Andrederiverttesten

Minner om: Alle egenverdiene til $Hf(\bar{a})$ positive: lokalt minimum

——— " ——— negative: —"— maksimum

Begge deler: Saddpunkt

For 2 variable:

$$Hf(\bar{a}) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(\bar{a}) & \frac{\partial^2 f}{\partial x \partial y}(\bar{a}) \\ \frac{\partial^2 f}{\partial x \partial y}(\bar{a}) & \frac{\partial^2 f}{\partial y^2}(\bar{a}) \end{pmatrix}$$

$\det\left(Hf(\bar{a})\right) = \lambda_1 \lambda_2 > 0$ når $\lambda_1, \lambda_2 > 0$ eller 1)
$\lambda_1, \lambda_2 < 0$

< 0 når vi har en av hver +/− 2)

se på 1)

$\frac{\partial^2 f}{\partial x^2}(\bar{a}) > 0$ lokalt minimum

< 0 —"— maksimum

Eks $f(x,y) = 4y - 2x + xy - 1$

$\frac{\partial f}{\partial x} = -2 + y \qquad \frac{\partial f}{\partial y} = 4 + x$

$\frac{\partial^2 f}{\partial x^2} = 0$

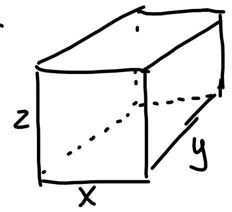
$\frac{\partial^2 f}{\partial x \partial y} = 1$

$\frac{\partial^2 f}{\partial y^2} = 0$

Hesse-determinanten: $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 < 0$

sadelpunkt.

Eks



Gitt volum $V = xyz$

Minimer areal

$A = 2xy + 2xz + 2yz$

Bruker: $z = \frac{V}{xy}$

$$A(x,y) = 2xy + 2x\frac{V}{xy} + 2y\frac{V}{xy}$$

$$= 2xy + \frac{2V}{y} + \frac{2V}{x}$$

stasjonære punkter:

$$\frac{\partial A}{\partial x} = 2y - \frac{2V}{x^2} = 0 \qquad \frac{\partial A}{\partial y} = 2x - \frac{2V}{y^2} = 0$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$y = \frac{V}{x^2} \qquad\qquad x = \frac{V}{y^2}$$

setter den ene inn i den andre

$$y = \frac{V}{\left(\frac{V}{y^2}\right)^2} \Rightarrow y = \frac{y^4}{V} \Rightarrow y = V^{\frac{1}{3}}$$

stasjonært punkt $x = y = z = V^{\frac{1}{3}}$ Kube.

$x = V^{\frac{1}{3}}$

$$\boxed{\frac{\partial^2 A}{\partial x^2} = \frac{4V}{x^3} \quad \frac{\partial^2 A}{\partial x \partial y} = 2 \quad \frac{\partial^2 A}{\partial y^2} = \frac{4V}{y^3}}$$

$$\det HA\left(V^{\frac{1}{3}}, V^{\frac{1}{3}}, V^{\frac{1}{3}}\right) = \begin{vmatrix} 4 & 2 \\ 2 & 4 \end{vmatrix} = 12 > 0$$

Ekstremalpunkt, siden $4 > 0$ må det være et lokalt minimumspunkt.

Lagrange metode (Finne maks/min under bibetingelser)

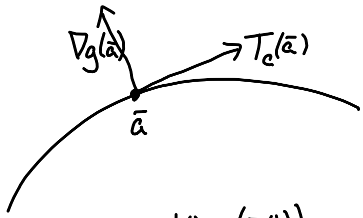
Funksjon: $f = f(x,y)$

Bibetingelse: $g(x,y) = 0$ (konstant)

Nivåkurve

Restriksjon av f til C: $\tilde{f}$

$C : g(x,y) = 0$



$C : g(x,y) = 0$

$\bar{r}(t) = (x(t), y(t))$

Ser på $h(t) = g(\bar{r}(t)) = g(x(t), y(t)) = \text{konstant}$

$$\Rightarrow h'(t) = \frac{\partial g}{\partial x} \cdot x'(t) + \frac{\partial g}{\partial y} y'(t) = \nabla g \cdot \underbrace{\bar{r}'(t)}_{T_c} = 0$$

Alt: $\nabla f(\bar{a}) \cdot T_c(\bar{a})$ måler endring av f langs med C i $\bar{a}$.

Skal finne maks/min til f langs C: $\nabla f(\bar{a}) \cdot T_c(\bar{a}) = 0$

Vi vet at $\nabla g(\bar{a}) \cdot T_c(\bar{a}) = 0$

$\Rightarrow \nabla g(\bar{a})$ og $\nabla f(\bar{a})$ er parallelle

$$\nabla g(\bar{a}) \parallel \nabla f(\bar{a})$$

Eks

$f(x,y) = xy$ langs $g(x,y) = x^2 + y^2 - 1 = 0$

$\nabla f = (y \; x)$ $\quad \nabla g = (2x, 2y)$

$$\begin{vmatrix} y & x \\ 2x & 2y \end{vmatrix} = 2y^2 - 2x^2 = 0 \quad \text{betyr } y = \pm x$$

Vi har og så $x^2 + y^2 - 1 = 0$, dvs $x^2 + x^2 - 1 = 2x^2 - 1 = 0$

dvs $x = \pm\frac{\sqrt{2}}{2}$ $\quad y = \pm\frac{\sqrt{2}}{2}$

4 ekstremalpunkter: $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$

Teorem 5.10.2

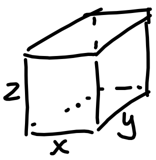
$f, g : U \to \mathbb{R}$ (C^1), $U \subset \mathbb{R}^m$ åpen, $b \in \mathbb{R}$

$\bar{x} \in U$ er et lokalt ekstremalpunkt for f på $A = \{\bar{x} \in U \mid g(\bar{x}) = b\}$

Da er $\nabla g(\bar{x}) = 0$ eller $\exists \lambda \in \mathbb{R}$ slik at

$$\nabla f(\bar{x}) = \lambda \cdot \nabla g(\bar{x})$$

Eks



$$A(x,y,z) = 2xy + 2xz + 2yz$$

$$\nabla A = (2y+2z,\ 2x+2z,\ 2x+2y)$$

$V = xyz$ (bibetingelse)

$$\nabla V = (yz,\ xz,\ xy)$$

Minimer areal

$\nabla A \parallel \nabla V$:

$$\left.\begin{aligned} 2y+2z &= \lambda \cdot yz \\ 2x+2z &= \lambda\, xz \\ 2x+2y &= \lambda\, xy \end{aligned}\right\} \quad \frac{y+z}{yz} = \frac{x+z}{xz} = \frac{x+y}{xy}$$

$$xyz = V \quad \Rightarrow \quad z = \frac{V}{xy}$$

gir
$$\frac{y + \frac{V}{xy}}{y \cdot \frac{V}{xy}} = \frac{x + \frac{V}{xy}}{x\,\frac{V}{xy}} = \frac{x+y}{xy} \implies \underline{\underline{x = y = z}} = V^{\frac{1}{3}}$$

Teorem 5.10.5

$f, g_1, .., g_k : U \longrightarrow \mathbb{R}$ (C^1), $U \subseteq \mathbb{R}^m$ åpen

$b_1, \dots, b_k \in \mathbb{R}$

$\bar{a} \in U$ ekstremalpunkt for f på

$$A = \{\bar{x} \in U \mid g_1(\bar{x}) = b_1, \dots, g_k(\bar{x}) = b_k\}$$

$\Rightarrow$ $\nabla g_1(\bar{a}), \dots, \nabla g_k(\bar{a})$ lineært avhengige

eller $\exists \lambda_i \in \mathbb{R}$ slik at $\nabla f(\bar{a}) = \sum_{i=1}^{k} \lambda_i \nabla g_i(\bar{a})$

Eks. $f(x,y,z) = x^2 + y^2 + z^2$, $\quad g_1(x,y,z) = x + 2y - z = 2$

$g_2(x,y,z) = -x + y + 2z = 1$

$\nabla f = (2x, 2y, 2z) \qquad \nabla g_1 = (1, 2, -1)$

$\nabla g_2 = (-1, 1, 2)$

$$\left.\begin{aligned}
2x &= \lambda_1 \cdot 1 + \lambda_2(-1) = \lambda_1 - \lambda_2 \\
2y &= \lambda_1 \cdot 2 + \lambda_2 \cdot 1 = 2\lambda_1 + \lambda_2 \\
2z &= \lambda_1(-1) + \lambda_2 \cdot 2 = -\lambda_1 + 2\lambda_2 \\
& x + 2y - z = 2 \\
& -x + y + 2z = 1
\end{aligned}\right\}$$

5 likninger med 5 ukjente: $x, y, z, \lambda_1, \lambda_2$

Løsning: $x = \frac{1}{7}$ $\quad y = \frac{34}{35}$ $\quad z = \frac{3}{35}$