Diagonalisering av matriser

Setning 4.10.8 La A være en $n \times n$ matrise med egenvektorer $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ som danner en basis for $\mathbb{R}^n$, med tilhørende egenverdier $\lambda_1, \lambda_2, \ldots, \lambda_n$.

La

$$M = [\vec{v}_1 \ \vec{v}_2 \ \ldots \ \vec{v}_n]$$

være matrisen med egenvektorene som kolonner.

Da er

$$AM = MD$$

der $D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$ er en diagonal-matrise.

M er invertibel, så $M^{-1}AM = D$

og $A = MDM^{-1}$.

<u>Bevis</u> av $AM = MD$

$$M = [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n]$$

$$A\vec{v}_j = \lambda_j \vec{v}_j$$

$$AM = A[\vec{v}_1 \cdots \vec{v}_n] = [A\vec{v}_1 \ \cdots \ A\vec{v}_n]$$

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = [\lambda_1 \vec{e}_1 \ \cdots \ \lambda_n \vec{e}_n] \qquad \begin{array}{cc} & e_1 \\ [\quad M \quad] & \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{array} = [\quad]$$

$$MD = M[\lambda_1 \vec{e}_1 \cdots \lambda_n \vec{e}_n] = [\lambda_1 \vec{v}_1 \ \cdots \ \lambda_n \vec{v}_n]$$

$\square$

<u>Korollar 4.10.10</u>

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$$

<u>Bevis</u>

$$AM = MD \qquad A = MDM^{-1}$$

$$\det(A) = \cancel{\det(M)} \det(D) \cancel{\det}(M)^{-1}$$

$$= \det \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} = \lambda_1 \cdots \lambda_n$$

$\square$.

LH 4.11 „I praksis"

Oppgave 4.11.3 = Eksamen MAT1110 2008
Oppgave 3

År n År $n+1$

$\vec{r}_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix} \in \mathbb{R}^2$

x_n byttedyr

y_n rovdyr

$\rightsquigarrow$

x_{n+1} byttedyr

y_{n+1} rovdyr

$\vec{r}_{n+1} = \begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} \in \mathbb{R}^2$

Modell

$$x_{n+1} = 1.1\, x_n - 0.2\, y_n$$
$$y_{n+1} = 0.1\, x_n + 0.8\, y_n$$

$$A = \begin{bmatrix} 1.1 & -0.2 \\ 0.1 & 0.8 \end{bmatrix} \qquad \vec{r}_{n+1} = A\vec{r}_n$$

$$\vec{r}_n = A^n \vec{r}_0$$

Diagonaliserer A:

$$p_A(\lambda) = \begin{vmatrix} \lambda - 1.1 & 0.2 \\ -0.1 & \lambda - 0.8 \end{vmatrix} = \lambda^2 - 1.9\lambda + 0.9$$

har nullpunkter

$$\lambda = \frac{1{,}9 \pm \sqrt{1{,}9^2 - 4 \cdot 0{,}9}}{2} = 0{,}9 \text{ eller } 1{,}0$$

to egenverdier:

$$\lambda_1 = 0.9, \quad \lambda_2 = 1.0$$

Egenvektor $\vec{v}_1 = \begin{pmatrix} x \\ y \end{pmatrix}$ for $\lambda_1 = 0.9$:

$$\left[\begin{array}{cc|c} 0.9-1.1 & 0.2 & 0 \\ -0.1 & 0.9-0.8 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} -0.2 & 0.2 & 0 \\ -0.1 & 0.1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\uparrow$ fri $\quad x - y = 0$

$$\vec{v} = \begin{pmatrix} y \\ y \end{pmatrix} = y\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad y \neq 0$$

$$\underline{\underline{\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}}$$

Egenvektor $\vec{v}_2$ for $\lambda_2 = 1$:

$$(\lambda I_2 - A)\vec{v} = \vec{0}$$

$$\left[\begin{array}{cc|c} 1.0-1.1 & 0.2 & 0 \\ -0.1 & 1.0-0.8 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} -0.1 & 0.2 & 0 \\ -0.1 & 0.2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x - 2y = 0$$

$$v_2 = \begin{bmatrix} 2y \\ y \end{bmatrix} = y\begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \underline{\underline{\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}}}$$

$y \neq 0$

$$M = [\vec{v}_1 \ \vec{v}_2] = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

Gitt $x_0 = 3000$ $\vec{r}_0 = \begin{bmatrix} 3000 \\ 1000 \end{bmatrix}$

$y_0 = 1000$

Skriver $\vec{r}_0 = c_1\vec{v}_1 + c_2\vec{v}_2$ som lineær-kombinasjon av egenvektorene.

$$c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3000 \\ 1000 \end{bmatrix} = \vec{r}_0$$

$$\overset{\shortparallel}{M\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}} \qquad \left(\vec{c}_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad M\vec{c}_0 = \vec{r}_0 \right)$$

$$M\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \vec{r}_0$$

$$\left[\begin{array}{cc|c} 1 & 2 & 3000 \\ 1 & 1 & 1000 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -1000 \\ 0 & 1 & +2000 \end{array}\right] \leftarrow \vec{c}_0$$

$c_1 = -1000$

$c_2 = +2000$

$$\vec{r}_0 = (-1000)\vec{v}_1 + 2000\,\vec{v}_2 .$$

Etter n år:

$$\vec{r}_n = A^n\vec{r}_0 = A^n(-1000\vec{v}_1 + 2000\vec{v}_2)$$

$$= -1000\,A^n\vec{v}_1 + 2000\,A^n\vec{v}_2$$

$$= -1000\,\lambda_1^n\vec{v}_1 + 2000\,\lambda_2^n\vec{v}_2$$

$\lambda_1 = 0.9$

$\lambda_2 = 1.0$

$$= -1000\,(0.9)^n\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2000\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} -1000(0.9)^n + 4000 \\ -1000(0.9)^n + 2000 \end{bmatrix}$$

$$\boxed{\begin{aligned} x_n &= 4000 - 1000\,(0.9)^n \\ y_n &= 2000 - 1000\,(0.9)^n \end{aligned}}$$

2000 4000

når $n \to \infty$.

Generelle former

$\vec{r}_0$ kjent $\quad A$ kjent $n \times n$ matrise $\quad \boxed{\vec{r}_{k+1} = A\vec{r}_k}$

Antar A kan diagonaliseres: $AM = MD$

$$M = [\vec{v}_1 \cdots \vec{v}_n]$$

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$\vec{r}_k = c_{k,1}\vec{v}_1 + \dots + c_{k,n}\vec{v}_n$$

$$\vec{c}_k = \begin{bmatrix} c_{k,1} \\ \vdots \\ c_{k,n} \end{bmatrix} \qquad \boxed{\vec{r}_k = M\vec{c}_k}$$

$$M\vec{c}_{k+1} = \vec{r}_{k+1} = A\vec{r}_k = AM\vec{c}_k$$

$$\boxed{\vec{c}_{k+1} = M^{-1}AM\vec{c}_k = D\vec{c}_k}$$

$$\vec{c}_k = D^k\vec{c}_0 \qquad \begin{cases} c_{k,1} = \lambda_1^k c_{0,1} \\ \vdots \\ c_{k,n} = \lambda_n^k c_{0,n} \end{cases}$$

$$\vec{r}_k = M\vec{c}_k = MD^k\vec{c}_0 = MD^kM^{-1}\vec{r}_0$$

$$\vec{r}_k = [\vec{v}_1 \cdots \vec{v}_n]\begin{bmatrix} \lambda_1^k c_{0,1} \\ \vdots \\ \lambda_n^k c_{0,n} \end{bmatrix}$$

Opgs 4.11.10

t (år)

$x(t)$ = antall rovdyr ved tiden t

$y(t)$ = — " — byttedyr — " —

$x(0) = 500$

$y(0) = 1000$

(a) Modell A:

$$\begin{cases} x'(t) = x(t) + y(t) \\ y'(t) = -x(t) + y(t) \end{cases}$$

$$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \qquad \vec{r}\,'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad \vec{r}\,'(t) = A\vec{r}(t)$$

Diagonaliserer A: $AM = MD$

$$M = [\vec{v}_1 \ \vec{v}_2] \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\vec{r}(t) = c_1(t)\vec{v}_1 + c_2(t)\vec{v}_2 = M \begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix}$$

$$\vec{c}(t) = \begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix} \qquad \vec{r}(t) = M\vec{c}(t)$$

$$M\vec{c}\,'(t) = \vec{r}\,'(t) = A\vec{r}(t) = AM\vec{c}(t)$$

$$\vec{c}\,'(t) = \underbrace{M^{-1}AM}\,\vec{c}(t) = D\vec{c}(t)$$

$$\begin{cases} c_1'(t) = \lambda_1 c_1(t) \\ c_2'(t) = \lambda_2 c_2(t) \end{cases}$$

$y(t) = c_1(t)$

$y'(t) = \lambda_1 y(t)$

$y(t) = e^{\lambda_1 t} y(0)$

$$\Downarrow$$

$$\begin{cases} c_1(t) = c_1(0)\, e^{\lambda_1 t} \\ c_2(t) = c_2(0)\, e^{\lambda_2 t} \end{cases}$$

$$\vec{c}(t) = e^{tD}\vec{c}(0)$$

$$\vec{r}(t) = M\vec{c}(t) \qquad e^{tD} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_2 t} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \vec{r}(0) = \begin{bmatrix} 500 \\ 1000 \end{bmatrix}$$

$$P_A(\lambda) = \det(\lambda I_2 - A) = \begin{vmatrix} \lambda-1 & -1 \\ 1 & \lambda-1 \end{vmatrix} = \lambda^2 - 2\lambda + 2$$

$i = \sqrt{-1}$

egenverdiene til A er

$$\lambda = \frac{2 \pm \sqrt{(-2)^2 - 4\cdot 2}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1+i \text{ eller } 1-i$$

$$\boxed{\lambda_1 = 1+i \quad \text{og} \quad \lambda_2 = 1-i}$$

Egenvektorer for $\lambda_1 = 1+i$:

$$A\vec{v} = (1+i)\vec{v} \qquad A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$((1+i)I_2 - A)\vec{v} = \vec{0}$$

$$\left[\begin{array}{cc|c} 1+i-1 & -1 & 0 \\ 1 & 1+i-1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right] \cdot (-i)$$

$$\sim \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \begin{array}{l} x + iy = 0 \\ x = -iy \end{array}$$

fri

$$\vec{v} = \begin{bmatrix} -iy \\ y \end{bmatrix} = y\begin{bmatrix} -i \\ 1 \end{bmatrix} \quad y \neq 0$$

$$\underline{\vec{v}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}}$$

<u>Egenvektorer for $\lambda_2 = 1-i$</u>

$$\dots \quad \underline{\vec{v}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}}$$

$$M = [\vec{v}_1 \ \vec{v}_2] = \begin{bmatrix} -i & +i \\ 1 & 1 \end{bmatrix}$$

diagonaliserer A

$$AM = MD \qquad D = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}$$

$$\vec{r}(t) = M\vec{c}(t) = \underline{c_1(t)}\,\vec{v}_1 + \underline{c_2(t)}\,\vec{v}_2$$

$t=0$

$$M\vec{c}(0) = \vec{r}(0)$$

$$+i\uparrow \left[\begin{array}{cc|c} -i & i & 500 \\ 1 & 1 & 1000 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 1000 \\ 0 & 2i & 500+1000i \end{array}\right] \frac{1}{2i}$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 500+250i \\ 0 & 1 & 500-250i \end{array}\right]$$

$$\begin{cases} c_1(0) = 500+250i \\ c_2(0) = 500-250i \end{cases}$$

$$\vec{r}(0) = \begin{bmatrix} 500 \\ 1000 \end{bmatrix} = (500+250i)\begin{bmatrix} -i \\ 1 \end{bmatrix} + (500-250i)\begin{bmatrix} +i \\ 1 \end{bmatrix}$$

$$\vec{r}\,'(t) = A\,\vec{r}(t) \qquad \vec{r}(t) = M\vec{c}(t)$$

$$AM = MD$$

$$M c'(t) = r'(t) = A\vec{r}(t) = AM\vec{c}(t)$$

$$c'(t) = M^{-1}AM\,\vec{c}(t) = D\,\vec{c}(t)$$

$$\begin{cases} c_1'(t) = \lambda_1 c_1(t) = (1+i)\,c_1(t) \\ c_2'(t) = \lambda_2 c_2(t) = (1-i)\,c_2(t) \end{cases}$$

$$\begin{cases} c_1(t) = c_1(0)\,e^{(1+i)t} \\ \qquad = e^{(1+i)t}(500+250i) \\ c_2(t) = c_2(0)\,e^{(1-i)t} \\ \qquad = e^{(1-i)t}(500-250i) \end{cases}$$

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = M\vec{c}(t)$$

$$= \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{(1+i)t}(500+250i) \\ e^{(1-i)t}(500-250i) \end{bmatrix}$$

$$= \begin{bmatrix} e^{(1+i)t}(+250-500i) + e^{(1-i)t}(250+500i) \\ e^{(1+i)t}(500+250i) + e^{(1-i)t}(500-250i) \end{bmatrix}$$

$$e^{a+ib} = e^{a}(\cos b + i \sin b)$$

$$= \begin{bmatrix} e^{t}(\cos t + i\sin t)(250-500i) + e^{t}(\cos t - i \sin t)(250+500i) \\ \ldots \end{bmatrix}$$

$$= \begin{bmatrix} e^{t}(500\cos t + 1000 \sin t) \\ e^{t}(1000 \cos t - 500 \sin t) \end{bmatrix}$$

$$\boxed{\begin{aligned} x(t) &= e^{t}(500\cos t + 1000\sin t) \\ y(t) &= e^{t}(1000\cos t - 500\sin t) \end{aligned}}$$

Antallet byttedyr

$y(t) = e^{t}(1000\cos t - 500\sin t)$

$y(t) = 0$ når $1000\cos t = 500\sin t$

$2 = \tan t$

$t = \arctan 2 \approx 1{,}107$ år

$\approx 13{,}3$ måneder.

(b) Modell $\mathcal{B}$

$$x'(t) = -2x(t) + 4y(t)$$
$$y'(t) = x(t) - 2y(t)$$

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \quad \vec{r}\,'(t) = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} \vec{r}(t)$$

$$\mathcal{B} = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}$$

$$p_{\mathcal{B}}(\lambda) = \begin{vmatrix} \lambda+2 & -4 \\ -1 & \lambda+2 \end{vmatrix} = \lambda^2 + 4\lambda$$

$\lambda_1 = -4$ eller $\lambda_2 = 0$

egenvektor $\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ egenvektor $\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$M = \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix} \quad AM = MD$$
$$D = \begin{bmatrix} -4 & 0 \\ 0 & 0 \end{bmatrix}$$

$\vec{r}(t) = M\vec{c}(t)$ $\quad \vec{c}\,'(t) = D\vec{c}(t)$

$$\begin{cases} c_1'(t) = -4c_1(t) \\ c_2'(t) = 0 \end{cases} \qquad \begin{cases} c_1(t) = e^{-4t}c_1(0) \\ c_2(t) = c_2(0) \end{cases}$$

$$M\vec{c}(0) = \vec{r}(0) = \begin{bmatrix} 500 \\ 1000 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -2 & 2 & 500 \\ 1 & 1 & 1000 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 375 \\ 0 & 1 & 625 \end{array}\right] \begin{array}{l} = c_1(0) \\ = c_2(0) \end{array}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix}$$
$$= \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix} \begin{bmatrix} e^{-4t} \cdot 375 \\ 625 \end{bmatrix}$$

$$\begin{cases} x(t) = 1250 - 750e^{-4t} \longrightarrow 1250 \\ y(t) = 625 + 375e^{-4t} \longrightarrow 625 \end{cases}$$

når $t \to \infty$.