b) Velger $P(x,y) = -y$, $Q(x,y) = x^2$

$$\rightarrow \frac{\partial Q}{\partial x} = 2x, \quad \frac{\partial P}{\partial y} = -1.$$

$y = 1 - x^2$ skjærer x-akse i $x = -1$ og $x = 1$

$\Downarrow$

$$\int_{\mathcal{C}} -y\,dx + x^2\,dy$$

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

Greens

Stykkevis glatt, pos. orientert

området avgr. av $\mathcal{C}$

$$= \int_{-1}^{1} \int_{0}^{1-x^2} (2x - 1)\,dy\,dx$$

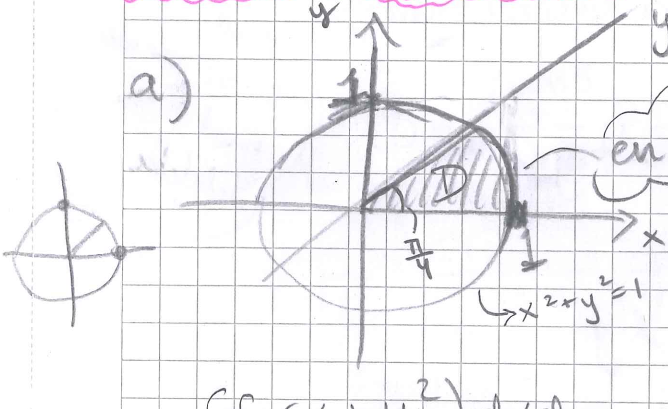
$$= \int_{-1}^{1} (2x-1)(1-x^2)\,dx = \int_{-1}^{1} (2x - 2x^3 + 1 - x^2)\,dx$$

$$= \left[x^2 - \frac{1}{2}x^4 + x - \frac{1}{3}x^3 \right]_{x=-1}^{1} = 1 + 1 - \frac{1}{3} - \frac{1}{3} = \underline{\underline{\frac{4}{3}}}$$

10.) $D : \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1,\ x \geq 0,\ y \geq 0,\ 0 \leq y \leq x\}$

SKISSER D og finn $\iint_D (x + y^2)\,dx\,dy$:

a)



Polarkoord:

$x = r\cos\theta$

$y = r\sin\theta$

$r \in [0, 1]$

$\theta \in [0, \frac{\pi}{4}]$ ← Fra fig

$y \leq x$

$r\sin\theta \leq r\cos\theta$

$\sin\theta \leq \cos\theta$

$\theta \in [0, \frac{\pi}{4}]$

Pga. D er i 1. kvadrant

Kan regne, men mer tungvint

$$\iint_D (x + y^2)\,dx\,dy$$

$$= \int_0^{\frac{\pi}{4}} \int_0^1 r(r\cos\theta + r^2\sin^2\theta)\,dr\,d\theta$$

Polar

OBS: HUSK!

[Trig.formel: se Kalk]

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{3}\cos\theta + \frac{1}{4}\sin^2\theta\right) d\theta = \frac{1}{3}\left[\sin\theta\right]_{\theta=0}^{\frac{\pi}{4}}$$

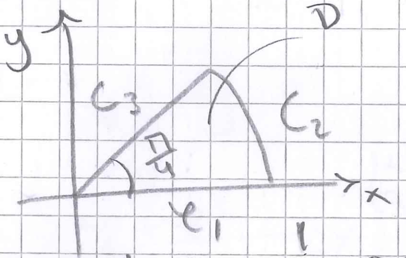
$$+ \frac{1}{4}\int_0^{\frac{\pi}{4}} \frac{1-\cos 2\theta}{2} d\theta = \frac{1}{3}\left(\frac{\sqrt{2}}{2} - 0\right) + \frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{4}\cdot\frac{1}{2}\left[\frac{1}{2}\sin(2\theta)\right]_{\theta=0}^{\frac{\pi}{4}}$$

$$= \frac{\sqrt{2}}{6} + \frac{\pi}{32} - \frac{1}{16}(1-0) = \underline{\underline{\frac{\sqrt{2}}{6} + \frac{\pi}{32} - \frac{1}{16}}}$$

Beregn I ved å regne ut et kurveintegral $\int_\mathcal{C} P\,dx + Q\,dy$ av et passelig vektorfelt langs randen til D:

Fra fig. forrige side:

b)



Int. over randen til D, $\mathcal{C}$:

$$\int_\mathcal{C} = \int_{\mathcal{C}_1} + \int_{\mathcal{C}_2} + \int_{\mathcal{C}_3}$$

Parametriserer alle kurvene orientert mot klokka (pos. orientert):

$\vec{r}_1(t) = \mathcal{C}_1 = (t, 0), \quad t \in [0, 1]$

$\vec{r}_2(t) = \mathcal{C}_2 = (\cos t, \sin t), \quad t \in [0, \frac{\pi}{4}]$

$\vec{r}_3(t) = \mathcal{C}_3 = (t, t), \quad t \in [\frac{\sqrt{2}}{2}, 0]$

dette er $\cos\frac{\pi}{4} = \sin\frac{\pi}{4}$

MERK: $\iint_D (x+y^2)\,dx\,dy = \int_\mathcal{C} P\,dx + Q\,dy$

hvis $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = x + y^2$ fra Greens thm.

randa til D er enkel, lukket, stykkvis glatt kurve orientert mot klokka

Velger (f.eks.) $P = 0$, $\frac{\partial Q}{\partial x} = x + y^2$

Mange muligheter

$$\Downarrow$$

$$Q = \frac{1}{2}x^2 + xy^2$$

Så

$$\iint_D (x+y^2)\,dx\,dy = \int_\mathcal{C} \left(\frac{1}{2}x^2 + xy^2\right) dy$$

Greens teorem

$$= \int_{\mathcal{C}_1} \left(\frac{1}{2}x^2 + xy^2\right) dy + \int_{\mathcal{C}_2} \left(\frac{1}{2}x^2 + xy^2\right) dy$$

$$+ \int_{\mathcal{C}_3} \left(\frac{1}{2}x^2 + xy^2\right) dy$$

$dy = 0\,dt$

$\mathcal{C}_1$: $I_1 = \int_0^1 \left(\frac{1}{2}t^2 + t \cdot 0^2\right) \cdot 0\,dt = 0$

$dy = \cos t\,dt$

$\mathcal{C}_2$: $I_2 = \int_0^{\frac{\pi}{4}} \left(\frac{1}{2}\cos^2 t + \cos t \sin^2 t\right) \cos t\,dt$

Del i 2. Tas v/ hhv. $\cos^2 = 1 - \sin^2$ og substitusjon $u = \sin t$ samt $\cos^2 t = \frac{1+\cos 2t}{2}$ og $\sin^2 t = \frac{1-\cos 2t}{2}$ og $\cos^2 2t = \frac{1+\cos 4t}{2}$

$dy = 1\,dt$

$\mathcal{C}_3$: $I_3 = \int_{\frac{\sqrt{2}}{2}}^{0} \left(\frac{1}{2}t^2 + t^3\right) \cdot 1\,dt = \ldots$

13.) Vis: $\vec{F}(x,y) = (P(x,y), Q(x,y))$ konservativ

$$\Downarrow$$

$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$ for enkle, lukkede, stykkevis glatte kurver.

Pf: $\vec{F}$ konservativ $\Rightarrow \exists\ \phi : \mathbb{R}^2 \to \mathbb{R}$ s.a. $\vec{F} = \nabla\phi$

$$= \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right) \Rightarrow P = \frac{\partial\phi}{\partial x},\ Q = \frac{\partial\phi}{\partial y}.$$

La $\vec{r}(t)$ være en parametrisering av $\mathcal{C}$.

6.5: 10) b) (Forts.:)

C_2: $I_2 = \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos^3 t\,dt + \int_0^{\frac{\pi}{4}} \cos^2 t \sin^2 t\,dt$

$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos t\,(1-\sin^2 t)\,dt + \int_0^{\frac{\pi}{4}} \frac{1+\cos(2t)}{2}\,\frac{1-\cos 2t}{2}\,dt$

trig formel, kalkulus

$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos t\,dt - \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos t \sin^2 t\,dt + \frac{1}{4}\int_0^{\frac{\pi}{4}} (1-\cos^2(2t))\,dt$

$= \frac{1}{2}[\sin t]_{t=0}^{\frac{\pi}{4}} - \frac{1}{2}\int_0^{\frac{\sqrt{2}}{2}} u^2\,du + \frac{1}{4}\left(\frac{\pi}{4}-0\right) - \frac{1}{4}\int_0^{\frac{\pi}{4}} \frac{1+\cos 4t}{2}\,dt$

$u = \sin t$

$du = \cos t\,dt$

$t=0 \Rightarrow u=0$

$t=\frac{\pi}{4} \Rightarrow u=\frac{\sqrt{2}}{2}$

$= \frac{1}{2}\frac{\sqrt{2}}{2} - \frac{1}{\cancel{2}}\frac{1}{3}\frac{(\sqrt{2})^3}{2^3} + \frac{1}{4}\frac{\pi}{4} - \frac{1}{4}\frac{1}{2}\frac{\pi}{4}$

$- \frac{1}{4}\frac{1}{2}\left[\frac{1}{4}\sin(4t)\right]_{t=0}^{\frac{\pi}{4}}$

$= \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{24} + \frac{\pi}{16} - \frac{\pi}{32} - \frac{1}{32}\,0$

$= \frac{6\sqrt{2}-\sqrt{2}}{24} + \frac{2\pi-\pi}{32} = \underline{\frac{5\sqrt{2}}{24} + \frac{\pi}{32}}$

I_3: $\int_{\frac{\sqrt{2}}{2}}^{0} \left(\frac{1}{2}t^2 + t^3\right)\cdot \underbrace{1\,dt}_{dy=1\,dt} = \left[\frac{1}{2}\frac{1}{3}t^3\right]_{t=\frac{\sqrt{2}}{2}}^{0} + \left[\frac{1}{4}t^4\right]_{t=\frac{\sqrt{2}}{2}}^{0}$

$= \frac{1}{6}\left(0 - \frac{(\sqrt{2})^3}{2^3}\right) + \frac{1}{4}\left(0 - \left(\frac{\sqrt{2}}{2}\right)^4\right)$

$= \frac{-1}{6}\frac{\cancel{2}\sqrt{2}}{2^{\cancel{3}2}} - \frac{1}{4}\frac{\cancel{4}2}{2^{\cancel{4}2}} = -\frac{\sqrt{2}}{24} - \frac{1}{16}$

Totalt: $I = I_1 + I_2 + I_3 = \frac{5\sqrt{2}}{24} + \frac{\pi}{32} - \frac{\sqrt{2}}{24} - \frac{1}{16}$

$= \frac{4\sqrt{2}}{24} + \frac{\pi}{32} - \frac{1}{16} = \underline{\underline{\frac{\sqrt{2}}{6} + \frac{\pi}{32} - \frac{1}{16}}}$