Fasit til utvalgte oppgaver MAT1110, uka 22-26/2

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Oppgave 6.3.1

a)

Første kvadrant svarer i polarkoordinater til at $0 \leq \theta \leq \frac{\pi}{2}$, og området innenfor sirkelen $x^2 + y^2 = 9$ svarer til at $0 \leq r \leq 3$. Integralet blir derfor

$$\begin{aligned}
\int\int_R xy^2 dxdy &= \int_0^{\pi/2} \int_0^3 xy^2 r dr d\theta = \int_0^{\pi/2} \int_0^3 r^4 \cos\theta \sin^2\theta dr d\theta \\
&= \int_0^{\pi/2} \left[\frac{1}{5} r^5 \cos\theta \sin^2\theta \right]_0^3 d\theta \\
&= \int_0^{\pi/2} \frac{243}{5} \cos\theta \sin^2\theta d\theta \\
&= \frac{243}{5} \left[\frac{1}{3} \sin^3\theta \right]_0^{\pi/2} = \frac{81}{5}.
\end{aligned}$$

b)

Området i første kvadrant innenfor sirkelen $x^2 + y^2 = 25$ svarer til at $0 \leq \theta \leq \frac{\pi}{2}$ og $0 \leq r \leq 5$, og området mellom linjene $y = 0$ og $y = x$ begrenser området ytterligere til $0 \leq \theta \leq \frac{\pi}{4}$. Integralet blir derfor

$$\begin{aligned}
\int\int_R (x^2 + y^2) dxdy &= \int_0^{\pi/4} \int_0^5 (x^2 + y^2) r dr d\theta = \int_0^{\pi/4} \int_0^5 r^3 dr d\theta \\
&= \int_0^{\pi/4} \left[\frac{1}{4} r^4 \right]_0^5 d\theta \\
&= \int_0^{\pi/4} \frac{625}{4} d\theta = \frac{625\pi}{16}.
\end{aligned}$$

e)

Tredje kvadrant svarer i polarkoordinater til at $\pi \leq \theta \leq \frac{3\pi}{2}$. Linjen $y = \sqrt{3}x$ danner vinkel $\frac{\pi}{3}$ med x-aksen, og delen av denne som ligger i tredje kvadrant danner derfor en vinkel $\frac{\pi}{3} + \pi = \frac{4\pi}{3}$ med x-aksen. Linjen $y = \frac{\sqrt{3}}{3}x$ danner vinkel $\frac{\pi}{6}$ med x-aksen. og delen av denne som ligger i tredje kvadrant danner derfor en vinkel $\frac{\pi}{6} + \pi = \frac{7\pi}{6}$ med x-aksen. Siden området innenfor $x^2 + y^2 = 1$ svarer til at $0 \leq r \leq 1$ så blir integralet

$$\begin{aligned}
\int\int_R (x^2-y^2)dxdy &= \int_{7\pi/6}^{4\pi/3}\int_0^1 (x^2-y^2)rdrd\theta \\
&= \int_{7\pi/6}^{4\pi/3}\int_0^1 r^3(\cos^2\theta-\sin^2\theta)drd\theta \\
&= \int_{7\pi/6}^{4\pi/3}\left[\frac{1}{4}r^4\cos(2\theta)\right]_0^1 d\theta \\
&= \int_{7\pi/6}^{4\pi/3}\frac{1}{4}\cos(2\theta)d\theta = \left[\frac{1}{8}\sin(2\theta)\right]_{7\pi/6}^{4\pi/3} \\
&= \frac{1}{8}(\sin(8\pi/3)-\sin(7\pi/3)) = \frac{1}{8}(\sqrt{3}/2-\sqrt{3}/2) = 0.
\end{aligned}$$

f)

Området i første kvadrant innenfor sirkelen $x^2+y^2=1$ svarer til at $0\le\theta\le\frac{\pi}{2}$ og $0\le r\le 1$. Integralet blir derfor

$$\begin{aligned}
\int\int_R \sqrt{2-x^2-y^2}dxdy &= \int_0^{\pi/2}\int_0^1 \sqrt{2-r^2}rdrd\theta \\
&= \int_0^{\pi/2}\left[-\frac{1}{3}(2-r^2)^{3/2}\right]_0^1 \\
&= \int_0^{\pi/2}\left(-\frac{1}{3}+\frac{2\sqrt{2}}{3}\right)d\theta \\
&= \frac{\pi}{6}(2\sqrt{2}-1).
\end{aligned}$$

g)

$(x-1)^2+y^2\le 1$ betyr at $r^2-2r\cos\theta+1\le 1$, som er det samme som at $r\le 2\cos\theta$. Tegner vi opp sirkelen ser vi at den ligger i første og fjerde kvadrant, og at $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$. Integralet blir derfor

$$\begin{aligned}
\int\int_R (x^2+y^2)^{3/2}dxdy &= \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}(x^2+y^2)^{3/2}rdrd\theta \\
&= \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^4rdrd\theta \\
&= \int_{-\pi/2}^{\pi/2}\left[\frac{1}{5}r^5\right]_0^{2\cos\theta}d\theta \\
&= \int_{-\pi/2}^{\pi/2}\frac{32}{5}\cos^5\theta d\theta = \int_{-\pi/2}^{\pi/2}\frac{32}{5}(1-\sin^2\theta)^2\cos\theta d\theta \\
&= \frac{32}{5}\int_{-1}^{1}(1-u^2)^2du = \frac{32}{5}\int_{-1}^{1}(u^4-2u^2+1)du \\
&= \frac{32}{5}\left[\frac{1}{5}u^5-\frac{2}{3}u^3+u\right]_{-1}^{1} \\
&= \frac{64}{5}\left(\frac{1}{5}-\frac{2}{3}+1\right) = \frac{64}{5}\frac{3-10+15}{15} = \frac{64}{5}\frac{8}{15} = \frac{512}{75},
\end{aligned}$$

der vi har gjort subsitusjonen $u=\sin\theta$.

Oppgave 6.3.3

a)

Sirkelskiven med sentrum i $(0, 1)$ og radius 1 svarer til at $x^2 + (y-1)^2 \leq 1$, som i polarkoordinater blir $r^2 \cos^2\theta + (r\sin\theta - 1)^2 \leq 1$, eller $r^2 \leq 2r\sin\theta$, som igjen svarer til at $r \leq 2\sin\theta$. Sirkelen er helt inneholdt i første og andre kvadrant, og vi ser ved opptegning at $0 \leq \theta \leq \pi$. Derfor er området beskrevet i polarkoordinater ved at $0 \leq \theta \leq \pi$ og $r \leq 2\sin\theta$, slik at det gitte integralet blir

$$\int\int_R f(x,y)dxdy = \int_0^{\pi}\left[\int_0^{2\sin\theta} f(r\cos\theta, r\sin\theta)rdr\right]d\theta.$$

b)

Vi får at

$$\begin{aligned}
\int\int_R \sqrt{x^2+y^2} &= \int_0^{\pi}\left[\int_0^{2\sin\theta} r^2 dr\right]d\theta \\
&= \int_0^{\pi}\left[\frac{1}{3}r^3\right]_0^{2\sin\theta} d\theta \\
&= \int_0^{\pi}\frac{8}{3}\sin^3\theta d\theta \\
&= \frac{8}{3}\int_0^{\pi}(1-\cos^2\theta)\sin\theta d\theta \\
&= \frac{8}{3}\left[-(\cos\theta - \frac{1}{3}\cos^3\theta\right]_0^{\pi} \\
&= \frac{8}{3}\left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right) \\
&= \frac{8}{3}\frac{4}{3} = \frac{32}{9}.
\end{aligned}$$

Oppgave 6.3.4

$$\begin{aligned}
|A| &= \int_{\alpha}^{\beta}\int_0^{r(\theta)} 1rdrd\theta = \int_{\alpha}^{\beta}\left[\frac{1}{2}r^2\right]_0^{r(\theta)} d\theta \\
&= \int_{\alpha}^{\beta}\frac{1}{2}r(\theta)^2 d\theta.
\end{aligned}$$

Hvis $r(\theta) = \sin(2\theta)$ får vi

$$\begin{aligned}
\int_0^{\pi/2}\frac{1}{2}\sin(2\theta)^2 d\theta &= \int_0^{\pi/2}\frac{1}{4}(1-\cos(4\theta))d\theta \\
&= \frac{1}{4}\left[\theta - \frac{1}{4}\sin(4\theta)\right]_0^{\pi/2} \\
&= \frac{1}{4}(\pi/2) = \frac{\pi}{8}.
\end{aligned}$$

Oppgave 6.4.1

a)

$$
\begin{aligned}
V &= \int_0^2 \int_0^1 \int_0^{x+y^2} dzdydx \\
&= \int_0^2 \int_0^1 [z]_0^{x+y^2} dydx \\
&= \int_0^2 \int_0^1 (x+y^2)dydx \\
&= \int_0^2 \left[xy + \frac{1}{3}y^3\right]_0^1 dx \\
&= \int_0^2 \left(x + \frac{1}{3}\right) dx \\
&= \left[\frac{1}{2}x^2 + \frac{1}{3}x\right]_0^2 \\
&= 2 + \frac{2}{3} = \frac{8}{3}.
\end{aligned}
$$

d)

Grafen $z = \sqrt{32 - 2x^2 - 2y^2} = \sqrt{32 - 2r^2}$ skjærer xy-planet for $r = 4$. Volumet kan derfor regnes ut ved hjelp av sylinderkoordinater:

$$
\begin{aligned}
V &= \int_0^{2\pi} \int_0^4 \int_0^{\sqrt{32-2r^2}} rdzdrd\theta \\
&= \int_0^{2\pi} \int_0^4 [rz]_0^{\sqrt{32-2r^2}} drd\theta \\
&= \int_0^{2\pi} \int_0^4 r\sqrt{32-2r^2}drd\theta \\
&= \int_0^{2\pi} \left[-\frac{1}{6}(32-2r^2)^{3/2}\right]_0^4 d\theta \\
&= \int_0^{2\pi} \frac{1}{6}32^{3/2}d\theta = \frac{128\pi\sqrt{2}}{3}.
\end{aligned}
$$

Oppgave 6.4.2

Trekanten med hjørner $(0,0)$, $(1,0)$, og $(1,1)$ er beskrevet ved $0 \leq x \leq 1$, $0 \leq y \leq x$. Massemiddelpunktet blir dermed

$$\begin{aligned}
\bar{x} &= \frac{\int\int_A xf(x,y)dxdy}{\int\int_A f(x,y)dxdy} = \frac{\int_0^1\int_0^x x^2dydx}{\int_0^1\int_0^x xdydx} \\
&= \frac{\int_0^1 \left[x^2y\right]_0^x dx}{\int_0^1 \left[xy\right]_0^x dx} \\
&= \frac{\int_0^1 x^3dx}{\int_0^1 x^2dx} = \frac{1/4}{1/3} = \frac{3}{4} \\
\bar{y} &= \frac{\int_0^1\int_0^x xydydx}{1/3} \\
&= 3\int_0^1 \left[\frac{1}{2}xy^2\right]_0^x dx = 3\int_0^1 \frac{1}{2}x^3dx = 3\times\frac{1}{8} = \frac{3}{8}.
\end{aligned}$$

Massemiddelpunktet er derfor $(\frac{3}{4}, \frac{3}{8})$.

Oppgave 6.4.5

Flaten $z = f(x,y) = x^2 - y^2$ har partiellderiverte $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = -2y$. Arealet er derfor

$$\begin{aligned}
\int\int_A \sqrt{1+4x^2+4y^2}dxdy &= \int_0^{2\pi}\int_0^2 \sqrt{1+4r^2}rdrd\theta \\
&= \frac{1}{8}\int_0^{2\pi}\left[\frac{2}{3}(1+4r^2)^{3/2}\right]_0^2 d\theta \\
&= \frac{1}{8}\int_0^{2\pi}\left(\frac{34\sqrt{17}}{3} - \frac{2}{3}\right)d\theta \\
&= \frac{\pi}{4}\frac{34\sqrt{17}-2}{3} = \frac{\pi}{6}(17\sqrt{17}-1),
\end{aligned}$$

der vi har brukt av området A lar seg beskrive i polarkoordinater ved $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 2$.

Oppgave 6.4.7

Sirkelen $\left(x - \frac{1}{2}\right)^2 + y^2 \leq \frac{1}{4}$ betyr i polarkoordinater at

$$\begin{aligned}
\left(r\cos\theta - \frac{1}{2}\right)^2 + r^2\sin^2\theta &\leq \frac{1}{4} \\
r^2\cos^2\theta + r^2\sin^2\theta - r\cos\theta + \frac{1}{4} &\leq \frac{1}{4} \\
r^2 &\leq r\cos\theta \\
r &\leq \cos\theta.
\end{aligned}$$

Flaten er beskrevet ved $z = f(x,y) = \sqrt{1-x^2-y^2} = \sqrt{1-r^2}$. Vi har at

$$\begin{aligned}
\frac{\partial f}{\partial x} &= -\frac{x}{\sqrt{1-x^2-y^2}} \\
\frac{\partial f}{\partial y} &= -\frac{y}{\sqrt{1-x^2-y^2}}.
\end{aligned}$$

Dermed blir

$$\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}=\sqrt{1+\frac{x^2}{1-x^2-y^2}+\frac{y^2}{1-x^2-y^2}}=\frac{1}{\sqrt{1-x^2-y^2}}.$$

Flateintegralet for arealet blir dermed

$$\begin{aligned}
\int\int_A \frac{1}{\sqrt{1-x^2-y^2}}dxdy &= \int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta}\frac{1}{\sqrt{1-x^2-y^2}}rdrd\theta \\
&= \int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta}\frac{1}{\sqrt{1-r^2}}rdrd\theta \\
&= \int_{-\pi/2}^{\pi/2}\left[-\sqrt{1-r^2}\right]_0^{\cos\theta}d\theta \\
&= \int_{-\pi/2}^{\pi/2}\left(1-\sqrt{1-\cos^2\theta}\right)d\theta \\
&= \int_{-\pi/2}^{\pi/2}(1-|\sin\theta|)\,d\theta \\
&= \int_{-\pi/2}^{\pi/2}1d\theta+\int_{-\pi/2}^{0}\sin\theta d\theta+\int_0^{\pi/2}(-\sin\theta)d\theta \\
&= \pi+[-\cos\theta]_{-\pi/2}^{0}+[\cos\theta]_0^{\pi/2} \\
&= \pi-1-1=\pi-2.
\end{aligned}$$

Oppgave 6.4.11

Vi har at

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial u} &= (-3\sin u\cos v,-3\sin u\sin v,3\cos u) \\
\frac{\partial \mathbf{r}}{\partial v} &= (-(5+3\cos u)\sin v,(5+3\cos u)\cos v,0) \\
\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v} &= (-3(5+3\cos u)\cos u\cos v, \\
&\quad -3(5+3\cos u)\cos u\sin v, \\
&\quad -3(5+3\cos u)\sin u\cos^2 v-3(5+3\cos u)\sin u\sin^2 v) \\
&= (-3(5+3\cos u)\cos u\cos v,-3(5+3\cos u)\cos u\sin v,-3(5+3\cos u)\sin u) \\
\left|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right| &= \sqrt{9(5+3\cos u)^2\cos^2 u\cos^2 v+9(5+3\cos u)^2\cos^2 u\sin^2 v+9(5+3\cos u)^2\sin^2 u} \\
&= 3(5+3\cos u)\sqrt{\cos^2 u\cos^2 v+\cos^2 u\sin^2 v+\sin^2 u} \\
&= 3(5+3\cos u).
\end{aligned}$$

Vi får derfor

$$
\begin{aligned}
\int\int_T z^2 dS &= \int_0^{2\pi}\int_0^{2\pi} z^2 \left|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right| dudv \\
&= \int_0^{2\pi}\int_0^{2\pi} 9\cos^2 u 3(5+3\cos u)dudv \\
&= 135\int_0^{2\pi}\int_0^{2\pi}\cos^2 ududv + 27\int_0^{2\pi}\int_0^{2\pi}\cos^3 ududv \\
&= 135\int_0^{2\pi}\int_0^{2\pi}\frac{1}{2}(1+\cos(2u))dudv + 27\int_0^{2\pi}\int_0^{2\pi}(1-\sin^2 u)\cos ududv \\
&= 135\int_0^{2\pi}\left[\frac{1}{2}u+\frac{1}{4}\sin(2u)\right]_0^{2\pi} dv + 27\int_0^{2\pi}\left[\sin u - \frac{1}{3}\sin^3 u\right]_0^{2\pi} dv \\
&= 135\int_0^{2\pi}\pi dv = 270\pi^2.
\end{aligned}
$$

Oppgave 6.4.18

a)

Vi finner skjæringen mellom paraboloiden og planet først.

$$
\begin{aligned}
2x+4y+4 &= x^2+y^2 \\
x^2-2x+1+y^2-4y+4 &= 1+4+4 \\
(x-1)^2+(y-2)^2 &= 3^2.
\end{aligned}
$$

Det er klart at dette gir en sirkel med sentrum i (1,2) med radius 3. Det er klart at planet ligger ovenfor paraboloiden og innenfor denne sirkelen. Volumet må da bli

$$
\int\int_D (2x+4y+4-x^2-y^2)dxdy,
$$

som er det uttrykket vi skulle frem til.

b)

Gjør variabelskiftet

$$
u = x-1, v = y-2.
$$

Omårdet vårt er da beskrevet ved at $u^2+v^2 \leq 9$, eller $0 \leq r \leq 3$, $0 \leq \theta \leq 2\pi$ i polarkoordinater. Vi regner ut at Jacobideterminanten blir 1, slik at integralet også kan skrives

$$
\begin{aligned}
&\int\int_D (2x+4y+4-x^2-y^2)dxdy \\
&= \int\int_E (2(u+1)+4(v+2)+4-(u+1)^2-(v+2)^2)dudv \\
&= \int\int_E (9-u^2-v^2)dudv.
\end{aligned}
$$

der E er sirkelen om origo med radius 3. I polarkoordinater blir dette integralet

$$
\begin{aligned}
\int_0^{2\pi}\int_0^3 (9-r^2)rdrd\theta &= 2\pi\left[\frac{9}{2}r^2-\frac{1}{4}r^4\right]_0^3 \\
&= 2\pi\left(\frac{81}{2}-\frac{81}{4}\right) = \frac{81\pi}{2}.
\end{aligned}
$$

Oppgave 6.5.1

a)

Vi skal regne ut $\int_{\mathcal{C}}(x^2+y)dx + x^2ydy$. Vi regner ut

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2xy - 1,$$

og Greens teorem gir derfor

$$\begin{aligned}
\int_{\mathcal{C}}(x^2+y)dx + x^2ydy &= \int_0^2\int_0^2(2xy-1)dxdy \\
&= \int_0^2\left[x^2y - x\right]_0^2 dy \\
&= \int_0^2(4y-2)dy \\
&= \left[2y^2 - 2y\right]_0^2 = 8 - 4 = 4.
\end{aligned}$$

b)

Vi skal regne ut $\int_{\mathcal{C}}(x^2y^3)dx + x^3y^2dy$. Vi regner ut

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3x^2y^2 - 3x^2y^2 = 0.$$

Linjeintegralet må derfor bli 0.

Oppgave 6.5.2

Skisserer vi kurven ser vi at orienteringen er mot klokka, slik at vi kan bruke Greens teorem direkte.

$$\begin{aligned}
A &= \int_{\mathcal{C}}xdy = \int_0^{2\pi} t\sin t(2\pi - 2t)dt \\
&= 2\pi\int_0^{2\pi} t\sin t - 2\int_0^{2\pi} t^2\sin tdt \\
&= 2\pi\left[-t\cos t + \int\cos tdt\right]_0^{2\pi} - 2\left[-t^2\cos t + \int 2t\cos tdt\right]_0^{2\pi} \\
&= 2\pi\left[-t\cos t + \sin t\right]_0^{2\pi} - 2\left[-t^2\cos t + 2t\sin t + 2\cos t\right]_0^{2\pi} \\
&= -4\pi^2 - 2(-4\pi^2 + 2) + 4 = 4\pi^2
\end{aligned}$$

Oppgave 6.5.4

Vi setter $P(x,y) = 0, Q(x,y) = x$ i Greens teorem og får

$$
\begin{aligned}
A &= \int_{\mathcal{C}} Pdx + Qdy = \int_{\mathcal{C}} xdy \\
&= \int_0^{2\pi} a\cos^3 t b 3\sin^2 t \cos t dt \\
&= \int_0^{2\pi} 3ab\cos^4 t \sin^2 t dt \\
&= \frac{3}{4}ab\int_0^{2\pi} \cos^2 t 4\sin^2 t\cos^2 t dt \\
&= \frac{3}{4}ab\int_0^{2\pi} \frac{1}{2}(\cos(2t)+1)\sin^2(2t)dt \\
&= \frac{3}{8}ab\left(\int_0^{2\pi} \sin^2(2t)dt + \int_0^{2\pi} \sin^2(2t)\cos(2t)dt\right) \\
&= \frac{3}{8}ab\left(\int_0^{2\pi} \frac{1}{2}(1-\cos(4t))dt + \left[\frac{1}{6}\sin^3(2t)\right]_0^{2\pi}\right) \\
&= \frac{3}{16}ab\left[t - \frac{1}{4}\sin(4t)\right]_0^{2\pi} \\
&= \frac{3}{16}ab2\pi = \frac{3\pi ab}{8}.
\end{aligned}
$$

Matlab-kode

```
% Oppgave 6.5.2
t=linspace(0,2*pi,100);
x=t.*sin(t);
y=2*pi*t-t.^2;
plot(x,y)
```

Python-kode

```
# Oppgave 6.5.2
t=linspace(0,2*pi,100)
x=t*sin(t)
y=2*pi*t-t**2
plot(x,y)
```