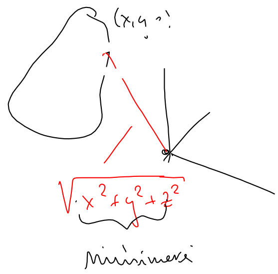
5.10: 3, 13, 8, 14

5.10.3: Finn punktene på skjæringskurven mellom flatene $x^2+y^2=1$ og $x^2-xy+y^2-z^2=1$ som ligger nærmest origo



Minimere:

$f(x,y,z)=x^2+y^2+z^2$

Under bibetingelsene

$g_1(x,y,z)=x^2+y^2=1$

$g_2(x,y,z)=x^2-xy+y^2-z^2=1$

To tilfeller:

1 Hovedtilfellet: $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$

2 Bitilfellet: ∇g_1 og ∇g_2 er lin. avh.

Ser på 1: $\nabla f = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$, $\nabla g_1 = \begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix}$, $\nabla g_2 = \begin{pmatrix} 2x-y \\ 2y-x \\ -2z \end{pmatrix}$

$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$ blir nå: $\begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2x-y \\ 2y-x \\ -2z \end{pmatrix}$

dvs

$2x = 2\lambda x + 2\mu x - \mu y$

$2y = 2\lambda y + 2\mu y - \mu x$

$2z = -2\mu z$ $\Rightarrow$
- $z=0$:
  - $x=0 \Rightarrow y=\pm 1$ $(\pm 1, 0, 0)$
  - $y=0 \Rightarrow x=\pm 1$ $(0,\pm 1,0)$
- $z \neq 0 \Rightarrow \mu = -1$:

$2x = 2\lambda x - 2x + y \Rightarrow y = (4-2\lambda)x$

$2y = 2\lambda y - 2y + x \Rightarrow x = (4-2\lambda)y$

$\Downarrow$

$\frac{y}{x} = \frac{\cancel{(4-2\lambda)}x}{\cancel{(4-2\lambda)}y}$

$y^2 = x^2$

$y = \pm x$

$\begin{cases} x^2+y^2=1 \\ x^2-xy+y^2-z^2=1 \end{cases}$

Forenklende regning.

$\boxed{\begin{matrix} xy+z^2=0 \\ z^2=-xy \end{matrix}}$

Bruker at $x^2+y^2=1$

$x^2+x^2=1$

$x^2=\frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt2} = \pm\frac{\sqrt2}{2}$

$y = \pm\frac{\sqrt2}{2}$

$z^2 = -xy = -\left(\pm\frac{1}{\sqrt2}\right)\left(\mp\frac{1}{\sqrt2}\right) = \frac{1}{2}$

Kandidater til minimalpunkt:

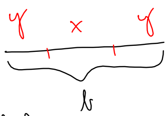
$(\pm1,0,0)$, $(0,\pm1,0)$, $\left(\pm\frac{\sqrt2}{2}, \mp\frac{\sqrt2}{2}, \pm\frac{\sqrt2}{2}\right)$

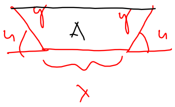
Sjekker funksjonsverdier: $f(x,y,z)=x^2+y^2+z^2$

$(\pm1,0,0)$, $(0,\pm1,0)$: Funksjonsverdi 1: Minimal avstand: $(\pm1,0,0)$ $(0,\pm1,0)$

$\left(\pm\frac{\sqrt2}{2}, \mp\frac{\sqrt2}{2}, \pm\frac{\sqrt2}{2}\right)$: Funksjonsverdi: $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=\frac{3}{2}$

5.10.13:

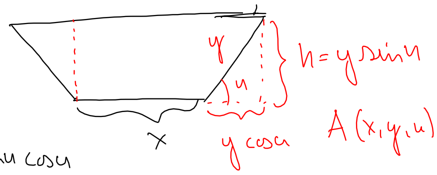




Vil få størst mulig areal

Bibetingelse: $\underline{x+2y=b}$

Areal:



$A(x,y,u) = xy\sin u + y^2\sin u\cos u$

$A(x,y,u) = (x+y\cos u)(y\sin u)$
$= xy\sin u + y^2\sin u\cos u$

Bibetingelsen

$$g(x,y,u) = x+2y = b$$

Ser etter punkter der: $\nabla A = \lambda \nabla g$

$$\nabla A = \begin{pmatrix} y\sin u \\ x\sin u + 2y\sin u\cos u \\ xy\cos u + y^2(\cos^2 u - \sin^2 u) \end{pmatrix} \qquad \nabla g = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

dvs

$$y\sin u = \lambda$$
$$x\sin u + 2y\sin u\cos u = 2\lambda$$
$$x\cancel{y}\cos u + y^{\cancel{2}}(\cos^2 u - \underbrace{\sin^2 u}_{(1-\cos^2 u)}) = 0$$
$$x+2y = b$$

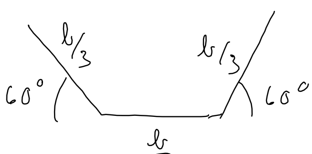
Setter $\lambda = y\sin u$ inn i ligning 2:

$$x\cancel{\sin u} + 2y\cancel{\sin u}\cos u = 2y\cancel{\sin u}$$
$$x + 2y\cos u = 2y \Rightarrow \underline{x = 2y(1-\cos u)}$$
$$x\cos u + y(2\cos^2 u - 1) = 0$$
$$x+2y = b$$

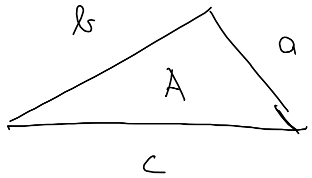
$$x\cos u = y(1-2\cos^2 u)$$
$$\cos u = \frac{\cancel{y}(1-2\cos^2 u)}{2\cancel{y}(1-\cos u)}$$

$$2(1-\cos u)\cos u = 1 - 2\cos^2 u$$
$$2\cos u - \cancel{2\cos^2 u} = 1 - \cancel{2\cos^2 u}$$
$$\cos u = \tfrac{1}{2} \Rightarrow \underline{u = \tfrac{\pi}{3}}$$

$$x = 2y(1-\overbrace{\cos u}^{\frac{1}{2}}) \Rightarrow x = y$$
$$x+2y = b \Rightarrow 3x = b, \ \underline{x = \tfrac{b}{3}}$$
$$\underline{y = \tfrac{b}{3}}$$



5.10.14 Heronsformel



$$s = \frac{a+b+c}{2}$$

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

Gitt omkretsen (dvs s), finn den trekanten som har maksimalt areal

Variable: a, b, c Bibetingelse $\overbrace{a+b+c}^{g(a,b,c)} = 2s$

Maksimere $A(a,b,c) = s(s-a)(s-b)(s-c)$

Lagrange: $\nabla A = \lambda \nabla g$

$$\nabla A = \begin{pmatrix} -s(s-b)(s-c) \\ -s(s-a)(s-c) \\ -s(s-a)(s-b) \end{pmatrix} \qquad \nabla g = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\left.\begin{aligned} -s(s-b)(s-c) &= \lambda \\ -s(s-a)(s-c) &= \lambda \\ -s(s-a)(s-b) &= \lambda \end{aligned}\right\} \Rightarrow \frac{\cancel{-s}(s-b)\cancel{(s-c)}}{\cancel{-s}(s-a)\cancel{(s-c)}} = \frac{\cancel{\lambda}}{\cancel{\lambda}} \Rightarrow \frac{s-b}{s-a} = 1$$

$$s-b = s-a \Rightarrow \underline{\underline{a=b}}$$

$$\frac{\cancel{-s}(\cancel{s-a})(s-c)}{\cancel{-s}(\cancel{s-a})(s-b)} = \frac{\cancel{\lambda}}{\cancel{\lambda}} \qquad \frac{s-c}{s-b} = 1$$

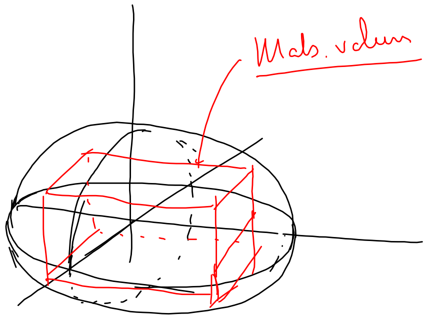
$$s-c = s-b \Downarrow b = c$$

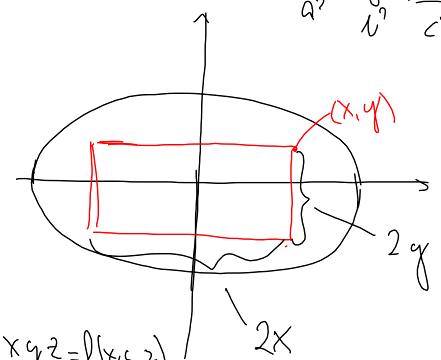
$$\underbrace{a+b+c = 2s}$$

$$\underline{3a = 2s \Rightarrow a = \frac{2}{3}s}$$

Altså: $a=b=c$, likesidet trekant.

5.10.11 Ellipsoide: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$





Volum $2x\cdot 2y\cdot 2z = 8xyz = f(x,y,z)$

Bibetingelse: $\underbrace{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}_{g(x,y,z)}=1$

$\nabla f=\lambda\nabla g$:

$$\nabla f=\begin{pmatrix}8yz\\8xz\\8xy\end{pmatrix},\quad \nabla g=\begin{pmatrix}\frac{2x}{a^2}\\\frac{2y}{b^2}\\\frac{2z}{c^2}\end{pmatrix}$$

$$8yz=2\lambda\frac{x}{a^2}$$
$$8xz=2\lambda\frac{y}{b^2}$$
$$8xy=2\lambda\frac{z}{c^2}$$
$$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1}$$

$$\Longrightarrow \underbrace{\frac{y}{x}=\frac{\cancel{\frac{x}{a^2}}}{\cancel{\frac{y}{b^2}}}=\frac{xb^2}{ya^2}}$$

$$\Downarrow$$

$$\frac{y^2}{b^2}=\frac{x^2}{a^2}$$

$$\frac{z}{y}=\frac{\cancel{\frac{y}{b^2}}}{\cancel{\frac{z}{c^2}}}=\frac{yc^2}{zb^2}$$

$$\frac{z^2}{c^2}=\frac{y^2}{b^2}=\frac{x^2}{a^2}$$

$$\Downarrow$$

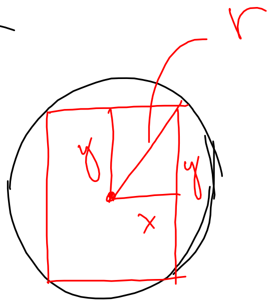
$$\frac{3x^2}{a^2}=1$$

$$x^2=\frac{a^2}{3}\Rightarrow x=\frac{a}{\sqrt{3}}$$

$$y=\frac{b}{\sqrt{3}}$$

$$z=\frac{c}{\sqrt{3}}$$

Oppgave 12



Bæreevne $B(x,y) = kxy^2$

Bibetingelse: $\underbrace{x^2+y^2}_{g(x,y)} = r^2$

$$\nabla B = \lambda \nabla g$$

$$\nabla B = \begin{pmatrix} ky^2 \\ 2kxy \end{pmatrix} \qquad \nabla g = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

$$ky^2 = 2\lambda x$$

$$2kxy = 2\lambda y \Longrightarrow kx = \lambda$$

$$x^2+y^2=r^2$$

$$\left.\begin{matrix} ky^2 = 2\lambda x \\ kx = \lambda \end{matrix}\right\} \Rightarrow \cancel{k}y^2 = 2(\cancel{k}x)x$$

$$x^2+y^2=r^2 \qquad y^2 = 2x^2$$

$$x^2+2x^2=r^2 \qquad 3x^2=r^2 \Rightarrow x = \frac{r}{\sqrt{3}}$$

$$y = \sqrt{2}x = \frac{\sqrt{2}r}{\sqrt{3}}$$