)^4=\cdots=\underline{\underline{\frac{216}{5}}}$$

6.10.1 c

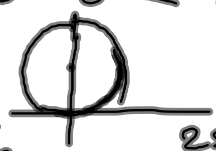
$\iiint_A z\sqrt{x^2+y^2}$

$x^2+(y-1)^2 \le 1 \rightarrow x = r\cos\theta,\ y = r\sin\theta$ $\qquad 0 \le z \le 2$

$r^2\cos^2\theta + r^2\sin^2\theta - 2r\sin\theta + 1 \le 1$

$r^2 - 2r\sin\theta \le 0$

$0 \le r \le 2\sin\theta \qquad 0 \le \theta \le \pi$



$$= \int_0^{\pi}\int_0^{2\sin\theta}\int_0^{2} z\cdot r\cdot r\,dz\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{2\sin\theta}\left[\frac{1}{2}z^2r^2\right]_0^2 dr\,d\theta = \int_0^{\pi}\int_0^{2\sin\theta} 2r^2\,dr\,d\theta = \int_0^{\pi}\left[\frac{2}{3}r^3\right]_0^{2\sin\theta} d\theta$$

$$= \int_0^{\pi}\frac{16}{3}\sin^3\theta\,d\theta = \int_0^{\pi}\frac{16}{3}(1-\cos^2\theta)\sin\theta\,d\theta = \left[-\frac{16}{3}\left(\cos\theta - \frac{1}{3}\cos^3\theta\right)\right]_0^{\pi}$$

$u = \cos\theta$

$$= -\frac{16}{3}\left(-1+\frac{1}{3}\right) + \frac{16}{3}\left(1-\frac{1}{3}\right) = \underline{\underline{\frac{64}{9}}}$$

6.10.2 a

$$\iiint_A (x^2+y^2)\,dx\,dy\,dz = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^2\sin^2\phi\,\rho^2\sin\phi \quad d\rho\,d\phi\,d\theta$$

$$x^2+y^2=\rho^2\sin^2\phi$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^4\sin^3\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{1}{5}\sin^3\phi\,d\phi\,d\theta$$

$$\left(\int \sin^3\phi\,d\phi = \int (1-\cos^2\phi)\sin\phi\,d\phi \right.$$

$$u=\cos\phi$$

6.10.2 c

$$\iiint_A 1 \, dxydz$$

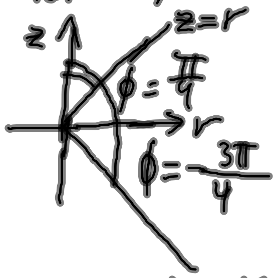
A: $z^2 \le \underbrace{x^2+y^2}_{\rho^2 \sin^2\phi \le 1} \le 1, \quad 3y^2 \le x^2, \quad x \ge 0$

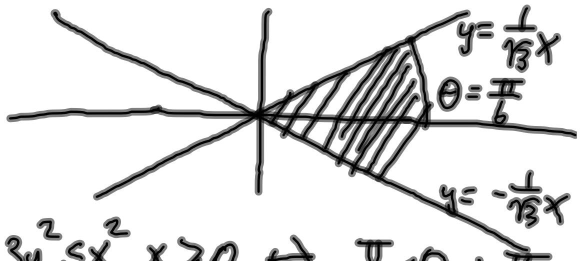
$z^2 \le r^2$

$|z| \le |r| = r$

$\rho \le \frac{1}{\sin\phi}$

$|\sqrt{3}y| \le |x|$





$3y^2 \le x^2, x \ge 0 \Leftrightarrow -\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^{\frac{1}{\sin\phi}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{3\sin^2\phi} \, d\phi \, d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \left[-\frac{1}{3} \frac{\cos\phi}{\sin\phi} \right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\theta$$

$$= \cdots = \frac{2}{9}\pi$$

6.10.7

$$\iiint_A \frac{1}{\sqrt{x^2+y^2+(z-a)^2}}\,dx\,dy\,dz$$

A kule om origo, radius R

$a > R$

Følger eksempel 4: $\sqrt{x^2+y^2+(z-a)^2} = \sqrt{\rho^2 - 2\rho a\cos\phi + a^2}$

$$I = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \frac{\rho^2 \sin\phi}{\underbrace{\sqrt{\rho^2-2a\rho\cos\phi+a^2}}_{u}}\,d\rho\,d\phi\,d\theta \longrightarrow \frac{\rho}{2a}\frac{1}{\sqrt{u}}\,du$$

$du = 2a\rho\sin\phi$

$\frac{\rho}{2a}du = \rho^2 \sin\phi$

$$= \int_0^{2\pi}\int_0^{R} \left[\frac{\rho}{2a} 2\sqrt{\rho^2-2a\rho\cos\phi+a^2}\right]_0^{\pi} d\rho\,d\theta$$

$$= \int_0^{2\pi}\int_0^{R} \frac{\rho}{a}\left(\sqrt{\rho^2+2a\rho+a^2} - \sqrt{\rho^2-2a\rho+a^2}\right) d\rho\,d\theta = \int_0^{2\pi}\int_0^{R} \frac{\rho}{a}\left(a+\rho-(a-\rho)\right) d\rho\,d\theta$$

$$= \int_0^{2\pi}\int_0^{R} \frac{2\rho^2}{a}\,d\rho\,d\theta = \int_0^{2\pi}\left[\frac{2}{3}\frac{\rho^3}{a}\right]_0^R d\theta = \frac{2}{3}\frac{R^3}{a}2\pi = \underline{\underline{\frac{4\pi}{3}\frac{R^3}{a}}}$$

6.11.4 $\quad x^2+y^2=1 \iff r=1$

$$x^2+y^2+z^2=4 \iff r^2+z^2=4$$

$$z=\pm\sqrt{4-r^2}$$

$$V=\int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^1\left[rz\right]_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}}dr\,d\theta,$$

$$=\int_0^{2\pi}\int_0^1 2r\underbrace{\sqrt{4-r^2}}_{u}\,dr\,d\theta=\int_0^{2\pi}\left[-\frac{2}{3}(4-r^2)^{\frac{3}{2}}\right]_0 d\theta$$

$$=2\pi\left(-\frac{2}{3}3^{3/2}+\frac{2}{3}4^{\frac{3}{2}}\right)=\cdots=\underline{\underline{\frac{4}{3}\pi(8-3\sqrt{3})}}$$

6.11.5

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$\Updownarrow$$

$$u^2+v^2+w^2=1$$

$$u=\frac{x}{a} \quad v=\frac{y}{b} \quad w=\frac{z}{c}$$

$$\vec{T}(x,y,z)=(u,v,w)=\left(\frac{x}{a},\frac{y}{b},\frac{z}{c}\right)$$

$$\vec{T}'(x,y,z)=\begin{pmatrix}\frac{1}{a} & 0 & 0\\ 0 & \frac{1}{b} & 0\\ 0 & 0 & \frac{1}{c}\end{pmatrix}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=\frac{1}{abc} \Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)}=abc$$

$$V=\iiint_E dx\,dy\,dz=\iiint_K \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw=\iiint_K abc\,du\,dv\,dw$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\cdots$$

$$=\frac{4}{3}\pi\,abc$$

6.11.6 $\quad x^2+y^2 \le 1 \quad 0 \le z \le 1 \quad$ tetthet: $f(x,y,z) = \frac{1}{x^2+y^2+z^2}$

$$M = \iiint_V \frac{1}{x^2+y^2+z^2}\,dx\,dy\,dz$$

$$= \int_0^{2\pi}\int_0^1\int_0^1 \frac{1}{r^2+z^2}\, r\,dz\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^1\int_0^1 \frac{2r}{\underbrace{r^2+z^2}_{u}}\,dr\,dz\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^1 \Big[\ln(r^2+z^2)\Big]_0^1\,dz\,d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^1 \big(\ln(1+z^2) - \ln(z^2)\big)\,dz\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^1 \ln(1+z^{-2})\,dz\,d\theta = \frac{1}{2}\int_0^{2\pi}\left(\Big[z\ln(1+z^{-2})\Big]_{0 \leftarrow \text{NB!}}^1 - \int z\,\frac{-2z^{-3}}{1+z^{-2}}\,dz\right)d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(\ln 2 + \int_0^1 \frac{2}{1+z^2}\,dz\right)d\theta = \frac{1}{2}\cdot 2\pi\left(\ln 2 + 2\Big[\arctan z\Big]_0^1\right)$$

$$= \pi\left(\ln 2 + 2\cdot\frac{\pi}{4}\right) = \underline{\underline{\pi\ln 2 + \frac{\pi^2}{2}}}$$