6.3.1 g)

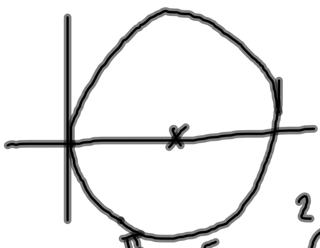
$$I=\iint_R (x^2+y^2)^{\frac{3}{2}}\,dx\,dy \qquad R:\ (x-1)^2+y^2\le 1$$

$$x=r\cos\theta,\ y=r\sin\theta:\ (r\cos\theta-1)^2+(r\sin\theta)^2\le 1$$

$$r^2\cos^2\theta-2r\cos\theta+1+r^2\sin^2\theta\le 1$$

$$r^2-2r\cos\theta\le 0$$

$$r\le 2\cos\theta$$



$$-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$$

$$I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int_0^{2\cos\theta}(\underbrace{x^2+y^2}_{r^2})^{\frac{3}{2}}\,r\,dr\right]d\theta=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int_0^{2\cos\theta}r^4\,dr\right]d\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac{1}{5}r^5\right]_0^{2\cos\theta}d\theta=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{32}{5}\cos^5\theta\,d\theta=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{32}{5}\cos^4\theta\cos\theta\,d\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{32}{5}(1-\sin^2\theta)^2\cos\theta\,d\theta=\int_{-1}^{1}\frac{32}{5}(1-u^2)^2\,du=\int_{-1}^{1}\frac{32}{5}(1-2u^2+u^4)\,du$$

$u=\sin\theta$

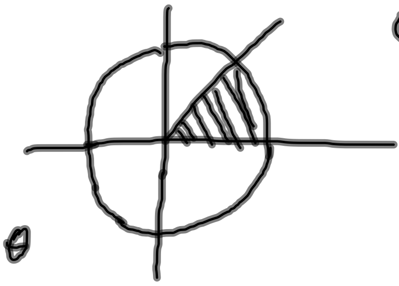
$$=\dots=\frac{512}{75}$$

6.3.1 d)

$$\iint_R xy\,dx\,dy$$

$$= \int_0^{\frac{\pi}{4}} \left[\int_0^1 r\cos\theta\, r\sin\theta\, r\, dr \right] d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left[\int_0^1 r^3 \frac{1}{2}\sin 2\theta\, dr \right] d\theta = \cdots = \frac{1}{16}$$



$0 \le \theta \le \frac{\pi}{4}$

$0 \le r \le 1$

6.3.4 $\alpha \le \theta \le \beta \qquad 0 \le r \le r(\theta)$

$$A = \iint_R dx\,dy = \int_\alpha^\beta \left[\int_0^{r(\theta)} r\,dr\right] d\theta = \int_\alpha^\beta \left[\frac{1}{2} r^2\right]_0^{r(\theta)} d\theta$$

$$= \int_\alpha^\beta \frac{1}{2} r(\theta)^2\, d\theta$$

$2\cos^2 x - 1$

$\cos 2x = 1 - 2\sin^2 x$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$r(\theta) = \sin 2\theta \qquad \theta \in \left[0, \frac{\pi}{2}\right]$

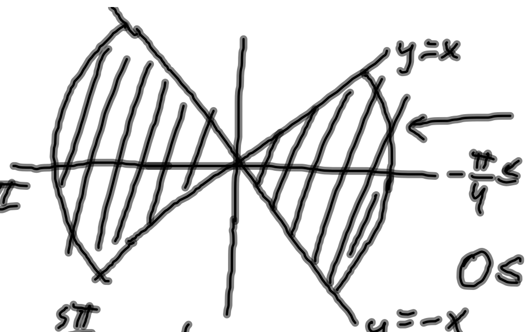
$$A = \int_0^{\frac{\pi}{2}} \frac{1}{2} \sin^2 2\theta\, d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2} \cdot \frac{1}{2}(1 - \cos 4\theta)\, d\theta =$$

$\sin 2x = 2\sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$$= \frac{1}{4}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{2}} = \frac{1}{4} \cdot \frac{\pi}{2} = \underline{\underline{\frac{\pi}{8}}}$$

6.4.1 e) $0 \le z \le x^2 - y^2$



$\frac{3\pi}{4} \le \theta \le \frac{5\pi}{4}$

$0 \le r \le 1$

$-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$

$0 \le r \le 1$

$$V = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\int_0^1 (x^2 - y^2) r \, dr \right] d\theta + \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \left[\int_0^1 (x^2 - y^2) r \, dr \right] d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\int_0^1 r^3 \cos 2\theta \, dr \right] d\theta + \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \left[\int_0^r r^3 \cos 2\theta \, dr \right] d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{4} \cos 2\theta \, d\theta + \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \frac{1}{4} \cos 2\theta \, d\theta = \left[\frac{1}{8} \sin 2\theta \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} + \left[\frac{1}{8} \sin 2\theta \right]_{\frac{3\pi}{4}}^{\frac{5\pi}{4}}$$

$$= \underline{\underline{\frac{1}{8}}}$$

6.4.5 $z = x^2 - y^2$ $\qquad x^2 + y^2 \leq 4$

$$\frac{\partial f}{\partial x} = 2x \qquad \frac{\partial f}{\partial y} = -2y \qquad \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} = \sqrt{1 + 4x^2 + 4y^2}$$

$$A = \iint_R \sqrt{1 + 4x^2 + 4y^2}\, dx\, dy = \frac{1}{8}\int_0^{2\pi} \left[\int_0^2 \sqrt{\underbrace{1 + 4r^2}_{u = 1+4r^2}}\, 8r\, dr \right] d\theta \qquad du = 8r\,dr$$

$$= \frac{1}{8}\int_0^{2\pi} \left[\frac{2}{3}(1 + 4r^2)^{\frac{3}{2}} \right]_0^2 d\theta = \frac{1}{8}\int_0^{2\pi} \left(\frac{2}{3} 17^{\frac{3}{2}} - \frac{2}{3} \right) d\theta$$

$$= \frac{1}{8} 2\pi \left(\frac{34}{3}\sqrt{17} - \frac{2}{3} \right) = \underline{\underline{\frac{\pi}{6}(17\sqrt{17} - 1)}}$$

6.4.6 $z^2 = x^2 + y^2$ $\quad 0 \le z \le 1 \Rightarrow 0 \le r \le 1$

$z = \pm r$

$$\vec{r}(r,\theta) = (r\cos\theta,\ r\sin\theta,\ r)$$

$$\frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \theta} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = -r\cos\theta\,\vec{i} - r\sin\theta\,\vec{j} + r\vec{k}$$

$$\left| \frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \theta} \right| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{r^2 + r^2} = r\sqrt{2}$$

$$A = \int_0^{2\pi} \left[\int_0^1 r\sqrt{2}\, dr \right] d\theta = \int_0^{2\pi} \left[\frac{1}{2} r^2 \sqrt{2} \right]_0^1 d\theta = 2\pi \cdot \frac{1}{2}\sqrt{2} = \underline{\underline{\pi\sqrt{2}}}$$

6.4.8

$$\vec{r}(r,\theta) = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} + f(r,\theta)\,\vec{k}$$

$$\frac{\partial\vec{r}}{\partial r}\times\frac{\partial\vec{r}}{\partial\theta} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & \frac{\partial f}{\partial r} \\ -r\sin\theta & r\cos\theta & \frac{\partial f}{\partial\theta} \end{vmatrix} = \left(\frac{\partial f}{\partial\theta}\sin\theta - \frac{\partial f}{\partial r}r\cos\theta\right)\vec{i} + \left(-\frac{\partial f}{\partial r}r\sin\theta - \frac{\partial f}{\partial\theta}\cos\theta\right)\vec{j} + r\vec{k}$$

$$\left|\frac{\partial\vec{r}}{\partial r}\times\frac{\partial\vec{r}}{\partial\theta}\right| = \sqrt{\left(\frac{\partial f}{\partial\theta}\sin\theta - \frac{\partial f}{\partial r}r\cos\theta\right)^2 + \left(-\frac{\partial f}{\partial r}r\sin\theta - \frac{\partial f}{\partial\theta}\cos\theta\right)^2 + r^2}$$

$$= \sqrt{\underline{\left(\frac{\partial f}{\partial\theta}\right)^2\sin^2\theta} + r^2\left(\frac{\partial f}{\partial r}\right)^2\cos^2\theta + r^2\left(\frac{\partial f}{\partial r}\right)^2\sin^2\theta + \underline{\left(\frac{\partial f}{\partial\theta}\right)^2\cos^2\theta} + r^2}$$

$$= \sqrt{\left(\frac{\partial f}{\partial\theta}\right)^2 + r^2\left(\frac{\partial f}{\partial r}\right)^2 + r^2} = \underline{\underline{\sqrt{1 + \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial\theta}\right)^2}\; r}}$$

6.4.9

$z = x^2 + y^2 = r^2 \qquad x^2+y^2 \le 1$

$\frac{\partial f}{\partial r} = 2r \qquad \frac{\partial f}{\partial \theta} = 0$

$\sqrt{1 + \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2}\, r = \sqrt{1+4r^2}\, r$

$\iint_T x^2\, dS = \int_0^{2\pi} \left[\int_0^1 x^2 \sqrt{1+4r^2}\, r\, dr \right] d\theta$

$= \int_0^{2\pi} \left[\int_0^1 \sqrt{1+4r^2}\, r^3 \cos^2\theta\, dr \right] d\theta$

$= \int_0^{2\pi} \left[\int_0^1 8r\sqrt{1+4r^2}\, \frac{1}{8} r^2 \cos^2\theta\, dr \right] d\theta$

$= \int_0^{2\pi} \left[\left[\frac{2}{3}(1+4r^2)^{\frac{3}{2}} \frac{1}{8} r^2 \cos^2\theta \right]_0^1 - \int_0^1 \frac{2}{3}(1+4r^2)^{\frac{3}{2}} \frac{1}{8} 2r \cos^2\theta \right] d\theta$

$= \int_0^{2\pi} \left(\frac{2}{3} 5\sqrt{5} \cdot \frac{1}{8} \cos^2\theta - \int_0^1 \frac{1}{6 \cdot 8} (1+4r^2)^{\frac{3}{2}} 8r \cos^2\theta \right] d\theta$

$= \int_0^{2\pi} \left(\frac{5}{12}\sqrt{5}\cos^2\theta - \left[\frac{1}{48} \cdot \frac{2}{5} (1+4r^2)^{\frac{5}{2}} \cos^2\theta \right]_0^1 \right) d\theta \qquad \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$= \cdots = \frac{\pi}{120}(25\sqrt{5}+1)$

6.4.18 $2x + 4y - z = -4$ $\quad z = x^2 + y^2$

$z = 2x + 4y + 4$

skjæring: $2x + 4y + 4 = x^2 + y^2$

$x^2 - 2x + 1 + y^2 - 4 + 4 = 4 + 1 + 4$

$(x-1)^2 + (y-2)^2 = 9$

sirkel med radius 3, senter i $(1,2)$

= området D fra oppgaven ($(x-1)^2 + (y-2)^2 \leq 9$)

sett inn $(x = y = 0)$: $1^2 + 2^2 = 5 \leq 9$, så $(0,0)$ ligger i D.

planet i $x = y = 0$: $z = 2x + 4y + 4 = 4$

paraboloiden: $z = x^2 + y^2 = 0$, så planet er øverst.

$$\text{Volum} = \iint_D \big(\underbrace{2x + 4y + 4}_{\text{plan}} \underbrace{- x^2 - y^2}_{\text{paraboloide}} \big)\, dx\, dy$$

b) Gjør variabelskifte $u = x-1 \qquad v = y-2$

$D: (x-1)^2 + (y-2)^2 \leq 9 \quad \Leftrightarrow \quad u^2 + v^2 \leq 9$

Jacobideterminant for $(x,y) \to (u,v)$ blir 1.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$u = r\cos\theta$
$v = r\sin\theta$

bruk polarkoordinater:

$$V = \int_0^{2\pi}\int_0^3 (2x+4y+4-x^2-y^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^3 2(u+1)+4(v+2)+4-(u+1)^2-(v+2)^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^3 (9-u^2-v^2)\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^3 (9-r^2)\, r\, dr\, d\theta$$

$$= \dots = \frac{81\pi}{2}$$