Litt repetisjon:

$$F(x_1,x_2) = \begin{pmatrix} e^{x_1} + x_2 \\ x_2 \cos x_1 \end{pmatrix} \quad i \quad (0,0) \in \mathbb{R}^2$$

- $F(0,0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- $F'(x_1, x_2) = \begin{pmatrix} e^{x_1} & 1 \\ -x_2 \sin x_1 & \cos x_1 \end{pmatrix}$ $\quad F'(0,0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ invertibel

$$G = F^{-1} \quad \Rightarrow \quad G'(1,0) = F'(0,0)^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$(0,0) \in U$ omegn

- $\overline{F(x_1,x_2) = F(0,0)}$ $\quad (x_1,x_2) \neq (0,0)$

  Skrumper U til ikke å inneholde (x_1,x_2)

- $\exists$ kurve C gjennom origo slik at $F(\bar{x}) = F(0,0)$ for alle $\bar{x} \in C$.

  $C: \bar{r}(t) = (x(t), y(t))$, $\quad F(\bar{r}(t)) = \text{konstant}$

  Kjerneregelen:

  $$F'(\bar{r}(t)) \cdot \underset{\neq 0}{\bar{r}'(t)} = 0$$

  Ikke mulig dersom $F'(\bar{x})$ er invertibel

Ekstremalverdi-setningen

Def $f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}^m$

- f er BEGRENSET på A dersom $\exists K, M \in \mathbb{R}$ slik at
  $K \leq f(\bar{x}) \leq M \quad \forall \bar{x} \in A$
- $\bar{c} \in A$: GLOBALT MAKSIMUM for f dersom
  $f(\bar{x}) \leq f(\bar{c}) \quad \forall \bar{x} \in A$
- $\bar{d} \in A$: GLOBALT MINIMUM for f dersom
  $f(\bar{d}) \leq f(\bar{x}) \quad \forall x \in A$

Teorem 5.8.2

$f : A \to \mathbb{R}$ (C^0), $A \subseteq \mathbb{R}^m$

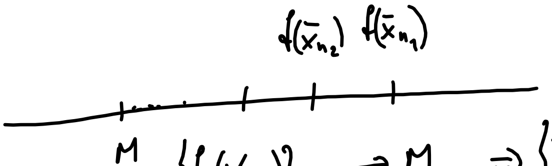
A: lukket og begrenset

Da $\exists$ globale ekstremalpunkter og f er begrenset.

Bevis:

$$M = \inf \{ f(\bar{x}) \mid \bar{x} \in A \}$$

$(M = -\infty)$ mulig. ikke her !!

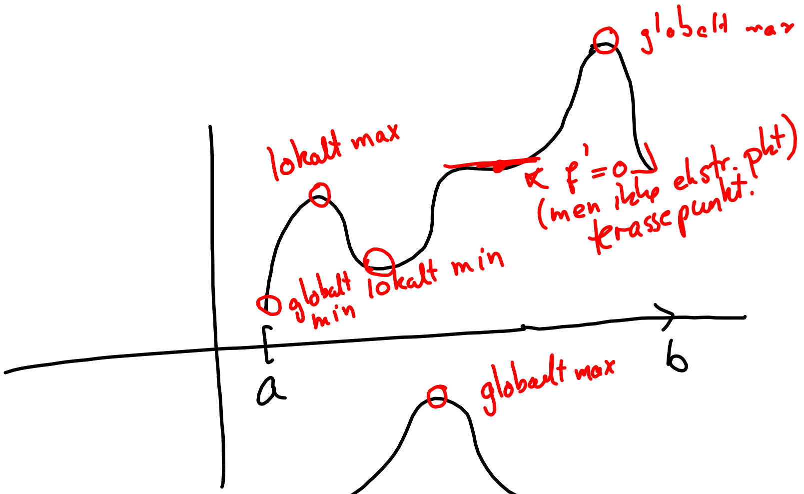


$\{f(X_{n_i})\} \to M \Rightarrow \{f(\bar{x}_{n_i})\} \to f(\bar{a})$

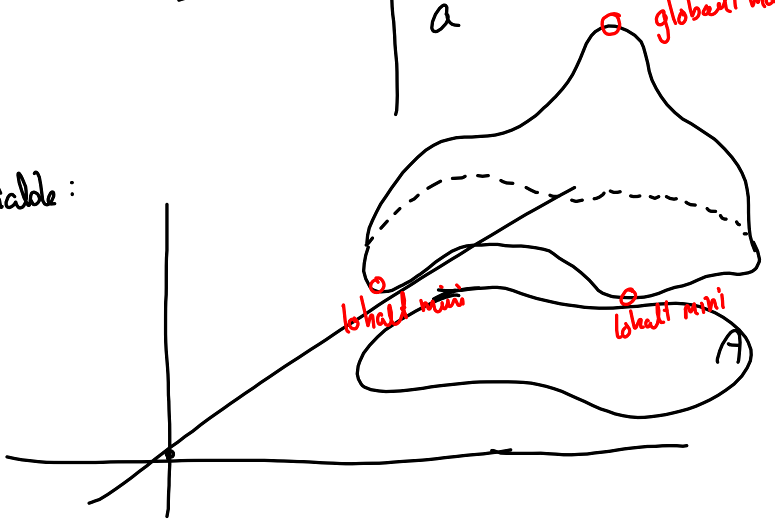
$\bar{x}_{n_i} \to \bar{a} \qquad \Rightarrow M = f(\bar{a}) \in A$

Ekstremalpunkter

Én variabel:



Flere variable:



Def.

$f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}^m$

$\bar{a} \in A$ lokalt max (min) for f dersom $\exists\, B(\bar{a}, r)$ slik at

$$f(x) \leq f(\bar{a}) \quad \forall x \in B(\bar{a}, r)$$
$$(\geq)$$

Felles betegnelse: Ekstremalpunkter.

Finne dem (i det indre): $\frac{\partial f}{\partial x_i}(\bar{a}) = 0 \quad \forall\, i = 1, \dots, m$

$\Leftrightarrow \quad \nabla f(\bar{a}) = 0$ STASJONÆRT PUNKT.

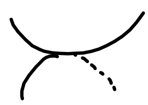
Eks $f(x,y) = 4y - 2x + xy - 1$

$\frac{\partial f}{\partial x} = -2 + y \qquad \frac{\partial f}{\partial y} = 4 + x \qquad$ Stasjonært pkt $(x,y) = (-4, 2)$

$$f(-4+\varepsilon_1, 2+\varepsilon_2) = 4(2+\varepsilon_2) - 2(-4+\varepsilon_1) + (2+\varepsilon_2)(-4+\varepsilon_1) - 1$$

$$= 7 + \varepsilon_1 \varepsilon_2$$

Sadelpunkt



$\varepsilon_1 \varepsilon_2 > 0 \qquad > 7$

$\varepsilon_1 > 0 \; \varepsilon_2 < 0 \qquad < 7$

Kommer til å få bruk for:

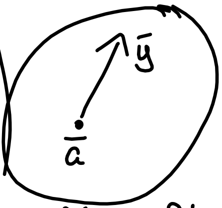
Hesse-matrisen:

$$\bar a \in A \subset \mathbb{R}^m, \qquad f : A \to \mathbb{R} \quad (C^2)$$

$$Hf(\bar a) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(\bar a) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_m}(\bar a) \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_m \partial x_1}(\bar a) & \cdots & \frac{\partial^2 f}{\partial x_m^2}(\bar a) \end{pmatrix}$$

Se på f i nærheten av et stasjonært punkt

$$\bar y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$



$$g(t) = f(\bar a + t\cdot\bar y)$$

$$g'(t) = \frac{\partial f}{\partial x_1}(\bar a + t\bar y)\cdot y_1 + \cdots + \frac{\partial f}{\partial x_m}(\bar a + t\bar y)\cdot y_m = \nabla f(\bar a + t\bar y)\cdot \bar y$$

$$\nabla f(\bar a) = \left(\frac{\partial f}{\partial x_1}(\bar a) \cdots \frac{\partial f}{\partial x_m}(\bar a)\right)$$

$$g''(t) = \sum_j \frac{\partial}{\partial x_j} \sum_i \frac{\partial f}{\partial x_i}(\bar a + t\bar y)\cdot y_i\, y_j$$

$$= \sum_{i,j} \frac{\partial^2 f}{\partial x_j \partial x_i}(\bar a + t\cdot\bar y)\cdot y_i y_j = \bar y^T\cdot Hf(\bar a + t\bar y)\cdot \bar y$$

$$\bar y^T = (y_1 \cdots y_m)$$

Taylor: $g(t) = g(0) + t\cdot g'(0) + \frac{1}{2}t^2\cdot g''(c) \qquad 0 \le c \le t$

Sett $t=1$: $g(1) = g(0) + g'(0) + \frac{1}{2} g''(c)$ $\qquad \boxed{g(t) = f(\bar a + t\bar y)}$

$$f(\bar a + \bar y) = f(\bar a) + \nabla f(\bar a)\cdot\bar y + \frac{1}{2}\bar y^T Hf(\bar a + c\bar y)\cdot \bar y$$

$$= f(\bar a) + \nabla f(\bar a)\cdot \bar y + \overbrace{\frac{1}{2}\bar y^T Hf(\bar a + c\bar y)\,\bar y - \frac{1}{2}\bar y^T\cdot Hf(\bar a)\cdot\bar y} + \frac{1}{2}\bar y^T\cdot Hf(\bar a)\cdot \bar y$$

$$\boxed{f(\bar a + \bar y) = f(\bar a) + \nabla f(\bar a)\cdot\bar y + \frac{1}{2}\bar y^T\cdot Hf(\bar a)\cdot\bar y + \varepsilon(\bar y)\cdot|\bar y|^2}$$

hvor $\varepsilon(\bar y) = \frac{1}{2}\bar y^T\big(\underbrace{Hf(\bar a + c\bar y) - Hf(\bar a)}_{K(\bar y)}\big)\cdot\bar y\ \frac{1}{|\bar y|^2} = \frac{1}{2}\underbrace{\frac{\bar y^T}{|\bar y|}\cdot K(\bar y)\cdot\frac{\bar y}{|\bar y|}}_{\to 0 \text{ når } \bar y \to 0.}$

Hesse-matrisen er symmetrisk!

Kan ortogonalt diagonaliseres:

$\exists P$ slik $P \cdot Hf(\bar{a}) \cdot P^T = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix}$

ortonormal matrise $P^{-1} = P^T$

$$\bar{y}^T \cdot Hf(\bar{a}) \cdot \bar{y} = \bar{y}^T P^T P \, Hf(\bar{a}) \, P^T P \, \bar{y}$$

$$= (\bar{y}^T \cdot P^T) P \, Hf(\bar{a}) P^T (P\bar{y})$$

$$= (P\bar{y})^T \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix} P\bar{y} =$$

$$= {y'_1}^2 \cdot \lambda_1 + {y'_2}^2 \cdot \lambda_2 + \cdots + {y'_m}^2 \cdot \lambda_m$$

$$\boxed{P\bar{y} = \begin{pmatrix} y'_1 \\ \vdots \\ y'_m \end{pmatrix}}$$

Alle egenverdiene er positive: $\bar{y}^T \cdot Hf(\bar{a}) \cdot \bar{y} \geq 0$

— " — negative: ≤ 0

Andrederivert test

$f: A \to \mathbb{R}$ (C^2), $A \subseteq \mathbb{R}^m$

$a \in A$, $\nabla f(\bar{a}) = 0$

Alle egenverdiene til $Hf(\bar{a})$ er positive: lokalt minimum

Alle egenverdiene til $Hf(\bar{a})$ er negative: lokalt maksimum

Både negative og positive: sadelpunkt.

Bevis.

$$f(a + \bar{y}) = f(\bar{a}) + \underbrace{\nabla f(\bar{a})}_{=0} \cdot \bar{y} + \frac{1}{2} \bar{y}^T \cdot Hf(\bar{a}) \cdot \bar{y} + \varepsilon(\bar{y}) \cdot |\bar{y}|^2$$

$|y|$ liten