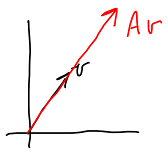
Egenvektorer

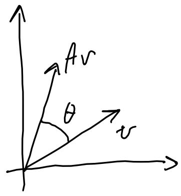
A $n \times n$ matrise

$v \neq 0$ egenvektor hvis $Av = \lambda v$ λ tall.



Har alle matriser egenvektorer?

A rotasjonsmatrise i planet. $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.



Prøver å finne egenverdiene. Løser $\det(A - \lambda I) = 0$.

$A - \lambda I = \begin{pmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{pmatrix}$ $\det(A - \lambda I) = (\cos\theta - \lambda)^2 + \sin^2\theta = 0$

$\lambda = \cos\theta \pm \sqrt{-\sin^2\theta} = \cos\theta \pm i \sin\theta$ $\leftarrow$ Komplekse egenverdier!

Generellt: $\det(A - \lambda I)$ er et $n$te grads polynom.

Da vet vi (fra fundamentalteoremet) at det fins n - komplekse røtter

Eksempel

$A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$. Finn egenvektorer/verdier.

$0 = \begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 4 = 0 \qquad (\lambda - 1)^2 = -2^2 \qquad \lambda = 1 \pm \sqrt{-2^2} = 1 \pm 2i$

Egenvektorene:

Løser $Av = \lambda v \qquad v = \begin{pmatrix} x \\ y \end{pmatrix} \qquad (A - \lambda I)v = 0 \qquad \begin{pmatrix} 1-(1+2i) & -2 \\ 2 & 1-(1+2i) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Utvidet matrise:

$$\begin{pmatrix} -2i & -2 & 0 \\ 2 & -2i & 0 \end{pmatrix} \overset{\substack{I/(-2i) \\ II \to II - iI}}{\sim} \begin{pmatrix} 1 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

y fri $\quad x = iy \qquad v = \begin{pmatrix} iy \\ y \end{pmatrix}$

$y = -i \qquad v = \begin{pmatrix} 1 \\ -i \end{pmatrix}$.

$\lambda_1 = 1 + 2i$

$\lambda_2 = 1 - 2i = \overline{\lambda_1}$ ← Komplekskonjugert.

(λ_1, v)

$(\overline{\lambda}_1, w)$.

$\overline{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$.

Det er alltid slik at $w = \overline{v}$.

A reell $n \times n$ matrise $\Rightarrow \det(A - \lambda I)$ $n$te grad, reelle koeffisienter.

Hvis λ er en rot/egenverdi så er også $\overline{\lambda}$ en rot/egenverdi

(v, λ) egenvektor/egenverdi $\lambda v = Av$

Sjekker om $\overline{v}$ er egenvektor.

$A\overline{v} = \overline{A}\,\overline{v} = \overline{Av} = \overline{\lambda v} = \overline{\lambda}\,\overline{v}$

Symmetriske matriser $n \times n$

$$S = \begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ s_{21} & s_{22} & & \vdots \\ s_{31} & & & \vdots \\ \vdots & & & \\ s_{n1} & \cdots & & s_{nn} \end{pmatrix}$$ er symmetrisk hvis $s_{ij} = s_{ji}$ $i,j = 1,\dots,n$.

$S^T = S$.

Eks

$$\boxed{\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}} \qquad \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 4 \\ -1 & 4 & 3 \end{pmatrix}$$

Egenverdier/vektorer.

$$\det\begin{pmatrix} 1-\lambda & -2 \\ -2 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda) - 4 = \lambda^2 - 4\lambda + 3 - 4 = \lambda^2 - 4\lambda - 1 = 0$$

$$\lambda = \frac{1}{2}\Big(4 \pm \sqrt{\underbrace{16+4}_{4\cdot 5}}\Big) = \frac{1}{2}(4 \pm 2\sqrt{5}) = \begin{cases} 2+\sqrt{5} \\ 2-\sqrt{5} \end{cases}$$

Egenvektor $2+\sqrt{5}$

$$\begin{pmatrix} -1-\sqrt{5} & -2 & 0 \\ -2 & 1-\sqrt{5} & 0 \end{pmatrix} \overset{\mathrm{I}/(1+\sqrt{5})}{\sim} \begin{pmatrix} -1 & -\frac{2}{1+\sqrt{5}} & 0 \\ -2 & 1-\sqrt{5} & 0 \end{pmatrix} \overset{\mathrm{II}-2\mathrm{I}}{\sim} \begin{pmatrix} +1 & \frac{2}{1+\sqrt{5}} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

y fri $\quad x = -\frac{2}{1+\sqrt{5}} y \qquad v_1 = \begin{pmatrix} 2 \\ -1-\sqrt{5} \end{pmatrix}$.

Egenvektor $2-\sqrt{5}$

$$\begin{pmatrix} -1+\sqrt{5} & -2 & 0 \\ -2 & 1+\sqrt{5} & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{2}{-1+\sqrt{5}} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

y fri $\quad x = \frac{2}{-1+\sqrt{5}} y \qquad v_2 = \begin{pmatrix} 2 \\ -1+\sqrt{5} \end{pmatrix}$.

$$v_1 \cdot v_2 = \begin{pmatrix} 2 \\ -1+\sqrt{5} \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1-\sqrt{5} \end{pmatrix} = 4 \mp (\sqrt{5}-1)(+\sqrt{5}+1)$$

$$= 4 - (\sqrt{5}^2 - 1^2) = 4 - 4 = 0$$

$v_1 \perp v_2$!

Spektralteoremet for symmetriske matriser.

S $n \times n$ symmetrisk matrise.
Da har S n reelle egenvektorer, og det fins n ortonormale egenvektorer.
$\{v_1, \dots v_n\}$ basis for $\mathbb{R}^n$.

Observasjon:
$\{v_1, \dots v_k\}$ ortogonale $\Rightarrow \{v_1, \dots v_k\}$ er l.u.

Hvis
$v_j \cdot \big| \quad 0 = c_1 v_1 + c_2 v_2 + \dots c_k v_k \quad c_k$ tall.
prikker med v_j på begge sider.

$$\underline{0} = c_1 v_j \cdot v_1 + \dots c_j v_j \cdot v_j + \dots + c_k v_j \cdot v_k$$
$$= 0 + \dots c_j |v_j|^2 + 0 + \dots + 0 = \underline{c_j |v_j|^2} \Rightarrow c_j = 0. \Rightarrow \underline{l.u.}$$

Diagonalisere matriser.

A $n \times n$ matrise, anta A har n lin. uavh. egenvektorer $\{v_1, \dots, v_n\}$.

$M = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$, M er invertibel.

Da har vi at

$$* \quad \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} = M^{-1} A M . \qquad \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

Bevis:

$*$ venstremultiplisert med M gir $M\Lambda = MM^{-1}AM = AM$

$* \Leftrightarrow M\Lambda = AM$

$$M\Lambda = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 & & \\ 0 & \lambda_2 & 0 & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_n \end{pmatrix} = \begin{pmatrix} | & | & & | \\ \lambda v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \\ | & | & & | \end{pmatrix}$$

$$\shortparallel$$

$$AM = A \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ Av_1 & Av_2 & \cdots & Av_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \\ | & | & & | \end{pmatrix}$$

$$\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix} = M^{-1}AM$$

Tar det() på begge sider: $\det\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix} = \det(M^{-1}AM).$

Triangular.

$$\det(M^{-1}AM) = \det(M^{-1})\det(A)\det(M) = \frac{1}{\cancel{\det(M)}}\det(A)\cancel{\det(M)}$$

$$\det\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix} = \lambda_1 \cdot \lambda_2 \cdots \lambda_n = \det(A).$$

A er invertibel $\Leftrightarrow \det(A) \neq 0$

$\Updownarrow$

Alle egenverdiene er $\neq 0$.

$$\boxed{\begin{matrix} v \mapsto Av \\ A^{-1}v \leftarrow w \end{matrix}}$$

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} = M^{-1} A M.$$

Stressomt å finne M^{-1}, men hvis $\{v_1, \dots v_n\}$ ortonormale...

$M^{-1} = M^T$.

$$M = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \qquad M^T = \begin{pmatrix} — v_1 — \\ — v_2 — \\ \vdots \\ — v_n — \end{pmatrix}.$$

ortonormale

Sjekke om $M^T = M^{-1}$ så multipliserer vi $M^T M$.

$$M^T M = \begin{pmatrix} — v_1 — \\ — v_2 — \\ \vdots \\ — v_n — \end{pmatrix} \begin{pmatrix} | & | & & | \\ v_1 & v_2 & & v_n \\ | & | & & | \end{pmatrix} = C \qquad c_{ij} = v_i \cdot v_j = \begin{cases} 1 & i = j \\ 0 & \text{ellers} \end{cases}$$

$$\Rightarrow C = I.$$

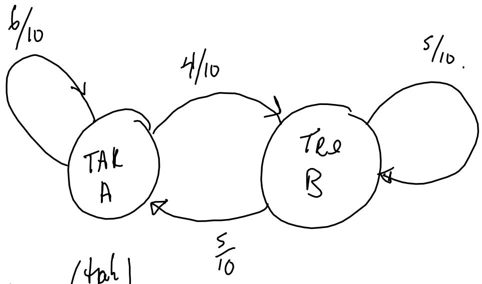
Fra Spektralteoremet: En basis av o.n. e.v. hvis A symmetrisk.

A sym. $\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = M^T A M$.

Eksempel:

Tak

Tre



$X_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ ← antall i $\begin{pmatrix} \text{tak} \\ \text{tre} \end{pmatrix}$ etter n dager.

x_n = # på tak

y_n = # i tre.

Kjenner $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 100 \end{pmatrix} = X_0$

$$\left.\begin{aligned} x_{n+1} &= \tfrac{6}{10}x_n + \tfrac{5}{10}y_n \\ y_{n+1} &= \tfrac{4}{10}x_n + \tfrac{5}{10}y_n \end{aligned}\right\} \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 6/10 & 5/10 \\ 4/10 & 5/10 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}. \qquad M = \frac{1}{10}\begin{pmatrix} 6 & 5 \\ 4 & 5 \end{pmatrix}.$$

$$X_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix} = M^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Finne egenverdier/vektorer.

Hvis v_1 og v_2 er lineært uavh. e.v.

$M v_1 = \lambda_1 v_1 \quad M v_2 = \lambda_2 v_2$

$$X_0 = a v_1 + b v_2.$$

$$\begin{aligned} X_n &= M^n X_0 \\ &= M^n (a v_1 + b v_2) \\ &= a M^n v_1 + b M^n v_2 \end{aligned}$$

$$X_n = a \lambda_1^n v_1 + b \lambda_2^n v_2$$

$\lambda_1 = 1 \quad \lambda_2 < 1.$

$$\boxed{X_n = a v_1} + b \lambda_2^n v_2$$

$\downarrow \qquad \downarrow$

$\qquad\quad 0$