Lagranges multiplikatormetode

Husk:

Maks/minimere $f(x_1,x_2,\dots,x_m)$

under bibetingelsene

$$g_1(x_1,x_2,\dots,x_m)=b_1$$
$$g_2(x_1,x_2,\dots,x_m)=b_2$$
$$\vdots$$
$$g_k(x_1,x_2,\dots,x_m)=b_k$$

Må løse ligningene:

$$\nabla f(\vec{x})=\lambda_1\nabla g_1(\vec{x})+\dots+\lambda_k\nabla g_k(\vec{x})$$
$$g_1(\vec{x})=b_1$$
$$\vdots$$
$$g_k(\vec{x})=b_k$$

m-ligninger

m+k ukjente

k ligninger.

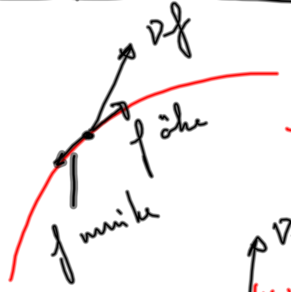
Hvorfor er det slik?

Se på tilfellet

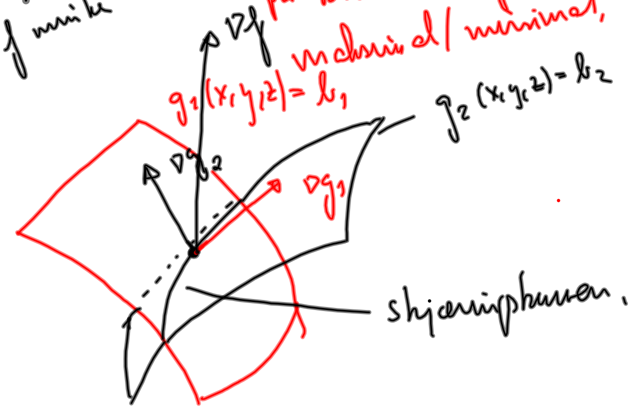
$f(x,y,z)$

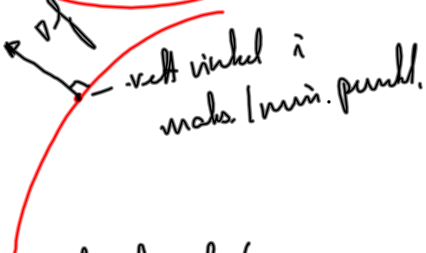
$g_1(x,y,z)=b_1$

$g_2(x,y,z)=b_2$



I hvilket punkt på kurven er f maksimal/minimal,





For å få et maks/min må ∇f ligge i normalplanet utspent av ∇g_1 og ∇g_2; dvs ∇f er en lin. komb. av ∇g_1 og ∇g_2, dvs. at det finnes tall λ_1 og λ_2 slik at

$$\nabla f=\lambda_1\nabla g_1+\lambda_2\nabla g_2$$

Eksempel: Finn mulige maks./min punkter for

$$f(x,y,z) = x^2 - 2x + 2y^2 + z^2 + z$$

under bibetingelsene
$x+y+z=1$ $\quad g_1(x,y,z) = x+y+z$
$2x-y+z=5$ $\quad g_2(x,y,z) = 2x-y+z$

Gradientene:

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} 2x-2 \\ 4y \\ 2z+1 \end{pmatrix}, \quad \underbrace{\nabla g_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \nabla g_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}}_{\text{lin. uavh.}}$$

Ligninger: $\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$, dvs

$$\left.\begin{aligned} 2x-2 &= \lambda_1 + 2\lambda_2 \\ 4y &= \lambda_1 - \lambda_2 \\ 2z+1 &= \lambda_1 + \lambda_2 \\ x+y+z &= 1 \\ 2x-y+z &= 5 \end{aligned}\right\}$$

fem ligninger med fem ukjente $x, y, z, \lambda_1, \lambda_2$

Bruker ligning (2) og (3) til å eliminere λ_1 og λ_2:

$$\left.\begin{aligned} (2)+(3):&\quad 4y+2z+1 = 2\lambda_1 \Rightarrow \lambda_1 = 2y+z+\tfrac{1}{2} \\ (3)-(2):&\quad 2z+1-4y = 2\lambda_2 \end{aligned}\right\}$$ setter inn i ligning (1):

$$2x-2 = 2y+z+\tfrac{1}{2} + 2z+1-4y$$

$$2x+2y-3z = \frac{7}{2}$$

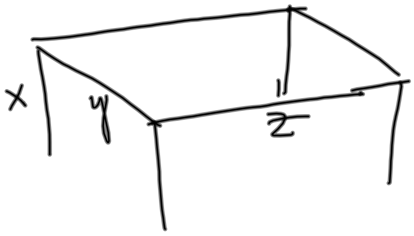
Sitter igjen med tre lineære ligninger:

$$\left.\begin{aligned} x+y+z &= 1 \\ 2x-y+z &= 5 \\ 2x+2y-3z &= \tfrac{7}{2} \end{aligned}\right\}$$ Gjør'e sjøl!

Eksempel: Teltet:

Volum: 500 m^3

Minimer rørlengden:



$$L(x,y,z) = 4x + 2y + 2z$$

$$V(x,y,z) = xyz = 500 \quad \text{bibetingelsen}$$

Jakter på punkter slik at $\nabla L = \lambda \nabla V$.

$$\nabla L = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}, \quad \nabla V = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix}$$

Ligninger:

$$\begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix} = \lambda \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} \iff \left.\begin{array}{l} 4 = \lambda yz \\ 2 = \lambda xz \\ 2 = \lambda xy \end{array}\right\} \Rightarrow \begin{array}{l} 4x = \lambda xyz = 500\lambda \\ 2y = \lambda xyz = 500\lambda \\ 2z = \lambda xyz = 500\lambda \end{array}$$

$$\underbrace{xyz = 500}$$

$$x = 125\lambda, \quad y = 250\lambda, \quad z = 250\lambda$$

$$(125\lambda)(250\lambda)(250\lambda) = 500$$

$$\lambda^3 = \frac{\cancel{500}\;2}{125 \cdot \cancel{250 \cdot 250}\;125} \Rightarrow \lambda^3 = \frac{1}{5^3 5^3} \Rightarrow \lambda = \frac{1}{25}$$

$$x = 125 \cdot \frac{1}{25} = \underline{5}$$

$$y = 2x = \underline{10}$$

$$z = 2z = \underline{10}$$

5.11 selvstudium!

Kalkulus kap 12.