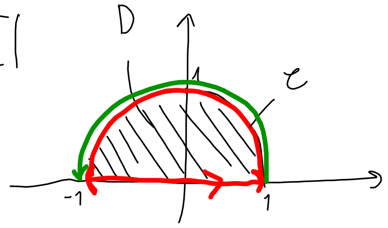
Plenum 10/5

$\boxed{6.5.8}$



$\mathcal{C}_1$: $\vec{r}_1(x) = (x, 0)$, $x \in [-1, 1]$

$\mathcal{C}_2$: $\vec{r}_2(x) = (x, 1-x^2)$, $x : 1 \to -1$

$(1, 0)$

$(-1, 0)$

a) $$I = \int_{\mathcal{C}} -y\,dx + x^2\,dy$$

$$= \int_{\mathcal{C}_1} -y\,dx + x^2\,dy \; + \int_{\mathcal{C}_2} -y\,dx + x^2\,dy$$

$\left[\begin{array}{l} y = 0 \\ dy = 0 \cdot dx \end{array}\right]$ $\left[\begin{array}{l} y = 1-x^2 \\ dy = -2x\,dx \end{array}\right]$

$$= \int_{-1}^{1} -0 \cdot dx + x^2 \cdot 0 \cdot dx \; + \int_{1}^{-1} -(1-x^2)\,dx + x^2(-2x\,dx)$$

$$= \underbrace{\int_{-1}^{1} -0 + 0\,dx}_{0} \; + \underbrace{\int_{1}^{-1} -2x^3 + x^2 - 1\,dx}_{4/3} = \underline{\underline{4/3}}$$

b) $$I = \iint_D f(x,y)\,dx\,dy$$

Green's teorem : $$\iint_D \frac{\partial Q}{\partial x}(x,y) - \frac{\partial P}{\partial y}(x,y)\,dx\,dy = \int_{\mathcal{C}} F(P(x,y), Q(x,y))\,d\vec{r}$$

$(\vec{r}(t) = (x(t), y(t)))$

$$\left.\begin{array}{l} P(x,y) = -y \\ Q(x,y) = x^2 \end{array}\right\} \quad \begin{array}{l} \frac{\partial Q}{\partial x} = 2x \\ \frac{\partial P}{\partial y} = -1 \end{array}$$

$$= \int_{\mathcal{C}} F \cdot d\vec{r}$$

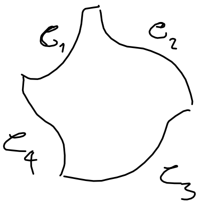
$$I = \iint_D 2x + 1\,dx\,dy$$

$$= \int_{\mathcal{C}} P\,dx + Q\,dy$$

$$= \int_{-1}^{1}\int_{0}^{1-x^2} 2x + 1\,dy\,dx = \int_{-1}^{1} (2x+1)(1-x^2)\,dx = \ldots = \underline{\underline{4/3}}$$

$\boxed{6.5.13}$ $\quad F(x,y) = \big(P(x,y),\, Q(x,y)\big)$ Konservativt felt

$$\oint_{\mathcal{C}} F \cdot d\vec{r} = 0$$



F konservativt $\Rightarrow \exists\, \varphi : \mathbb{R}^2 \to \mathbb{R}$ slik at

$F = \nabla \varphi$. Dvs. $F(x,y) = (P,\, Q) = \left(\frac{\partial \varphi}{\partial x},\, \frac{\partial \varphi}{\partial y} \right) = \nabla \varphi$

$P = \frac{\partial \varphi}{\partial x}$, $Q = \frac{\partial \varphi}{\partial y}$

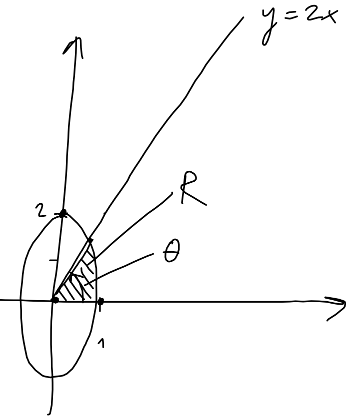
$$\int_{\mathcal{C}} F \cdot d\vec{r} \underset{\text{Greens}}{=} \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\, dy \overset{\text{konservativt felt.}}{=} \iint_D \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial x}\right)}_{\frac{\partial^2}{\partial x \partial y}\varphi - \frac{\partial^2}{\partial y \partial x}\varphi = 0} dx\, dy$$

$$= \iint_D 0\, dx\, dy = 0$$

$\boxed{6.7.8}$

$x = u\cos v$ | $x = r\cos\theta$

$y = 2u\sin v$ | $y = 2r\sin\theta$



$y = 2x$

$\cancel{2r}\sin\theta = \cancel{2r}\cos\theta$ $\qquad \theta = \frac{\pi}{4} + n\cdot\pi,\ n\in\mathbb{Z}$

$x^2 + y^2/4 = 1$

$r^2\cos^2\theta + \frac{\cancel{4}r^2\sin^2\theta}{\cancel{4}} = 1$

$r^2 = 1 \to r = 1$

$T(r,\theta) = (r\cos\theta,\ 2r\sin\theta)$

$|\det T'|$

$R = T(A)$

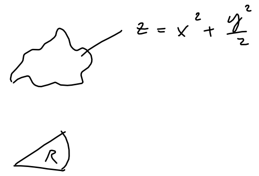
$$\iint_R 1\,dx\,dy = \iint_{T^{-1}(R)=A} |\det T'|\,d\theta\,dr$$

$$T'(r,\theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ 2\sin\theta & 2r\cos\theta \end{pmatrix}$$

$$\det T'(r,\theta) = 2r\cos^2\theta + 2r\sin^2\theta = 2r$$

$$= \int_0^1\int_0^{\frac{\pi}{4}} 2r\,d\theta\,dr = 2\frac{\pi}{4}\int_0^1 r\,dr = \frac{\pi}{4}\left[r^2\right]_0^1 = \underline{\underline{\frac{\pi}{4}}}$$

b) $z = x^2 + \frac{y^2}{2} \qquad (x,y) \in R$



$$z = (r\cos\theta)^2 + \frac{(2r\sin\theta)^2}{2}$$
$$= r^2\cos^2\theta + \frac{4r^2\sin^2\theta}{2}$$
$$= r^2(\underbrace{\cos^2\theta}_{1-\sin^2\theta} + 2\sin^2\theta)$$
$$= r^2(1+\sin^2\theta)$$

$$\vec{r}(r,\theta) = (r\cos\theta,\ 2r\sin\theta,\ r^2(1+\sin^2\theta))$$

$$\iint_R \left|\frac{\partial\vec{r}}{\partial r} \times \frac{\partial\vec{r}}{\partial\theta}\right| dr\,d\theta$$

$$\frac{\partial\vec{r}}{\partial r} = (\cos\theta,\ 2\sin\theta,\ 2r(1+\sin^2\theta))$$
$$\frac{\partial\vec{r}}{\partial\theta} = (-r\sin\theta,\ 2r\cos\theta,\ 2r^2\sin\theta\cos\theta)$$

$$\frac{\partial\vec{r}}{\partial r} \times \frac{\partial\vec{r}}{\partial\theta} = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & 2\sin\theta & 2r(1+\sin^2\theta) \\ -r\sin\theta & 2r\cos\theta & 2r^2\sin\theta\cos\theta \end{pmatrix}$$

$$= -4r^2\cos\theta\,\vec{i} - 4r^2\sin\theta\,\vec{j} + 2r\,\vec{k}$$

$$\left|\frac{\partial\vec{r}}{\partial r} \times \frac{\partial\vec{r}}{\partial\theta}\right| = \left(16r^4\cos^2\theta + 16r^4\sin^2\theta + 4r^2\right)^{\frac{1}{2}}$$
$$= 2r\left(4r^2\underbrace{(\cos^2\theta+\sin^2\theta)}_{1} + 1\right)^{\frac{1}{2}}$$
$$= 2r(4r^2+1)^{\frac{1}{2}}$$

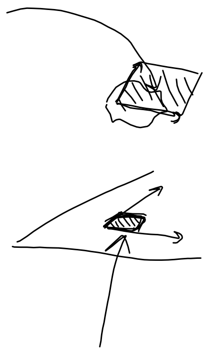
$$\iint_R \left|\frac{\partial\vec{r}}{\partial r} \times \frac{\partial\vec{r}}{\partial\theta}\right| dr\,d\theta = \int_0^1\int_0^{\frac{\pi}{4}} 2r(4r^2+1)^{\frac{1}{2}}\,d\theta\,dr$$

$u = 4r^2+1$; $r=0 \to u=1$, $r=1 \to u=5$

$du = 8r\,dr$

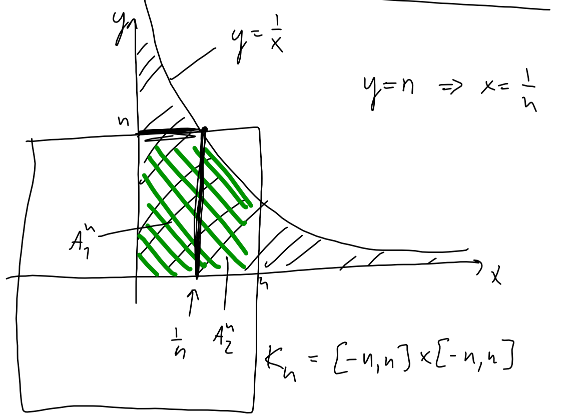
$$= \frac{\pi}{4}\int_0^1 2r(4r^2+1)^{\frac{1}{2}}\,dr$$
$$= \frac{\pi}{4}\frac{1}{8}\int_1^5 2u^{\frac{1}{2}}\,du$$
$$= \frac{\pi}{16}\int_1^5 u^{\frac{1}{2}}\,du = \frac{\pi}{16}\frac{2}{3}\left[u^{\frac{3}{2}}\right]_1^5 = \frac{\pi}{24}(5^{\frac{3}{2}}-1)$$
$$= \frac{\pi}{24}(5\sqrt{5}-1)$$

$$\iint_A f\, dS = \iint_A f(r) \underbrace{\left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right|}\, du\, dv$$



$\boxed{6.8.4}$ $\displaystyle\iint_A xy\, dx dy$

$$= \lim_{n\to\infty} \iint_{\underbrace{A \cap K_n}} xy\, dx dy$$



$y = n \Rightarrow x = \frac{1}{n}$

$A_1^n = [0, \frac{1}{n}] \times [0, n]$

$A_2^n = [\frac{1}{n}, n] \times [0, \frac{1}{x}]$

$\Rightarrow A \cap K_n = A_1^n \cup A_2^n$

$$\lim_{n\to\infty} \iint_{A\cap K_n} xy\, dx dy = \lim_{n\to\infty} \left(\iint_{A_1^n} xy\, dx dy + \iint_{A_2^n} xy\, dx dy \right)$$

$$= \lim_{n\to\infty} \left(\int_0^{\frac{1}{n}} \int_0^n xy\, dy dx + \int_{\frac{1}{n}}^n \int_0^{\frac{1}{x}} xy\, dy dx \right)$$

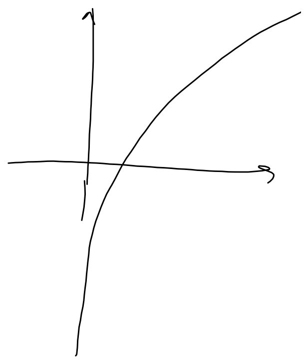
$$= \lim_{n\to\infty} \left(\frac{1}{2} n^2 \int_0^{\frac{1}{n}} x\, dx + \frac{1}{2} \int_{\frac{1}{n}}^n \frac{1}{x}\, dx \right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{4} + \frac{1}{2}\left(\ln(n) - \ln\left(\frac{1}{n}\right)\right) \right)$$

$\ln\left(\frac{1}{n}\right) = \underbrace{\ln 1}_{0} - \ln n$

$$= \lim_{n\to\infty} \left(\frac{1}{4} + \cancel{\frac{1}{2}}\, \cancel{2} \ln(n) \right)$$

$= \infty$, divergerer.



Eks. 2012

5 $$V = \iiint 1 \, dz \, dx \, dy$$

$$= \iint \left[z \right]_{x^2+4x+y^2-8y}^{8-x^2-y^2} dx \, dy$$

$$8 - x^2 - y^2 = x^2 + 4x + y^2 - 8y$$

$$8 \underbrace{-2x^2 - 4x}_{-2(x^2+2x+1)} -2 + 2 \underbrace{-2y^2 + 8y}_{-2(y^2-4y+4)} -8 + 8 = 0$$

$$18 - 2(x+1)^2 - 2(y-2)^2 = 0$$

$$3^2 = 9 = (x+1)^2 + (y-2)^2$$

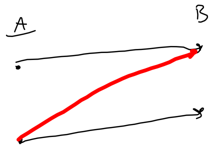
$r = 3$ $(-1, 2)$ ← sentrum

b) $x+1 = r \cdot \cos\theta$
$y-2 = r \cdot \sin\theta$

$$\iint_S 18 - 2\underbrace{(x+1)^2}_{r^2\cos^2\theta} - 2\underbrace{(y-2)^2}_{r^2\sin^2\theta} \overset{\cdot r}{\vee} dx\,dy = \int_0^{2\pi}\int_0^3 (18 - 2r^2) \cdot r \, dr \, d\theta = \dots = \underline{\underline{81\pi}}$$

Eks 2013

⑤ 1-1:



På: $B \subset F(A)$

På: La $(x_B, y_B) \in B$ være et vilkårlig pkt. i B. Konstruér et nytt pkt. (x_A, y_A)

$$x_A = \frac{x_B - x_0}{a}, \quad y_A = \frac{y_B - y_0}{b}$$

Da vil:

$$x_A^2 + y_A^2 = \left(\frac{x_B - x_0}{a}\right)^2 + \left(\frac{y_B - y_0}{b}\right)^2 \overset{(x_B, y_B) \in B}{\le} 1$$

$\Rightarrow (x_A, y_A) \in A$

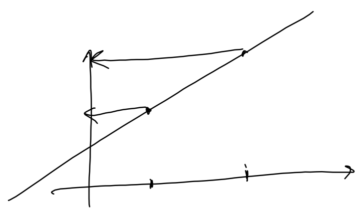
$x_B = x_0 + a x_A$

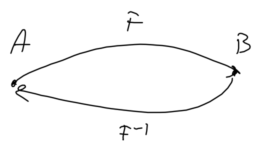
$\frac{x_B - x_0}{a} = x_A$

$$F(x_A, y_A) = (\overbrace{x_0 + a \cdot x_A}^{x_B}, y_0 + b \cdot y_A)$$

$$= \left(x_0 + a \frac{x_B - x_0}{a}, y_0 + b \frac{y_B - y_0}{b}\right)$$

$$= (x_B, y_B)$$





$$F^{-1} = \left(\frac{x - x_0}{a}, \frac{y - y_0}{b}\right)$$

$$F(x,y) = (x_0 + ax; y_0 + by)$$

$$\det(F') = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$$|\det F'| = ab$$

$x = r\cos\theta$ $\Rightarrow$ $\cdot r$

$y = r\sin\theta$