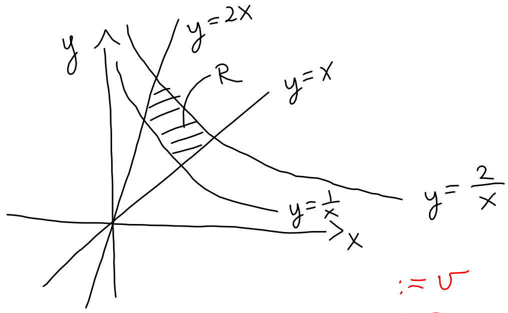
Plenum 29/3-16

6.7: 3c, 9
6.8: 2, 6
6.9: 2b
6.10: 2b, 3a

6.7:

3) c) $\iint_R (y^2 - yx)\,dx\,dy$; R er avgrenset av

$y = x$, $y = 2x$, $y = \frac{2}{x}$, $y = \frac{1}{x}$.



$$x \le y \le 2x$$
og
$$\frac{1}{x} \le y \le \frac{2}{x}$$
$\Downarrow$ ($x > 0$ fra figur)

$$1 \le \underbrace{\frac{y}{x}}_{:= v} \le 2$$

og $1 \le \underbrace{yx}_{:= u} \le 2$

Så: $\iint_R (y^2 - yx)\,dx\,dy = \int_1^2\int_1^2 (uv - u)\left|\frac{1}{2v}\right| du\,dv$

$u = xy$

$uv = xy\,\frac{y}{\cancel{x}} = y^2$

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix} = \begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x}\end{vmatrix} = \frac{y}{x} + \frac{y}{x} = 2\frac{y}{x} = 2v$$

$$= \int_1^2\int_1^2 (uv - u)\frac{1}{2v}\,du\,dv$$

$$= \frac{1}{2}\int_1^2\int_1^2 u\left(1 - \frac{1}{v}\right) du\,dv$$

$$= \frac{1}{2}\int_1^2 \left(1 - \frac{1}{v}\right)\left[\frac{1}{2}u^2\right]_{u=1}^2 dv$$

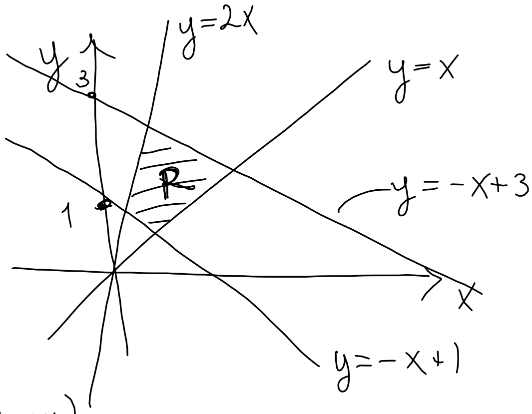
$$= \frac{1}{4}\int_1^2 \left(1 - \frac{1}{v}\right) 3\,dv$$

$$= \frac{3}{4}\left[v - \ln(v)\right]_{v=1}^2 = \frac{3}{4}\left(2 - \ln(2) - 1 + \underbrace{\ln(1)}_{0}\right)$$

$$= \frac{3}{4} - \frac{3}{4}\ln(2)$$

9) R; avgr. $y = x$, $y = 2x$, $y = -x+1$, $y = -x+3$

FINN $\iint_R \frac{x+y}{x^2}\,dxdy$



$x \le y \le 2x$

og

$-x+1 \le y \le -x+3$

$\Downarrow$ $(x>0$ fra figur$)$

$1 \le \underbrace{\frac{y}{x}}_{:=u} \le 2$

og $1 \le \underbrace{y+x}_{:=v} \le 3$

Jacobideterminant: $\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ 1 & 1 \end{vmatrix}$

$= -\frac{y}{x^2} - \frac{1}{x}$

Så:

$$\iint_R \frac{x+y}{x^2}\,dxdy = \int_1^3\int_1^2 \frac{x+y}{x^2}\left|\det\left(\frac{\partial(x,y)}{\partial(u,v)}\right)\right| dudv$$

$$= \int_1^3\int_1^2 \frac{v}{x^2}\left|\frac{1}{-\frac{y}{x^2}-\frac{1}{x}}\right| du\,dv = \int_1^3\int_1^2 v\left|\frac{1}{-y-x}\right| du\,dv$$

$x^2>0$

Kan tas innenfor $|\cdot|$

$$= \int_1^3\int_1^2 v\left|\frac{1}{-v}\right| du\,dv = \int_1^3\int_1^2 \frac{v}{|v|}\,du\,dv = \int_1^3\int_1^2 1\,dudv$$

$|-v| = |v|$

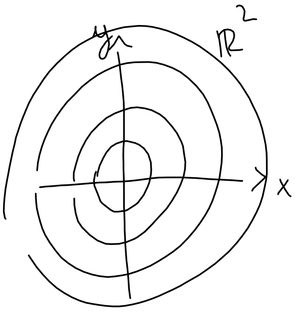
$v>0$ så $|v| = v$

$= (2-1)(3-1)$

$= \underline{\underline{2}}$

6.8

2.) $$\iint_{\mathbb{R}^2} \frac{1}{1+x^2+y^2}\,dx\,dy = \lim_{n\to\infty} \iint_{B(0,n)} \frac{1}{1+x^2+y^2}\,dx\,dy$$



polar-koordinater: $x^2+y^2=r^2$

$$= \lim_{n\to\infty} \int_0^{2\pi}\int_0^{n} \frac{r}{1+r^2}\,dr\,d\theta$$

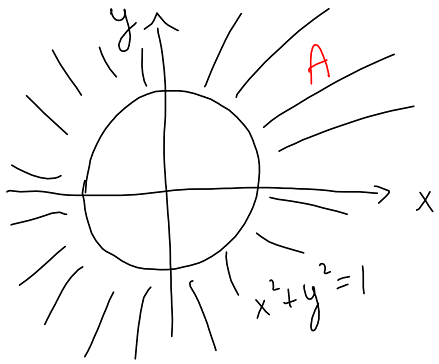
$$= \lim_{n\to\infty} \int_0^{2\pi} \left[\tfrac{1}{2}\ln(1+r^2)\right]_{r=0}^{n} d\theta$$

direkte el. substitusjon

$$= \lim_{n\to\infty} \int_0^{2\pi} \frac{1}{2}\ln(n^2+1)\,d\theta$$

$$= \lim_{n\to\infty} \pi \ln(1+n^2) = \underline{\infty}$$; Integralet divergerer!

6.) $A = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 \geq 1\}$; $\displaystyle\iint_A \frac{1}{(x^2+y^2)^{p}}\,dx\,dy$



Anta $p\neq 1$:

$$\iint_A \frac{1}{(x^2+y^2)^p}\,dx\,dy = \lim_{n\to\infty} \iint_{A\cap B(0,n)} \frac{1}{(x^2+y^2)^p}\,dx\,dy$$

$$= \lim_{n\to\infty} \int_0^{2\pi}\int_1^n \frac{r}{r^{2p}}\,dr\,d\theta$$

polarkoord: $x^2+y^2=r^2$

$$= \lim_{n\to\infty} \int_0^{2\pi}\int_1^n r^{1-2p}\,dr\,d\theta$$

$$= \lim_{n\to\infty} \int_0^{2\pi} \left[\frac{1}{2-2p} r^{2-2p}\right]_{r=1}^{n} d\theta$$

Her skiller $p=1$ seg fra resten!

$$= \lim_{n\to\infty} \int_0^{2\pi} \left(\frac{1}{2-2p} n^{2-2p} - \frac{1}{2-2p}\right) d\theta$$

$$= \lim_{n\to\infty} \frac{\pi}{1-p}\left(n^{2-2p} - 1\right) = \begin{cases} \frac{\pi}{p-1}, & p>1 \\ \infty, & p<1 \end{cases}$$

Konvergerer!

Divergerer!

Hvis $p=1$: $\lim_{n\to\infty} \int_0^{2\pi}\int_1^n \frac{1}{r}\,dr\,d\theta$

$$= \lim_{n\to\infty} \int_0^{2\pi} [\ln(r)]_{r=1}^{n} d\theta = \lim_{n\to\infty} \int_0^{2\pi} \ln(n)\,d\theta$$

$$= \lim_{n\to\infty} 2\pi \ln(n) = \underline{\infty}$$ Divergerer!

Integralet konvergerer for $\underline{\underline{p>1}}$, og divergerer ellers.

6.9: $A = \{(x,y,z) : 0 \le x \le 2,\ 0 \le y \le \sqrt{x},\ -y^2 \le z \le xy\}$

2) b) $\iiint_A z\, dx\, dy\, dz = \int_0^2 \int_0^{\sqrt{x}} \int_{-y^2}^{xy} z\, dz\, dy\, dx$

$= \int_0^2 \int_0^{\sqrt{x}} \left[\frac{1}{2} z^2\right]_{z=-y^2}^{xy} dy\, dx = \int_0^2 \int_0^{\sqrt{x}} \left(\frac{1}{2} x^2 y^2 - \frac{1}{2} y^4\right) dy\, dx$

$= \int_0^2 \left[\frac{1}{6} x^2 y^3 - \frac{1}{10} y^5\right]_{y=0}^{\sqrt{x}} dx$

$\sqrt{x} = x^{\frac{1}{2}}$

$= \int_0^2 \left(\frac{1}{6} x^{\frac{7}{2}} - \frac{1}{10} x^{\frac{5}{2}}\right) dx = \left[\frac{1}{27} x^{\frac{9}{2}} - \frac{1}{35} x^{\frac{7}{2}}\right]_{x=0}^{2}$

$= \frac{1}{27} 2^{\frac{9}{2}} - \frac{1}{35} 2^{\frac{7}{2}} = \frac{16\sqrt{2}}{27} - \frac{8\sqrt{2}}{35} = 8\sqrt{2}\left(\frac{2}{27} - \frac{1}{35}\right)$

$2^{\frac{9}{2}} = \sqrt{2}^9 = \sqrt{2}^8 \sqrt{2} = (\sqrt{2}^2)^4 \sqrt{2} = 2^4\sqrt{2} = 16\sqrt{2}$

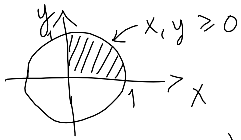
$= \frac{344\sqrt{2}}{945}$

6.10:

2b) $\iiint_A x\, dx\, dy\, dz$, $A = \{(x,y,z) \mid x, y \ge 0,\ z \ge \frac{1}{2},\ x^2 + y^2 + z^2 \le 1\}$

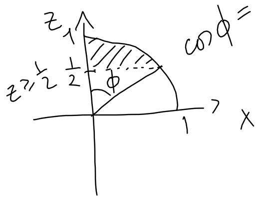
Finner A i kulekoordinater:

xy-plan ($z=0$):



$\theta \in [0, \frac{\pi}{2}]$

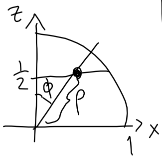
xz-plan ($y=0$):
(Tilsv. yz-plan



$\cos\phi = \frac{1}{2}$
$\Downarrow$
$\phi = \frac{\pi}{3}$

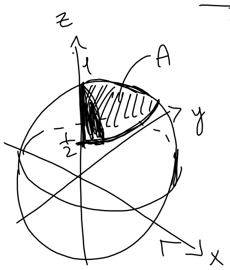
$\phi \in [0, \frac{\pi}{3}]$

Hva med ρ:



$z = \rho\cos\phi \ge \frac{1}{2} \longrightarrow \cos\phi > 0$

$\rho \ge \frac{1}{2\cos\phi}$



I tillegg: $\rho \le 1$ → innenfor kule m/ radius 1

$\rho \in \left[\frac{1}{2\cos\phi}, 1\right]$

$\iiint_A x\, dx\, dy\, dz = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{3}} \int_{\frac{1}{2\cos\phi}}^{1} \overbrace{\rho\cos\theta\sin\phi}^{x}\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

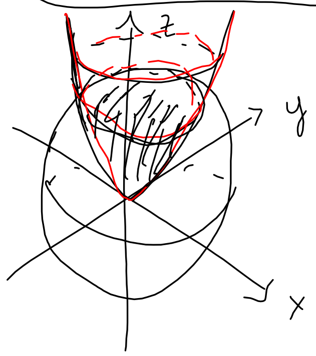
$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{3}} \left[\frac{1}{4}\rho^4\right]_{\rho=\frac{1}{2\cos\phi}}^{1} \sin^2\phi \cos\theta\, d\phi\, d\theta$

$$= \int_0^{\frac{\pi}{2}} \cos\theta \left(\frac{1}{4}\underbrace{\int_0^{\frac{\pi}{3}} \sin^2\phi \, d\phi}_{\cos 2x = 1 - 2\sin^2 x} - \frac{1}{4\cdot 2^4}\underbrace{\int_0^{\frac{\pi}{3}} \frac{\sin^2\phi}{\cos^4\phi} d\phi}_{\frac{\sin^2}{\cos^4} = \frac{\tan^2}{\cos^2}} \right) d\theta$$

Substituer $u = \tan\phi$

3a) $\iiint_A z \, dx\, dy\, dz$

Over paraboloide: $z = x^2 + y^2$
Under kule: $x^2 + y^2 + z^2 = 2$



Skjæring mellom paraboloide & kule:

$$x^2 + y^2 = \sqrt{2 - x^2 - y^2}$$
$$r^2 = \sqrt{2 - r^2}$$
$$r^4 = 2 - r^2$$
$$r^4 + r^2 - 2 = 0$$

$$r^2 = \frac{-1 \pm \sqrt{1 + 4\cdot 2}}{2} = \frac{-1 \pm \sqrt{9}}{2} = \frac{-1 \pm 3}{2} = \begin{cases} 1 \\ -2 \end{cases}$$

$$r^2 = 1 \Rightarrow r = 1 \Rightarrow r \in [0, 1]$$

$$\theta \in [0, 2\pi), \quad z \in [x^2 + y^2, \sqrt{2 - x^2 - y^2}] = [r^2, \sqrt{2 - r^2}]$$

Sylinder koord: $\int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} z \, r \, dz\, dr\, d\theta$