Plenum 5/4

$\boxed{5.4.7}$ $f(\sqrt{2}) = \sqrt{2}$. $f(\sqrt{2}) = \sqrt{2}^2 + \sqrt{2} - 2 = 2 + \sqrt{2} - 2 = \underline{\sqrt{2}}$

$\boxed{5.4.9}$ $A = \begin{pmatrix} 1.3 & -0.2 \\ 0.1 & 1 \end{pmatrix}$

a) $$p(\lambda) = \det\big(\underbrace{A - \lambda I_2}\big) = (1.3-\lambda)(1-\lambda) + 0.02$$
$$= \lambda^2 - 2.3\lambda + 1.32 \quad \Rightarrow \lambda = 1.1, 1.2$$
$$= (\lambda - 1.1)(\lambda - 1.2)$$

$\underline{\lambda = 1.1}$: $\begin{pmatrix} 0.2 & -0.2 \\ 0.1 & -0.1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\underline{\lambda_2 = 1.2}$: $\begin{pmatrix} 0.1 & -0.2 \\ 0.1 & -0.2 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \Rightarrow \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

b) $$\begin{array}{cc} \vec{v}_1 & \vec{v}_2 \end{array}$$
$$\left(\begin{array}{cc|cc} 1 & 2 & 2 & 4 \\ 1 & 1 & 0 & 3 \end{array}\right) \overset{II - I}{\sim} \begin{pmatrix} 1 & 2 & 2 & 4 \\ 0 & -1 & -2 & -1 \end{pmatrix}$$
$\sim I_2$
$$\underset{I - 2II}{\overset{II\cdot(-1)}{\sim}} \begin{pmatrix} 1 & 0 & -2 & 2 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

$\begin{pmatrix} 2 \\ 0 \end{pmatrix} = -2\vec{v}_1 + 2\vec{v}_2$, $\begin{pmatrix} 4 \\ 3 \end{pmatrix} = 2\vec{v}_1 + \vec{v}_2$

c) $f(x) = \lambda x + k$, $\lambda \neq 1$

$\underbrace{x_{n+1} = \lambda x_n + k}$, $x_0 \rightarrow x_n = \lambda^n\left(x_0 - \frac{k}{1-\lambda}\right) + \frac{k}{1-\lambda}$

$$x_n = \underbrace{\lambda x_{n-1}} + k = \lambda(\lambda x_{n-2} + k) + k = \lambda^2 x_{n-2} + (\lambda+1)k$$
$x_{n-1} = \lambda x_{n-2} + k$
$$= \dots$$
$$= \lambda^n x_0 + \underbrace{(\lambda^{n-1} + \lambda^{n-2} + \dots + \lambda + 1)}_{\frac{1-\lambda^n}{1-\lambda}} k$$
$$= \lambda^n x_0 + \frac{1-\lambda^n}{1-\lambda} k$$
$$= \lambda^n x_0 + \frac{1}{1-\lambda} k - \frac{\lambda^n}{1-\lambda} k$$
$$= \underline{\lambda^n\left(x_0 - \frac{k}{1-\lambda}\right) + \frac{k}{1-\lambda}}$$

d) $\vec{r}_0 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $\vec{r}_{n+1} = A\vec{r}_n + \vec{b}$, $\vec{b} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

Finn $\vec{r}_n$:

$$\vec{r}_n = a_n\vec{v}_1 + b_n\vec{v}_2, \quad \vec{b} = 2\vec{v}_1 + \vec{v}_2, \quad \vec{r}_0 = \underbrace{-2}_{a_0}\vec{v}_1 + \underbrace{2}_{b_0}\vec{v}_2$$

$$\underbrace{a_{n+1}\vec{v}_1 + b_{n+1}\vec{v}_2}_{\vec{r}_{n+1}} = A\underbrace{(a_n\vec{v}_1 + b_n\vec{v}_2)}_{\vec{r}_n} + \underbrace{2\vec{v}_1 + \vec{v}_2}_{\vec{b}}$$
$$= a_n\lambda_1\vec{v}_1 + b_n\lambda_2\vec{v}_2 + 2\vec{v}_1 + \vec{v}_2$$
$$= (a_n\lambda_1 + 2)\vec{v}_1 + (b_n\lambda_2 + 1)\vec{v}_2$$

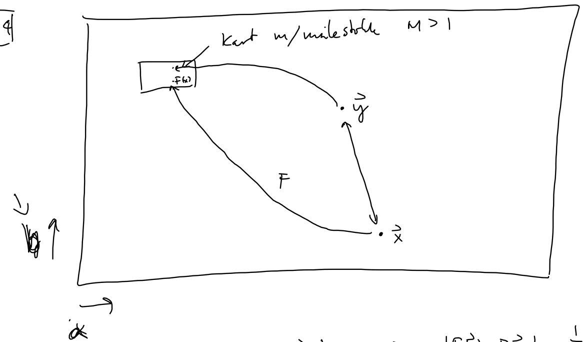
$a_{n+1} = a_n\lambda_1 + 2$ & $b_{n+1} = b_n\lambda_2 + 1$

$$a_n = \lambda_1^n\left(-2 - \frac{2}{1-\lambda_1}\right) + \frac{2}{1-\lambda_1} = 1.1^n\left(-2 - \frac{2}{-0.1}\right) + \frac{2}{-0.1}$$
$$b_n = \lambda_2^n\left(2 - \frac{1}{1-\lambda_2}\right) + \frac{1}{1-\lambda_2} = 1.2^n\left(2 - \frac{1}{-0.2}\right) + \frac{1}{-0.2}$$

$a_n = 18 \cdot 1.1^n - 20$, $b_n = 7 \cdot 1.2^n - 5$

$$\vec{r}_n = (18 \cdot 1.1^n - 20)\vec{v}_1 + (7 \cdot 1.2^n - 5)\vec{v}_2$$

$\boxed{5.5.4}$



$$|\vec{x}-\vec{y}| = M\cdot|F(\vec{x})-F(\vec{y})| \Rightarrow |F(\vec{x})-F(\vec{y})| = \frac{1}{M}|\vec{x}-\vec{y}| \quad \forall\, \overset{(a_1,b_1)}{\vec{x}},\vec{y}\in\mathbb{R}^2$$

$\Rightarrow F$ kontraksjon $\qquad M>1 \Rightarrow \frac{1}{M}<1$

$\boxed{5.5.5}$ A ikke-tom, lukket delmengde av $\mathbb{R}^m$

$F: A\to A$, $F^{\circ k}$ er en kontraksjon for $k\in\mathbb{N}$

Da har F et unikt fikspunkt $\vec{x}\in A$.

Ny $\to$ $|F^{\circ k}(x)-F^{\circ k}(y)| \leq C|x-y| \quad \forall\, x,y\in A.$

Banach $\to$ $|F(x)-F(y)| \leq C|x-y|$

---

a) $F^{\circ k}(x) = \underbrace{F(F(F(\dots(F(x)))))}_{k \text{ ganger}}$

x er et f.p. for $F \Rightarrow x$ er et f.p. for $F^{\circ k}$

$F^{\circ k}(x)=x$?

$F^{\circ k}(x) = F^{\circ(k-1)}(\underbrace{F(x)}_{x}) = F^{\circ(k-1)}(x) = \dots = F(x) = x$

b) Vet: $F^{\circ k}(x)=x$

Vis: $F^{\circ k}(F(x)) = F(x)$

$F^{\circ k}(F(x)) = F^{\circ(k+1)}(x) = F(\underbrace{F^{\circ k}(x)}_{x}) = F(x)$

a) F har maks ett fikspunkt

b) $F(x)$ også er et fikspunkt for $F^{\circ k}$ dersom x er et fikspunkt for $F^{\circ k}$.

x : $F^{\circ k}(x) = x \Rightarrow F(x)=x$

$F(x)$: $F^{\circ k}(F(x)) = F(x)$

Eks. 2008

③ a) $A = \begin{pmatrix} 1.1 & -0.2 \\ 0.1 & 0.8 \end{pmatrix} \rightarrow \lambda_1 = 1, \ \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\lambda_2 = 0.9, \ \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

b) $x_{n+1} = 1.1x_n - 0.2y_n$

$y_{n+1} = 0.1x_n + 0.8y_n$, $x_0 = 3000$, $y_0 = 1000$

$$\begin{array}{ccc} \vec{v}_2 & \vec{v}_1 & \\ \downarrow & \downarrow & \end{array}$$

$$\left(\begin{array}{cc|c} 1 & 2 & 3000 \\ 1 & 1 & 1000 \end{array}\right) \underset{I-2II}{\overset{II-I}{\sim}} \left(\begin{array}{cc|c} 1 & 0 & -1000 \\ 0 & 1 & 2000 \end{array}\right)$$

$F(\vec{x}) = A\vec{x} \rightarrow \begin{pmatrix} x_n \\ y_n \end{pmatrix} = A \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}$, $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 3000 \\ 1000 \end{pmatrix} = 2000\vec{v}_1 - 1000\vec{v}_2$

$= \ldots$

$= A^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

$= A^n\left(2000\vec{v}_1 - 1000\vec{v}_2\right) = 2000\lambda_1^n\vec{v}_1 - 1000\lambda_2^n\vec{v}_2$

$= 2000\vec{v}_1 - 1000 \cdot 0.9^n\vec{v}_2$

$$\lim_{n\to\infty}\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \lim_{n\to\infty}\left(2000\vec{v}_1 - 1000 \cdot 0.9^n\vec{v}_2\right) = 2000\vec{v}_1 = \begin{pmatrix} 4000 \\ 2000 \end{pmatrix}$$

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⑦ A $m \times m$-matrise, $\{\vec{x}_n\}$ i $\mathbb{R}^m$ $\lim_{n\to\infty} \vec{x}_n = \vec{0}$

$\{A\vec{x}_n\} \to \vec{0}$ $\quad (Ax_n)_j \to 0$ $\quad Ax_n = \begin{pmatrix} (Ax_n)_1 \\ (Ax_n)_2 \\ \vdots \\ (Ax_n)_m \end{pmatrix}$

$\mathbb{R}^m$

$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots \\ a_{m1} & \cdots & \cdots & a_{mm} \end{pmatrix}$, $\vec{x}_n = \begin{pmatrix} x_n^{(1)} \\ x_n^{(2)} \\ \vdots \\ x_n^{(m)} \end{pmatrix}$

$(Ax_n)_j = \sum_{k=1}^{m} a_{jk} x_n^{(k)}$

$$\sum_{k=1}^{m} |a_{j^*k}| \geq \sum_{k=1}^{m} |a_{jk}| \quad \forall \ j \in \{1,2,3,\dots,m\}$$

$$\forall \ k \in \{1,2,\dots,m\} : \exists \ N_k \in \mathbb{N} : \ |x_n^{(k)} - 0| < \frac{\varepsilon}{\sum_{k=1}^{m} |a_{j^*k}|}, \quad \varepsilon \in (0,1)$$

La $N = \text{maks}(N_1, N_2, \dots, N_k)$, da vil for vilkårlig $j \in \{1,2,\dots,m\}$

$\forall \ n \geq N$:

$$|(Ax_n)_j| = \left| \sum_{k=1}^{m} a_{jk} x_n^{(k)} \right| < \left| \sum_{k=1}^{m} a_{jk} \cdot \Bigg(\overbrace{\frac{\varepsilon}{\sum_{r=1}^{m} a_{j^*r}}}^{\text{uavh. av } k} \Bigg) \right| \qquad |a+b| \leq |a| + |b|$$

$$= \varepsilon \left| \frac{\sum_{k=1}^{m} a_{jk}}{\sum_{r=1}^{m} |a_{j^*r}|} \right| \leq \varepsilon \frac{\sum_{k=1}^{m} |a_{jk}|}{\sum_{r=1}^{m} |a_{j^*r}|} \leq \varepsilon$$

$$\Rightarrow \quad |(Ax_n)_j| < \varepsilon \quad \forall \ n \geq N \ \forall \ j \in \{1,2,\dots,m\}$$

$$\Rightarrow \quad Ax_n \to \vec{0}$$

b) B inverterbar. $\{Bx_n\} \to \vec{0}$

B^{-1} finnes.

$x_n \to \vec{0} \Rightarrow A\vec{x}_n \to \vec{0}$

$Bx_n \to \vec{0} \Rightarrow \underbrace{B^{-1}(Bx_n)}_{x_n} \to 0$

$x_n \to 0$

c)