Kapittel 4 : Lineær algebra i $\mathbb{R}^n$

lineære likningssystemer:

$x_1, x_2, \ldots, x_n$ ukjente tall i $\mathbb{R}$

kjente tall $a_{11}, \ldots, a_{mn}, b_1, \ldots, b_m$

$$\circledast \begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases}$$

m (lineære) likninger i n ukjente

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\circledast \qquad A\vec{x} = \vec{b}$$ matriseform

lineærtransformasjon

$$\vec{T} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$\vec{x} \longmapsto A\vec{x}$$

$$\text{ukjent} \longmapsto \overset{\shortparallel}{\vec{b}}$$

Matematisk modell

beskriver tilstanden

$$\begin{tikzcd} \vec{x} \arrow[r, mapsto] \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & \vec{F}(\vec{x}) = \vec{c} \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] \\ \mathbb{R}^n \arrow[r, "\vec{F}"] & \mathbb{R}^m \end{tikzcd}$$

målbar

ofte ikke lineær

men ofte deriverbar

vil løse $\vec{F}(\vec{x}) = \vec{c}$

Hvis $\vec{F}(\vec{a}) \approx \vec{c}$ i nærheten

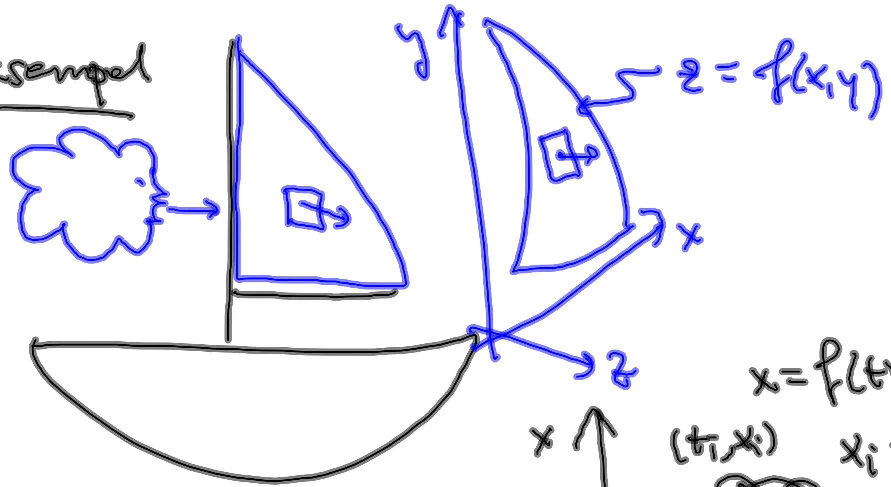
bør $\vec{x} \approx \vec{a}$. Da er

$$\vec{c} = \vec{F}(\vec{x}) \approx T_{\vec{a}}\vec{F}(\vec{x}) = \vec{F}'(\vec{a})(\vec{x} - \vec{a}) + \vec{F}(\vec{a})$$

$$\underbrace{\vec{F}'(a)}_{A}\vec{x} = \underbrace{\vec{F}'(\vec{a})\vec{a} - \vec{F}(\vec{a}) + \vec{c}}_{\vec{b}}$$

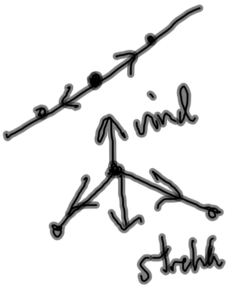
$$A\vec{x} = \vec{b}$$

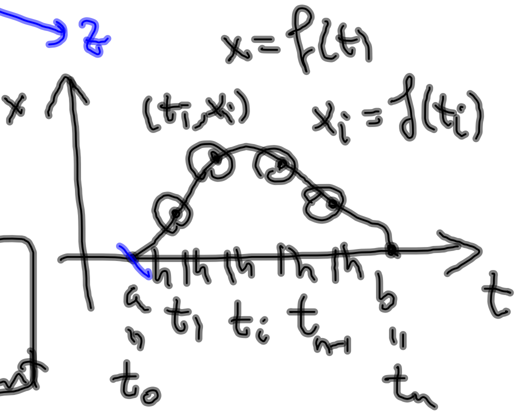
Eksempel



modell

$$\boxed{f''(t) = \underset{\text{kjent}}{g(t)}}$$





$f(t_0) = 0$

$f(t_n) = 0$

$x_0 = f(t_0) = 0$

$x_n = f(t_n) = 0$

$$\begin{cases} x_1 = f(t_1) \\ \vdots \\ x_{n-1} = f(t_{n-1}) \end{cases}$$

Hva er $f''(t)$?

$$f'(t_i) \approx \frac{f(t_{i+1}) - f(t_i)}{t_{i+1} - t_i}$$

$$\frac{x_{i+1} - x_i}{h}$$

$h = \frac{b-a}{n}$

$t_i = a + ih$

$t_{i+1} - t_i = h$

$0 \le n < n$

$$f''(t_i) \approx \frac{f'(t_i) - f'(t_{i-1})}{t_i - t_{i-1}}$$

$$\approx \frac{1}{h}\left(\frac{x_{i+1} - x_i}{h} - \frac{x_i - x_{i-1}}{h} \right)$$

$$= \frac{1}{h^2}\left(x_{i-1} - 2x_i + x_{i+1} \right)$$

$$\left.\begin{array}{l} h^2 f''(t_i) = h^2 g(t_i) = b_i \\ \quad\approx \\ \cancel{\frac{1}{h^2}}(x_{i-1} - 2x_i + x_{i-1}) \end{array}\right\} 0<i<n$$

$$\begin{array}{l} i=1 \\ x_0=0 \\ \\ \\ i=n-1 \\ x_n=0 \end{array} \left\{\begin{array}{llll} -2x_1 + x_2 & & & = b_1 \\ x_1 - 2x_2 + x_3 & & & = b_2 \\ & x_2 - 2x_3 + x_4 & & = b_3 \\ & & & \\ & & x_{n-2} - 2x_{n-1} & = b_{n-1} \end{array}\right.$$

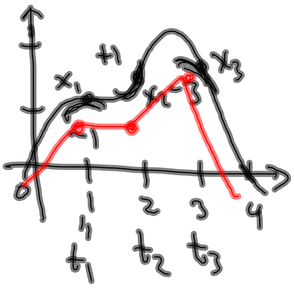
$$A = \begin{bmatrix} -2 & 1 & 0 & & & \\ 1 & -2 & 1 & 0 & & 0 \\ 0 & 1 & -2 & & & \\ & & & \ddots & 1 & \\ 0 & & 0 & 1 & -2 & 1 \\ & & & 0 & 1 & -2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} h^2 g(t_1) \\ \vdots \\ h^2 g(t_{n-1}) \end{bmatrix}$$

$$A\vec{x} = \vec{b} \qquad \vec{x} = \begin{bmatrix} f(t_1) \\ \vdots \\ f(t_{n-1}) \end{bmatrix}$$

Gauß-eliminasjon

$n=4$

$t_i = i$



$$\begin{cases} -2x_1 + x_2 = -1 \\ x_1 - 2x_2 + x_3 = +1 \\ x_2 - 2x_3 = -3 \end{cases}$$

$$\begin{array}{l} \text{I} \\ \text{II} \\ \text{III} \end{array} \begin{cases} x_1 - 2x_2 + x_3 = +1 \\ -2x_1 + x_2 = -1 \\ x_2 - 2x_3 = -3 \end{cases} \quad +2$$

$$\begin{array}{rl} 2x_1 - 4x_2 + 2x_3 = 2 & 2\text{I} \\ -2x_1 + x_2 = -1 & + \text{II} \\ \hline -3x_2 + 2x_3 = 1 & \end{array}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = +1 \\ -3x_2 + 2x_3 = +1 \\ x_2 - 2x_3 = -3 \end{cases}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 1 \\ x_2 - 2x_3 = -3 \\ -3x_2 + 2x_3 = 1 \end{cases} \quad +3$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 1 \\ x_2 - 2x_3 = -3 \\ -4x_3 = -8 \end{cases} \quad -\frac{1}{4}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 1 \\ x_2 - 2x_3 = -3 \\ x_3 = 2 \end{cases}$$

$$x_2 = 2x_3 - 3 = 2 \cdot 2 - 3 = 1$$

$$x_1 = 2x_2 - x_3 + 1 = 2 \cdot 1 - 2 + 1 = 1$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}}$$

På matriseform $\underset{\text{koeffisientmatrisen}}{A}\vec{x} = \vec{b}$ $\quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

utvidet matrise: $[A \mid b]$

$$\left[\begin{array}{ccc|c} -2 & 1 & 0 & -1 \\ 1 & -2 & 1 & 1 \\ 0 & 1 & -2 & -3 \end{array}\right] \circlearrowright$$

$$\updownarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ -2 & 1 & 0 & -1 \\ 0 & 1 & -2 & -3 \end{array}\right] \circlearrowright{+2} \quad \begin{matrix} \text{I} \\ \text{II} \end{matrix} \mapsto \begin{matrix} \text{I} \\ \text{II} + 2\text{I} \end{matrix}$$

$$-2 \downarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & -3 & 2 & 1 \\ 0 & 1 & -2 & -3 \end{array}\right] \circlearrowright$$

$$\text{II}$$
$$(\text{III} + \cancel{3\text{II}}) \cancel{-3\text{II}} = \text{III}$$

$$\updownarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 1 & -2 & -3 \\ 0 & -3 & 2 & 1 \end{array}\right] \circlearrowright{+3} \quad \begin{matrix} \text{II} \\ \text{III} \end{matrix} \mapsto \begin{matrix} \text{II} \\ \text{III} + 3\text{II} \end{matrix}$$

$$-3 \downarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & -8 \end{array}\right] \times\left(-\tfrac{1}{4}\right)$$

$$\times(-4) \left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$x_1 - 2x_2 + x_3 = 1 \qquad x_1 = 1$$
$$x_2 - 2x_3 = -3 \qquad x_2 = 1$$
$$x_3 = 2 \quad \leftrightarrow$$

Radoperasjoner

(1) Bytt om to rader $\quad \begin{matrix} I \\ II \end{matrix} \circlearrowright \begin{matrix} II \\ I \end{matrix}$

(2) Multipliser én rad med et tall $\neq 0$

$$I \xrightarrow{k \neq 0} kI$$

(3) Legg et multiplum av en rad til en annen rad (og behold den første raden) $\quad \begin{matrix} I \\ II \end{matrix} \Big\}{+a} \quad \begin{matrix} I \\ II + aI \end{matrix}$

Hvis vi kan komme fra $[A \mid b]$ til $[C \mid d]$ ved radoperasjoner sier vi at $[A \mid b]$ og $[C \mid d]$ er <u>radekvivalente</u>.

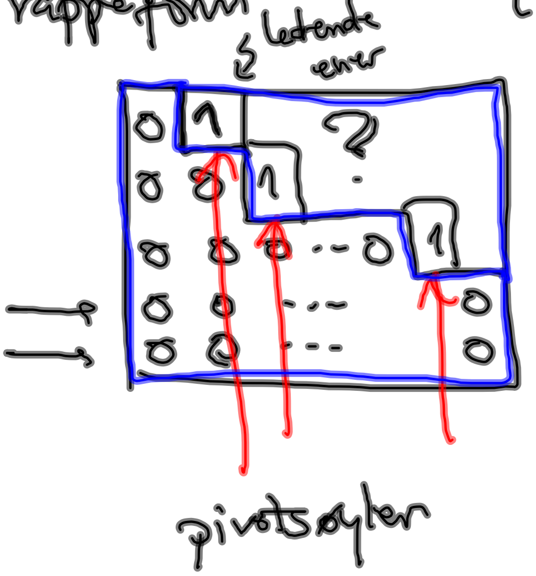
<u>Sats</u> Hvis $[A \mid b]$ og $[C \mid d]$ er radekvivalente, så har likningssystemet

$$A\vec{x} = b$$

de samme løsningene $\vec{x}$ som

$$C\vec{x} = d.$$

Trappeform (row echelon form)



antallet ledende nuller er strengt voksende

<u>Setning 4.2.3</u> Enhver matrise A (evnt. $[A|b]$) er radekvivalent med en matrise C (evnt. $[C|d]$) på trappeform.

Eks $A = \begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$ $\quad b = \begin{bmatrix}4\\7\\\cancel{10}\ 11\end{bmatrix}$ 11

$$x_1+2x_2+3x_3=4 \qquad A\vec{x}=b$$
$$\vdots$$

$$[A|b] = \left[\begin{array}{ccc|c}1&2&3&4\\4&5&6&7\\7&8&9&\cancel{10}\ 11\end{array}\right] \begin{matrix}-4\\-7\end{matrix} \sim \left[\begin{array}{ccc|c}1&2&3&4\\0&-3&-6&-9\\0&-6&-12&\cancel{-18}\ -17\end{array}\right]\left(-\tfrac{1}{3}\right)$$

$$\sim \left[\begin{array}{ccc|c}1&2&3&4\\0&1&2&3\\0&-6&-12&\cancel{-18}\ -17\end{array}\right] +6 \sim \left[\begin{array}{ccc|c}1&2&3&4\\0&1&2&3\\0&0&0&\cancel{0}\ 1\end{array}\right]$$

x_3 fri variabel

$$\ast \quad \begin{cases} x_1+2x_2+3x_3=4\\ \qquad x_2+2x_3=3\\ \qquad\qquad 0=\cancel{0}\ 1\end{cases}$$

$$x_1 = -2x_2-3x_3+4 = -2(3-2t)-3t+4 = t-2$$
$$x_2 = -2x_3+3 = 3-2t$$
$$x_3 = t$$

$$\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}t-2\\3-2t\\t\end{bmatrix} \text{ for } t\in\mathbb{R}$$