12.1

$$\sum_{n=0}^{\infty} a x^n = \frac{a}{1-x}$$

$|x| < 1$

d) $2 - \frac{3}{2} + \frac{9}{8} - \frac{27}{32} + \cdots$

$$= \sum_{n=0}^{\infty} 2\left(-\frac{3}{4}\right)^n = \frac{2}{1-\left(-\frac{3}{4}\right)} = \frac{2}{\frac{7}{4}} = \frac{8}{7}$$

3a) $1 - x + x^2 - x^3 + \cdots$ $\quad |x| < 1$

$$= \sum_{n=0}^{\infty} (-x)^n = \frac{1}{1-(-x)} = \underline{\underline{\frac{1}{1+x}}}$$

12.1

3 c) $a^2 - 4a^4 + 16a^6 - \dots$ $\quad |a| < \frac{1}{2}$

$\Downarrow$

$$= \sum_{n=0}^{\infty} a^2(-4a^2)^n \qquad |-4a^2| < 1$$

$$= \underline{\underline{\frac{a^2}{1+4a^2}}}$$

3 f) $\frac{1}{y} - \frac{3}{y\sqrt{y}} + \frac{9}{y^2} - \frac{27}{y^2\sqrt{y}} + \dots$

$$y > 9 \qquad = \sum_{n=0}^{\infty} \frac{1}{y}\left(\frac{-3}{\sqrt{y}}\right)^n$$

<u>konvergent når</u> $\left|\frac{-3}{\sqrt{y}}\right| < 1 \Leftrightarrow \sqrt{y} > 3$

$\Leftrightarrow y > 9$

$$\sum_{n=0}^{\infty} \frac{1}{y}\left(\frac{-3}{\sqrt{y}}\right)^n = \frac{\frac{1}{y}}{1+\frac{3}{\sqrt{y}}} = \underline{\underline{\frac{1}{y+3\sqrt{y}}}}$$

12.1

4)

a) $\sum_{n=0}^{\infty} \arctan n$ er divergent

siden $\arctan n \xrightarrow[n\to\infty]{} \frac{\pi}{2} \neq 0$

b) $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$ divergent siden

$\cos\left(\frac{1}{n}\right) \xrightarrow[n\to\infty]{} \cos 0 = 1 \neq 0$

c)

$$\sum_{n=1}^{\infty} \left(1 - \sin\frac{1}{n}\right)^n$$

$$\left(1-\sin\frac{1}{n}\right)^n = e^{n \ln\left(1-\sin\frac{1}{n}\right)}$$

$$\lim_{n\to\infty} n \ln\left(1-\sin\frac{1}{n}\right) =$$

$$= \lim_{x\to\infty} \frac{\ln\left(1-\sin\frac{1}{x}\right)}{\frac{1}{x}} = \text{„}\frac{0}{0}\text{"}$$

L'Hop.

$$= \lim_{x\to\infty} \frac{\frac{1}{1-\sin\frac{1}{x}}\left(-\cos\frac{1}{x}\right)\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}} =$$

$$= \lim_{x\to\infty} \frac{-\cos\left(\frac{1}{x}\right)}{1-\sin\frac{1}{x}} = \frac{-1}{1-0} = -1$$

$\left(1-\sin\frac{1}{n}\right)^n \xrightarrow[n\to\infty]{} e^{-1} = \frac{1}{e} \neq 0$

Vi får divergens.

12.1

5) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$

a) sadavise $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{(A+B)x+A}{x(x+1)}$$

$A+B=0$, $A=1$, $\Rightarrow B=-1$ som gir.

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

b) $$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$= \left(1-\cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}\right) + \cdots$$

$$+ \left(\cancel{\tfrac{1}{n-1}} - \cancel{\tfrac{1}{n}}\right) + \left(\cancel{\tfrac{1}{n}} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

c) $$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k(k+1)}$$

$$= \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$$

<u>12.2</u>

1b) $\sum_{n=0}^{\infty} \frac{1}{n^2+1} = 1 + \sum_{n=1}^{\infty} f(n)$

$f(x) = \frac{1}{x^2+1}$, rekka konvergerer

hvis og bare hvis $\int_1^{\infty} f(x)\,dx$

konvergerer. $\int_1^{\infty} \frac{1}{x^2+1}\,dx =$

$= \lim_{t \to \infty} \left[\arctan x \right]_1^t$

$= \lim_{t \to \infty} \left(\arctan t - \frac{\pi}{4} \right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \neq \infty$

Integralet og derfor rekka vil konvergere.

12.2

1 e) $$\sum_{n=0}^{\infty}\left(\frac{\pi}{2}-\arctan n\right)$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty}\left(\frac{\pi}{2}-\arctan n\right)$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty} f(n), \quad f(x) = \frac{\pi}{2} - \arctan x$$

$$\int_1^{\infty}\left(\frac{\pi}{2}-\arctan x\right)dx = \,?$$

Delvis integrasjon:

$$\int 1\cdot\arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}\,dx$$

$$= x\arctan x - \frac{1}{2}\ln(1+x^2) + C$$

$$\int_1^t \left(\frac{\pi}{2} - \arctan x\right) dx =$$

$$= \left[\frac{\pi}{2}x - (x \arctan x - \frac{1}{2}\ln(1+x^2)\right]_1^t$$

$$= \frac{\pi}{2}t - t\arctan t + \frac{1}{2}\ln(1+t^2) - \frac{\pi}{2}\ln 2 = t\underbrace{\left(\frac{\pi}{2} - \arctan t\right)}_{>0} + \frac{1}{2}\ln(1+t^2) - \frac{1}{2}\ln 2 > \frac{1}{2}\ln(1+t^2) - \frac{1}{2}\ln 2 \xrightarrow[t\to\infty]{} \infty$$

integralet og derfor rekka divergerer

<u>12.2</u>

3 a)

$$\sum_{n=1}^{\infty} \frac{7n^2+3}{4n^3-2}$$

$$\frac{7n^2+3}{4n^3-2} \Big/ \frac{1}{n} = \frac{7n^3+3n}{4n^3-2} =$$

$$= \frac{7+3/n^2}{4-2/n^3} \xrightarrow[n\to\infty]{} \frac{7}{4}$$

Får divergens ved grensesammenlikningskriteriet siden $\sum_{n=1}^{\infty} \frac{1}{n}$ er divergent.

12.2

3 b) $\sum_{n=1}^{\infty} \frac{2n-7}{4n^3+8}$

$$\frac{2n-7}{4n^3+8} \Big/ \frac{1}{n^2} = \frac{2n^3-7n^2}{4n^3+8} = \frac{2-\frac{7}{n}}{4+\frac{8}{n^3}} \xrightarrow[n\to\infty]{} \frac{1}{2}$$

konvergent, siden $\sum_{n=1}^{\infty} \frac{1}{n^2}$ er konvergent.

e) $\sum_{n=1}^{\infty} (1-\cos\frac{1}{n})$

$$\lim_{x\to\infty} \frac{1-\cos\frac{1}{x}}{\frac{1}{x^2}} \,\text{„}\frac{0}{0}\text{"} \overset{\text{l'Hop.}}{=} \lim \frac{\left(\sin\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}{-\frac{2}{x^3}}$$

$$= \lim_{x\to\infty} \frac{1}{2} \frac{\sin\frac{1}{x}}{\frac{1}{x}} = \lim_{t\to 0^+} \frac{1}{2}\frac{\sin t}{t} = \frac{1}{2}$$

K. $(1-\cos\frac{1}{n}) \Big/ \frac{1}{n^2} \xrightarrow[n\to\infty]{} \frac{1}{2}$

Konvergens siden $\sum \frac{1}{n^2}$ konvergerer.

f) $\sum_{n=1}^{\infty} \sin\frac{1}{n}$

$\lim_{n\to\infty} \sin\frac{1}{n} \Big/ \frac{1}{n} = 1 \left(= \lim_{t\to 0} \frac{\sin t}{t} \right)$

divergens siden $\sum_{n=1}^{\infty} \frac{1}{n}$ divergerer.

4) Anta $\lim_{n\to\infty} a_n = 0$

a) $\sum_{n=1}^{\infty} \sin(a_n)$ konvergerer hviss

$\sum_{n=1}^{\infty} a_n$ konvergerer fordi

$\lim_{n\to\infty} \frac{\sin(a_n)}{a_n} = \lim_{t\to 0} \frac{\sin t}{t} = 1$

b) $\sum_{n=1}^{\infty} \sin(\sin a_n)$ konvergerer

hviss $\sum_{n=1}^{\infty} a_n$ konvergerer.

Vet at også $\sin a_n \to 0$ når $n \to \infty$

får da fra a)

$\sum_{n=1}^{\infty} \sin(\sin a_n)$ konvergent

$\overset{(a)}{\Leftrightarrow} \sum_{n=1}^{\infty} \sin(a_n)$ —"—

$\overset{(a)}{\Leftrightarrow} \sum_{n=1}^{\infty} a_n$ —"—

12.2

5 a)

$$\sum_{n=0}^{\infty} \frac{n}{3^n} \quad \text{forholdstest}$$

$$\lim_{n\to\infty} \frac{n+1}{3^{n+1}} \Big/ \frac{n}{3^n} = \lim_{n\to\infty} \frac{n+1}{n} \frac{3^n}{3^{n+1}}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)\frac{1}{3} = \frac{1}{3} < 1 \quad \underline{\underline{\text{konvergens}}}$$

b)

$$\sum_{n=1}^{\infty} \frac{3^n}{n^{10}}$$

$$\lim_{n\to\infty} \frac{3^{n+1}}{(n+1)^{10}} \Big/ \frac{3^n}{n^{10}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{10} 3$$

$$= \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^{10} \cdot 3 = 3 > 1$$

får divergens.

12.2

5 c)

$$\sum_{n=1}^{\infty} \underbrace{\left(1-\frac{1}{n}\right)^{n^2}}_{a_n} \quad \text{Rottest}$$

$$a_n^{\frac{1}{n}} = \left(1-\frac{1}{n}\right)^{\frac{n^2}{n}} = \left(1-\frac{1}{n}\right)^n$$

$$= e^{n\ln\left(1-\frac{1}{n}\right)} \longrightarrow e^{-1} = \frac{1}{e} < 1$$

$$\lim_{x\to\infty} x\ln\left(1-\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1-\frac{1}{x}\right)}{\frac{1}{x}} = \frac{0}{0}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{1-\frac{1}{x}} \cdot \frac{1}{x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{-1}{1-\frac{1}{x}} = -1$$

Får altså konvergens ved rottest.

12.25

d) $\sum_{n=1}^{\infty} \frac{e^n}{n!}$. Forholdstest

$$\lim_{n\to\infty} \frac{e^{n+1}}{(n+1)!} \Big/ \frac{e^n}{n!} = \lim_{n\to\infty} \frac{e \cdot n!}{\underbrace{(n+1)!}_{(n+1)n!}}$$

$$= \lim_{n\to\infty} \frac{e}{n+1} = 0 < 1$$ får konvergens.

6a) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n^2+1}$ forholdstest:

$$\frac{n+1}{(n+1)^2+1} \Big/ \frac{n}{n^2+1} = \frac{n^2+1 \cdot n+1}{((n+1)^2+1)\, n}$$

$$= \frac{\left(1+\frac{1}{n^2}\right)\left(1+\frac{1}{n}\right)}{\left(\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}\right)\cdot 1} \xrightarrow[n\to\infty]{} 1$$ kan ikke si noe.

Grense sammenligningskriteriet:

$$\frac{n}{n^2+1} \Big/ \frac{1}{n} = \frac{n^2}{n^2+1} = \frac{1}{1+\frac{1}{n^2}} \xrightarrow[n\to\infty]{} 1$$

får divergens siden $\sum \frac{1}{n}$ divergerer

7 e)

$$\sum_{n=0}^{\infty} n e^{-n^2} \quad \text{Forholdstest}$$

$$\frac{(n+1)e^{-(n+1)^2}}{n e^{-n^2}} = \left(1+\frac{1}{n}\right)e^{n^2-(n+1)^2}$$

$$= \left(1+\frac{1}{n}\right)e^{-2n-1} \xrightarrow[n\to\infty]{} 0 \text{ . } \underline{\text{Får konvergens}}$$

12.3

1a) $$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n^2+1} = \sum_{n=0}^{\infty} (-1)^{n+1} a_n$$

$a_n = \frac{1}{n^2+1}$ går monotont mot 0

så rekka konvergerer.

1c) $$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

der $a_n = \frac{1}{\sqrt{n}}$ går monotont mot 0.

så rekka konvergerer.

(Men $\sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{1}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ er

$\frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$ $\left(\frac{1}{2} < 1 \right)$

divergent)

12.3

3 a) $$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)^2}$$

Skal finne summen med tilnærmelse bedre enn $\varepsilon = 0.05$.

Vet rekke konvergerer ved alternerende rekke test. Setter n:

$$S_N = \sum_{n=1}^{N} \frac{(-1)^n}{(n+1)^2} \quad \text{så blir}$$

$$|\text{feil}| < \frac{1}{((N+1)+1)^2} = \frac{1}{(N+2)^2}$$

Må finne N slik at

$$\frac{1}{(N+2)^2} < 0.05 \Rightarrow (N+2)^2 > \frac{1}{0.05} = 20$$

$N = 3$ er ok. $S_N = \sum_{n=1}^{3} \frac{(-1)^n}{(n+1)^2}$

$$= \frac{-1}{4} + \frac{1}{9} - \frac{1}{16} = \frac{-29}{\underline{144}} \text{ gir}$$

estimat med ønsket feil.

12.4

12.1

a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$

er konvergent ved alternerende rekke test.

men rekke $\sum_{n=0}^{\infty} \frac{1}{n+1}$ er divergent

Rekka er altså betinget konvergent

b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+4}$, $\frac{1}{n^2+4} < \frac{1}{n^2}$

ved sammenlikningskriteriet
er rekke absolutt konvergent

e) $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$

$f(x) = \frac{\sqrt{x}}{x+1}$, $f'(x) = \frac{\frac{1}{2}\frac{1}{\sqrt{x}}(x+1) - \sqrt{x}}{(x+1)^2} =$

$= \frac{1-x}{2\sqrt{x}(x+1)^2} < 0$ for $x >$

$f(x)$ er avtagende $\Rightarrow$ $\frac{\sqrt{n}}{n+1}$ er avtagende

og vi har også at $\frac{\sqrt{n}}{n+1} = \frac{1}{\sqrt{n}+\frac{1}{\sqrt{n}}} \xrightarrow[n\to\infty]{} 0$

alternerende rekke test $\Rightarrow$ rekka er konvergent. Men $\frac{\sqrt{n}}{n+1} \Big/ \frac{1}{\sqrt{n}} =$

$= \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \xrightarrow[n\to\infty]{} 1$

og siden $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ er divergent

er rekka <u>ikke</u> absolutt konvergent.
Dvs. <u>betinget</u> konvergent.