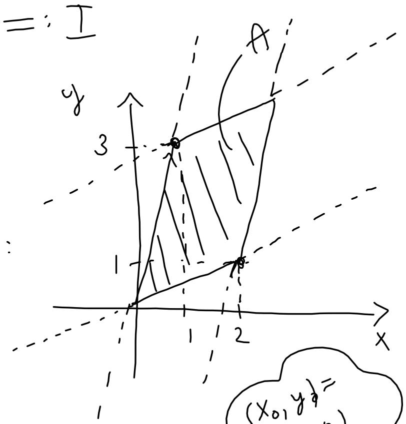
Plenum 18/3

6.7: 5, 8, 9 (ode)

6.8: 3, 6

6.7: Skifte av variabel i dobbeltintegral

5) d) $\iint_A (3x - 2y)\,dx\,dy =: I$



A: utspent av (2,1) & (1,3)

Vil beskrive A vha. ligninger:

(2,1): Stigningstall?

$$\frac{\Delta y}{\Delta x} = \frac{1}{2}$$

Ettpunktsformelen: $y = \frac{1}{2}(x - x_0) + y_0 = \frac{1}{2}x$ $\quad (x_0, y_0) = (0,0)$

Øvre linja parallell med (2,1): Startpkt (1,3) og stigningstall $\frac{1}{2}$.

Ettpkt. formel: $y = \frac{1}{2}(x-1) + 3 = \frac{1}{2}x + \frac{5}{2}$

(1,3): Stigningstall $\frac{\Delta y}{\Delta x} = \frac{3}{1} = 3$

Ettpkt. formel: $y = 3x$ $\quad x_0 = y_0 = 0$

Siste linje (parallell med $(1,3)$): Startpkt. i $(2,1)$

og har stign.tall 3 $\Rightarrow$ $y = 3(x-2)+1 = 3x-5$

ettpkt.-formel

Dvs. A er beskrevet ved:

$$\frac{1}{2}x \le y \le \frac{1}{2}x + \frac{5}{2}$$

$$3x-5 \le y \le 3x$$

Kan omskrives:

$$\left.\begin{aligned} 0 &\le \overbrace{y-\tfrac{1}{2}x}^{:=v} \le \frac{5}{2} \\ -5 &\le \underbrace{y-3x}_{:=u} \le 0 \end{aligned}\right\}$$

Bra! Har et rektangel.

Jacobideterminant:

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -3 & 1 \\ -\frac{1}{2} & 1 \end{vmatrix}$$

$$= -3 + \frac{1}{2} = -\frac{5}{2}$$

$$I = \iint_A (3x-2y)\,dx\,dy = \int_{-5}^{0}\int_{0}^{\frac{5}{2}} \left(\frac{6}{5}(v-u) - \frac{2}{5}(6v-u)\right)\left|\frac{1}{-\frac{5}{2}}\right| dv\,du$$

Variabelskifte:

$u = y-3x \Rightarrow 3x = y-u \Rightarrow \frac{1}{2}x = \frac{y-u}{6}$

$v = y-\frac{1}{2}x \Rightarrow v = y - \frac{y-u}{6} \Rightarrow \dots \Rightarrow y = \frac{6}{5}\left(v-\frac{1}{6}u\right)$

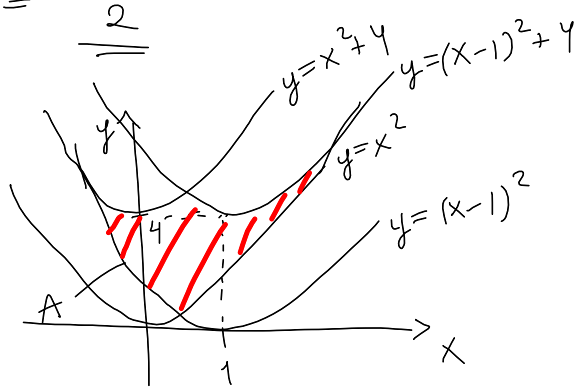
$3x = y-u = \dots = \frac{6}{5}(v-u)$

$$= \frac{4}{25} \int_{-5}^{0} \int_{0}^{\frac{5}{2}} (-3v - 2u)\, dv\, du$$

$$= \ldots = \frac{4}{5} \int_{-3}^{0} \left(-\frac{15}{8} - u\right) du$$

$$= \ldots = \underline{\underline{\frac{5}{2}}}$$

e) $\iint_A x \, dx\,dy$



Ser: $x^2 \leq y \leq x^2 + 4$ & $(x-1)^2 \leq y \leq (x-1)^2 + 4$

$$0 \leq \underbrace{y - x^2}_{:= u} \leq 4 \quad \text{og} \quad 0 \leq \underbrace{y - (x-1)^2}_{:= v} \leq 4$$

$$\left| \frac{\partial(u,v)}{\partial(x,y)} \right| = \begin{vmatrix} -2x & 1 \\ -2(x-1) & 1 \end{vmatrix} = -\cancel{2x} + 2(\cancel{x} - 1) = \underline{-2}$$

$$\iint_A x \, dx\,dy = \int_0^4 \int_0^4 \frac{v-u+1}{2} \left| \frac{1}{-2} \right| du\,dv = \frac{1}{4} \int_0^4 \int_0^4 (v - u + 1)\, du\,dv$$

Var. skifte:

$u - v = -x^2 + (x-1)^2$

$= -\cancel{x^2} + \cancel{x^2} - 2x + 1$

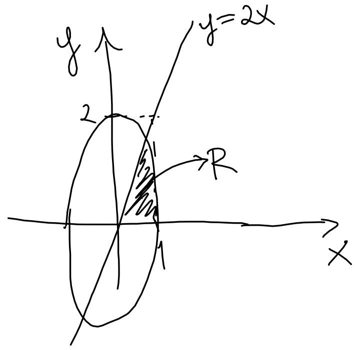
$= 1 - 2x \Rightarrow x = \frac{v-u+1}{2}$

$$= \ldots = \int_0^4 (v-1)\, dv$$

$$= \ldots = 8 - 4 = \underline{\underline{4}}$$

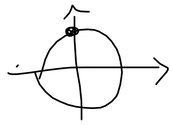
8.) $x = u\cos v \;,\; y = 2u\sin v$

a)



Ellipse: $x^2 + \frac{y^2}{4} = 1$

$$\frac{x^2}{1^2} + \frac{y^2}{2^2} = 1$$



Merk: • Holder å se på $u > 0$ og $v \in [0, 2\pi]$.

• 1. kvadrant i xy-planet: $v \in [0, \frac{\pi}{2}]$

• y=2x: $2u\sin v = 2u\cos v$

NB: Hvis $\cos v = 0$, så er ikke $\sin v = \cos v$ ($v = \frac{\pi}{2}$ i dette tilfellet)

$$\Downarrow$$

$u = 0$ eller $\underbrace{\frac{\sin v}{\cos v}}_{\tan v} = 1 \Rightarrow v = \frac{\pi}{4} + \underbrace{k\pi}$, $k \in \mathbb{Z}$

er ikke i 1. kvadrant for $k \neq 0$

• Ellipsen $x^2 + \frac{y^2}{4} = 1$:

$$1 = u^2\cos^2 v + \frac{\cancel{4}u^2\sin^2 v}{\cancel{4}} = u^2\overbrace{(\cos^2 v + \sin^2 v)}^{1} = u^2$$

$$\Downarrow$$

$u = \pm 1 \xRightarrow{u>0} u = 1$ er det som er interessant for oss.

Er interessert i det indre av ellipsen $\Rightarrow 0 \le u \le 1$.

<u>Så:</u> R er beskrevet ved

$$0 \leq u \leq 1 \quad \text{og} \quad 0 \leq v \leq \frac{\pi}{4}$$

<u>Jacobideterminant:</u> $\vec{T}(u,v) = (u\cos v,\ 2u\sin v)$

$$\vec{T}'(u,v) = \begin{bmatrix} \cos v & -u\sin v \\ 2\sin v & 2u\cos v \end{bmatrix}$$

$$|\det(\vec{T}'(u,v))| = |2u\cos^2 v + 2u\sin^2 v| \overset{u \geq 0}{=} \underline{2u}$$

$$\Downarrow$$

$$\text{Areal}_R = \iint_R 1\,dx\,dy \overset{\text{var. skifte}}{=} \iint_D 1\,|\det(\vec{T}'(u,v))|\,du\,dv$$

$$= \int_0^{\frac{\pi}{4}}\int_0^1 2u\,du\,dv = \int_0^{\frac{\pi}{4}} 1\,dv = \underline{\underline{\frac{\pi}{4}}}$$

b) $$z = x^2 + \frac{y^2}{2} \overset{\text{variabelskifte}}{=} u^2\cos^2 v + 2u^2\sin^2 v$$

$$= u^2(1 + \sin^2 v)$$

<u>Parametrisering:</u>

$$\vec{r}(u,v) = (u\cos v,\ 2u\sin v,\ u^2(1+\sin^2 v))$$

$$\frac{\partial \vec{r}}{\partial u} = (\cos v,\ 2\sin v,\ 2u(1+\sin^2 v))$$

$$\frac{\partial \vec{r}}{\partial v} = (-u\sin v,\ 2u\cos v,\ 2u^2\sin v\cos v)$$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = (-4u^2 \cos v, -4u^2 \sin v, 2u)$$

$$\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| = \sqrt{16u^4 + 4u^2} \underset{u \geq 0}{=} 2u\sqrt{4u^2+1}$$

$$\text{Areal av flaten} = \iint_R \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du\,dv$$

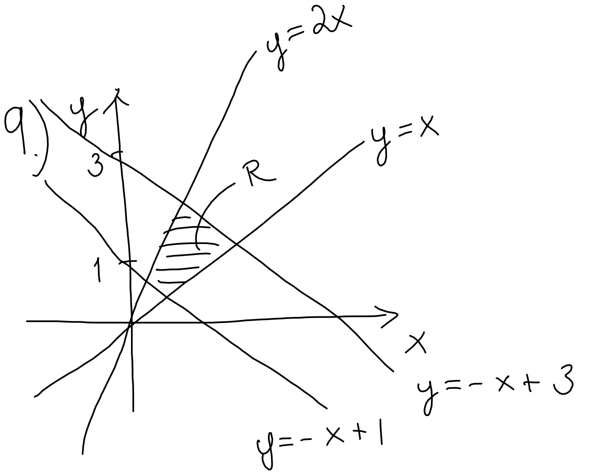
$$\text{a)} = \int_0^{\frac{\pi}{4}} \int_0^1 2u \underbrace{\sqrt{4u^2+1}}_{(4u^2+1)^{\frac{1}{2}}} \, du\, dv$$

$$= \int_0^{\frac{\pi}{4}} \left[\frac{1}{6}(4u^2+1)^{\frac{3}{2}} \right]_{u=0}^{1} dv$$

$$= \frac{1}{6} \int_0^{\frac{\pi}{4}} (5^{\frac{3}{2}} - 1)\, dv = \frac{1}{6} \frac{\pi}{4} (5\sqrt{5} - 1)$$

$$= \underline{\underline{\frac{\pi(5\sqrt{5}-1)}{24}}}$$

9.)



R kan beskrives ved:

$$x \leq y \leq 2x$$

$$-x+1 \leq y \leq -x+3$$

Omskriver:

$$\left. \begin{aligned} 1 \leq \underbrace{\frac{y}{x}}_{:=u} \leq 2 \\ 1 \leq \underbrace{y+x}_{:=v} \leq 3 \end{aligned} \right\} \text{"Rektangel"}$$

$$\left| \frac{\partial(u,v)}{\partial(x,y)} \right| = \begin{vmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ 1 & 1 \end{vmatrix} = \underline{-\frac{y}{x^2} - \frac{1}{x}}$$

$$\iint_R \frac{x+y}{x^2}\,dx\,dy = \int_1^3\int_1^2 \frac{v}{x^2}\left|\frac{1}{-\frac{y}{x^2} - \frac{1}{x}}\right| du\,dv$$

$$= \int_1^3\int_1^2 v\left|\frac{1}{-y-x}\right| du\,dv = \int_1^3\int_1^2 \frac{v}{|-v|}\,du\,dv$$

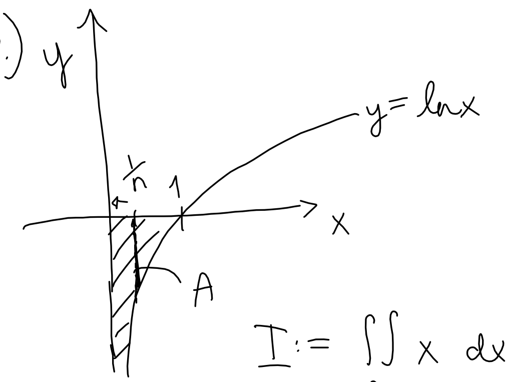
$|-5| = 5 = |5|$

$$= \int_1^3\int_1^2 \frac{v}{|v|}\,du\,dv = \int_1^3\int_1^2 \frac{v}{v}\,du\,dv = \int_1^3\int_1^2 1\,du\,dv$$

$$= (2-1)(3-1) = \underline{\underline{2}}$$

6.8: Uegentlige int. i planet

3)



Begrenser området:

Lar $A_n = \{ (x,y) \in \mathbb{R}^2 : \frac{1}{n} \le x \le 1,\ \ln x \le y \le 0 \}$

$$I := \iint_A x\,dx\,dy = \lim_{n\to\infty} \iint_{A_n} x\,dx\,dy$$

$$= \lim_{n\to\infty} \int_{\frac{1}{n}}^{1}\int_{\ln x}^{0} x\,dy\,dx = \lim_{n\to\infty} \int_{\frac{1}{n}}^{1} -x\ln x\,dx$$

$$= \lim_{n\to\infty}\left(\left[-\frac{1}{2}x^2\ln x\right]_{x=\frac{1}{n}}^{1} + \int_{\frac{1}{n}}^{1}\frac{x}{2}\,dx \right)$$

Delvis int:
$u = \ln x,\ v' = x$
$u' = \frac{1}{x},\ v = \frac{1}{2}x^2$

$$= \lim_{n\to\infty}\left(\frac{1}{2}\overbrace{\left(\frac{1}{n}\right)^2}^{\to 0}\overbrace{\ln\left(\frac{1}{n}\right)}^{\to -\infty} + \int_{\frac{1}{n}}^{1}\frac{x}{2}\,dx \right)$$

M: $\lim\limits_{n\to\infty} \frac{1}{2}\left(\frac{1}{n}\right)^2 \ln\left(\frac{1}{n}\right) = \frac{1}{2}\lim\limits_{n\to\infty} \frac{\ln\left(\frac{1}{n}\right)}{n^2}$

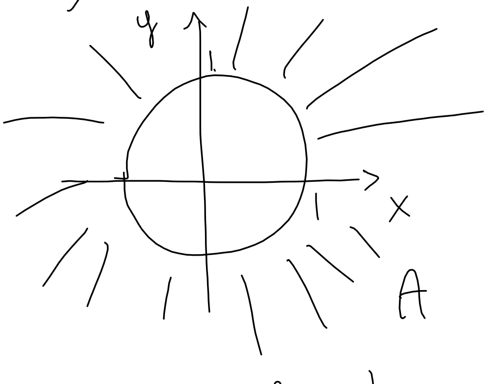
$$= \frac{1}{2}\lim_{n\to\infty} \frac{\frac{1}{\frac{1}{n}}\left(-\frac{1}{n^2}\right)}{2n} = -\frac{1}{2}\lim_{n\to\infty}\frac{1}{n^2} = 0$$

"$\frac{\infty}{\infty}$" : L'H

$$I = \lim_{n\to\infty}\int_{\frac{1}{n}}^{1} \frac{1}{2}x\,dx = \int_0^1 \frac{1}{2}x\,dx = \frac{1}{2}\left[\frac{1}{2}x^2\right]_{x=0}^{1}$$

$$= \underline{\underline{\frac{1}{4}}}$$ Konvergerer!

6.) $A = \{(x,y)\in\mathbb{R}^2 : x^2+y^2 \geq 1\}$



$$\iint_A \frac{1}{(x^2+y^2)^p}\,dx\,dy$$

Anta at $p \neq 1$:

$$\iint_A \frac{1}{(x^2+y^2)^p}\,dx\,dy = \lim_{n\to\infty}\iint_{A\cap B(0,n)} \frac{1}{(x^2+y^2)^p}\,dx\,dy$$

Polarkoord.

$$= \lim_{n\to\infty}\int_0^{2\pi}\int_1^n \frac{1}{r^{2p}}\,r\,dr\,d\theta$$

$$= \lim_{n\to\infty}\int_0^{2\pi}\int_1^n r^{1-2p}\,dr\,d\theta$$

$$= \lim_{n\to\infty} \int_0^{2\pi} \left[\frac{1}{2-2p} r^{2-2p}\right]_{r=1}^{n} d\theta$$

$$= \lim_{n\to\infty} \int_0^{2\pi} \left(\frac{1}{2-2p} n^{2-2p} - \frac{1}{2-2p}\right) d\theta$$

$$= \lim_{n\to\infty} \frac{\pi}{1-p}\left(n^{2-2p} - 1\right) = \begin{cases} \frac{\pi}{p-1}, & p>1 \\ \infty, & p<1 \end{cases}$$

Konvergerer!

Divergerer!

$p=1$:

$$\lim_{n\to\infty} \int_0^{2\pi} [\ln r]_{r=1}^{n} d\theta$$

$$= \lim_{n\to\infty} \int_0^{2\pi} \ln(n)\, d\theta = \lim_{n\to\infty} 2\pi \ln(n) = \infty$$

Divergerer!

Så: Integralet divergerer for $p \leq 1$ og konvergerer (mot $\frac{\pi}{p-1}$) ellers (dvs. $p>1$).