$\boxed{5.1.2}$ Setning 5.1.6: $\vec{x}_n \to \vec{x} \iff x_n^{(i)} \to x^{(i)} \quad \forall\, i \in \{1,2,\dots,m\}$

a) $\vec{x}_n = \begin{pmatrix} \frac{2n^2+1}{n^2+3n} \\ \frac{3n}{1-2n} \end{pmatrix} \xrightarrow{?} \vec{x}$ $\quad x_n^{(1)}$

$$\lim_{n\to\infty} x_n^{(1)} = \lim_{n\to\infty} \frac{2n^2+1}{n^2+3n} \overset{L'H}{=} \lim_{n\to\infty} \frac{4n}{2n+3} \overset{L'H}{=} \lim_{n\to\infty} \frac{4}{2} = \underline{\underline{2}}$$

$$\lim_{n\to\infty} x_n^{(2)} = \lim_{n\to\infty} \frac{3n}{1-2n} = \underline{-\frac{3}{2}}$$

$$\vec{x}_n \to \begin{pmatrix} 2 \\ -\frac{3}{2} \end{pmatrix}$$

$m = \frac{1}{n} \quad m \to 0: \lim_{m\to 0} \frac{\sin(m)}{m} \overset{L'H}{=} \frac{\cos(m)}{1} = 1$

b) $\vec{x}_n = \begin{pmatrix} n \sin\left(\frac{1}{n}\right) \\ n\left(1-e^{2/n}\right) \end{pmatrix}$

$$\lim_{n\to\infty} x_n^{(1)} = \lim_{n\to\infty} n \sin\left(\tfrac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \overset{L'H}{=} \lim_{n\to\infty} \frac{\cos\left(\frac{1}{n}\right)\cancel{\left(-\frac{1}{n^2}\right)}}{\cancel{\left(-\frac{1}{n^2}\right)}} = \lim_{n\to\infty} \cos\left(\tfrac{1}{n}\right) = 1$$

$$\lim_{n\to\infty} x_n^{(2)} = \lim_{n\to\infty} \underset{\infty}{n}\underbrace{\left(1-e^{2/n}\right)}_{0} = \lim_{n\to\infty} \frac{1-e^{2/n}}{1/n} = \lim_{m\to 0} \frac{1-e^{2m}}{m} \overset{L'H}{=} \lim_{m\to 0} \frac{-2e^{2m}}{1} = -2$$

$$\vec{x}_n \to \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

c) $x_n = \begin{pmatrix} \sqrt{n^2+2n} - n \\ \cos\frac{1}{n} \\ \left(\cos\frac{1}{n}\right)^{n^2} \end{pmatrix}$

$$\lim_{n\to\infty} x_n^{(1)} = \lim_{n\to\infty} n\sqrt{1+\tfrac{2}{n}} - n = \lim_{n\to\infty} \underset{\infty}{n}\underbrace{\left(\sqrt{1+\tfrac{2}{n}}-1\right)}_{0} = \lim_{n\to\infty} \frac{\sqrt{1+\frac{2}{n}}-1}{1/n}$$

$$= \lim_{m\to 0} \frac{\sqrt{1+2m}-1}{m} \overset{L'H}{=} \lim_{m\to 0} \frac{\cancel{\frac{1}{2}}(1+2m)^{-\frac{1}{2}}\cdot \cancel{2}}{1} = \lim_{m\to 0} \frac{1}{\sqrt{1+2m}} = 1$$

$$\lim_{n\to\infty} x_n^{(3)} = \lim_{n\to\infty} \underbrace{\left(\cos\tfrac{1}{n}\right)}_{e^{\ln(\cos\frac{1}{n})}}^{n^2} = \lim_{n\to\infty} e^{\ln(\cos\frac{1}{n})\cdot n^2} = e^{\lim_{n\to\infty} \ln(\cos\frac{1}{n})\cdot n^2}$$

$a = e^{\ln a}$

$$\lim_{n\to\infty} \underbrace{\ln\left(\cos\tfrac{1}{n}\right)}_{0}\cdot \underset{\infty}{n^2} = \lim_{n\to\infty} \frac{\ln\left(\cos\frac{1}{n}\right)}{1/n^2} = \lim_{m\to 0} \frac{\ln(\cos m)}{m^2}$$

$$\overset{L'H}{=} \lim_{m\to 0} \frac{\frac{1}{\cos m}(-\sin m)}{2m} = \lim_{m\to 0} \frac{-\tan m}{2m} = -\frac{1}{2} \lim_{m\to 0} \frac{\tan m}{m}$$

$$\overset{L'H}{=} -\frac{1}{2} \lim_{m\to 0} \frac{\frac{1}{\cos^2 m}}{1} = -\frac{1}{2} \lim_{m\to 0} \frac{1}{\cos^2 m} = -\frac{1}{2}$$

$x_n^{(3)} \to e^{-\frac{1}{2}}$

$$\vec{x}_n \to \begin{pmatrix} 1 \\ 1 \\ e^{-\frac{1}{2}} \end{pmatrix}$$