Rekker

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots = \frac{\pi}{4}$$

$$S_0=1 \quad S_1=1-\frac{1}{3} \quad S_2=1-\frac{1}{3}+\frac{1}{5}, \ \ldots$$

Betyr: $\lim_{n\to\infty} S_n = \frac{\pi}{4}$

Vanlig navn: $\sum_{n=0}^{\infty} a_n$ $\quad S_m=\sum_{n=0}^{m} a_n$ delsummer $\quad S_m \to S = \sum_{n=0}^{\infty} a_n$

anta, konvergerer følgen $\{S_n\}$

$\Downarrow$

Rekka konvergerer

Hva mener vi med $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} b_n$?

Motsatt: Divergere

Geometriske rekker: $\sum_{n=0}^{\infty} a_0 r^n = a_0 + a_0 r + a_0 r^2 + \cdots$

Konvergerer $|r|<1$ $\quad = \frac{a_0}{1-r}$

Divergerer $|r|\geq 1$

En annen type rekke: Binomialformelen: $(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \cdots$
$\cdots + \binom{n}{n-1}xy^{n-1} + y^n$

Sett $x=1$, $y=\frac{a}{n}$ $\quad$ hvor $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$$\left(1+\frac{a}{n}\right)^n = 1 + \cancel{n}\cdot 1\cdot\frac{a}{\cancel{n}} + \frac{1}{2}n(n-1)\cdot 1\left(\frac{a}{n}\right)^2 + \frac{1}{2\cdot 3}n(n-1)(n-2)\cdot 1\left(\frac{a}{n}\right)^3 + \cdots$$

$$= 1 + a + \frac{1}{2}\frac{n-1}{n}a^2 + \frac{1}{6}\frac{(n-1)}{n}\frac{(n-2)}{n}a^3 + \cdots$$

$$\xrightarrow[n\to\infty]{} 1 + a + \frac{1}{2}a^2 + \frac{1}{6}a^3 + \cdots = e^a$$

$$\boxed{\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = \sum_{n=0}^{\infty}\frac{1}{n!}a^n = e^a}$$

Hvordan avgjøre konvergens/divergens?

Divergenstest: Hvis $\sum a_n$ konvergerer, så $a_n \to 0$. Dersom a_n ikke går mot 0, så $\sum a_n$ divergere.

Eks $\sum a_0 r^n$ $\quad |r|<1 \Rightarrow a_0 r^n \to 0$

$\sum \left(\frac{n}{n+1}\right)^n \quad \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(\frac{n+1}{n}\right)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \longrightarrow \frac{1}{e} \neq 0$

Mest berømt divergent rekke:

Harmoniske rekka $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$

$$\left.\begin{array}{ll} 1 & \geq \frac{1}{2} \\ +\frac{1}{2} & \geq \frac{1}{2} \\ +\frac{1}{3}+\frac{1}{4} & \geq \frac{1}{2} \\ +\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} & \geq \frac{1}{2} \\ \vdots & \end{array}\right\} \text{så stort man vil.}$$

Konvergens tester:

1) Generelle regler $\quad \sum a_n, \sum b_n$ konv. $\Rightarrow \sum a_n + b_n$ konv.

2) Sammenlikningsregler

skille mellom
1) alle ledd er positive
2) ikke.

Eks. positive ledd: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots = 1$

$$\frac{1}{2}+\frac{1}{8}+\frac{1}{16}+\frac{1}{4}+\frac{1}{64}+\frac{1}{128}+\frac{1}{32}+\cdots = 1$$

$$\frac{1}{2}+\frac{1}{8}+\frac{1}{16}+\frac{1}{4}+\frac{1}{64}+\frac{1}{128}+\frac{1}{256}+\frac{1}{32}+\cdots = 1$$

Rekkefølge er uvesentlig.

Eks Ikke bare positive ledd:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots = \ln 2$$

Se på

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}-\frac{1}{4}+\frac{1}{7}+\frac{1}{9}-\frac{1}{6}+\frac{1}{11}+\frac{1}{13}-\frac{1}{8}$$
$$+\frac{1}{15}+\frac{1}{17}-\frac{1}{10}+\frac{1}{19}+\frac{1}{21}-\frac{1}{12}+\cdots = 1$$

Delsummer: $1+\frac{1}{3}>1$, $1+\frac{1}{3}-\frac{1}{2}<1$, $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}>1$

$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}-\frac{1}{4}<1$, $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}-\frac{1}{4}+\frac{1}{7}+\frac{1}{9}>1$

Rekkefølgen er vesentlig

Integraltesten:

$f:[1,\infty\rangle \to \mathbb{R}$ positiv, kontinuerlig, avtagende

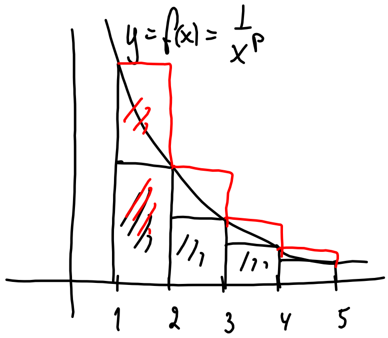
$\sum_{n=1}^{\infty} f(n)$ konverger $\iff$ $\int_1^{\infty} f(x)\,dx$ konverger

Eks. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $p>1$ Konverger

fordi $\int_1^{\infty} \frac{1}{x^p}\,dx = \left[\frac{1}{1-p} x^{1-p}\right]_1^{\infty} = -\frac{1}{1-p} 1^{1-p} = \underline{\underline{\frac{1}{p-1}}}$

Bevis for integraltest.

$$\sum_{n=2}^{\infty} f(n) \le \int_1^{\infty} f(x)\,dx \le \sum_{n=1}^{\infty} f(n)$$
$$\qquad\qquad\qquad\qquad\qquad\qquad \| $$
$$\qquad\qquad\qquad\qquad\qquad\qquad M$$



Lemma:
En positiv rekke konverger hviss den er begrenset.

Enklere tester:

$\sum a_n, \sum b_n$ positive

1) $b_n \le c \cdot a_n$ $\quad \sum a_n$ konv. $\Rightarrow$ $\sum b_n$ konv.

2) $b_n \ge d \cdot a_n$ $\quad \sum a_n$ div $\Rightarrow$ $\sum b_n$ div.

Eks. $\sum \frac{n^2+1}{2n^3-1}$

Vi har $\frac{n^2+1}{2n^3-1} > \frac{1}{2} \cdot \frac{1}{n}$ fordi $2n^3+2n > 2n^3-1$

$\sum \frac{3n^2+4n}{8n^4-2}$

$$\frac{3n^2+4n}{8n^4-2} < c\,\frac{1}{n^2}$$

Tester:
$(3n^2+4n)\,8n^2 < 3(8n^4-2)$
$\cancel{24n^4} + 32n^3 < \cancel{24n^4} - 6$

$\cancel{c < \frac{3}{8}}$ $\quad c > \frac{3}{8}$

f.eks $c = \cancel{\frac{2}{8} = \frac{1}{4}}$ $\frac{4}{8}$

$(3n^2+4n)\,8n^2 < 4(8n^4-2)$

$24n^4 + 32n^3 < 32n^4 - 8$ $\quad$ $\underline{\text{ok}}$