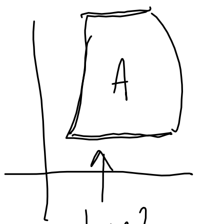
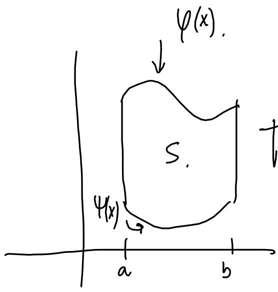
Dobbeltintegraler.



type 2

$$\iint_A f(x,y)\,dx\,dy$$

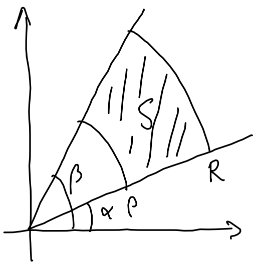


Type 1.

$$S = \{(x,y) \mid a \leq x \leq b \quad \psi(x) \leq y \leq \varphi(x)\}.$$

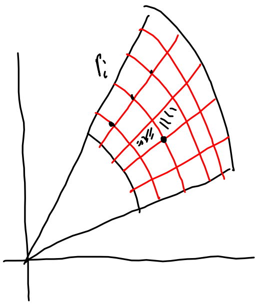
$$\iint_S f(x,y)\,dx\,dy = \int_a^b \left[\int_{\psi(x)}^{\varphi(x)} f(x,y)\,dy\right] dx$$

Polarkoordinater i dobbeltintegral.



$$S = \{(r,\theta) \mid \rho \le r \le R,\ \alpha \le \theta \le \beta\}.$$

$$\iint_S f(x,y)\,dx\,dy\ ?$$



Lager "rutenett".

$$r_i = \rho + i\Delta r,\quad \Delta r = \frac{R-\rho}{N} \qquad i = 0,\dots,N-1.$$

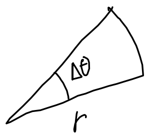
$$\theta_j = \alpha + j\Delta\theta,\quad \Delta\theta = \frac{\beta-\alpha}{M} \qquad j = 0,\dots,M-1.$$

$$x = r\cos\theta \quad y = r\sin\theta.$$

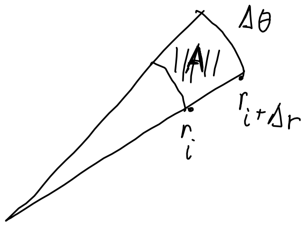
$$\tilde{f}(r,\theta) = f(r\cos\theta, r\sin\theta)$$

$$\iint_S f(x,y)\,dx\,dy \approx \sum_{i=0}^{N-1}\sum_{j=0}^{M-1} \tilde{f}(r_i,\theta_j)\cdot(\text{"areal av ◇"}).$$

"Areal av ◇"



$\leftarrow$ areal $= \frac{1}{2}\Delta\theta\, r^2$



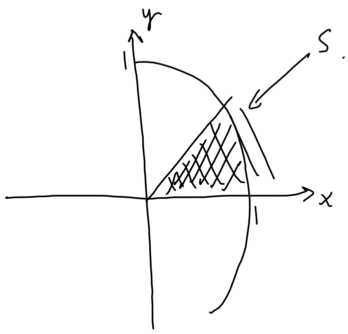
$$A = \frac{1}{2}\Delta\theta(r_i+\Delta r)^2 - \frac{1}{2}\Delta\theta\, r_i^2 = r_i\Delta r\Delta\theta + \frac{1}{2}\Delta\theta(\Delta r)^2$$

$$\iint_S f(x,y)\,dx\,dy \approx \sum_{i=0}^{N-1}\sum_{j=0}^{M-1} \tilde{f}(r_i,\theta_j)\left(r_i\Delta r\cdot\Delta\theta + \frac{1}{2}\Delta\theta\underbrace{(\Delta r)^2}_{\approx 0}\right)$$

$$\approx \int_\rho^R\int_\alpha^\beta \tilde{f}(r,\theta)\, r\, dr\, d\theta.$$

$$\boxed{\iint_S f(x,y)\,dx\,dy := \int_\rho^R\int_\alpha^\beta f(r\cos\theta, r\sin\theta)\, r\, d\theta\, dr}$$

<u>Eksempel.</u>



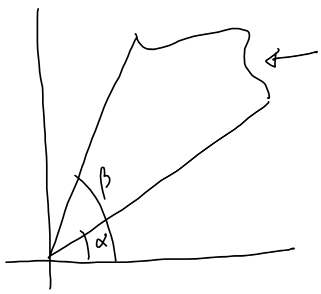
$$S = \{(x,y) \quad 0 \le y \le x \ , \ x^2+y^2 \le 1\}.$$

$$= \{(r,\theta) \ ; \ 0 \le \theta \le \pi/4 \ , \ 0 \le r \le 1\}.$$

$$f(x,y) = \ln(x^2+y^2) = \ln(r^2) = 2\ln(r).$$

$$\iint_S \ln(x^2+y^2)\,dx\,dy = \int_0^{\pi/4}\int_0^1 2\ln(r)\cdot r\,dr\,d\theta = 2\frac{\pi}{4}\int_0^1 \ln(r)\,r\,dr$$

$$= \frac{\pi}{2}\left(\frac{1}{2}r^2\ln(r)\Big|_0^1 - \frac{1}{2}\int_0^1 r^2\frac{1}{r}\,dr\right) = \frac{\pi}{2}\left(0 - \frac{1}{4}r^2\Big|_0^1\right) = -\frac{\pi}{8}$$

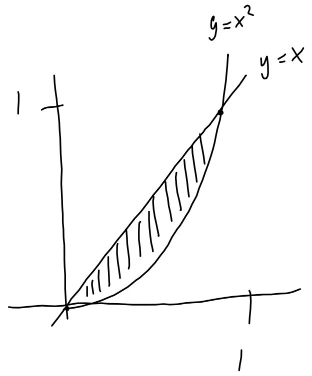


$$S = \{(r,\theta) ; \alpha \le \theta \le \beta , 0 \le r \le g(\theta)\}.$$

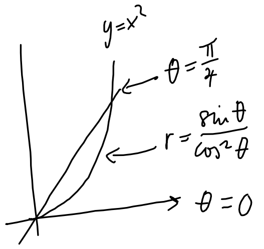
$$\iint_S f(x,y)\,dx\,dy = \int_\alpha^\beta \int_0^{g(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Eksempel

$$S = \{(x,y) \mid x^2 \le y \le x\}$$



S i polarkoordinater?



$$\left.\begin{array}{l} y = x^2 \\ r\sin\theta = r^2\cos^2\theta \end{array}\right\} \rightarrow r = \frac{\sin\theta}{\cos^2\theta}$$

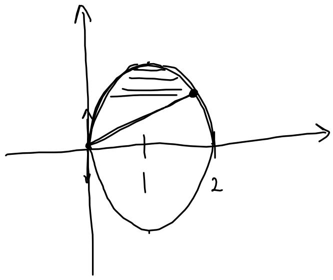
$$f(x,y) = \frac{y}{\sqrt{x^2+y^2}} = \frac{r\sin\theta}{r} = \sin\theta.$$

$$\iint_S f(x,y)\,dx\,dy = \int_0^{\pi/4}\int_0^{\frac{\sin\theta}{\cos^2\theta}} \sin\theta \; r\; dr\, d\theta = \int_0^{\pi/4} \sin\theta \int_0^{\frac{\sin\theta}{\cos^2\theta}} r\,dr\; d\theta = \frac{1}{2}\int_0^{\pi/4} \sin\theta \,\frac{\sin^2\theta}{\cos^4\theta}\, d\theta$$

$$\frac{1}{2}\int_0^{\pi/4} \sin\theta\, \frac{1-\cos^2\theta}{\cos^4\theta}\, d\theta \qquad = \frac{-1}{2}\int_1^{1/\sqrt{2}} \frac{1-u^2}{u^4}\, du = \frac{1}{2}\int_{1/\sqrt{2}}^{1} \frac{1}{u^4} - \frac{1}{u^2}\, du = 0$$

$u = \cos\theta$
$du = -\sin\theta\, d\theta$
$\theta = 0 \quad u = 1$
$\theta = \pi/4 \quad u = \frac{1}{\sqrt{2}}$

$S = \{(x,y) \mid (x-1)^2 + y^2 \leq 1\}.$



$$\iint_S \sqrt{x^2+y^2}\, dx\,dy$$

$y^2 = 1 - (x-1)^2$

$= \cancel{1} - x^2 + 2x - \cancel{1}$

$S = \{(r,\theta) \mid 0 \leq r \leq 2\cos\theta \quad -\pi/2 \leq \theta \leq \frac{\pi}{2}\}.$

$\underbrace{x^2+y^2}_{} = 2x$

$r^2 = 2r\cos\theta$

$r = 2\cos\theta.$

$$\iint_S \sqrt{x^2+y^2}\, dx\,dy = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r\,r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{3} 8\cos^3\theta\, d\theta$$

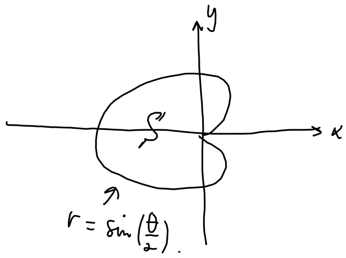
$$= \frac{8}{3}\int_{-\pi/2}^{\pi/2} \cos\theta(1-\sin^2\theta)\,d\theta \qquad = \frac{8}{3}\int_{-1}^{1} 1-u^2\,du =$$

$u = \sin\theta \quad \theta = \pm\pi/2 \quad u = \pm 1$

$du = \cos\theta\, d\theta$

Eksempel.

En kurve gitt i polarkoord. ved $r(t) = \sin\left(\frac{t}{2}\right)$ $t \in [0, 2\pi]$.



$$\text{Areal av } S = \iint_S 1 \, dx\,dy$$

$$= \int_0^{2\pi} \int_0^{\sin(\theta/2)} r \, dr \, d\theta$$

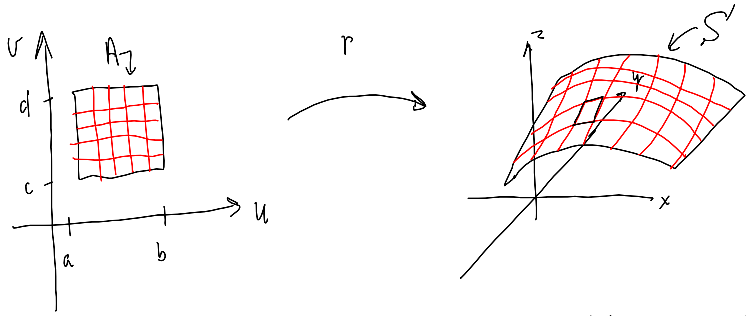
$$= \int_0^{2\pi} \frac{1}{2} \sin^2\left(\frac{\theta}{2}\right) d\theta$$

$$= \frac{1}{4} \int_0^{2\pi} 1 - \cos(\theta) \, d\theta = \frac{1}{4} \int_0^{2\pi} 1 \, d\theta = \frac{\pi}{2}$$

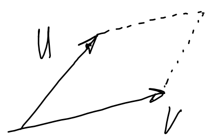
$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$$
$$= 1 - 2\sin^2(\alpha).$$
$$\cos(\theta) = 1 - 2\sin^2(\theta/2)$$
$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}\left(1 - \cos(\theta)\right).$$

Arealet av flater.

Parametrisert flate $r(u,v) = X(u,v)\,i + Y(u,v)\,j + Z(u,v)\,k$
$= \big(X(u,v), Y(u,v), Z(u,v)\big)$



Husk:



Hva er arealet av parallellogrammet utspent av U og V?

$|U \times V|$

$$S = \{(X(u,v), Y(u,v), Z(u,v)) \mid a \le u \le b \quad c \le v \le d\}.$$

Hva er arealet av S?

$\Rightarrow A \approx \sum_{i=0}^{N-1}\sum_{j=0}^{M-1}$ "areal av liten bit"

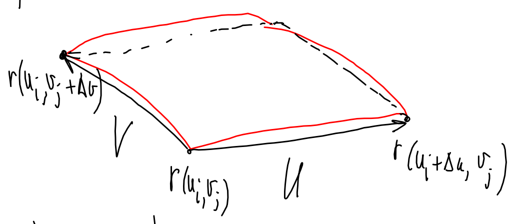
$u_i = a + i\Delta u \quad \Delta u = \frac{b-a}{N} \quad i = 0,\dots,N-1$

$v_j = c + j\Delta v \quad \Delta v = \frac{d-c}{M} \quad j = 0,\dots,M-1.$

"areal av liten bit" $\approx$ "areal utspent av U og V"

$= |U \times V|$

$= \Big|\big(r(u_i+\Delta u, v_j) - r(u_i,v_j)\big) \times \big(r(u_i, v_j+\Delta v) - r(u_i,v_j)\big)\Big|$



$$A \approx \sum_{j=1}^{M-1}\sum_{i=1}^{N-1} \left|\left(\frac{r(u_i+\Delta u, v_j) - r(u_i,v_j)}{\Delta u}\right) \times \left(\frac{r(u_i, v_j+\Delta v) - r(u_i,v_j)}{\Delta v}\right)\right| \Delta u \Delta v$$

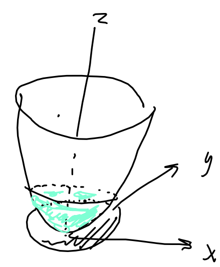
$\sum\sum \to \int_c^d\int_a^b \qquad \to \frac{\partial r}{\partial u} \qquad \to \frac{\partial r}{\partial v} \qquad \Delta u\Delta v \to du\,dv$

$N, M \to \infty, \quad \Delta u, \Delta v \to 0.$

$$\boxed{A = \iint_A \left|\frac{\partial r}{\partial u}(u,v) \times \frac{\partial r}{\partial v}(u,v)\right| du\,dv}$$

Eksempel.

Areal av paraboloide $z = x^2 + y^2$ og $x^2 + y^2 \leq 1$.



Parametrisering av flaten:

$A = \{(x,y) \mid x^2 + y^2 \leq 1\}$, $\quad$ Areal $= \iint_A \left| \frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} \right| dx\,dy$

$r(x,y) = x i + y j + (x^2+y^2) k$

$= (x, y, x^2+y^2)$

$\frac{\partial r}{\partial x} = (1, 0, 2x) \quad \frac{\partial r}{\partial y} = (0, 1, 2y) \quad \frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} = \begin{vmatrix} i & j & k \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix} = (-2x, -2y, 1)$

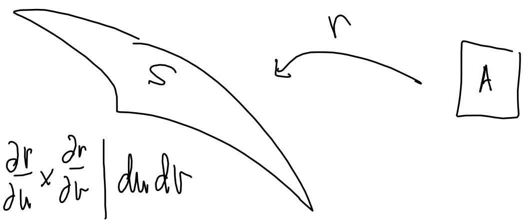
$\left| \frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} \right| = \sqrt{1 + 4x^2 + 4y^2}$

$u = 1+4r^2 \quad r=0 \quad u=1$
$du = 8r\,dr \quad r=1 \quad u=5$

$$\iint_A \sqrt{1+4(x^2+y^2)}\,dx\,dy = \int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}\; r\,dr\,d\theta = 2\pi \int_0^1 r\sqrt{1+4r^2}\,dr = \frac{\pi}{4}\int_1^5 \sqrt{u}\,du =$$

Integrasjon av skalarer på parametriserte flater.

$S = \{ (X, Y, Z)(u,v) \mid (u,v) \in A \}$.



$$\iint_S f(x,y,z)\, dS := \iint_A f(X(u,v), Y(u,v), Z(u,v)) \left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right| du\, dv$$