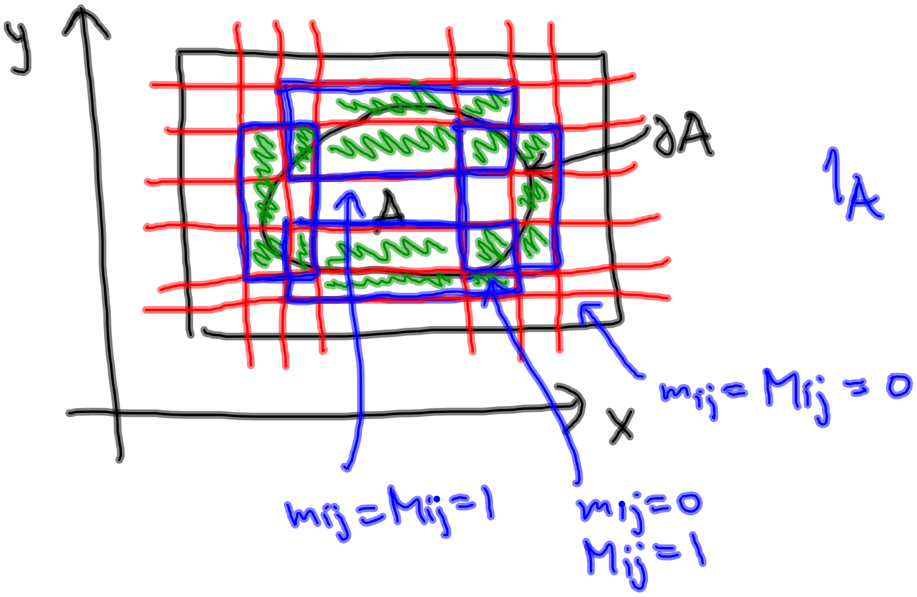
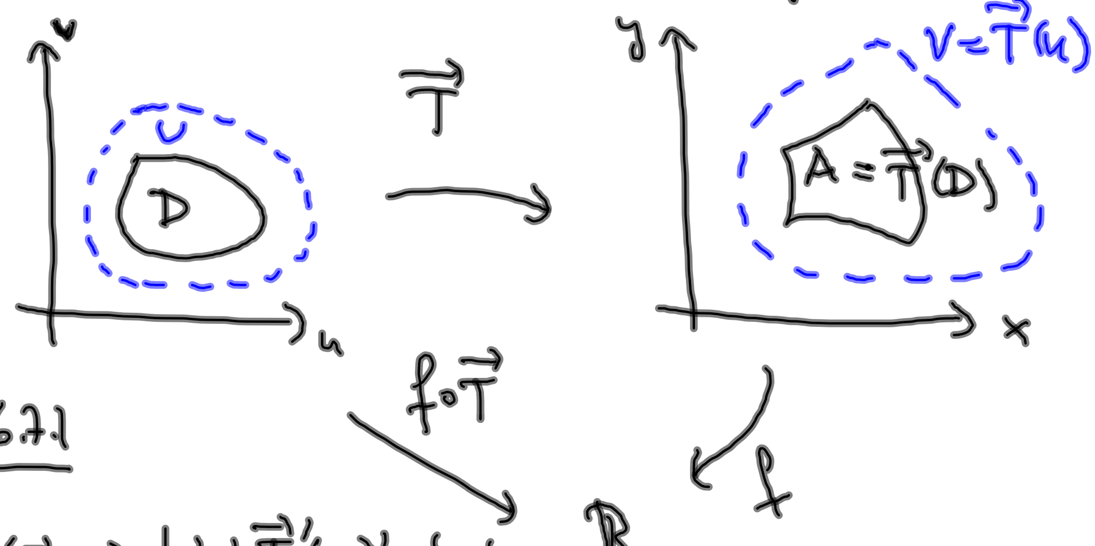

y
x
A
∂A
1_A
$m_{ij} = M_{ij} = 0$
$m_{ij} = M_{ij} = 1$
$m_{ij} = 0$
$M_{ij} = 1$

LH 6.7 Skifte av variable i dobbeltintegral



Thm 6.7.1

$$\iint_D f(\vec{T}(u,v))\,|\det \vec{T}'(u,v)|\,du\,dv = \iint_A f(x,y)\,dx\,dy$$

$\Uparrow$

$U \subseteq \mathbb{R}^2$ åpen, begrenset

$\vec{T} : U \longrightarrow \mathbb{R}^2$ injektiv med kontinuerlige partielle deriverte

$(\Rightarrow V = \vec{T}(U)$ åpen$)$

med $\det \vec{T}'(u,v) \neq 0$ for alle $(u,v) \in U$

$D \subset U$ lukket (begrenset) Jordan-målbar

$(A = \vec{T}(D) \subset V)$

$f : A = \vec{T}(D) \longrightarrow \mathbb{R}$ kontinuerlig.

$$D \xrightarrow{\vec{T}} A \qquad \vec{T}(u,v)$$

$D \subset \mathbb{R}^2$ (u,v), $A \subset \mathbb{R}^2$ (x,y)

$\vec{T}(u,v) = (x(u,v), y(u,v))$

$$\vec{T}'(u,v) = \begin{bmatrix} \frac{\partial x}{\partial u}(u,v) & \frac{\partial x}{\partial v}(u,v) \\ \frac{\partial y}{\partial u}(u,v) & \frac{\partial y}{\partial v}(u,v) \end{bmatrix}$$

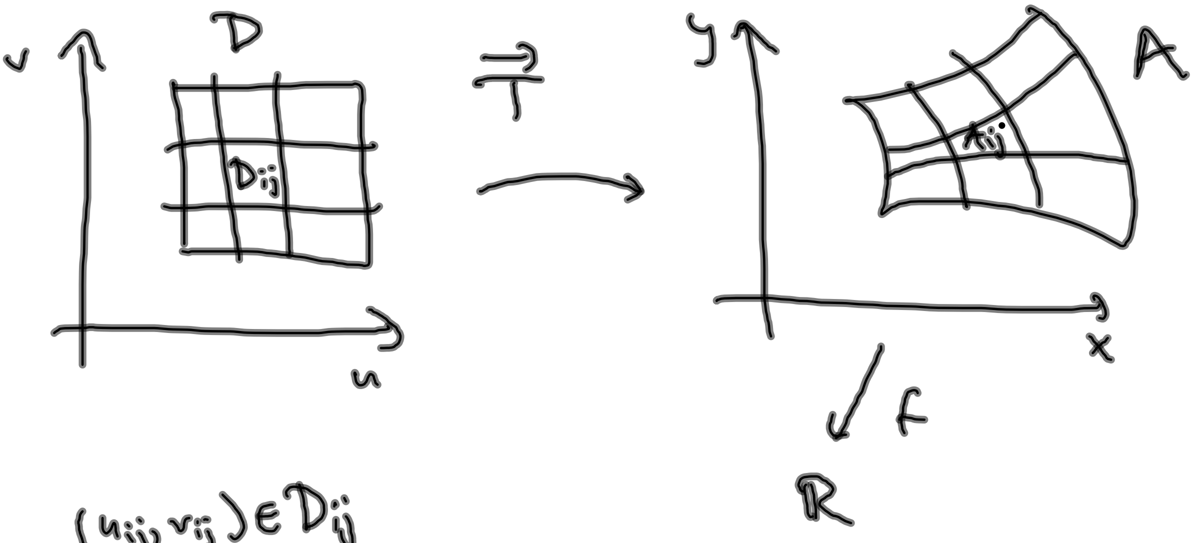
$$\det \vec{T}'(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} (u,v) \overset{\text{def}}{=} \frac{\partial(x,y)}{\partial(u,v)}(u,v)$$

$$\iint_A f(x,y)\,dx\,dy = \iint_D f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right|^{(u,v)} du\,dv$$

$\vec{T}(D) = A$ " $dx\,dy = \left| \frac{\partial(x,y)}{\partial(u,v)}(u,v) \right| du\,dv$ "

$t([c,d]) = [a,b]$ $\qquad x = x(u)$

$$\int_a^b f(x)\,dx = \int_c^d f(x(u)) \left| \frac{\partial x}{\partial u}(u) \right| du$$



$(u_{ij}, v_{ij}) \in D_{ij}$

$\vec{T}$ nær (u_{ij}, v_{ij}) er "nær" lineariseringen med matrise $\vec{T}'(u_{ij}, v_{ij})$ så

$$|A_{ij}| \approx |\det \vec{T}'(u_{ij}, v_{ij})| \; |D_{ij}|$$

$$(x_{ij}, y_{ij}) = \vec{T}(u_{ij}, v_{ij})$$

og
$$\iint_A f(x,y)\,dx\,dy \xleftarrow{\text{fin}} \sum_{i,j} f(\cancel{x_{ij}}, \cancel{y_{ij}})\,|A_{ij}|$$

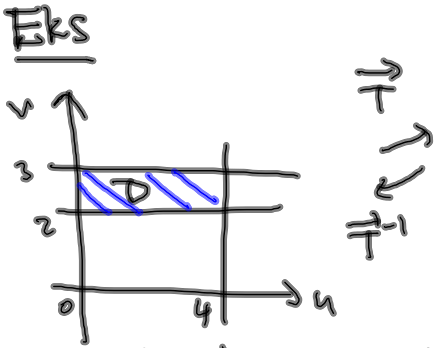
$$\approx$$

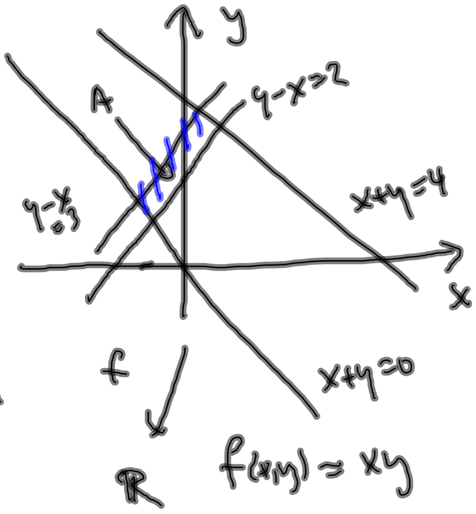
$$\sum_{i,j} f(\vec{T}(u_{ij}, v_{ij}))\,|\det \vec{T}'(u_{ij}, v_{ij})|\,|D_{ij}|$$

$$\swarrow \text{fin}$$

$$\iint_D f(\vec{T}(u,v))\,|\det \vec{T}'(u,v)|\;du\,dv$$

Eks





$u = y+x \quad 0 \le u \le 4$

$v = y-x \quad 2 \le v \le 3$

$$\iint_A xy \, dx dy =$$

$0 \le y+x \le 4$

$2 \le y-x \le 3$

$x = \frac{u-v}{2}$

$y = \frac{u+v}{2}$

$$\iint_D \underline{x(u,v)}\,\underline{y(u,v)} \left| \underline{\frac{\partial(x,y)}{\partial(u,v)}} \right| du dv$$

$\vec{T}(u,v) = \left(\frac{u-v}{2}, \frac{u+v}{2}\right) = (x(u,v), y(u,v))$

$$\vec{T}'(u,v) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\frac{\partial(x,y)}{\partial(u,v)}(u,v) = \det \vec{T}'(u,v) = \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix}$$

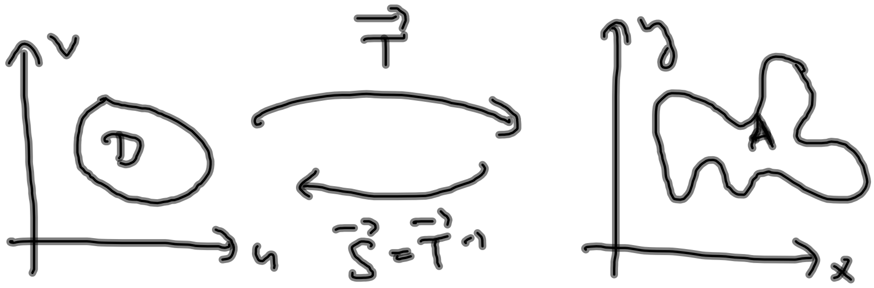
$$= 1/2 \cdot 1/2 - (-1/2 \cdot 1/2) = 1/2 > 0.$$

$$\int_2^3 \int_0^4 \left(\frac{u-v}{2}\right)\left(\frac{u+v}{2}\right) \left|\frac{1}{2}\right| du dv$$

$$= \frac{1}{8} \int_2^3 \int_0^4 u^2 - v^2 \, du \, dv = \frac{1}{8} \int_2^3 \left[\frac{1}{3}u^3 - uv^2\right]_0^4 dv$$

$$= \frac{1}{8} \int_2^3 \frac{64}{3} - 4v^2 \, dv = \frac{1}{8}\left[\frac{64}{3}v - \frac{4}{3}v^3\right]_2^3$$

$$= \frac{1}{8}\left(\frac{64}{3} - \frac{4}{3}19\right) = \frac{-12}{24} = -\frac{1}{2}$$

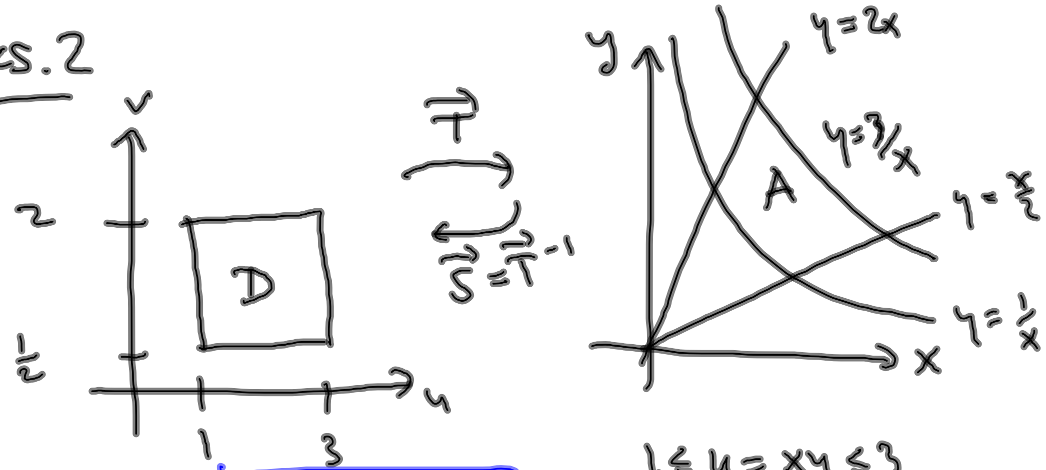


$$\det \vec{T}'(u,v) = \frac{\partial(x,y)}{\partial(u,v)} \quad \text{og}$$

$$\det \vec{S}'(x,y) = \frac{\partial(u,v)}{\partial(x,y)}$$

oppfyller $$\frac{\partial(x,y)}{\partial(u,v)} = 1 \Big/ \frac{\partial(u,v)}{\partial(x,y)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1}$$

Eks. 2



$$1 \le u = xy \le 3$$

$$(u,v) = \boxed{\vec{S}(x,y) = \left(xy, \frac{y}{x}\right)}$$

$$\frac{1}{2} \le v = \frac{y}{x} \le 2$$

$$\left[\begin{array}{l} (x,y) = \vec{T}(u,v) = \dots = \left(\sqrt{\frac{u}{v}}, \sqrt{uv}\right) \\ \dfrac{\partial(x,y)}{\partial(u,v)} = \dots = \dfrac{1}{2v} \end{array}\right]$$

$$\vec{S}'(x,y) = \begin{bmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{bmatrix}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \det \vec{S}'(x,y) = y\left(\frac{1}{x}\right) - x\left(-\frac{y}{x^2}\right) = \frac{2y}{x} = 2v$$

$$\iint_A \underbrace{\frac{x}{y}}_{= 1/v} dx\,dy = \iint_D \frac{x(u,v)}{y(u,v)} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \int_{1/2}^{2}\int_{1}^{3} \frac{1}{v} \cdot |2v|^{-1}\, du\, dv$$

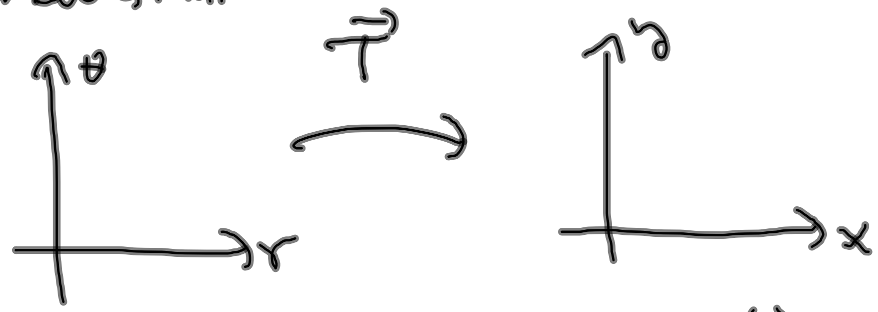
feil i boken!

$$= \int_{1/2}^{2}\int_{1}^{3} \frac{1}{2v^2}\, du\, dv = (3-1)\left[-\frac{1}{2v}\right]_{1/2}^{2}$$

$$= 2 \cdot \left(-\frac{1}{4} + 1\right) = \frac{3}{2}$$

RETTET 15/3

Polarkoordinater



$$\vec{T}(r,\theta) = (r\cos\theta, r\sin\theta)$$

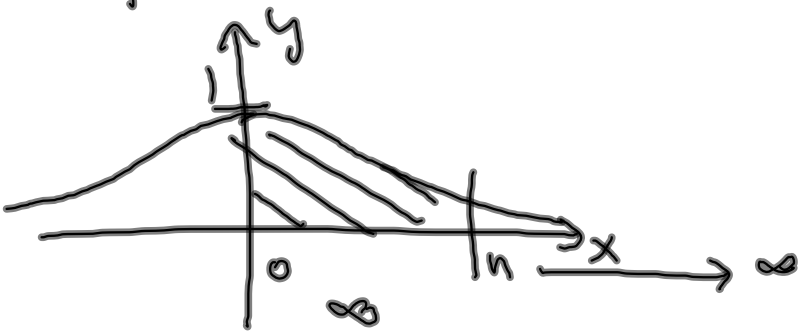
$$\vec{T}'(r,\theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$$

$$= r \geq 0$$

LH 6.8 Uegentlige integraler i planet

$f : \mathbb{R} \to \mathbb{R} \quad f(x) = e^{-x^2/2}$

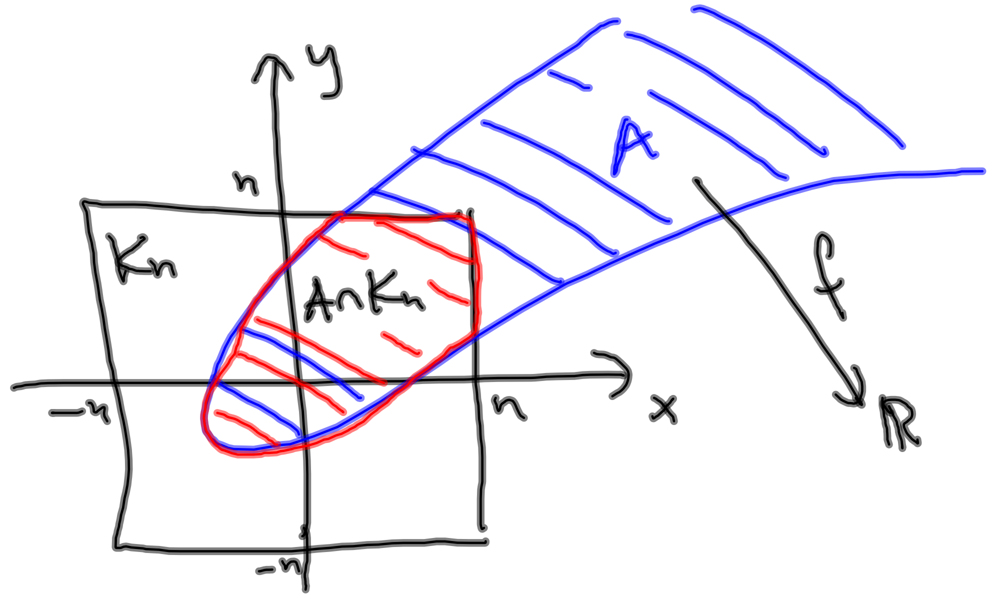


$$\int_0^n e^{-x^2/2}\,dx \xrightarrow[n\to\infty]{} \int_0^\infty e^{-x^2/2}\,dx$$

uegentlig

$K_n = [-n, n] \times [-n, n]$

$B_n = B(0, n) = \{(x,y) \mid x^2 + y^2 \leq n^2\}$



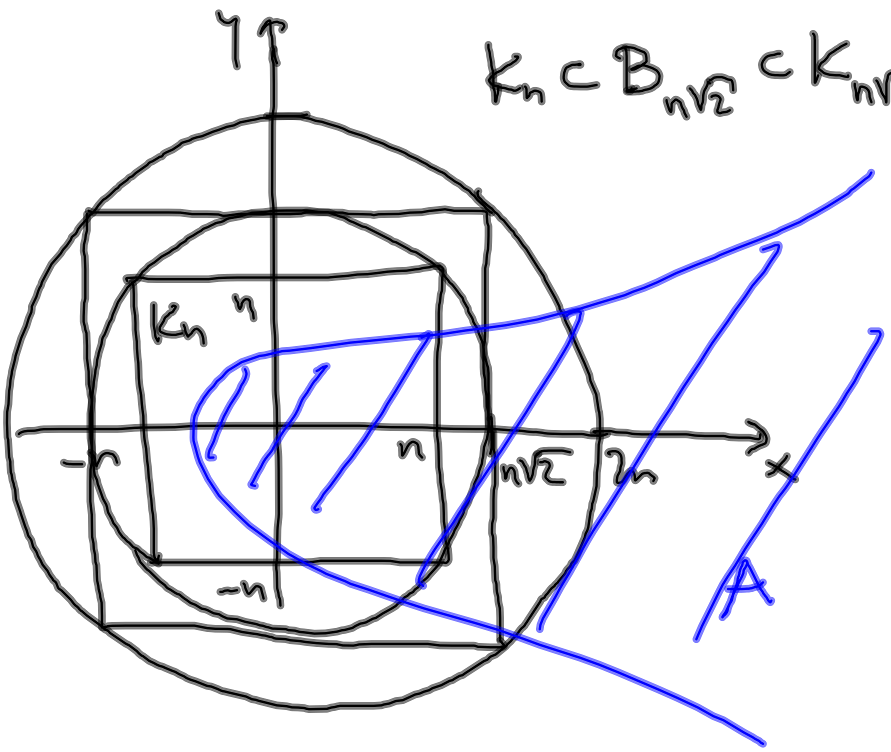
Hvis $f(x,y) \geq 0$ (f er ikke-negativ)

definerer vi

$$\iint_A f(x,y)\,dx\,dy = \lim_{n\to\infty} \iint_{A\cap K_n} f(x,y)\,dx\,dy$$

hvis grensen eksisterer.

$$K_n \subset B_{n\sqrt{2}} \subset K_{n\sqrt{2}} \subset B_{2n}$$



$$K_n \cap A \subset B_{n\sqrt{2}} \cap A \subset K_{n\sqrt{2}} \cap A \subset B_{2n} \cap A$$

$f \geq 0$

$$\iint_{K_n \cap A} f(x,y)\,dx\,dy \leq \iint_{B_{n\sqrt{2}} \cap A} f(x,y)\,dx\,dy$$

$$\leq \iint_{K_{n\sqrt{2}} \cap A} f(x,y)\,dx\,dy \leq \iint_{B_{2n} \cap A} f(x,y)\,dx\,dy$$

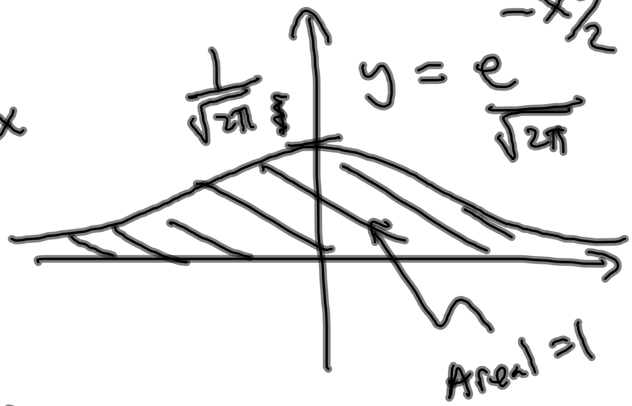
Hvis $\iint_A f(x,y)\,dx\,dy = \lim_{n \to \infty} \iint_{K_n \cap A} f(x,y)\,dx\,dy$

konvergerer $(< \infty)$

vil $\iint_A f(x,y)\,dx\,dy = \lim_{n \to \infty} \iint_{B_n \cap A} f(x,y)\,dx\,dy$

Eks

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}\,dx$$



$A = \mathbb{R}^2$

$f(x,y) = e^{-(x^2+y^2)/2}$

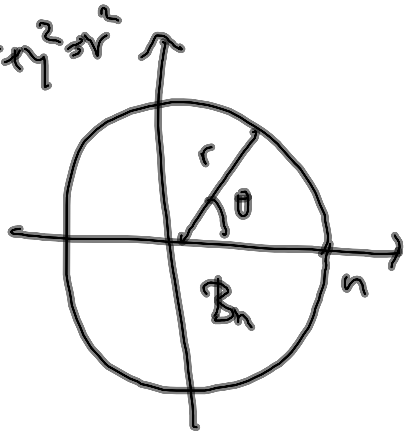
$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)/2}\,dx\,dy$$

$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2=r^2$

$$= \lim_{n\to\infty}\iint_{B_n} e^{-\frac{(x^2+y^2)}{2}}\,dx\,dy$$



$$= \lim_{n\to\infty}\int_0^{2\pi}\int_0^n e^{-r^2/2}\,r\,dr\,d\theta$$

$$= \lim_{n\to\infty} 2\pi\left[-e^{-r^2/2}\right]_0^n$$

$$= \lim_{n\to\infty} 2\pi\left(-e^{-n^2/2}+1\right) = \underline{\underline{2\pi}}$$

La $I_n = \int_{-n}^{n} e^{-x^2/2}\,dx$

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)/2}\,dx\,dy = \lim_{n\to\infty} \iint_{K_n} e^{-\frac{(x^2+y^2)}{2}}\,dx\,dy$$

$$= \lim_{n\to\infty} \int_{-n}^{n} \left(\int_{-n}^{n} e^{-\frac{(x^2+y^2)}{2}}\,dx \right) dy \qquad e^{-\frac{(x^2+y^2)}{2}} = e^{-x^2/2}\cdot e^{-y^2/2}$$

$$= \lim_{n\to\infty} \int_{-n}^{n} e^{-y^2/2} \underbrace{\left(\int_{-n}^{n} e^{-x^2/2}\,dx \right)}_{} dy$$

$$= \lim_{n\to\infty} \underbrace{\int_{-n}^{n} e^{-y^2/2} \cdot I_n \, dy}_{I_n} = \lim_{n\to\infty} I_n^2$$

$$\therefore\ 2\pi = \lim_{n\to\infty} I_n^2$$

$$\Longrightarrow \quad \lim_{n\to\infty} I_n = \sqrt{2\pi}$$

$$\lim_{n\to\infty} \int_{-n}^{n} e^{-x^2/2}\,dx = \int_{-\infty}^{\infty} e^{-x^2/2}\,dx$$

Uegentlige integraler av ikke nødv. ikke-neg. f

$$f(x,y) = f_+(x,y) - f_-(x,y)$$

$$|f(x,y)| = f_+(x,y) + f_-(x,y)$$

$$f_+(x,y) = \begin{cases} f(x,y) & \text{hvis } f(x,y) \geq 0 \\ 0 & \text{ellers} \end{cases}$$

$\iint_A f \, dx dy$ konvergerer

$\Updownarrow$

$\iint_A f_+ \, dx dy$ konv. $(<\infty)$ og $\iint_A f_- \, dx dy$ konv. $(<\infty)$

$\Updownarrow$

$\iint_A |f| \, dx dy$ konv. $(<\infty)$

Da er

$$\iint_A f(x,y)\, dx dy = \iint_A f_+(x,y)\, dx dy - \iint_A f_-(x,y)\, dx dy$$