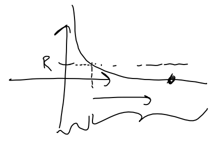
Plenum 26/4



$\boxed{5.8.2}$ $f: \mathbb{R}^m \to \mathbb{R}$ $\qquad \lim_{|x|\to\infty} f(x) = 0$

Siden $\lim_{|x|\to\infty} f(x)=0$ betyr det at $\exists\ R \in \mathbb{R}_+$ slik at $f(x) < \varepsilon$ $(\varepsilon>0)$ for alle $\underbrace{\{x : |x| > R\}}$. På $\{x : |x| \le R\}$ (som er lukket og begrenset) vil f være begrenset av setn. 5.8.2

På $\{x: |x|\le R\}$: $f(x) \le M$. $\Rightarrow$ $f(x) \le \max(M, \varepsilon)$ På hele $\mathbb{R}^m$.

$\boxed{5.8.3}$ $||u|-|v|| \le |u-v|$

$f: A\to\mathbb{R}$ gitt ved $f(x) = |x - F(x)|$.

a) $A \subset \mathbb{R}^m$ (lukket, begrenset) $\quad F: A\to A$ kontinuerlig.

La $\varepsilon>0$ være gitt. Siden F er kont. finnes det for hver $x\in A$ en $\delta(x)>0$ slik at $|F(x)-F(y)|<\varepsilon/2$ dersom $|x-y|<\delta(x)$. Derfor finn for $x\in A$ en $y\in A$ slik at $|x-y| < \min(\frac{\varepsilon}{2}, \delta(x))$. Da vil

$$|f(x)-f(y)| = ||x-F(x)| - |y-F(y)|| \le |x-F(x) - (y-F(y))|$$
$$= |x-y+F(y)-F(x)| \le \underbrace{|x-y|}_{<\frac{\varepsilon}{2}} + \underbrace{|F(y)-F(x)|}_{<\frac{\varepsilon}{2}} < \varepsilon$$

b) Anta at $|F(x)-F(y)| < |x-y| \quad \forall\, x,y\in A,\ x\ne y$

Entydighet: Anta at det finnes 2 fikspunkt, $x, y \in A$.

$F(x)=x, \quad F(y)=y$

$$\underset{\substack{\uparrow \\ \text{antakelse}}}{|x-y|} > |F(x)-F(y)| \underset{\substack{\uparrow \\ \text{fikspunkt}}}{=} |x-y| \quad \Rightarrow \quad |x-y|<|x-y| \quad \perp$$

$\Rightarrow \quad x=y$

Eksistens: Bruker (c). La x være mini.pkt. for f. Husk at $f: A\to\mathbb{R}$

$F(x)\in A, \quad F: A\to A.$

$$f(F(x)) \underset{\substack{\uparrow \\ \text{def. av } f}}{=} |F(x) - F(F(x))| \underset{\substack{\uparrow \\ \text{antakelse}}}{<} |x - F(x)| \underset{\substack{\uparrow \\ \text{def. av } f}}{=} f(x)$$

$$f(F(x)) < f(x)$$

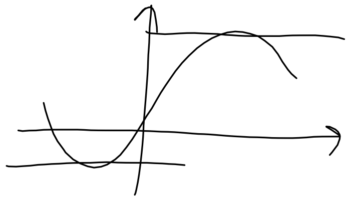
Ulikheten gjelder kun dersom $x\ne F(x)$. Siden x er minimumspunkt for f kan ikke $f(y)<f(x)$

$\Rightarrow F(x)=x$. fikspunkt.

c) La $A = \mathbb{R}_+\setminus\{0\}$ åpent, ikke-begrenset område

$F(x)=\frac{x}{2}$. $\quad x=0$ er fikspkt, men $x\notin A$.

$\boxed{5.9.5}$ $f(x,y) = x^3 + 3x^2y + 3y^2$



$$\nabla f(x,y) = \left(3x^2 + 6xy,\ \underbrace{3x^2 + 6y}\right)$$

$$3x^2 + 6y = 0$$

$$\boxed{6y = -3x^2} \leftarrow \quad 6y = -3\cdot 1^2$$

$$y = -\frac{1}{2}$$

$$3x^2 + 6xy = 0$$

$$3x^2 + x\cdot(-3x^2) = 0$$

$$3x^2(1-x) = 0 \quad \Rightarrow \quad x = 1 \quad \vee \quad x = 0$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$y = -\frac{1}{2} \qquad y = 0$$

stasjonære pkt. $(0,0)$ og $(1,-\frac{1}{2})$

$$Hf(x,y) = \begin{pmatrix} \frac{\partial^2}{\partial x^2}f & \frac{\partial^2}{\partial x \partial y}f \\ \frac{\partial^2}{\partial y \partial x}f & \frac{\partial^2}{\partial y^2}f \end{pmatrix} = \begin{pmatrix} 6x+6y & 6x \\ 6x & 6 \end{pmatrix}$$

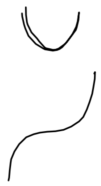
$$Hf(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix}, \quad Hf(1,-\tfrac{1}{2}) = \begin{pmatrix} 6-3 & 6 \\ 6 & 6 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 6 & 6 \end{pmatrix}$$

$$\det\left(Hf(0,0)\right) = \begin{vmatrix} 0 & 0 \\ 0 & 6 \end{vmatrix} = 0$$

$$f(0,0) = 0^3 + 3\cdot 0^2\cdot 0 + 3\cdot 0^2 = 0$$

$$f(0,s) = 0^3 + 3\cdot 0^2\cdot s + 3\cdot s^2 = \underline{3s^2 > 0}$$

$$f(s,0) = s^3 + 3\cdot s^2\cdot 0 + 3\cdot 0^2 = s^3 \begin{matrix} < 0 \\ > 0 \end{matrix}$$



$(0,0)$ er et sadelpkt.

$$\det\begin{pmatrix} 3 & 6 \\ 6 & 6 \end{pmatrix} = 3\cdot 6 - 6\cdot 6 = -18 < 0 \quad \Rightarrow \text{sadelpkt.}$$

$\boxed{5.10.2}$ $z^2 - xy = 1$ nærmest origo.

$$|(x,y,z) - (0,0,0)| = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$$
$$= \sqrt{\underbrace{x^2 + y^2 + z^2}}$$
$$f(x,y,z) = x^2 + y^2 + z^2$$

$$\min_{(x,y,z)} f(x,y,z)$$

gitt $g(x,y,z) = 1$ der $g(x,y,z) = z^2 - xy$.

$\cancel{\nabla g = 0}$ eller $\nabla f = \lambda \nabla g$

$\nabla g(x,y,z) = (-y, -x, 2z)$

$\nabla g(0,0,0) = (0, 0, 0)$.

$\nabla f(x,y,z) = (2x, 2y, 2z)$

I	$2x = -\lambda y$
II	$2y = -\lambda x$
III	$2z = \lambda 2z \longrightarrow z = 0 \vee \lambda = 1$
IV	$z^2 - xy = 1$

$\underline{z=0}$:

I $2x = -\lambda y \Rightarrow x = -\frac{\lambda}{2} y$
II $2y = -\lambda x \Rightarrow y = -\frac{\lambda}{2} x$

$\frac{2x}{y} = \frac{2y}{x} \Rightarrow x = \pm y$

IV $z^2 - xy = 1 \Rightarrow -xy = 1 \longrightarrow xy = -1 \Rightarrow x = -\frac{1}{y}$

$\Rightarrow$ $x = -1 \Rightarrow y = 1$
eller $y = -1 \Rightarrow x = 1$

$\Rightarrow$ $(-1, 1, 0) \to \sqrt{(-1)^2 + 1^2 + 0^2} = \sqrt{2}$
$(1, -1, 0) \to \sqrt{2}$

$\underline{\lambda = 1}$:

I $2x = -y$
II $2y = -x$

$x = -2y \Rightarrow x = 0$

$2(-2y) = -y \Leftrightarrow -4y = -y \Rightarrow y = 0$

IV $z^2 - xy = 1$

$z^2 = 1 \Rightarrow z = \pm 1$

$\boxed{(0,0,1) \text{ og } (0,0,-1)}$

$\downarrow$ 1 $\quad$ $\downarrow$ 1