Plenum 9/2-16

3.1: 9, 17, 21, (10), (14)
3.2: 3
3.3: 11
A9: 1

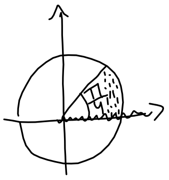
3.1: Parametriserte kurver

9) $\vec{r}(t) = (t, \ln(\cos(t)))$, $t \in [0, \frac{\pi}{4}]$

a) Hastighet: $\vec{v}(t) = \vec{r}\,'(t) = \left(1, \frac{1}{\cos(t)}(-\sin(t))\right)$

$$= (1, -\tan(t))$$

Fart: $v(t) = |\vec{v}(t)| = \sqrt{1^2 + (-\tan(t))^2} = \sqrt{1 + \tan^2(t)}$



$$= \sqrt{1 + \left(\frac{\sin(t)}{\cos(t)}\right)^2} = \sqrt{\frac{\cos^2(t) + \sin^2(t)}{\cos^2(t)}} = \sqrt{\frac{1}{\cos^2(t)}}$$

$$= \frac{1}{|\cos(t)|} = \frac{1}{\cos(t)}$$

$t \in [0, \frac{\pi}{4}]$

b) $L(0, \frac{\pi}{4}) \underset{\text{Def. 3.1.5}}{=} \int_0^{\frac{\pi}{4}} v(t)\,dt \underset{\text{a)}}{=} \int_0^{\frac{\pi}{4}} \frac{1}{\cos(t)}\,dt$

$= \int_0^{\frac{\pi}{4}} \frac{\cos(t)}{1-\sin^2(t)}\,dt = \int_0^{\frac{\sqrt{2}}{2}} \frac{1}{1-u^2}\,du$

$\frac{1}{\cos(t)} = \frac{\cos(t)}{1-\sin^2(t)}$

Hvorfor? $\frac{\cos(t)}{1-\sin^2(t)} = \frac{\cancel{\cos(t)}}{\cos^{\cancel{2}}(t)} = \frac{1}{\cos t}$

$u = \sin(t)$

$du = \cos(t)\,dt$

$\frac{du}{\cos(t)} = dt$

$t = 0 \Rightarrow u = \sin(0) = 0$

$t = \frac{\pi}{4} \Rightarrow u = \sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

Kalkulus-perm

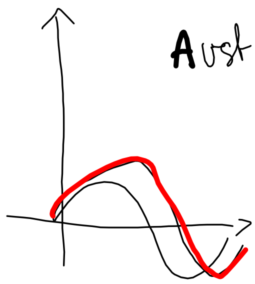
eller DBOS

$= \left[\operatorname{arctanh}(u)\right]_{u=0}^{\frac{\sqrt{2}}{2}} = \operatorname{arctanh}(\frac{\sqrt{2}}{2}) - \operatorname{arctanh}(0)$

$\underset{\boxed{\text{Def}}}{=} \frac{1}{2}\ln\left(\frac{1+\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}}\right) - \underbrace{\frac{1}{2}\ln\Big(\underbrace{\frac{1+0}{1-0}}_{1}\Big)}_{0} = \frac{1}{2}\ln\left(\frac{1+\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}}\right)$

$=$

17.)

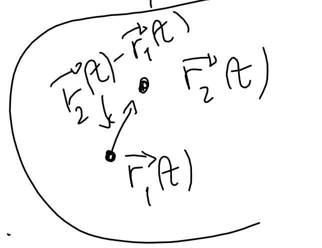


Avst. for og bakhjul: 1 m

$\vec{T}_1(t)$ = spor bakhjul

$\vec{r}_2(t)$ = spor forhjul

a) Fartsretningen til bakhjul er mot forhjulet $\Rightarrow \vec{v}_1(t)$ har samme retning som $\vec{r}_2(t) - \vec{r}_1(t)$. Dette $\vec{r}_2(t) - \vec{r}_1(t)$ har lengde 1 (fra oppg.). Dette er det samme som å være enhetsvektor i fartsretningen.



Dette er def. av $\vec{T}_1(t)$ (s. 167):

$\vec{T}_1(t) = \vec{r}_2(t) - \vec{r}_1(t)$

$\vec{r}_2(t) = \vec{r}_1(t) + \vec{T}_1(t)$

b) $\vec{r}_1(t) = (t, \sin(t))$

Fra a): $\vec{r}_2(t) = (t, \sin(t)) + \vec{T}_1(t)$ (*)

M: $\vec{T}_1(t) \underset{\boxed{\text{def.}}}{=} \dfrac{\vec{v}_1(t)}{v_1(t)}$

$\vec{v}_1(t) = \vec{r}_1{}'(t) = (1, \cos(t))$

$v_1(t) = |\vec{v}_1(t)| = \sqrt{1 + \cos^2(t)}$

$\vec{T}_1(t) = \left(\dfrac{1}{\sqrt{1+\cos^2(t)}}, \dfrac{\cos(t)}{\sqrt{1+\cos^2(t)}}\right)$

$\vec{r}_2(t) \underset{(*)}{=} \left(t + \dfrac{1}{\sqrt{1+\cos^2(t)}}, \underline{\underline{\sin(t) + \dfrac{\cos(t)}{\sqrt{1+\cos^2(t)}}}}\right)$

d) Venstre mot høyre: Når t øker, så flytter sykkelen seg positivt langs x-aksen.

21.) $\vec{a}(t) = k(t)\vec{r}(t)$

a) $\dfrac{d}{dt}\left(\vec{r}(t) \times \vec{v}(t)\right) \underset{\text{Set. 3.1.10 (iv)}}{=} \vec{r}\,'(t) \times \vec{v}(t) + \vec{r}(t) \times \vec{v}\,'(t)$

$= \vec{v}(t) \times \vec{v}(t) + \vec{r}(t) \times \vec{a}(t)$

$= \vec{v}(t) \times \vec{v}(t) + \vec{r}(t) \times (k(t)\vec{r}(t))$

$\underset{\text{Set. 1.4.2}}{=} \vec{v}(t) \times \vec{v}(t) + k(t)[\vec{r}(t) \times \vec{r}(t)]$

$= \vec{0} + \vec{0} = \vec{0}$

Set. 1.4.3: $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin(v)$

$|\vec{a} \times \vec{a}| = |\vec{a}|^2 \sin(0) = 0$

$\Downarrow$

$\vec{a} \times \vec{a} = \vec{0}$

b) a) $\Rightarrow$ derivert av hver komponent i $\vec{r}(t)\times\vec{v}(t)$ er 0 $\Rightarrow$ hver komponent i $\vec{r}(t)\times\vec{v}(t)$ er konstant $\Rightarrow$ $\vec{r}(t)\times\vec{v}(t) = \vec{c}$.

c) Planet som inneholder $\vec{0}$, $\vec{r}(0)$, $\vec{v}(0)$ er unikt gitt ved normalvektoren $\vec{r}(0)\times\vec{v}(0)$ og at det går gjennom $\vec{0}$.

Å vise at $\vec{r}(t)$ er i dette planet er ekvivalent m/ å vise:

$$\vec{r}(t)\cdot(\vec{r}(0)\times\vec{v}(0)) = 0 \text{ for alle } t \text{ (siden plan gj. } \vec{0}\text{)}$$

$$\vec{r}(t)\cdot(\vec{r}(0)\times\vec{v}(0)) \underset{(b)}{=} \vec{r}(t)\cdot\vec{c} \underset{(b)}{=} \vec{r}(t)\cdot(\vec{r}(t)\times\vec{v}(t)) \underset{(\square)}{=} 0$$

$(\square)$: Vektorprod. $\vec{r}(t)\times\vec{v}(t)$ er normal på $\vec{r}(t)$.

3.2: Kjerneregel for parametriserte kurver

3) $f(x,y,z) = x^2z - y\sin(yz)$, $\vec{r}(t) = e^t\vec{i} + t\vec{j} + \cos(t^2)\vec{k} = (e^t, t, \cos(t^2))$

$g(t) = f(\vec{r}(t))$;

$$g'(t) \underset{\text{set. 3.2.1}}{=} \nabla f(\vec{r}(t))\cdot\vec{r}\,'(t)$$

$$= (2e^t\cos(t^2),\ -\sin(t\cos(t^2)) - t\cos(t\cos(t^2))\cos(t^2),\ e^{2t} - t^2\cos(t\cos(t^2))).\ (e^t, 1, -\sin(t^2)2t)$$

$$\nabla f(x,y,z) = \begin{bmatrix} 2xz \\ -\sin(yz) - y\cos(yz)z \\ x^2 - y\cos(yz)y \end{bmatrix}$$

$= \ldots$ stygt uttrykk!

3.3: Linjeint. for skalarfelt

11) $\vec{r}(t) = \left(\frac{t^2}{2}, \frac{2\sqrt{2}}{9} t^{\frac{3}{2}}, \frac{t}{9}\right), \quad 1 \leq t \leq 7$

avh. bratt vei

Bensinforbruk: $\frac{1}{15} + \frac{1}{2}\frac{dz}{ds}$ per km.

strekning — buelengde

Merk: $\vec{v}(t) = \vec{r}\,'(t) = \left(t, \frac{\sqrt{2}}{3} t^{\frac{1}{2}}, \frac{1}{9}\right)$

$$v(t) = |\vec{v}(t)| = \sqrt{t^2 + \frac{2}{9}t + \frac{1}{81}} = \sqrt{\left(t+\frac{1}{9}\right)^2}$$

$$= \left|t + \frac{1}{9}\right| = t + \frac{1}{9}$$

$1 \leq t \leq 7$

Husk: $\text{fart} = \frac{\Delta\,\text{strekning}}{\Delta\,\text{tid}}$

$\boxed{\Delta \quad \delta}$

$$t + \frac{1}{9} = v(t) = \frac{ds}{dt}$$

I tillegg: $\frac{dz}{dt} = \frac{1}{9}$

Så: $\frac{dz}{dt} = \frac{dz}{ds}\frac{ds}{dt}$

"Forkort"

$$\frac{1}{9} = \frac{dz}{ds}\left(t + \frac{1}{9}\right)$$

$$\frac{dz}{ds} = \frac{1}{9t+1}$$

Totalt bensinforbruk:

$$\int_{\mathcal{C}} f\,ds = \int_1^7 f(\vec{r}(t))\,v(t)\,dt$$

Def. 3.3.1

$$= \int_1^7 \left(\frac{1}{15} + \frac{1}{2}\frac{1}{9t+1}\right)\left(t + \frac{1}{9}\right)dt$$

$$= \int_1^7 \left(\frac{1}{15}t + \frac{1}{135} + \frac{1}{18}\right)dt$$

$$= \left[\frac{1}{30}t^2 + \left(\frac{1}{135} + \frac{1}{18}\right)t\right]_{t=1}^{7}$$

$$= \dots = \frac{8}{5} + \frac{2}{45} + \frac{1}{3} = \frac{72 + 2 + 15}{45} = \underline{\underline{\frac{89}{45}}}$$

Totalt bensin<u>forbuk</u> $\frac{89}{45}$ ℓ.