Plenum 16/2-16

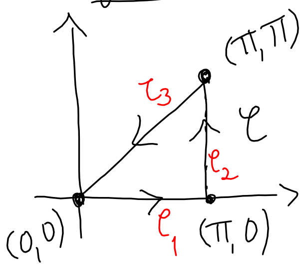
3.4: 8, 14, 15 a)-c), (12)

3.5: 9, (11)

3.6: 9, 10

3.4: Linjeintegraler for vektorfelt

8) $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$;

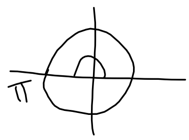


$\vec{F}(x,y) = (\cos(x)\sin(y), x)$

$= \cos(x)\sin(y)\,\vec{i} + x\,\vec{j}$

$\mathcal{C}_1$: $\vec{r}_1(t) = (t, 0)$, $t \in [0,\pi]$ $(0 \leq t \leq \pi)$

$\mathcal{C}_2$: $\vec{r}_2(t) = (\pi, t)$, $0 \leq t \leq \pi$



$\mathcal{C}_3$: $\vec{r}_3(t) = (\pi - t, \pi - t)$, $0 \leq t \leq \pi$

$\vec{F}(\vec{r}_1(t)) = (0, t)$, $\vec{F}(\vec{r}_2(t)) = (-\sin t, \pi)$,

$$\vec{F}(\vec{r}_3(t)) = (\cos(\pi - t)\sin(\pi - t), \pi - t)$$
$$= (-\cos(t)\sin(t), \pi - t)$$

$$\cos(\pi - t) = \cos(\pi)\cos(t) + \underbrace{\sin(\pi)}_{0}\sin(t)$$
$$= -\cos(t)$$
$$\sin(\pi - t) = \underbrace{\sin(\pi)}_{0}\cos(t) - \sin(t)\cos(\pi)$$
$$= \sin(t)$$

$$\vec{r}_1\,'(t) = (1,0), \quad \vec{r}_2\,'(t) = (0,1), \quad \vec{r}_3\,'(t) = (-1,-1)$$

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_{\mathcal{C}_1} \vec{F} \cdot d\vec{r} + \int_{\mathcal{C}_2} \vec{F} \cdot d\vec{r} + \int_{\mathcal{C}_3} \vec{F} \cdot d\vec{r}$$

Set. 3.4.4

Def. 3.4.1

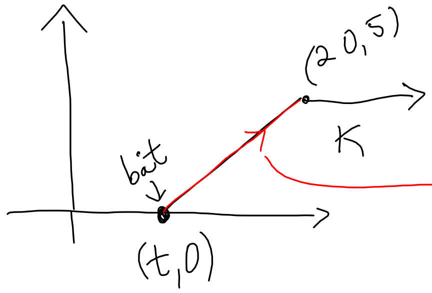
$$= \int_0^{\pi} \vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1\,'(t)\,dt + \int_0^{\pi} \vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2\,'(t)\,dt + \int_0^{\pi} \vec{F}(\vec{r}_3(t)) \cdot \vec{r}_3\,'(t)\,dt$$

$$= \int_0^{\pi} \left(0 + \cancel{\pi} + \cos(t)\sin(t) - \cancel{\pi} + t\right)dt = \int_0^{\pi}\left(\cos(t)\sin(t) + t\right)dt$$

$$= \int_0^{\pi}\left(\frac{1}{2}\sin(2t) + t\right)dt = \left[-\frac{1}{4}\cos(2t) + \frac{1}{2}t^2\right]_{t=0}^{\pi}$$

$$= -\cancel{\frac{1}{4}} + \frac{1}{2}\pi^2 + \cancel{\frac{1}{4}} = \underline{\underline{\frac{\pi^2}{2}}}$$

14)



a) Retning trekkraft: Retning til denne vektoren; $(20,5)-(t,0) = (20-t,5)$

Enhetsvektor i denne retningen:

$$\left(\frac{20-t}{\sqrt{(20-t)^2+5^2}}, \frac{5}{\sqrt{(20-t)^2+5^2}}\right)$$

Kraftvektor: $\vec{K}(t) = \left(\frac{K(20-t)}{\sqrt{(20-t)^2+5^2}}, \frac{5K}{\sqrt{(20-t)^2+5^2}}\right)$

$\vec{r}(t) = (t,0)$, $0 \le t \le 20$, $\vec{r}\,'(t) = (1,0)$

Arbeidet kraften utfører

$$\int_0^{20} \vec{K}(t)\cdot\vec{r}\,'(t)\,dt = \int_0^{20} \frac{K(20-t)}{\sqrt{(20-t)^2+25}}\,dt$$

$$= K\int_0^{20}\frac{20-t}{\sqrt{(20-t)^2+25}}\,dt$$

b) $K\int_0^{20}\frac{20-t}{\sqrt{(20-t)^2+25}}\,dt = K\int_{425}^{25} -\frac{1}{2\sqrt{u}}\,du$

$u = (20-t)^2+25$

$du = -2(20-t)\,dt$

$-\frac{du}{2(20-t)} = dt$

$t=0 \Rightarrow u = 425$

$t=20 \Rightarrow u = 25$

$= K\int_{25}^{425}\frac{1}{2\sqrt{u}}\,du$

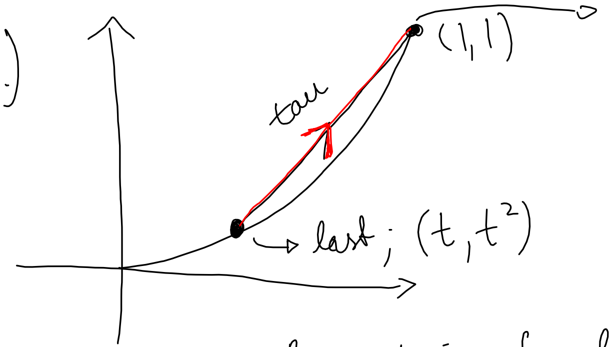
$= K\left[\sqrt{u}\right]_{u=25}^{425}$

$\sqrt{u} = u^{\frac{1}{2}}$

$= K(\sqrt{425}-5)$

$= K(\sqrt{25\cdot 17}-5) = \underline{\underline{5K(\sqrt{17}-1)}}$

15)



a) Trekkraften har retning fra lasten mot taljen:

Det er vektoren $\underbrace{(1,1)}_{\text{taljen}} - \underbrace{(t,t^2)}_{\text{lasten}} = (1-t,\ \underline{1-t^2})$

Lengden til denne vektoren:

$$\sqrt{(1-t)^2+(1-t^2)^2} = \sqrt{(1-t)^2+(1-t)^2(1+t)^2}$$

$$= (1-t)\sqrt{1+(1+t)^2} = (1-t)\sqrt{\underline{2}+2t+t^2}$$

Så enhetsvektoren i trekkraftretningen:

$$\frac{1}{(1-t)\sqrt{2+2t+t^2}}\left(1-t,\ \underbrace{1-t^2}_{(1-t)(1+t)}\right) = \frac{\cancel{1-t}}{\cancel{(1-t)}\sqrt{2+2t+t^2}}(1,\ 1+t)$$

$$= \frac{1}{\sqrt{2+2t+t^2}}(1,\ 1+t)$$

Trekkraft er konstant lik K, så kraftvektoren er:

$$\vec{K}(t) = \frac{K}{\sqrt{2+2t+t^2}}(1, 1+t)$$

b) Arbeid totalt $= \int_{\mathcal{C}} \vec{K}(t)\cdot d\vec{r} = \int_0^1 \vec{K}(t)\cdot \vec{r}\,'(t)\,dt$

$$= \int_0^1 \frac{K}{\sqrt{2+2t+t^2}}(1, 1+t)\cdot(1, 2t)\,dt$$

$$= \int_0^1 \left(\frac{K}{\sqrt{2+2t+t^2}} + \frac{2Kt(1+t)}{\sqrt{2+2t+t^2}} \right) dt$$

$$= K\int_0^1 \frac{1+2t+2t^2}{\sqrt{2+2t+t^2}}\,dt$$

c) $$\left((t-1)\sqrt{t^2+2t+2}\right)' = 1\cdot\sqrt{t^2+2t+2} + \frac{(t-1)(2t+2)}{2\sqrt{t^2+2t+2}}$$

$\sqrt{u} = u^{\frac{1}{2}}$

$$= \frac{t^2+2t+2+\overbrace{(t-1)(t+1)}^{t^2-1}}{\sqrt{t^2+2t+2}} = \frac{2t^2+2t+1}{\sqrt{t^2+2t+2}}$$

3.5: Gradienter og konservative felt

9.) $\int_{\mathcal{C}} \vec{F}\cdot d\vec{r}$, $\vec{F}(x,y) = (y^2e^{xy^2}, 2xye^{xy^2}+1)$

sirkel, $r=5$, $(1,-1)$

$$\frac{\partial F_1}{\partial y}(x,y) = 2ye^{xy^2} + y^2e^{xy^2}2xy = 2ye^{xy^2} + 2xy^3e^{xy^2}$$

$$\frac{\partial F_2}{\partial x}(x,y) = 2ye^{xy^2} + 2xye^{xy^2}y^2 = 2ye^{xy^2} + 2xy^3e^{xy^2}$$

Så $\frac{\partial F_1}{\partial y}(x,y) = \frac{\partial F_2}{\partial x}(x,y) \underset{\text{Teorem 3.5.7}}{\Longrightarrow} \vec{F}$ er konservativt.

$\Rightarrow \vec{F} = \nabla\phi$ for en eller annen funk. ϕ.

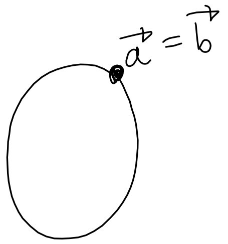
$$\Rightarrow \int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \int_{\mathcal{C}} \nabla\phi \cdot d\vec{r} = \phi(\vec{b}) - \phi(\vec{a})$$

$$= \phi(\vec{b}) - \phi(\vec{b}) = \underline{\underline{0}}$$

Set. 3.5.1

$\mathcal{C}$ sirkel

lukket



3.6: Kjeglesnitt

9.) $y^2 = 4ax$, (x_0, y_0)

$$\Downarrow$$

$$y = \pm\sqrt{4ax}$$

Anta at $y_0 > 0$ og $a > 0$: Da er $y_0 = \sqrt{4ax_0}$. Tangentlinja i

(x_0, y_0): $\quad y = \frac{2a}{\sqrt{4ax_0}}(x - x_0) + \sqrt{4ax_0}$

Hvorfor? I x_0 skal linja gj. $\sqrt{4ax_0}$ og ha stigning $y'(x_0)$.

$$y'(x_0) = \frac{1}{2\sqrt{4ax_0}}\,4a = \frac{2a}{\sqrt{4ax_0}}$$: Ettpkts.formel:

$$y = \frac{2a}{\sqrt{4ax_0}}(x - x_0) + \sqrt{4ax_0}.$$

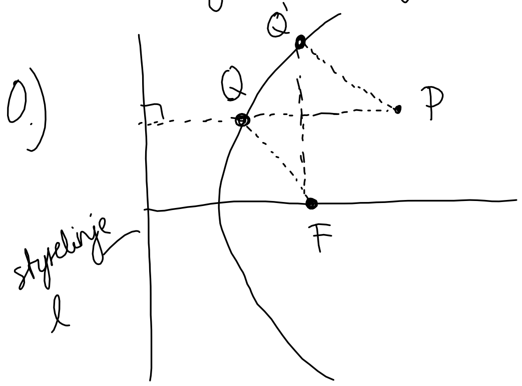
Skjæring x-akse: $\frac{2a}{\sqrt{4ax_0}}(x - x_0) + \sqrt{4ax_0} = 0$

$$x - x_0 = -\frac{2\cancel{4}ax_0}{\cancel{2}a} = 2x_0 \Rightarrow \underline{x = -x_0}$$

Skjæring med x-aksen $(-x_0, 0)$.

($a < 0$ og/eller $y_0 < 0$; tilsvarende)

10)



Korteste vei til br. pkt via parabel.

Korteste vei: Gå ut til parabelen vinkelrett på l. Gå rett fra Q til F.

Hvorfor? Lengde vei: $|PQ| + |QF| \geq |Pl|$

Korteste vei mellom pkt. og linje er vinkelrett på linja

Holder med = kun hvis P og Q er på linje parallell med aksen, dvs. vinkelrett på styrelinja.