1.9 Lineæravbildninger

$\vec{T}: \mathbb{R}^n \to \mathbb{R}^m$ kalles en lineæravbildning dersom

(i) $\vec{T}(c\vec{x}) = c\vec{T}(\vec{x})$ for alle $c \in \mathbb{R}$, $\vec{x} \in \mathbb{R}^n$

(ii) $\vec{T}(\vec{x}+\vec{y}) = \vec{T}(\vec{x}) + \vec{T}(\vec{y})$ for alle $\vec{x}, \vec{y} \in \mathbb{R}^n$

Setning: Hvis $\vec{T}: \mathbb{R}^n \to \mathbb{R}^m$ er en lineæravbildning, så er

$$\vec{T}(c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_n\vec{x}_n) = c_1\vec{T}(\vec{x}_1) + c_2\vec{T}(\vec{x}_2) + \dots + c_n\vec{T}(\vec{x}_n)$$

Bevis: $\vec{T}(\underbrace{c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_{n-1}\vec{x}_{n-1}}_{\vec{x}} + \underbrace{c_n\vec{x}_n}_{\vec{y}}) \overset{(ii)}{=}$

$\vec{T}(c_1\vec{x}_1 + \dots + c_{n-1}\vec{x}_{n-1}) + \vec{T}(c_n\vec{x}_n) \overset{(i)}{=}$

$= \vec{T}(\underbrace{c_1\vec{x}_1 + \dots}_{\vec{x}} + \underbrace{c_{n-1}\vec{x}_{n-1}}_{\vec{y}}) + c_n\vec{T}(\vec{x}_n) \overset{\text{osv.}}{=}$

$\vdots$

$= c_1\vec{T}(\vec{x}_1) + \dots + c_n\vec{T}(\vec{x}_n)$

Eksempel: La A være en m×n-matrise, og la

$$\vec{T}(\vec{x}) = A\underset{\uparrow \atop \mathbb{R}^n}{\vec{x}} \in \mathbb{R}^m \qquad \vec{T}(\vec{x}) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$\vec{T}$ er lineær:

(i) $\vec{T}(c\vec{x}) = A(c\vec{x}) = cA\vec{x} = c\vec{T}(\vec{x})$

(ii) $\vec{T}(\vec{x}+\vec{y}) = A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} = \vec{T}(\vec{x}) + \vec{T}(\vec{y})$

Hurra, T er lineær!

Setning: Hvis $T:\mathbb{R}^n \to \mathbb{R}^m$ er en lineæravbildning, så finnes det en $m\times n$-matrise A slik at $\vec{T}(\vec{x}) = A\vec{x}$ for alle $\vec{x}\in\mathbb{R}^n$. Den i-te søylen til A er

$$\vec{T}(\vec{e}_i) = T\begin{pmatrix}0\\ \vdots\\ 1\\ 0\\ \vdots\\ 0\end{pmatrix} \leftarrow \text{i-te posisjon.}$$

Bevis: La

$$\vec{x} = \begin{pmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{pmatrix} = x_1\begin{pmatrix}1\\ 0\\ \vdots\\ 0\end{pmatrix} + x_2\begin{pmatrix}0\\ 1\\ \vdots\\ 0\end{pmatrix} + \cdots + x_n\begin{pmatrix}0\\ \vdots\\ \vdots\\ 1\end{pmatrix}$$

$$= x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n$$

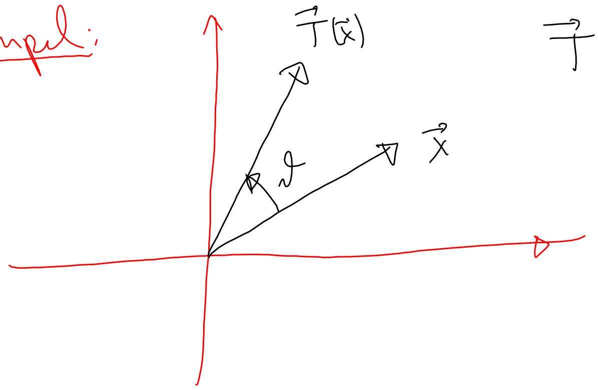
Altså

$$\vec{T}(\vec{x}) = \vec{T}(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n) = x_1\vec{T}(\vec{e}_1) + x_2\vec{T}(\vec{e}_2) + \cdots + x_n\vec{T}(\vec{e}_n)$$

$$= x_1\begin{pmatrix}a_{11}\\ a_{21}\\ \vdots\\ a_{m1}\end{pmatrix} + x_2\begin{pmatrix}a_{12}\\ a_{22}\\ \vdots\\ a_{m2}\end{pmatrix} + \cdots + x_n\begin{pmatrix}a_{1n}\\ a_{2n}\\ \vdots\\ a_{mn}\end{pmatrix}$$

$$= \begin{pmatrix}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n\\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n\\ \vdots\\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n\end{pmatrix} = \underbrace{\begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{pmatrix}}_{A}\underbrace{\begin{pmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{pmatrix}}_{\vec{x}} = A\vec{x}.$$

Eksempel:



$\vec{T}$ drejer enhver vektor en vinkel θ.

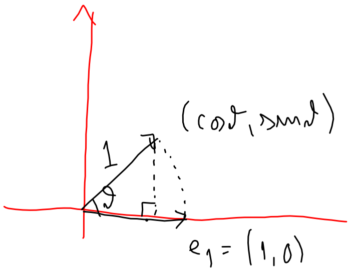
$$\vec{T}(c\vec{x}) = c\vec{T}(\vec{x})$$

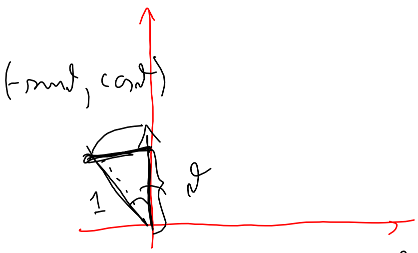
$$\underbrace{\vec{T}(\vec{x}+\vec{y})} = \underbrace{\vec{T}(\vec{x})+\vec{T}(\vec{y})}$$

Kan vi finde matricen A_θ til T?

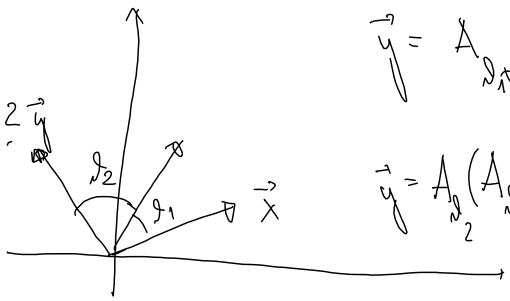
$$A_\theta = [\vec{T}(\vec{e}_1), \vec{T}(\vec{e}_2)] \qquad \vec{e}_1 = \begin{pmatrix}1\\0\end{pmatrix}, \ \vec{e}_2 = \begin{pmatrix}0\\1\end{pmatrix}$$

$$A_\theta = \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$$





$$\vec{T}(\vec{x}) = A_\theta\vec{x} = \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta\end{pmatrix}$$



$$\vec{y} = A_{\theta_1+\theta_2}\vec{x} = \begin{pmatrix}\cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2)\end{pmatrix}$$

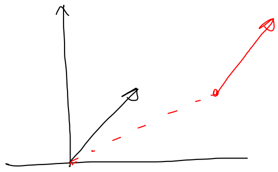
$$\vec{y} = A_{\theta_2}(A_{\theta_1}\vec{x}) = (A_{\theta_2}A_{\theta_1})\vec{x}$$

$$\boxed{\begin{aligned}&\cos(\theta_1+\theta_2) \\ &= \cos\theta_2\cos\theta_1 - \sin\theta_2\sin\theta_1 \\ &\sin(\theta_1+\theta_2) \\ &= \cos\theta_2\sin\theta_1 + \sin\theta_2\cos\theta_1\end{aligned}}$$

$$A_{\theta_2}A_{\theta_1} = \begin{pmatrix}\cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2\end{pmatrix}\begin{pmatrix}\cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1\end{pmatrix}$$

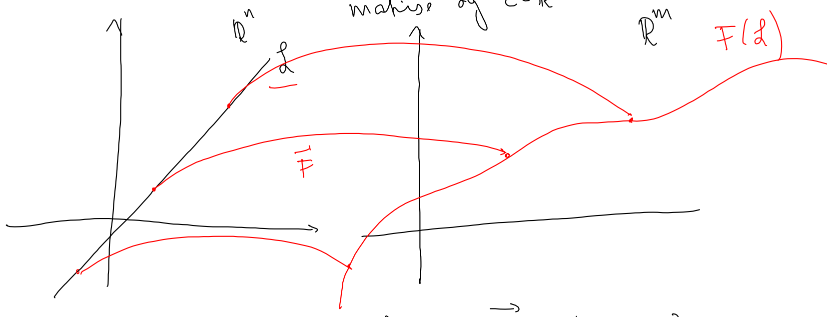
$$= \begin{pmatrix}\cos\theta_2\cos\theta_1 - \sin\theta_2\sin\theta_1 & -\cos\theta_2\sin\theta_1 - \sin\theta_2\cos\theta_1 \\ & \end{pmatrix}$$

1.10 Affinavbildninger



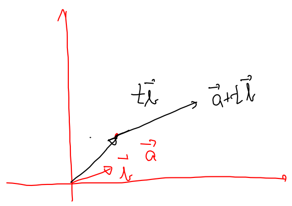
$\vec T(\vec 0)=\vec T(0\vec 0)=0\vec T(\vec 0)=\vec 0$

Definisjon: En affinavbildning $\vec F:\mathbb{R}^n\to\mathbb{R}^m$ er en funksjon på formen $\vec F(\vec x)=A\vec x+\vec c$ der A er en $m\times n$-matrise og $\vec c\in\mathbb{R}^m$



Rett linje gjennom $\vec a$ i retningen $\vec b$: $\vec r(t)=\vec a+t\vec b$

$\vec F(\vec r(t))$



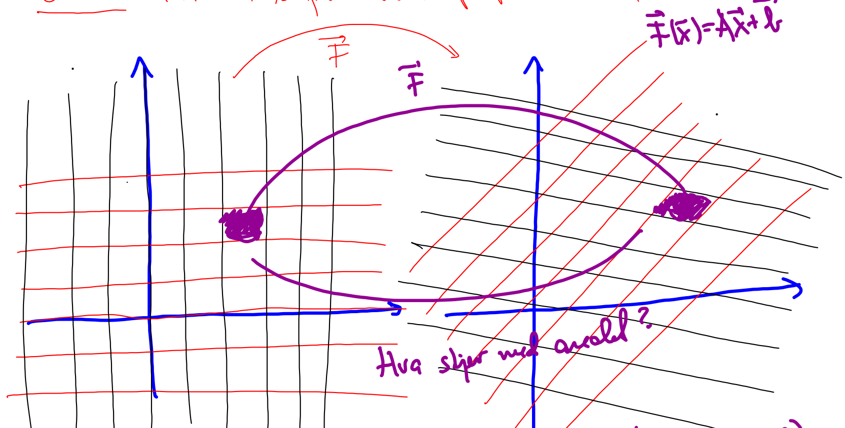
Setning: Anta at ℓ er linjen $\vec r(t)=\vec a+t\vec b$ i $\mathbb{R}^n$, og at $\vec F(\vec x)=A\vec x+\vec c$ er en affinavbildning fra $\mathbb{R}^n$ til $\mathbb{R}^m$. Hvis $A\vec b\neq\vec 0$, så er bildet av ℓ en rett linje med retningsvektor $A\vec b$.

Bevis: $\vec F(\vec r(t))=A\vec r(t)+\vec c=A(\vec a+t\vec b)+\vec c=A\vec a+tA\vec b+\vec c$

$=\underbrace{(A\vec a+\vec c)}+t\underbrace{A\vec b}$

linjen gjennom $A\vec a+\vec c$ i retning $A\vec b$

Observer: Parallelle linjer avbildes på parallelle linjer.



Nytt areal $=$ det(A)(gamle areal)