Green's teorem.

Analog til $f(b)-f(a)=\int_a^b f'(x)\,dx$.

$=$

Vektorintegral langs kurver. $r(t)=(x(t),y(t))$ $t\in[a,b]$ kurve C

$$\int_C F\cdot dr=\int_a^b \underline{F(r(t))\cdot r'(t)}\,dt$$

$F=Pi+Qj=(P,Q)$ $P=P(x,y)$ $Q=Q(x,y)$.

$F(r(t))=(P(x(t),y(t)),Q(x(t),y(t)))$ $r'(t)=(x'(t),y'(t))$.

$F(r(t))\cdot r'(t)=P(x(t),y(t))x'(t)+Q(x(t),y(t))y'(t)$

$$\int_C F\cdot dr=\int_a^b P(x(t),y(t))x'(t)+Q(x(t),y(t))y'(t)\,dt$$

$$=\int P(x,y)dx+Q(x,y)dy$$

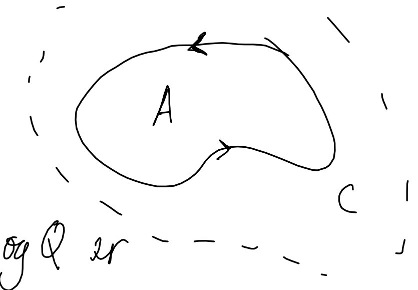
$$=:\int_C Pdx+Qdy$$

" $x'(t)dt=\frac{dx}{dt}dt=dx$ "

" $y'(t)dt=\frac{dy}{dt}dt=dy$ "

Greene's teorem:

C enkeltlukket kurve i $\mathbb{R}^2$, orientert mot klokka.



A er området inne i C.

$F(x,y) = (P(x,y), Q(x,y))$ de partiellderiverte til P og Q er kontinuerlige i et område som omfatter A og C.

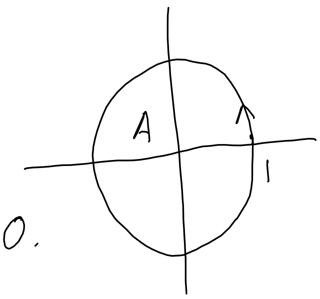
$$\int_C P\,dx + Q\,dy = \iint_A \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$$

$$f(b) - f(a) = \int_a^b \frac{df}{dx}\,dx$$

Eksempel

$F = (xy^2, yx^2) \qquad P = xy^2 \quad Q = x^2 y$

C gitt ved $r(t) = (\cos(t), \sin(t)) \quad t \in [0, 2\pi]$.

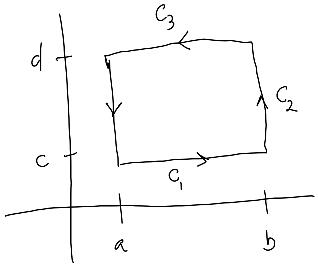


$$\int_C xy^2\,dx + x^2y\,dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_A 0\,dx\,dy = 0.$$

$$\frac{\partial Q}{\partial x} = 2xy \quad \frac{\partial P}{\partial y} = 2xy$$

Argumentasjon for at Greene holder.

Anta A rektangel $A=[a,b]\times[c,d]$



C består av 4 deler.

$C_1: r_1(x)=(x,c) \quad x\in[a,b]$

$C_2: r_2(y)=(b,y) \quad y\in[c,d]$

$C_3: r_3(x)=(x,d) \quad x\in[b,a]$ x går fra b til a!

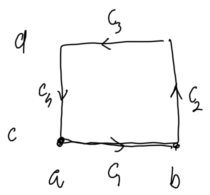
$C_4: r_4(y)=(a,y) \quad y\in[c,d]$ y går fra d til c!

$r_1'(x)=(1,0) \quad r_2'(y)=(0,1) \quad r_3'(x)=(-1,0) \quad r_4'(y)=(0,-1)$

$$\iint_A \frac{\partial Q}{\partial x}\,dx\,dy = \int_c^d\left[\int_a^b \frac{\partial Q}{\partial x}(x,y)\,dx\right]dy = \int_c^d Q(b,y)-Q(a,y)\,dy = \int_c^d Q(b,y)\,dy - \int_c^d Q(a,y)\,dy$$

$$-\iint_A \frac{\partial P}{\partial y}\,dx\,dy = -\int_a^b\left[\int_c^d \frac{\partial P}{\partial y}(x,y)\,dy\right]dx = -\int_a^b P(x,d)-P(x,c)\,dx = \int_a^b P(x,c)\,dx - \int_a^b P(x,d)\,dx$$

$$\iint_A \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\,dx\,dy = \int_a^b P(x,c)\,dx + \int_c^d Q(b,y)\,dy - \int_a^b P(x,d)\,dx - \int_c^d Q(a,y)\,dy$$

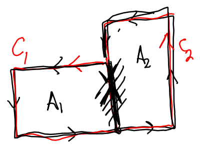


$$= \int_{C_1} P\,dx + Q\,\overset{0}{dy} + \int_{C_2} \overset{0}{P\,dx} + Q\,dy + \int_{C_3} P\,dx + \overset{0}{Q\,dy} + \int_{C_4} \overset{0}{P\,dx} + Q\,dy$$

$$= \int_C P\,dx + Q\,dy.$$

OK for rekt.

Område sammensatt av 2 rekt.



$A = A_1 \cup A_2$

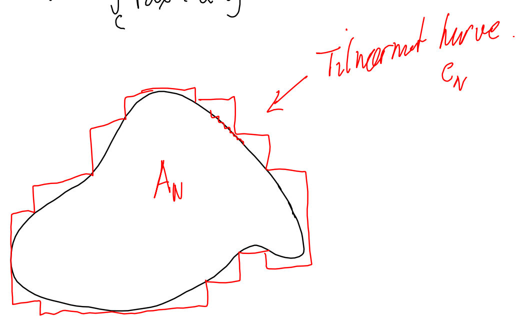
$$\iint_A \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\,dx\,dy = \iint_{A_1} \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\,dx\,dy + \iint_{A_2} \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\,dx\,dy$$

$$= \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy$$

$$= \int_C P\,dx + Q\,dy$$

Kan skjøte på med fler rekt.

Gjelder med N rekt.



$$\iint_{A_N} \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\,dx\,dy = \int_{C_N} P\,dx + Q\,dy$$

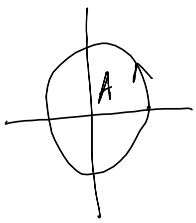
$\downarrow N\to\infty$ bedre og bedre tilnærming.

$$\iint_A \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\,dx\,dy = \int_C P\,dx + Q\,dy.$$

Eks

$$F = \left(\underbrace{\frac{-y}{x^2+y^2}}_{P}, \underbrace{\frac{x}{x^2+y^2}}_{Q} \right)$$

C gitt ved $r(t) = (\cos(t), \sin(t))$ $t \in [0, 2\pi]$



$$F(r(t)) = \left(-\frac{\cancel{1}\sin t}{\cancel{1}}, \frac{\cancel{1}\cos t}{\cancel{1}} \right)$$

$$r'(t) = (-\sin t, \cos(t)) \qquad F(r(t)) \cdot r'(t) = 1.$$

$$\int_C F \cdot dr = \int_0^{2\pi} 1 \, dt = \underline{\underline{2\pi}}.$$

$$\parallel$$

$$\iint_A \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dx dy = 0 \quad !$$

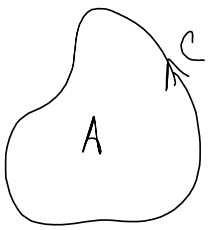
Greenes teorem?

$$\frac{\partial Q}{\partial x} = \frac{x^2+y^2 - x\,2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(\quad)^2}$$

$$\frac{\partial P}{\partial y} = -\frac{x^2-y^2}{(\quad)^2} = \frac{y^2-x^2}{(\quad)^2}$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

Arealer med Greens teorem.



Areal av $(A) = \iint_A dx\,dy$

$= \iint_A \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$ hvis $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$.

$= \int_C P\,dx + Q\,dy$

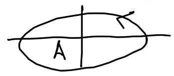
F.eks $P = 0$ og $Q(x,y) = x$

eller $P(x,y) = -y$ $Q(x,y) = 0$

eller $Q = \frac{1}{2}x$ $P = -\frac{1}{2}y$

$= \int_C x\,dy = \int_C -y\,dx = \int_C -\frac{1}{2}y\,dx + \frac{1}{2}x\,dy$

Eks Areal av ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$C: r(t) = (\underbrace{a\cos(t)}_{x}, \underbrace{b\sin(t)}_{y})$

$dx = -a\sin t\,dt$

$dy = b\cos(t)\,dt$

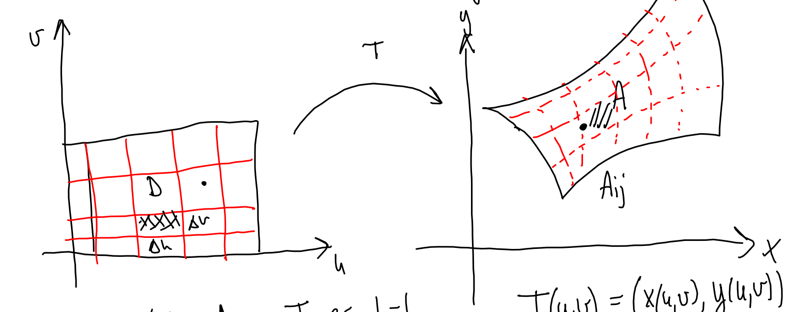
$t \in [0, 2\pi]$

Areal $= \iint_A dx\,dy = \int_C -\frac{1}{2}y\,dx + \frac{1}{2}x\,dy$

$= \int_0^{2\pi} \left(-\frac{b\sin(t)}{2}\right)(-a\sin(t)) + \frac{1}{2}a\cos(t)\,b\cos(t)\,dt$

$= \frac{1}{2}ab\int_0^{2\pi} 1\,dt = \underline{\underline{\pi ab}}$

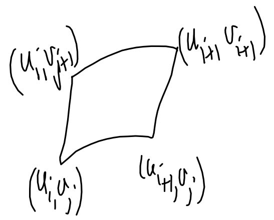
Skifte av variabel i dobbeltintegraler.



$$\iint_A f(x,y)\,dx\,dy$$

$T(D) = A$, T er 1-1 $\qquad T(u,v) = (x(u,v), y(u,v))$

Rutenett: (u_{ij}, v_{ij}) hjørner. Rektangler D_{ij} med hjørner (u_i, v_j) (u_{i+1}, v_j) (u_{i+1}, v_{j+1})



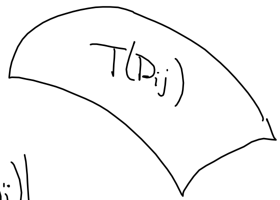
$$\iint_A f(x,y)\,dx\,dy \approx \sum_{ij} f(x_{ij}, y_{ij})\,|A_{ij}|$$

$x_{ij} = x(u_{ij}, v_{ij})$

$y_{ij} = y(u_{ij}, v_{ij})$

$$\sum_{ij} f(x_{ij}, y_{ij})\,|A_{ij}| = \sum_{ij} f(T(u_{ij}, v_{ij}))\,|T(D_{ij})|$$

$$\approx \sum_{ij} f(T(u_{ij}, v_{ij}))\,|\det(T'(u_{ij}, v_{ij}))|\,\Delta v\,\Delta u$$



$$|T(D_{ij})| \approx |\det(T'(u_{ij}, v_{ij}))|\,|D_{ij}|$$

$\Delta u, \Delta v \to 0$.

$$\boxed{\iint_{A=T(D)} f(x,y)\,dx\,dy = \iint_D f(T(u,v))\,|\det(T'(u,v))|\,du\,dv}$$

Skrivemåte

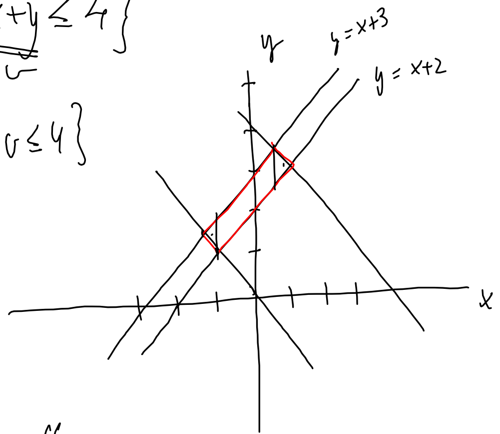
$$T(u,v) = (x(u,v), y(u,v)) \qquad T'(u,v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

$$\boxed{|\det(T'(u,v))| =: \frac{\partial(x,y)}{\partial(u,v)}}$$

$$\iint_A f(x,y)\,dx\,dy = \iint_D f(T(u,v))\,\frac{\partial(x,y)}{\partial(u,v)}\,du\,dv$$

<u>Eks.</u>

$$A = \{(x,y) \mid 2 \le \underbrace{y - x}_{u} \le 3,\ 0 \le \underbrace{x+y}_{v} \le 4\}$$



$$\iint_A xy \, dx dy \qquad D = \{(u,v) \mid 2 \le u \le 3,\ 0 \le v \le 4\}$$

Sett $\left.\begin{aligned} u &= -x+y \\ v &= x+y \end{aligned}\right\}$ $\left\{\begin{aligned} x &= \frac{v-u}{2} \\ y &= \frac{u+v}{2} \end{aligned}\right.$

$$xy = \frac{1}{2}(u+v)(v-u)\frac{1}{2} = \frac{1}{4}(v^2-u^2).$$

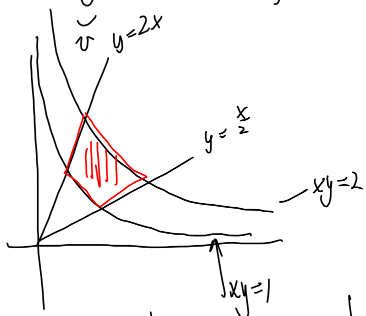
$$\frac{\partial(x,y)}{\partial(u,v)} = \left| \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} \right| = \frac{1}{2}$$

$$\iint_A xy \, dx dy = \iint_D \frac{1}{4}(v^2-u^2)\frac{1}{2} \, du \, dv$$

$$= \int_2^3 \left[\int_0^4 \frac{1}{8} v^2 - u^2 \, dv\right] du = \frac{1}{8}\int_2^3 \left(\frac{v^3}{3} - u^2 v\right)\Big|_0^4 du$$

$$= \frac{1}{8}\int_2^3 \frac{64}{3} - 4u^2 \, du = \frac{1}{8}\left(\frac{64}{3} - \frac{4}{3}(3^3 - 2^3)\right)$$

$A = \{(x,y) \mid \quad 1 \le \underbrace{xy}_{u} \le 2 \quad \frac{1}{2} \le \underbrace{\frac{x}{y}}_{v} \le 2 \quad x,y > 0\}.$



$\iint_A dx\,dy$

$u = xy$

$v = \frac{x}{y}$

$\det\left(T'(u,v)\right) = \frac{1}{\det\left(T^{-1\prime}(x,y)\right)}.$

$\frac{\partial(x,y)}{\partial(u,v)} = \left[\frac{\partial(u,v)}{\partial(x,y)}\right]^{-1}$

$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \cdot & \cdot \end{vmatrix}$

$= \begin{vmatrix} y & x \\ \frac{1}{y} & -\frac{x}{y^2} \end{vmatrix} = \left|-\frac{x}{y} - \frac{x}{y}\right| = 2\frac{x}{y} = \underline{\underline{2v}}.$