Plenum 11/2

3.1: 10, 12, 14, 17, 21

3.2: 7

3.3: 11

3.1: Parametriserte kurver

10) $\vec{r}(t) = (2\cos t, \sqrt{2}\sin t, \sqrt{2}\sin t)$

a) $\vec{v}(t) = \vec{r}\,'(t) = (-2\sin t, \sqrt{2}\cos t, \sqrt{2}\cos t)$

$$v(t) = |\vec{v}(t)| = \sqrt{4\sin^2 t + 2\cos^2 t + 2\cos^2 t}$$

$$= \underline{\underline{2}}$$

$$\vec{a}(t) = \vec{v}\,'(t) = (-2\cos t, -\sqrt{2}\sin t, -\sqrt{2}\sin t)$$

b) $L(0, 2\pi) \underset{\text{Def. 3.1.2}}{=} \int_0^{2\pi} v(t)\,dt \underset{a)}{=} \int_0^{2\pi} 2\,dt = 2 \cdot 2\pi = \underline{\underline{4\pi}}$

c) Husk: Kule med sentrum i origo & radius r;

$$x^2 + y^2 + z^2 = r^2$$

Merk at:

$$(2\cos t)^2 + (\sqrt{2}\sin t)^2 + (\sqrt{2}\sin t)^2$$

$$= 4\cos^2 t + 2\sin^2 t + 2\sin^2 t = 4 = 2^2$$

Så kurven ligger på ei kule, sentrum origo, radius 2.

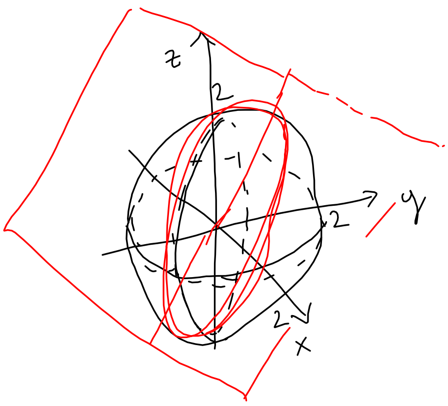
d) $y - z = \sqrt{2}\sin t - \sqrt{2}\sin t = 0$

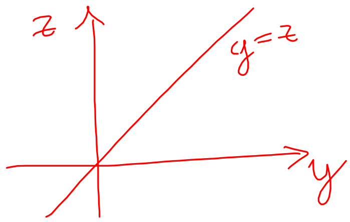
for kurven gitt v/ $\vec{r}$

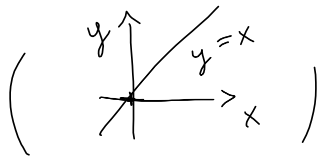
$\Downarrow$

Kurven ligger i planet $y - z = 0$.

e) Kurven er en sirkel med radius 2, sentrum i origo som ligger på planet $y-z=0$: Fordi kurven ligger på kula fra c) & i planet fra d).







12) a) $\vec{r}(t) = (rt - r\sin t,\ r - r\cos t)$

$$\vec{v}(t) = \vec{r}\,'(t) = (r - r\cos t,\ r\sin t)$$

$$v(t) = |\vec{v}(t)| = r\sqrt{2}\sqrt{1-\cos t}$$

$$\vec{a}(t) = \vec{v}\,'(t) = (r\sin t,\ r\cos t)$$

b) $$L(0,2\pi) = \int_0^{2\pi} v(t)\,dt \overset{a)}{=} \int_0^{2\pi} r\sqrt{2}\sqrt{1-\cos t}\,dt$$

$$= r\sqrt{2}\int_0^{2\pi}\sqrt{1-\cos t}\,dt$$

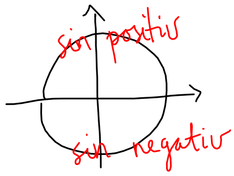
$\sin^2 t + \cos^2 t = 1$

c) $$\sqrt{1-\cos t} = \sqrt{\frac{(1-\cos t)(1+\cos t)}{1+\cos t}} = \sqrt{\frac{1-\cos^2 t}{1+\cos t}} = \sqrt{\frac{\sin^2 t}{1+\cos t}}$$

$\sqrt{\ }$ betyr positiv rot

$$= \frac{|\sin t|}{\sqrt{1+\cos t}}$$

d) $$L(0,2\pi) \overset{b)}{=} r\sqrt{2}\int_0^{2\pi}\sqrt{1-\cos t}\,dt \overset{c)}{=} r\sqrt{2}\int_0^{2\pi}\frac{|\sin t|}{\sqrt{1+\cos t}}\,dt$$



$$= r\sqrt{2}\left(\int_0^{\pi} \frac{\sin t}{\sqrt{1+\cos t}}\,dt - \int_{\pi}^{2\pi} \frac{\sin t}{\sqrt{1+\cos t}}\,dt\right)$$

$$= r\sqrt{2}\left(-\int_2^0 \frac{1}{\sqrt{u}}\,du + \int_0^2 \frac{1}{\sqrt{u}}\,du\right)$$

$u = 1+\cos t$

$du = -\sin t\,dt$

$-\frac{du}{\sin t} = dt$

$t = 0 \Rightarrow u = 2$

$t = \pi \Rightarrow u = 0$

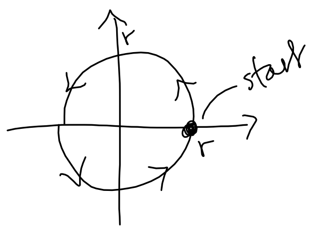
$t = 2\pi \Rightarrow u = 2$

$$= r\sqrt{2}\,2\left[2\sqrt{u}\right]_{u=0}^{2}$$

$\sqrt{u} = u^{\frac{1}{2}}$

$\frac{1}{\sqrt{u}} = u^{-\frac{1}{2}}$

$$= \underline{\underline{8r}}$$

14) 

a) Parametrisering av sirkel m/ radius r om origo:

$(r\cos t, r\sin t)$, $t \in [0, 2\pi]$

Hva er <u>farten</u>?

$$v(t) = |\vec{v}(t)| = |r'(t)| = |(-r\sin t, r\cos t)|$$

$$= \sqrt{r^2(\underbrace{\sin^2 t + \cos^2 t}_{1})} = r$$

Vi vil ha farten v, ikke r. Hvordan fikse dette?

Hvis vi har:

$\left(r\cos\left(\frac{vt}{r}\right), r\sin\left(\frac{vt}{r}\right)\right)$, så vil

$$v(t) = \left|\left(-\cancel{r}\,\frac{v}{\cancel{r}}\sin\left(\frac{vt}{r}\right), \cancel{r}\,\frac{v}{\cancel{r}}\cos\left(\frac{vt}{r}\right)\right)\right|$$

$$= \sqrt{v^2\Big(\underbrace{\sin^2\left(\tfrac{vt}{r}\right) + \cos^2\left(\tfrac{vt}{r}\right)}_{1}\Big)} = v$$

Dette er fortsatt en sirkel om origo m/ radius r mot klokka, så må posisjonenen v/ tid t være

$$\underline{\underline{\left(r\cos\left(\frac{vt}{r}\right), r\sin\left(\frac{vt}{r}\right)\right)}}$$

b) $$\vec{a}(t) = \vec{v}'(t) = \left(-\frac{v^2}{r}\cos\left(\frac{vt}{r}\right), -\frac{v^2}{r}\sin\left(\frac{vt}{r}\right)\right)$$
$$= -\frac{v^2}{r^2}\left(r\cos\left(\frac{vt}{r}\right), r\sin\left(\frac{vt}{r}\right)\right)$$
$$= \underline{\underline{-\frac{v^2}{r^2}\vec{r}(t)}}$$

17) $\vec{r}_1(t) =$ spor bakhjul, $\vec{r}_2(t) =$ spor forhjul

a)

a) Fartsretningen til bakhjulet er i retning av forhjulet $\Rightarrow \vec{v}_1(t)$ har samme retning som $\vec{r}_2(t) - \vec{r}_1(t)$. Lengden til $\vec{r}_2(t) - \vec{r}_1(t)$ er 1 (fra oppg.) $\Rightarrow$ dette er en enhetsvektor i fartsretningen.

Å være enhetsvektor i fartsretningen er def. av $\vec{T}_1(t)$.

$$\vec{T}_1(t) = \vec{r}_2(t) - \vec{r}_1(t)$$

$$\underline{\underline{\vec{r}_2(t) = \vec{r}_1(t) + \vec{T}_1(t)}}$$

b) $\vec{r}_1(t) = (t, \sin t) \Rightarrow \vec{r}_2(t) \overset{a)}{=} (t, \sin t) + \vec{T}_1(t) \quad : (\star)$

$$\vec{T}_1(t) = \frac{\vec{v}_1(t)}{v_1(t)}$$

$$\vec{v}_1(t) = \vec{r}_1\,'(t) = \underline{(1, \cos t)}$$

$$v_1(t) = |\vec{v}_1(t)| = \underline{\sqrt{1 + \cos^2 t}}$$

$$\Rightarrow \vec{T}_1(t) = \left(\frac{1}{\sqrt{1 + \cos^2 t}}, \frac{\cos t}{\sqrt{1 + \cos^2 t}} \right)$$

$$r_2(t) \overset{(\star)}{=} (t, \sin t) + \left(\frac{1}{\sqrt{1 + \cos^2 t}}, \frac{\cos t}{\sqrt{1 + \cos^2 t}} \right)$$

$$= \left(t + \frac{1}{\sqrt{1+\cos^2 t}} \, , \, \sin t + \frac{\cos t}{\sqrt{1+\cos^2 t}} \right)$$

d) Fra plot i c) ser vi at dette er likt plottet i d), så sykkelen flytter seg fra venstre mot høyre.

21) $\vec{a}(t) = k(t)\,\vec{r}(t)$

sat. 3.1.4 (iv)

a) $\frac{d}{dt}\left(\vec{r}(t) \times \vec{v}(t)\right) = \vec{r}\,'(t) \times \vec{v}(t) + \vec{r}(t) \times \vec{v}\,'(t)$

$= \vec{v}(t) \times \vec{v}(t) + \vec{r}(t) \times \vec{a}(t)$

oppg. $= \vec{v}(t) \times \vec{v}(t) + \vec{r}(t) \times \left(k(t)\,\vec{r}(t)\right)$

sat. 1.4.1 $= \vec{v}(t) \times \vec{v}(t) + k(t)\left[\vec{r}(t) \times \vec{r}(t)\right]$

$= \vec{0} + k(t) \cdot \vec{0} = \underline{\underline{\vec{0}}}$

b) a) $\Rightarrow$ derivert av hver komponent i $\vec{r}(t) \times \vec{v}(t)$ er 0 $\Rightarrow$ hver komponent i $\vec{r}(t) \times \vec{v}(t)$ er konstant $\Rightarrow \underline{\underline{\vec{r}(t) \times \vec{v}(t) = \vec{c}}}$ ($\vec{c}$ konstant).

c) Planet som inneh. $\vec{0}$, $\vec{r}(0)$ og $\vec{v}(0)$ er unikt gitt ved normalvektoren $\vec{r}(0) \times \vec{v}(0)$ og at planet går gjennom $\vec{0}$

Å vise at $\vec{r}(t)$ er i dette planet er ekvivalent med å vise at:

$$\vec{r}(t) \cdot (\vec{r}(0) \times \vec{v}(0)) = 0$$

Dette er OK siden:

$$\vec{r}(t) \cdot (\vec{r}(0) \times \vec{v}(0)) \underset{b)}{=} \vec{r}(t) \cdot (\vec{r}(t) \times \vec{v}(t))$$

$$= 0$$

$\vec{r}(t) \times \vec{v}(t)$ er normal på $\vec{r}(t)$ fra Set. 1.4.1 d)

3.2: Kjerneregel for parametriserte kurver

7.) $f(x, y, t) = 20 + 2t - x^2 + y^2$

$$\vec{s}(t) = \left(3t - \frac{t^2}{4}, 2t + \frac{t^2}{8}\right)$$

$$\vec{r}(t) = (\vec{s}(t), t)$$

$$\nabla f(x, y, t) = (-2x, 2y, 2)$$

$$\nabla f(\vec{r}(t)) = \left(-2\left(3t - \frac{t^2}{4}\right), 2\left(2t + \frac{t^2}{8}\right), 2\right)$$

$$= \left(-6t + \frac{t^2}{2}, 4t + \frac{t^2}{4}, 2\right)$$

$$\nabla f(\vec{r}(1)) = \left(-\frac{11}{2}, \frac{17}{4}, 2\right)$$

$$\vec{r}'(t) = \left(3 - \frac{t}{2}, 2 + \frac{t}{4}, 1\right)$$

$$\vec{r}'(1) = \left(\frac{5}{2}, \frac{9}{4}, 1\right)$$

Temp. pers. opplever v/ tid t: $g(t) := f(\vec{r}(t))$

$$g'(1) = \nabla f(\vec{r}(1)) \cdot \vec{r}'(1) = \left(-\frac{11}{2}, \frac{17}{4}, 2\right) \cdot \left(\frac{5}{2}, \frac{9}{4}, 1\right)$$

Kjerne-regelen for parametriserte kurver

$$= \dots = -\frac{35}{16} < 0$$

$$\Downarrow$$

Temperaturen er avtagende.

3.3: Linjeint.

11) $\vec{r}(t) = \left(\frac{t^2}{2}, \frac{2\sqrt{2}}{9} t^{\frac{3}{2}}, \frac{t}{9}\right)$, $t \in [1, 7]$

Bensin: $\frac{1}{15} + \frac{1}{2} \frac{dz}{ds}$ per km.

$$\vec{v}(t) = \vec{r}\,'(t) = \left(t, \frac{\sqrt{2}}{3} t^{\frac{1}{2}}, \frac{1}{9}\right)$$

$$v(t) = |\vec{v}(t)| = \underline{t + \frac{1}{9}}$$

HUSK: $\quad$ fart $= \dfrac{\Delta \text{ strekning}}{\Delta \text{ tid}}$

$$v(t) = \frac{ds}{dt} = t + \frac{1}{9}$$

I tillegg: $\quad \dfrac{dz}{dt} = \dfrac{1}{9}$

Så: $\quad \dfrac{dz}{dt} = \dfrac{dz}{ds} \dfrac{ds}{dt} \Rightarrow \dfrac{1}{9} = \dfrac{dz}{ds}\left(t + \dfrac{1}{9}\right)$

$$\Downarrow$$

$$\frac{dz}{ds} = \frac{1}{9t+1}$$

Tot. bensin $= \int_{\mathcal{C}} f\, ds = \int_1^7 \left(\frac{1}{15} + \frac{1}{2} \frac{1}{9t+1}\right)\left(t + \frac{1}{9}\right) dt$

$$= \ldots = \underline{\frac{89}{45}\, \ell}$$