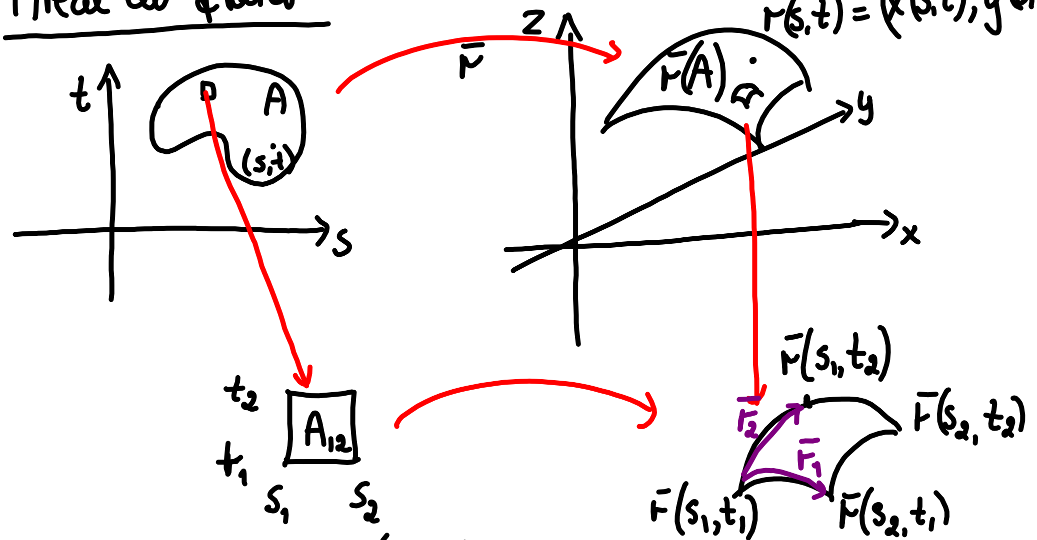
Areal av flater

$\bar r(s,t) = \big(x(s,t), y(s,t)\, z(s,t)\big)$



areal : $(s_2-s_1)(t_2-t_1)$

areal ≈ $\bar r(s_2,t_1)-\bar r(s_1,t_1) = \bar r_1$

$\bar r(s_1,t_2)- r(s_1,t_1) = \bar r_2$

$$\text{areal} \approx |\bar r_1 \times \bar r_2| = \left|\frac{\partial \bar r}{\partial s}(s_2-s_1) \times \frac{\partial \bar r}{\partial t}(t_2-t_1)\right|$$

$$\bar r_1 \approx \frac{\partial \bar r}{\partial s}(s_1,t_1)(s_2-s_1)$$

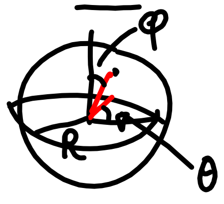
$$\bar r_2 \approx \frac{\partial \bar r}{\partial t}(s_1,t_1)(t_2-t_1)$$

$$= \left|\frac{\partial \bar r}{\partial s} \times \frac{\partial \bar r}{\partial t}\right| (s_2-s_1)(t_2-t_1)$$

$$= \left|\frac{\partial \bar r}{\partial s} \times \frac{\partial \bar r}{\partial t}\right| \text{areal}(A_{12})$$

$$\Rightarrow \quad \text{areal}(\bar r(A)) = \iint_A \left|\frac{\partial \bar r}{\partial s} \times \frac{\partial \bar r}{\partial t}\right| ds\,dt$$

Eks.



$$\bar{r}(\theta,\varphi) = (R\cos\theta \sin\varphi, R\sin\theta\sin\varphi, R\cdot\cos\varphi)$$

$$0 \le \theta \le 2\pi$$
$$0 \le \varphi \le \pi$$

$$\frac{\partial \bar{r}}{\partial \theta} = (-R\sin\theta\sin\varphi, R\cos\theta\sin\varphi, 0)$$

$$\frac{\partial \bar{r}}{\partial \varphi} = (R\cos\theta\cos\varphi, R\sin\theta\cos\varphi, -R\sin\varphi)$$

$$\frac{\partial \bar{r}}{\partial \theta} \times \frac{\partial \bar{r}}{\partial \varphi} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -R\sin\theta\sin\varphi & R\cos\theta\sin\varphi & 0 \\ R\cos\theta\cos\varphi & R\sin\theta\cos\varphi & -R\sin\varphi \end{vmatrix}$$

$$= \left(-R^2\cos\theta\sin^2\varphi, -R^2\sin\theta\sin^2\varphi, -R^2(\sin^2\theta\sin\varphi\cos\varphi + \cos^2\theta\sin\varphi\cos\varphi)\right)$$

$$= R^2(-\cos\theta\sin^2\varphi, -\sin\theta\sin^2\varphi, -\sin\varphi\cos\varphi)$$

$$\left|\frac{\partial \bar{r}}{\partial \theta} \times \frac{\partial \bar{r}}{\partial \varphi}\right| = R^2\sqrt{\cos^2\theta\sin^4\varphi + \sin^2\theta\sin^4\varphi + \sin^2\varphi\cos^2\varphi}$$

$$= R^2\sqrt{\sin^4\varphi + \sin^2\varphi\cos^2\varphi}$$

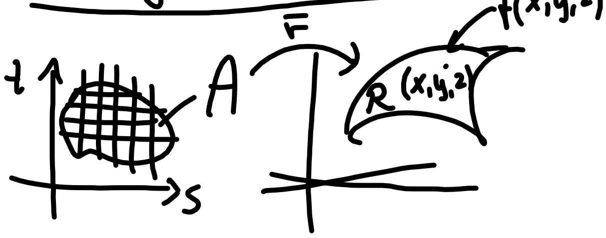
$$= R^2\sin\varphi\sqrt{\sin^2\varphi + \cos^2\varphi} = R^2\sin\varphi$$

$$\text{areal} = \int_0^{2\pi}\int_0^{\pi} R^2\sin\varphi\, d\varphi\, d\theta = R^2\int_0^{2\pi}\left(\int_0^{\pi}\sin\varphi\, d\varphi\right) d\theta$$

$$= R^2\int_0^{2\pi}\left[-\cos\varphi\right]_{\varphi=0}^{\varphi=\pi} d\theta = R^2\int_0^{2\pi}(1+1)\, d\theta$$

$$= R^2\cdot 2\cdot(2\pi - 0) = \underline{\underline{4\pi R^2}}$$

Flateintegral av et skalarfelt ← Funksjon



Flateintegralet av f over R

$$\iint f(\bar{r}(s,t))\left|\frac{\partial \bar{r}}{\partial s}\times\frac{\partial \bar{r}}{\partial t}\right| ds\, dt$$

Eks.

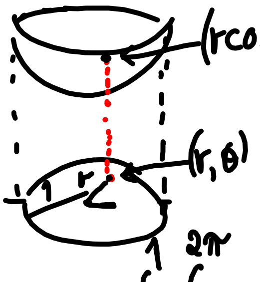
Flaten: Den delen av paraboloiden $z=x^2+y^2$ over disk $A: x^2+y^2 \le 1$

Parametrisering:

$\bar{r}(\cancel{s,t}) =$

$\bar{r}(r,\theta)=(r\cos\theta, r\sin\theta, r^2)$

Funksjon: $f(x,y,z) = xy$



Funksjonen:

$f(\bar{r}(r,\theta)) = r^2\cos\theta\sin\theta$

$$\iint_R f\, dA = \int_0^1\int_0^{2\pi} r^2\cos\theta\sin\theta\left|\frac{\partial \bar{r}}{\partial r}\times\frac{\partial \bar{r}}{\partial \theta}\right| d\theta\, dr$$

$$\frac{\partial \bar{r}}{\partial r} = (\cos\theta, \sin\theta, 2r)$$

$$\frac{\partial \bar{r}}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0)$$

$$\left|\frac{\partial \bar{r}}{\partial r}\times\frac{\partial \bar{r}}{\partial \theta}\right| = \left\|\begin{matrix} \bar{i} & \bar{j} & \bar{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0\end{matrix}\right\|$$

$$= \left|(-2r^2\cos\theta, -2r^2\sin\theta, r)\right|$$

$$= \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2}$$

$$= \sqrt{4r^4+r^2} = r\sqrt{4r^2+1}$$

$$= \int_0^1\int_0^{2\pi} r^2\cos\theta\sin\theta\, r\cdot\sqrt{4r^2+1}\, d\theta\, dr$$

$$= \int_0^1 r^3\sqrt{4r^2+1}\left(\int_0^{2\pi}\cos\theta\sin\theta\, d\theta\right) dr$$

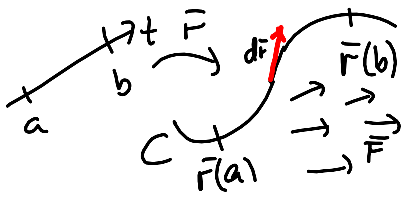
$$= \int_0^1 r^3\sqrt{4r^2+1}\int_0^{2\pi}\frac{1}{2}\sin 2\theta\, d\theta\, dr$$

$$= \int_0^1 r^3\sqrt{4r^2+1}\left[\frac{1}{4}(-\cos 2\theta)\right]_{\theta=0}^{\theta=2\pi} dr$$

$$= \int_0^1 r^3\sqrt{4r^2+1}\left(-\frac{1}{4}\right)(\cos 4\pi - \cos 0)\, dr = \underline{\underline{0}}$$

Greens teorem

Linjeintegral: $\int_C \bar{F}\cdot d\bar{r} = \int_a^b \bar{F}(\bar{r}(t))\,\bar{r}'(t)\,dt$



I planet $\bar{F}(x,y) = P(x,y)\,\bar{i} + Q(x,y)\,\bar{j}$

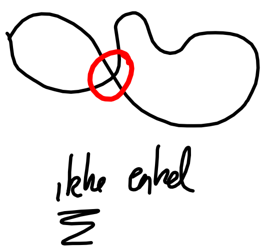
$\bar{r}(t) = x(t)\,\bar{i} + y(t)\,\bar{j}$

$$= \int_C (P,Q)(x',y')\,dt \qquad x' = \frac{dx}{dt},\; y' = \frac{dy}{dt}$$

$$= \int_C \left(P\cdot\frac{dx}{dt} + Q\,\frac{dy}{dt}\right)dt = \int_C P\,dx + Q\,dy$$

Enkel, lukket kurve:

Orientering mot/med klokka



Greens teorem

C

enkel, lukket kurve i planet

stykkevis glatt parametrisering

$\partial R = C$ (C er randa til området R)

orientert mot klokka

P, Q partielt deriverte er kontinuerlig i et område som inneholder R.

Da gjelder:

$$\boxed{\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy}$$

Én kommentar:

Greens generaliserer fundamentalsetningen

$$F(b) - F(a) = \int_a^b F'(x)\,dx$$

Ned en dimensjon

F funksjon

VS: $F(b) - F(a)$

HS $\int_a^b F'(x)\,dx$