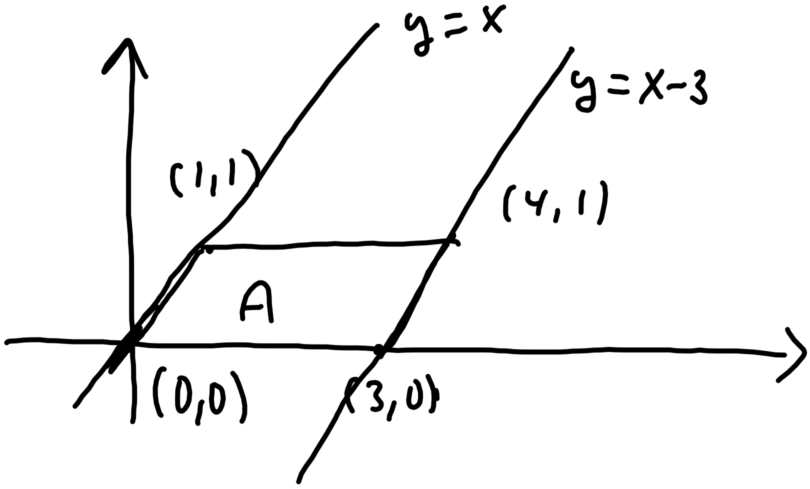
1 b) 6.7

$$I = \iint_A x \, dx\,dy$$



$u = x - y$, $y = x \Leftrightarrow u = 0$, $y = x-3 \Leftrightarrow u = 3$

$v = y$, $x = u + v$, $y = v$

$(x,y) = T(u,v) = (u+v, v)$, $T'(u,v) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

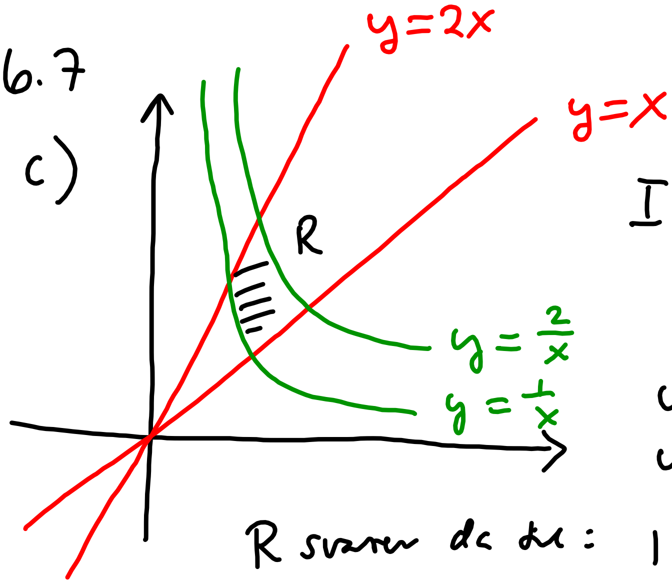
$|\det T'(u,v)| = 1$, $I = \int_0^3 \int_0^1 (u+v) \cdot 1 \, dv\,du$

$$= \int_0^3 \left[uv + \frac{1}{2}v^2 \right]_0^1 du = \int_0^3 \left(u + \frac{1}{2}\right) du$$

$$= \left[\frac{1}{2}u^2 + \frac{1}{2}u \right]_0^3 = \frac{9}{2} + \frac{3}{2} = \underline{\underline{6}}$$

6.7

3 c)



$$I = \iint_R (y^2 - yx)\,dx\,dy$$

$$u = yx$$
$$v = \frac{y}{x}$$

R svarer da til: $1 \le u \le 2$

$1 \le v \le 2$

$$(u,v) = S(x,y) = \left(yx, \frac{y}{x}\right)$$

$$S'(x,y) = \begin{bmatrix} y & x \\ -\frac{y}{x^2}, & \frac{1}{x} \end{bmatrix}, \quad \det S'(x,y) = 2\frac{y}{x}$$

$$T(u,v) = (x,y), \quad \det T'(u,v) = \frac{1}{\det S'(x,y)} =$$

$$= \frac{x}{2y} = \frac{1}{2v},$$

$$I = \int_1^2 \int_1^2 (uv - u)\frac{1}{2v}\,du\,dv =$$

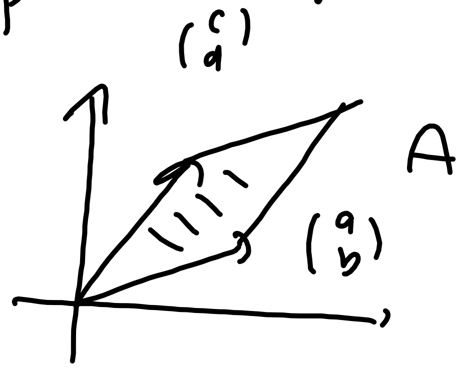
$$= \int_1^2 \int_1^2 \frac{(v-1)}{2v} u\,du\,dv = \frac{1}{2}\int_1^2 \left[\frac{(v-1)}{v}\frac{1}{2}u^2\right]_1^2$$

$$= \frac{3}{2}\int_1^2 \frac{1}{2}\left(1 - \frac{1}{v}\right)dv = \frac{3}{4}\left[v - \ln v\right]_1^2$$

$$= \frac{3}{4}\left[1 - \ln 2\right]$$

6.7.7

A parallellogram utspent $\begin{pmatrix} a \\ b \end{pmatrix}$ og $\begin{pmatrix} c \\ d \end{pmatrix}$



$, \quad M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = M \begin{pmatrix} u \\ v \end{pmatrix} = T\begin{pmatrix} u \\ v \end{pmatrix}$$

$K = [0,1] \times [0,1], \quad T(K) = A$

$$A = \left\{ u \begin{pmatrix} a \\ b \end{pmatrix} + v \begin{pmatrix} c \\ d \end{pmatrix} \;\middle|\; u \in [0,1], v \in [0,1] \right\}$$

$$K = \{ u\vec{e}_1 + v\vec{e}_2 \mid u, v \in [0,1] \}$$

Siden T er lineær er

$$T(K) = \{ T(u\vec{e}_1 + v\vec{e}_2) \mid u, v \in [0,1] \}$$

$$= \{ uT(e_1) + vT(e_2) \mid u, v \in [0,1] \}$$

$$= \left\{ u \begin{pmatrix} a \\ b \end{pmatrix} + v \begin{pmatrix} c \\ d \end{pmatrix} \;\middle|\; u, v \in [0,1] \right\} = A$$

b) (6.7.7 forts.)

$f(x,y)$ kontinuerlig,

$$\iint_A f(x,y)\,dx\,dy = |\det M| \int_0^1\!\!\int_0^1 f(au+cv,\; bu+dv)\,du\,dv$$

$$(x,y) = T(u,v) = \underset{\overset{\shortparallel}{M}}{\begin{pmatrix} a & c \\ b & d \end{pmatrix}} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} au+cv \\ bu+dv \end{pmatrix}$$

$A = T(K)$,

$$T'(u,v) = \begin{pmatrix} a & c \\ b & a \end{pmatrix}, \quad |\det T'(u,v)| = |\det M|$$

Variabelskifte formelen gir direkte:

$$\iint_A f(x,y)\,dx\,dy = \iint_{[0,1]\times[0,1]} f(au+cv,\; bu+dv)\,\underbrace{|\det M|}_{\text{konstant}}\,du\,dv$$

6.7. 7 forts.

c) $f(x,y) = e^{2x-3y}$

$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$M = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}, \ \det M = 6+1 = 7$

$x = 2u+v, \quad y = -u+3v$

$f(x,y) = e^{(4u+2v)-(-3u+9v)} = e^{7u-7v}$

$$\iint_A f(x,y)\,dx\,dy = 7\int_0^1\int_0^1 e^{7u-7v}\,du\,dv$$

$$= 7\int_0^1\int_0^1 e^{7u}\cdot e^{-7v}\,du\,dv =$$

$$= 7\int_0^1 e^{-7v}\left(\int_0^1 e^{7u}\,du\right)dv =$$

$$= 7\int_0^1 e^{-7v}\,dv\int_0^1 e^{7u}\,du =$$

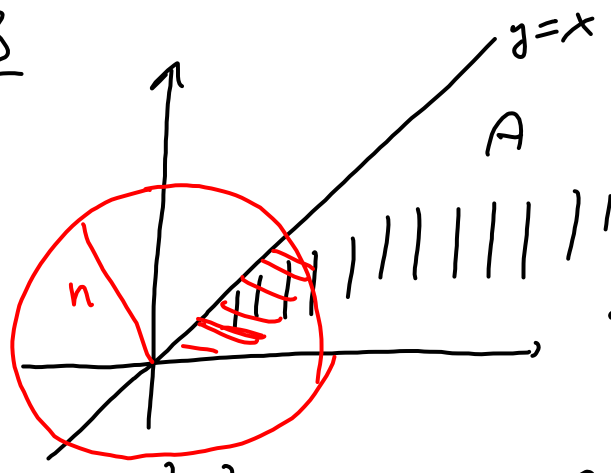
$$= 7\left[-\frac{1}{7}e^{-7v}\right]_0^1\left[\frac{1}{7}e^{7u}\right]_0^1 =$$

$$= \frac{1}{7}(e^7-1)(1-e^{-7}) =$$

$$= \frac{1}{7}(e^7-1-1+e^{-7}) = \underline{\underline{\frac{1}{7}(e^7+e^{-7}-2)}}$$

6.8

1)



$\{(x,y) \mid x,y \geq 0$
$y \leq x\}$

$$I = \iint_A e^{-x^2-y^2}\,dx\,dy = \lim_{n\to\infty} \iint_{A\cap B(0,n)} e^{-x^2-y^2}\,dx\,dy$$

$B(0,n) = \{ \cancel{\text{Hex.}}\ (x,y) \mid \|(x,y)\| < n\}$

$x = r\cos\theta$
$y = r\sin\theta$

$B(0,n) \cap A$

$$I = \lim_{n\to\infty} \int_0^{\frac{\pi}{4}} \int_0^n e^{-r^2}\, r\,dr\,d\theta =$$

$$= \lim_{n\to\infty} \int_0^{\pi/4} \left[-\frac{1}{2}e^{-r^2}\right]_0^n d\theta$$

$$= \lim_{n\to\infty} \int_0^{\pi/4} \frac{1}{2}\left(1-e^{-n^2}\right) d\theta =$$

$$= \lim_{n\to\infty} \frac{\pi}{8}\left(1-e^{-n^2}\right) = \frac{\pi}{8}$$

Integralet konvergerer og har verdi:

$\frac{\pi}{8}$

2) $$\iint_{\mathbb{R}^2} \frac{1}{1+x^2+y^2}\,dx\,dy =$$

$$= \lim_{n\to\infty} \iint_{D(0,n)} \frac{1}{1+x^2+y^2}\,dx\,dy =$$

$$= \lim_{n\to\infty} \int_0^{2\pi}\int_0^{n} \frac{r}{1+r^2}\,dr\,d\theta$$

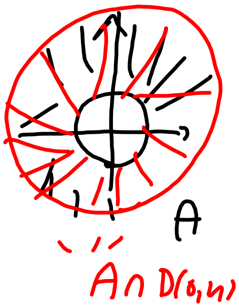
$$= \lim_{n\to\infty} 2\pi \left[\frac{1}{2}\ln(1+r^2)\right]_0^n =$$

$$= \lim_{n\to\infty} 2\pi \left[\frac{1}{2}\ln(1+n^2)\right] \longrightarrow \infty$$

dvs. integralet divergerer.

6) $A = \{ (x,y) \mid x^2+y^2 \geq 1 \}$

$$I_p = \iint_A \frac{1}{(x^2+y^2)^p}\, dx\,dy$$



For hvilke p konvergerer dette.

$$I_p = \lim_{n\to\infty} \iint_{A\cap D(0,n)} \frac{1}{(x^2+y^2)^p}\, dx\,dy =$$

$$= \lim_{n\to\infty} \int_0^{2\pi} \int_1^n \frac{1}{(r^2)^p}\, r\,dr\,d\theta$$

$$= \lim_{n\to\infty} 2\pi \int_1^n (r^{1-2p})\,dr$$

$p \neq 1$ $$\int_1^n r^{1-2p}\,dr = \left[\frac{1}{2-2p} r^{2-2p} \right]_1^n$$

$$= \frac{1}{2-2p} \left[n^{2-2p} - 1 \right] \xrightarrow[n\to\infty]{} \begin{cases} \infty & p<1 \ (2-2p>0) \\ \frac{1}{2p-2} & p>1 \ (2-2p<0) \end{cases}$$

$p=1$ $$\int_1^n \frac{1}{r}\,dr = \ln n \xrightarrow[n\to\infty]{} \infty$$

Integralet konvergerer altså hvis og bare hvis $p>1$ og har da verdi:

$$2\pi \frac{1}{2p-2} = \underline{\underline{\frac{\pi}{p-1}}}$$

6.9.1

c) $$\iiint_{[1,2]\times[\pi,2\pi]\times[0,1]} zy\cos(xy)\,dx\,dy\,dz =$$

$$= \int_{\pi}^{2\pi}\int_{1}^{2}\int_{0}^{1} zy\cos xy\,dz\,dx\,dy =$$

$$= \frac{1}{2}\int_{\pi}^{2\pi}\int_{1}^{2} y\cos xy\,dx\,dy = \frac{1}{2}\int_{\pi}^{2\pi}\Big[\sin(xy)\Big]_{1}^{2}dy$$

$$= \frac{1}{2}\int_{\pi}^{2\pi}(\sin 2y - \sin y)\,dy$$

$$= \frac{1}{2}\Big[-\frac{1}{2}\cos 2y + \cos y\Big]_{\pi}^{2\pi} =$$

$$= \frac{1}{2}\Big[\Big(-\frac{1}{2}+1\Big)-\Big(-\frac{1}{2}-1\Big)\Big] = \underline{\underline{1}}\,.$$

6.7.2

a) $A = \{ (x,y,z) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 2,\ 0 \leq z \leq x^2 y \}$

$$\iiint_A (xy+z)\,dx\,dy\,dz = \int_0^1 \int_0^2 \int_0^{x^2y} (xy+z)\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^2 \left[xyz + \frac{1}{2} z^2 \right]_0^{x^2y} dy\,dx =$$

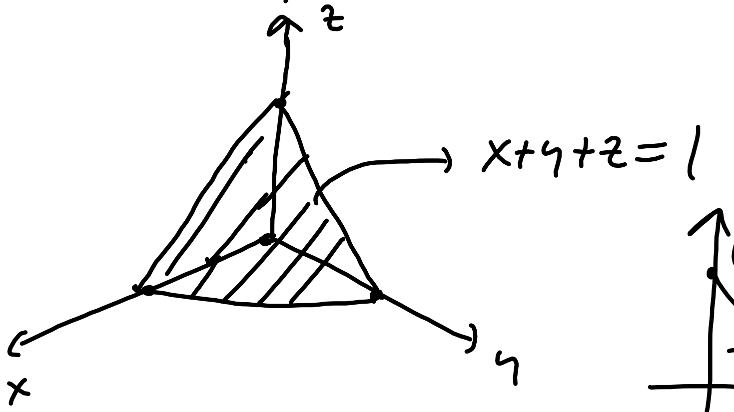
$$= \int_0^1 \int_0^2 \left[x^3y^2 + \frac{1}{2} x^4 y^2 \right] dy\,dx =$$

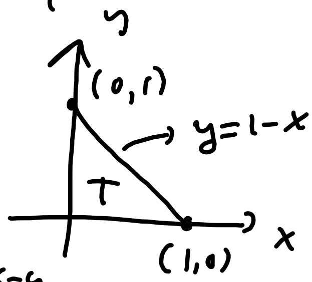
$$= \int_0^1 \left[\left(x^3 + \frac{1}{2} x^4 \right) \frac{y^3}{3} \right]_0^2 dx = \int_0^1 \left(\frac{8}{3} x^3 + \frac{4}{3} x^4 \right) dx$$

$$= \left[\frac{8}{3} \frac{x^4}{4} + \frac{4}{3} \frac{x^5}{5} \right]_0^1 = \frac{2}{3} + \frac{4}{15} = \underline{\underline{\frac{14}{15}}} \quad ✓$$

2 e) A er pyramiden med

hjörner (0,0,0), (1,0,0), (0,1,0), (0,0,1)





$$\iiint_A xy\,dx\,dy\,dz = \iint_T \int_0^{1-x-y} xy\,dz\,dy\,dx$$

$$= \iint_T xy(1-x-y)\,dy\,dx = \iint_T (xy - x^2y - xy^2)\,dy\,dx$$

$$= \iint_T (xy - 2x^2y)\,dy\,dx =$$

$$= \int_0^1 \int_0^{1-x} (xy - 2x^2y)\,dy\,dx =$$

litt regning

$$= \int_0^1 \Big[(x-2x^2)\tfrac{1}{2}y^2\Big]_0^{1-x} dx =$$

litt regning

$$= \frac{1}{2}\int_0^1 (x - 4x^2 + 5x^3 - 2x^4)\,dx = \underline{\underline{\frac{1}{120}}}$$