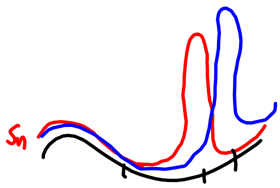
Rekker av funksjoner

Skal se på $\sum_{n=0}^{\infty} \underbrace{a_n (x-a)^n}_{V_n(x)}$ $a_i, a \in \mathbb{R}$

Veldig ofte : $a = 0$

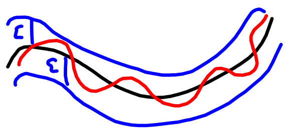
n-te delsum: $\sum_{m=0}^{n} V_m(x)$

Konvergens : Punktvis konv. $\underbrace{\sum_{n=0}^{N} V_n(x)}_{S_N(x)} \to V(x)$



$\forall x \in A, \forall \varepsilon > 0 ; \exists N$ s.a.
$|S_n(x) - V(x)| < \varepsilon$ når $n \geq N$

Uniform konv. $S_N(x) \to V(x)$



$\forall \varepsilon > 0 ; \exists N$ s.a $\forall x \in A$
$|S_n(x) - V(x)| < \varepsilon$ når $n \geq N$

Konvergens av potensrekker

$\sum_{n=0}^{\infty} a_n x^n$

Tre muligheter: 1) Konvergens overalt
2) Konvergens kun for $x = 0$
3) $\exists r > 0$, konvergensradius
$|x| < r$ Absolutt konvergens
$|x| > r$ Divergens

Eks.

1) $\sum (n+1)x^n$

Forholdstesten $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+2)\,x^{n+1}}{n+1\;\;x^n}\right| = \frac{n+2}{n+1}|x| \xrightarrow[n\to\infty]{} |x| < 1$

Tilfelle 3)

Konvergens-radius $r=1$

2) $\sum \frac{x^n}{n!}$

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)\cancel{!}} \cdot \frac{\cancel{n!}}{x^n}\right| = \frac{1}{n+1}|x| \xrightarrow[n\to\infty]{} 0 < 1$

Konvergens overalt

Tilfelle 1)

3) $\sum n^n x^n$

$|a_n|^{\frac{1}{n}} = |nx^n|^{\frac{1}{n}} = n\,|x| \xrightarrow[n\to\infty]{} \infty$

Konvergens kun for $x=0$

Tilfelle 2)

Samlet:	$\sum(n+1)x^n$,	$\sum \frac{x^n}{n!}$,	$\sum n^n x^n$
	$\|x\|<1$	alle x	$x=0$

Konsekvenser:

1) Konvergensradius $r>0$ + uniform konvergens for $|x|<r$
$\Rightarrow S(x) = \sum_{n=0}^{\infty} a_n x^n$ er kontinuerlig på $|x|<r$

2) $f(x) = \sum_{n=0}^{\infty} a_n x^n$ med konv. radius r ($|x|<r$ ok)
$\Rightarrow \int_a^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ for $|x|<r$

3) $f(x) = \sum_{n=0}^{\infty} a_n x^n$ $|x|<r$ (konv. radius)
$\Rightarrow f$ er deriverbar og $f'(x) = \sum_{n=1\,(n=0)}^{\infty} n a_n x^{n-1}$ $|x|<r$

Eks

$$f(x) = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots = \sum_{n=0}^{\infty} (-1)^n x^n \qquad |x|<1$$

Integrasjon:

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^n\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

Sett $x=1$

$$\ln(1+1) = \ln 2 = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \dots$$

Ta utgangspunkt i $g(x) = \dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

Konv. område $|x|<1$

$$\int_0^x g(t)\,dt = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n}\,dt = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n}\,dt$$

$$\arctan(x) = \int_0^x \frac{dt}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1} \qquad |x|<1$$

setter $x=1$ (Konvergens)

$$\arctan(1) = \boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots}$$

Taylor-rekker

Gitt $f : \mathbb{R} \to \mathbb{R}$ (C^∞)

$$Tf(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$ Taylorrekka til f.

$x=0$

Eks

$f(x) = \frac{1}{1+x} = (1+x)^{-1}$ $f(0) = 1$

$f'(x) = -(1+x)^{-2}$ $f'(0) = -1$

$f''(x) = 2(1+x)^{-3}$ $f''(0) = 2$

$f^{(3)}(x) = -2\cdot 3(1+x)^{-4}$ $f^{(3)}(0) = -2\cdot3 = -6$

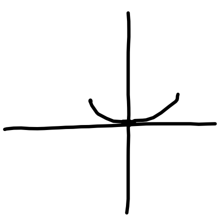
$\vdots$

$f^{(k)}(x) = (-1)^k (1+x)^{-(k+1)} \cdot k!$ $f^{(k)}(0) = (-1)^k k!$

$$\Rightarrow Tf(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \cancel{k!}}{\cancel{k!}} x^k = \sum_{k=0} (-1)^k x^k = \sum_{k=0} (-x)^k = f(x)$$

Liten oppgave:
La $f(x) = \sum_{n=0}^{\infty} a_n x^n$.
Vis at $Tf(x) = f(x)$

Nesten alltid
$Tf(x) = f(x)$
(men ikke alltid!)



Motekse mpel:
$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Men $Tf(x) = 0$

Hvorfor?

$$\frac{d}{dx} e^{-\frac{1}{x^2}} = \frac{2}{x^3} e^{-\frac{1}{x^2}} \qquad x \neq 0$$

$$\lim_{x \to 0} \frac{2}{x^3} e^{-\frac{1}{x^2}} = ?$$

Setter $y = \frac{1}{x}$

$$\lim_{y \to \infty} 2y^3 e^{-y^2} = \lim_{y \to \infty} \frac{2y^3}{e^{y^2}}$$

$$= \lim_{y \to \infty} \frac{6y^2}{2y e^{y^2}} = \lim_{y \to \infty} \frac{3y}{e^{y^2}}$$

$$= \lim_{y \to \infty} \frac{3}{2y e^{y^2}} = 0$$