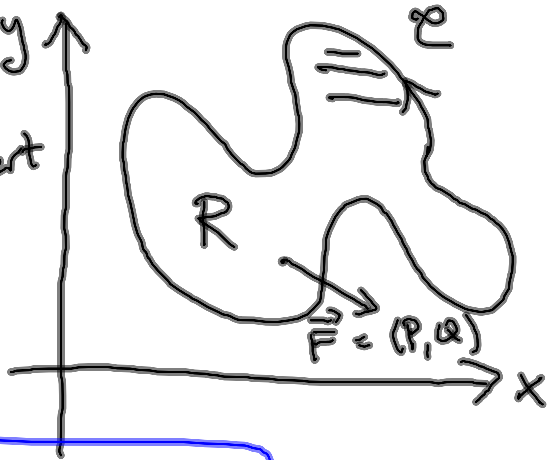
LH 6.5 Greens teorem



Teorem $\mathcal{C} = \partial R$ pos. orientert

$\frac{\partial Q}{\partial x}, \frac{\partial P}{\partial y}$ kontinuerlige

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} \overset{\text{def}}{=} \boxed{\int_{\mathcal{C}} P\,dx + Q\,dy \overset{\text{setn.}}{=} \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy}$$

Kor 6.5.2

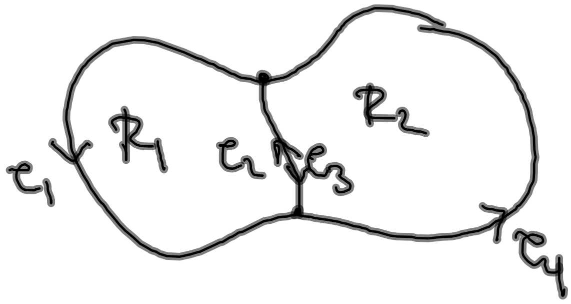
$$\text{areal}(R) = \int_{\mathcal{C}} x\,dy = \int_{\mathcal{C}} -y\,dx$$

$(P,Q) = (0,x)$ $\qquad$ $(P,Q) = (-y,0)$

$$= \frac{1}{2} \int_{\mathcal{C}} -y\,dx + x\,dy$$

Bevis av Greens teorem

. Forenkler R til type I og II



$$R = R_1 \cup R_2$$
$$\partial R = C_1 \cup C_4 = C$$
$$\partial R_1 = C_1 \cup C_2$$
$$\partial R_2 = C_3 \cup C_4$$

Hvis vi vet at

$$\int_{C_1 \cup C_2} P\,dx + Q\,dy = \iint_{R_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

og

$$\int_{C_3 \cup C_4} P\,dx + Q\,dy = \iint_{R_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

må

$$\int_{C_1 \cup C_2 \cup C_3 \cup C_4} P\,dx + Q\,dy = \iint_{R_1 \cup R_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \iint_{R} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

$$= \int_{C_1} (\;) + \cancel{\int_{C_2} (\;)} + \cancel{\int_{C_3} (\;)} + \int_{C_4} (\;) = \int_{C} P\,dx + Q\,dy$$

like, men med motsatt fortegn

Nok å vise Greens teorem når R er av type I og av type II.

$$\vec{F} = (P, Q) = (P, 0) + (0, Q)$$

$$P\,dx + Q\,dy = P\,dx + Q\,dy$$

Nok å vise at

$$\int_{C} P\,dx = \iint_{R} \left(-\frac{\partial P}{\partial y}\right) dx\,dy$$

lett å vise når R har type I

og

$$\int_{C} Q\,dy = \iint_{R} \left(\frac{\partial Q}{\partial x}\right) dx\,dy$$

lett å vise når R har type II

$*$

$$\Downarrow$$

$$\int_{C} P\,dx + Q\,dy = \iint_{R} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

ok når R er av type I og av type II

Viser (4)

$$\int_C Q\,dy = \int_C \vec{F}\cdot d\vec{r} \qquad \vec{F}(x,y) = (0, Q(x,y))$$

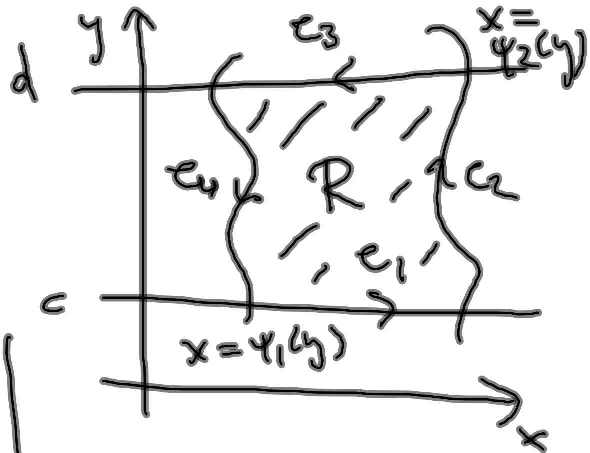
C_1: $\vec{r}_1(t) = (t, c)$

$$\psi_1(c) \le t \le \psi_2(c)$$

$$\vec{r}_1{}'(t) = (1,0)$$

$$\vec{F}(\vec{r}_1(t))\cdot \vec{r}_1{}'(t) = (0, Q(x,y))\cdot(1,0) = 0$$

$$\int_{C_1} Q\,dy = \int_{\psi_1(c)}^{\psi_2(c)} (\ \ ) \, dt = 0$$



$$\psi_1, \psi_2 : [c,d] \longrightarrow \mathbb{R}$$

$$\psi_1(y) \le \psi_2(y)$$

$$\partial R = C_1 \cup C_2 \cup C_3 \cup C_4$$

$$\int_{C_3} Q\,dy = 0.$$

Parametriserer $\mathcal{C}_2$:

$$\vec{r}_2(t) = (\psi_2(t), t) \qquad c \le t \le d$$

$$\vec{r}_2\,'(t) = (\psi_2'(t), 1)$$

$$\int_{\mathcal{C}_2} Q\,dy = \int_{\mathcal{C}_2} (0, Q) \cdot d\vec{r}_2$$

$$= \int_c^d (0, Q(\psi_2(t), t)) \cdot (\psi_2'(t), 1)\,dt$$

$$= \int_c^d Q(\psi_2(t), t)\,dt.$$

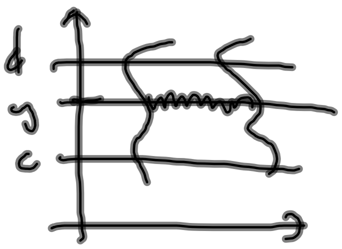
$$\boxed{\int_{\mathcal{C}_4} Q\,dy = -\int_c^d Q(\psi_1(t), t)\,dt}$$

$$\int_{\mathcal{C}} Q\,dy = \int_c^d \Big(Q(\psi_2(t), t) - Q(\psi_1(t), t)\Big)\,dt. \qquad \star$$

Høyre side

$$\iint_R \frac{\partial Q}{\partial x}\,dx\,dy = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} \frac{\partial Q}{\partial x}\,dx \right) dy$$

type II



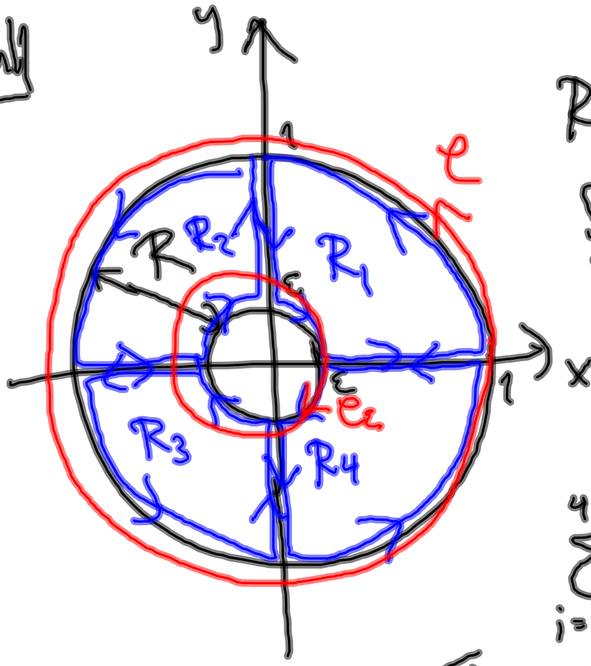
$\qquad \| \quad$ an. fund. thm.

$$\int_c^d \Big[Q(x,y) \Big]_{x=\psi_1(y)}^{x=\psi_2(y)} dy$$

$\|$

$$\int_c^d \Big(Q(\psi_2(y), y) - Q(\psi_1(y), y) \Big)\,dy \qquad \star$$

Indre hull



$$R = R_1 \cup R_2 \cup R_3 \cup R_4$$

$$\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\,dy$$

$$\shortparallel$$

$$\sum_{i=1}^{4} \iint_{R_i} (\ )$$

$$\shortparallel$$

$$\sum_{i=1}^{4} \int_{\partial R_i} P\,dx + Q\,dy$$

$$=$$

$$\int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy$$

$$\shortparallel$$

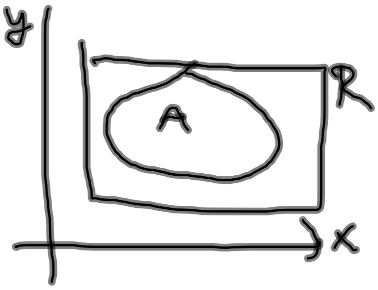
$$\int_{C} P\,dx + Q\,dy - \int_{C_i} P\,dx + Q\,dy$$

C: sirkel radius 1 pos. orientert

C_i: indre rand = sirkel radius ε pos. orientert

LH 6.6 Jordan-målbare mengder

<u>Kor 6.1.8</u> $R = [a,b] \times [c,d] \subset \mathbb{R}^2$ lukket, begrenset rektangel, $f : R \to \mathbb{R}$ kontinuerlig. Da er f integrerbar på R, så $\iint_R f(x,y)\,dx\,dy$ er definert.



$A \subset \mathbb{R}^2$ lukket, begrenset område

$f : A \to \mathbb{R}$ kontinuerlig

$f_A : R \to \mathbb{R}$ $\quad f_A(x,y) = \begin{cases} f(x,y) & (x,y) \in A \\ 0 & \text{ellers} \end{cases}$

(f_A <u>ikke</u> kontinuerlig!)

$$\iint_A f(x,y)\,dx\,dy = \iint_R f_A(x,y)\,dx\,dy$$

<u>Teorem 6.6.6</u> Hvis A er <u>Jordan-målbar</u> er f_A integrerbar på R, (så f er integrerbar på A.)

Spesialtilfelle hvor $f \equiv 1$:

$$\text{areal}(A) = \iint_A 1 \, dx\,dy = \iint_R 1_A \, dx\,dy$$

der $1_A(x,y) = \begin{cases} 1 & (x,y) \in A \\ 0 & \text{ellers} \end{cases}$

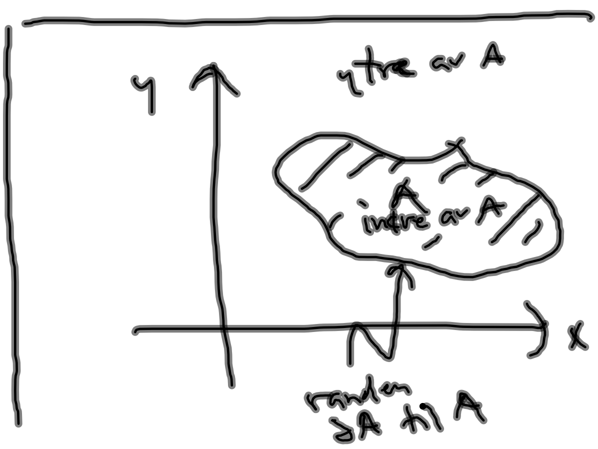
∂A innhold null

$\Updownarrow$

A er Jordan-målbar

$\Updownarrow$ def

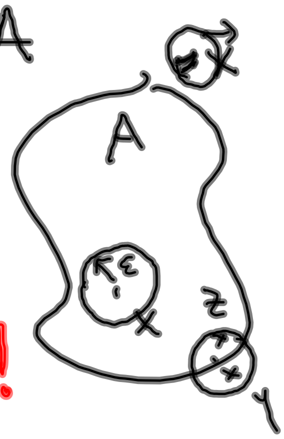
1_A er integrerbar



Randen ∂A til A

$A \subset \mathbb{R}^2$

Et punkt $\vec{x} \in \mathbb{R}^2$ er et indre punkt i A hvis $\exists\ \varepsilon > 0$ slik at $B(\vec{x}, \varepsilon) \subseteq A$



Et punkt $\vec{x} \in \mathbb{R}^2$ er et ytre punkt for A hvis $\exists\ \varepsilon > 0$ slik at

$$B(\vec{x}, \varepsilon) \cap A = \emptyset.$$

RETTET! 9/3/11

Resten av $\mathbb{R}^2$ kalles randen til A, ∂A

$\vec{x} \in \partial A \iff \forall\ \varepsilon > 0$ vil

finnes $y \in B(\vec{x}, \varepsilon)$ med $y \notin A$

og $z \in B(\vec{x}, \varepsilon)$ med $z \in A$.

<u>Def</u> ∂A har <u>innhold 0</u>

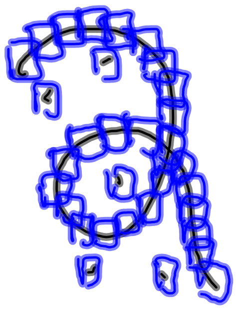
$\iff$ for enhver $\varepsilon > 0$ finnes endelig mange rektangler

$S_1 = [a_1, b_1] \times [c_1, d_1], S_2, \ldots, S_\ell$

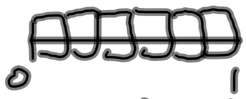
slik at $\partial A \subseteq S_1 \cup S_2 \cup \ldots \cup S_\ell$

og $|S_1| + |S_2| + \ldots + |S_\ell| < \varepsilon$

der $|S_k| = (b_k - a_k)(d_k - c_k)$.



A



$n \cdot \left(\frac{1}{n}\right)^2 = \frac{1}{n}$

<u>Teorem 6.6.3</u> $A \subset \mathbb{R}^2$ er Jordan-målbar ($\Leftrightarrow 1_A$ er integrerbar) hvis og bare hvis ∂A har innhold 0.

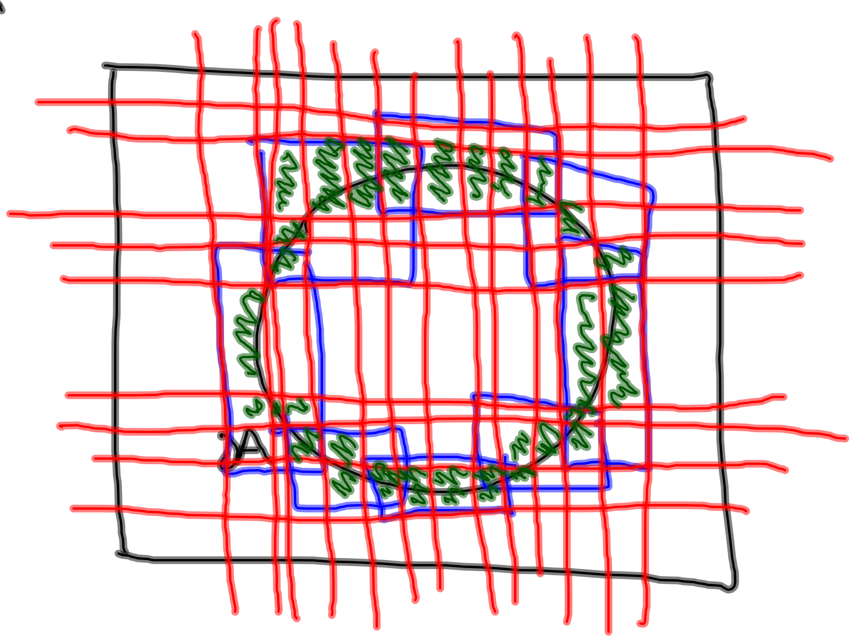
∂A innhold 0 $\Longrightarrow 1_A$ integrerbar

La $\varepsilon > 0$. Søker Π partisjon av

$$A \subset R = [a,b] \times [c,d]$$

$$N(\Pi) \leq \varnothing(\Pi) < N(\Pi) + \varepsilon$$

trappesum for $f = 1_A$

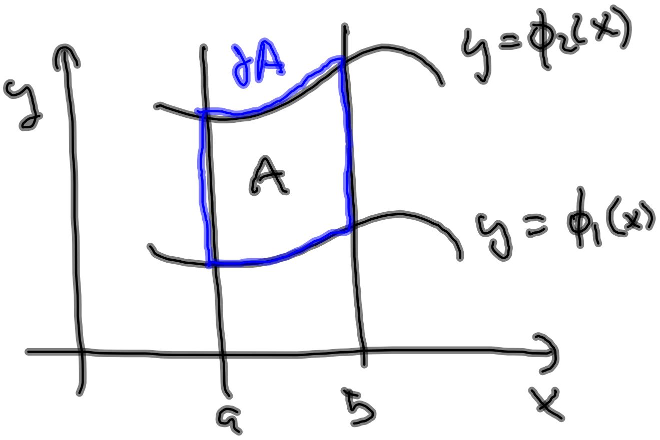


$$\varnothing(\Pi) - N(\Pi) = \sum_{\substack{\text{rekt. } R_{ij} \\ \text{møter } \partial A}} (1-0)|R_{ij}|$$

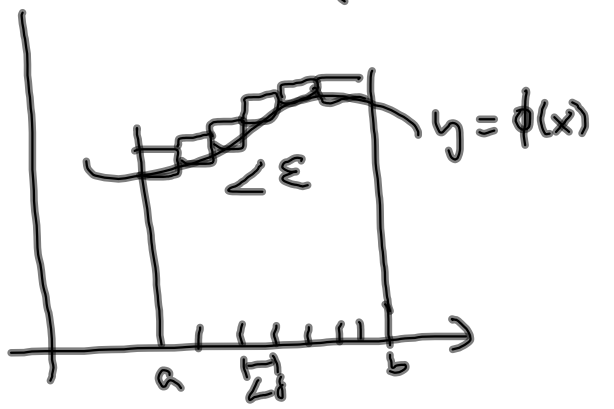
$$\leq \sum_{k=1}^{l} |S_k| < \varepsilon$$

Lemma Hvis A har type I eller type II så har ∂A innhold 0, (så A er Jordan-målbar og 1_A er integrerbar).

Bevis



ϕ_1, ϕ_2 kontinuerlige



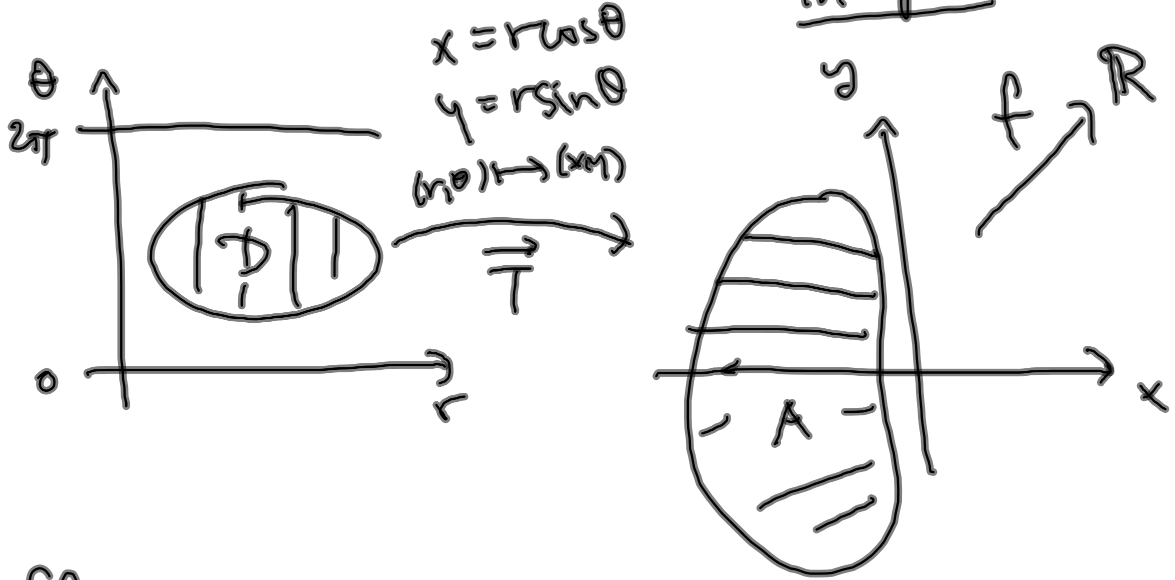
$\varepsilon > 0$

$\delta > 0$

$|x_2 - x_1| < \delta$

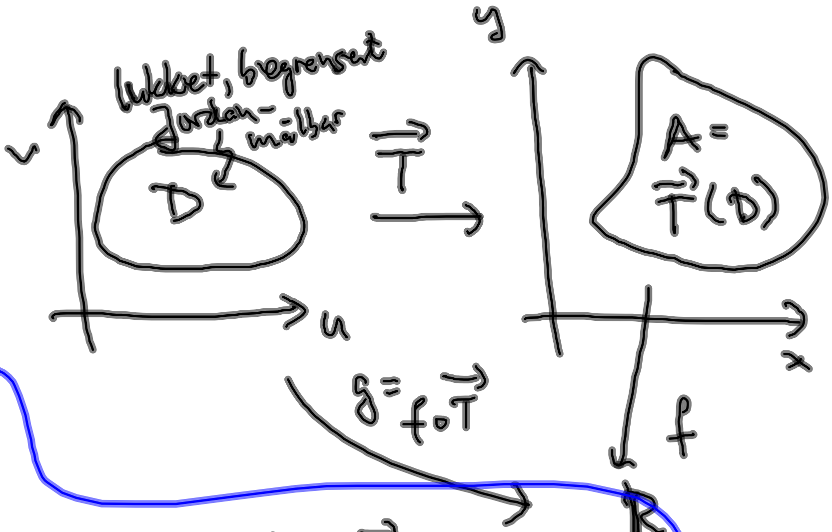
$\Rightarrow |\phi(x_2) - \phi(x_1)| < \frac{\varepsilon}{2(b-a)}$

LH 6.7 Skifte av variabel i dobbelt-integral



$$\iint_A f(x,y)\,dx\,dy = \iint_D f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Teorem 6.7.1



$$\boxed{\iint_A f(x,y)\,dx\,dy = \iint_D f(\vec{T}(u,v))\,\left|\det \vec{T}'(u,v)\right|\,du\,dv}$$