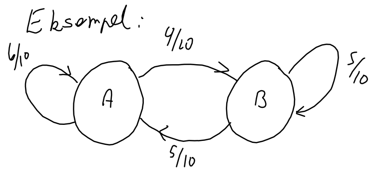
Eksempel:



x_n = # fugler i A etter n dager
y_n = # fugler i B etter n dager.

$$x_{n+1} = \frac{6}{10}x_n + \frac{5}{10}y_n$$
$$y_{n+1} = \frac{4}{10}x_n + \frac{5}{10}y_n$$

$$X = \begin{pmatrix} x \\ y \end{pmatrix} \quad X_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix} \qquad X_{n+1} = \frac{1}{10}\begin{pmatrix} 6 & 5 \\ 4 & 5 \end{pmatrix} X_n$$

Anta at X_0 kjent: $X_0 = \begin{pmatrix} 100 \\ 0 \end{pmatrix}$ $\qquad X_{n+1} = \frac{1}{10} M X_n$

$$X_1 = \frac{1}{10}MX_0, \quad X_2 = \frac{1}{10}MX_1 = X_2 = \frac{1}{10^2}M^2X_0 \; \dots \quad \boxed{X_n = \frac{1}{10^n}M^n X_0}$$

Prøve å finne egenverdier/egenvektorer til M.

Egenverdiene: $\det(M-\lambda I) = 0$.

$$\begin{vmatrix} 6-\lambda & 5 \\ 4 & 5-\lambda \end{vmatrix} = (6-\lambda)(5-\lambda) - 20 = \lambda^2 - 11\lambda + 30 - 20 = \lambda^2 - 11\lambda + 10 = 0.$$

$$\lambda = \frac{1}{2}\left(11 \pm \sqrt{121-40}\right) = \frac{1}{2}\left(11 \pm \sqrt{81}\right) = \frac{1}{2}(11 \pm 9) = \begin{cases} 10 \\ 1 \end{cases}$$

$\lambda = 10$ gir: $\left(Mv = \lambda v \text{ eller } (M-\lambda I)v = 0, \; v \neq 0 \right)$.

Utvidet matrise:

$$\begin{pmatrix} -4 & 5 & 0 \\ 4 & -5 & 0 \end{pmatrix} \sim \begin{pmatrix} -4 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad y \text{ fri } x = \frac{5}{4}y \qquad v_1 = \begin{pmatrix} 5 \\ 4 \end{pmatrix}. \quad \lambda_1 = 10$$

$\lambda = 1$. Utvidet matrise:

$$\begin{pmatrix} 5 & 5 & 0 \\ 4 & 4 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad y \text{ fri } x = -y \qquad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad \lambda_2 = 1$$

$X_n = \frac{1}{10^n}M^n X_0$.

Skriver $X_0 = av_1 + bv_2$, setter inn i formel:

$$X_n = \frac{1}{10^n}M^n(av_1 + bv_2) = \frac{1}{10^n}aM^n v_1 + \frac{b}{10^n}M^n v_2 = \frac{a}{10^n}\lambda_1^n v_1 + \frac{b}{10^n}\lambda_2^n v_2 \quad \text{, vet at } \lambda_1 = 10, \; \lambda_2 = 1.$$

$$= a\frac{10^n}{10^n}v_1 + \frac{b}{10^n}v_2 = \boxed{av_1 + b\frac{1}{10^n}v_2 = X_n}$$

$v_1 = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ $\quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. $\quad X_0 = \begin{pmatrix} 100 \\ 0 \end{pmatrix}$.

$$av_1 + bv_2 = X_0 \longrightarrow \begin{pmatrix} 5 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 100 \\ 0 \end{pmatrix}.$$ Utvidet matrise

$$\begin{pmatrix} 5 & 1 & 100 \\ 4 & -1 & 0 \end{pmatrix} \overset{I-II}{\sim} \begin{pmatrix} 1 & 2 & 100 \\ 4 & -1 & 0 \end{pmatrix} \overset{II-4I}{\sim} \begin{pmatrix} 1 & 2 & 100 \\ 0 & -9 & -400 \end{pmatrix} \overset{II/-9}{\sim} \begin{pmatrix} 1 & 2 & 100 \\ 0 & 1 & \frac{400}{9} \end{pmatrix}$$

$$\overset{I-2II}{\sim} \begin{pmatrix} 1 & 0 & \frac{100}{9} \\ 0 & 1 & \frac{400}{9} \end{pmatrix} \qquad a = \frac{100}{9}, \quad b = \frac{400}{9} \qquad \boxed{X_n = \frac{100}{9}\begin{pmatrix} 5 \\ 4 \end{pmatrix} + \frac{400}{9}\frac{1}{10^n}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

$$x_n = \frac{500}{9} + \frac{400}{9}\frac{1}{10^n}$$
$$y_n = \frac{400}{9} - \frac{400}{9}\frac{1}{10^n}$$

$$\lim_{n\to\infty}\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} \frac{500}{9} \\ \frac{400}{9} \end{pmatrix}$$

$$M \mid Mv = \lambda v$$

$$M \cdot Mv = M\lambda v$$

$$\underline{M^2 v} = \lambda M v = \lambda \lambda v = \underline{\lambda^2 v}$$

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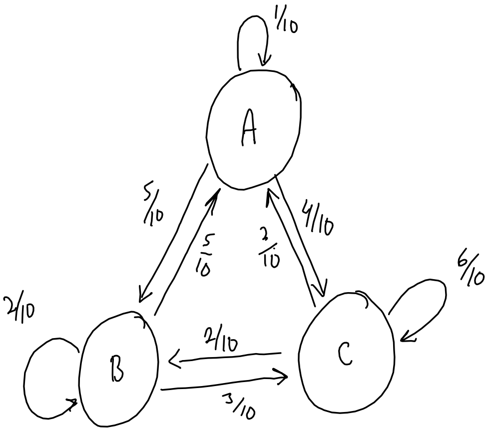
Bysykler, 2 stativer A, B eller ukjent oppholdssted C.



x_n = # sykler i A etter n dager

y_n = # sykler i B etter n dager

z_n = # sykler i C etter n dager.

$$\boxed{\begin{aligned} x_{n+1} &= \tfrac{1}{10}x_n + \tfrac{5}{10}y_n + \tfrac{2}{10}z_n \\ y_{n+1} &= \tfrac{5}{10}x_n + \tfrac{2}{10}y_n + \tfrac{2}{10}z_n \\ z_{n+1} &= \tfrac{4}{10}x_n + \tfrac{3}{10}y_n + \tfrac{6}{10}z_n \end{aligned}}$$

Kjenner x_0, y_0, z_0

$x_0 = y_0 = 50 \quad z_0 = 0.$

$$X_n = \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$$

$$X_{n+1} = \frac{1}{10} M X_n \; ; \quad M = \begin{pmatrix} 1 & 5 & 2 \\ 5 & 2 & 2 \\ 4 & 3 & 6 \end{pmatrix}$$

$$\boxed{X_n = \frac{1}{10^n} M^n X_0.}$$

$$M = \begin{pmatrix} 1 & 5 & 2 \\ 5 & 2 & 2 \\ 4 & 3 & 6 \end{pmatrix}$$

<u>Lemma</u>: Hvis alle kolonnene i en $n\times n$ matrise M har samme sum; S, så er S en egenverdi

$$\det(M-\lambda I) = \begin{vmatrix} 1-\lambda & 5 & 2 \\ 5 & 2-\lambda & 2 \\ 4 & 3 & 6-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & 2 \\ 3 & 6-\lambda \end{vmatrix} - 5\begin{vmatrix} 5 & 2 \\ 4 & 6-\lambda \end{vmatrix} + 2\begin{vmatrix} 5 & 2-\lambda \\ 4 & 3 \end{vmatrix}$$

$$= (1-\lambda)\Big((2-\lambda)(6-\lambda)-6\Big) - 5\Big(5(6-\lambda)-8\Big) + 2\Big(15-4(2-\lambda)\Big)$$

$$= (1-\lambda)\left(\lambda^2-8\lambda+6\right) - 5\left(22-5\lambda\right) + 2\left(7+4\lambda\right)$$

$$= -\lambda^3+8\lambda^2-6\lambda+\lambda^2-8\lambda+6-110+25\lambda+14+8\lambda$$

$$= -\lambda^3+9\lambda^2+19\lambda-90 = p(\lambda) \qquad \lambda=10 \quad p(10) = -10^3+9\cdot 10^2+19\cdot 10-90$$
$$= -1000+900+190-90 = 0.$$

$p(\lambda) = (\lambda-10)\,q(\lambda)$

Polynomdivisjon for å finne q.

$$-\lambda^3+9\lambda^2+19\lambda-90 : \lambda-10 = -\lambda^2-\lambda+9 = q(\lambda).$$
$$\underline{-\lambda^3+10\lambda^2}$$
$$-\lambda^2+19\lambda$$
$$\underline{-\lambda^2+10\lambda}$$
$$9\lambda-90$$
$$\underline{\underline{9\lambda-90}}$$

$q(\lambda)=0$ gir $\lambda^2+\lambda-9=0 \quad \lambda = \frac{1}{2}\left(-1\pm\sqrt{1+36}\right) = \frac{1}{2}\left(-1\pm\sqrt{37}\right)$

$$\lambda_1 = 10 \quad \lambda_2 = \tfrac{1}{2}\left(-1+\sqrt{37}\right) \quad \lambda_3 = \tfrac{1}{2}\left(-1-\sqrt{37}\right). \qquad x_0 = a v_1 + b v_2 + c v_3.$$

$$x_n = \frac{1}{10^n} a \overset{10^n}{\lambda_1^n} v_1 + \frac{1}{10^n} b \lambda_2^n v_2 + c\frac{1}{10^n}\lambda_3^n v_3$$

$$= \boxed{a v_1} + \underbrace{\left(\frac{\lambda_2}{10}\right)^n b v_2 + \left(\frac{\lambda_3}{10}\right)^n c v_3}_{\text{blir fort små}}$$

Merk $|\lambda_2| < 10$, $|\lambda_3| < 10$.

$\Rightarrow \left(\frac{\lambda_{2,3}}{10}\right)^n \to 0.$ når $n\to\infty$.

$$M = \begin{pmatrix} 1 & 5 & 2 \\ 5 & 2 & 2 \\ 4 & 3 & 6 \end{pmatrix}$$ Egenvektor for $\lambda = 10$.

Utvidet matrise
$$\begin{pmatrix} -9 & 5 & 2 & 0 \\ 5 & -8 & 2 & 0 \\ 4 & 3 & -4 & 0 \end{pmatrix} \overset{I/-9}{\sim} \begin{pmatrix} 1 & -5/9 & -2/9 & 0 \\ 5 & -8 & 2 & 0 \\ 4 & 3 & -4 & 0 \end{pmatrix}$$

$$-\frac{72}{9} + \frac{25}{9} = -\frac{47}{9}$$

$$\underset{\sim}{\substack{II-5I \\ III-4I}} \begin{pmatrix} 1 & -\frac{5}{9} & -\frac{2}{9} & 0 \\ 0 & -8 & \frac{5\cdot 5}{9} & 2+\frac{10}{9} & 0 \\ 0 & 3+\frac{20}{9} & -4+\frac{8}{9} & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{5}{9} & -\frac{2}{9} & 0 \\ 0 & -\frac{47}{9} & \frac{28}{9} & 0 \\ 0 & \frac{47}{9} & -\frac{28}{9} & 0 \end{pmatrix} \overset{II+III}{\sim} \begin{pmatrix} 1 & -5/9 & -2/9 & 0 \\ 0 & +47 & -28 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\underset{\sim}{I+\frac{II}{14}} \begin{pmatrix} 1 & -\frac{5}{9}+\frac{47}{9\cdot 14} = -\frac{23}{14\cdot 9} & 0 & 0 \\ 0 & -\frac{47}{9} & \frac{28}{9} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$v = z \begin{pmatrix} -\frac{23}{14}\cdot\frac{28}{48} \\ \frac{47}{28} \\ 9 \end{pmatrix} \approx \begin{pmatrix} \frac{26}{58} \\ \frac{47}{28} \\ 9 \end{pmatrix}.$$

z fri!

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