Uegentlige...

1) Beskrive metoden for positive funksjoner

2)

$$f(x) = f_+(x) - f_-(x)$$

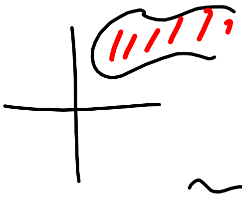
hvor

$$f_+(x) = \begin{cases} f(x) & f(x) > 0 \\ 0 & \text{ellers} \end{cases}$$

og

$$f_-(x) = \begin{cases} 0 & \text{ellers} \\ -f(x) & f(x) < 0 \end{cases}$$

positive funksjoner



$$\rightsquigarrow \iint_A f \, dx\, dy = \iint_A f_+ \, dx\, dy - \iint_A f_- \, dx\, dy$$

$A \subseteq \mathbb{R}^2$

$A \cap K_n$ Jordan-målbar $\forall n$

$f : A \to \mathbb{R}$ begrenset kontinuerlig

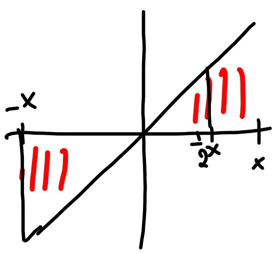
Konvergens, dersom begge konvergerer

Alt.

Konvergens dersom $\iint_A |f| \, dx\, dy$ konvergerer.

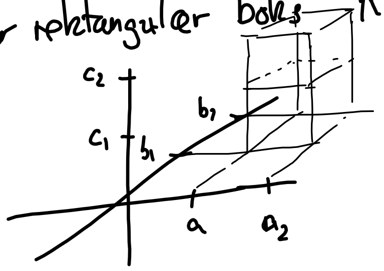
$$|f| = f_+ + f_-$$

Illustrasjon



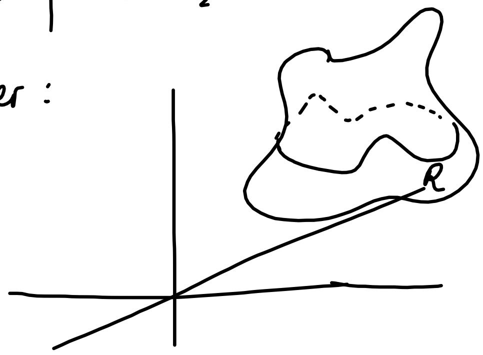
Trippel-integraler

Integral over rektangulær boks $R: [a_1, a_2] \times [b_1, b_2] \times [c_1, c_2]$



Bruker Riemann-sum teknikk.

Generelle områder :



Eks $\quad R: [0,1] \times [1,2] \times [0,2] \qquad f(x,y,z) = x - y^2 + z^3$

$$\iiint_R f \, dx\,dy\,dz = \int_0^1 \left(\int_1^2 \left(\int_0^2 x - y^2 + z^3 \, dz \right) dy \right) dx$$

$$= \int_0^1 \int_1^2 \left[xz - y^2 z + \tfrac{1}{4} z^4 \right]_{z=0}^{z=2} dy\, dx$$

$$= \int_0^1 \int_1^2 2x - 2y^2 + 4 \; dy\, dx$$

$$= \int_0^1 \left[2xy - \tfrac{2}{3} y^3 + 4y \right]_{y=1}^{y=2} dx = \int_0^1 2x - \tfrac{2}{3} \, dx$$

$$= \left[x^2 - \tfrac{2}{3} x \right]_0^1 = \underline{\underline{\tfrac{1}{3}}}$$

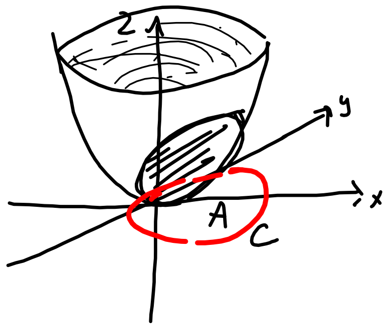
Trippelintegraler over mer generelle områder

$f(x,y,z) = xy$ over S: mellom $z = 2x+4y$ og $z = x^2+y^2$

$\uparrow$ plan $\qquad \uparrow$ paraboloide

$2x+4y-z=0$

$\bar{n}: (2,4,-1)$



skjæringen: $2x+4y = x^2+y^2$

$\Updownarrow$

$(x-1)^2+(y-2)^2 = 5$

sirkel med sentrum i $(1,2)$ og radius $\sqrt{5}$

parametrisering av C: $x = 1 + r\cdot\cos\theta$, $y = 2 + r\sin\theta$

hvor $0 \le \theta \le 2\pi$, $0 \le r \le \sqrt{5}$

$$\iiint_S xy \, dx\, dy\, dz = \iint_A \left(\int_{z=x^2+y^2}^{z=2x+4y} xy \, dz \right) dx\, dy$$

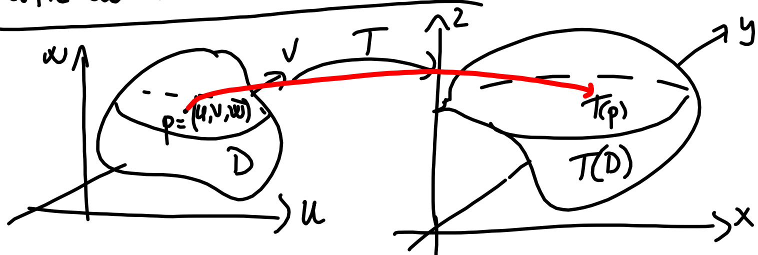
$$= \iint_A \left[xyz \right]_{z=x^2+y^2}^{z=2x+4y} dx\, dy = \iint_A xy\left(2x+4y-x^2-y^2\right) dx\,dy$$

$$= \int_0^{2\pi}\int_0^{\sqrt{5}} (1+r\cos\theta)(2+r\sin\theta)(5-r^2)\, r\, dr\, d\theta$$

$$\boxed{\begin{aligned} &2x+4y-x^2-y^2 \\ &= -\left((x-1)^2+(y-2)^2-5\right) \\ &= -(r^2-5) \\ &= \underline{5-r^2} \end{aligned}}$$

$$= \underline{\underline{25\pi}}$$

Skifte av variable i rommet



$$T(u,v,w) = \left(T_1(u,v,w),\, T_2(u,v,w),\, T_3(u,v,w)\right)$$

Jacobi: $\begin{vmatrix} \frac{\partial T_1}{\partial u} & \frac{\partial T_1}{\partial v} & \frac{\partial T_1}{\partial w} \\ \frac{\partial T_2}{\partial u} & \frac{\partial T_2}{\partial v} & \frac{\partial T_2}{\partial w} \\ \frac{\partial T_3}{\partial u} & \frac{\partial T_3}{\partial v} & \frac{\partial T_3}{\partial w} \end{vmatrix} = \frac{\partial(T_1,T_2,T_3)}{\partial(u,v,w)}$

Da vil:

$$\iiint_{T(D)} f(x,y,z)\,dx\,dy\,dz = \iiint_{D} f(T(u,v,w)) \left|\frac{\partial(T_1,T_2,T_3)}{\partial(u,v,w)}\right| du\,dv\,dw$$

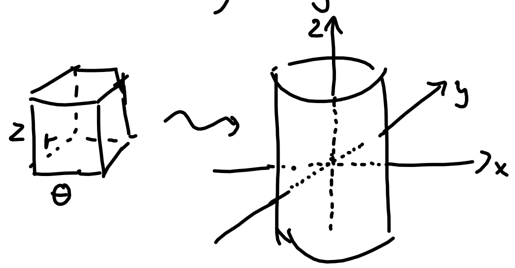
Tre standard metoder:

1) T linear avbildning: $T\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} a_{11}u + a_{12}v + a_{13}w \\ a_{21}u + a_{22}v + a_{23}w \\ a_{31}u + a_{32}v + a_{33}w \end{pmatrix} \begin{matrix} T_1 \\ T_2 \\ T_3 \end{matrix}$

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\left|\frac{\partial T}{\partial(u,v,w)}\right| = \left|\frac{\partial(T_1,T_2,T_3)}{\partial(u,v,w)}\right| = \left|\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}\right|$$

2) Sylinderkoordinater:



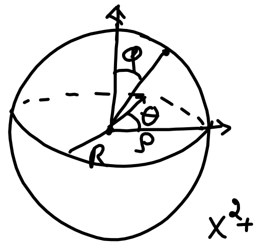
$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \underline{\underline{r}}$$

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

3) Kulekoordinater (sfæriske)



$x = \rho \cdot \cos\theta \cdot \sin\varphi$
$y = \rho \cdot \sin\theta \cdot \sin\varphi$
$z = \rho\cos\varphi$

$\underline{x^2+y^2+z^2 = \rho^2}$

Parameterverdier:
$0 \le \rho \le R$
$0 \le \theta \le 2\pi$
$0 \le \varphi \le \pi$

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\right| = \left|\begin{vmatrix} \cos\theta\sin\varphi & \sin\theta\sin\varphi & \cos\varphi \\ -\rho\sin\theta\sin\varphi & \rho\cos\theta\cdot\sin\varphi & 0 \\ \rho\cos\theta\cos\varphi & \rho\sin\theta\cos\varphi & -\rho\sin\varphi \end{vmatrix}\right|$$

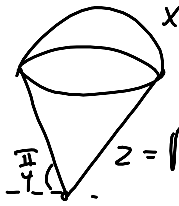
$$= \left| \cos\varphi \begin{vmatrix} -\rho\sin\theta\sin\varphi & \rho\cos\theta\sin\varphi \\ \rho\cos\theta\cos\varphi & \rho\sin\theta\cos\varphi \end{vmatrix} - \rho\sin\varphi \begin{vmatrix} \cos\theta\sin\varphi & \sin\theta\sin\varphi \\ -\rho\sin\theta\sin\varphi & \rho\cos\theta\sin\varphi \end{vmatrix} \right|$$

$$= \left|\rho^2\cos\varphi(-\sin\varphi\cos\varphi) - \rho^2\sin\varphi(\cos^2\theta\sin^2\varphi + \sin^2\theta\sin^2\varphi)\right|$$

$$= \left|\rho^2(-\sin\varphi\cos^2\varphi - \sin\varphi\cdot\sin^2\varphi)\right|$$

$$= \left|\rho^2\sin\varphi\right| = \underline{\underline{\rho^2\cdot\sin\varphi}}$$

Eks



Kulekoordinater med
$0 \le \rho \le 1$
$0 \le \theta \le 2\pi$
$0 \le \varphi \le \frac{\pi}{4}$

$$V = \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^1 1\cdot\rho^2\cdot\sin\varphi \, d\rho\, d\theta\, d\varphi$$

$$= \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\left[\frac{1}{3}\rho^3\cdot\sin\varphi\right]_{\rho=0}^{\rho=1} d\theta\, d\varphi = \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\frac{1}{3}\sin\varphi\, d\theta\, d\varphi$$

$$= \int_0^{\frac{\pi}{4}}\left[\frac{1}{3}\sin\varphi\cdot\theta\right]_{\theta=0}^{\theta=2\pi} d\varphi = \int_0^{\frac{\pi}{4}}\frac{2\pi}{3}\cdot\sin\varphi\, d\varphi$$

$$= \left[\frac{2\pi}{3}(-\cos\varphi)\right]_0^{\frac{\pi}{4}} = \frac{2\pi}{3}\left(-\frac{\sqrt{2}}{2}+1\right) = \underline{\underline{\frac{\pi}{3}(2-\sqrt{2})}}$$

Liten oppgave:

$$\bar{z} = \frac{1}{\text{Volum}}\iiint_R \underset{\uparrow \atop \rho\cos\varphi}{z}\, dx\,dy\,dz$$

Svar: $\frac{3}{8(2-\sqrt{2})}$ (?)