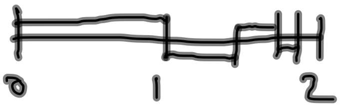
Lindstrøms Kalkulus XII : Rekker

Eksempler

$$0{,}9999\ldots = 1$$

$$\| $$

$$\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10{,}000} + \ldots$$



$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} + \ldots = 2$$

geometrisk rekke

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} + \ldots = +\infty$$

divergent rekke

harmonisk rekke

$$\zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots + \frac{1}{n^2} + \ldots = \frac{\pi^2}{6}$$

L. Euler 1735

$$\zeta(3) = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \ldots + \frac{1}{n^3} + \ldots$$

konvergerer

Apéry 1978
irrasjonalt tall

Geometrisk rekke $r \in \mathbb{R}$

$$1+r+r^2+\dots+r^n = \begin{cases} \frac{1-r^{n+1}}{1-r} & \text{hvis } r \neq 1 \\ n+1. & \text{hvis } r=1 \end{cases}$$

Bevis ved induksjon for $n \geq 0$

$\underline{n=0}$ $1 = \frac{1-r^1}{1-r}$ ✓

$$1+r+\dots+r^{n-1} + r^n = \frac{1-r^n}{1-r} + r^n = \frac{1-r^n-(1-r)r^n}{1-r} = \frac{1-r^{n+1}}{1-r} \quad \square.$$

$n \geq 1.$

$$\lim_{n\to\infty}(1+r+\dots+r^n) = \begin{cases} \frac{1}{1-r} & \text{hvis } |r|<1 \\ \text{divergerer} & \text{hvis } |r| \geq 1. \end{cases}$$

$$\lim_{n\to\infty}\sum_{k=0}^{n} r^k = \sum_{k=0}^{\infty} r^k$$

$$1+r+r^2+\dots+r^n+\dots = \frac{1}{1-r} \quad \text{for } |r|<1$$

$r = -x^2$

$$1-x^2+x^4-\dots+(-x^2)^n+\dots = \frac{1}{1+x^2} \quad \text{for } |x|<1.$$

$$\int_0^x 1-x^2+\dots+(-x^2)^n\,dx = \int_0^x \frac{1-(-x^2)^{n+1}}{1+x^2}\,dx$$

$$x-\frac{x^3}{3}+\frac{x^5}{5}-\dots\pm\frac{x^{2n+1}}{2n+1} = \dots \quad \text{Kap 11}$$

$$x-\frac{x^3}{3}+\frac{x^5}{5}-\dots+(-1)^n\frac{x^{2n+1}}{2n+1}+\dots = \int_0^x \frac{dx}{1+x^2} \quad |x| \leq 1$$

$= \arctan x$

$x=1$ Nils Abel 12.6.9

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots+\frac{(-1)^n}{2n+1}+\dots = \arctan 1 = \frac{\pi}{4}$$

$$\int_0^x \frac{dx}{\sqrt{x^3-x-1}}$$

La $\{a_n\}_{n=0}^{\infty}$ være en <u>følge</u> i $\mathbb{R}$

$(a_0, a_1, a_2, \ldots)$ sequence

Danner summene

$$s_0 = a_0$$
$$s_1 = a_0 + a_1$$
$$s_2 = a_0 + a_1 + a_2$$
$$\vdots$$
$$s_k = a_0 + a_1 + \ldots + a_k = \sum_{n=0}^{k} a_n$$

series

Følgen $\{s_k\}_{k=0}^{\infty}$ av summer kalles en <u>rekke</u>

Hva betyr $\sum_{n=0}^{\infty} a_n$? $\lim_{k\to\infty} s_k = \lim_{k\to\infty} \sum_{n=0}^{k} a_n$

<u>Def</u> Hvis $s_k = \sum_{n=0}^{k} a_n \longrightarrow A$ når $k \to \infty$

sier vi at rekken $\sum_{n=0}^{\infty} a_n$ <u>konvergerer</u>

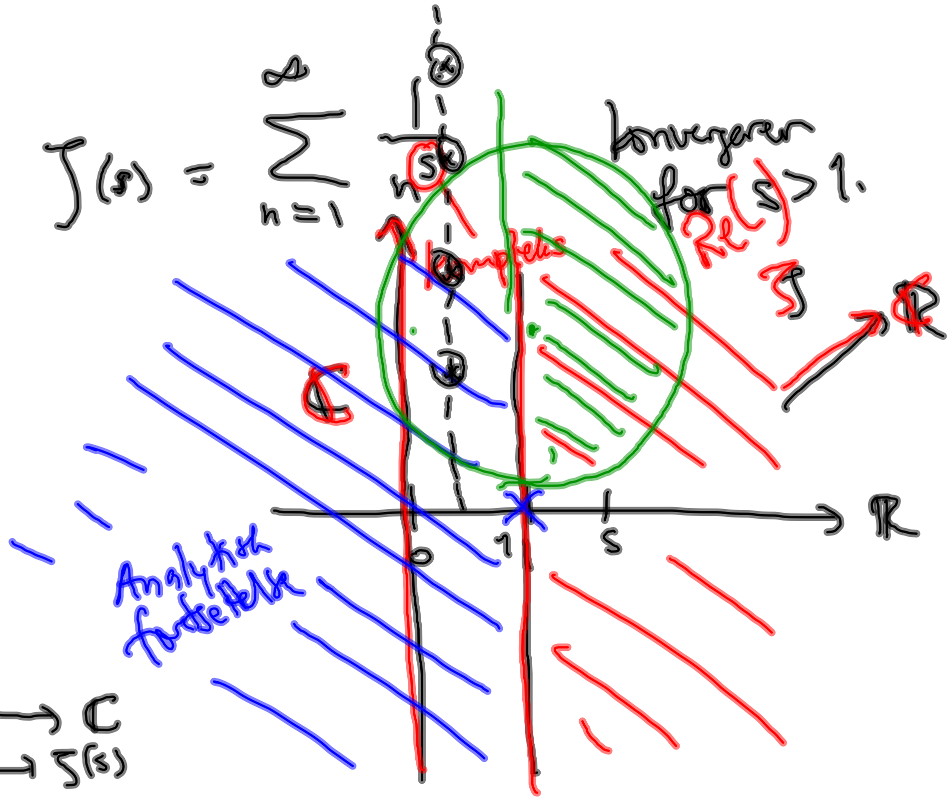
og skriver $\sum_{n=0}^{\infty} a_n = A$.

Ellers sier vi at rekken <u>divergerer</u>

Eksempel

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$ konvergerer for $\mathrm{Re}(s) > 1$.

Riemann



$$\zeta : \mathbb{C} \setminus \{1\} \longrightarrow \mathbb{C}$$
$$s \longmapsto \zeta(s)$$

Riemann-hypotesen

Hvis $\zeta(s) = 0$ og $\mathrm{Re}(s) \in (0,1)$

er $\mathrm{Re}(s) = \frac{1}{2}$.

Setning Hvis $\sum_{n=0}^{\infty} a_n = A$ konvergerer og $c \in \mathbb{R}$ vil $\sum_{n=0}^{\infty} ca_n = cA$ konvergere.

Hvis også $\sum_{n=0}^{\infty} b_n = B$ konvergerer vil

$$\sum_{n=0}^{\infty} (a_n + b_n) = A + B$$ konvergere.

Lemma Hvis $\sum_{n=0}^{\infty} a_n = A$ konvergerer

må $a_n \longrightarrow 0$ når $n \to \infty$.

Bevis

$$s_k = \sum_{n=0}^{k} a_k \longrightarrow A \quad \text{når } k \to \infty$$

$$s_{k-1} = \sum_{n=0}^{k-1} a_k \longrightarrow A \quad \text{når } k \to \infty.$$

$$a_k = s_k - s_{k-1} \longrightarrow A - A = 0 \quad \text{når } k \to \infty$$

$\square$

<u>Merk</u> Omvendingen er <u>ikke</u> generelt riktig:

Kan ha $a_n \to 0$ uten at $\sum_{n=0}^{\infty} a_n$ konvergerer.

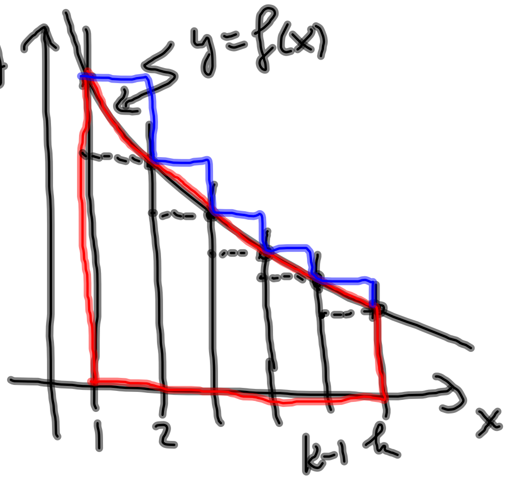
<u>Sats</u> $\sum_{n=1}^{\infty} \frac{1}{n}$ divergerer.

$$1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{\geq \frac{2}{4}} + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \dots + \frac{1}{16}\right) + \dots$$

$$\geq 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \frac{8}{16} + \dots$$ som vokser over alle grenser.

Integraltesten

La $f : [1, \infty) \longrightarrow \mathbb{R}$
være en positiv, kontinuerlig og avtagende. Da er



$$\sum_{n=2}^{k} f(n) \leq \int_1^k f(x)dx \leq \sum_{n=1}^{k-1} f(n)$$

så rekken $\sum_{n=1}^{\infty} f(n)$ konvergerer hvis og bare hvis $\int_1^{\infty} f(x)dx$ konvergerer.

<u>Eks</u> $f(x) = \frac{1}{x^s} = x^{-s}$ der $s > 0$.

$$\int_1^k f(x)\,dx = \int_1^k x^{-s}\,dx = \left[\frac{1}{1-s} x^{1-s}\right]_1^k \quad s \neq 1$$

$$= \frac{1}{1-s} k^{1-s} - \frac{1}{1-s}$$

$$\longrightarrow \begin{cases} \text{divergerer} & 1-s>0 \\ \frac{1}{s-1} & 1-s<0, \Updownarrow\ s>1 \end{cases}$$

<u>Setning</u>

$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ konvergerer hvis og bare hvis $s > 1$.

Integraltesten for $s=1$

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ div.} \iff \int_1^{\infty} \frac{1}{x}dx \text{ div.}$$

$$= \lim_{k\to\infty} \left[\ln x\right]_1^k$$

$$= \lim_{k\to\infty} \ln k = +\infty.$$

Forholdstesten

La $\sum_{n=0}^{\infty} a_n$ være en rekke og anta

at $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = b$ eksisterer.

Dersom $\boxed{b<1 \text{ er rekken konvergent.}}$

Dersom $b>1$ er rekken divergent.

Dersom $b=1$ gir testen ingen konklusjon.

Anta $a_n > 0$. Hvis $b<1$ finnes r med

$b < r < 1$ og en N slik at

$$\frac{a_{n+1}}{a_n} < r \quad \text{for alle } n \geq N.$$

$$a_{N+1} \leq r a_N, \quad a_{N+2} \leq r a_{N+1} \leq r^2 a_N$$

$$a_{N+i} \leq r^i a_N \quad \text{for } i \geq 0.$$

$\sum_{n=0}^{\infty} a_n$ konvergerer $\iff$ $\sum_{n=N}^{\infty} a_n$ konvergerer

$$\sum_{n=N}^{N+k} a_n = \sum_{i=0}^{k} a_{N+i} \leq \sum_{i=0}^{k} r^i a_N \longrightarrow \frac{a_N}{1-r} \text{ når } k\to\infty$$

$\downarrow$

konvergerer når $k\to\infty$

<u>Eks</u> (MAT1110, 2009)

$x \in \mathbb{R}$

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{\sqrt{n^2+n}}$$ for hvilke x konvergerer rekken?

$$= \underbrace{\frac{(x-2)}{\sqrt{2}}}_{a_1(x)} + \underbrace{\frac{(x-2)^2}{\sqrt{6}}}_{a_2(x)} + \underbrace{\frac{(x-2)^3}{\sqrt{12}}}_{a_3(x)} + \dots$$

Forholdene er

$$\left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \left|\frac{(x-2)^{n+1}}{\sqrt{n^2+3n+2}} \frac{\sqrt{n^2+n}}{(x-2)^n}\right|$$

$$= |x-2| \sqrt{\frac{n^2+n}{n^2+3n+2}} \to |x-2| = b$$ når $n \to \infty$.

$$\left(\frac{n^2+n}{n^2+3n+2} = \frac{1+1/n}{1+3/n+2/n^2} \to \frac{1+0}{1+0+0} = 1 \text{ når } n \to \infty \right)$$

$\therefore$ Hvis $1 < x < 3 \iff |x-2| < 1$ konvergerer rekken.

Hvis $x < 1$ eller $x > 3 \iff |x-2| > 1$ divergerer rekken.

For $x = 3$ er rekken

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}} \geq \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{1}{n} \to \infty$$

$$\frac{1}{\sqrt{n^2+n}} \overset{?\checkmark}{\geq} \frac{1}{\sqrt{2}\,n}$$

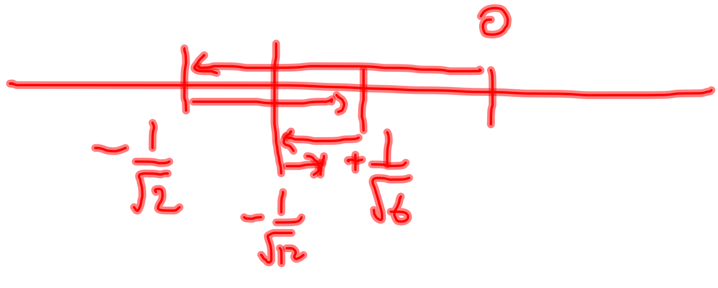
$$n^2+n \overset{\checkmark}{\leq} 2n^2$$

divergent

For $x = 1$ er rekken

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+n}} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{12}} + - + \ldots$$

alternerende med ledd med absoluttverdi som avtar mot null $\Rightarrow$ konvergent



$\therefore$ Konvergensområdet er $[1,3)$.