3.3.9

$\vec{r}_1$ og $\vec{r}_2$ har motsatt orientering: $\vec{r}_1 : [a,b] \to \mathbb{R}^n$

$\vec{r}_2 : [c,d] \to \mathbb{R}^n$

$\vec{r}_1(a) = \vec{r}_2(d) \qquad \vec{r}_1(b) = \vec{r}_2(c)$

$\vec{r}_2(\phi(t)) = \vec{r}_1(t) \Rightarrow \phi(a) = d\ ,\ \phi(b) = c$

$\Rightarrow \phi$ avtagende $\Rightarrow \phi'(t) \leq 0$

Bevis for Setning 3.3.5

$$I_1 = \int_C f\,ds = \int_a^b f(\vec{r}_1(t))\,v_1(t)\,dt$$

Som i beviset: $\vec{r}_2(\phi(t)) = \vec{r}_1(t) \Rightarrow \vec{v}_2(\phi(t))\phi'(t) = \vec{v}_1(t)$

$$\Rightarrow v_1(t) = |\phi'(t)|\,v_2(\phi(t)) \Rightarrow v_1(t) = -v_2(\phi(t))\phi'(t)$$

$$I_1 = \int_a^b f(\vec{r}_1(t))\,v_1(t)\,dt = -\int_a^b f(\vec{r}_2(\phi(t)))\,v_2(\phi(t))\,\phi'(t)\,dt$$

$u = \phi(t)$: $du = \phi'(t)\,dt$

$$= -\int_d^c f(\vec{r}_2(u))\,v_2(u)\,du = \int_c^d f(\vec{r}_2(u))\,v_2(u)\,du$$

$= I_2 \Rightarrow$ orienteringer som er motsatte gir samme svar.

3.3.12

a) $x = r(\theta)\cos\theta = f(\theta)\cos\theta$

$y = r(\theta)\sin\theta = f(\theta)\sin\theta$

$\Rightarrow \vec{r}(\theta) = \underline{\underline{f(\theta)\cos\theta\,\vec{i} + f(\theta)\sin\theta\,\vec{j}}} \quad \theta \in [a, b]$

b) $v(\theta) = \sqrt{(x'(\theta))^2 + (y'(\theta))^2}$

$= \sqrt{(f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2}$

$= \sqrt{f'(\theta)^2\cos^2\theta + f(\theta)^2\sin^2\theta \underline{- 2f(\theta)f'(\theta)\sin\theta\cos\theta} + f'(\theta)^2\sin^2\theta + f(\theta)^2\cos^2\theta + \underline{2f(\theta)f'(\theta)\sin\theta\cos\theta}}$

$= \sqrt{f'(\theta)^2(\cos^2\theta + \sin^2\theta) + f(\theta)^2(\sin^2\theta + \cos^2\theta)}$

$= \sqrt{f'(\theta)^2 + f(\theta)^2}$

c) $f(\theta) = \sin\theta \qquad f'(\theta) = \cos\theta$

$v(\theta) = \sqrt{f(\theta)^2 + f'(\theta)^2} = \sqrt{\sin^2\theta + \cos^2\theta} = 1$

$L = \int_0^{\pi} v(\theta)\, d\theta = \int_0^{\pi} d\theta = \underline{\underline{\pi}}$

d) $g(x,y) = xy \qquad x = f(\theta)\cos\theta = \sin\theta\cos\theta$

$y = f(\theta)\sin\theta = \sin^2\theta$

$\int_C g\, ds = \int_0^{\pi} xy \cdot \underbrace{1}_{v(\theta)}\, d\theta = \int_0^{\pi} \sin^3\theta \cos\theta\, d\theta \qquad u = \sin\theta$

$du = \cos\theta\, d\theta$

$\int \sin^3\theta \cos\theta\, d\theta = \int u^3\, du = \frac{1}{4}u^4 + C$

$\left[\frac{1}{4}\sin^4(\theta)\right]_0^{\pi} = \underline{\underline{0}}$

3.4.5 $\vec{F}(x,y,z) = yz\,\vec{i} + x\,\vec{j} + xy\,\vec{k}$

$\vec{r}(t) = t\,\vec{i} + \arctan t\,\vec{j} + t\,\vec{k} \qquad t \in [0,1]$

$\vec{r}\,'(t) = \left(1, \frac{1}{1+t^2}, 1\right)$

$\vec{F}(\vec{r}(t)) = (t\arctan t, t, t\arctan t)$

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = t\arctan t + \frac{t}{1+t^2} + t\arctan t$

$= 2t\arctan t + \frac{t}{1+t^2}$

$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_0^1 \left(\underline{2t}\,\underline{\arctan t} + \frac{t}{1+t^2}\right)dt$

$= \left[t^2\arctan t\right]_0^1 - \int_0^1 \frac{t^2}{1+t^2}dt + \int_0^1 \frac{t}{1+t^2}dt \qquad u = 1+t^2,\ du = \underline{2t}\,dt$

$= \frac{\pi}{4} - \int_0^1 \left(1 - \frac{1}{1+t^2}\right)dt + \left[\frac{1}{2}\ln|1+t^2|\right]_0^1$

$= \frac{\pi}{4} - 1 + \left[\arctan t\right]_0^1 + \frac{1}{2}\ln 2 = \frac{\pi}{4} - 1 + \frac{\pi}{4} + \frac{1}{2}\ln 2 = \underline{\underline{\frac{\pi}{2} - 1 + \frac{1}{2}\ln 2}}$

3.4.7 $\vec{F}(x,y) = (x^2 y, xy)$

$\mathcal{C}$: del av $y = x^2$, $x \in [-2, 2]$ $\quad t = x$

$\mathcal{C}$: $\vec{r}(t) = (t, t^2)$ $\quad \vec{r}\,'(t) = (1, 2t)$

$\vec{F}(\vec{r}(t)) = (t^2 \cdot t^2, t \cdot t^2) = (t^4, t^3)$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = (t^4, t^3) \cdot (1, 2t) = t^4 + 2t^4 = 3t^4$

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_{-2}^{2} (3t^4)\,dt = \left[\frac{3}{5}t^5\right]_{-2}^{2} = \frac{96}{5} + \frac{96}{5}$$

$$= \underline{\underline{\frac{192}{5}}}$$

3.4.12

$\vec{r}_1(t): [a,b] \to \mathbb{R}^n \qquad \vec{r}_1(a) = \vec{r}_1(b) = \vec{X}_0$

$\vec{r}_2(t): [c,d] \to \mathbb{R}^n \qquad \vec{r}_2(c) = \vec{r}_2(d) = \vec{X}_1$

Finnes en t_0, $a < t_0 < b$ s.a. $\vec{r}_1(t_0) = \vec{X}_1$

Finnes en t_1, $c < t_1 < d$ s.a. $\vec{r}_2(t_1) = \vec{X}_0$

Vi regner ut $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{r}_1$:

$$\int_a^b \vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1{}'(t)\,dt = \underbrace{\int_a^{t_0} \vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1{}'(t)\,dt}_{= \int_{C_1} \vec{F} \cdot d\vec{r}} + \underbrace{\int_{t_0}^{b} \vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1{}'(t)\,dt}_{\int_{C_2} \vec{F} \cdot d\vec{r}}$$

der C_1 er den delen av C som går fra $\vec{X}_0$ til $\vec{X}_1$
der C_2 er den delen av C som går fra $\vec{X}_1$ til $\vec{X}_0$

for $\vec{r}_2$ blir $\int \vec{F} \cdot d\vec{r}$

$$\int_c^d F(\vec{r}_2(t)) \cdot \vec{r}_2{}'(t)\,dt = \underbrace{\int_c^{t_1} \vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2{}'(t)\,dt}_{\int_{C_2} \vec{F} \cdot d\vec{r}} + \underbrace{\int_{t_1}^{d} \vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2{}'(t)\,dt}_{\int_{C_1} \vec{F} \cdot d\vec{r}}$$

Vi får at $\int_a^b \vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1{}'(t)\,dt = \int_c^d \vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2{}'(t)\,dt$,

siden $\int_{C_1} \vec{F} \cdot d\vec{r}$, $\int_{C_2} \vec{F} \cdot d\vec{r}$ er uavhengig av parametrisering.

3.4.13 $\quad f = \vec{F} \cdot \vec{T}$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (\vec{F} \cdot \vec{T})\, ds$$

Påstår:
Når vi endrer orientering på en kurve, endrer enhetstangentvektoren retning:
Som i Oppgave 3.3.9: $\quad \vec{r}_2(\phi(t)) = \vec{r}_1(t)$

$$\Downarrow$$

$$\vec{v}_2(\phi(t))\,\phi'(t) = \vec{v}_1(t)$$

$\Rightarrow \vec{v}_1(t)$ og $\vec{v}_2(\phi(t))$ har motsatt retning siden $\phi'(t) \leq 0$

$\Rightarrow \vec{T}_1(t) = -T_2(\phi(t))$.

$\int_C f\, ds$ avhenger jo ikke av orientering, men

$\int_C (\vec{F} \cdot \vec{T})\, ds$ avhenger <u>likevel</u> av orientering, siden integranden forandrer seg ved skifte av orientering.

$\Rightarrow$ siden $\vec{T}$ skifter fortegn, skifter også $\int_C (\vec{F} \cdot \vec{T})\, ds$ fortegn, slik som $\int_C \vec{F} \cdot d\vec{r}$ også gjør det.

3.5.12

a) $\phi_1(x) = \arctan\frac{y}{x} + C$

$$\nabla\phi_1(x,y) = \frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2}\vec{i} + \frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2}\vec{j}$$

$$= -\frac{y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j} \overset{\text{def}}{=} \vec{F}(x,y) \quad (x \neq 0)$$

b) $\int_C \vec{F}\cdot d\vec{r} = \phi_1(3,3) - \phi_1(1,-1) = \arctan\frac{3}{3} - \arctan\frac{-1}{1}$

$= \arctan 1 - \arctan -1 = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$

c) $\phi_2(x,y) = -\arctan\frac{x}{y} + C$

$$\nabla\phi_2(x,y) = -\frac{\frac{1}{y}}{1+\left(\frac{x}{y}\right)^2}\vec{i} - \frac{-\frac{x}{y^2}}{1+\left(\frac{x}{y}\right)^2}\vec{j}$$

$$= -\frac{y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j} = \vec{F}(x,y) \qquad (y \neq 0)$$

d)

e) ϕ_1, ϕ_2 er begge kont. for $x, y \neq 0$
Spesielt er de kont. i hver kvadrant, der skiller de seg fra hverandre men en konstant, siden de har de samme partielle deriverte.

finner C der $\phi_1(x,y) = \phi_2(x,y) + C$

første kvadrant: $(x,y) = (1,1)$ $\Downarrow$

$$\arctan\frac{y}{x} = -\arctan\frac{x}{y} + C$$
$$\arctan 1 = -\arctan 1 + C \Rightarrow C = 2\cdot\frac{\pi}{4} = \frac{\pi}{2}$$

I første kvadrant: $\arctan\frac{y}{x} = -\arctan\frac{x}{y} + \frac{\pi}{2}$
I tredje kvadrant finner vi samme C-verdi (sett inn $(x,y) = (-1,-1)$)
Andre kvadrant: $(x,y) = (-1,1)$: $\arctan\frac{y}{x} = -\arctan\frac{x}{y} + C$

$$\Downarrow$$
$$\arctan(-1) = -\arctan -1 + C$$

Får samme C i fjerde kvadrant. $\Rightarrow C = 2\arctan(-1) = -\frac{\pi}{2}$.
Vi får altså:

$\arctan\frac{y}{x} = -\arctan\frac{x}{y} + \frac{\pi}{2}$ (1. og 3 kvadrant) $(xy > 0)$

$\arctan\frac{y}{x} = -\arctan\frac{x}{y} - \frac{\pi}{2}$ (2. og 4. kvadrant) $(xy < 0)$

f) Rekker ikke, legges ut som egen pdf.