Plenum 13/4-15

4.3: 5, 7

4.4: 3, 4, 5

4.5: 3ab, 6, 7

4.3: Redusert trappeform

5.) $\begin{cases} 2x - y + z = b_1 \\ -x + 3y + 2z = b_2 \\ 3x - 4y - z = b_3 \end{cases}$

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 3 & 2 \\ 3 & -4 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 3 & 2 \\ 2 & -1 & 1 \\ 3 & -4 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 3 & 2 \\ 0 & 5 & 5 \\ 0 & 5 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \text{Ikke identitets-matrisen.}$$

Dermed har ligningssystemet ikke entydig løsning for alle mulige b_1, b_2, b_3 (fra Set. 4.3.3).

7) $x + y - z + 2u - v = 1$

$-2x - 2y + z - u + v = 2$

$3x + 3y \quad - 2u + 2v = 1$

Utvidet matrise:

$$\begin{bmatrix} 1 & 1 & -1 & 2 & -1 & 1 \\ -2 & -2 & 1 & -1 & 1 & 2 \\ 3 & 3 & 0 & -2 & 2 & 1 \end{bmatrix} \overset{\text{MATLAB}}{\sim} \begin{bmatrix} 1 & 1 & 0 & 0 & 2 & 7 \\ 0 & 0 & 1 & 0 & 7 & 26 \\ 0 & 0 & 0 & 1 & 2 & 10 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} x + y + 2v &= 7 \\ z + 7v &= 26 \\ u + 2v &= 10 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= 7 - y - 2v \\ z &= 26 - 7v \\ u &= 10 - 2v \end{aligned}$$

, y og v er frie variable.

$$\begin{bmatrix} x \\ y \\ z \\ u \\ v \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 26 \\ 10 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + v \begin{bmatrix} -2 \\ 0 \\ -7 \\ -2 \\ 1 \end{bmatrix}$$

4.4: Matriseligninger

3) Løs: $A\vec{x}_1 = \vec{b}_1$ og $A\vec{x}_2 = \vec{b}_2$ der $A = \begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & -1 \\ 0 & 2 & 1 \end{bmatrix}$, $\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

og $\vec{b}_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

Utvidet matrise; Simultanløsning

$$\begin{bmatrix} -2 & 1 & 3 & 1 & 2 \\ 1 & 2 & -1 & 0 & 2 \\ 0 & 2 & 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ -2 & 1 & 3 & 1 & 2 \\ 0 & 2 & 1 & -1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 5 & 1 & 1 & 6 \\ 0 & 2 & 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} & \frac{6}{5} \\ 0 & 2 & 1 & -1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -\frac{7}{5} & -\frac{2}{5} & -\frac{7}{5} \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} & \frac{6}{5} \\ 0 & 0 & \frac{3}{5} & -\frac{7}{5} & -\frac{7}{5} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{7}{5} & -\frac{2}{5} & -\frac{7}{5} \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} & \frac{6}{5} \\ 0 & 0 & 1 & -\frac{7}{3} & -\frac{7}{3} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -\frac{11}{3} & -\frac{11}{3} \\ 0 & 1 & 0 & \frac{2}{3} & \frac{5}{3} \\ 0 & 0 & 1 & -\frac{7}{3} & -\frac{7}{3} \end{bmatrix}$$

$$\vec{x}_1 = \begin{bmatrix} -\frac{11}{3} \\ \frac{2}{3} \\ -\frac{7}{3} \end{bmatrix} = \underline{\underline{\frac{1}{3}\begin{bmatrix} -11 \\ 2 \\ -7 \end{bmatrix}}} \qquad \vec{x}_2 = \begin{bmatrix} -\frac{11}{3} \\ \frac{5}{3} \\ -\frac{7}{3} \end{bmatrix} = \underline{\underline{\frac{1}{3}\begin{bmatrix} -11 \\ 5 \\ -7 \end{bmatrix}}}$$

4) $A = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & -1 & 1 \\ 6 & 0 & -6 & 7 \\ 2 & 1 & 0 & 0 \end{bmatrix}$, $A\vec{x} = \vec{b}$; $\vec{b} = \begin{bmatrix} 0 \\ 1 \\ h \\ 0 \end{bmatrix}$

a) og b)

Utvidet matrise:

$$\begin{bmatrix} 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & -1 & 1 & 1 \\ 6 & 0 & -6 & 7 & h \\ 2 & 1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 6 & 0 & -6 & 7 & h \\ 2 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & h-6 \\ 0 & 1 & 2 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & h-6 \\ 0 & 0 & 0 & -2 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & h-6 \\ 0 & 0 & 0 & 0 & 2h-14 \end{bmatrix}$$

A på trappeform

siste ligning

Hvis $2h-14 \neq 0$, dvs. $h \neq 7 \Rightarrow 0 =$ noe som ikke er null

$\Rightarrow$ Systemet har ingen løsninger.

Hvis $h=7$: Systemet har ∞ mange løsninger siden søyle 3 ikke er en pivotsøyle, dvs. x_3 er en fri variabel.

$$\underline{x_4} = h-6 = 7-6 = \underline{1}$$

$$\underline{x_2} = 0 - 2x_3 = \underline{-2x_3}$$

$$\underline{x_1} = 1 + x_3 - x_4 = 1 + x_3 - 1 = \underline{x_3} \quad \text{og } x_3 \text{ er fri.}$$

Dvs.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

5.) a) og b)

$$C = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 2 & 1 & a^2-a & 3 \\ -1 & 1 & -3 & a \end{array}\right] \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & a^2-a-2 & 1 \\ 0 & 1 & -2 & a+1 \end{bmatrix}$$

$$\underbrace{\begin{matrix}\end{matrix}}_{A} \qquad \underbrace{}_{\vec{b}}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & a^2-a-2 & 1 \\ 0 & 0 & -a^2+a & a \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & a^2-a-2 & 1 \\ 0 & 0 & -a(a-1) & a \end{bmatrix}}_{\text{Trappeform}}$$

3 tilfeller: • $a \notin \{0, 1\}$: Dvs. at $a \neq 0$, og $-a(a-1) \neq 0$

$$C \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & a^2-a-2 & 1 \\ 0 & 0 & 1 & -\frac{1}{a-1} \end{bmatrix}$$

Dvs: Entydig løsning (3 pivotsøyler).

• $a = 0$: $C \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Dvs: ∞ mange løsninger (Søyle 3 er ikke pivot $\Rightarrow$ z er en fri variabel)

• $a = 1$: $C \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ Dvs: Siste ligning; $0 = 1$; usant! Systemet har ingen løsninger.

4.5: Inverse matriser

$$\begin{aligned} x + 2y \quad &= 5 \\ y + z &= 3 \\ -2y + z &= 3 \end{aligned}$$

6.) a) & b)

$$\left[\begin{array}{cccccc|c} 1 & 2 & 0 & 1 & 0 & 0 & 5 \\ 0 & 1 & 1 & 0 & 1 & 0 & 3 \\ 0 & -2 & 1 & 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccccccc} 1 & 2 & 0 & 1 & 0 & 0 & 5 \\ 0 & 1 & 1 & 0 & 1 & 0 & 3 \\ 0 & 0 & 3 & 0 & 2 & 1 & 9 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccccc} 1 & 2 & 0 & 1 & 0 & 0 & 5 \\ 0 & 1 & 1 & 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 & \frac{2}{3} & \frac{1}{3} & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccccc} 1 & 0 & 0 & 1 & -\frac{2}{3} & \frac{2}{3} & 5 \\ 0 & 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & \frac{2}{3} & \frac{1}{3} & 3 \end{array}\right]$$

$$B^{-1} = \underline{\underline{\begin{pmatrix} 1 & -\frac{2}{3} & \frac{2}{3} \\ 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}}} \quad \text{og} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underline{\underline{\begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix}}}$$

<u>Alt</u>: b) $B\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix} \Rightarrow \underbrace{B^{-1}B}_{I}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = B^{-1}\begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = B^{-1}\begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -\frac{2}{3} & \frac{2}{3} \\ 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 - \cancel{2} + \cancel{2} \\ 0 + \cancel{1} - \cancel{1} \\ 0 + 2 + 1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix}}}$$

c) $\begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & -2 & a+1 & b^2-10 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & a+3 & b^2-4 \end{bmatrix}$

- Hvis $a \neq -3$: 3 pivotsøyler $\Rightarrow$ Systemet har entydig løsning.

- Hvis $a = -3$: $\sim \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & b^2-4 \end{bmatrix}$

  $\Rightarrow$ Hvis $b^2 - 4 = 0$, dvs. $b \in \{-2, 2\}$: Har 2 pivotsøyler (søyle 3 er ikke pivot) $\Rightarrow$ z er fri variabel og det er ∞ mange løsninger.

  $\Rightarrow$ $b \notin \{-2, 2\}$: Da er $b^2 - 4 \neq 0 \Rightarrow$ Siste ligning: $0 =$ noe ikke-null $\Rightarrow$ Usant! Så systemet har ingen løsninger.

7) $\vec{x} A = \vec{b}$

Vis: $\vec{x} = \vec{b} A^{-1}$ er unik løsning av $\vec{x} A = \vec{b}$. $(\star)$

Bevis: i) $\vec{x}$ er løsning: Setter inn:

Def. av $\vec{x}$

Def. av invers

VS. i $(\star)$ $= (\vec{b} A^{-1}) A = \vec{b} (A^{-1} A) = \vec{b} I = \vec{b} =$ HS. i $(\star)$

Så $\vec{x}$ løser $(\star)$.

ii) Fins ingen andre løsninger: Antar at vi har to løsninger, dvs at $\vec{x}$ og $\vec{y}$ løser $(\star)$.

Da er:

$$\vec{x} A - \vec{y} A = \vec{b} - \vec{b} = \vec{0} \qquad \text{(I)}$$

løser $(\star)$

Men:

$$\vec{x} A - \vec{y} A = (\vec{x} - \vec{y}) A \qquad \text{(II)}$$

Fra (I) og (II):

$$(\vec{x} - \vec{y}) A = \vec{0}$$

$\Downarrow$ (A inverterbar)

$$(\vec{x} - \vec{y}) A A^{-1} = \vec{0} A^{-1}$$

$$(\vec{x} - \vec{y}) I = \vec{0}$$

$$\vec{x} - \vec{y} = 0$$

$$\vec{x} = \vec{y}$$

Dvs: Fins kun en løsning.