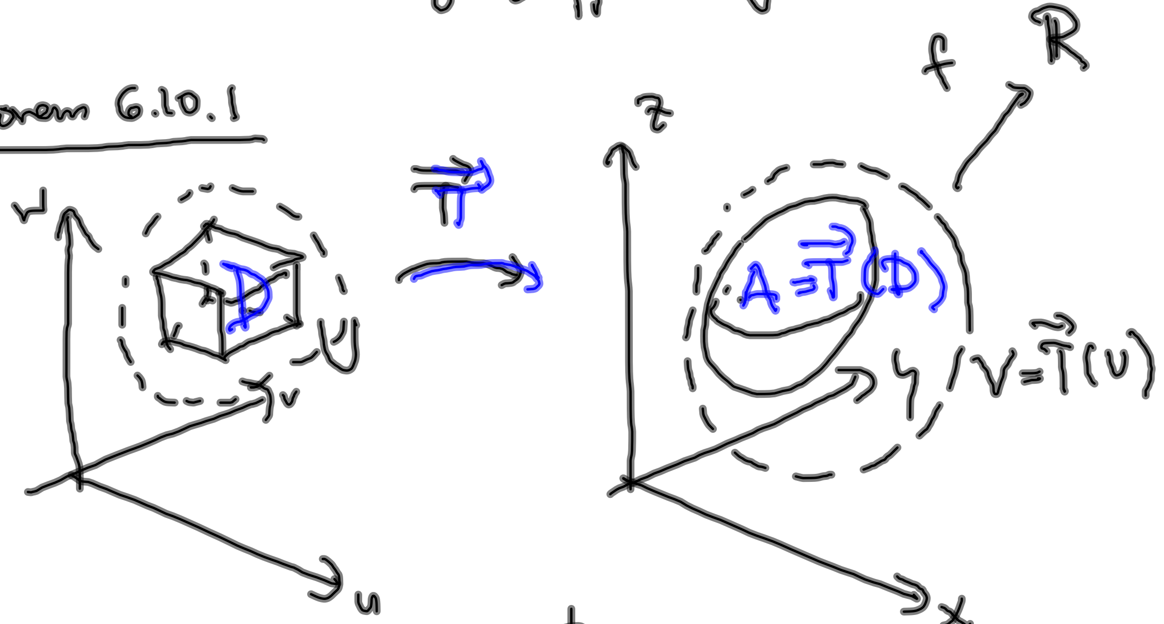
LH 6.10 Skifte av variabel i trippelintegraler

Teorem 6.10.1



$U \subset \mathbb{R}^3$ åpen, begrenset

$\vec{T} : U \longrightarrow \mathbb{R}^3$ injektiv, kontinuerlige partielle deriverte

$\vec{T}(u,v,w) = (x,y,z)$

$$\vec{T}'(u,v,w) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}(u,v,w)$$

der $\boxed{\det \vec{T}'(u,v,w) \neq 0}$ for alle $(u,v,w) \in U$

$D \subset U$ lukket, Jordan-målbar mengde

$f : A = \vec{T}(D) \longrightarrow \mathbb{R}$ kontinuerlig

Da er

$$\boxed{\iiint_A f(x,y,z)\,dx\,dy\,dz = \iiint_D f(\vec{T}(u,v,w))\,\left|\det \vec{T}'(u,v,w)\right|\,du\,dv\,dw}$$

$$\vec{T}(u,v,w) = (x,y,z) \qquad x = x(u,v,w), \ldots$$

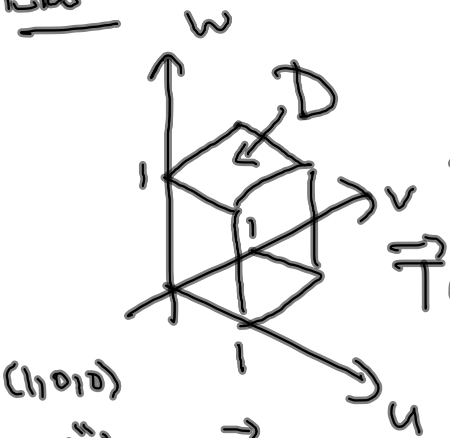
$$\vec{T}' = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

$$\det \vec{T}' = \frac{\partial x}{\partial u} \begin{vmatrix} \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} - \frac{\partial x}{\partial v} \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial w} \end{vmatrix} + \frac{\partial x}{\partial w} \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix}$$

$$= \frac{\partial(x,y,z)}{\partial(u,v,w)}$$

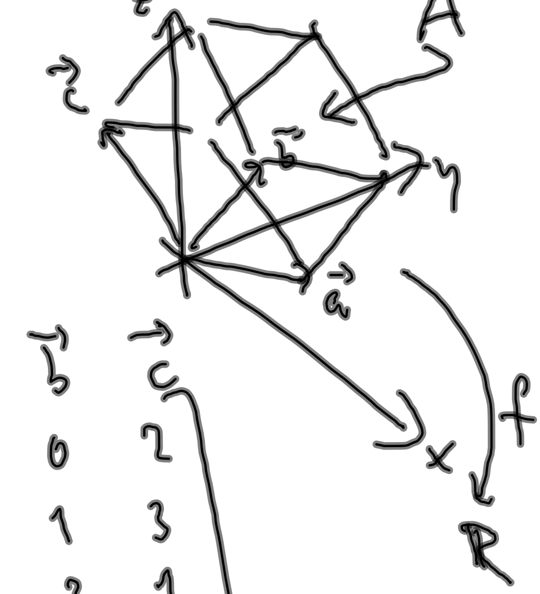
$$dx\,dy\,dz = \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$$

Eks 1



$\vec{T}$

Parallellepiped



$$\vec{T}\vec{u} = M\vec{u}$$

(1,0,0)

$\vec{e}_1 \longmapsto \vec{a}$

$\vec{e}_2 \longmapsto \vec{b}$

$\vec{e}_3 \longmapsto \vec{c}$

$$M = \vec{T}'(\vec{u}) = \begin{array}{c} \begin{array}{ccc} \vec{a} & \vec{b} & \vec{c} \end{array} \\ \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ -1 & 2 & 1 \end{bmatrix} \end{array}$$

$$\vec{T}(u,v,w) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = (u+2w,\ v+3w,\ -u+2v+w)$$

$$f(x,y,z) = 4x - y + 2z$$

$$\iiint_A f(x,y,z)\,dx\,dy\,dz = \,?$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det M = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ -1 & 2 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} - 0\begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} + 2\begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix}$$

$$= 1(-5) - 0 + 2(1) = -3 \neq 0$$

$$\iiint_A (4x - y + 2z)\, dx\,dy\,dz$$

$$= \iiint_D \Big(4(u+2w) - (v+3w) + 2(-u+2v+w)\Big) \cdot |-3| \, du\,dv\,dw$$

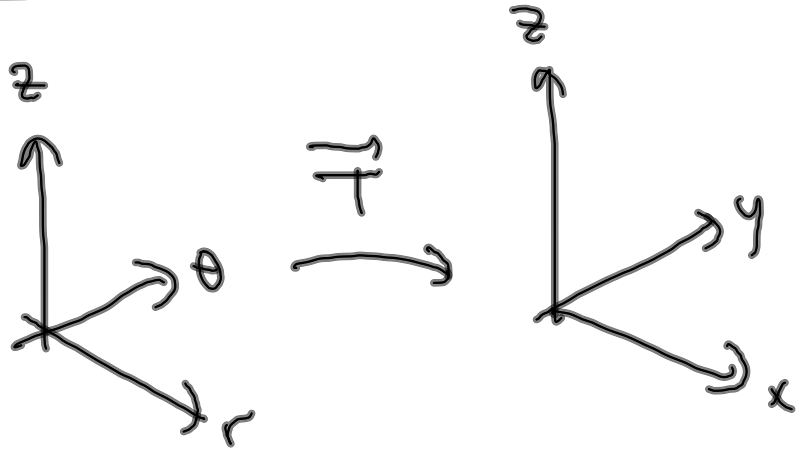
$$= \int_0^1\int_0^1\int_0^1 (2u + 3v + 7w) \cdot 3 \, du\,dv\,dw$$

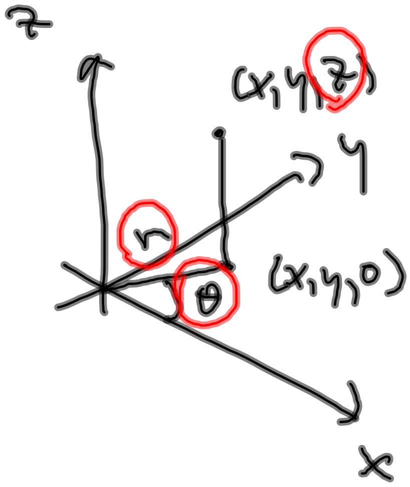
$$= 3 \int_0^1\int_0^1 1 + 3v + 7w \, dv\,dw$$

$$= 3\int_0^1 \left(1 + \frac{3}{2} + 7w\right) dw = 3\int$$

$$= 3\left(1 + \frac{3}{2} + \frac{7}{2}\right) = \underline{\underline{18}}$$

Sylinderkoordinater





$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

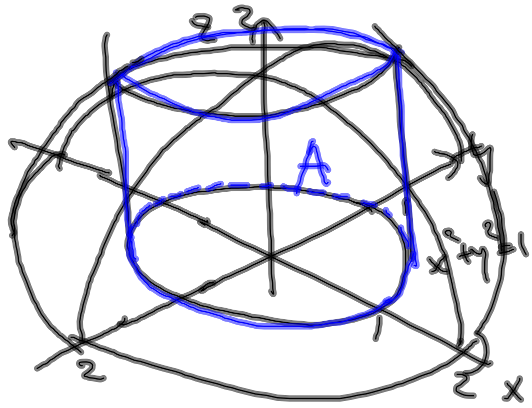
$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \cos\theta\,(r\cos\theta) + r\sin\theta\,(\sin\theta)$$

$$= r(\cos^2\theta + \sin^2\theta) = \underline{\underline{r}} \geq 0$$

$$\boxed{\iiint_A f(x,y,z)\,dx\,dy\,dz = \iiint_D f(r\cos\theta, r\sin\theta, z)\, r\, dr\,d\theta\,dz}$$

Eks 2



$0 \le z \le \sqrt{4-x^2-y^2}$

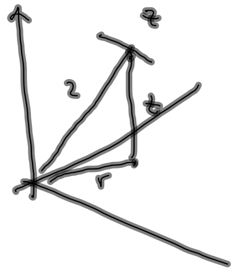
$(x,y):\ x^2+y^2=1$

$$A \begin{cases} x^2+y^2 \le 1 \\ z \ge 0 \\ x^2+y^2+z^2 \le 4 \end{cases}$$

$$f(x,y,z) = x^2 z$$

$$\iiint_A x^2 z \, dx\,dy\,dz = \iiint_D (r\cos\theta)^2 \underline{z}\ r \, dr\,d\theta\,dz$$

$$D: \begin{cases} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \\ 0 \le z \le \sqrt{4-r^2} \end{cases}$$

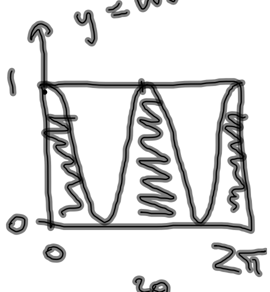


$$= \int_0^1 \int_0^{2\pi} \int_0^{\sqrt{4-r^2}} z\, r^3 \cos^2\theta \, dz\,d\theta\,dr$$

$$= \int_0^1 \int_0^{2\pi} \left[\tfrac{1}{2} z^2 r^3 \cos^2\theta \right]_{z=0}^{z=\sqrt{4-r^2}} d\theta\,dr$$

$$= \int_0^1 \int_0^{2\pi} \tfrac{1}{2}(4-r^2) r^3 \cos^2\theta \, d\theta\,dr$$

$$= \int_0^1 \tfrac{1}{2} \underbrace{(4-r^2) r^3}_{4r^3 - r^5} \underbrace{\left(\int_0^{2\pi} \cos^2\theta \, d\theta \right)}_{\pi} dr$$



$\cos 2\theta = \cos^2\theta - \sin^2\theta$

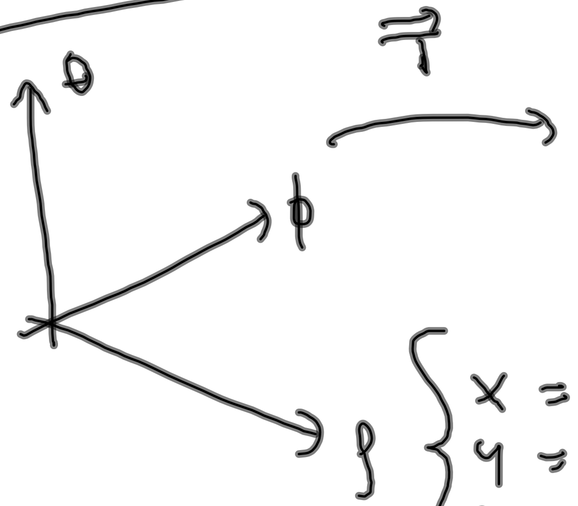
$= 2\cos^2\theta - 1$

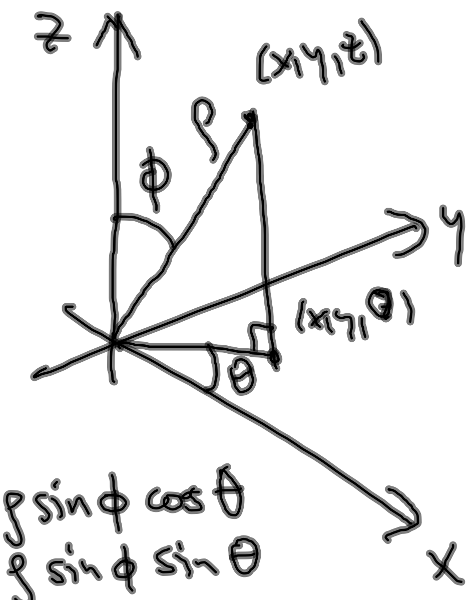
$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= \tfrac{1}{2}\pi \left[r^4 - \tfrac{1}{6} r^6 \right]_0^1$$

$$= \tfrac{1}{2}\pi \left(1 - \tfrac{1}{6} \right) = \frac{5\pi}{12}$$

Kulekoordinater





$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$$

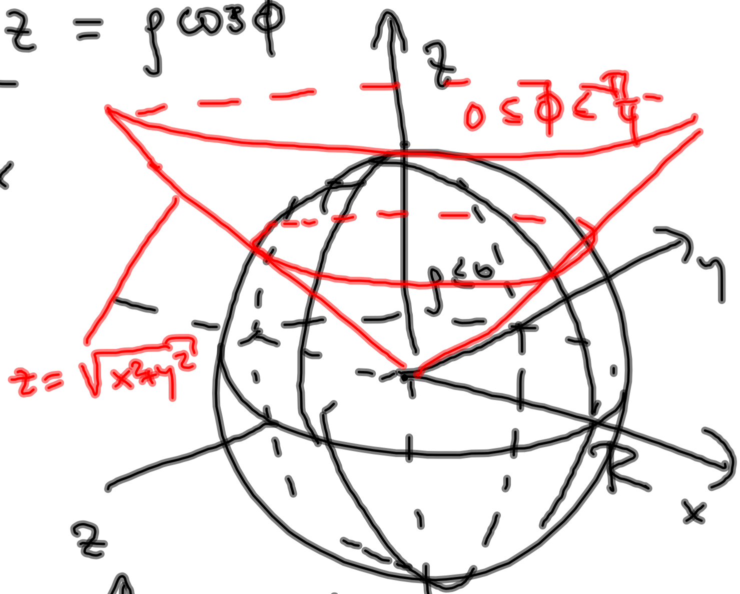
$\vec{T}(\rho, \phi, \theta) = (x$

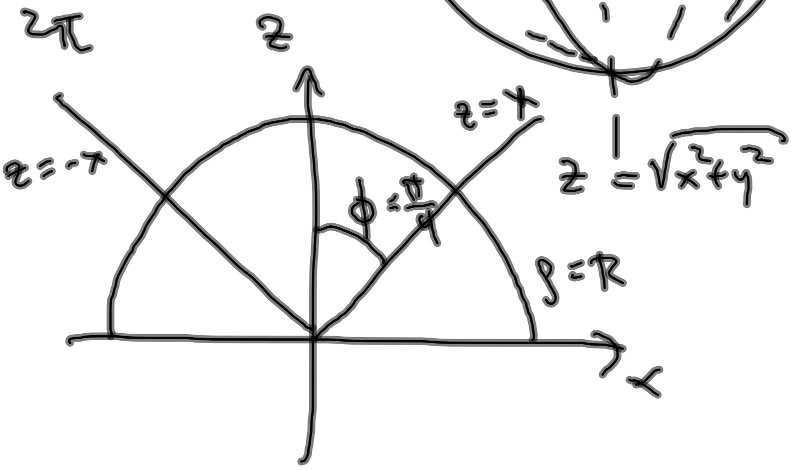
rho phi theta

$0 \le \rho \le R$

$0 \le \phi \le \pi$

$0 \le \theta \le 2\pi$





$x = \rho \sin\phi \cos\theta \leftarrow$

$y = \rho \sin\phi \sin\theta \leftarrow$

$z = \rho \cos\phi \leftarrow$

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$\shortparallel$

$\rho^2\sin\phi$

$$= \sin\phi\cos\theta\,\rho\sin\phi\cos\theta\,\rho\sin\phi$$
$$+ \rho\cos\phi\cos\theta\,\rho\sin\phi\cos\theta\,\cos\phi$$
$$+ \rho\sin\phi\sin\theta(+\underbrace{\rho\sin^2\phi\sin\theta + \rho\cos^2\phi\sin\theta}_{\rho\sin\theta})$$

$$= \rho^2\sin\phi\cos^2\theta\sin^2\phi$$
$$+ \rho^2\sin\phi\cos^2\phi\cos^2\theta$$
$$+ \rho^2\sin\phi\sin^2\theta$$

$$= \rho^2\sin\phi\cos^2\theta + \rho^2\sin\phi\sin^2\theta$$

$$= \underline{\underline{\rho^2\sin\phi}}$$

D $\rho\phi\theta$-rommet

$\downarrow$ kulekoord.

A xyz-rommet

$$\iiint_A f(x,y,z)\,dxdydz$$

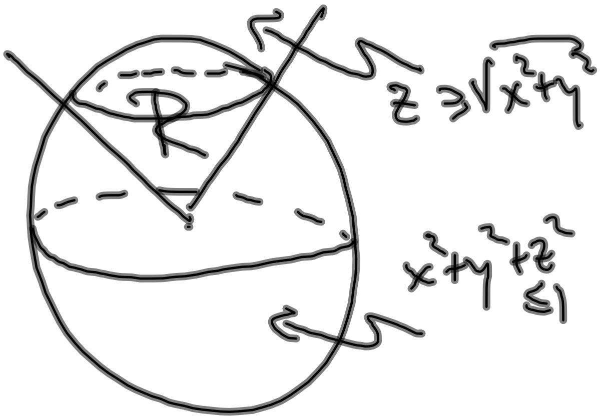
$$= \iiint_D f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) \cdot \rho^2\sin\phi \; d\rho d\phi d\theta$$

<u>Eks</u> R inne i kjeglen $z = \sqrt{x^2+y^2}$

$(z \geq \sqrt{x^2+y^2})$

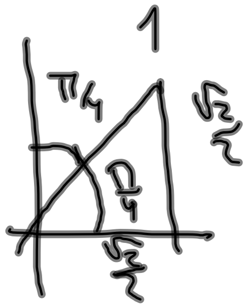
og i kulen $x^2+y^2+z^2 = 1$

$$D \quad \begin{cases} 0 \leq \rho \leq 1 \\ 0 \leq \phi \leq \pi/4 \\ 0 \leq \theta \leq 2\pi \end{cases}$$



$$\iiint_R z \, dx\,dy\,dz =$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho\cos\phi \cdot \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$



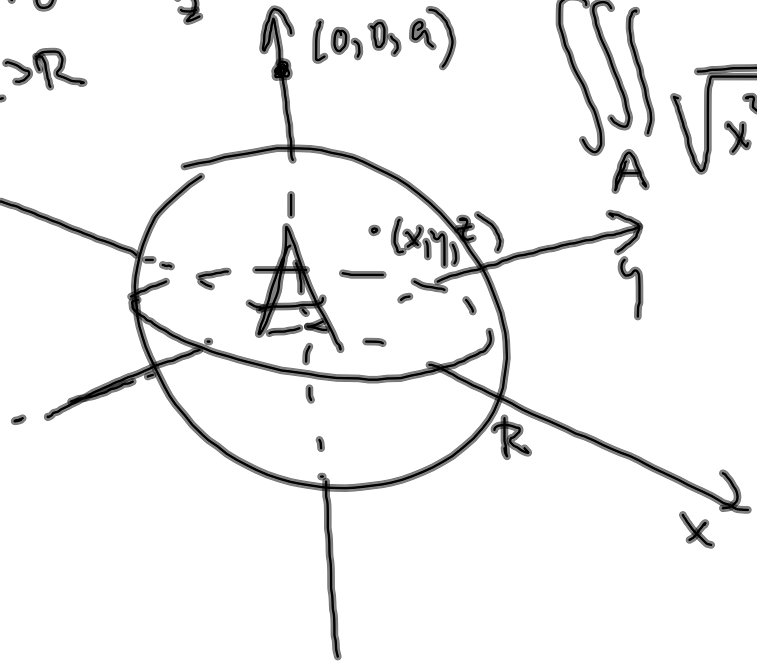
$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{1}{4}\sin\phi\cos\phi \; d\phi\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{8}\sin^2\phi\right]_0^{\pi/4} d\theta = \int_0^{2\pi} \frac{1}{8}\left(\frac{1}{2} - 0\right) d\theta$$

$$= \underline{\underline{\pi/8}}$$

Oppgave 6.10.7

$a > R$



$$\iiint_A \frac{1}{\sqrt{x^2+y^2+(z-a)^2}}\,dx\,dy\,dz$$

$$= \frac{4}{3}\pi R^3 \cdot \frac{1}{a}$$

LH 6.11 Anvendelser

$A \subseteq R = [a,b] \times [c,d] \times [e,f]$

$A \subset \mathbb{R}^3$

$$\text{volum}(A) = \iiint_A 1 \, dx\,dy\,dz = \iiint_R 1_A \, dx\,dy\,dz$$

Hvis legemet A har tetthet $f(x,y,z)$ i (x,y,z), så en liten bit av A nær (x,y,z) har masse $\underset{\text{tilnærmet}}{\wedge}$ lik $f(x,y,z)$ ganger volumet til biten

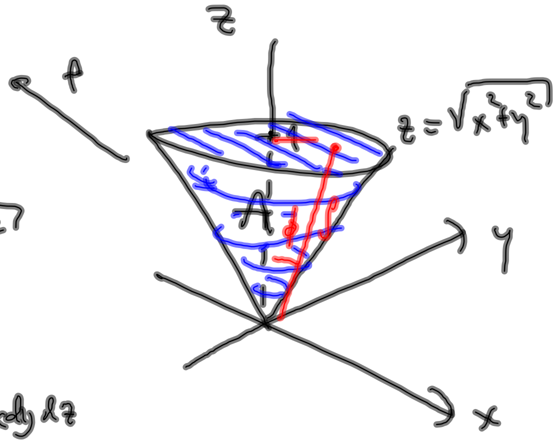
er den totale massen

$$\iiint_A f(x,y,z) \, dx\,dy\,dz$$

Eks 2

tetthet

$f(x,y,z) = k\sqrt{x^2+y^2+z^2}$



masse

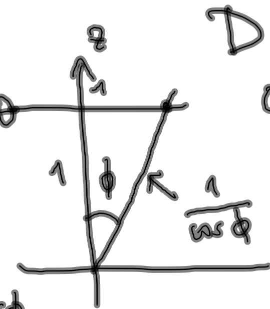
$$M = \iiint_A f(x,y,z)\,dx\,dy\,dz$$

$$= \iiint_A k\sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$$

kulekoordinater

D ρφθ-rommet

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\frac{1}{\cos\phi}} k\rho\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$



$0 \le \phi \le \pi/4$

$0 \le \theta \le 2\pi$

$0 \le \rho \le \frac{1}{\cos\phi}$

$$\int_0^{2\pi}\int_0^{\pi/4}\left[\frac{k}{4}\rho^4\sin\phi\right]_{\rho=0}^{\rho=1/\cos\phi} d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_{\phi=0}^{\phi=\pi/4} \frac{k}{4}\cdot\frac{\sin\phi}{\cos^4\phi}\,d\phi\,d\theta$$

$u = \cos\phi$

$du = -\sin\phi\,d\phi$

$u \quad \cos 0 = 1$

$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$$= \int_0^{2\pi}\int_1^{\sqrt{2}/2} \frac{k}{4}\,\frac{1}{u^4}\,(-du)\,d\theta$$

$u^{-4} \quad -\frac{u^{-3}}{3}$

$= -\frac{1}{3u^3}$

$$= \int_0^{2\pi}\left[\frac{k}{4}\,\frac{1}{3u^3}\right]_1^{\sqrt{2}/2} du\,d\theta$$

$$= 2\pi\frac{k}{3\cdot 4}(2\sqrt{2}-1) = \underline{\underline{\frac{k\pi}{6}(2\sqrt{2}-1)}}$$