2.7: Kjerneregelen

8.) $T = f(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$

$$T = f(r\cos\theta, r\sin\theta)$$

a) $$\frac{\partial T}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x(r,\theta)}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y(r,\theta)}{\partial r}$$

Kjerneregelen på komponentform

$$= \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$

$$\frac{\partial T}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x(r,\theta)}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y(r,\theta)}{\partial \theta}$$

$$= -\frac{\partial f}{\partial x} r\sin\theta + \frac{\partial f}{\partial y} r\cos\theta$$

b) $r = g(t)$, $\theta = h(t)$ $\quad T(r,\theta) \rightsquigarrow T(t)$

$$T'(t) = \frac{\partial T(r,\theta)}{\partial r}\underbrace{\frac{\partial r(t)}{\partial t}}_{= r'(t)} + \frac{\partial T(r,\theta)}{\partial \theta}\underbrace{\frac{\partial \theta(t)}{\partial t}}_{= \theta'(t)}$$

Kjerneregel på komponentform

$$\overset{a)}{=} \left(\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right) g'(t)$$
$$+ \left(-\frac{\partial f}{\partial x} r\sin\theta + \frac{\partial f}{\partial y} r\cos\theta\right) h'(t)$$

9.) Anta: $\exists$ derivbar. $g : \mathbb{R}^n \to \mathbb{R}$ s.a.

$$f(x_1, x_2, \dots, x_n, g(x_1, x_2, \dots, x_n)) = 0$$

a) La $\vec{G}(x_1, \dots, x_n) = (x_1, \dots, x_n, g(x_1, \dots, x_n))$ og

$$H(x_1, \dots, x_n) = f(\vec{G}(x_1, \dots, x_n)) = 0$$

Da er:

Kjerneregel på matriseform

$$0 = H'(x_1, \dots, x_n) = \left(\frac{\partial f}{\partial u_1}, \dots, \frac{\partial f}{\partial u_{n+1}}\right) \cdot \vec{G}'(x_1, \dots, x_n)$$

$$= \left(\frac{\partial f}{\partial u_1}, \dots, \frac{\partial f}{\partial u_{n+1}}\right) \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \\ & & \ddots & \\ & 0 & & 1 \\ \frac{\partial g}{\partial x_1} & \cdots & & \frac{\partial g}{\partial x_n} \end{bmatrix}$$

$n \times n$ identitets-matrise

$$= \left(\frac{\partial f}{\partial u_1} + \frac{\partial f}{\partial u_{n+1}}\frac{\partial g}{\partial x_1}, \dots, \frac{\partial f}{\partial u_n} + \frac{\partial f}{\partial u_{n+1}}\frac{\partial g}{\partial x_n}\right)$$

i'te komponent: $\frac{\partial f}{\partial u_i} + \frac{\partial f}{\partial u_{n+1}}\frac{\partial g}{\partial x_i}$

$$\frac{\partial f}{\partial u_i} + \frac{\partial f}{\partial u_{n+1}}\frac{\partial g}{\partial x_i} = 0 \text{ for alle } i = 1, \dots, n$$

$$\frac{\partial g}{\partial x_i} = -\frac{\frac{\partial f}{\partial u_i}}{\frac{\partial f}{\partial u_{n+1}}}$$

Setter inn det aktuelle pkt:

$$\frac{\partial g}{\partial x_i}(x_1,\dots,x_n) = - \frac{\frac{\partial f}{\partial x_i}(x_1,\dots,x_n,g(x_1,\dots,x_n))}{\frac{\partial f}{\partial x_{n+1}}(x_1,\dots,x_n,g(x_1,\dots,x_n))}$$

b) $f(x,y) = x^2 + y^2 - R^2$ $\qquad (f(x,g(x))=0)$

Svarer til a) med $n=1$, så $\vec{G}(x) = (x, g(x))$

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y \underset{\uparrow}{=} 2g(x)$$

I det aktuelle pkt

Fra a):

$$g'(x) = \frac{\partial g}{\partial x}(x) = - \frac{\frac{\partial f}{\partial x}(x,g(x))}{\frac{\partial f}{\partial y}(x,g(x))} = - \frac{\cancel{2}x}{\cancel{2}g(x)} = -\frac{x}{g(x)}$$

$$\Downarrow$$

$$\underline{\underline{g'(x) = -\frac{x}{g(x)}}}$$

Geometrisk tolk: $f(x, g(x)) = 0$

$$x^2 + g(x)^2 - R^2 = 0$$

$$x^2 + g(x)^2 = R^2$$

Dvs. at $y = g(x)$ er på sirkelen $x^2 + y^2 = R^2$

Fra uttrykket på forrige side omskrevet:

$$2x + 2g(x)g'(x) = 0$$

$$x + g(x)g'(x) = 0$$

$$\underbrace{(x, g(x))}_{H(x)} \cdot \underbrace{(1, g'(x))}_{H'(x)} = 0$$

punkt på sirkelen

Derivert, dvs. tangenten i pkt.

Dvs: Vektorene som gir pkt'er på sirkelen står vinkelrett på tangenten sin (siden prikkprodukt = 0).

Dette er klart sant for sirkler: