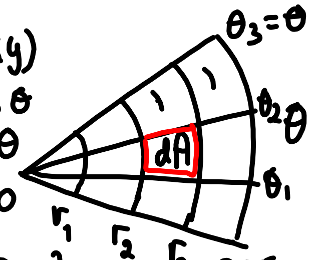
Dobbeltintegral i polarkoordinater

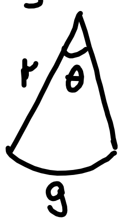
$z = f(x,y)$

$x = r \cdot \cos\theta$

$y = r \cdot \sin\theta$



$T: [0,R] \times [0,\theta]$



areal: $\frac{r \cdot g}{2}$

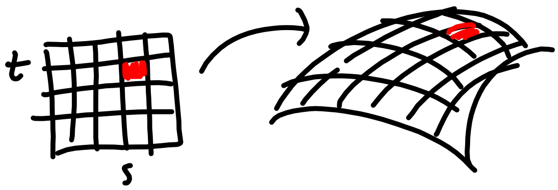
$g = r \cdot \theta$

Areal $dA = \frac{1}{2} r_i (\theta_j - \theta_{j-1}) r_i - \frac{1}{2} r_{i-1} (\theta_j - \theta_{j-1}) \cdot r_{i-1}$

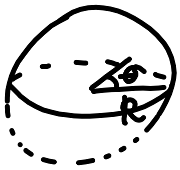
$= \underbrace{\frac{1}{2}(r_i + r_{i-1})}_{r_i^*} (r_i - r_{i-1})(\theta_j - \theta_{j-1})$

$$\iint_S f(x,y)\,dx\,dy = \iint_T f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Et spesialtilfelle av integrasjon over mer generelle flater



Eks.



$$x^2+y^2+z^2=R^2$$

$$z=\sqrt{R^2-x^2-y^2}$$

Polar koordinater

$$x=r\cos\theta$$

$$y=r\sin\theta$$

$$T:[0,R]\times[0,2\pi]$$

$$z=f(x,y)=\sqrt{R^2-(r\cos\theta)^2-(r\sin\theta)^2}$$

$$=\sqrt{R^2-r^2}$$

$$\text{Volum}=2\int_0^R\int_0^{2\pi}\sqrt{R^2-r^2}\,r\,d\theta\,dr$$

—o—o—o—

$$\int\sqrt{R^2-r^2}\cdot r\,dr=-\frac{1}{2}\int\sqrt{u}\,du$$

$$\boxed{\begin{aligned}u&=R^2-r^2\\ du&=-2r\,dr\end{aligned}}$$

$$=-\frac{1}{\cancel{2}}\frac{\cancel{2}}{3}u^{\frac{3}{2}}=-\frac{1}{3}(R^2-r^2)^{\frac{3}{2}}$$

—o—o—o—

$$=2\int_0^{2\pi}\left(\int_0^R\sqrt{R^2-r^2}\,r\,dr\right)d\theta$$

$$=2\int_0^{2\pi}\left[-\frac{1}{3}(R^2-r^2)^{\frac{3}{2}}\right]_{r=0}^{r=R}d\theta$$

$$=2\int_0^{2\pi}\left(\cancel{-\frac{1}{3}0^{\frac{3}{2}}}+\frac{1}{3}(R^2)^{\frac{3}{2}}\right)d\theta$$

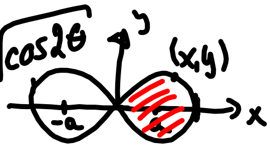
$$=\frac{2}{3}\int_0^{2\pi}R^3\,d\theta$$

$$=\frac{2R^3}{3}\int_0^{2\pi}d\theta=\frac{2R^3}{3}\cdot 2\pi$$

$$=\underline{\underline{\frac{4}{3}\pi R^3}}$$

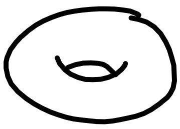
$0 \le r \le \sqrt{2}\, a \sqrt{\cos 2\theta}$

$-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$



Lemniscate.

Produktet av avstandene til to gitte punkter er konstant



Likning $(x^2+y^2)^2 = 2a^2(x^2-y^2)$

$x = r\cos\theta$

$y = r\sin\theta$

$r^4 = 2a^2 r^2 \cos 2\theta \quad \Big| \frac{1}{r^2}$

$r^2 = 2a^2 \cos 2\theta$

$$\text{Areal (av hele)} = 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\sqrt{2}a\sqrt{\cos 2\theta}} 1 \cdot r \cdot dr\, d\theta = 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\int_0^{\sqrt{2}a\sqrt{\cos 2\theta}} r\, dr \right) d\theta$$

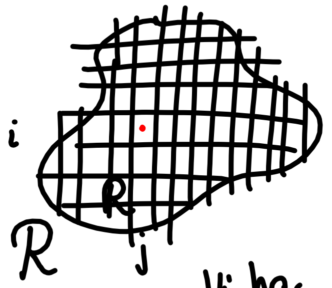
$$= 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\frac{1}{2} r^2\right]_{r=0}^{r=\sqrt{2}a\sqrt{\cos 2\theta}} d\theta$$

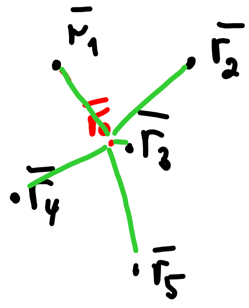
$$= 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{\cancel{2}} \cancel{2} \cdot a^2 \cos 2\theta \; d\theta = 2a^2 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2\theta\, d\theta$$

$$= 2a^2 \left[\frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \underline{\underline{2a^2}}$$

Anvendelser

Massemiddelpunkt.





$$\sum_{i=1}^{5}(\bar r_i - \bar r_0) = 0$$

$$\sum_{i=1}^{5}\bar r_i - \sum_{i=1}^{5}\bar r_0 = 0$$

$$\sum_{i=1}^{5}\bar r_i - 5\cdot\bar r_0 = 0$$

Vi har $\underline{\bar r_0 = \frac{1}{5}\sum_{i=1}^{5}\bar r_i}$

$$\sum_{i,j}\bar r_{ij}\,\Delta x_j\,\Delta y_i = \bar r_0\cdot\sum_{i,j}\Delta x_j\,\Delta y_i \approx \text{areal}(R)$$

$\bar r_0 = (x_0, y_0)$

$$\sum_{i,j}(x_0)_{ij}\,\Delta x_j\,\Delta y_i = x_0\cdot\text{areal}(R)$$

tilsv. for y.

$$\sum_{i,j}(y_0)_{ij}\,\Delta x_j\,\Delta y_i = y_0\,\text{areal}(R)$$

Forfiner partisjoner, antar integrabilitet

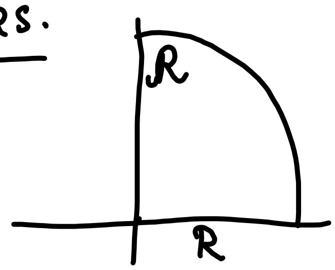
$$\iint_R x\,dx\,dy = x_0\cdot\text{areal}(R)$$

Vanlig notasjon: $\bar x$

Tyngdepkt: $$\bar x = \frac{1}{\text{areal}(R)}\iint_R x\,dx\,dy$$

$$\bar y = \frac{1}{\text{areal}(R)}\iint_R y\,dx\,dy$$

Eks.



Ved symmetri:

$$\bar{x} = \bar{y}$$

Areal: $\frac{\pi R^2}{4}$

$$\bar{x} = \frac{1}{\frac{\pi R^2}{4}} \int_0^{\frac{\pi}{2}} \int_0^{R} r \cdot \cos\theta \, r \, dr \, d\theta$$

$$= \frac{4}{\pi R^2} \int_0^{\frac{\pi}{2}} \frac{1}{3} R^3 \cos\theta \, d\theta$$

$$= \frac{4}{3\pi} R \int_0^{\frac{\pi}{2}} \cos\theta \, d\theta$$

$$= \frac{4}{3\pi} R \left[\sin\theta \right]_0^{\frac{\pi}{2}} = \underline{\underline{\frac{4}{3\pi} R}}$$