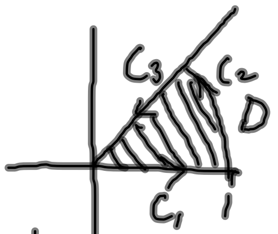
6.5.10 $x^2+y^2\le 1$ $\quad x\ge 0 \quad y\ge 0 \quad y\le x$



$0\le\theta\le\frac{\pi}{4} \qquad 0\le r\le 1$

a/ $$\iint_D (x+y^2)\,dx\,dy = \int_0^{\frac{\pi}{4}}\int_0^1 (r\cos\theta + r^2\sin^2\theta)\,r\,dr\,d\theta$$

$$= \dots \quad \left(\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)\right) \quad = \dots = \frac{\sqrt{2}}{6}+\frac{\pi}{32}-\frac{1}{16}$$

b/ Greens teorem: $\quad x+y^2 \qquad P(x,y) = -xy$

$$\underbrace{\frac{\partial Q}{\partial x}}_{y^2} - \underbrace{\frac{\partial P}{\partial y}}_{x} \qquad Q(x,y) = y^2x$$

$$\left(\begin{array}{l} Q(x,y) = \frac{1}{2}x^2+xy^2 \\ P(x,y)=0 \end{array}\right)$$

C_1: $\vec{r}_1(t) = (t,0) \quad 0\le t\le 1 \qquad \vec{F}(\vec{r}(t))$: $P(\vec{r}(t)) = 0$, $Q(\vec{r}(t)) = 0$

$$\Rightarrow \int_{C_1} P\,dx + Q\,dy = 0$$

$C_2: \quad \vec{r}_2(t) = (\cos t, \sin t) \qquad 0 \leq t \leq \frac{\pi}{4} \qquad \vec{F} = (\underbrace{-xy}_{P}, \underbrace{\cancel{x}y^2 x}_{Q})$

$\quad\;\; \vec{r}_2{}'(t) = (-\sin t, \cos t)$

$$\int_{C_2} P\,dx + Q\,dy = \int_0^{\frac{\pi}{4}} \vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2{}'(t)\,dt$$

$$= \int_0^{\frac{\pi}{4}} -\cos t \sin t\,(-\sin t)\,dt + \int_0^{\frac{\pi}{4}} \sin^2 t \cos t \cos t\,dt$$

$$= \int_0^{\frac{\pi}{4}} \sin^2 t \cos t\,dt + \frac{1}{4}\int_0^{\frac{\pi}{4}} 4\sin^2 t \cos^2 t\,dt$$

$$= \left[\frac{1}{3}\sin^3 t\right]_0^{\frac{\pi}{4}} + \frac{1}{4}\int_0^{\frac{\pi}{4}} \sin^2 2t\,dt = \frac{\sqrt{2}}{12} + \frac{1}{8}\int_0^{\frac{\pi}{4}} (1 - \cos 4t)\,dt$$

$$= \frac{\sqrt{2}}{12} + \frac{1}{8}\left[t - \frac{1}{4}\sin 4t\right]_0^{\frac{\pi}{4}} = \underline{\frac{\sqrt{2}}{12} + \frac{\pi}{32}}$$

$C_3: \quad \vec{r}_3(t) = (t, t) \qquad \vec{r}_3{}'(t) = (1, 1)$

$t = \frac{\sqrt{2}}{2}$

$$= \int_{\frac{\sqrt{2}}{2}}^{0} \underbrace{-t^2}_{P}\,dt + \int_{\frac{\sqrt{2}}{2}}^{0} t^3\,dt = \left[-\frac{1}{3}t^3\right]_{\frac{\sqrt{2}}{2}}^{0} + \left[\frac{1}{4}t^4\right]_{\frac{\sqrt{2}}{2}}^{0}$$

$$= \cdots = \frac{\sqrt{2}}{12} - \frac{1}{16}$$

$$I = \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy + \int_{C_3} P\,dx + Q\,dy$$

$$= 0 + \frac{\sqrt{2}}{12} + \frac{\pi}{32} + \frac{\sqrt{2}}{12} - \frac{1}{16} = \underline{\underline{\frac{\sqrt{2}}{6} + \frac{\pi}{32} - \frac{1}{16}}}$$

6.5.12

a) $9x^2+4y^2-18x+16y=11$

$$9x^2-18x+9 \quad +4y^2+16y+16 = 11+9+16$$

$$\frac{(x-1)^2}{2^2}+\frac{(y+2)^2}{3^2}=1$$

sentrum i $(1,-2)$, store halvakse $= b = 3$

lille halvakse $= a = 2$

b) Vi setter inn $x = 1+2\cos t \quad y=-2+3\sin t$:

$$\frac{(x-1)^2}{2^2}+\frac{(y+2)^2}{3^2} = \frac{(2\cos t)^2}{2^2}+\frac{(3\sin t)^2}{3^2} = \cos^2 t+\sin^2 t = 1$$

slik at $\vec{r}(t)$ ligger på ellipsen. Det er klart at, når t går fra 0 til 2π, gjennomløper $\vec{r}(t)$ <u>hele</u> ellipsen

b) forts. $\int_C \vec{F} \cdot dr$

$\vec{F}(x,y) = (y^2, x)$

$\vec{F}(\vec{r}(t)) = ((-2+3\sin t)^2, 1+2\cos t)$

$\vec{r}'(t) = (-2\sin t, 3\cos t)$

$$= \int_0^{2\pi} \left((-2+3\sin t)^2 (-2\sin t) + (1+2\cos t) 3\cos t \right) dt$$

$$= \int_0^{2\pi} \left(\underbrace{-18\sin^3 t}_{0} + 24\sin^2 t \underline{-8\sin t} + \underline{3\cos t} + 6\cos^2 t \right) dt$$

$$= \int_0^{2\pi} \left(24\sin^2 t + 6\cos^2 t \right) dt = \int_0^{2\pi} \left(18\sin^2 t + 6 \right) dt$$

$$= \ldots = \underline{\underline{30\pi}}$$

c) Ser at $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - 2y$

Derfor er $\iint_R (1-2y)\, dx\, dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$

$$= \int_C P\, dx + Q\, dy = \underline{\underline{30\pi}}$$

$$\iint_R (1-2y)\,dx\,dy$$

sentrum i $(1,-2)$

$$x = 1 + 2\cos t$$
$$y = -2 + 3\sin t$$

$$\frac{\partial(x,y)}{\partial(r,t)} = \begin{vmatrix} 2\cos t & -2r\sin t \\ 3\sin t & 3r\cos t \end{vmatrix} = 6r$$

$$\boxed{\begin{array}{ll} x = 1 + 2r\cos t & 0 \le r \le 1 \\ y = -2 + 3r\sin t & 0 \le t \le 2\pi \end{array}}$$

$$\iint (1-2y)6r\,dr\,dt$$
$$= \iint (1-2(-2+3r\sin t))6r\,dr\,dt$$
$$= \iint (5-6r\sin t)6r\,dr\,dt = \int_0^{2\pi}\int_0^1 (30r-36r\sin t)\,dr\,dt = 2\pi\cdot 15$$
$$= 30\pi$$

6.6.1

Gitt $\varepsilon > 0$ finnes n_i rektangler s.a.

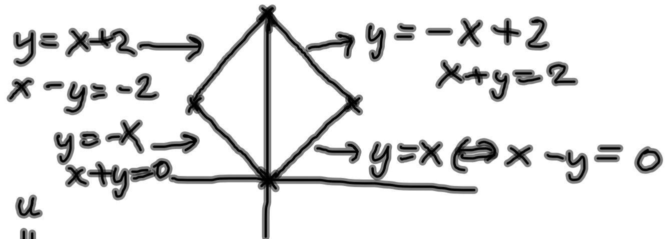
$$A_i \subset R_{i1} \cup \dots \cup R_{in_i}$$, summen av arealene $\left(\sum_{k=1}^{n_i} |R_{ik}| < \frac{\varepsilon}{m} \right)$

$\{R_{ij}\}_{1 \le i \le m,\ 1 \le j \le n_i}$ er en overdekning av $A_1 \cup \dots \cup A_m$, med areal $\sum_{i=1}^{m} \sum_{k=1}^{n_i} |R_{ik}| = \sum_{i=1}^{m} \frac{\varepsilon}{m} = \varepsilon$

$\Rightarrow$ $\underline{\underline{A_1 \cup \dots \cup A_m \text{ innhold null}}}$.

6.7.3

b) $\iint_R (x^2-y^2)e^{x+y}\,dx\,dy$



$$0 \le \underset{u}{\underbrace{x+y}} \le 2$$

$$-2 \le \underset{v}{\underbrace{x-y}} \le 0$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

$$\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2}$$

$$= \int_0^2 \int_{-2}^0 (x^2-y^2)e^{x+y}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\,du$$

$$= \int_0^2 \int_{-2}^0 (x-y)(x+y)e^{x+y}\,\frac{1}{2}\,dv\,du$$

$$= \int_0^2 \int_{-2}^0 \frac{1}{2}uv\,e^u\,dv\,du = \int_0^2 \left[\frac{1}{4}uv^2e^u\right]_{-2}^0 du$$

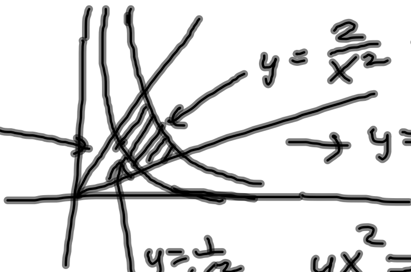
$$= \int_0^2 -ue^u\,du = \left[-ue^u\right]_0^2 + \int_0^2 e^u\,du = -2e^2+e^2-1$$

$$= -e^2-1$$

6.7.5

c) $\iint\limits_A y\,dx\,dy$

$y=2x \Leftrightarrow \frac{y}{x}=2$



$y=\frac{2}{x^2} \Rightarrow yx^2=2$

$\rightarrow y=\frac{x}{2} \Leftrightarrow \frac{y}{x}=\frac{1}{2}$

$y=\frac{1}{x^2}$ $\quad yx^2=1$

$$= \int_{\frac12}^{2}\int_{1}^{2} y \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\,du$$

$$= \int_{\frac12}^{2}\int_{1}^{2} y\,\frac{1}{3y}\,dv\,du$$

$$= \frac{1}{3}\int_{\frac12}^{2}\int_{1}^{2} dv\,du$$

$$= \frac{1}{3}\cdot\left(2-\frac{1}{2}\right)(2-1) = \frac{1}{3}\cdot\frac{3}{2} = \underline{\underline{\frac{1}{2}}}$$

$\frac{1}{2} \le \underbrace{\frac{y}{x}}_{u} \le 2$

$1 \le \underbrace{x^2y}_{v} \le 2$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ 2xy & x^2 \end{vmatrix} = -y-2y = -3y$$

$$\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{3y}$$

6.8.6

$$\iint_A \frac{1}{(x^2+y^2)^p}\,dx\,dy = \lim_{n\to\infty} \iint_{A\cap B(0,n)} \frac{1}{(x^2+y^2)^p}\,dx\,dy$$

$$= \lim_{n\to\infty} \int_0^{2\pi}\int_1^n \frac{1}{(r^2)^p}\, r\,dr\,d\theta = \lim_{n\to\infty} \int_0^{2\pi}\int_1^n r^{1-2p}\,dr\,d\theta$$

$1-2p \neq -1$ $(p\neq 1)$: $$= \lim_{n\to\infty} \int_0^{2\pi} \left[\frac{1}{1-2p+1} r^{1-2p+1}\right]_1^n d\theta$$

$$= \lim_{n\to\infty} 2\pi\left(\frac{1}{2-2p} n^{2-2p} - \frac{1}{2-2p}\right)$$

$$= \lim_{n\to\infty} \frac{\pi}{1-p}\left(n^{2-2p} - 1\right)$$

konvergerer hvis og bare hvis $2-2p<0 \Leftrightarrow \underline{\underline{p>1}}$

grensen blir da $-\frac{\pi}{1-p} = \underline{\frac{\pi}{p-1}}$

$p=1$:

$$\lim_{n\to\infty} \int_0^{2\pi} \int_1^n \frac{1}{r}\, dr\, d\theta = \lim_{n\to\infty} \int_0^{2\pi} \left[\ln r \right]_1^n d\theta$$

$$= \lim_{n\to\infty} \int_0^{2\pi} \ln n \, d\theta = \lim_{n\to\infty} 2\pi \ln n = \infty$$

så integralet konvergerer hvis og bare hvis $\underline{\underline{p > 1}}$