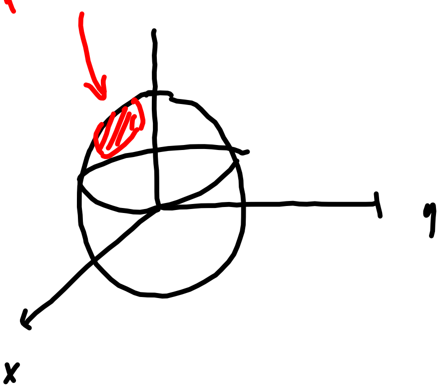
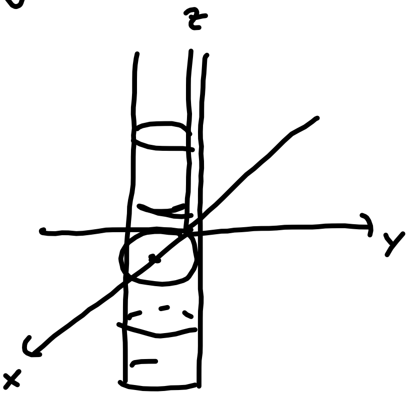
6.4.7 Finn arealet av den delen av kuleflaten $x^2+y^2+z^2=1$ som ligger over sirkelen $(x-\frac{1}{2})^2+y^2 \le \frac{1}{4}$. Kall denne biten C.

Vi skal finne arealet av dette.





Fra forelesning (3/3):

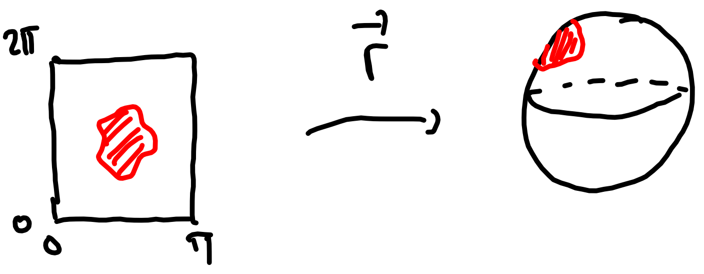
Kulekoordinater: $\vec{r}(\phi,\theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$.

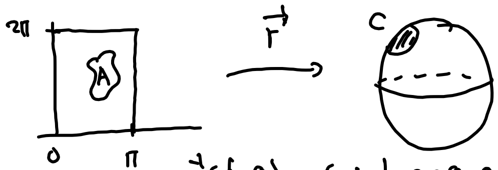
Regnet ut: $\left|\frac{\partial \vec{r}}{\partial \phi} \times \frac{\partial r}{\partial \theta}\right|(\phi,\theta) = \sin\phi$.

Vil finne et område $A \subset [0,\pi] \times [0,2\pi]$

s.a. $\vec{r}(A) = C$. Da vet vi at arealet er

$$\iint_A \sin\phi \, d\phi d\theta.$$



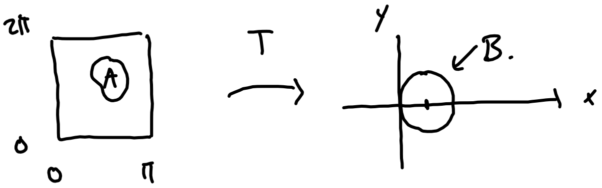


$\vec{r}(\phi,\theta) = (\sin\phi\cdot\cos\theta, \sin\phi\sin\theta, \cos\phi)$.

Se at avbildingen $T(\phi,\theta) = (\sin\phi\cos\theta, \sin\phi\cdot\sin\theta)$

ville avbildet A på området

$B = \{(x,y) : (x-\frac{1}{2})^2 + y^2 \leq \frac{1}{4}\}$.



Vil finne: $\iint_A \sin\phi \, d\phi d\theta$.

Strategi: Forsøk å finne en funksjon $f(x,y)$ s.a. $\iint_B f(x,y)\,dx\,dy = \iint_A \sin\phi\, d\phi d\theta$

$$= I = \iint_A f(T(\phi,\theta)) \cdot |\det T'(\phi,\theta)| \, d\phi d\theta.$$

En utregning gir: $\det T'(\phi,\theta) = \cos\phi\sin\phi$.

$$I = \iint_A f(T(\phi,\theta)) \cdot \cos\phi\cdot\sin\phi \, d\phi d\theta$$

Vil ha $f(T(\phi,\theta))\cdot\cos\phi\cdot\sin\phi = \sin\phi$

så vi må ha $f(T(\phi,\theta)) = \frac{1}{\cos\phi}$.

$T(\phi,\theta) = (\sin\phi\cdot\cos\theta, \sin\phi\cdot\sin\theta)$

ser at $\sqrt{1-(\sin\phi\cos\theta)^2-(\sin\phi\sin\theta)^2} = \sqrt{1-\sin^2\phi}$

$= \sqrt{\cos^2\phi} = \cos\phi$.

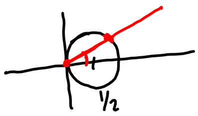
Se da at vi kan sette $f(x,y) = \frac{1}{\sqrt{1-x^2-y^2}}$

mar 19-10:26

Har funnet ut at

$$\iint_A \sin\phi \, d\phi \, d\theta = \iint_B \frac{1}{\sqrt{1-x^2-y^2}}\, dx\,dy,$$

der $B = \{(x,y) : (x-\frac{1}{2})^2 + y^2 \le \frac{1}{4}\}$.



Skifter til polarkoordinater

$x = r\cdot\cos t$, $y = r\cdot\sin t$,

t kan variere i $[-\frac{\pi}{2}, \frac{\pi}{2}]$, men vi må finne integrasjonsgrensen $r(t)$. Må finne ut hvor linja med vinkel t til x-aksen snitter sirkelen:

$$(r\cdot\cos t - \tfrac{1}{2})^2 + r^2\sin^2 t = 1/4$$

$$r^2\cos^2 t - r\cos t + \cancel{\tfrac{1}{4}} + r^2\sin^2 t = \cancel{1/4}$$

$$r^2 \div r\cos t = 0$$

$$r^2 = r\cos t$$

$$r = \cos t.$$

$$\iint_B \frac{1}{\sqrt{1-x^2-y^2}}\, dx\,dy = \int_{-\frac{\pi}{2}}^{\pi/2}\int_0^{\cos t} \frac{1}{\sqrt{1-r^2}}\cdot r \, dr\,dt.$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^{\cos t} (1-r^2)^{-1/2}\cdot r\, dr\,dt.$$

$$\tfrac{1}{2}(1-r^2)^{-1/2}\cdot(-2r) = -(1-r^2)^{-1/2}\cdot r$$

$$= \int_{-\pi/2}^{\pi/2} \left[-(1-r^2)^{1/2}\right]_0^{\cos t} dt.$$

$$= \int_{-\pi/2}^{\pi/2} -(1-\cos^2 t)^{1/2} + 1 \, dt$$

$$= \int_{-\pi/2}^{\pi/2} -|\sin t| + 1 \, dt$$

$$= 2\cdot\Big(\underbrace{\int_0^{\pi/2} -\sin t + 1 \, dt}_{\left[\cos t + t\right]_0^{\pi/2} = -1 + \frac{\pi}{2}}\Big).$$

$$= \underline{\underline{\pi - 2.}}$$

4.9 Regn ut integralet $\iint_T x^2 dS$ når

T er flaten gitt ved

$z = x^2+y^2$ og $x^2+y^2 \leq 1$.

(i) Må finne en parametrisering:

Polarkoordinater i x og y,

$x = r\cos t$, $y = r\cdot\sin t$ og $z = r^2$.

$T(r,t) = (r\cdot\cos t, r\sin t, r^2)$, $0 \leq r \leq 1$, $0 \leq t \leq 2\pi$

parametrisere flaten.

$$\text{Da er } \iint_T r^2\cos^2 t \cdot \left|\left(\frac{\partial T}{\partial r} \times \frac{\partial T}{\partial t}\right)(r,t)\right| \, dr\, dt$$

$$\left(\text{Fra forrige oppgave: } \left(\frac{\partial T}{\partial r} \times \frac{\partial T}{\partial t}\right)(r,t) = \sqrt{1+4r^2}\cdot r.\right)$$

$$= \iint_T \sqrt{1+4r^2}\cdot r^3 \cdot \cos^2 t \, dr\, dt$$

$$= \int_0^1 \int_0^{2\pi} \sqrt{1+4r^2}\, r^3 \cdot \cos^2 t \, dt\, dr$$

$$= \underbrace{\left(\int_0^1 \sqrt{1+4r^2}\cdot r^3 dr\right)}_{I_1} \cdot \underbrace{\left(\int_0^{2\pi} \cos^2 t\, dt\right)}_{I_2}.$$

I_1: $s = r^2$

$ds = 2r\,dr$

$r\cdot dr = \frac{1}{2} ds$.

$$I_1 = \frac{1}{2}\int_0^1 \sqrt{1+4s}\cdot s \, ds$$

Delvis integrasjon: $u = s$, $v' = (1+4s)^{1/2}$

$u' = 1$, $v = \frac{2}{3}\frac{1}{4}(1+4s)^{3/2}$

$= \frac{1}{6}\cdot(1+4s)^{3/2}$.

$$I_1 = \frac{1}{2}\left[\left[\frac{1}{6}s(1+4s)^{3/2}\right]_0^1 - \frac{1}{6}\int_0^1 (1+4s)^{3/2} ds\right]$$

$$= \frac{1}{2}\cdot\frac{1}{6}\cdot 5^{3/2} - \frac{1}{6}\cdot\left[\frac{2}{5}\frac{1}{4}(1+4s)^{5/2}\right]_0^1$$

$$= \frac{1}{12}\cdot 5^{3/2} - \frac{1}{120}\left[5^{5/2} - 1\right].$$

mar 19-11:11

$$I_2 = \int_0^{2\pi} \cos^2 t \, dt = \int_0^{2\pi} \frac{1+\cos(2t)}{2} \, dt .$$

$$= \pi .$$

Så $\underline{\underline{I = \pi \cdot \left[\frac{1}{12} 5^{3/2} - \frac{1}{120} \cdot (5^{5/2} - 1) \right]}}$.

$\underline{6.4.4}$: Se forelesningsnotater 3/3.

$\underline{6.5.7}$: La D være området i $\mathbb{R}^2$ som består av punkter (x,y) som oppfyller $x^2+y^2 \leq 1$ og $y \geq 0$. La C være randa til D orientert mot urviseren.

Regn ut

$$\int_C \underbrace{(xy + \ln(x^2+1))}_{P(x,y)} \, dx + \underbrace{(4x + e^{y^2} + 3\arctan y)}_{Q(x,y)} \, dy .$$

Green: $\int_C P\,dx + Q\,dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$$= \iint_A (4 - x) \, dx \, dy .$$

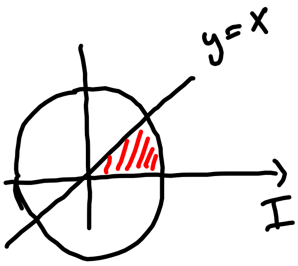
$\left(\text{se } \iint_A x \, dx\,dy = 0\right)$.

$$= \underline{\underline{2\pi}} .$$

(10) La D være området av punkter i $\mathbb{R}^2$ som oppfyller ulikhetene

$x^2+y^2 \leq 1$, $x \geq 0$, $y \geq 0$, $0 \leq y \leq x$.

(a)



Regn ut

$$I = \iint_D (x+y^2)\,dx\,dy$$

ved å innføre polarkoordinater.

$x = r\cos t$, $y = r\cdot\sin t$, $0 \leq r \leq 1$, $0 \leq t \leq \frac{\pi}{4}$.

$$I = \int_0^1 \int_0^{\pi/4} (r\cdot\cos t + r^2\sin^2 t)\,r\cdot dr\,dt.$$

$$= \frac{1}{3}\int_0^{\pi/4} \cos t\,dt + \frac{1}{4}\int_0^{\pi/4} \sin^2 t\,dt$$

$$= \frac{1}{3}\left(\sin\frac{\pi}{4} - \sin 0\right) + \frac{1}{4}\int_0^{\pi/4} \frac{1-\cos(2t)}{2}\,dt.$$

$$= \frac{\sqrt{2}}{6} + \frac{1}{8}\cdot\left[t - \frac{1}{2}\sin(2t)\right]_0^{\pi/4}$$

$$= \frac{\sqrt{2}}{6} + \frac{\pi}{32} - \frac{1}{16}.$$

(b) Regn ut I ved regn ut integralet av et passelig vektorfelt $F=(P,Q)$ lang kurven C som avgrense D.

$$I=\iint_D (x+y^2)\,dx\,dy.$$

Vil finne $P\,dx+Q\,dy$

s.a. $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=x+y^2.$

$P(x,y)=-xy$

$Q(x,y)=y^2x$

mar 19-11:52