$\boxed{2.7 \text{ Kjerneregelen}}$

Anta at vi har to mengder $A \subseteq \mathbb{R}^n$ og $B \subseteq \mathbb{R}^m$ og funksjoner $\vec{G} : A \to B$, $\vec{F} : B \to \mathbb{R}^k$.
Hvis $\vec{G}$ er deriverbar i $\vec{a} \in A$ og $\vec{F}$ er deriverbar i $\vec{b} = G(\vec{a})$, så er den sammensatte funksjonen

$$\vec{H}(\vec{x}) = \vec{F}(\vec{G}(\vec{x}))$$

deriverbar i $\vec{a}$, og Jacobimatrisene oppfyller

$$\vec{H}'(\vec{a}) = \vec{F}'(\vec{G}(\vec{a})) \cdot \vec{G}'(\vec{a})$$

Spesialtilfelle 1: $u, v : \mathbb{R}^3 \to \mathbb{R}$ og $f : \mathbb{R}^2 \to \mathbb{R}$

$k(x,y,z) = f(u(x,y,z), v(x,y,z))$. Kjerneregelen:

$$\begin{bmatrix} \frac{\partial k}{\partial x} & \frac{\partial k}{\partial y} & \frac{\partial k}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{bmatrix}$$

Altså:

$$\begin{cases} \frac{\partial k}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} \\ \frac{\partial k}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} \\ \frac{\partial k}{\partial z} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial z} \end{cases}$$

eks. $f(u,v) = u^2 v$

$u(x,y,z) = x^2 + 3y^2 - 5z$

$v(x,y,z) = xyz$

Oppgave 2.7.1: Disse heter g og h (dóp evt. om)

Hvis $k(x,y,z) = f(u(x,y,z), v(x,y,z))$, gir kjerneregelen

$$\frac{\partial k}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= 2uv \cdot 2x + u^2 \cdot yz$$

$$= \underline{\underline{2(x^2+3y^2-5z) \cdot xyz \cdot 2x + (x^2+3y^2-5z)^2 \cdot yz}}$$

Spesialtilfelle 2: Tur i landskap med høyde $f(x,y)$

$h(t) = f(\vec{p}(t))$, der $\vec{p}(t) = (x(t), y(t))$: Vår høyde ved tid t

Kjerneregelen:

$$h'(t) = f'(\vec{p}(t)) \cdot \vec{p}\,'(t)$$

$$= \begin{bmatrix} \frac{\partial f}{\partial x}(\vec{p}(t)) & \frac{\partial f}{\partial y}(\vec{p}(t)) \end{bmatrix} \cdot \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}$$

$$= \frac{\partial f}{\partial x}(\vec{p}(t)) \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y}(\vec{p}(t)) \cdot \frac{dy}{dt} . \quad \square$$

2.8 Linearisering

Definisjon 2.8.2

Anta at $\vec{F}: A \to \mathbb{R}^n$ er en funksjon av n variable som er deriverbar i $\vec{a}$. Affinavbildningen $T_{\vec{a}}\vec{F}: \mathbb{R}^n \to \mathbb{R}^m$ gitt ved

$$T_{\vec{a}}\vec{F}(\vec{x}) = \vec{F}(\vec{a}) + \vec{F}'(\vec{a})\cdot(\vec{x}-\vec{a})$$

kalles lineariseringen til $\vec{F}$ i punktet $\vec{a}$.

eks. Finn lineariseringen til $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^3$ gitt ved

$$\vec{F}(x,y) = \begin{pmatrix} \cos(xy) \\ \sin(xy) \\ xy \end{pmatrix} \quad \text{i punktet } \left(\tfrac{\pi}{2}, 1\right)$$

Løsn.

$$\vec{F}' = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} \end{pmatrix} = \begin{pmatrix} -\sin(xy)\cdot y & -\sin(xy)\cdot x \\ \cos(xy)\cdot y & \cos(xy)\cdot x \\ y & x \end{pmatrix}$$

$$\boxed{\vec{a} = (x,y) = \left(\tfrac{\pi}{2}, 1\right)} \quad = \begin{pmatrix} -1 & -\frac{\pi}{2} \\ 0 & 0 \\ 1 & \frac{\pi}{2} \end{pmatrix}$$

Så:

$$T_{(\frac{\pi}{2},1)}\vec{F}(\vec{x}) = \vec{F}\left(\tfrac{\pi}{2},1\right) + \vec{F}'\left(\tfrac{\pi}{2},1\right)\cdot\left(\vec{x} - \begin{pmatrix} \frac{\pi}{2} \\ 1 \end{pmatrix}\right)$$

$$\boxed{\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}} \quad = \begin{pmatrix} 0 \\ 1 \\ \frac{\pi}{2} \end{pmatrix} + \begin{pmatrix} -1 & -\frac{\pi}{2} \\ 0 & 0 \\ 1 & \frac{\pi}{2} \end{pmatrix}\cdot\begin{pmatrix} x - \frac{\pi}{2} \\ y - 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ \frac{\pi}{2} \end{pmatrix} + \begin{pmatrix} -1\cdot\left(x-\frac{\pi}{2}\right) - \frac{\pi}{2}(y-1) \\ 0 \\ 1\cdot\left(x-\frac{\pi}{2}\right) + \frac{\pi}{2}(y-1) \end{pmatrix}$$

$$= \underline{\begin{pmatrix} 0 - 1\left(x-\frac{\pi}{2}\right) - \frac{\pi}{2}(y-1) \\ 1 \\ \frac{\pi}{2} + \left(x-\frac{\pi}{2}\right) + \frac{\pi}{2}(y-1) \end{pmatrix}}$$