Greens teorem

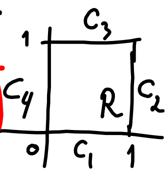
C

enkel, lukket, plan kurve
stykkevis glatt parametrisering
$\partial R = C$
orientert mot klokka
P, Q: kont. part. deriverte

$$\oint_C P\,dx + Q\,dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$$

Eks.

$\boxed{C = C_1 \cup C_2 \cup C_3 \cup C_4}$



$P(x,y) = x + y \;;\; \frac{\partial P}{\partial y} = 1$

$Q(x,y) = x^2 y \;;\; \frac{\partial Q}{\partial x} = 2xy$

$$\int_0^1 \int_0^1 2xy - 1\,dx\,dy = \int_0^1 \left[x^2 y - x\right]_{x=0}^{x=1} dy$$

$$= \int_0^1 y - 1\,dy$$

$$= \left[\tfrac{1}{2}y^2 - y\right]_0^1 = \frac{1}{2} - 1 = \underline{\underline{-\frac{1}{2}}}$$

C_1: $P(x,0) = x$, $Q(x,0) = 0$
dx, $dy = 0$

C_2: $P(1,y) = 1 + y$, $Q(1,y) = y$
$dx = 0$, dy

C_3: $P(x,1) = x + 1$, $Q(x,1) = x^2$
dx, $dy = 0$

C_4: $P(0,y) = y$, $Q(0,y) = 0$
$dx = 0$, dy

$$\int_C P\,dx + Q\,dy = \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy + \int_{C_3} P\,dx + Q\,dy + \int_{C} P\,dx + Q\,dy$$

$$= \int_0^1 x\,dx + \int_0^1 (1+\cancel{y})\cdot 0 + y\,dy + \int_1^0 (x+1)\,dx + \int_1^0 0\,dy$$

$$= \left[\tfrac{1}{2}x^2\right]_0^1 + \left[\tfrac{1}{2}y^2\right]_0^1 + \left[\tfrac{1}{2}x^2 + x\right]_1^0 + 0$$

$$= \frac{1}{2} + \frac{1}{2} + 0 - \frac{1}{2} - 1 = \underline{\underline{-\frac{1}{2}}}$$

Korollar

$$\text{areal}(R) = \int_C x\,dy = -\int_C y\,dx = \frac{1}{2}\int_C -y\,dx + x\,dy$$

Bevis.

$$\int_C x\,dy = \iint_R \underbrace{\frac{\partial x}{\partial x}}_{\frac{\partial Q}{\partial x}} - 0\,dx\,dy = \iint_R dx\,dy = \text{areal}(R)$$

$P = 0$ $Q = x$

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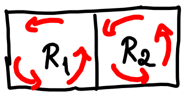
Skisse av bevis

$\boxed{\partial R : \text{randa til } R}$

1.



$$\int_{\partial R_1} P\,dx + Q\,dy = \iint_{R_1} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$$

$R = R_1 \cup R_2$

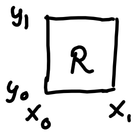
$\partial R = \partial R_1 \cup \partial R_2 \setminus (R_1 \cap R_2)$

$$\int_{\partial R_2} P\,dx + Q\,dy = \iint_{R_2} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy \quad +$$

$$\int_{\partial R} P\,dx + Q\,dx = \iint_{R} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$$

Begge sider i Greens teorem er "additive"

2.



$$\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy = \int_{x_0}^{x_1}\int_{y_0}^{y_1} -\frac{\partial P}{\partial y}\,dy\,dx + \int_{y_0}^{y_1}\int_{x_0}^{x_1} \frac{\partial Q}{\partial x}\,dx\,dy$$

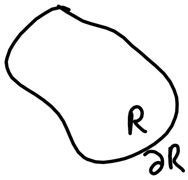
$$= \int_{x_0}^{x_1} \left[-P\right]_{y=y_0}^{y=y_1} dx + \int_{y_0}^{y_1} \left[Q\right]_{x=x_0}^{x=x_1} dy$$

$$= -\int_{x_0}^{x_1} P(x,y_1) - P(x,y_0)\,dx + \int_{y_0}^{y_1} Q(x_1,y) - Q(x_0,y)\,dy$$

$$= \int_{x_0}^{x_1} P(x,y_0)\,dx + \int_{y_0}^{y_1} Q(x_1,y)\,dy + \int_{x_1}^{x_0} P(x,y_1)\,dx + \int_{y_1}^{y_0} Q(x_0,y)\,dy$$

$$= \oint_{\partial R} P\,dx + Q\,dy$$

Ulike områder:



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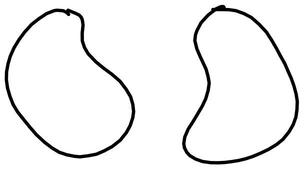

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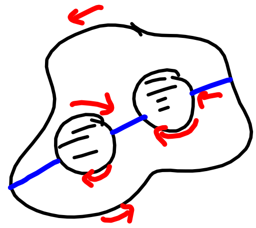
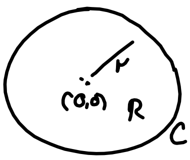


Eks
$$\vec{F}(x,y) = -\underbrace{\frac{y}{x^2+y^2}}_{=P}\,\vec{i} + \underbrace{\frac{x}{x^2+y^2}}_{=Q}\,\vec{j} \qquad (x,y)\neq(0,0)$$



$$\int_0^{2\pi} P\,dx + Q\,dy = \int_0^{2\pi} -r\frac{\sin\theta}{r^2}(-r\sin\theta\,d\theta) + \frac{r\cos\theta}{r^2}(r\cos\theta\,d\theta)$$

$x = r\cos\theta \quad dx = -r\sin\theta\,d\theta$

$y = r\sin\theta \quad dy = r\cos\theta\,d\theta$

$$= \int_0^{2\pi} \sin^2\theta + \cos^2\theta \; d\theta = \int_0^{2\pi} d\theta = 2\pi$$

$x^2+y^2 = r^2$

$$\frac{\partial Q}{\partial x} = \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

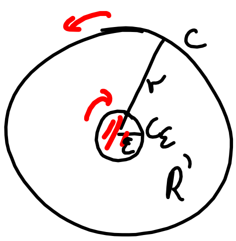
$$\frac{\partial P}{\partial y} = \frac{-(x^2+y^2)\cdot 1 + y\cdot 2y}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

$$\Rightarrow \iint_R 0\,dx\,dy = 0$$

Green?

Forklaring:

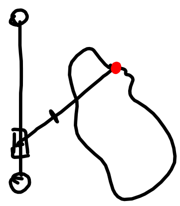


$$\int_{R'} P\,dx + Q\,dy = \int_C P\,dx + Q\,dy - \int_{C_\varepsilon} P\,dx + Q\,dy$$

$$= 2\pi - 2\pi = 0$$

La $\varepsilon \to 0$

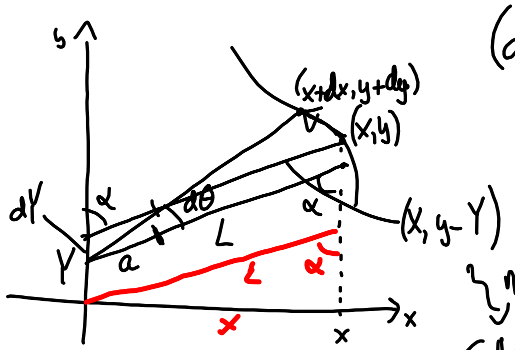
Planimeter



målingen på tellevarhet

$$ds = \sin\alpha \cdot dY + a \cdot d\theta = \frac{x}{L}\,dY + a\,d\theta$$

$$\underset{1.}{S} = \frac{1}{L}\int_{C'} x\,dY \qquad \underline{\int a\,d\theta = 0}$$



normalt

Tellevarhet $(Y-y, x)$

$$(dx, dy)\cdot\frac{1}{L}(Y-y, x) = \frac{1}{L}\left(Y-y\,dx + x\,dy\right)$$

$$\underset{2.}{S} = \frac{1}{L}\int_C Y-y\,dx + x\,dy$$

$$= \frac{1}{L}\int_{C'} Y\,dx - \frac{1}{L}\int y\,dx + \frac{1}{L}\int x\,dy$$

$$\underset{1.}{S} = \underset{2.}{S} \quad \text{som betyr}$$

$$\frac{1}{L}\int_{C'} Y\,dx - \frac{1}{L}\int y\,dx + \frac{1}{L}\int x\,dy = \frac{1}{L}\int_{C'} x\,dY$$

Flytter rundt

$$-\frac{2}{L}\int_{C'} x\,dY = \frac{1}{L}\int_{C'} Y\,dx - x\,dY = \frac{1}{L}\int y\,dx - \frac{1}{L}\int x\,dy = -\frac{2}{L}\int_C x\,dy$$

$$-\frac{2}{L}\int_{C'} x\,dY = -2S, \qquad -\frac{2}{L}\int_C x\,dy = \frac{-2}{L}\,\text{areal}(R)$$

Greens teorem

$$\Rightarrow \quad S = \frac{1}{L}\,\text{areal}(R)$$