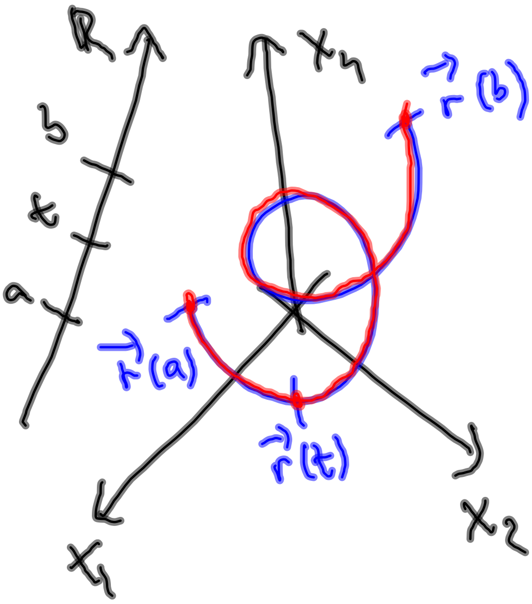
LH 3.1: PARAMETRISERTE KURVER

$I \subseteq \mathbb{R}$ intervall

$\mathbb{R}^n$ n-rommet

$$\vec{r} : I \longrightarrow \mathbb{R}^n$$
$$t \longmapsto \vec{r}(t)$$



parametrisering av kurven

Skriver

$$\vec{r}(t) = (x_1(t), \dots, x_n(t))$$

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$$= (x(t), y(t)) \qquad n=2$$
$$= (x(t), y(t), z(t)) \qquad n=3$$

Eks: $\vec{r}(t) = (\cos(t), \sin(t), t)$

for $t \in I = [0, 6\pi]$

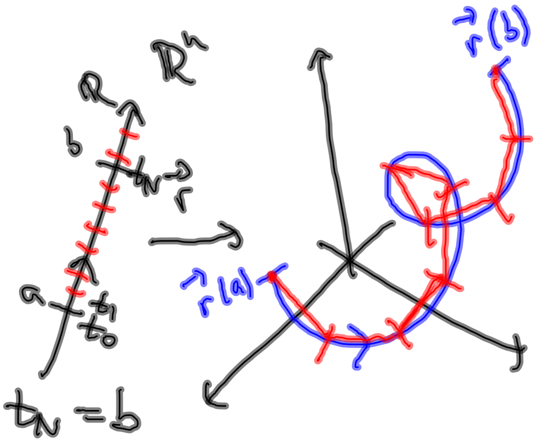
DEF: En parametrisert kurve er en kontinuerlig funksjon

$$\vec{r} : I \to \mathbb{R}^n$$

der $I \subset \mathbb{R}$ er et intervall.

BUELENGDE:

Deler $I = [a,b]$ i N biter:



$$a = t_0 < t_1 < t_2 < \cdots < t_N = b$$

Kurven går innom

$\vec{r}(a), \vec{r}(t_1), \vec{r}(t_2), \ldots, \vec{r}(t_N) = \vec{r}(b)$

$\vec{r}(t_0)$

Den stykkevis lineære kurven har lengde

$$|\vec{r}(t_1) - \vec{r}(t_0)| + |\vec{r}(t_2) - \vec{r}(t_1)| + \ldots + |\vec{r}(t_N) - \vec{r}(t_{N-1})|$$

$$= \sum_{i=1}^{N} |\vec{r}(t_i) - \vec{r}(t_{i-1})|$$

går mot

$L(a,b) =$ lengden til kurven fra $t=a$ til $t=b$

når partisjonen $\{\underset{a}{t_0} < t_1 < \cdot < \underset{b}{t_N}\}$

blir „fin" $\left(\max\limits_{1 \le i \le N} (t_i - t_{i-1}) \to 0\right)$.

$$\vec{r}(t) = (x_1(t), \dots, x_n(t))$$

$$\sum_{i=1}^{N} |\vec{r}(t_i) - \vec{r}(t_{i-1})|$$

$$= \sum_{i=1}^{N} |(x_1(t_i) - x_1(t_{i-1}), \dots, x_n(t_i) - x_n(t_{i-1}))|$$

$$= \sum_{i=1}^{N} \sqrt{(x_1(t_i) - x_1(t_{i-1}))^2 + \dots + (x_n(t_i) - x_n(t_{i-1}))^2}$$

$$= \sum_{i=1}^{N} \sqrt{\left(\frac{x_1(t_i) - x_1(t_{i-1})}{t_i - t_{i-1}}\right)^2 + \dots + \left(\frac{x_n(t_i) - x_n(t_{i-1})}{t_i - t_{i-1}}\right)^2}\,(t_i - t_{i-1})$$

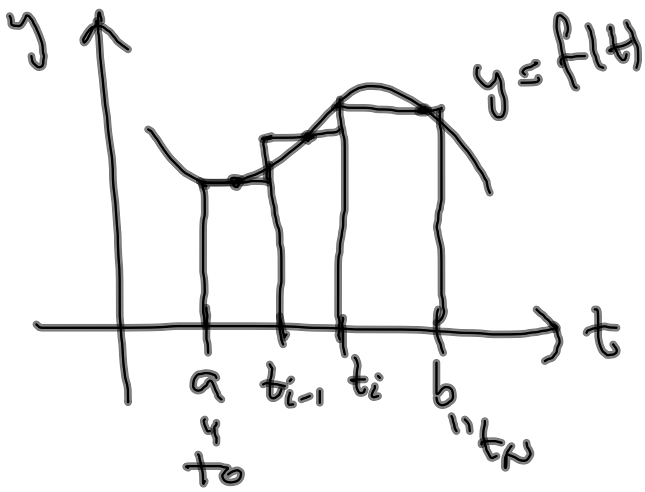
$$\underset{t_i - t_{i-1} \to 0}{\approx} \sum_{i=1}^{N} \sqrt{x_1'(t_i)^2 + \dots + x_n'(t_i)^2}\,(t_i - t_{i-1})$$

er en Riemann-sum for integralet

$$\int_a^b \sqrt{x_1'(t)^2 + \dots + x_n'(t)^2}\, dt$$

[Riemann-integralet

$f : I = [a,b] \to \mathbb{R}$ kontinuerlig



$$\int_a^b f(t)\,dt$$

$\uparrow$ $t_i - t_{i-1} \to 0$

$$\sum_{i=1}^{N} f(t_i)(t_i - t_{i-1})$$]

DEF La $\vec{r}: I \to \mathbb{R}^n$ med

$$\vec{r}(t) = (x_1(t), \dots, x_n(t))$$

ha kontinuerlig derivert

$$\vec{r}\,'(t) = (x_1'(t), \dots, x_n'(t)).$$

Da er buelengden av $\vec{r}$ fra $t=a$ til $t=b$ (= fra $\vec{r}(a)$ til $\vec{r}(b)$) lik

$$L(a,b) = \int_a^b |\vec{r}\,'(t)|\,dt$$

$$= \int_a^b \sqrt{x_1'(t)^2 + \dots + x_n'(t)^2}\,dt.$$

Eks: $\vec{r}(t) = (\cos(t), \sin(t), t)$

$0 \leq t \leq 6\pi$

$$L(0, 6\pi) = \int_0^{6\pi} |\vec{r}\,'(t)|\, dt$$

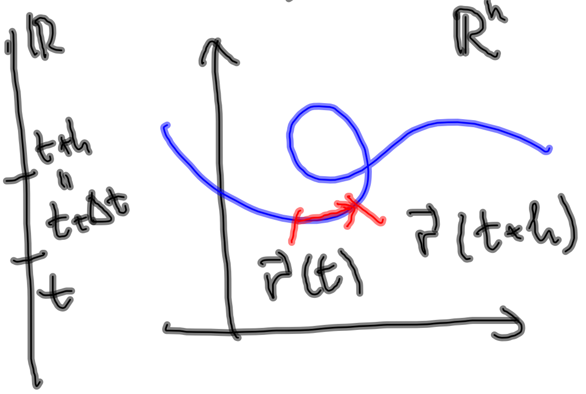
$$\vec{r}\,'(t) = (-\sin(t), \cos(t), 1)$$

$$|r'(t)| = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + (1)^2}$$

$$= \sqrt{\underbrace{\sin^2(t) + \cos^2(t)}_{=1} + 1} = \sqrt{2}$$

$$= \int_0^{6\pi} \sqrt{2}\, dt = \underline{\underline{6\pi \cdot \sqrt{2}}}.$$

HASTIGHET, FART



Gj. hastighet i $[t, t+h]$

$$= \frac{\vec r(t+h) - \vec r(t)}{h}$$

når $h \to 0$

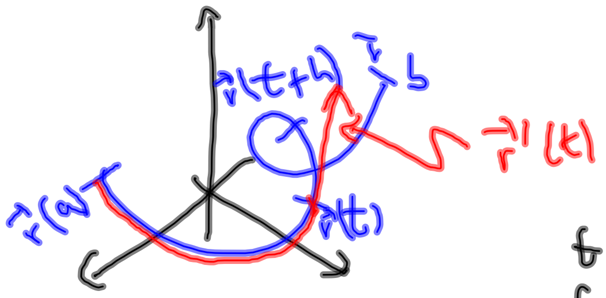
Hastigheten i t: $\vec r\,'(t) = \vec v(t)$

(Velocity)

Farten i t (Speed)

$$v(t) = |\vec v(t)| = |\vec r\,'(t)|$$

skalar vektor



La $s(t) = L(a,t) = \int_a^t |\vec r\,'(t)|\, dt$

være buelengden fra $\vec r(a)$ til $\vec r(t)$.

Farten er

$$\lim_{h \to 0} \frac{s(t+h) - s(t)}{h} = s'(t)$$

(Fundamentalteoremet)

$$= |r'(t)| = v(t) = |\vec v(t)|$$

REGNEREGLER FOR $\vec{r}\,'(t)$:

SETNING 3.1.4:
La $\vec{r}_1, \vec{r}_2 : I \to \mathbb{R}^n$ være (deriverbare) parametriserte kurver i $\mathbb{R}^n$.

(i) $(\vec{r}_1 + \vec{r}_2)'(t) = \vec{r}_1\,'(t) + \vec{r}_2\,'(t)$

(ii) $(\vec{r}_1 - \vec{r}_2)'(t) = \vec{r}_1\,'(t) - \vec{r}_2\,'(t)$

(iii) $(\vec{r}_1 \cdot \vec{r}_2)'(t) = \vec{r}_1\,'(t) \cdot \vec{r}_2(t) + \vec{r}_1(t) \cdot \vec{r}_2\,'(t)$

(iv) $(n=3)$

$$(\vec{r}_1 \times \vec{r}_2)'(t) = \vec{r}_1\,'(t) \times \vec{r}_2(t) + \vec{r}_1(t) \times \vec{r}_2\,'(t)$$

(v) $u : I \to \mathbb{R}$ deriverbar

$$(u\vec{r}_1)(t) = u(t)\,\vec{r}_1(t)$$

$$(u\vec{r}_1)'(t) = u'(t)\,\vec{r}_1(t) + u(t)\,\vec{r}_1\,'(t)$$

BEVIS FOR (v):

$$\vec{r}_1(t) = (x_1(t), \dots, x_n(t))$$

$$\vec{r}_1\,'(t) = (x_1'(t), \dots, x_n'(t))$$

$$u(t) \qquad u'(t)$$

$$(u\vec{r}_1)(t) = u(t)\,\vec{r}_1(t) = u(t)(x_1(t), \dots, x_n(t)) = (u(t)x_1(t), \dots, u(t)x_n(t))$$

$$(u\vec{r}_1)'(t) = ((u(t)x_1(t))', \dots (u(t)x_n(t)$$

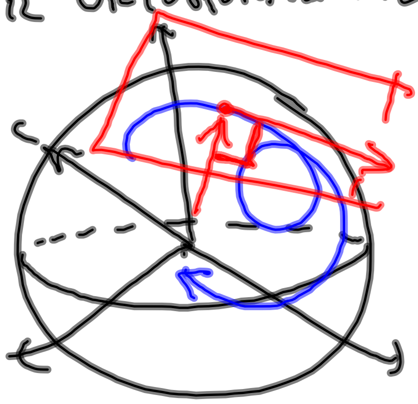
$$= (u'(t)x_1(t) + u(t)x_1'(t), \dots, u'(t)x_n(t) + u(t)x_n'(t$$

$$= (u'(t)x_1(t), \dots)$$

<u>KOROLLAR</u> Hvis $|\vec{r}(t)| = c$ ER KONSTANT, ER

$$\vec{r}\,'(t) \cdot \vec{r}(t) = 0$$

SÅ $\vec{r}\,'(t) = \vec{v}(t)$ ER ORTOGONAL TIL $\vec{r}(t)$.



<u>BEVIS</u>:

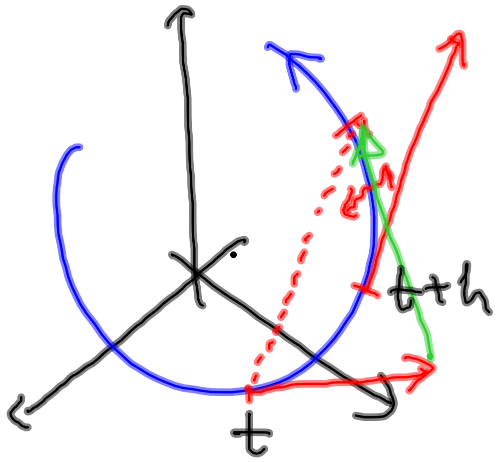
$$c^2 = |\vec{r}(t)|^2 = \vec{r}(t) \cdot \vec{r}(t).$$

DERIVERER MHP t:

$$0 = (c^2)'(t) = (\vec{r}(t) \cdot \vec{r}(t))'$$
$$= \vec{r}\,'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}\,'(t)$$
$$= 2\,\vec{r}\,'(t) \cdot \vec{r}(t)$$

SÅ $\vec{r}(t) \cdot \vec{r}\,'(t) = 0.$

AKSELERASJON ETC.



$$\vec{a}(t) = \lim_{h \to 0} \frac{\vec{v}(t+h) - \vec{v}(t)}{h} = \vec{v}\,'(t)$$

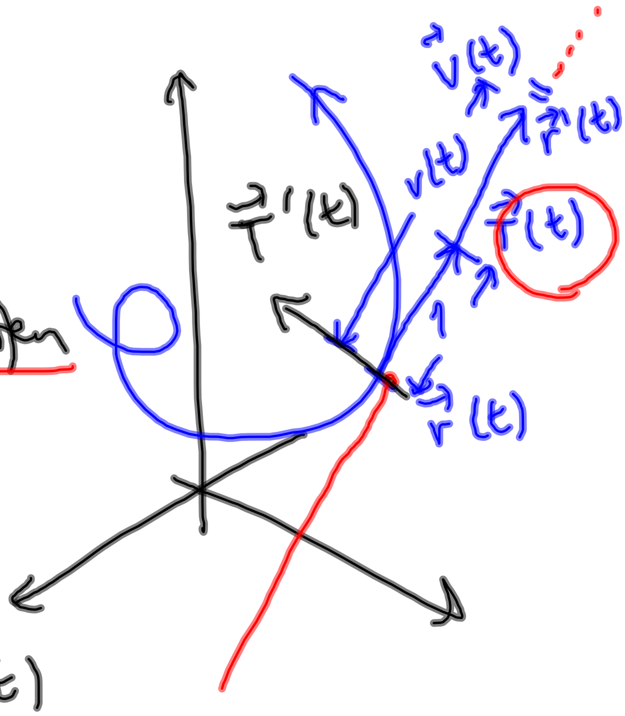
akselerasjon = den deriverte av hastigheten

Baneakselerasjon = den deriverte av farten

$$a(t) = v'(t)$$

La

$$\vec{T}(t) = \frac{\vec{v}(t)}{v(t)}$$



være enhetstangent(vektoren) til den parametriserte kurven

Eks

$$\vec{r}(t) = (\cos t, \sin t, t)$$

$$\vec{v}(t) = \vec{r}'(t) = (-\sin t, \cos t, 1)$$

$$v(t) = |\vec{v}(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$\vec{T}(t) = \frac{\vec{v}(t)}{v(t)} = \frac{(-\sin t, \cos t, 1)}{\sqrt{2}}$$

$$= \left(-\frac{\sqrt{2}}{2}\sin t, \frac{\sqrt{2}}{2}\cos t, \frac{\sqrt{2}}{2}\right)$$

$$|\vec{T}(t)| = 1.$$

$\nearrow$

$$\vec{T} : I \longrightarrow \mathbb{R}^n$$

$$t \longmapsto \vec{T}(t)$$

VED KOROLLARET ER

$$\vec{T}'(t) \cdot \vec{T}(t) = 0.$$

RELASJON MELLOM $\vec{a}(t)$ og $a(t)$:

$$\vec{T}(t) = \frac{\vec{v}(t)}{v(t)}$$

$$\vec{v}(t) = \underbrace{v(t)\vec{T}(t)}_{(v\cdot\vec{T})(t)}$$

deriverer mhp t

$$\vec{a}(t) = \vec{v}\,'(t)$$
$$= (v\cdot\vec{T})'(t)$$
$$= v'(t)\vec{T}(t) + v(t)\vec{T}\,'(t)$$
$$= a(t)\vec{T}(t) + v(t)\vec{T}\,'(t)$$

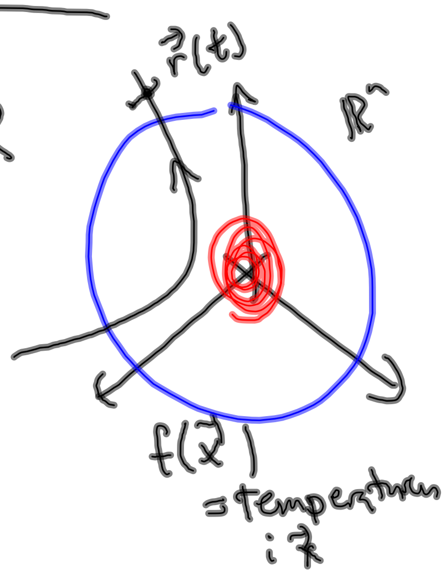
SETNING:
DERSOM $v(t) \neq 0$ ER AKSELERASJONEN

$$\vec{a}(t) = a(t)\vec{T}(t) + v(t)\vec{T}\,'(t)$$

DER $a(t)\vec{T}(t)$ ER PARALELL MED TANGENTEN OG $v(t)\vec{T}\,'(t)$ STÅR NORMALT PÅ TANGENTEN.

LH 3.5: KJERNEREGELEN II

$$\begin{tikzcd} \mathbb{R} \supset I \arrow[rr, bend left, "u(t) = f(\vec{r}(t))"] \arrow[dr, "\vec{r}"'] & & \mathbb{R} \\ & \mathbb{R}^n \arrow[ur, "f"'] & \end{tikzcd}$$



$$\begin{aligned} u'(t) &= f'(\vec{r}(t))\,\vec{r}\,'(t) \\ &= \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) \\ &= \nabla f(\vec{r}(t)) \cdot \vec{v}(t) \\ &= \left(\frac{\partial f}{\partial x_1}(\vec{r}(t)), \dots, \frac{\partial f}{\partial x_n}(\vec{r}(t))\right) \cdot \\ &\qquad (x_1'(t), \dots, x_n'(t)) \\ &= \frac{\partial f}{\partial x_1}(\vec{r}(t))\,x_1'(t) + \dots + \frac{\partial f}{\partial x_n}(\vec{r}(t))\,x_n'(t) \end{aligned}$$

MERK: Hvis $u(t) = f(\vec{r}(t)) = C$

ER KONSTANT ER

$$0 = u'(t) = \nabla f(\vec{r}(t)) \cdot \vec{v}(t)$$

SÅ GRADIENTEN TIL f i $\vec{r}(t)$

STÅR NORMALT PÅ HASTIGHETEN