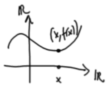
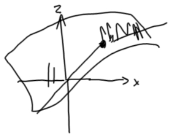
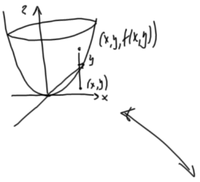
Hvordan "plotte" funksjoner $f: \mathbb{R}^2 \to \mathbb{R}$.





$(x, y, f(x,y)) \leftarrow$ grafen

Eks $f(x,y) = x^2 + y^2$



Konturkurver

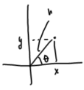
$$N_c = \{(x,y) \mid f(x,y) = c\}$$

$$N_c = \{(x,y) \mid x^2 + y^2 = c\} = \begin{cases} \emptyset & c < 0 \\ \text{sirkel med radius } \sqrt{c} & \text{ellers} \end{cases}$$

Polarkoordinater:

$x = r\cos\theta$

$y = r\sin\theta$

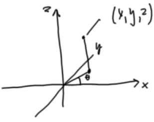


$r = \sqrt{x^2+y^2}$ $\quad r > 0$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$. $\quad \theta \in [0, 2\pi]$.

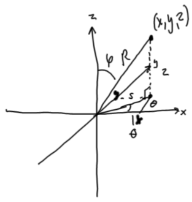
I 3d

Sylinderkoordinater



$(x,y,z) = (r\cos\theta, r\sin\theta, z)$

Kulekoordinater



$x = s\cos\theta$

$y = s\sin\theta$

$s = R\sin\varphi$

$z = R\cos\varphi$

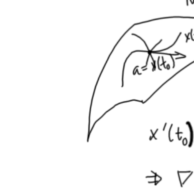
$R^2 = x^2 + y^2 + z^2$

$x = R\sin\varphi\cos\theta$ $\quad R \geq 0$

$y = R\sin\varphi\sin\theta$ $\quad \theta \in [0, 2\pi]$

$z = R\cos\varphi$ $\quad \varphi \in [0, \pi]$

Tangentplan

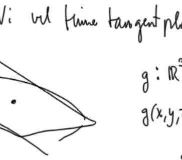


Tangentplan. Hvis vi finner $\bar{n}$ normalvektor i a, så har tangentplanet ligning: $\boxed{\bar{n} \cdot (\bar{x} - \bar{a}) = 0.}$

Hvordan finne $\bar{n}$?

Anta $\bar{x}(t)$ er en kurve som ligger "inne i" mengden $\{f(x) = c\} = N_c$.

$f: \mathbb{R}^n \to \mathbb{R}$



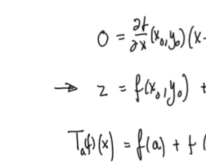
$f(x(t)) = c. \quad \Big| \frac{d}{dt}$

$\nabla f(x(t)) \cdot x'(t) = 0.$

$x'(t_0)$ ligger i tangentplant i a.

$\Rightarrow \nabla f(x(t))$ er normal på alle vektorer i tangentplanet til N_c.

Vi vil finne tangentplan til $(x, y, f(x,y))$ i $(x_0, y_0, f(x_0, y_0))$



$g: \mathbb{R}^3 \to \mathbb{R}$

$g(x,y,z) = f(x,y) - z$; Da blir flaten lik $\{(x,y,z) \mid g(x,y,z) = 0\}$

$$\nabla g(x,y,z) = \frac{\partial g}{\partial x} i + \frac{\partial g}{\partial y} j + \frac{\partial g}{\partial z} k$$

$$= \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j - k.$$

$\bar{n} \cdot (\bar{x} - \bar{a}) = 0.$

$\bar{n} = \nabla g(x_0, y_0, z_0)$

$\bar{x} = xi + yj + zk$

$\bar{a} = x_0 i + y_0 j + f(x_0, y_0) k$

$$0 = \frac{\partial f}{\partial x}(x_0,y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) - (z - f(x_0, y_0))$$

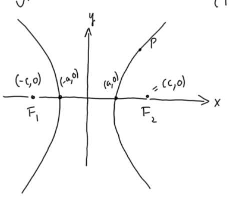
$$\Rightarrow z = f(x_0, y_0) + \nabla f(x_0, y_0) \cdot (x - x_0, y - y_0)$$

$T_a f(x) = f(a) + f'(a)(x - a) \quad \leftarrow$ Tangentplan

Linearisering.

Hyperbel

$\{P \mid |PF_1| - |PF_2| = \pm k\}$



$|(a,0),(c,0)| = c - a = |PF_1|$ $\quad$ $|PF_1| - |PF_2| = c - a - c - a = -2a = \pm K$

$|(a,0),(-c,0)| = c + a = |PF_2|$ $\quad$ Kan velge $K = 2a$.

<u>Hjelpestørrelse</u> $b^2 = c^2 - a^2$; $a^2 + b^2 = c^2$

Regner ut formel: $P = (x,y)$ på hyperbelen.

$|PF_1| = \sqrt{(x+c)^2 + y^2}$ $\quad$ $|PF_2| = \sqrt{(x-c)^2 + y^2}$

ligning: $|PF_1| - |PF_2| = \pm 2a$.

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a$$

$$(x+c)^2 + \cancel{y^2} = \left(\sqrt{(x-c)^2 + y^2} \pm 2a\right)^2 = (x-c)^2 + \cancel{y^2} \pm 4a\sqrt{(x-c)^2 + y^2} + 4a^2$$

$$(x+c)^2 - (x-c)^2 = \cancel{4}cx = \cancel{4}a\sqrt{(x-c)^2 + y^2} + \cancel{4}a^2$$

$$cx - a^2 = a\sqrt{(x-c)^2 + y^2} \quad \Big| \cdot^2$$

$$c^2x^2 - \cancel{2a^2cx} + a^4 = a^2\left((x-c)^2 + y^2\right) = a^2x^2 - \cancel{2a^2cx} + a^2c^2 + a^2y^2$$

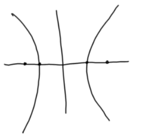
$$a^4 - a^2c^2 = (a^2 - c^2)x^2 + a^2y^2$$

$$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2) \qquad b^2 = c^2 - a^2$$

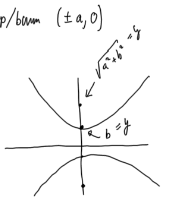
$$b^2x^2 - a^2y^2 = a^2b^2 \quad \Big| \frac{1}{a^2b^2}$$

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$

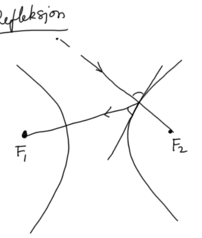
← Hyperbel med senter $(0,0)$ Brennpunkter $(\pm\sqrt{a^2+b^2}, 0)$ topp/bunn $(\pm a, 0)$



$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$



<u>Refleksjon</u>



$$\frac{(x-n)^2}{a^2} - \frac{(y-m)^2}{b^2} = \pm 1$$

<u>Eks</u>

ligning: $-3x^2 + 4y^2 + 6x + 32y + 49 = 0$

Vis at hyperbel, finn bren-, topp-, bunnpunkt.

Komplettere kvadrat.

$$-3(x^2 - 2x + 1) + 3 + 4(y^2 + 8y + 16) - 64 + 49$$

$$-3(x-1)^2 + 4(y+4)^2 + \underbrace{3 + 49 - 64}_{-12} = 0$$

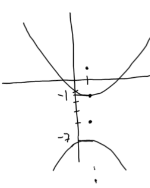
$$-3(x-1)^2 + 4(y+4)^2 = 12$$

$$-\frac{(x-1)^2}{4} + \frac{(y+4)^2}{3} = 1$$

Senter i $(1,-4)$

Brennpunkt $(1, -4 \pm \sqrt{16+9}) = (-1, -4 \pm 5)$

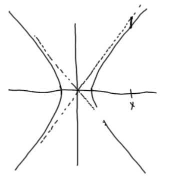
Topp/bunn $(1, -4 \pm 3)$



$y = \frac{1}{x}$ $\quad \lim_{x \to 0} y(x)$ asymptote

$\lim_{x \to \infty} y(x) = 0$ horisontal asymptote.

Hyperbel med senter i $(0,0)$.



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2} \underset{|x| \text{ stor}}{\approx} \pm\frac{b}{a}|x|.$$

Hyperbelen har skrå asymptoter $y = \pm\frac{b}{a}x$ når $|x|$ stor.

Må vise at $\lim_{x \to \infty}\left(\frac{b}{a}x - \frac{b}{a}\sqrt{x^2 - a^2}\right) = 0$.

$$\frac{b}{a}\lim_{x \to \infty}\left(x - \sqrt{x^2 - a^2}\right) = \frac{b}{a}\lim_{x \to \infty}\left(\frac{(x - \sqrt{x^2 - a^2})(x + \sqrt{x^2 - a^2})}{(x + \sqrt{x^2 - a^2})}\right)$$

$$= \frac{b}{a}\lim_{x \to \infty}\left(\frac{\cancel{x^2} - (\cancel{x^2} - a^2)}{x + \sqrt{x^2 - a^2}}\right) = \frac{b}{a}\lim_{x \to \infty}\left(\frac{a^2}{x + \sqrt{x^2 - a^2}}\right) = 0.$$

Parametrisering av hyperbel

$$\frac{(x-n)^2}{a^2} - \frac{(y-m)^2}{b^2} = 1$$

Hyperbolsk sin / cos

$$\left.\begin{aligned} \cosh(t) &= \frac{e^t + e^{-t}}{2} \\ \sinh(t) &= \frac{e^t - e^{-t}}{2} \end{aligned}\right\} \cosh^2(t) - \sinh^2(t) = 1$$

$$\frac{x-n}{a} = \cosh(t)$$

$$\frac{y-m}{b} = \sinh(t)$$

$x(t) = n + a\cosh(t)$ $\quad t \in \langle -\infty, \infty \rangle$

$y(t) = m + b\sinh(t)$