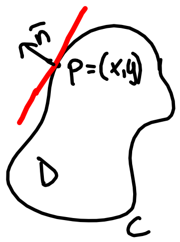
Eks (Green)



$\bar n_p = \bar n(x,y)$: utadpekende normal i p

i) $|\bar n| = 1$

ii) $\bar n$ står normalt på tangenten til C i p

$\bar n \perp \bar t_p$

iii) $\bar n$ peker ut av D i p

$F(x,y) = P(x,y)\,\bar i + Q(x,y)\,\bar j$

(kont. part. der)

$$\iint_D \underbrace{\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}}_{\mathrm{div}(\bar F)}\, dx\,dy = \underbrace{\int_C \bar F\cdot\bar n\, ds}_{\text{Flux av } \bar F \text{ ut av D / gjennom C}}$$

Vi har

$$\bar F\cdot d\bar r = \bar F\cdot \underset{\text{tangent, lengde 1}}{\bar t_p}\cdot ds = \bar F\cdot \bar n^{\perp}\, ds = \bar F^{\perp}\cdot \bar n\, ds$$

$\bar F = P\,\bar i + Q\,\bar j$

$\bar F^{\perp} = -Q\,\bar i + P\,\bar j$

$$\int_C \bar F\cdot d\bar r = \int_C \bar F^{\perp}\cdot\bar n\, ds \quad \text{eller} \quad \int_C \bar F^{\perp}\cdot d\bar r = \boxed{\int_C \bar F\cdot\bar n\, ds}$$

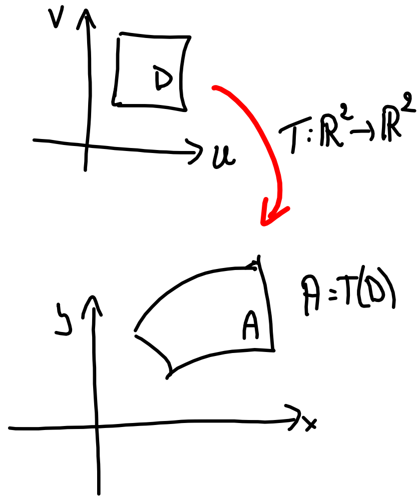
Greens

$$= \iint_D \frac{\partial P}{\partial x} - \left(-\frac{\partial Q}{\partial y}\right) dx\,dy$$

$$= \boxed{\iint_D \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\, dx\,dy}$$

(Divergensteoremet)

Variabel skifte



$$\iint_A f(x,y) = \iint_D f(T(u,v)) \left|\frac{\partial T}{\partial u} \times \frac{\partial T}{\partial v}\right| du\, dv$$

$$T(u,v) = T_1(u,v)\,\bar{i} + T_2(u,v)\,\bar{j} + 0\,\bar{k}$$

$$\left|\frac{\partial T}{\partial u} \times \frac{\partial T}{\partial v}\right| = \left|\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial T_1}{\partial u} & \frac{\partial T_2}{\partial u} & 0 \\ \frac{\partial T_1}{\partial v} & \frac{\partial T_2}{\partial v} & 0 \end{vmatrix}\right| = \left|0\,\bar{i} + 0\,\bar{j} + \begin{vmatrix} \frac{\partial T_1}{\partial u} & \frac{\partial T_2}{\partial u} \\ \frac{\partial T_1}{\partial v} & \frac{\partial T_2}{\partial v} \end{vmatrix} \bar{k}\right|$$

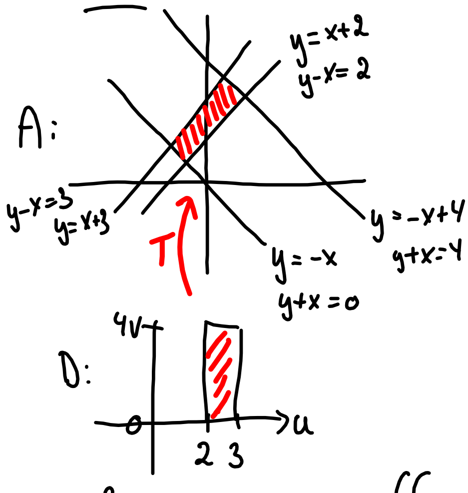
$$= \left|\begin{vmatrix} \frac{\partial T_1}{\partial u} & \frac{\partial T_2}{\partial u} \\ \frac{\partial T_1}{\partial v} & \frac{\partial T_2}{\partial v} \end{vmatrix}\right|$$

Kan skrives

$$\frac{\partial(T_1, T_2)}{\partial(u,v)}$$

Jacobi-determinanten

Eks



$f(x,y) = x \cdot y$

A: $2 \le \underbrace{y-x}_{u} \le 3 \qquad 0 \le \underbrace{y+x}_{v} \le 4$

$$\Rightarrow \quad \frac{u+v}{2} = y \qquad \frac{v-u}{2} = x$$

$T: D \to A$

$$(u,v) \longmapsto (x,y) = \left(\tfrac{1}{2}(v-u), \tfrac{1}{2}(v+u)\right)$$

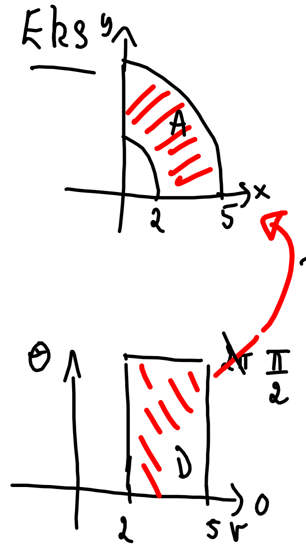
$$T_1(u,v) = -\tfrac{1}{2}u + \tfrac{1}{2}v \qquad T_2(u,v) = \tfrac{1}{2}u + \tfrac{1}{2}v$$

$$\text{Jacobi: } \left|\begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix}\right| = \left|-\tfrac{1}{2}\right| = \tfrac{1}{2}$$

$$\iint_A xy\, dx\, dy = \iint_D \tfrac{1}{2}(v-u) \cdot \tfrac{1}{2}(u+v)\, \tfrac{1}{2}\, du\, dv$$

$$= \int_2^3 \int_0^4 \tfrac{1}{8}(v^2 - u^2)\, dv\, du \quad = \underline{\underline{-\tfrac{1}{2}}}$$

Eks



$T: \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}$

$f(x,y) = x+y$

Jacobi: $\left| \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} \right| = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}$

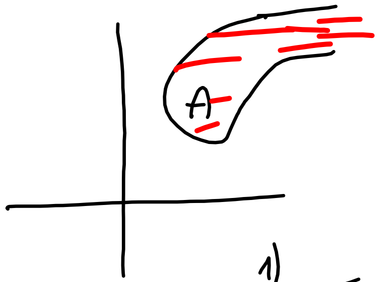
$= r\cos^2\theta + r\sin^2\theta = \underline{\underline{r}}$

$$\iint_A f\,dx\,dy = \iint_D f(r\cos\theta, r\sin\theta)\cdot r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_2^5 (r\cos\theta + r\sin\theta) r\,dr\,d\theta$$

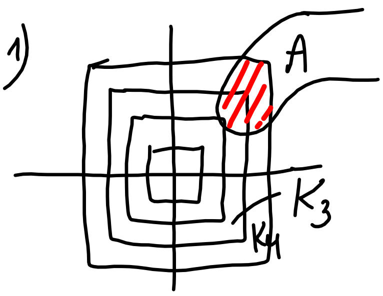
$$= \underline{\underline{78}}$$

Uegentlige integraler i planet



Studerer kun ikke-negative funksjoner

1)

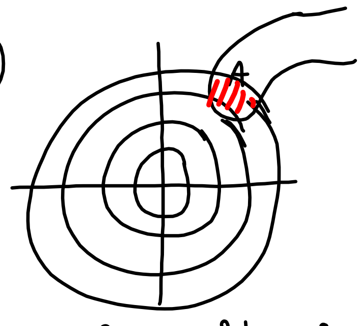


$K_n = \{(x,y) \in \mathbb{R}^2 \mid |x|, |y| \leq n\}$

$$\iint_A f\,dx\,dy = \lim_{n\to\infty} \iint_{A\cap K_n} f\,dx\,dy$$

dersom ∪ har konvergen.

2)

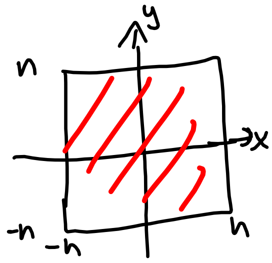


$B(0,n) = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq n^2\}$

$$\iint_A f\,dx\,dy = \lim_{n\to\infty} \iint_{A\cap B(0,n)} f\,dx\,dy$$

Eks $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx = 1$

Skal regne ut $\iint_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}}\,dx\,dy$



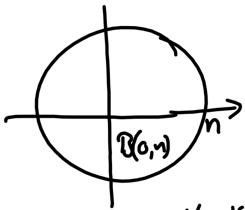
1) $\int_{-n}^{n}\int_{-n}^{n} e^{-\frac{x^2}{2}} e^{-\frac{y^2}{2}}\,dx\,dy$

$$= \int_{-n}^{n} e^{-\frac{y^2}{2}} \int_{-n}^{n} e^{-\frac{x^2}{2}}\,dx\,dy = \int_{-n}^{n} e^{-\frac{x^2}{2}}dx \cdot \int_{-n}^{n} e^{-\frac{y^2}{2}}dy$$

$$= \left(\int_{-n}^{n} e^{-\frac{x^2}{2}}dx\right)^2$$

2) $\iint_{B(0,n)} e^{-\frac{x^2+y^2}{2}}\,dx\,dy$

Merk: $x^2+y^2=r^2$



$0 \le \theta \le 2\pi$

$0 \le r \le n$

$x = r\cos\theta$
$y = r\sin\theta$
$J = r$ (Jacobi)

$$= \int_0^n \int_0^{2\pi} e^{-\frac{r^2}{2}}\cdot r\,d\theta\,dr$$

$$= 2\pi \int_0^n e^{-\frac{r^2}{2}}\cdot r\,dr$$

Mellomregning →

$$= 2\pi\left[-e^{-\frac{r^2}{2}}\right]_0^n$$

$$= 2\pi\left(-e^{-\frac{n^2}{2}} + e^0\right)$$

$$= 2\pi\left(1 - e^{-\frac{n^2}{2}}\right) \xrightarrow[n\to\infty]{} \underline{\underline{2\pi}}$$

Mellomregning:

$\int e^{-\frac{r^2}{2}}\,r\,dr =$ $\boxed{u = \frac{r^2}{2} \quad du = r\,dr}$

$= \int e^{-u}\,du = -e^{-u}$

$= -e^{-\frac{r^2}{2}}$

Dette gir $\left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}dx\right)^2 = 2\pi$

$$\Rightarrow \underline{\underline{\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}dx = \sqrt{2\pi}}}$$