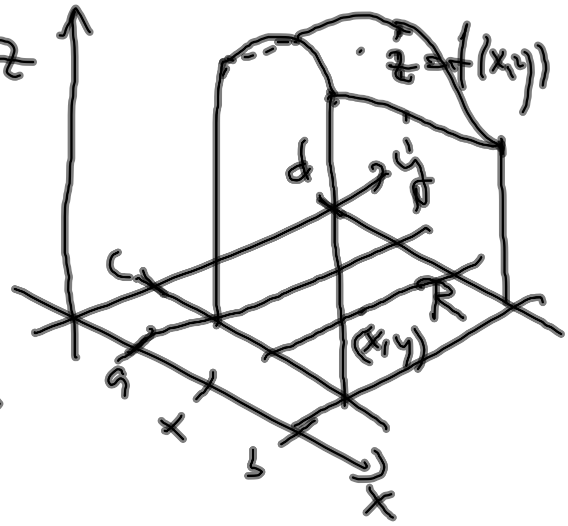
Teorem 6.1.7

$R = [a,b] \times [c,d] \subset \mathbb{R}^2$

$f : R \longrightarrow \mathbb{R}$



Hvis f er integrerbar på R

og

$$y \longmapsto f(x,y)$$

er integrerbar på $[c,d]$ for hver $x \in [a,b]$

så er

$$x \longmapsto F(x) = \int_c^d f(x,y)\,dy$$

en integrerbar funksjon på $[a,b]$, og

$$\iint_R f(x,y)\,dx\,dy = \int_a^b F(x)\,dx$$

$$= \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx \,.$$

Korollar 6.1.8

Hvis $f: R \to \mathbb{R}$ er kontinuerlig så er

$$\iint_R f \, dxdy = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

$$= \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

$A \subset \mathbb{R}^2$ begrenset område ; finnes rektangel

$A \subseteq R = [a,b] \times [c,d]$

$f: A \longrightarrow \mathbb{R}$

utvider med 0 dvs $f_A : R \longrightarrow \mathbb{R}$

$$f_A(x,y) = \begin{cases} f(x,y) & \text{hvis } (x,y) \in A \\ 0 & \text{ellers} \end{cases}$$

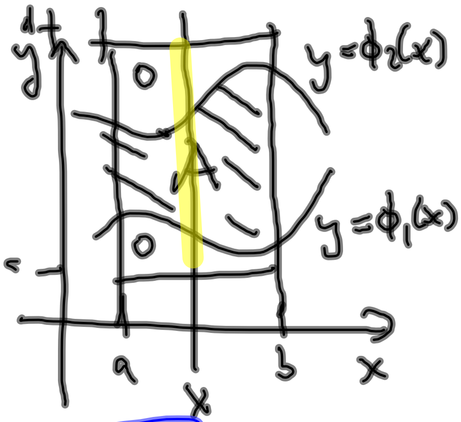
Def

$$\iint_A f(x,y)\,dxdy = \iint_R f_A(x,y)\,dxdy$$

6.2 $A \subset \mathbb{R}^2$ type I:

$\phi_1, \phi_2 : [a,b] \longrightarrow \mathbb{R}$
kontinuerlige

$\phi_1(x) \leq \phi_2(x)$ for alle $x \in [a,b]$



$f : A \longrightarrow \mathbb{R}$

$\cap$

$R = [a,b] \times [c,d]$

$$\boxed{\iint_A f(x,y)\,dx\,dy}$$

$$\iint_R f_A(x,y)\,dx\,dy$$

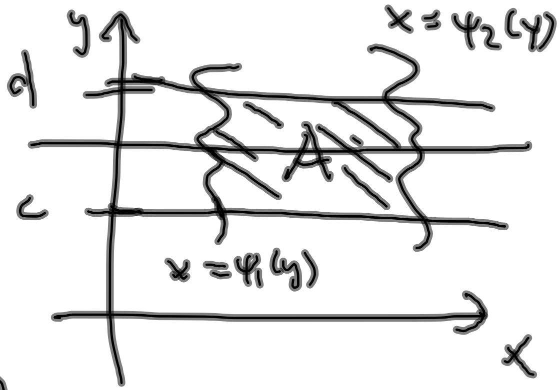
$$= \int_a^b \left(\int_c^d \underline{\underline{f_A(x,y)}}\,dy \right) dx$$

$$\boxed{= \int_a^b \left(\int_{y=\phi_1(x)}^{y=\phi_2(x)} f(x,y)\,dy \right) dx}$$

Type II område

$\psi_1, \psi_2 : [c,d] \longrightarrow \mathbb{R}$
kontinuerlige

$\psi_1(y) \leq \psi_2(y)$ for alle
$y \in [c,d]$



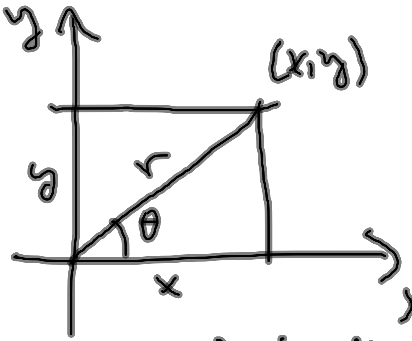
ψ psi

$f : A \to \mathbb{R}$ kontinuerlig

$$\iint_A f(x,y)\,dx\,dy = \int_c^d \left(\int_{x=\psi_1(y)}^{x=\psi_2(y)} f(x,y)\,dx \right) dy$$

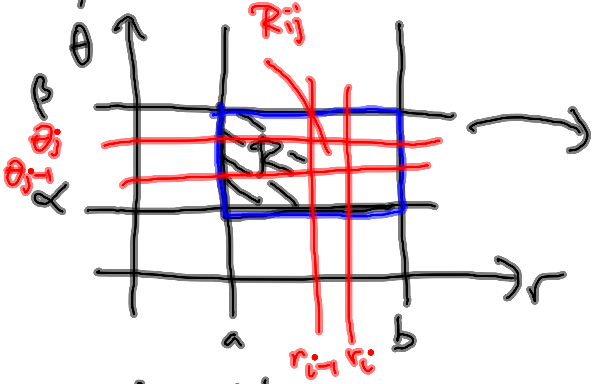
LH 6.3 Polarkoordinater & $\iint$

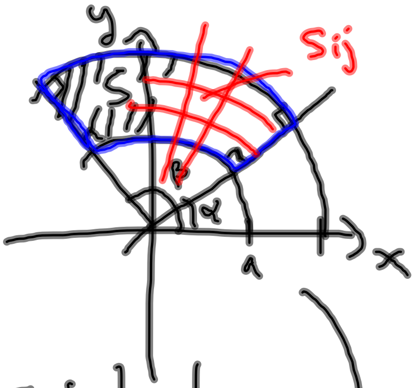
$$(r,\theta) \longmapsto (x,y) = (r\cos\theta, r\sin\theta)$$



kartesiske koordinater

polarkoordinater





$0 \le a \le b$

$\alpha \le \beta \le \alpha + 2\pi$

S = polart rektangel

f kontinuerlig

$\mathbb{R}$

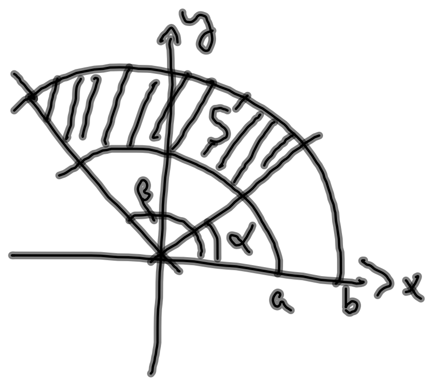
Setning 6.3.1

$$\iint_S f(x,y)\,dx\,dy = \iint_R f(r\cos\theta, r\sin\theta)\, r \quad dr\,d\theta$$

"$dx\,dy = r\,dr\,d\theta$"

Lemma

$$\text{areal}(S) = \frac{a+b}{2}(b-a)(\beta-\alpha)$$
$$= |S|$$



$$\left|\begin{matrix}\text{Sirkelskiven med} \\ \text{radius } b\end{matrix}\right| = \pi b^2$$

$$\left|\begin{matrix}\text{ringen mellom} \\ \text{radius } a \text{ og radius } b\end{matrix}\right| = \pi b^2 - \pi a^2$$

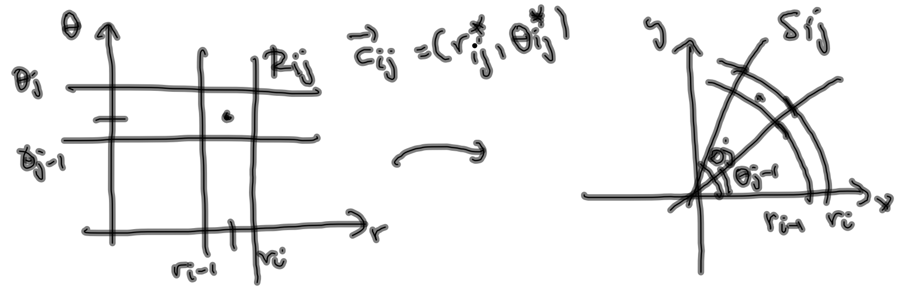
$$|S| = \frac{\beta-\alpha}{2\pi}(\pi b^2 - \pi a^2) = \frac{a+b}{2}(b-a)(\beta-\alpha)$$

$\boxtimes$

Beregner volumet under grafen til $f: S \to \mathbb{R}$ ved å dele S i små polare rektangler S_{ij}:

$$a = r_0 < r_1 < \dots < r_{n-1} < r_n = b$$

$$\alpha = \theta_0 < \theta_1 < \dots < \theta_{m-1} < \theta_m = \beta$$



$$r_{ij}^* = \frac{r_{i-1} + r_i}{2} \qquad x_{ij}^* = r_{ij}^* \cos \theta_{ij}^*$$

$$\theta_{ij}^* = \frac{\theta_{j-1} + \theta_j}{2} \qquad y_{ij}^* = r_{ij}^* \sin \theta_{ij}^*$$

Volumet

$$\approx \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_{ij}^*, y_{ij}^*) \, |S_{ij}| \quad \text{i } xy\text{-planet}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} f(r_{ij}^* \cos \theta_{ij}^*, r_{ij}^* \sin \theta_{ij}^*) \, r_{ij}^* \underbrace{(r_i - r_{i-1})(\theta_j - \theta_{j-1})}_{|R_{ij}| \text{ i } r\theta\text{-planet}}$$

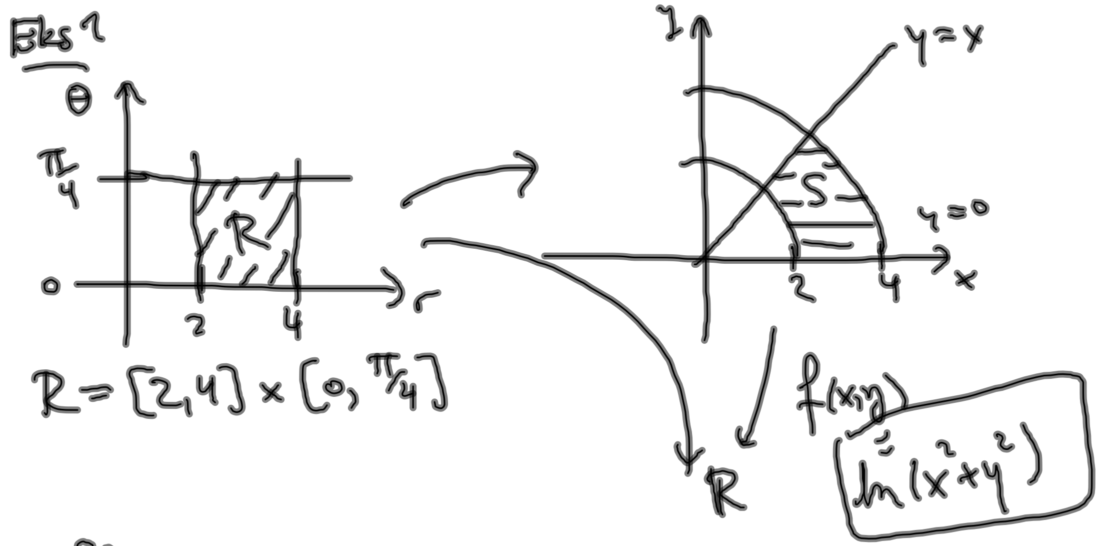
Riemann-sum for $(r, \theta) \mapsto f(r\cos\theta, r\sin\theta)\, r$ på $R = [a,b] \times [\alpha, \beta]$

når $r_i - r_{i-1} \to 0$ og $\theta_j - \theta_{j-1} \to 0$

$$\boxed{\iint_R f(r\cos\theta, r\sin\theta)\, r \, dr\, d\theta.}$$

$$\boxed{= \iint_S f(x,y)\, dx\, dy}$$

Eks 1



$R = [2,4] \times [0, \frac{\pi}{4}]$

$f(x,y) = \boxed{\ln(x^2+y^2)}$

$$\iint_S f(x,y)\,dx\,dy = \iint_S \ln(x^2+y^2)\,dx\,dy$$

$[\ln(e) = 1 \quad e = 2{,}718281828459045]$

$$= \iint_R \ln(r^2)\, r\, dr\, d\theta$$

$$= \int_2^4 \left(\int_0^{\pi/4} 2r \ln(r)\, d\theta \right) dr$$

$$= \int_2^4 \Big[2r \ln r\, \theta \Big]_0^{\pi/4} dr$$

$$= \int_2^4 \frac{\pi}{4} \cdot 2r \ln r\, dr$$

$\int 2r \ln r\, dr$

$\shortparallel$

$r^2 \ln r - \frac{1}{2} r^2 + C$

$u' = 2r$
$v = \ln r$
$u = r^2$
$v' = \frac{1}{r}$

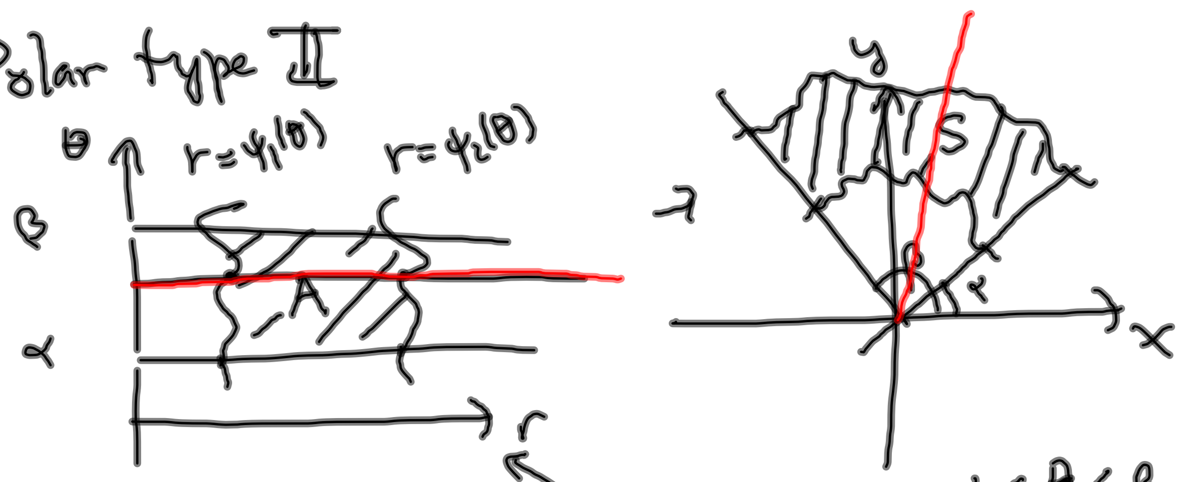
$$= \frac{\pi}{4} \left[r^2 \ln r - \frac{1}{2} r^2 \right]_2^4$$

$\ln 4 = 2 \ln 2$

$$= \frac{\pi}{4} (16 \ln 4 - 8 - 4 \ln 2 + 2)$$

$$\boxed{= 7\pi \ln 2 - \frac{3\pi}{2}}$$

Polar type II



$\alpha \le \beta \le \alpha + 2\pi$

$\psi_1, \psi_2 : [\alpha, \beta] \longrightarrow \mathbb{R}$

kontinuerlige

$0 \le \psi_1(\theta) \le \psi_2(\theta)$ for alle $\theta \in [\alpha, \beta]$

$A = \{(r, \theta) : \alpha \le \theta \le \beta, \ \psi_1(\theta) \le r \le \psi_2(\theta)\}$

$S = \{(r\cos\theta, r\sin\theta) : (r, \theta) \in A\}$

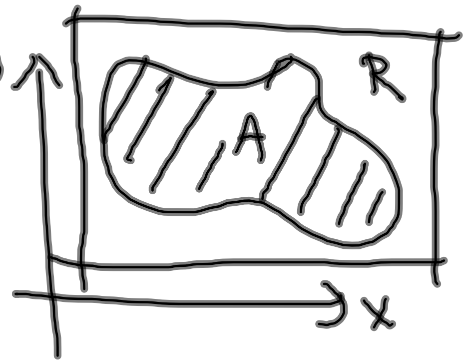
$f : S \longrightarrow \mathbb{R}$ kontinuerlig

6.3.2

$$\iint_S f(x,y)\,dx\,dy = \iint_A f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_\alpha^\beta \left(\int_{r=\psi_1(\theta)}^{r=\psi_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr \right) d\theta$$

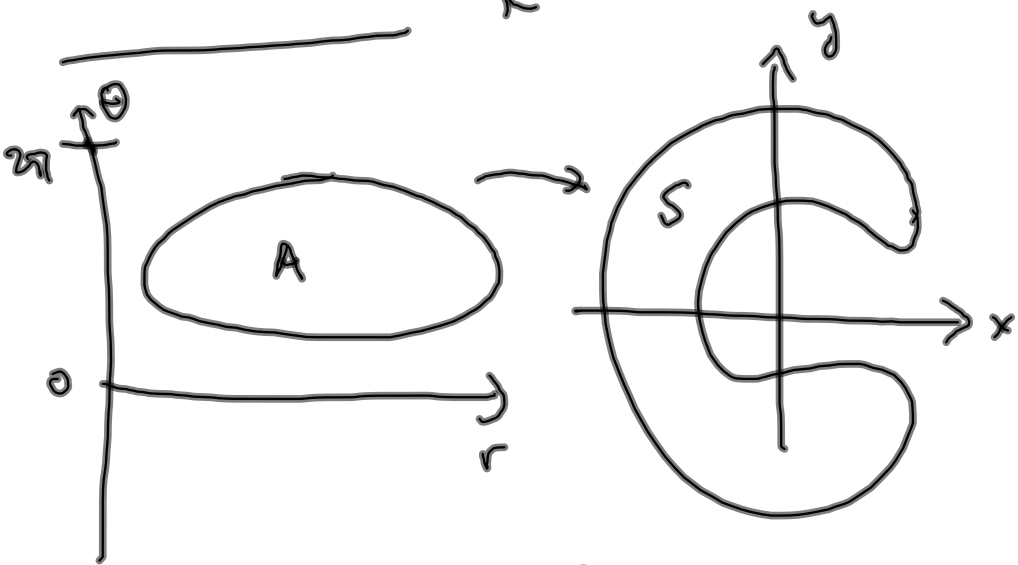
6.4. Anvendelser



areal(A)

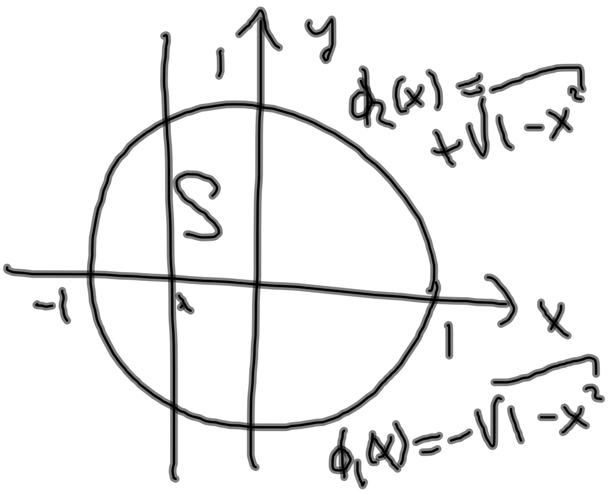
$$= \iint_A 1 \, dx\,dy = \iint_R 1_A \, dx\,dy$$

$1_A = 1$ på A
$\ 0$ ellers



$$\text{areal}(S) = \iint_S 1 \, dx\,dy = \iint_A r \, dr\,d\theta$$

Eks enhetssirkelen



$$\text{areal}(S) = \iint_S 1 \, dx\,dy = \int_{-1}^{1} \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1 \, dy \right) dx = \int_{-1}^{1} 2\sqrt{1-x^2} \, dx$$

$x = \sin t$
$t = \arcsin x$
$dx = \cos t \, dt$

$$= \int_{-\pi/2}^{\pi/2} 2\sqrt{1-\sin^2 t} \, \cos t \, dt$$

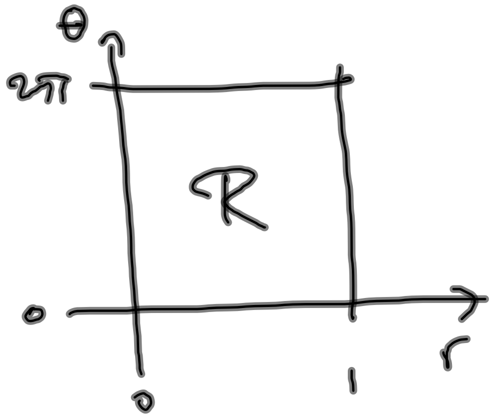
$\cos 2t = 2\cos^2 t - 1$

$$= \int_{-\pi/2}^{\pi/2} 2\cos^2 t \, dt$$

$$= \int_{-\pi/2}^{\pi/2} 1 + \cos 2t \, dt = \left[t + \frac{1}{2}\sin 2t \right]_{-\pi/2}^{\pi/2}$$

$$= \frac{\pi}{2} + 0 - \left(-\frac{\pi}{2}\right) - 0 = \underline{\underline{\pi}}$$

Alt. m. polarkoordinater:



areal(S) $= \iint_S 1\,dx\,dy = \iint_R r\,dr\,d\theta$

$0 \leq r \leq 1$
$0 \leq \theta \leq 2\pi$

$$= \int_0^1 \left(\int_0^{2\pi} r\,d\theta \right) dr$$

$$= \int_0^1 \left[r\theta \right]_0^{2\pi} dr = \int_0^1 2\pi r\,dr = \left[\pi r^2 \right]_0^1 = \underline{\underline{\pi}}$$