Plenum 6/5-15

4.10: 1f, 2b, 7, 9, 12

4.11: 10

4.10: Egenvektorer

1) f) $A = \begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix}$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 5 & -2 \\ 1 & \lambda - 3 \end{vmatrix} = (\lambda-5)(\lambda-3)+2$$

$$= \lambda^2 - 8\lambda + 17 = 0$$

$$\lambda = \frac{8 \pm \sqrt{64 - 4\cdot 17}}{2} = \frac{8 \pm \sqrt{-4}}{2} = \frac{8 \pm 2i}{2} = \underline{4 \pm i}$$

Egenverdier: $\underline{\lambda_1 = 4+i}$, $\underline{\lambda_2 = 4-i}$

Egenvektorer:

$\vec{v}_1$: $A\vec{v}_1 = \lambda_1 \vec{v}_1 \Rightarrow \lambda_1 \vec{v}_1 - A\vec{v}_1 = 0 \Rightarrow (\lambda_1 I - A)\vec{v}_1 = 0$

$$\begin{bmatrix} -1+i & -2 & 0 \\ 1 & 1+i & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1+i & 0 \\ -1+i & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1+i & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$-2 - (-1+i)(1+i)$
$= -2 - (-1 - \cancel{i} + \cancel{i} - 1)$
$= 0$

$\Rightarrow x = -(1+i)y$, y fri

$$\vec{v}_1 = \begin{bmatrix} -1-i \\ 1 \end{bmatrix}$$

F.eks. $y=1$

$$\tilde{\vec{v}}_1 = \begin{bmatrix} 2 \\ -1+i \end{bmatrix}$$

SJEKK:

$$A\vec{v}_1 = \begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -1-i \\ 1 \end{bmatrix} = \begin{bmatrix} -5-5i+2 \\ 1+i+3 \end{bmatrix} = \begin{bmatrix} -3-5i \\ 4+i \end{bmatrix}$$

$$\lambda_1 \vec{v}_1 = (4+i) \begin{bmatrix} -1-i \\ 1 \end{bmatrix} = \begin{bmatrix} -4-4i-i+1 \\ 4+i \end{bmatrix} = \begin{bmatrix} -3-5i \\ 4+i \end{bmatrix}$$

Tilsv: $\vec{v}_2$: $(\lambda_2 I - A)\vec{v}_2 = 0$

$$\begin{bmatrix} -1-i & -2 & 0 \\ 1 & 1-i & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1-i & 0 \\ -1-i & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1-i & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow x = (-1+i)y$, y fri $\underset{y=1}{\Rightarrow}$ $\vec{v}_2 = \begin{bmatrix} i-1 \\ 1 \end{bmatrix}$

2)b) $A = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 0 & 1 \\ -1 & -1 & 2 \end{pmatrix}$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & -3 & 1 \\ -2 & \lambda & -1 \\ 1 & 1 & \lambda-2 \end{vmatrix}$$

$$= (\lambda-1)\begin{vmatrix} \lambda & -1 \\ 1 & \lambda-2 \end{vmatrix} + 3\begin{vmatrix} -2 & -1 \\ 1 & \lambda-2 \end{vmatrix} + \begin{vmatrix} -2 & \lambda \\ 1 & 1 \end{vmatrix}$$

$$= (\lambda-1)\left(\lambda(\lambda-2)+1\right) + 3\left(-2\lambda+4+1\right) - 2 - \lambda$$

$$= \lambda^3 - 3\lambda^2 - 4\lambda + 12 = 0$$

Gjett: $\lambda = 2$ er en rot $\left(2^3 - 3\cdot 2^2 - 4\cdot 2 + 12 = \cancel{8} - \cancel{12} - \cancel{8} + \cancel{12} = 0\right)$

Polynomdividerer med $\lambda - 2$:

$$\lambda^3 - 3\lambda^2 - 4\lambda + 12 : \lambda - 2 = \underline{\lambda^2 - \lambda - 6}$$
$$\underline{-(\lambda^3 - 2\lambda^2)}$$
$$-\lambda^2 - 4\lambda + 12$$
$$\underline{-(-\lambda^2 + 2\lambda)}$$
$$-6\lambda + 12$$
$$\underline{-(-6\lambda + 12)}$$
$$0$$

abc-formel

$\lambda = -2$
$\lambda = 3$

Så

$$\lambda^3 - 3\lambda^2 - 4\lambda + 12 = (\lambda-2)(\lambda^2 - \lambda - 6) = (\lambda-2)(\lambda+2)(\lambda-3) = 0$$

Egenverdier:

$$\Rightarrow \lambda_1 = -2,\ \underline{\underline{\lambda_2 = 2}},\ \lambda_3 = 3$$

$\vec{v}_1$: $(-2I - A)\vec{v}_1 = 0 \rightarrow \boxed{A\vec{v} = \lambda\vec{v}}$

$$\begin{bmatrix} -3 & -3 & 1 \\ -2 & -2 & -1 \\ 1 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -4 \\ -2 & -2 & -1 \\ -3 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -4 \\ 0 & 0 & -9 \\ 0 & 0 & -11 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} z = 0 \\ x = -y, \ y \text{ er fri} \end{array} \Rightarrow \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$\boxed{y = 1}$

$\vec{v}_2$: $(2I - A)\vec{v}_2 = 0$

$$\begin{bmatrix} 1 & -3 & 1 \\ -2 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 1 \\ 0 & -4 & 1 \\ 0 & 4 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{1}{4} \\ 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} y = \frac{1}{4}z \quad (y - \frac{1}{4}z = 0) \\ x = -\frac{1}{4}z \quad , z \text{ fri} \end{array}$$

$\boxed{\text{F.eks}}$ $z = 4$ $\Rightarrow \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$

$\vec{v}_3$: Løser $(3I - A)\vec{v}_3 = 0$

... $x = -\frac{4}{5}z$, $y = -\frac{1}{5}z$

z er fri

$z = 5$ $\Rightarrow \vec{v}_3 = \begin{bmatrix} -4 \\ -1 \\ 5 \end{bmatrix}$

7) Vis: A og A^T har samme egenverdier.

Bevis: Anta at λ er en egenverdi for A: Da er

$$0 = \det(\lambda I - A) = \det((\lambda I - A)^T) = \det(\lambda I - A^T)$$

def. egenverdi

$\det(B) = \det(B^T)$

$I^T = I$

$\Rightarrow \lambda$ er en egenverdi for A^T. Så alle egenverdier for A er også egenverdier for A^T.

Anta at λ er en egenverdi for A^T: Då er

$$0 = \det(\lambda I - A^T) = \det((\lambda I - A)^T) = \det(\lambda I - A)$$

$\Rightarrow \lambda$ er en egenverdi for A. Så alle egenverdier for A^T er ~~også~~ egenverdier for A.

$\Downarrow$

A og A^T har samme egenverdier.

Ingen grunn til at A og A^T skal ha samme egenvektorer fordi man får to helt ulike ligningssys. å løse

$$\left(\begin{array}{l} (\lambda I - A)\vec{v} = 0 \\ (\lambda I - A^T)\vec{v} = 0 \end{array} \right)$$

Moteks: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ er egenvektor for $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ men ikke for $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

def. egenvektor

9) Bevis: $\vec{v}$ er egenvektor for $A \Rightarrow$ Fins λ_a s.a. $A\vec{v} = \lambda_a \vec{v}$

— " — $B \Rightarrow$ Fins λ_b s.a. $B\vec{v} = \lambda_b \vec{v}$

Men da er: $(AB)\vec{v} = A(B\vec{v}) = A(\lambda_b \vec{v}) = \lambda_b (A\vec{v})$

$$= \lambda_b (\lambda_a \vec{v}) = \lambda_b \lambda_a \vec{v}$$

Så fra def. av egenvektor så er $\vec{v}$ egenvektor for AB med egenverdi $\lambda_b \lambda_a$. ▨

12) Bevis: La A være en $n \times n$ matrise der alle søylene har samme sum:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \quad \text{og} \quad \sum_{j=1}^{n} a_{ji} = \overbrace{k}^{\text{en konst.}} \text{ for alle } i = 1, \dots, n.$$

Ser på:

$$\lambda I - A = \begin{bmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & & & \\ -a_{n1} & \cdots & \cdots & \lambda - a_{nn} \end{bmatrix}$$

Radoperasjoner: Legg alle radene til siste rad i $\lambda I - A$. $\underline{\text{MERK}}$: Dette endrer ikke determinanten!

$$\lambda I - A \sim \begin{bmatrix} \lambda - a_{11} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} \ \cdots & -a_{2n} \\ \vdots & \vdots & \vdots \\ \lambda - \underbrace{\sum_{j=1}^{n} a_{j1}}_{k} & \lambda - \underbrace{\sum_{j=1}^{n} a_{j2}}_{k} \ \cdots & \lambda - \underbrace{\sum_{j=1}^{n} a_{jn}}_{k} \end{bmatrix}$$

$$= \begin{bmatrix} & \text{SOM FØR!} & \\ & & \\ \lambda - k & \lambda - k \ \cdots & \lambda - k \end{bmatrix} \sim \begin{bmatrix} & \text{SOM FØR!} & \\ & & \\ 1 & \cdots & 1 \end{bmatrix} =: C$$

endrer determinant m/ faktor $\lambda - k$

Så:

$$\det(\lambda I - A) = (\lambda - k) \det(C).$$

Dvs. at:

$$0 = \det(\lambda I - A) = (\lambda - k)\det(C)$$

$\lambda - k$ er en faktor i det karakteristiske polynomet til $A \Rightarrow$

k er en egenverdi til A $\boxtimes$

4.11:

10) $\underline{\text{Rov}}$: $x(t)$, $x(0) = 500$
$\underline{\text{Bytte}}$: $y(t)$, $y(0) = 1000$

$$\rightsquigarrow (*) \begin{cases} x'(t) = x(t) + y(t) \\ y'(t) = -x(t) + y(t) \end{cases}$$

$$\vec{r}(t) := \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

Fra ($\star$):

$$\vec{r}\,'(t) = \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}_{:=A} \vec{r}(t)$$

Egenverdier & egenvektorer:

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & -1 \\ 1 & \lambda-1 \end{vmatrix} = \ldots = \lambda^2 - 2\lambda + 2 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2} = \begin{cases} 1+i \to \lambda_1 \\ 1-i \to \lambda_2 \end{cases}$$

Egenvektor:

$\lambda_1 I - A = 0$: $\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \sim \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \Rightarrow$ ($x = -iy$, y fri) ($\boxed{y=1}$) $\vec{v}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$\lambda_2 I - A = 0$: $\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \sim \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix} \Rightarrow$ ($x = iy$, y fri) ($\boxed{y=1}$) $\vec{v}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

Fra arg. s. 292 ($\boxed{?}$): Sett: $\vec{r}(t) = c_1(t)\vec{v}_1 + c_2(t)\vec{v}_2 \quad : (\square)$

$$\Downarrow$$

$c_1'(t) = \lambda_1 c_1(t)$ og $c_2'(t) = \lambda_2 c_2(t)$

$\Downarrow \qquad \Downarrow$

$c_1(t) = C_1 e^{(1+i)t}$ og $c_2(t) = C_2 e^{(1-i)t}$

Fra startbetingelse:

$$\vec{r}(0) = \begin{bmatrix} 500 \\ 1000 \end{bmatrix} = C_1 e^{\overset{1}{0}} \vec{v}_1 + C_2 e^{\overset{1}{0}} \vec{v}_2$$

(☐)

$$= \begin{bmatrix} -C_1 i + C_2 i \\ C_1 + C_2 \end{bmatrix}$$

$$\frac{1}{i} = \frac{i}{i \cdot i} = \frac{i}{-1} = -i$$

$$C_1 = 1000 - C_2 \Rightarrow -(1000 - C_2)i + C_2 i = 500$$

$$(-1000 + 2C_2)i = 500$$

$$-1000 + 2C_2 = -500i$$

$$\underline{C_1 = 500 + 250i} \Leftarrow \underline{C_2 = 500 - 250i}$$

Fra (☐): $x(t) = -(500 + 250i)e^t(\cos t + i\sin t)i$

$$+ (500 - 250i)e^t(\cos t - i\sin t)i$$

$$= \ldots = \underline{\underline{e^t(1000\sin t + 500\cos t)}}$$

$e^{(1+i)t} = e^t(\cos t + i\sin t)$

$e^{(1-i)t} = e^t(\cos(-t) + i\sin(-t)) = e^t(\cos t - i\sin t)$

$$y(t) = (500 + 250i)e^t(\cos t + i\sin t) + (500 - 250i)e^t \cdot$$

$$(\cos t - i\sin t) = \ldots = \underline{\underline{e^t(1000\cos t - 500\sin t)}}$$

b) $\begin{cases} x'(t) = -2x(t) + 4y(t) \\ y'(t) = x(t) - 2y(t) \end{cases}$

$$\vec{r}'(t) = \overbrace{\begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}}^{A} \vec{r}(t)$$

$$\det(\lambda I - A) = \dots = \lambda(\lambda+4) \Rightarrow \underline{\lambda_1 = 0} \text{ og } \underline{\lambda_2 = -4}$$

<u>Egenvektor</u>: $\underline{\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$, $\underline{\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$