2.5.2

b) $f(x,y,z) = x^2 y^3 \cos(xyz)$ $\qquad \cos \to -\sin \to -\cos$

Setning 2.5.1: Rekkefølgen under derivasjon spiller ingen rolle.
Deriver to ganger med tanke på z først:

$$\frac{\partial^2 f}{\partial z^2} = x^2 y^3 (-\cos(xyz))(xy)^2 = -x^4 y^5 \cos(xyz)$$

$$\frac{\partial^3 f}{\partial z \partial z \partial x} = -4x^3 y^5 \cos xyz + x^4 y^5 \sin(xyz) yz$$
$$= \underline{-4x^3 y^5 \cos xyz + x^4 y^6 z \sin(xyz)}$$

$$\frac{\partial^4 f}{\partial z \partial z \partial x \partial y} = -20x^3 y^4 \cos(xyz) + 4x^4 y^5 z \sin(xyz)$$
$$+ 6x^4 y^5 z \sin(xyz) + x^5 y^6 z^2 \cos(xyz)$$
$$= \underline{\underline{x^3 y^4 (-20 + x^2 y^2 z^2) \cos(xyz) + 10x^4 y^5 z \sin(xyz)}}$$

antall ledd under derivasjon: $1 \xrightarrow{\frac{\partial}{\partial z}} 1 \xrightarrow{\frac{\partial}{\partial z}} 1 \xrightarrow{\frac{\partial}{\partial x}} 2 \xrightarrow{\frac{\partial}{\partial y}} 4$

rekkefølgen i oppgaven: $1 \xrightarrow{\frac{\partial}{\partial y}} 2 \xrightarrow{\frac{\partial}{\partial z}} 2 \xrightarrow{\frac{\partial}{\partial x}} 4 \xrightarrow{\frac{\partial}{\partial z}} 4$

Det var altså lurt å bytte rekkefølge under derivasjon

2.7.1 $\quad f(u,v) = u^2 + v \qquad g(x,y) = 2xy \qquad h(x,y) = x + y^2$

$$\frac{\partial f}{\partial u} = 2u \qquad \frac{\partial f}{\partial v} = 1 \qquad \frac{\partial g}{\partial x} = 2y \qquad \frac{\partial g}{\partial y} = 2x \qquad \Bigg| \quad u = g(x), \quad v = h(y)$$

$$\frac{\partial h}{\partial x} = 1 \qquad \frac{\partial h}{\partial y} = 2y$$

Sett $\vec{G}(x,y) = (g(x,y), h(x,y))$. Vi skal se på $k(x,y) = f(\vec{G}(x,y))$.

$$\vec{G}'(x,y) = \begin{pmatrix} 2y & 2x \\ 1 & 2y \end{pmatrix} \qquad f'(u,v) = \begin{pmatrix} 2u & 1 \end{pmatrix}$$

$$k'(x,y) = \begin{pmatrix} \frac{\partial k}{\partial x} & \frac{\partial k}{\partial y} \end{pmatrix} = f'(\underbrace{\vec{G}(x,y)}_{u,v})\, \vec{G}'(x,y) \quad (\text{kjerneregel})$$

$$= \begin{pmatrix} 2u & 1 \end{pmatrix} \begin{pmatrix} 2y & 2x \\ 1 & 2y \end{pmatrix} = \begin{pmatrix} 2g(x,y) & 1 \end{pmatrix} \begin{pmatrix} 2y & 2x \\ 1 & 2y \end{pmatrix}$$

$$= \begin{pmatrix} 4xy & 1 \end{pmatrix} \begin{pmatrix} 2y & 2x \\ 1 & 2y \end{pmatrix} = \begin{pmatrix} 8xy^2 + 1 & 8x^2y + 2y \end{pmatrix}$$

Derfor:

$$\frac{\partial k}{\partial x} = 8xy^2 + 1$$

$$\frac{\partial k}{\partial y} = 8x^2y + 2y$$

2.7.2 $\quad f(u,v) = ue^{-v} \qquad u = g(x,y,z) = 2xy + z$

$$v = h(x,y,z) = 2y(z+x)$$

sett $\vec{G}(x,y,z) = (g(x,y,z), h(x,y,z)) = (2xy+z, 2y(z+x))$

$$f'(u,v) = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{pmatrix} = \begin{pmatrix} e^{-v} & -ue^{-v} \end{pmatrix}$$

$$\vec{G}'(x,y,z) = \begin{pmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{pmatrix} = \begin{pmatrix} 2y & 2x & 1 \\ 2y & 2(z+x) & 2y \end{pmatrix}$$

Sett $k(x,y,z) = f(\vec{G}(x,y,z))$

$$k'(x,y,z) = f'(\vec{G}(x,y,z))\,\vec{G}'(x,y,z)$$

$$= \begin{pmatrix} e^{-v} & -ue^{-v} \end{pmatrix} \begin{pmatrix} 2y & 2x & 1 \\ 2y & 2(z+x) & 2y \end{pmatrix}$$

$$= \begin{pmatrix} 2ye^{-v}(1-u) & 2e^{-v}(x - u(z+x)) & e^{-v}(1-2yu) \end{pmatrix}$$

evt. sett $e^{-2y(z+x)}$ utenfor.

$$= \Big(2ye^{-2y(z+x)}(1-2xy-z), \quad = \frac{\partial k}{\partial x}$$

$$2e^{-2y(z+x)}(x-(2xy+z)(z+x)), \quad = \frac{\partial k}{\partial y}$$

$$\underline{\underline{e^{-2y(z+x)}(1-2y(2xy+z))\Big)}} \quad = \frac{\partial k}{\partial z}$$

2.7.3

2.7.3

$f(u_1, u_2) = 2u_1 u_2^2$

$g_1(x_1, x_2, x_3) = x_1 x_2 \operatorname{sen} x_3$

$g_2(x_1, x_2, x_3) = 3x_1^2 x_2 x_3$

$$\frac{\partial h}{\partial x_3} = \frac{\partial f}{\partial u_1}\frac{\partial g_1}{\partial x_3} + \frac{\partial f}{\partial u_2}\frac{\partial g_2}{\partial x_3}$$

$$= 2u_2^2 \, x_1 x_2 \cos x_3 + 4u_1 u_2 \, 3x_1^2 x_2$$

$$= 2(3x_1^2 x_2 x_3)^2 x_1 x_2 \cos x_3 + 4 x_1 x_2 \sin x_3 \, 3x_1^2 x_2 x_3 \, 3x_1^2 x_2$$

$$= \underline{\underline{18 x_1^5 x_2^3 x_3^2 \cos x_3 + 36 x_1^5 x_2^3 x_3 \operatorname{sen} x_3}}$$

2.7.8 a) $\vec{G}(r,\theta) = (r\cos\theta, r\sin\theta)$

$T(r,\theta) = f(r\cos\theta, r\sin\theta) = f(\vec{G}(r,\theta))$.

$$f'(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix}$$

$$\vec{G}'(r,\theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$T'(r,\theta) = f'(\vec{G}(r,\theta))\,\vec{G}'(r,\theta)$$

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$= \Big(\underbrace{\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta}_{\frac{\partial T}{\partial r}},\ \underbrace{-\frac{\partial f}{\partial x} r\sin\theta + \frac{\partial f}{\partial y} r\cos\theta}_{\frac{\partial T}{\partial \theta}} \Big)$$

b) $r = g(t) \qquad \theta = h(t)$

Definer $T_1(t) = T(r(t), \theta(t)) = T(\underbrace{g(t), h(t)}_{\vec{H}(t)}) = T(\vec{H}(t))$

$$T_1'(t) = T'(r, \theta)\vec{H}'(t)$$

$$= \begin{pmatrix} \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta & -\frac{\partial f}{\partial x} r\sin\theta + \frac{\partial f}{\partial y} r\cos\theta \end{pmatrix} \begin{pmatrix} g'(t) \\ h'(t) \end{pmatrix}$$

$$\underline{\underline{= \left(\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right) g'(t) + \left(-\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y} r\cos\theta\right) h'(t)}}$$

2.7.9 $\quad \underbrace{f(x_1, x_2, \dots, x_n, \underbrace{g(x_1, \dots, x_n)})}_{\vec{G}(x_1, \dots, x_n)} = 0 \qquad f : \mathbb{R}^{n+1} \to \mathbb{R}$

$\vec{G}(x_1, \dots, x_n) \; : \mathbb{R}^n \to \mathbb{R}^{n+1} \quad (f(\vec{G}(x_1, \dots, x_n)) = 0)$

$\vec{G}'(x_1, \dots, x_n)$ er en $(n+1) \times n$ - matrise

$$\vec{G}'(x_1, \dots, x_n) = \begin{pmatrix} 1 & 0 & 0 & \cdots & & \\ 0 & 1 & 0 & \cdots & & \\ \cdots & 0 & 1 & 0 & \cdots & \\ & & \vdots & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \cdots & & \cdots & \frac{\partial g}{\partial x_n} \end{pmatrix}$$

→ partielle der. av x_1
—"— av x_2
⋮
—"— av x_n

Siden $h(x_1, \dots, x_n) = f(\vec{G}(x_1, \dots, x_n)) = 0$ så er $h'(x_1, \dots, x_n) = 0$

$\Rightarrow \quad f'(\underbrace{u_1, \dots, u_{n+1}}_{\vec{G}(x_1, \dots, x_n)}) \vec{G}'(x_1, \dots, x_n) = 0$

$\Rightarrow$

$$\begin{pmatrix} \frac{\partial f}{\partial u_1} & \frac{\partial f}{\partial u_2} & \cdots & \frac{\partial f}{\partial u_{n+1}} \end{pmatrix} \begin{pmatrix} & I_n & \\ & & \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \cdots & \frac{\partial g}{\partial x_n} \end{pmatrix}$$

$$= \left(\frac{\partial f}{\partial u_1} + \frac{\partial f}{\partial u_{n+1}} \frac{\partial g}{\partial x_1}, \cdots, \frac{\partial f}{\partial u_n} + \frac{\partial f}{\partial u_{n+1}} \frac{\partial g}{\partial x_n} \right) = 0$$

$\Rightarrow$ for alle i har vi at $\quad \frac{\partial f}{\partial u_i} + \frac{\partial f}{\partial u_{n+1}} \frac{\partial g}{\partial x_i} = 0$

$$\Rightarrow \frac{\partial g}{\partial x_i} = - \frac{\frac{\partial f}{\partial u_i}}{\frac{\partial f}{\partial u_{n+1}}}$$

$$\Rightarrow \frac{\partial g}{\partial x_i}(x_1, \dots, x_n) = - \frac{\frac{\partial f}{\partial x_i}(x_1, \dots, x_n, g(x_1, \dots, x_n))}{\frac{\partial f}{\partial x_{n+1}}(x_1, \dots, x_n, g(x_1, \dots, x_n))}$$

b) $f(x,y) = x^2 + y^2 - R^2$. Finn g s.a. $f(x, g(x)) = 0$

Dette passer inn i a) med $n = 1$

$$\frac{\partial g}{\partial x} = g'(x) = -\frac{\frac{\partial f}{\partial x}(x, g(x))}{\frac{\partial f}{\partial y}(x, g(x))} = -\frac{2x}{2y} = -\underline{\underline{\frac{x}{g(x)}}}$$

$$\Rightarrow g'(x) = -\frac{x}{g(x)} \Leftrightarrow g(x)\,g'(x) = -x \Leftrightarrow g(x)\,g'(x) + x \cdot 1 = 0$$

$$\underbrace{(x, g(x))}_{\text{punkt på sirkel}} \cdot \underbrace{(1, g'(x))}_{\text{tangent}} = 0$$

Vi har altså vist at $\vec{r}(t)$ står normalt på tangenten når $\vec{r}$ beskriver en sirkel.

c) $f(x,y,z) = x^2+y^2+z^2-R^2 = 0 \qquad z = g(x,y)$

Passer inn i a) med $n=2$ $\left(f(x,y,g(x,y)) = 0\right.$

$$\frac{\partial g}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} = -\frac{2x}{2z} = -\underline{\underline{\frac{x}{g(x,y)}}}$$

$$\frac{\partial g}{\partial y} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}} = -\frac{2y}{2z} = -\underline{\underline{\frac{y}{g(x,y)}}}$$

$$\frac{\partial g}{\partial x} = -\frac{x}{g(x,y)} \iff (x,y,g(x,y)) \cdot \left(1,0,\frac{\partial g}{\partial x}(x,y)\right) = 0$$

$$\frac{\partial g}{\partial y} = -\frac{y}{g(x,y)} \iff (x,y,g(x,y)) \cdot \left(0,1,\frac{\partial g}{\partial y}(x,y)\right) = 0$$

$\Rightarrow$ $\vec{r}(t)$ står normalt på tangentene vi får når vi beveger oss med x konstant, eller y konstant.