4.9

2b)

$$\begin{vmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix} = (-1) \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 2 & 1 & 2 \end{vmatrix} \begin{matrix} -2 \quad -2 \\ \\ \\ \end{matrix}$$

$$= (-1) \begin{vmatrix} 1 & -1 & 1 \\ 0 & 3 & -3 \\ 0 & 3 & 0 \end{vmatrix} \begin{matrix} \\ (-1) \\ \\ \end{matrix} = (-1) \begin{vmatrix} 1 & -1 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 3 \end{vmatrix}$$

$$= (-1) \cdot 3 \cdot 3 = -9$$

4.9

3c)

$$\begin{vmatrix} 3 & 1 & 0 & 4 \\ 2 & 1 & 2 & 0 \\ 0 & 0 & 1 & -2 \\ 1 & 2 & 1 & 2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 3 & 1 & 4 \\ 2 & 1 & 0 \\ 1 & 2 & 2 \end{vmatrix}$$

$$+2 \begin{vmatrix} 3 & 1 & 0 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} = (-2) \begin{vmatrix} 1 & 4 \\ 2 & 2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$$

$$+2 \left(3 \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - \begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} \right) =$$

$$= (-2)(2-8) + (6-4)$$

$$+2(3(1-4) - (2-2))$$

$$= 12 + 2 - 18 = \underline{\underline{-4}}$$

4.9

10) U er ortogonal hvis

U er en inverterbar $n \times n$-matrise

med $U^{-1} = U^T$. Da er $\det(U) = \pm 1$.

$$I_n = U U^T$$

$1 = \det(I_n) = \det(U U^T) =$

$= \det(U) \det(U^T) = \det(U) \det(U)$

$= (\det(U))^2. \Rightarrow \det(U) = \pm 1.$

4.9

11 Cramers regel.

A inverterbar $n \times n$-matris

$$\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in \mathbb{R}^n$$

Skal løse $A\vec{x} = \vec{b}$, $\vec{x} \in \mathbb{R}^n$.

(Vet at vi har én entydig løsning)

$$A_i(b) = \begin{bmatrix} a_{11} & a_{12} & \cdots & b_1 & a_{1n} \\ a_{21} & a_{22} & \cdots & b_2 & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdot & b_n & a_{nn} \end{bmatrix}$$
$$\uparrow_i$$

($A_i(\vec{b})$ er matrisen vi får ved å la $\vec{b}$ være i-te søyle og beholde de andre søylene fra A)

Cramers regel Løsningen av $A\vec{x} = \vec{b}$

$$\vec{x} = \begin{pmatrix} x_1 \\ \\ x_n \end{pmatrix}$$ er gitt ved $$x_i = \frac{\det(A_i(\vec{b}))}{\det A}$$

4.9.11 forts.

a) Skal vise at $\det I_i(\vec{x}) = x_i$

$$I_i(\vec{x}) = \begin{array}{c} \begin{pmatrix} 1 & 0 & x_1 & & & 0 \\ 0 & 1 & x_2 & & & \vdots \\ \vdots & & & & & \\ 0 & \cdot\, 0 & x_i & 0 & \cdot\;\cdot & 0 \\ \vdots & & & & & \\ 0 & 0 & x_n & & & \\ & & & & & 1 \end{pmatrix} \\ \uparrow \\ i \end{array}$$

Utvikler

$\det(I_i(x))$ etter i-te rad

dette gir $x_i \cdot \underbrace{\det I_{n-1}}_{=1} = \underline{x_i}$

b) Skal vise at $AI_i(\vec{x}) = A_i(\vec{b})$

når $A\vec{x} = \vec{b}$.

$$AI_i(\vec{x}) = \begin{pmatrix} a_{11} & & a_{1n} \\ & & \\ a_{n1} & & a_{nn} \end{pmatrix} \begin{array}{cccc} & & \vec{x} & \\ & & \| & \\ \multicolumn{4}{c}{\begin{pmatrix} 1 & 0 & x_1 & 0 \\ 0 & 1 & x_2 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & x_n & 1 \end{pmatrix}} \\ \vec{e}_1 & \vec{e}_2 & \vec{x} & \vec{e}_n \end{array}$$

$$= (A\vec{e}_1, A\vec{e}_2, \ldots, \underset{= \vec{b}}{A\vec{x}}, \ldots A\vec{e}_n)$$

$$= \begin{pmatrix} a_{11} & a_{12} & b_1 & a_{1n} \\ a_{n1} & a_{n2} & b_n & a_{nn} \end{pmatrix} = A_i(\vec{b}).$$

c) Vise Cramers regel:

Hadde:

$$A I_i(\vec{x}) = A_i(\vec{b})$$

$$\det A \underbrace{\det I_i(\vec{x})}_{\substack{\parallel \ (c) \\ x_i}} = \det A_i(\vec{b})$$

$$(\det A) x_i = \det A_i(\vec{b}) \Rightarrow x_i = \frac{\det A_i(b)}{\det A}.$$

d) Skal løse
$$2x - 3y = 4$$
$$x - 4y = -2$$

$$A = \begin{pmatrix} 2 & -3 \\ 1 & -4 \end{pmatrix}, \quad \det A = -8+3 = -5$$

$$A_1(\vec{b}) = \begin{pmatrix} 4 & -3 \\ -2 & -4 \end{pmatrix}, \quad \det A_1(b) = -16-6 = -22$$

$$x = \frac{-22}{-5} = \frac{22}{5}. \quad A_2(\vec{b}) = \begin{pmatrix} 2 & 4 \\ 1 & -2 \end{pmatrix}$$

$$\det A_2(b) = -4-4 = -8, \quad y = \frac{-8}{-5} = \frac{8}{5}.$$

$$\underline{\underline{(x, y) = \left(\frac{22}{5}, \frac{8}{5}\right).}}$$

4.10.

1 f) Skal finne egenverdien og egenvektoren til $\begin{pmatrix} 5 & 2 \\ -1 & 3 \end{pmatrix} = A$

$$P_A(\lambda) = \det\left(\lambda I_2 - \begin{pmatrix} 5 & 2 \\ -1 & 3 \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} \lambda-5 & -2 \\ 1 & \lambda-3 \end{pmatrix} = (\lambda-5)(\lambda-3)+2$$

$$= \lambda^2 - 8\lambda + 17 = 0,$$

$$\lambda = \frac{8 \pm \sqrt{64-4\cdot17}}{2} = \frac{8 \pm \sqrt{-4}}{2}$$

$$= \underline{4 \pm i}. \quad \lambda_1 = 4+i, \; \lambda_2 = 4-i$$

Er egenverdiene.

Skal finne egenvektorer:

Løser , $\lambda_1 = 4+i$

$$(\lambda_1 I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} (4+i)-5 & -2 \\ 1 & (4+i)-3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{matrix} (-1+i)x - 2y = 0 \\ x + (1+i)y = 0 \end{matrix}\right\} \Rightarrow y = \frac{-1+i}{2}x$$

Får egenvektor $\underline{\underline{\begin{pmatrix} 2 \\ -1+i \end{pmatrix}}}$

$\lambda_2 = 4-i$ $\quad (-1-i)x - 2y = 0$

$\qquad\qquad\qquad x + (1-i)y = 0$

$y = \frac{-1-i}{2}x$, $\begin{pmatrix} 2 \\ -1-i \end{pmatrix}$ egenvektor

4.10

2b) Skal finne egenverdier og vektoren

for $A = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 0 & 1 \\ -1 & -1 & 2 \end{pmatrix}$

$$\det(\lambda I_3 - A) = \begin{vmatrix} \lambda-1 & -3 & 1 \\ -2 & \lambda & -1 \\ 1 & 1 & \lambda-2 \end{vmatrix}$$

$$= (\lambda-1)(\lambda(\lambda-2)+1) + 3(-2(\lambda-2)-1)$$

$$+ (-2-\lambda) = \lambda^3 - 3\lambda^2 - 4\lambda + 12 = 0$$

utregning

Ser at $\lambda = 2$ er en rot (prøver med hele tall som går opp i 12)

$(8 - 12 - 8 + 12 = 0)$

$$\lambda^3 - 3\lambda^2 - 4\lambda + 12 : \lambda - 2 = \underline{\lambda^2 - \lambda - 6}$$

$$\underline{\lambda^3 - 2\lambda^2}$$

$$-\lambda^2 - 4\lambda$$

$$\underline{-\lambda^2 + 2\lambda}$$

$$-6\lambda + 12$$

$$\underline{P_\lambda(A) = (\lambda^2 - \lambda - 6)(\lambda - 2)}$$

$$\lambda^2 - \lambda - 6 = 0 \quad \lambda = \frac{1 \pm \sqrt{1+24}}{2} = \frac{1 \pm 5}{2} = \begin{cases} -2 \\ 3 \end{cases}$$

Egenverdier $\lambda_1 = -2$, $\lambda_2 = 2$, $\lambda_3 = 3$.

4.10 2b forts.

$\lambda_1 = -2$ finner egenvektor. $A = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 0 & 1 \\ -1 & -1 & 2 \end{pmatrix}$

$$\left.\begin{array}{rl} -3x - 3y + z = 0 & I \\ -2x - 2y - z = 0 & II \\ x + y - 4z = 0 & III \end{array}\right\} \quad \begin{array}{l} I + II \\ -5x - 5y = 0 \\ y = -x \end{array}$$

$III \Rightarrow -4z = 0 \quad z = 0$

$v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ er egenvektor til $\lambda_1 = -2$

$\lambda_2 = 2$

$$\left.\begin{array}{r} x - 3y + z = 0 \\ -2x + 2y - z = 0 \\ x + y \quad = 0 \end{array}\right\} \quad \begin{array}{l} y = -x \\ z = 3y - x \\ = -4x \end{array}$$

$v_2 = \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix}$ er egenvektor til $\lambda_2 = 2$

$\lambda_2 = 3$

$$\left.\begin{array}{ll} I & 2x - 3y + z = 0 \\ II & -2x + 3y - z = 0 \\ III & x + y + z = 0 \end{array}\right\} \quad \begin{array}{l} II + III \\ -x + 4y = 0 \\ x = 4y \end{array}$$

$z = -x - y$

$= -5y$

$v_3 = \begin{pmatrix} 4 \\ 1 \\ -5 \end{pmatrix}$ blir egenvektor til $\lambda_3 = 3$.

4.10

7) Skal vise at A og A^T har samme egenverdier

$$P_\lambda(A) = \det(\lambda I_n - A)$$
$$= \det((\lambda I_n - A)^T) = \det(\lambda I_n^T - A^T)$$
$$= \det(\lambda I_n - A^T) = P_\lambda(A^T)$$

Siden de karakteristiske polynomene til A og A^T er like må egenverdiene også være like. Hva med egenvektorene

Eks $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

ser at $A\begin{pmatrix}1\\0\\0\end{pmatrix} = 0\begin{pmatrix}1\\0\\0\end{pmatrix}$ så $\begin{pmatrix}1\\0\\0\end{pmatrix}$ er egenvektor til $\lambda = 0$.

$$A^T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} \quad A^T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix}$$

Ser at $\begin{pmatrix}1\\0\\0\end{pmatrix}$ ikke egenvektor for A^T

(svar nei).

9) Anta $\vec{v}$ egenvektor for både

A og B $\Rightarrow$ $\vec{v}$ egenvektor for AB.

$A\vec{v} = \alpha\vec{v}$, $B\vec{v} = \beta\vec{v}$ for passende α, β

$(AB)(\vec{v}) = A(B\vec{v}) = A(\beta\vec{v})$

$= \beta A(\vec{v}) = \underline{\underline{\beta\alpha\vec{v}}}$ $\vec{v}$ egenvektor for AB med egenverdi $\beta\alpha$

4.11.

2) $x'(t) = x(t) + 8y(t)$

$y'(t) = 2x(t) + y(t)$

$$= \underbrace{\begin{pmatrix} 1 & 8 \\ 2 & 1 \end{pmatrix}}_{\overset{\shortparallel}{A}} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$P_A(\lambda) =$

$$= \begin{vmatrix} \lambda-1 & -8 \\ -2 & \lambda-1 \end{vmatrix} = (\lambda-1)^2 - 16 = (\lambda-5)(\lambda+3)$$

$\lambda_1 = -3$ $\quad -4x - 8y = 0 \quad x = -2y \quad \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\quad -2x - 4y = 0$

eigenvektor.

$\lambda_2 = 5$ $\quad \left.\begin{matrix} 4x - 8y = 0 \\ -2x + 4y = 0 \end{matrix}\right\} \quad x = 2y$

$v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ egenvektor.

Generell Lösung $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{-3t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$+ c_2 e^{5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Skal ha $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$

$c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$ $\quad -2c_1 + 2c_2 = 1 \quad \underline{I}$

$c_1 + c_2 = 6 \quad \underline{II}$

$I - 2II \quad -4c_1 = -11, \; c_1 = \frac{11}{4}$

$c_2 = 6 - c_1 = \frac{13}{4}$

$x(t) = -\frac{11}{2} e^{-3t} + \frac{13}{2} e^{5t}$

$y(t) = \frac{11}{4} e^{-3t} + 13/4\, e^{5t}.$

4.11

1) $$x_{n+1} = x_n + 3y_n$$
$$y_{n+1} = 2x_n + 2y_n$$

$x_0 = 5$, $y_0 = -5$, $x_n = ?$, $y_n = ?$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}, \quad P_A(\lambda) = \det \begin{pmatrix} \lambda-1 & -3 \\ -2 & \lambda-2 \end{pmatrix}$$

$$= \lambda^2 - 3\lambda - 4 = 0 \qquad \lambda = \begin{cases} 4 \\ -1 \end{cases}$$

$\lambda_1 = -1$, $\quad \begin{matrix} -2x - 3y = 0 \\ -2x - 3y = 0 \end{matrix} \qquad x = -\frac{3}{2} y$

$\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ egenvekt. for $\lambda = -1$

$\lambda = 4 \qquad \left.\begin{matrix} 3x - 3y = 0 \\ -2x + 2y = 0 \end{matrix}\right\} \Rightarrow x = y \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

egenvektor for $\lambda = 4$

4.11.1 forts. Hadde

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \end{pmatrix}$$

$$\begin{pmatrix} 5 \\ -5 \end{pmatrix} = c_1 \begin{pmatrix} -3 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\left.\begin{aligned} -3c_1 + c_2 &= 5 \\ 2c_1 + c_2 &= -5 \end{aligned}\right\} \quad \begin{aligned} &5c_1 = -10, \; c_1 = -2 \\ &c_2 = 5 + 3c_1 = -1 \end{aligned}$$

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = (-2)\begin{pmatrix} -3 \\ 2 \end{pmatrix} + (-1)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = M \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} = M^2 \begin{pmatrix} x_{n-2} \\ y_{n-2} \end{pmatrix}$$

$$= \cdots M^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = M^n \left((-2)\begin{pmatrix} -3 \\ 2 \end{pmatrix} + (-1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)$$

$$= (-2)\,(-1)^n \begin{pmatrix} -3 \\ 2 \end{pmatrix} + (-1)\, 4^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\left.\begin{aligned} x_n &= 6(-1)^n - 4^n \\ y_n &= -4(-1)^n - 4^n \end{aligned}\right\}$$