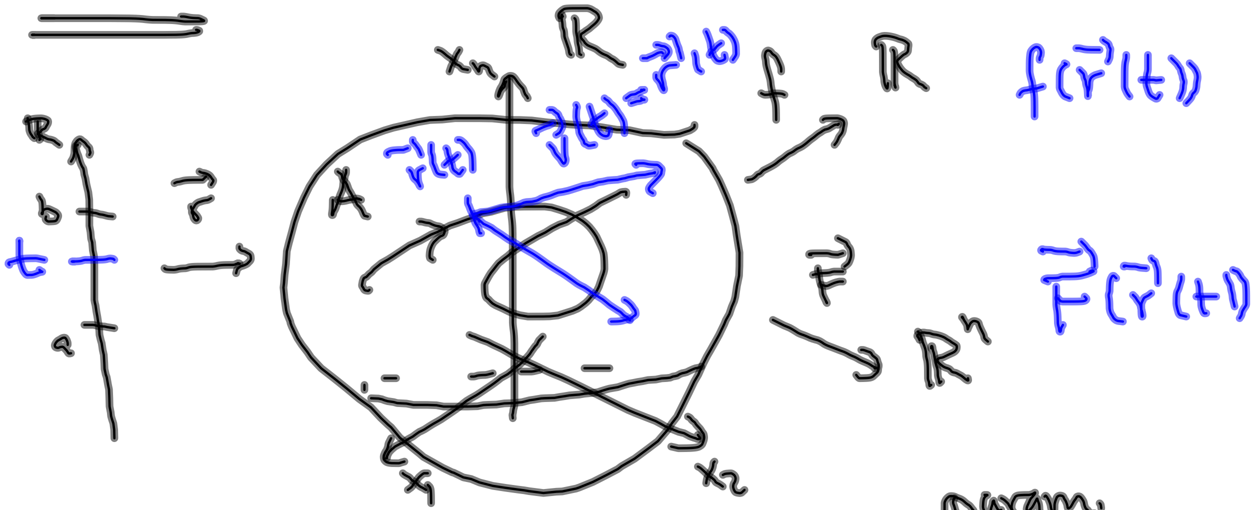
LH 3.4 - 3.5

Linjeintegral for vektorfelt

Gradientfelt = konservative vektorfelt



$\vec{r} : [a,b] \to \mathbb{R}^n$ deriverbar param. av $\mathcal{C}$

$\vec{v}(t) = \vec{r}'(t) : (a,b) \to \mathbb{R}^n$ kontinuerlig

$v = |\vec{v}|$

$$\int_{\mathcal{C}} f \, ds = \int_a^b f(\vec{r}(t)) \, v(t) \, dt$$

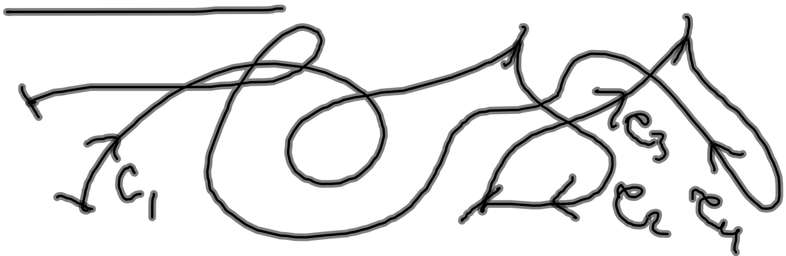
$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{v}(t) \, dt$$

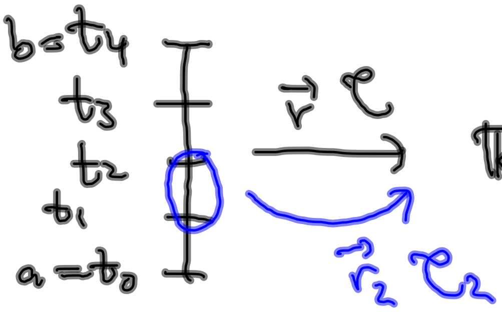
STANDARD EGENSKAPER:

$$[a,b] \xrightarrow{\vec{r}} A \underset{\vec{G}}{\overset{\vec{F}}{\rightrightarrows}} \mathbb{R}^n$$

$A \subset \mathbb{R}^n$ $\quad \lambda, \mu \in \mathbb{R}$

$$\int_{\mathcal{C}} (\lambda \vec{F} + \mu \vec{G}) \cdot d\vec{r} = \lambda \int_{\mathcal{C}} \vec{F} \cdot d\vec{r} + \mu \int_{\mathcal{C}} \vec{G} \cdot d\vec{r}$$



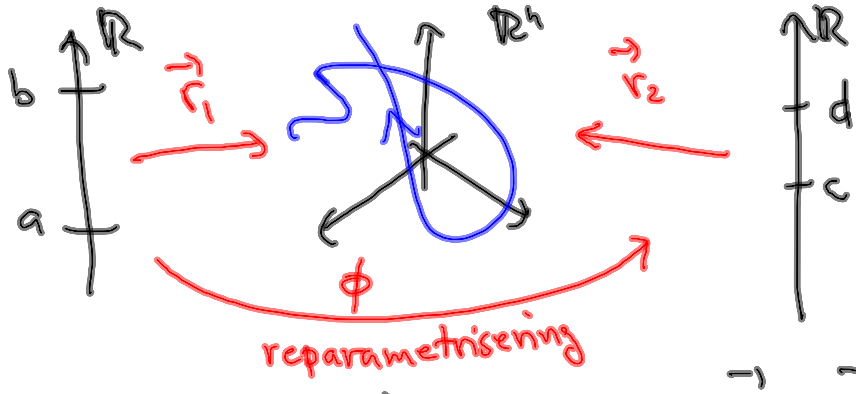


$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_{\mathcal{C}_1} \vec{F} \cdot d\vec{r}_1 + \ldots + \int_{\mathcal{C}_4} \vec{F} \cdot d\vec{r}_4$$

$a < b < c \quad f : [a, c] \longrightarrow \mathbb{R}$

$$\int_a^c f(t)\,dt = \int_a^b f(t)\,dt + \int_b^c f(t)\,dt$$

(U-)avhengighet av <u>orientert</u> parametrisering



$$\vec r_1(t) = \vec r_2(\phi(t)) \qquad \vec r_1 = \vec r_2 \circ \phi$$

$$\phi : [a,b] \longrightarrow [c,d]$$

kontinuerlig deriverbar med $\phi'(t) \neq 0$

$$\phi([a,b]) = [c,d]$$

<u>Setning 3.4.4</u> Hvis ϕ <u>er strengt voksende</u> ($\phi'(t) > 0$) er

$$I_1 = \int_a^b \vec F(\vec r_1(t)) \cdot \vec v_1(t)\, dt$$

lik

$$I_2 = \int_c^d \vec F(\vec r_2(u)) \cdot \vec v_2(u)\, du$$

$$\text{“} \int_{\mathcal{C}} \vec F \cdot d\vec r_1 = \int_{\mathcal{C}} \vec F \cdot d\vec r_2 \text{”}$$

Hvis $\phi'(t) < 0$ er

$$I_1 = -I_2 .$$

$$\text{“} \int_{\mathcal{C}} \vec F \cdot d\vec r_1 = -\int_{\mathcal{C}} \vec F \cdot d\vec r_2 \text{”}$$

<u>Bevis</u> Anta $\phi'(t) > 0$.

$$I_1 = \int_{\mathcal{C}} \vec{F} \cdot d\vec{r}_1 = \int_a^b \vec{F}(\vec{r}_1(t)) \cdot \vec{v}_1(t)\, dt$$

der $\vec{r}_1(t) = \vec{r}_2(\phi(t))$ så

$$\vec{v}_1(t) = \vec{r}_1{}'(t) = \vec{r}_2{}'(\phi(t))\, \phi'(t)$$
$$= \vec{v}_2(\phi(t))\, \phi'(t) .$$

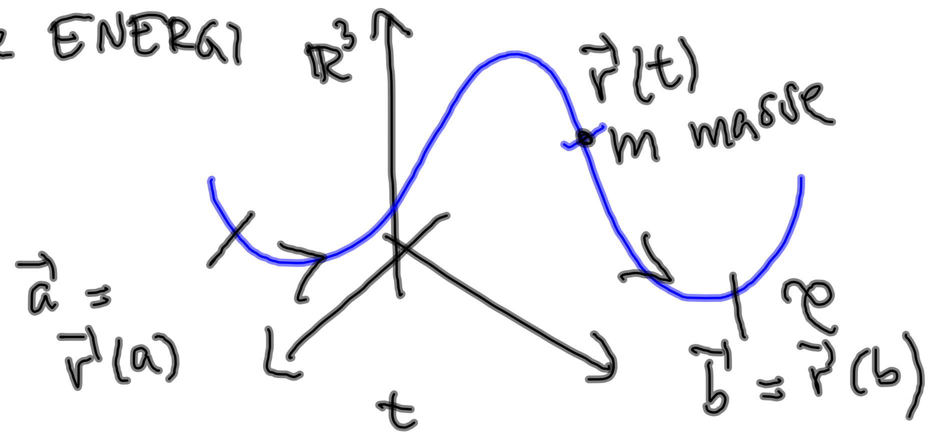
Derfor er

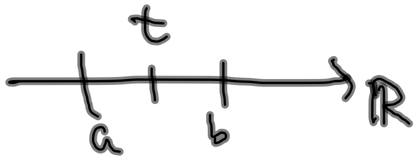
$$I_1 = \int_a^b \vec{F}(\vec{r}_2(\phi(t))) \cdot \vec{v}_2(\phi(t))\, \phi'(t)\, dt$$

$u = \phi(t)$
$du = \phi'(t)\, dt$
$c = \phi(a)$
$d = \phi(b)$

$$= \int_c^d \vec{F}(\vec{r}_2(u)) \cdot \vec{v}_2(u)\, du$$
$$= \int_{\mathcal{C}} \vec{F} \cdot d\vec{r}_2 = I_2$$

KINETISK ENERGI





$\vec{v}(t) = \vec{r}\,'(t)$

$\vec{a}(t) = \vec{r}\,''(t) = \vec{v}\,'(t)$

Newtons 2. lov: $\boxed{\vec{F}(t) = m\vec{a}(t)}$

Arbeid som utføres på partikkelen fra $t = a$ til $t = b$ er

$$W = \int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{v}(t)\, dt$$

<u>PÅSTAND</u>:

$$W = E_k(b) - E_k(a)$$

der $E_k(t)$ er den kinetiske energien i tiden t.

$$E_k(t) = \frac{1}{2} m\, v(t)^2$$

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \underline{E_k(b) - E_k(a)}$$

$$E_k(t) = \frac{1}{2} m v(t)^2 = \frac{1}{2} m \, \vec{v}(t) \cdot \vec{v}(t)$$

deriverer m.h.p t:

$$E_k'(t) = \frac{1}{2} m \left(\vec{v}'(t) \cdot \vec{v}(t) + \vec{v}(t) \cdot \vec{v}'(t) \right)$$

$$= \frac{1}{2} m \left(\vec{a}(t) \cdot \vec{v}(t) + \vec{v}(t) \cdot \vec{a}(t) \right)$$

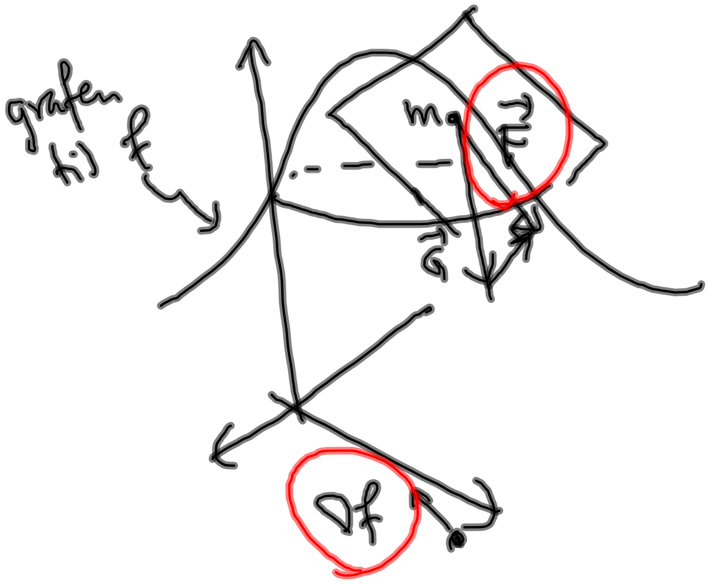
$$= m \, \vec{a}(t) \cdot \vec{v}(t) = \vec{F}(\vec{r}(t)) \cdot v(t)$$

$$\underset{\text{W}}{\overset{}{}} \int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{v}(t) \, dt$$

$$= \int_a^b E_k'(t) \, dt = \Big[E_k(t) \Big]_a^b$$

$$= \underline{E_k(b) - E_k(a)}$$

LH 3.5 Gradienter og konservative vektorfelt



Vil se på tilfellet der $\vec F : A \to \mathbb{R}^n$ ($A \subseteq \mathbb{R}^n$) er gradienten

$$\vec F = \nabla \phi$$

til en funksjon $\phi : A \to \mathbb{R}$.

$$\vec F(\vec x) = \left(\frac{\partial \phi}{\partial x_1}(\vec x), \dots, \frac{\partial \phi}{\partial x_n}(\vec x) \right).$$

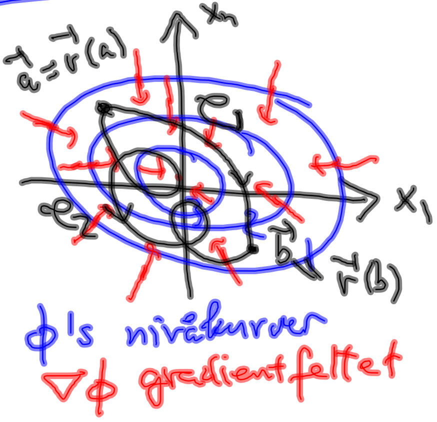
- $\vec F$ er et <u>gradientfelt</u>
- ϕ kalles <u>potensialet</u> til $\vec F$.

SETNING 3.5.1:

ANTA $\phi : A \to \mathbb{R}$, DER $A \subset \mathbb{R}^n$ ER ÅPENT, HAR KONTINUERLIG GRADIENT $\nabla\phi : A \to \mathbb{R}^n$.
LA $\vec{r} : [a,b] \to A$ PARAMETRISERE EN STYKKEVIS GLATT KURVE $\mathcal{C}$ I A, FRA $\vec{a} = \vec{r}(a)$ TIL $\vec{b} = \vec{r}(b)$. DA ER

$$\boxed{\int_{\mathcal{C}} \nabla\phi \cdot d\vec{r} = \phi(\vec{b}) - \phi(\vec{a})}$$

$$\begin{tikzcd} \underset{\subset \mathbb{R}}{[a,b]} \arrow[r, "\vec{r}"] \arrow[rr, bend right, "\phi\circ\vec{r}"'] & \underset{\subset \mathbb{R}^n}{A} \arrow[r, "\phi"] & \mathbb{R} \end{tikzcd}$$



BEVIS:

Antar først at $\mathcal{C}$ er glatt.

$$\int_{\mathcal{C}} \nabla\phi \cdot d\vec{r} = \int_a^b \nabla\phi(\vec{r}(t)) \cdot \vec{v}(t)\, dt$$

Deriverer $\phi\circ\vec{r} : t \mapsto \phi(\vec{r}(t))$ m.h.p. t:

$$(\phi\circ\vec{r})'(t) = \phi(\vec{r}(t))'$$
$$= \phi'(\vec{r}(t))\,\vec{r}\,'(t)$$
$$= \nabla\phi(\vec{r}(t)) \cdot \vec{v}(t)$$

$$= \int_a^b (\phi\circ\vec{r})'(t)\, dt$$

$$= \Big[(\phi\circ\vec{r})(t)\Big]_a^b = \phi(\vec{r}(b)) - \phi(\vec{r}(a))$$

$$\phi(\vec{b}) - \phi(\vec{a})$$

Eks. Hvis $\mathcal{C}$ er en løkke,
så $\vec{a} = \vec{r}(a) = \vec{r}(b) = \vec{b}$ vil

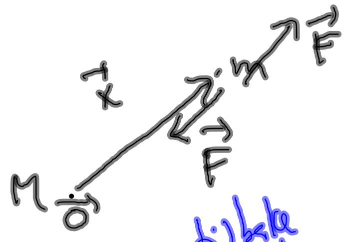
$$\int_{\mathcal{C}} \nabla\phi \cdot d\vec{r} = \phi(\vec{b}) - \phi(\vec{a}) = 0$$

så det „totale arbeidet" er null.

DEF. Et vektorfelt $\vec{F} : A \to \mathbb{R}^n$
på formen $\vec{F} = \nabla\phi$ der $\phi : A \to \mathbb{R}$
kalles et gradientfelt, eller sies
å være et konservativt vektorfelt.

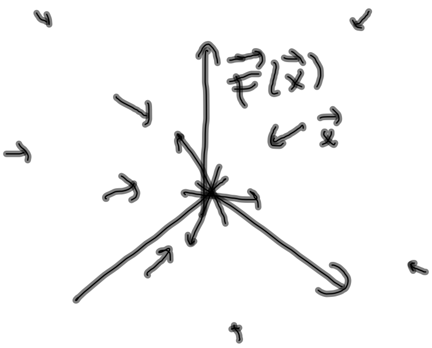
Eks Gravitasjon

$$\vec{F}(\vec{x}) = \frac{k}{|\vec{x}|^2}\cdot\left(+\frac{\vec{x}}{|\vec{x}|}\right)$$



enhetsvektoren tilbake mot M i $\vec{0}$

$$= +\frac{k\vec{x}}{|\vec{x}|^3}$$



$$\vec{F} = \nabla\phi$$

der

$$\phi(\vec{x}) = \frac{\div k}{|\vec{x}|}\,.$$

gravitasjons-potensialet

Regne ut $\nabla\phi$.

$$\psi(\vec{x}) = \vec{x}\cdot\vec{x} = |\vec{x}|^2$$

psi

$$= x_1^2 + x_2^2 + \dots + x_n^2$$

$$\nabla\psi(\vec{x}) = \left(\frac{\partial\psi}{\partial x_1}(\vec{x}), \dots, \frac{\partial\psi}{\partial x_n}(\vec{x})\right)$$

$$= (2x_1, 2x_2, \dots, 2x_n) = 2\vec{x}$$

$$\phi(\vec{x}) = -\frac{k}{|\vec{x}|} = -k\,\psi(\vec{x})^{-\frac{1}{2}}$$

$$\nabla\phi(\vec{x}) = (-k)\left(-\frac{1}{2}\right)\psi(\vec{x})^{-\frac{3}{2}}\,\nabla\psi(\vec{x})$$

$$= \frac{k}{2}\,\frac{1}{|\vec{x}|^3}\,2\vec{x} = k\,\frac{\vec{x}}{|\vec{x}|^3}$$

$$= \vec{F}(\vec{x}).$$

$$\vec{F} = \nabla\phi.$$

$\boxed{\checkmark}$

Hvilke vektorfelt er konservative?

Setning 3.5.3 Hvis $\vec{F}(\vec{x}) = (F_1(\vec{x}), \dots, F_n(\vec{x}))$ er konservativt i et område $A \subseteq \mathbb{R}^n$ vil

$$\boxed{\frac{\partial F_i}{\partial x_j}(\vec{x}) = \frac{\partial F_j}{\partial x_i}(\vec{x})}$$

Antar at alle $\partial F_i / \partial x_j$ er kontinuerlige

for alle $i, j \in \{1, \dots, n\}$ og $\vec{x} \in A$.

Bevis: $\vec{F} = \nabla \phi$ så $F_i = \frac{\partial \phi}{\partial x_i}$

og

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial \phi}{\partial x_i}\right) = \frac{\partial^2 \phi}{\partial x_j \partial x_i}$$

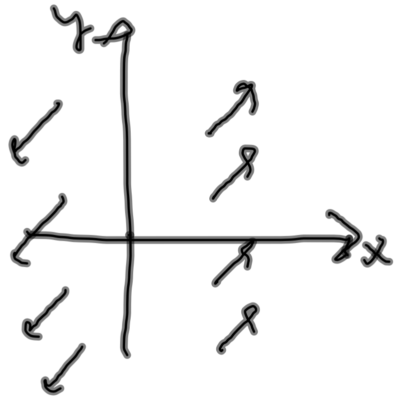
er like!

$$\frac{\partial F_j}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\partial \phi}{\partial x_j}\right) = \frac{\partial^2 \phi}{\partial x_i \partial x_j}$$

☑

Ikke-eksempel

$n=2$ $\quad \vec{F}(x,y) = (x,x)$



$F_1(x,y) = \underline{x}$, $F_2(x,y) = \underline{x}$

$\frac{\partial F_1}{\partial y} = 0$ og $\frac{\partial F_2}{\partial x} = 1$

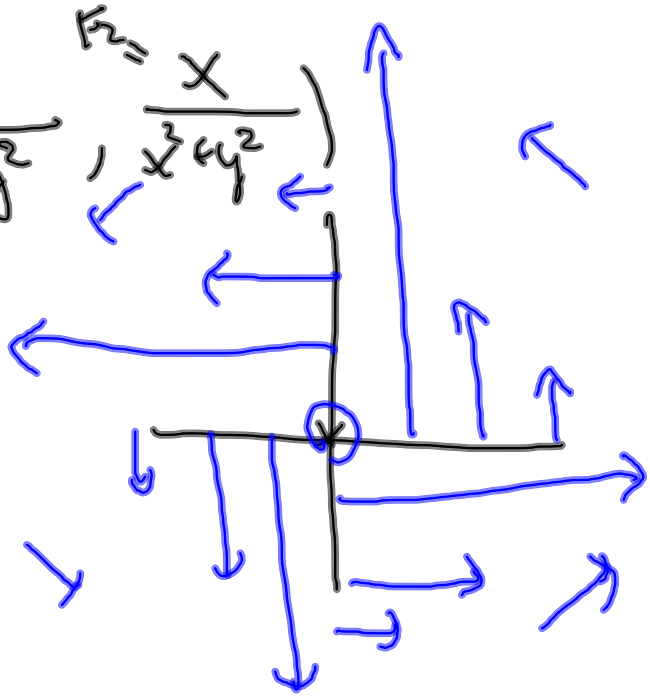
er forskjellige. Så $\vec{F} \neq \nabla \phi$.

Omvendingen til 3.5.3 gjelder ikke generelt.

Eks $n=2$

$$\vec{F}(x,y) = \Big(\underbrace{\frac{-y}{x^2+y^2}}_{F_1}, \overbrace{\frac{x}{x^2+y^2}}^{F_2 =} \Big)$$

for $(x,y) \neq (0,0)$.



(1)

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

(2) $F \neq \nabla \phi$.