Forhold mellom forholdstest og rottest

$x, y \in \mathbb{R}_+$ Aritmetisk middelverdi $\frac{x+y}{2}$, $\frac{x_1+\cdots+x_n}{n}$

Geometrisk middelverdi $\sqrt{xy}$, $\sqrt[n]{x_1 \cdot x_2 \cdot \cdots \cdot x_n}$

$$\left(\frac{x+y}{2}\right)^2 = \frac{x^2}{4} + \frac{xy}{2} + \frac{y^2}{4} \qquad (\sqrt{xy})^2 = xy$$

$$\left(\frac{x+y}{2}\right)^2 - (\sqrt{xy})^2 = \frac{x^2}{4} + \frac{xy}{2} + \frac{y^2}{4} - xy = \frac{x^2}{4} - \frac{xy}{2} + \frac{y^2}{4} = \left(\frac{x-y}{2}\right)^2 \geq 0$$

$$AM \geq GM$$

Forholdstest $\left|\frac{a_{n+1}}{a_n}\right| \xrightarrow[n\to\infty]{} ?$

Rottest $|a_n|^{\frac{1}{n}} \xrightarrow[n\to\infty]{} ?$

Se på: $|a_n|^{\frac{1}{n}} = \left|\frac{a_n}{a_{n-1}} \cdot \frac{a_{n-1}}{a_{n-2}} \cdot \cdots \cdot \frac{a_2}{a_1} \frac{a_1}{a_0} a_0\right|^{\frac{1}{n}}$

$$= \left|\frac{a_n}{a_{n-1}} \cdots \frac{a_1}{a_0}\right|^{\frac{1}{n}} \cdot \underbrace{|a_0|^{\frac{1}{n}}}_{\to 1}$$

$a_0 \neq 0$ betyr $|a_0|^{\frac{1}{n}} \xrightarrow[n\to\infty]{} 1$

Geometriske middelverdien av $\left|\frac{a_n}{a_{n-1}}\right|, \ldots, \left|\frac{a_1}{a_0}\right|$

Forholdstest vs rottest:
Rot: GM av forholdene
Forhold: forholdene

Eks. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$

$\frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{16} + \cdots$

Forholdstest: $\frac{1}{\frac{1}{2}} = 2$ annethvert forhold

$\frac{\frac{1}{8}}{1} = \frac{1}{8}$ — " —

Rottest ok.

AM-test: $\frac{1}{n}\left|\frac{a_n}{a_{n-1}} + \cdots + \frac{a_1}{a_0}\right| \xrightarrow[n\to\infty]{} ?$

sterkere enn forholdstesten

Taylorrekker

Eks. 1) Geometriske rekker $a_0 + a_0 k + a_0 k^2 + \cdots = \frac{a_0}{1-k}$

$|k| < 1$

2) Eksponensialfunksjon $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots = e^x$

$= \sum_{k=0}^{\infty} \frac{x^k}{k!}$

2.1) $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$

2.2) $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$

Fordi: $e^{ix} = \cos x + i \sin x$

Eks $\frac{1}{\sin x} = \frac{1}{x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots} = \frac{1}{x}(a_0 + a_1 x + a_2 x^2 + \cdots)$

Krever, hva blir $a_i = ?$

Må løse

$$1 = \frac{1}{x}(a_0 + a_1 x + a_2 x^2 + \cdots)(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots)$$

$$= a_0 + a_1 x + (-\frac{1}{6}a_0 + a_2)x^2 + \cdots$$

Dette betyr:

$a_0 = 1 \quad a_1 = 0 \quad -\frac{1}{6}a_0 + a_2 = 0$

$\Downarrow$

$a_2 = \frac{1}{6}$

$$\Longrightarrow \frac{1}{\sin x} = \frac{1}{x}\left(1 + \frac{1}{6}x^2 \cdots\right)$$

Liten oppgave

$\frac{1}{1-x} = 1 + x + x^2 + \cdots$

Hva er rekkeutviklingen til $\frac{1}{\frac{1}{1-x}} = 1 - x$?

Eks.

$$\sum_{n=1}^{\infty}\frac{x^n}{n} = \sum_{n=1}^{\infty}\int_0^x t^{n-1}\,dt = \int_0^x\Big(\sum_{n=1}^{\infty} t^{n-1}\Big)dt$$

Merk $\sum_{n=1}^{\infty} t^{n-1} = \sum_{n=0}^{\infty} t^n$

$$= \int_0^x \frac{1}{1-t}\,dt = \Big[-\ln|1-t|\Big]_0^x = \underline{\underline{-\ln|1-x|}}$$

$|x|<1$

$$\sum_{n=0}^{\infty}(n+1)x^n = \sum_{n=0}^{\infty}\frac{d}{dx}\left(x^{n+1}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty}x^{n+1}\right)$$

$$= \frac{d}{dx}\left(\frac{x}{1-x}\right) = \underline{\underline{\frac{1}{(1-x)^2}}}$$

$|x|<1$

Eks

$$\sum_{n=2}^{\infty}\frac{1}{n(n-1)2^n} = \,? = f\left(\tfrac{1}{2}\right) \quad \text{hvor } f(x) = \sum_{n=2}^{\infty}\frac{1}{n(n-1)}x^n$$

Vi har:

$$f(x) = \sum_{n=2}^{\infty}\frac{x^n}{n(n-1)} = \sum_{n=2}^{\infty}\int_0^x\Big(\int_0^u t^{n-2}\,dt\Big)du$$

$$= \int_0^x\Big[\frac{t^{n-1}}{n-1}\Big]_0^u du = \int_0^x \frac{u^{n-1}}{n-1}du = \Big[\frac{1}{n-1}\frac{1}{n}u^n\Big]_0^x = \frac{x^n}{n(n-1)}$$

$$= \int_0^x\Big(\int_0^u \sum_{n=2}^{\infty} t^{n-2}\,dt\Big)du$$

$$= \int_0^x\Big(\int_0^u \frac{1}{1-t}\,dt\Big)du$$

$$= \int_0^x -\ln|1-u|\,du$$

$$= -(x-1)\ln|1-x| + x$$

Som gir

$$f\left(\tfrac{1}{2}\right) = -\left(\tfrac{1}{2}-1\right)\ln\left|1-\tfrac{1}{2}\right| + \tfrac{1}{2} = \tfrac{1}{2}(-\ln 2) + \tfrac{1}{2} = \underline{\underline{\tfrac{1}{2}(1-\ln 2)}}$$

Men husk: $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

$$\frac{1}{2}(1-\ln 2) = \frac{1}{4} - \frac{1}{6} + \frac{1}{8} - \frac{1}{10} + \cdots$$

$$\frac{1}{2}(1-\ln 2) = \frac{1}{8} + \frac{1}{48} + \frac{1}{192} + \cdots$$

$$\frac{1}{4}\left(1-\frac{1}{2}\right) + \frac{1}{8}\left(\frac{1}{2}-\frac{1}{3}\right) + \cdots$$

Huske:

$$\sum_{n=2}^{\infty}\frac{1}{n(n-1)2^n}$$

$$\frac{1}{n(n-1)2^n} = \frac{1}{2^n}\left(\frac{1}{n-1} - \frac{1}{n}\right)$$

Eks

Løs diff.likningen $y'' + 2xy' + 2y = 0$

Prøv med $y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{\substack{n=0 \\ (n=1)}}^{\infty} n a_n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Likningen ser ut :

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \overbrace{2x \sum_{n=0}^{\infty} n a_n x^{n-1}}^{x^n} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$m = n-2$ { gjøre om til x^n } { $n = m$ }

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m + \sum_{m=0}^{\infty} 2m a_m x^m + \sum_{m=0}^{\infty} 2 a_m x^m = 0$$

$$\sum_{m=0}^{\infty} \left((m+2)(m+1) a_{m+2} + 2m a_m + 2 a_m \right) x^m = 0$$

Dette betyr $(m+2)\cancel{(m+1)} a_{m+2} + 2\cancel{(m+1)} a_m = 0$

$$(m+2) a_{m+2} + 2 a_m = 0$$

eller $a_{m+2} = -\frac{2}{m+2} a_m$

m partall

$a_0,\ a_2 = -a_0,\ a_4 = \frac{1}{2} a_0,\ \dots,\ a_{2n} = \frac{(-1)^n}{n!} a_0$

m oddetall

$a_{2n+1} = \frac{(-1)^n 4^n n!}{(2n+1)!} a_1$

Se på løsningen som har $\begin{cases} a_0 = 1 \\ a_1 = 0 \end{cases}$:

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} \qquad e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!}$$

$$= \underline{\underline{e^{-x^2}}}$$