Plenum 8/3

$\boxed{4.6.2}$ $\vec{a}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\vec{a}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\vec{a}_3 = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -2 \\ 5 \\ 1 \end{pmatrix}$

Finnes $x, y, z \in \mathbb{R}$ s.a.

$$x\vec{a}_1 + y\vec{a}_2 + z\vec{a}_3 = \vec{b} \quad ?$$

$$[\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3]\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{b}$$

$$(\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3 \ \vdots \ \vec{b})$$

$$\begin{pmatrix} 1 & 1 & 3 & -2 \\ 0 & 2 & -1 & 5 \\ -1 & 1 & 2 & 1 \end{pmatrix} \overset{\text{III}+\text{I}}{\sim} \begin{pmatrix} 1 & 1 & 3 & -2 \\ 0 & 2 & -1 & 5 \\ 0 & \boxed{2} & 5 & -1 \end{pmatrix}$$

$$\overset{\text{III}-\text{II}}{\sim} \begin{pmatrix} 1 & 1 & 3 & -2 \\ 0 & 2 & -1 & 5 \\ 0 & 0 & 6 & -6 \end{pmatrix} \overset{\text{III}\cdot\frac{1}{6}}{\sim} \begin{pmatrix} 1 & 1 & \boxed{3} & -2 \\ 0 & 2 & \boxed{-1} & 5 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\underset{\text{I}-3\text{III}}{\overset{\text{II}+\text{III}}{\sim}} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{pmatrix} \overset{\text{II}\cdot\frac{1}{2}}{\sim} \begin{pmatrix} 1 & \boxed{1} & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\overset{\text{I}-\text{II}}{\sim} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\underline{\underline{\vec{b} = -\vec{a}_1 + 2\vec{a}_2 - \vec{a}_3}}$$

$\boxed{4.6.8}$ a) $\begin{pmatrix}2\\-1\end{pmatrix}, \begin{pmatrix}-4\\2\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}$

$$\begin{pmatrix}2\\-1\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}-4\\2\end{pmatrix}$$

$$\begin{pmatrix}2\\-1\end{pmatrix} \neq c\begin{pmatrix}1\\3\end{pmatrix} \quad \forall\ c \in \mathbb{R}.$$

$$\mathbb{R}^2 = \operatorname{span}\left\{\begin{pmatrix}2\\-1\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}\right\}$$

b) $\overbrace{\begin{pmatrix}1\\3\\-1\end{pmatrix}, \begin{pmatrix}0\\2\\1\end{pmatrix}, \begin{pmatrix}2\\8\\0\end{pmatrix}}, \cancel{\begin{pmatrix}2\\3\\1\end{pmatrix}}$

$$\begin{pmatrix}1 & 0 & 2 & 2\\ 3 & 2 & 8 & 3\\ -1 & 1 & 0 & 1\end{pmatrix} \xrightarrow[\text{II}-3\text{I}]{\text{III}+\text{I}} \begin{pmatrix}1 & 0 & 2 & 2\\ 0 & 2 & 2 & -3\\ 0 & \boxed{1} & 2 & 3\end{pmatrix}$$

$$\xrightarrow{\text{III}\leftrightarrow\text{II}} \begin{pmatrix}1 & 0 & 2 & 2\\ 0 & 1 & 2 & 3\\ 0 & 2 & 2 & -3\end{pmatrix} \xrightarrow{\text{III}-2\text{II}} \begin{pmatrix}1 & 0 & 2 & 2\\ 0 & 1 & 2 & 3\\ 0 & 0 & -2 & -9\end{pmatrix}$$

$$\xrightarrow{\text{III}\cdot\left(-\frac{1}{2}\right)} \begin{pmatrix}1 & 0 & 2 & 2\\ 0 & 1 & 2 & 3\\ 0 & 0 & 1 & \frac{9}{2}\end{pmatrix}$$

$\boxed{4.6.10}$ a) $\begin{pmatrix}1\\2\\-1\end{pmatrix}, \begin{pmatrix}2\\4\\3\end{pmatrix}, \begin{pmatrix}x\\y\\z\end{pmatrix}$

$$\begin{pmatrix}1&2&x\\2&4&y\\-1&3&z\end{pmatrix} \sim \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$\begin{pmatrix}1&2\\ \boxed{2}&4\\ \boxed{-1}&3\end{pmatrix} \underset{II-2I}{\overset{III+I}{\sim}} \begin{pmatrix}1&2\\0&0\\0&5\end{pmatrix} \overset{II\leftrightarrow III}{\sim} \begin{pmatrix}1&2\\0&5\\0&0\end{pmatrix}$$

$$= \begin{pmatrix}1&2&0\\0&5&0\\0&0&1\end{pmatrix} \quad \vec{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\begin{pmatrix}1&2&0\\0&5&0\\0&0&1\end{pmatrix} \overset{II\leftrightarrow III}{\sim} \begin{pmatrix}1&2&0\\0&0&1\\0&5&0\end{pmatrix}$$

$$\underset{III-I}{\overset{II+2I}{\sim}} \begin{pmatrix}1&2&0\\2&4&1\\-1&3&0\end{pmatrix} \quad \vec{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\mathbb{R}^3 = \text{span}\left\{\begin{pmatrix}1\\2\\-1\end{pmatrix}, \begin{pmatrix}2\\4\\3\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$$

b) $\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}x\\y\\z\\u\end{pmatrix}$

$$\begin{pmatrix}1&2&0\\ \boxed{-1}&1&1\\0&1&0\\0&0&1\end{pmatrix} \overset{II+I}{\sim} \begin{pmatrix}1&2&0\\0&3&1\\0&1&0\\0&0&1\end{pmatrix}$$

$$\underset{II-IV}{\overset{II-3III}{\sim}} \begin{pmatrix}1&2&0\\0&0&0\\0&1&0\\0&0&1\end{pmatrix} \underset{III\leftrightarrow IV}{\overset{II\leftrightarrow III}{\sim}} \begin{pmatrix}1&2&0\\0&1&0\\0&0&1\\0&0&0\end{pmatrix}$$

$$= \begin{pmatrix}1&2&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$$\begin{pmatrix}1&2&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \underset{III\leftrightarrow II}{\overset{IV\leftrightarrow III}{\sim}} \begin{pmatrix}1&2&0&0\\0&0&0&1\\0&1&0&0\\0&0&1&0\end{pmatrix}$$

$$\sim \begin{pmatrix}1&2&0&0\\-1&1&1&1\\0&1&0&0\\0&0&1&0\end{pmatrix}$$

Legger til $\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$.

$\boxed{4.6.11}$ $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

a) og b)

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right) \overset{II-I}{\sim} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{array}\right)$$

$$\overset{II\cdot(-\frac{1}{2})}{\sim} \left(\begin{array}{cc|cc} 1 & \boxed{1} & 1 & 0 \\ 0 & 1 & 1/2 & -1/2 \end{array}\right) \overset{I-II}{\sim} \left(\begin{array}{cc|cc} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1/2 & -1/2 \end{array}\right)$$

(a) (b)

$\vec{e}_1 = \frac{1}{2}(\vec{v}_1 + \vec{v}_2)$

$\vec{e}_2 = \frac{1}{2}(\vec{v}_1 - \vec{v}_2)$

c) $T(\vec{v}_1) = \underbrace{2\vec{v}_1}_{\vec{w}_1}$, $T(\vec{v}_2) = \underbrace{-\vec{v}_2}_{\vec{w}_2}$

d) $T(\vec{x}) = A\vec{x}$ $\forall \vec{x} \in \mathbb{R}^2$

$A = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$

$T(\vec{e}_1) = \underbrace{A\vec{e}_1}$ $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$T(\vec{e}_2) = A\vec{e}_2$ $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$A = [\,T(\vec{e}_1) \quad T(\vec{e}_2)\,]$

$$T(\vec{e}_1) = T\left(\tfrac{1}{2}(\vec{v}_1 + \vec{v}_2)\right) = \tfrac{1}{2}T(\vec{v}_1 + \vec{v}_2)$$
$$= \tfrac{1}{2}\left(T(\vec{v}_1) + T(\vec{v}_2)\right)$$
$$= \tfrac{1}{2}(2\vec{v}_1 - v_2)$$
$$= \tfrac{1}{2}\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = \tfrac{1}{2}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 3/2 \end{pmatrix}$$

$$T(\vec{e}_2) = \tfrac{1}{2}\left(T(v_1) - T(v_2)\right)$$
$$= \tfrac{1}{2}\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = \tfrac{1}{2}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$$

$$\underline{\underline{A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}}}$$

$\boxed{4.6.12}$ $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \neq \vec{0}$

$\vec{v}_i \cdot \vec{v}_j = 0 \quad \forall \; i \neq j$

$\underbrace{\vec{v}_i \cdot \vec{v}_i}_{|\vec{v}_i|^2} \neq 0$

$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k = \vec{0}, \; c_i \neq 0$

for hvertfall én $i \in \{1, 2, \ldots, k\}$.

For vilkårlig $i \in \{1, 2, \ldots, k\}$

$c_1\vec{v}_1 \cdot \vec{v}_i + c_2\vec{v}_2 \cdot \vec{v}_i + \ldots + c_i\vec{v}_i \cdot \vec{v}_i$

$+ \ldots + c_k\vec{v}_k \cdot \vec{v}_i = \vec{0} \cdot \vec{v}_i$

$c_i \underbrace{|\vec{v}_i|^2}_{\neq 0} = 0 \implies c_i = 0$

$c_1 = c_2 = \ldots = c_k = 0 \quad \perp$

$\boxed{4.8.3}$ $\begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & 3 \\ 0 & 1 & 6 \end{pmatrix} \sim I_3$

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & 3 \\ 0 & 1 & 0 \end{pmatrix} \overset{\overbrace{II + I}^{E_1}}{\sim} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 4 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\overset{\overbrace{II \leftrightarrow III}^{E_2}}{\sim} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$\overset{\overbrace{III \cdot \frac{1}{4}}^{E_3}}{\sim} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\overset{\overbrace{I - 2II}^{E_4}}{\sim} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\overset{\overbrace{I - III}^{E_5}}{\sim} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}$

$E_4 = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$E_5 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\underbrace{E_5^{-1}E_5}_{I_3}E_4E_3E_2E_1A = E_5^{-1}I_3$$

$$(E_1 \vdots I_3) \sim (I_3 \vdots E_1^{-1})$$

$$E_4E_3E_2E_1A = E_5^{-1}$$

$\vdots$

$$\underline{A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}}$$

$\boxed{4.9.7}$ Vis: $\det(rA) = r^n \det(A)$

$$A = \begin{pmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{pmatrix} \leftarrow \text{radvektor.} \implies rA = \begin{pmatrix} r\vec{a}_1 \\ r\vec{a}_2 \\ \vdots \\ r\vec{a}_n \end{pmatrix}$$

4.9.10 (iii) får vi derfor at

$$\det(rA) = \det\begin{pmatrix} r\vec{a}_1 \\ r\vec{a}_2 \\ \vdots \\ r\vec{a}_n \end{pmatrix} = r \cdot \det\begin{pmatrix} \vec{a}_1 \\ r\vec{a}_2 \\ \vdots \\ r\vec{a}_n \end{pmatrix}$$

$$= \dots = r^n \det\begin{pmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{pmatrix} = r^n \det(A).$$

$\boxed{4.9.8}$ **Vis:** $\det(A^n) = \det(A)^n$ for $n \in \mathbb{N}$

setning 4.9.16: $\det(AB) = \det(A) \cdot \det(B)$

induksjon $n=1$: $\det(A^1) = \det(A)^1$

$n=2$: $\det(A^2) = \det(A \cdot A)$

$= \det(A) \cdot \det(A)$

$= \det(A)^2$

Anta derfor at det gjelder for

$n-1$: $\det(A^{n-1}) = \det(A)^{n-1}$

$\det(A^n) = \det(A^{n-1} \cdot A)$

$= \det(A^{n-1}) \cdot \det(A)$

$= \det(A)^{n-1} \cdot \det(A) = \det(A)^n$

$\square$

$\boxed{4.9.9}$ • Korollar 4.9.18

$$\det(A^T) = \det(A)$$

• Teorem 4.9.12 (v)

$\det(A) \neq 0 \iff$ søylene er lin. uavhengige.

$\boxed{4.9.10}$ $U^{-1} = U^T$. $\det(U) = 1$ eller -1

$$U^T U = U^{-1} U = I_n$$

$$1 = \det(I_n) = \det(U^T U) = \det(U^T) \cdot \det(U)$$

$$= \det(U) \cdot \det(U) = \det(U)^2$$

$$\underbrace{\det(U)}_{1 \text{ eller } -1}^2 = 1$$