Ordinære lineære systemer av differensiallikninger med konstante koffisienter.

Ukjente funksjoner $x_1(t), \dots, x_n(t)$.

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_n + \dots + a_{nn}x_n \end{aligned}$$

a_{ij} tall.

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & & \\ a_{n1} & & & a_{nn} \end{pmatrix}$$

$X'(t) = AX(t)$. Initialverdi $x_i(0)$ kjent $i=1,\dots,n$ eller. $X(0) = X_0 \leftarrow$ kjent.

$n=1$

$X'(t) = AX(t)$ (A tall) $X(0) = X_0$.

Løsning: $X(t) = X_0 e^{At}$, $X(0) = X_0 e^{0A} = X_0$, $X'(t) = AX_0 e^{At} = AX(t)$.

$n>1$

Gjetter på $X(t) = e^{\lambda t} v$ som løsning. v vektor i $\mathbb{R}^n$, λ tall.

Dette gir.

$$X'(t) = \lambda e^{\lambda t} v \overset{\text{ønsker}}{=} AX(t) = Ae^{\lambda t} v = e^{\lambda t} A v$$

$$\lambda v = Av$$

λ er en egenverdi og v tilhørende egenvektor

Da blir $ce^{\lambda t} v$ en løsning.

Hva med initialverdien/kravet $X(0) = X_0$? $X(0) = e^{\lambda 0} cv = cv$

c konstant.

Anta at vi kan finne n lineært uavhengige egenvektorer $\{v_1, v_2 \dots v_n\}$ og egenverdier $\{\lambda_1, \dots, \lambda_n\}$.

$\uparrow$ Basis for $\mathbb{R}^n$

$X_0 \in \mathbb{R}^n$ $X_0 = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$

Sett

$$X(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \dots + c_n e^{\lambda_n t} v_n$$

Viser at $X(t)$ løser $X'(t) = AX(t)$.

$$X'(t) = c_1 \lambda_1 e^{\lambda_1 t} v_1 + c_2 \lambda_2 e^{\lambda_2 t} v_2 + \dots + c_n \lambda_n e^{\lambda_n t} v_n$$

$$\begin{aligned} AX(t) &= A(c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \dots + c_n e^{\lambda_n t} v_n) \\ &= Ac_1 e^{\lambda_1 t} v_1 + \dots + Ac_n e^{\lambda_n t} v_n \\ &= c_1 e^{\lambda_1 t} Av_1 + c_2 e^{\lambda_2 t} Av_2 + \dots + c_n e^{\lambda_n t} Av_n \\ &= c_1 e^{\lambda_1 t} \lambda_1 v_1 + c_2 e^{\lambda_2 t} \lambda_2 v_2 + \dots + c_2 e^{\lambda_n t} \lambda_n v_n \\ &= X'(t). \end{aligned}$$

Initialverdi?

$X(0) = c_1 v_1 + \dots + c_n v_n$

$= X_0$

OK

Eksempel

$x' = 4x - y \qquad x(0) = 1$

$y' = 5x - 2y \qquad y(0) = 2$

$X = \begin{pmatrix} x \\ y \end{pmatrix}. \quad X' = \begin{pmatrix} 4 & -1 \\ 5 & -2 \end{pmatrix} X \quad X(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$

Finner egenverdier og egenvektorer.

$\det\left(\begin{pmatrix} 4 & -1 \\ 5 & -2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0 \qquad \begin{vmatrix} 4-\lambda & -1 \\ 5 & -2-\lambda \end{vmatrix} = (4-\lambda)(-2-\lambda) + 5 = \lambda^2 - 2\lambda - 8 + 5$

$= \lambda^2 - 2\lambda - 3 = 0.$

$\lambda = \frac{1}{2}\left(2 \pm \sqrt{4+12}\right) = \frac{1}{2}(2 \pm 4) = \begin{cases} 3 \\ -1 \end{cases}$

$\underline{\lambda_1} = 3 \quad v_1 = ?$

$(A - \lambda I)v = 0 \quad \begin{pmatrix} 1 & -1 & 0 \\ 5 & -5 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ c_2 fri $\quad c_1 = c_2 \quad \underline{v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}.$

$\underline{\lambda_2} = -1 \quad \begin{pmatrix} 5 & -1 & 0 \\ 5 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 5 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ c_2 fri $\quad 5c_1 = c_2 \quad \underline{v_2} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$

$X(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$ Finne c_1, c_2.

$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 5 & 2 \end{pmatrix} \overset{II-I}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 1 \end{pmatrix} \overset{II/4}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1/4 \end{pmatrix} \overset{I-II}{\sim} \begin{pmatrix} 1 & 0 & 3/4 \\ 0 & 1 & 1/4 \end{pmatrix} \quad \begin{matrix} c_1 = 3/4 \\ c_2 = 1/4 \end{matrix}$

Løsning:

$X(t) = \frac{1}{4} e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{3}{4} e^{-t} \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{1}{4}e^{3t} + \frac{3}{4}e^{-t} \\ \frac{1}{4}e^{3t} + \frac{15}{4}e^{-t} \end{pmatrix}$

$$X'(t) = \begin{pmatrix} 0 & -2 \\ -1 & 1 \end{pmatrix} X(t) \qquad X(0) = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

Egenverdier:

$$\det\begin{pmatrix} -\lambda & -2 \\ -1 & 1-\lambda \end{pmatrix} = 0 = -\lambda(1-\lambda) - 2 = \lambda^2 - \lambda - 2 = 0$$

$$\lambda = \frac{1}{2}\left(1 \pm \sqrt{1+8}\right) = \frac{1}{2}(1 \pm 3) = \begin{cases} 2 \\ -1 \end{cases}$$

$\lambda_1 = 2$
$$\begin{pmatrix} -2 & -2 & 0 \\ -1 & -1 & 0 \end{pmatrix} \quad x \text{ fri} \quad y = -x \quad v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\lambda_2 = -1$
$$\begin{pmatrix} 1 & -2 & 0 \\ -1 & 2 & 0 \end{pmatrix} \quad y \text{ fri} \quad x = 2y \quad v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 4 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 4 \end{pmatrix} \overset{II+I}{\sim} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 5/3 \end{pmatrix}$$

$$\overset{I-2II}{\sim} \begin{pmatrix} 1 & 0 & -7/3 \\ 0 & 1 & 5/3 \end{pmatrix} \qquad \begin{matrix} c_1 = -7/3 \\ c_2 = 5/3. \end{matrix}$$

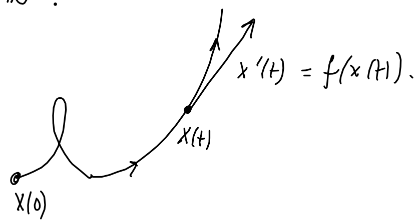
$$\underline{\underline{X(t) = -\frac{7}{3} e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{5}{3} e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}}}.$$

System av ordinære differensialligninger.

$f : \mathbb{R}^n \to \mathbb{R}^n$

$x'(t) = f(x(t))$, $x(0) = x_0$ $\quad$ $x(t)$ er i $\mathbb{R}^n$.

initialbetingelse



Eks

$f(x) = Ax$

$x' = Ax$

Hvordan finne tilnærmede løsninger?

Euler's metode.

$$x'(t) \approx \frac{x(t+\Delta t) - x(t)}{\Delta t}$$ erstatter x' med dette

$$\frac{x(t+\Delta t) - x(t)}{\Delta t} = f(x(t))$$

$\bar{x}(t+\Delta t) = \bar{x}(t) + \Delta t\, f(\bar{x}(t))$ $\quad$ $x(0)$ kjent!

$\bar{x}(\Delta t) = \bar{x}(0) + \Delta t\, f(\bar{x}(0))$

$\bar{x}(2\Delta t) = \bar{x}(\Delta t) + \Delta t\, f(\bar{x}(\Delta t))$

$\vdots$

$\bar{x}((n+1)\Delta t) = \bar{x}(n\Delta t) + \Delta t\, f(\bar{x}(n\Delta t))$

$\vdots$

x

Kan vise at dersom $n\Delta t = t$

så er $|x(t) - \bar{x}(n\Delta t)| \lesssim C(\Delta t)$.

$\Delta t \to 0$, $n \to \infty$ $\quad$ $n\Delta t$ fast $= t$.

for $i = 1, \dots, n$

$\quad x = x + \Delta t \cdot f(x)$

Velg n s.a. $n\Delta t = t$