Skifte av variable i dobbeltintegraler

$$\int_0^1 \left[\int_y^1 e^{x^2}\, dx \right] dy$$

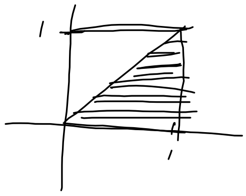
$0 \le y \le 1$

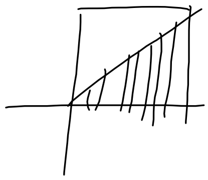
$y \le x \le 1$

$\Downarrow$

$0 \le x \le 1$

$0 \le y \le x$





$$= \int_0^1 \left[\int_0^x e^{x^2}\, dy \right] dx$$

$u = x^2$

$$= \int_0^1 \left[y e^{x^2} \right]_0^x dx = \frac{1}{2}\int_0^1 2x e^{x^2}\, dx = \left[\frac{1}{2} e^{x^2} \right]_0^1 = \frac{1}{2}(e-1)$$

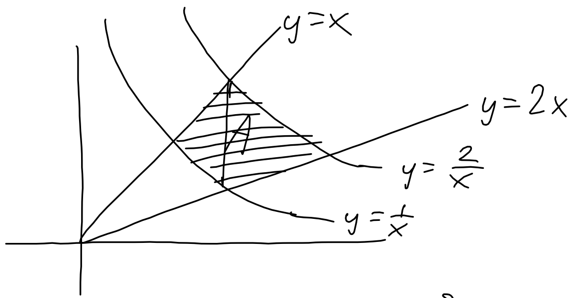
Teorem 6.7.1

$$\iint_A f(x,y)\, dx\, dy = \iint_D f(\vec{T}(u,v))\, |\det \vec{T}'(u,v)|\, du\, dv$$

$\uparrow$ Jacobimatr.

$\vec{T}(D) = A$

$(x,y) = \vec{T}(u,v)$



$$x \le y \le 2x \,, \quad \frac{1}{x} \le y \le \frac{2}{x}$$

$$1 \le \underbrace{\frac{y}{x}}_{v} \le 2 \qquad 1 \le \underbrace{xy}_{u} \le 2$$

areal $(A) = ?$

$$\vec{T}: (u,v) \to (x,y) \qquad |\det \vec{T}'| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$

$$u = xy \qquad (\vec{T}^{-1})' = \begin{pmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{pmatrix} \Rightarrow$$

$$v = \frac{y}{x}$$

$$\vec{T}^{-1}: (x,y) \to (u,v) \qquad |\det (\vec{T}^{-1})'| = \frac{y}{x} + \frac{yx}{x^2} = \frac{2y}{x} = 2v$$

fra bok $\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|^{-1}$

$$|\det (\vec{T}^{-1})'| = \frac{\partial(u,v)}{\partial(x,y)} \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2v}$$

$$\iint_A 1 \cdot dx\,dy = \iint_D 1 \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \iint_D \underbrace{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|}_{2v}^{-1} du\,dv = \iint_D \frac{1}{2v}\,du\,dv$$

$$= \int_1^2\int_1^2 \frac{1}{2v}\,du\,dv = \int_1^2 \frac{1}{2v}\,dv = \left[\frac{1}{2}\ln|v|\right]_1^2 = \underline{\underline{\frac{1}{2}\ln 2}}$$

Oppgs 2013

$A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$

$B = \{(x,y) \in \mathbb{R}^2 : \left(\frac{x-x_0}{a}\right)^2 + \left(\frac{y-y_0}{b}\right)^2 \leq 1\}$

$\vec{F}(x,y) = (x_0 + ax, y_0 + by)$

a) $\vec{F}(A) = B$. Anta $(x,y) \in A$: setter inn $\vec{F}(x,y)$:

$$\left(\frac{(x_0+ax) - x_0}{a}\right)^2 + \left(\frac{(y_0+by) - b_0}{b}\right)^2 = \left(\frac{ax}{a}\right)^2 + \left(\frac{by}{b}\right)^2 = x^2 + y^2 \leq 1$$

$\Rightarrow \vec{F}(x,y) \in B$

Anta $(z,w) \in B$ $\quad \boxed{z = x_0 + ax, \quad w = y_0 + by} \; \vec{F}$

$x = \frac{z - x_0}{a} \quad y = \frac{w - y_0}{b} \; \vec{F}^{-1}$

$x^2 + y^2 = \left(\frac{z-x_0}{a}\right)^2 + \left(\frac{w-y_0}{b}\right)^2 \leq 1$ siden antar at $(z,w) \in B$.

Vi gjør et variabelskifte: $\vec{T} = \vec{F}: A \to B$

$\vec{F}'(x,y) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \Rightarrow |\vec{F}'(x,y)| = ab$

teorem 6.7.1

$$\iint_B f(z,w)\,dz\,dw = \iint_A f(\vec{F}(x,y))\,|\vec{F}'(x,y)|\,dx\,dy$$

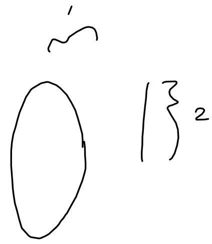
$$= \iint_A f(x_0+ax, y_0+by)\,ab\,dx\,dy \qquad \begin{array}{l} x = r\cos t \\ y = r\sin t \end{array}$$

$$= \int_0^{2\pi}\int_0^1 f(x_0 + ar\cos t, y_0 + br\sin t)\,ab\underline{r}\,dr\,dt$$

Oppg 6 2013

$$z + 4x^2 - 8x + 4 + y^2 - 4y = 0$$

a) $z = \underbrace{-4x^2 + 8x - 4 - y^2 + 4y}_{\to f(x,y)}$



$z = 0$: $\underline{4x^2 - 8x + 4} + y^2 - 4y = 0$

$$4(x-1)^2 + y^2 - 4y + 4 = 4$$

$$4(x-1)^2 + (y-2)^2 = 4$$

$$(x-1)^2 + \frac{(y-2)^2}{2^2} = 1$$

(ellipse med sentrum $(1,2)$, halvakser $1, 2$

b) Innenfor avgrenset område $(\underbrace{(x-1)^2 + \frac{(y-2)^2}{2^2} = 1}_{B})$, ligger flaten over eller under xy-planet?

$f(1,2) = -4 + 8 - 4 - 4 + 8 = 4 > 0 \Rightarrow$ flaten ligger øverst:

$$V = \iint_B (-4x^2 + 8x - 4 - y^2 + 4y)\, dx\, dy$$

$x \to x_0 + ax$
$y \to y_0 + bx$

$x_0 + ar\cos t$
$y_0 + br\sin t$
$1 + r\cos t$
$2 + 2r\sin t$

oppg. 5

$$= \int_0^1 \int_0^{2\pi} \left(-4(x-1)^2 - (y-2)^2 + 4\right) r \cdot \overset{a}{1} \cdot \overset{b}{2}\, dt\, dr$$

$$= \int_0^1 \int_0^{2\pi} \left(-4(1 + r\cos t - 1)^2 - (2 + 2r\sin t - 2)^2 + 4\right) 2r\, dt\, dr$$

$$= \int_0^1 \int_0^{2\pi} (-4r^2\cos^2 t - 4r^2\sin^2 t + 4) 2r\, dt\, dr = \int_0^1 \int_0^{2\pi} (4 - 4r^2) 2r\, dt\, dr$$

$$= \int_0^1 \int_0^{2\pi} (8r - 8r^3)\, dt\, dr = 2\pi \int_0^1 (8r - 8r^3)\, dr = 2\pi \left[4r^2 - 2r^4\right]_0^1 = \underline{\underline{4\pi}}$$

oppg 6 2014

$f(x,y,z)=0$: $\quad z = -x^2 + 2x - y^2 + 4y - 1 = -(x-1)^2 - (y-2)^2 + 4$

$z=0$: $(x-1)^2 + (y-2)^2 = 4 = 2^2$

sirkel med sentrum $(1,2)$ og radius 2.

er flaten over eller under, i avgrenset område?

$f(1,2) = -(1-1)^2 - (2-2)^2 + 4 = 4 > 0$

$\Rightarrow$ avgrenset område er <u>over</u> xy-planet.

Vi prøver translaterte polarkoord.

$x = 1 + r\cos\theta \qquad 0 \le r \le 2$

$y = 2 + r\sin\theta \qquad 0 \le \theta \le 2\pi$

flaten: $z = -(x-1)^2 - (y-2)^2 + 4 = -r^2\cos^2\theta - r^2\sin^2\theta + 4 = 4 - r^2$

$$\iiint_S z\,dx\,dy\,dz$$

$$= \int_0^{2\pi}\int_0^2\int_0^{\text{flaten}} zr\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2\int_0^{4-r^2} zr\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \left[\tfrac{1}{2}z^2 r\right]_0^{4-r^2} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \tfrac{1}{2}(4-r^2)^2 r\,dr\,d\theta \qquad u = 4-r^2,\; du = -2r\,dr \Rightarrow \tfrac{1}{2}r\,dr = -\tfrac{1}{4}du$$

$$= \int_0^{2\pi}\int_4^0 \left(-\tfrac{1}{4}\right)u^2\,du\,d\theta = \int_0^{2\pi}\left[-\tfrac{1}{12}u^3\right]_4^0 d\theta = \int_0^{2\pi}\tfrac{64}{12}\,d\theta$$

$$= 2\pi\cdot\frac{64}{12} = \underline{\frac{32\pi}{3}}$$