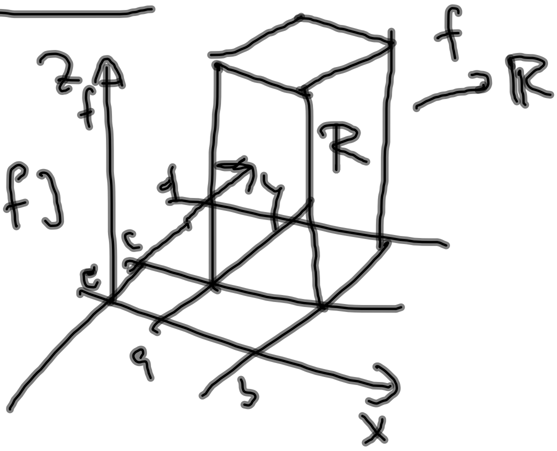
LH 6.9 Trippelintegraler



Boks

$B = R = [a,b] \times [c,d] \times [e,f]$

funksjon $\quad f \neq f!$

$f : R \longrightarrow \mathbb{R}$

partisjon

$$\left.\begin{aligned} a = x_0 < \dots < x_n = b \\ c = y_0 < \dots < y_m = d \\ e = z_0 < \dots < z_l = f \end{aligned}\right\} \Pi$$

$$R_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

$$R = \bigcup_{i,j,k} R_{ijk} \xrightarrow{\;f\;} \mathbb{R}$$

unionen av alle R_{ijk}

$$m_{ijk} = \inf \{ f(x,y,z) \mid (x,y,z) \in R_{ijk} \}$$

$$M_{ijk} = \sup \{ f(x,y,z) \mid (x,y,z) \in R_{ijk} \}$$

$$N(\Pi) = \sum_{i,j,k} m_{ijk} (x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1})$$

$$\leq \varnothing(\Pi) = \sum_{i,j,k} M_{ijk} (x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1})$$

nedreintegral

$$\underline{\iiint_R} f(x,y,z)\,dxdydz = \sup_{\Pi} N(\Pi)$$

øvreintegral

$$\overline{\iiint_R} f(x,y,z)\,dxdydz = \inf_{\Pi} \varnothing(\Pi)$$

__Def__ En begrenset $f: R \to \mathbb{R}$ er __integrerbar__ hvis

$$\underline{\iiint_R} f(x,y,z)\,dxdydz = \overline{\iiint_R} f(x,y,z)\,dxdydz$$

Da lar vi integralet

$$\iiint_R f(x,y,z)\,dxdydz$$ være den felles verdien.

Setning 6.9.2

$$R = [a,b] \times [c,d] \times [e,f] \xrightarrow{f} \mathbb{R}$$

kontinuerlig

$\Longrightarrow$ f er integrerbar på R

Setning 6.9.2

$$A = [a,b] \times [c,d]$$

$$R = A \times [e,f]$$

$$\iiint_R f(x,y,z)\,dx\,dy\,dx = \iint_A \left(\int_e^f f(x,y,z)\,dz \right) dx\,dy$$

$$= \int_a^b \left(\int_c^d \left(\int_e^f f(x,y,z)\,dz \right) dy \right) dx$$

$$xyz - xzy - yxz - yzx - zxy - zyx$$

Eksempel 1

$$R = [0,1] \times [1,3] \times [0,2] \subset \mathbb{R}^3$$

$$f(x,y,z) = x + ye^{2z}$$

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_R (x + ye^{2z})\,dx\,dy\,dz$$

$$= \int_{z=0}^{z=2} \left(\int_{y=1}^{y=3} \left(\int_{x=0}^{x=1} x + ye^{2z}\,dx \right) dy \right) dz$$

$$= \int_0^2 \left(\int_1^3 \left[\tfrac{1}{2}x^2 + xye^{2z} \right]_0^1 dy \right) dz$$

$$= \int_0^2 \left(\int_1^3 \left(\tfrac{1}{2} + ye^{2z} \right) dy \right) dz$$

$$= \int_0^2 \left[\tfrac{1}{2}y + \tfrac{1}{2}y^2 e^{2z} \right]_1^3 dz$$

$$= \int_0^2 (1 + 4e^{2z})\,dz = \left[z + 2e^{2z} \right]_0^2$$

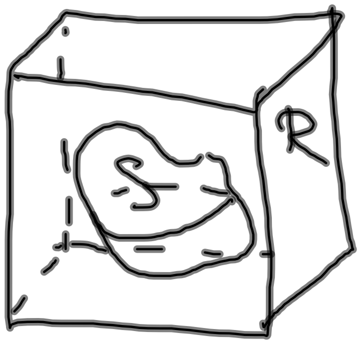
$$= (2 + 2e^4) - (0 + 2e^0) = \underline{\underline{2e^4}}$$

Mer generelle begrensede områder

$S \subset \mathbb{R}^3$

finner en boks $R = [a,b] \times [c,d] \times [e,f]$

med $S \subseteq R$



$f : S \longrightarrow \mathbb{R}$

utvider til

$f_S : R \longrightarrow \mathbb{R}$

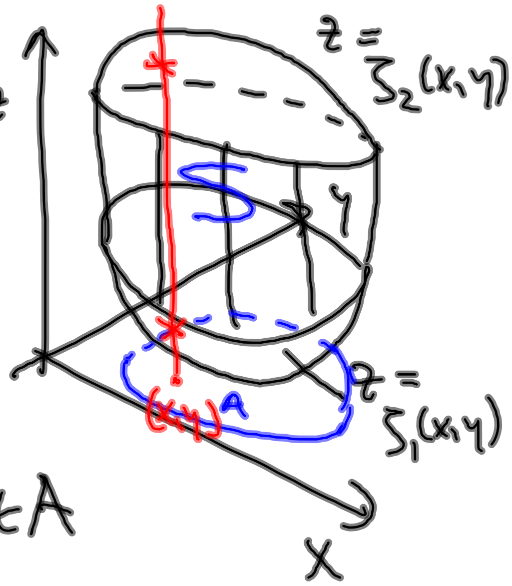
$$f_S(x,y,z) = \begin{cases} f(x,y,z) & (x,y,z) \in S \\ 0 & \text{ellers} \end{cases}$$

Hvis f_S er integrerbar lar vi

$$\iiint_S f(x,y,z)\,dx\,dy\,dz = \iiint_R f_S(x,y,z)\,dx\,dy\,dz$$

$A \subset \mathbb{R}^2$

lukket, begrenset, Jordan-målbart



$\zeta_1, \zeta_2 : A \longrightarrow \mathbb{R}$

kontinuerlige funksjoner

$\zeta_1(x,y) \leq \zeta_2(x,y)$ for $(x,y) \in A$

La $\qquad\qquad\qquad\qquad \zeta = \text{zeta}$

$$S = \{(x,y,z) \mid (x,y) \in A, \ \zeta_1(x,y) \leq z \leq \zeta_2(x,y)\}$$

være området mellom grafene til ζ_1 og ζ_2.

For $f : S \longrightarrow \mathbb{R}$ kontinuerlig er

$$\iiint_S f(x,y,z)\, dx\,dy\,dz = \iint_A \left(\int_{z=\zeta_1(x,y)}^{z=\zeta_2(x,y)} f(x,y,z)\, dz \right) dx\,dy$$

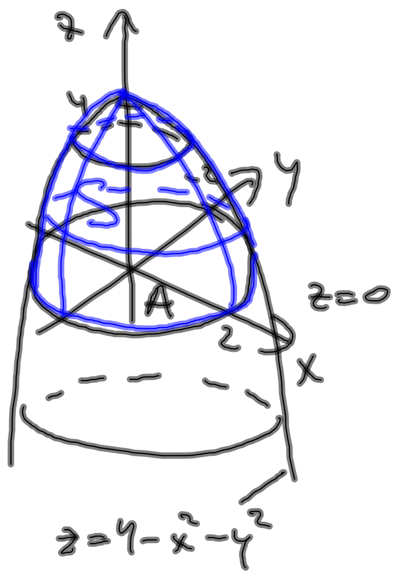
Eks 2

$$\iiint_S x \, dx\,dy\,dz$$

S ligger over A der

$0 \leq z \leq 4 - x^2 - y^2$

z finnes når $\left. x^2 + y^2 \leq 4 \right\} A$



$$= \iint_A \left(\int_{z=0}^{z=4-x^2-y^2} x \, dz \right) dx\,dy$$

$$= \iint_A \left[xz \right]_{z=0}^{z=4-x^2-y^2} dx\,dy$$

$$= \iint_A 4x - x(x^2+y^2) \, dx\,dy$$

$$= \int_0^{2\pi} \int_0^2 (4r\cos\theta - r\cos\theta \cdot r^2) \, r \, dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{4}{3} r^3 \cos\theta - \frac{1}{5} r^5 \cos\theta \right]_0^2 d\theta$$

$$= \int_0^{2\pi} \frac{32}{3}\cos\theta - \frac{32}{5}\cos\theta \, d\theta$$

$$\frac{32}{3} - \frac{32}{5} = 32\left(\frac{5-3}{15}\right) = \frac{64}{15}$$

$$= \frac{64}{15} \left[\sin\theta \right]_0^{2\pi} = \underline{\underline{0}}$$