Forelesning 27/04 - 17 kap. 6.8 - 6.10

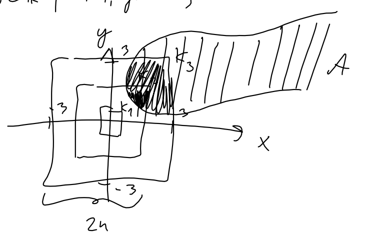
6.8 Uegentlige int. i planet.

- Integrering over ubegrensede områder i planet.

1-dim: $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to\infty} \int_{-a}^{a} f(x)\,dx$

2-dim: $\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy := \lim_{n\to\infty} \iint_{K_n} f(x,y)\,dx\,dy$

$K_n = \{(x,y)\in\mathbb{R}^2 \mid |x|, |y| \leq n\}$

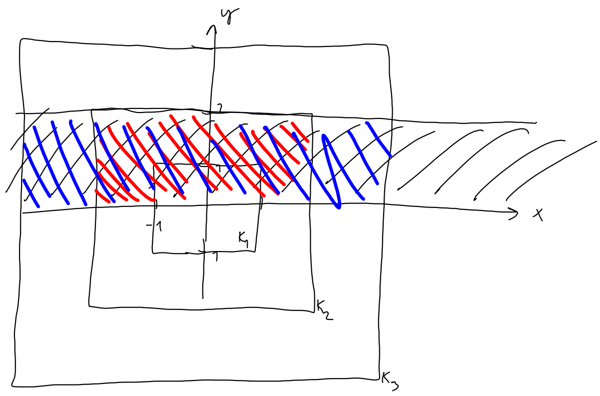


$$\iint_A f(x,y)\,dx\,dy := \lim_{n\to\infty} \iint_{A\cap K_n} f(x,y)\,dx\,dy$$

$A = \mathbb{R}\times[0,2]$

For $n \geq 2$:

$K_n \cap A = [-n,n]\times[0,2]$



Eks. $\iint_A \frac{y^2}{1+x^2}\,dx\,dy$, $A = \mathbb{R}\times[0,2]$

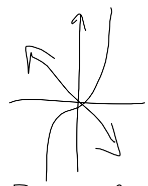
$$\iint_A \frac{y^2}{1+x^2}\,dx\,dy \overset{\text{def}}{=} \lim_{n\to\infty} \underbrace{\iint_{A\cap K_n} \frac{y^2}{1+x^2}\,dx\,dy}$$

$n \geq 2$:

$$\iint_{A\cap K_n} \frac{y^2}{1+x^2}\,dx\,dy = \int_{-n}^{n}\left(\int_0^2 \frac{y^2}{1+x^2}\,dy\right)dx$$

$$= \int_{-n}^{n} \frac{1}{3}\frac{1}{1+x^2}\left[y^3\right]_0^2 dx$$

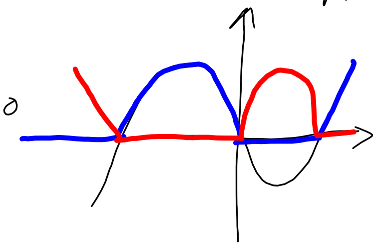
$$= \frac{8}{3}\int_{-n}^{n} \underbrace{\frac{1}{1+x^2}}\,dx = \frac{8}{3}\left[\arctan(n) - \arctan(-n)\right]$$



$$\lim_{n\to\infty} \frac{8}{3}\left(\underbrace{\arctan(n)}_{\to \frac{\pi}{2}} - \underbrace{\arctan(-n)}_{\to -\frac{\pi}{2}}\right) = \underline{\underline{\frac{8\pi}{3}}}$$

Teori

La $f(x,y)$ være en funksjon i $\mathbb{R}^2$. Vi ønsker å dele opp f i positiv og en negativ del.



$$f_+(x,y) = \begin{cases} f(x,y) & \text{dersom } f(x,y) > 0 \\ 0 & \text{ellers} \end{cases}$$

$$f_-(x,y) = \begin{cases} -f(x,y) & \text{dersom } f(x,y) < 0 \\ 0 & \text{ellers} \end{cases}$$

$$f(x,y) = f_+(x,y) - f_-(x,y)$$

$$|f(x,y)| = f_+(x,y) + f_-(x,y)$$

Def1: La $A \subset \mathbb{R}^2$: $A \cap K_n$ er Jordan-målbar (∂A er ganske regulær) for alle $n \in \mathbb{N}$. Hvis $f: A \to \mathbb{R}$ er en kont. funksjon definerer vi det uegentlige integralet av f over A som grensen:

$$\iint_A f(x,y)\,dx\,dy = \lim_{n\to\infty} \overbrace{\iint_{A\cap K_n} f(x,y)\,dx\,dy}^{\{a_n\}_{n\in\mathbb{N}}}$$

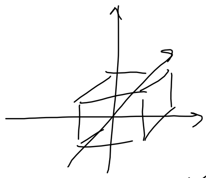
Def2: Gitt $f: A \to \mathbb{R}$ begrenset og kontinuerlig. Dersom

$\lim_{n\to\infty} \iint_{A\cap K_n} |f(x,y)|\,dx\,dy$ eksisterer (konvergerer), så eksisterer

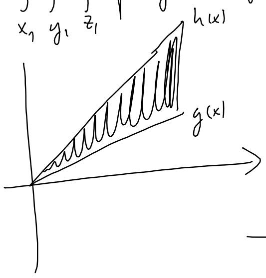
også $\lim_{n\to\infty} \iint_{A\cap K_n} f(x,y)\,dx\,dy$

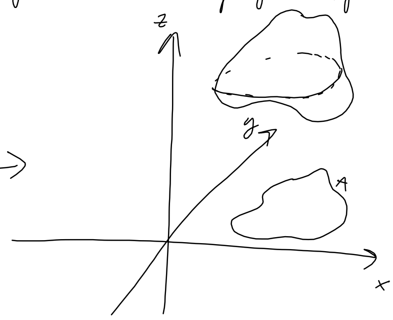
6.9 Trippelintegral

$$R = [x_1, x_2] \times [y_1, y_2] \times [z_1, z_2]$$

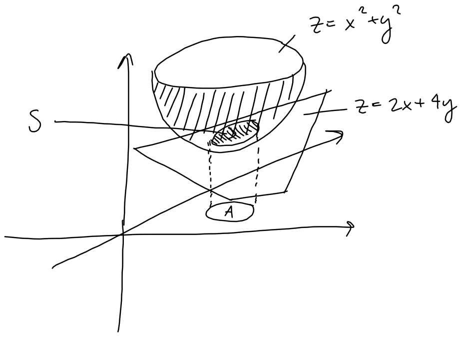


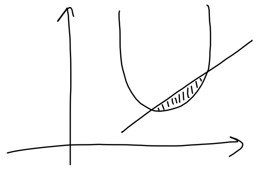
$$\iiint_R f(x,y,z)\, dx\,dy\,dz = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\, dx\,dy\,dz$$





Eksempel





$$\iiint_S xy\, dx\,dy\,dz = \iint_A \left(\int_{x^2+y^2}^{2x+4y} xy\, dz\right) dx\,dy$$

$$= \iint_A \Big[xyz \Big]_{z=x^2+y^2}^{z=2x+4y} dx\,dy$$

$$= \iint_A xy\underbrace{(2x+4y-x^2-y^2)}\, dx\,dy$$

Finne A:

$$x^2+y^2 = 2x+4y$$

$$\underbrace{x^2-2x+1}_{(x-1)^2} + \underbrace{y^2-4y+4}_{(y-2)^2} = 1+4$$

$$(x-1)^2+(y-2)^2 = \sqrt{5}^2$$

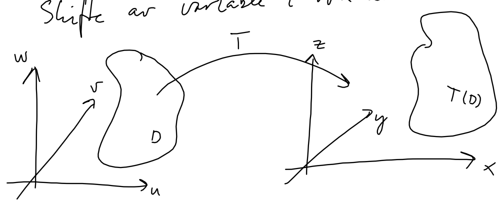
$$\left.\begin{aligned} x-1 &= r\cos\theta \\ y-2 &= r\sin\theta \end{aligned}\right\} \quad (x-1)^2+(y-2)^2 = r^2$$

$$2x+4y-x^2-y^2 = 5 - \underbrace{[(x-1)^2+(y-2)^2]}_{r^2} = 5-r^2$$

$$= \int_0^{\sqrt{5}}\int_0^{2\pi} (r\cos\theta+1)(r\sin\theta+2)(5-r^2)\, r\, d\theta\, dr$$

$$= \dots = \underline{\underline{25\pi}}$$

6.10 Skifte av variable i rommet



$$T(u,v,w) = (T_1(u,v,w), T_2(u,v,w), T_3(u,v,w))$$

Jacobi:

$$\begin{vmatrix} \frac{\partial T_1}{\partial u} & \frac{\partial T_1}{\partial v} & \frac{\partial T_1}{\partial w} \\ \frac{\partial T_2}{\partial u} & \frac{\partial T_2}{\partial v} & \frac{\partial T_2}{\partial w} \\ \frac{\partial T_3}{\partial u} & \frac{\partial T_3}{\partial v} & \frac{\partial T_3}{\partial w} \end{vmatrix} = \frac{\partial(T_1,T_2,T_3)}{\partial(u,v,w)}$$

$$\iiint_{T(D)} f(x,y,z)\,dx\,dy\,dz = \iiint_D f(T(u,v,w)) \left| \frac{\partial(T_1,T_2,T_3)}{\partial(u,v,w)} \right| du\,dv\,dw$$

Tre standard metoder:

1) Lineær avbildning:

$$T\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{array}{ll} \begin{pmatrix} a_{11}u + a_{12}v + a_{13}w \\ a_{21}u + a_{22}v + a_{23}w \\ a_{31}u + a_{32}v + a_{33}w \end{pmatrix} & \begin{array}{l} T_1 \\ T_2 \\ T_3 \end{array} \end{array}$$

$$= \underbrace{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}}_{A} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

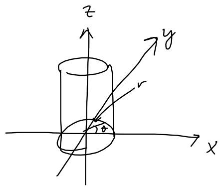
$$\left| \frac{\partial T}{\partial(u,v,w)} \right| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

2) Sylinderkoordinater:

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$



$0 \le \theta \le 2\pi$

$0 \le r \le R$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

3) Kulekoordinater:

$x = \rho\cos\theta\sin\varphi$

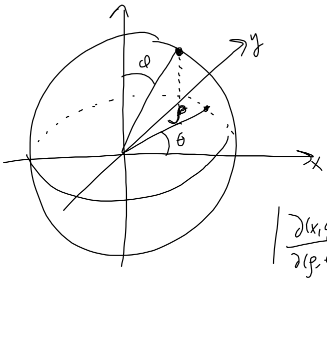
$y = \rho\sin\theta\sin\varphi$

$z = \rho\cos\varphi$

$0 \le \rho \le R$

$0 \le \theta \le 2\pi$

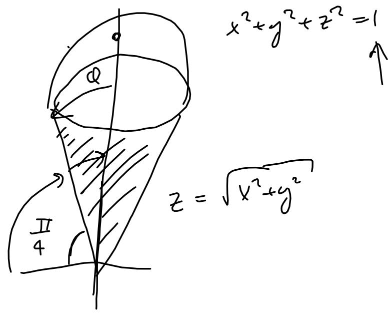
$0 \le \varphi \le \pi$



$$\left| \frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} \right| = \left| \begin{vmatrix} \cos\theta\sin\varphi & -\rho\sin\theta\sin\varphi & \rho\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi \\ \cos\varphi & 0 & -\rho\sin\varphi \end{vmatrix} \right|$$

$$= \rho^2\sin\varphi$$

Eks:



$0 \le \rho \le 1$

$0 \le \theta \le 2\pi$

$0 \le \varphi \le \pi/4$

$$V = \iiint_S 1 \, dx\,dy\,dz = \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^{1} \rho^2 \sin\phi \, d\rho\, d\theta\, d\varphi$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{4}}\int_0^{2\pi} \sin\phi \left[\rho^3\right]_0^1 d\theta\, d\varphi$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{4}}\int_0^{2\pi} \sin\phi \, d\theta\, d\varphi = \frac{1}{3}\int_0^{\frac{\pi}{4}} 2\pi \sin\varphi \, d\varphi$$

$$= \frac{2\pi}{3}\left[-\cos\phi\right]_0^{\frac{\pi}{4}} = \frac{2\pi}{3}\left(-\frac{\sqrt{2}}{2}+1\right) = \underline{\underline{\frac{\pi}{3}(2-\sqrt{2})}}$$