Plenum 19/04

$\boxed{5.6.1}$ $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - x$

a) $f(x) = x^3 - x = x(x^2 - 1) = x(x-1)(x+1)$

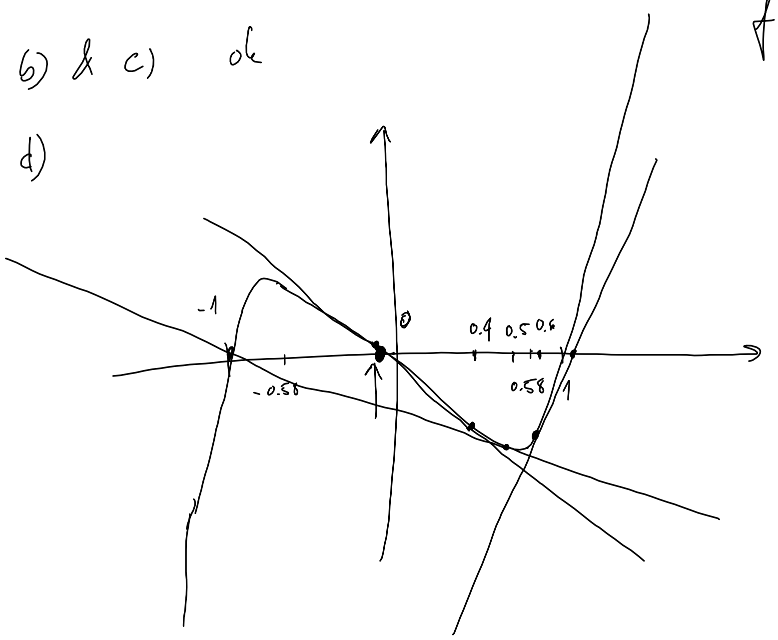
$x = -1, 0, 1$

$f' = 3x^2 - 1 = 0$

$x = \pm 0.58$

b) & c) ok

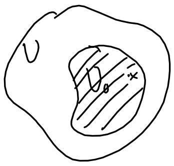
d)



$\boxed{5.7.1}$ $F(x,y) = \begin{pmatrix} x^2+y+1 \\ x-y-2 \end{pmatrix}$, $F: \mathbb{R}^2 \to \mathbb{R}^2$

$G(1,-2) = (0,0)$ $\quad (\Leftarrow F(0,0) = (1,-2))$

$\bar{x} \in U$: $F'(\bar{x})$ inverterbar $\Rightarrow \exists \ U \supset U_0 \ni \bar{x}$: F injektiv på U_0.



$\exists$ G på $F(U_0) = V$ og G er deriverbar i $F(\bar{x}) = \bar{y}$

med Jacobi-matrise $G'(\bar{y}) = F'(\bar{x})^{-1}$

Vis: $F'(0,0)$ er inverterbar , $F(0,0) = (1,-2)$

$F\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix} 0^2+0+1 \\ 0-0-2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ ok

$F'(x,y) = \begin{pmatrix} 2x & 1 \\ 1 & -1 \end{pmatrix}$, $F'(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$

$F'(0,0)$ er inverterbar $\Leftrightarrow$ $\det(F'(0,0)) \neq 0$.

$\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = 0 \cdot -1 - 1 \cdot 1 = \underline{\underline{-1}} \neq 0$

$\Rightarrow$ $G'(1,-2) = F'(0,0)^{-1}$

$\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right)$

$G'(1,-2) = F'(0,0)^{-1}$

$F\begin{pmatrix}-1\\-1\end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $F'\begin{pmatrix}-1\\-1\end{pmatrix}$ er inverterbar

$F'\begin{pmatrix}-1\\-1\end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}$, $\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} = 2 - 1 = 1 \neq 0$

$H\begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $H'\begin{pmatrix}1\\-2\end{pmatrix} = F'\begin{pmatrix}-1\\-1\end{pmatrix}^{-1}$

$\boxed{3.7.5}$ $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix}$

a) $(A \,\vdots\, I_3) \sim (I_3 \,\vdots\, A^{-1})$

$$A^{-1} = \begin{pmatrix} 2/3 & 0 & 1/3 \\ -5/3 & 1 & -1/3 \\ 1/3 & 0 & -1/3 \end{pmatrix}$$

b) $F(x,y,z) = \begin{pmatrix} x+z \\ x^2 + \frac{1}{2}y^2 + z \\ x + z^2 \end{pmatrix}$, $F : \mathbb{R}^3 \to \mathbb{R}^3$

$$F'(x,y,z) = \begin{pmatrix} 1 & 0 & 1 \\ 2x & y & 1 \\ 1 & 0 & 2z \end{pmatrix}$$

Vis: $G(0, \frac{1}{2}, 2) = (1, 1, -1)$, $G'(0, \frac{1}{2}, 2)$

$F(1,1,-1) = (0, \frac{1}{2}, 2)$, $F'(1,1,-1)$ er inverterbar

$$F\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1^2 + \frac{1}{2}1^2 - 1 \\ 1 + (-1)^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ 2 \end{pmatrix} \quad \text{ok}$$

$$\det\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix} = 1 \cdot \begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = 1 \cdot -2 + 1 \cdot (-1) = \underline{-3}$$

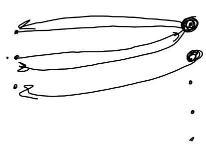
$$\left(F'(1,1,-1) \,\vdots\, I_3\right) \sim \left(I_3 \,\vdots\, F'(1,1,-1)^{-1}\right)$$

$G'(0, \frac{1}{2}, 2) = F'(1,1,-1)^{-1} = A^{-1}$ fra oppg. (a).

$f: \overset{\mathbb{R}^{n+1}}{U} \to \mathbb{R}$ kont. part. deriverte

gitt at det finnes et pkt. $(\bar{x}, \bar{y}) \in U$ $\quad (\bar{x} \in \mathbb{R}^n, \ \bar{y} \in \mathbb{R})$.

med $\frac{\partial f}{\partial y}(\bar{x}, \bar{y}) \neq 0$



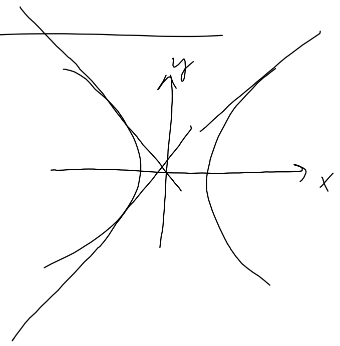
$$\Downarrow$$

$\exists$ omegn U_0 om $\bar{x}$: $g(x)$ injektiv på U_0 slik at

$f(x, g(x)) = 0$ på U_0. Videre er g deriverbar på U_0

med : $\quad \frac{\partial g}{\partial x_i}(x) = - \dfrac{\frac{\partial f}{\partial x_i}(x, g(x))}{\frac{\partial f}{\partial y}(x, g(x))}$

$\boxed{5.7.7}$ $\quad \underbrace{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$



$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$$

$$y^2 = \frac{b^2 x^2}{a^2} - b^2$$

$g(x) = y(x) = y = \pm |b| \underbrace{\sqrt{\frac{x^2}{a^2} - 1}}$, $\quad g'(x) = \frac{1}{2}|b| \frac{1}{\sqrt{\frac{x^2}{a^2} - 1}} \cdot \frac{2x}{a^2}$

$$= \frac{|b|\, x}{a^2 \underbrace{\sqrt{\frac{x^2}{a^2} - 1}}_{\frac{g(x)}{|b|}}} = \frac{b^2 x}{a^2 g(x)} = \frac{b^2 x}{a^2 y}$$

$|b|^2 = b^2$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

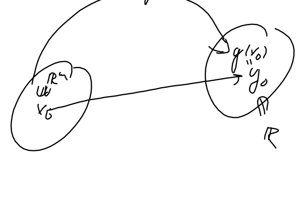
$$\underbrace{\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1}_{f(x,y)} = 0 \qquad f(x,y) = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$$

- $\frac{\partial f}{\partial x} = \frac{2x}{a^2}$, $\quad \frac{\partial f}{\partial y} = - \frac{2y}{b^2}$ $\quad$ åpenbart kont.

- $\frac{\partial f}{\partial y}(x_0, y_0) = - \frac{2y_0}{b^2} \neq 0$ $\quad$ siden $y_0 \neq 0$.

$$\frac{\partial g}{\partial x}(x_0) = g'(x_0) = - \frac{\frac{\partial f}{\partial x}(x_0, g(x_0))}{\frac{\partial f}{\partial y}(x_0, g(x_0))} = - \frac{\frac{\partial f}{\partial x}(x_0, y_0)}{\frac{\partial f}{\partial y}(x_0, y_0)} , \quad g(x_0) \mapsto y_0$$

$$= \cancel{-} \frac{\frac{\cancel{2} x_0}{a^2}}{\cancel{-}\frac{\cancel{2} y_0}{b^2}} = \frac{b^2 x_0}{a^2 y_0}$$



$\boxed{5.7.9}$

$$\phi(x, y(x)) = C \quad , \quad \underbrace{\frac{\partial \phi}{\partial y}(x, y(x)) \neq 0}$$

<u>Vis</u> : $$y'(x) = -\frac{\frac{\partial \phi}{\partial x}(x, y(x))}{\frac{\partial \phi}{\partial y}(x, y(x))}$$

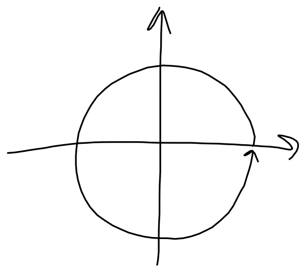
$$\psi(x, y(x)) = \phi(x, y(x)) - C = 0 \qquad \forall x \in \mathbb{R}$$

$$\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial x} \quad \& \quad \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial y}$$

$$\Rightarrow \frac{\partial y}{\partial x} = y'(x) = -\frac{\frac{\partial \psi}{\partial x}(x, y(x))}{\frac{\partial \psi}{\partial y}(x, y(x))} = -\frac{\frac{\partial \phi}{\partial x}(x, y(x))}{\frac{\partial \phi}{\partial y}(x, y(x))}$$

Eksamen 2009

② $\int_{\mathcal{C}} F \cdot \vec{r}$ $\qquad \vec{r}(t) = (\cos t, \sin t)$, $\quad t \in [0, 2\pi]$



$F(x,y) = \begin{pmatrix} xy^2 \\ x^2 y \end{pmatrix}$ $\begin{matrix} \leftarrow F_1 \\ \leftarrow F_2 \end{matrix}$

$\dfrac{\partial F_1}{\partial y} = \dfrac{\partial F_2}{\partial x}$ $\qquad \dfrac{\partial F_1}{\partial y} = 2xy$

$\dfrac{\partial F_2}{\partial x} = 2xy$

$F = \nabla \phi \qquad \int_{\mathcal{C}} \nabla \phi \cdot \vec{r} = \phi(b) - \phi(a) = \phi(a) - \phi(a) = 0$

3 a)
$$\begin{matrix} -4x+2y+2z=0 \\ 3x-6y+3z=0 \\ x+4y-5z=0 \end{matrix} \iff \underbrace{\begin{pmatrix} -4 & 2 & 2 \\ 3 & -6 & 3 \\ 1 & 4 & -5 \end{pmatrix}}_{B}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -4 & 2 & 2 & 0 \\ 3 & -6 & 3 & 0 \\ 1 & 4 & 5 & 0 \end{array}\right) \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{matrix} x - z = 0 \\ y - z = 0 \\ z = z \end{matrix}\right\} \quad \begin{matrix} x = z \\ y = z \\ z = z \end{matrix} \quad \to \vec{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ z \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} z$$

$z \in \mathbb{R}$

b)
$$A = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.1 & 0.4 & 0.5 \end{pmatrix}$$

$$A\vec{v} = 1\vec{v} \qquad A\vec{v} - \vec{v} = \vec{0}$$

$$\underbrace{(A - I_3)}\vec{v} = \vec{0}$$

$$\begin{pmatrix} -0.4 & 0.2 & 0.2 \\ 0.3 & -0.6 & 0.3 \\ 0.1 & 0.4 & -0.5 \end{pmatrix} = \frac{1}{10}B$$

$$\Rightarrow \lambda = 1, \quad v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

c)
$$[u,v] = \mathrm{eig}(A), \quad u = \begin{bmatrix} 0.57 & 0.7071 & 0 \\ 0.57 & 0 & -0.7071 \\ 0.57 & -0.7071 & 0.7071 \end{bmatrix}$$

$$v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$$

$$A\vec{v} = \lambda\vec{v}$$
$$\underbrace{A(a\vec{v})} = aA\vec{v} = a\lambda\vec{v} = \lambda\underbrace{(a\vec{v})}$$

$$\underset{v_1}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}$$

$$\begin{bmatrix} 120 \\ 0 \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 120 \\ 0 & 1 & -1 & 0 \\ -1 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 80 \\ 0 & 1 & 0 & 40 \\ 0 & 0 & 1 & 40 \end{array}\right]$$

$$\begin{bmatrix} 120 \\ 0 \\ 0 \end{bmatrix} = 40\, v_1 + 80\, v_2 + 40\, v_3$$

d)
$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{bmatrix} = A\begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} = A^2\begin{bmatrix} x_{n-1} \\ y_{n-1} \\ z_{n-1} \end{bmatrix} = \dots \qquad \begin{matrix} x_0 = 120 \\ y_0 = 0 \\ z_0 = 0 \end{matrix}$$

$$\vec{x}_n = \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} = A^n \underset{\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}}{\begin{bmatrix} 120 \\ 0 \\ 0 \end{bmatrix}} = A^n\left(40\,\vec{v}_1 + 80\,\vec{v}_2 + 40\,\vec{v}_3\right)$$

$$= 40\,\lambda_1^n\vec{v}_1 + 80\,\lambda_2^n\vec{v}_2 + 40\cdot\lambda_3^n\vec{v}_3$$

$$= 40\,\vec{v}_1 + 80\cdot\underbrace{0.4^n}\vec{v}_2 + 40\cdot\underbrace{0.1^n}_{\to 0}\vec{v}_3$$

$$\lim_{n\to\infty}\vec{x}_n = 40\,\vec{v}_1 = 40\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 40 \\ 40 \\ 40 \end{pmatrix}$$