Plenum 8/4-15

6.10: 3ae, 6

6.11: 6

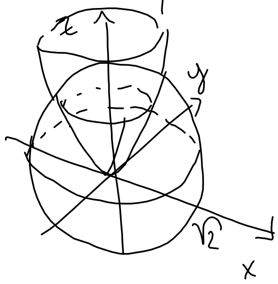
4.1: 4, 6

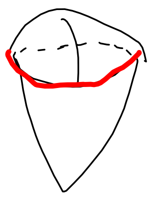
4.2: 4, 10

6.10: Skifte av variable i trippelintegraler

3) a) $\iiint_A z \, dx\,dy\,dz$

A: over paraboloiden $z = x^2 + y^2$, og under kula $x^2 + y^2 + z^2 = 2$:





Skjæring kule & paraboloide:

$$x^2 + y^2 = \sqrt{2 - x^2 - y^2}$$

Sylinderkoord: $\Updownarrow$

$$r^2 = \sqrt{2 - r^2} \iff r^4 + r^2 - 2 = 0$$

2. gradslign. i r^2: $r^2 = \dfrac{-1 \pm \sqrt{1+8}}{2} \Rightarrow r^2 = 1 \Rightarrow \underline{r = 1}$

pos. løsning

Fra parabol... til kule

$$\iiint_A z \, dx\,dy\,dz = \int_0^{2\pi} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} z r \, dz\, dr\, d\theta$$

Sylinder-koord.

$$= \int_0^{2\pi} \int_0^1 \left[\frac{1}{2} z^2 r \right]_{z=r^2}^{\sqrt{2-r^2}} dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(\frac{1}{2}(2-r^2)r - \frac{1}{2}r^5\right) dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(-\frac{1}{2}r^5 - \frac{1}{2}r^3 + r\right) dr\, d\theta$$

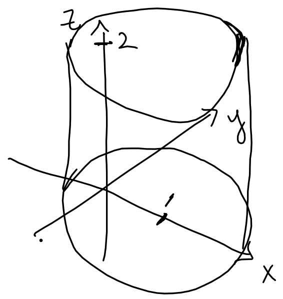
$$= \int_0^{2\pi} \left[-\frac{1}{12}r^6 - \frac{1}{8}r^4 + \frac{1}{2}r^2\right]_{r=0}^{1} d\theta$$

$$= \int_0^{2\pi} \frac{-2-3+12}{24}\, d\theta = \underline{\underline{\frac{7\pi}{12}}}$$

e) $\iiint_A (x^2+y^2)\, dx\, dy\, dz$

A: Sylinder $x^2 - 2x + y^2 = 1$ og $z=0$, $z=2$.

$x^2 - 2x + y^2 = 1 \iff (x-1)^2 + y^2 = \overset{(\sqrt{2})^2}{2}$, sirkel med sentrum i $(1,0)$ og radius $\sqrt{2}$.



Variabelskifte: $u = x-1$, $v = y$, $w = z$

$\Downarrow$

Jacobideterminant lik 1

La: D være sylinderen med sentrum i origo (mhp u,v,w) og radius $\sqrt{2}$ mellom $w=0$ og $w=2$.

$$\iiint_A (x^2+y^2)\,dx\,dy\,dz = \iiint_D \left((u+1)^2+v^2\right)du\,dv\,dw$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}}\int_0^2 \left((r\cos\theta+1)^2 + r^2\sin^2\theta\right) r\,dw\,dr\,d\theta$$

Sylinderkoord:
$u = r\cos\theta$
$v = r\sin\theta$
$w = w$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} \left(2r^3 + 4r^2\cos\theta + 2r\right) dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{2}r^4 + \frac{4}{3}r^3\cos\theta + r^2\right]_{r=0}^{\sqrt{2}} d\theta$$

$$= \int_0^{2\pi} \left(2 + \frac{4}{3}\cdot 2\sqrt{2}\cos\theta + 2\right) d\theta$$

$$= \left[4\theta + \frac{8}{3}\sqrt{2}\sin\theta\right]_{\theta=0}^{2\pi} = \underline{\underline{8\pi}}$$

6) $a > R$. VIS: $\displaystyle\iiint_A \frac{1}{\sqrt{x^2+y^2+(z-a)^2}}\,dx\,dy\,dz = \frac{4\pi R^3}{3a}$

<u>Kulekoordinater</u>: $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$, $\phi\in[0,\pi]$, $\theta\in[0,2\pi]$, $\rho\in[0,R]$

<u>Jacobideterminant</u>: $\rho^2\sin\phi$.

$$\iiint_A \frac{1}{\sqrt{x^2+y^2+(z-a)^2}}\,dx\,dy\,dz = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \frac{\rho^2\sin\phi}{\sqrt{\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta+(\rho\cos\phi-a)^2}}\,d\rho\,d\phi\,d\theta$$

$$=\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\frac{\rho^2\sin\phi}{\sqrt{\rho^2\sin^2\phi+\rho^2\cos^2\phi-2\rho\cos(\phi)a+a^2}}\,d\rho\,d\phi\,d\theta$$

$$=\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\frac{\rho^2\sin\phi}{\sqrt{\rho^2-2\rho\cos(\phi)a+a^2}}\,d\rho\,d\phi\,d\theta$$

$$=2\pi\int_0^R\int_{\rho^2-2\rho a+a^2}^{\rho^2+2\rho a+a^2}\frac{\rho}{2a}\,\frac{1}{\sqrt{u}}\,du\,d\rho$$

$u=\rho^2-2\rho\cos(\phi)a+a^2$

$du=2\rho\sin(\phi)a$

$\phi=0 \Rightarrow u=\rho^2-2\rho a+a^2$

$\phi=\pi \Rightarrow u=\rho^2+2\rho a+a^2$

$$=\frac{\pi}{a}\int_0^R\rho\left[2u^{\frac{1}{2}}\right]_{u=\rho^2-2\rho a+a^2}^{\rho^2+2\rho a+a^2}d\rho$$

$\frac{1}{\sqrt{u}}=u^{-\frac{1}{2}}$

$$=\frac{2\pi}{a}\int_0^R\rho\left(\sqrt{\rho^2+2\rho a+a^2}-\sqrt{\rho^2-2\rho a+a^2}\right)d\rho$$

$$=\frac{2\pi}{a}\int_0^R\rho\left(\sqrt{(\rho+a)^2}-\sqrt{(\rho-a)^2}\right)d\rho$$

$$= \frac{2\pi}{a}\int_0^R \rho\,(\rho+\cancel{a}-\cancel{a}+\rho)\,d\rho$$

$$= \frac{2\pi}{a}\int 2\rho^2\,d\rho = \frac{4\pi}{a}\left[\frac{1}{3}\rho^3\right]_{\rho=0}^{R}$$

$$= \frac{4\pi R^3}{3a}$$

$a > R > 0$;
$\underbrace{\rho + a}_{>0} > 0$

Vet: $\rho \in [0,R]$, $a > R$
$\rho - a < 0$
$\sqrt{(\rho-a)^2} = |\rho - a|$
$= -(\rho - a)$
$= a - \rho$

6.11: Anvendelser

6.) $x^2 + y^2 \leq 1$, $z \in [0,1]$.

$$M = \int_0^{2\pi}\int_0^1\int_0^1 \frac{1}{r^2+z^2}\, r\, dr\, dz\, d\theta$$

Variabel-skifte: sylinderkoord.

$$= \int_0^{2\pi}\int_0^1 \left[\frac{1}{2}\ln(r^2+z^2)\right]_{r=0}^{1} dz\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \frac{1}{2}\left(\ln(1+z^2) - \ln(z^2)\right) dz\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^1 \ln\left(1+\frac{1}{z^2}\right) dz\,d\theta$$

$$= \pi\int_0^1 1\cdot\ln\left(1+\frac{1}{z^2}\right) dz$$

$$= \pi \left[z \ln\left(1+\frac{1}{z^2}\right) + 2\int \frac{\frac{1}{z^2}}{1+\frac{1}{z^2}}\, dz \right]_{z=0}^{1}$$

Delvis int:

$v' = 1$

$v = z$

$u = \ln\left(1+\frac{1}{z^2}\right)$

$u' = \frac{1}{1+\frac{1}{z^2}}\left(-2\frac{1}{z^3}\right)$

$$= \pi \ln 2 + 2\pi \int_0^1 \frac{1}{1+z^2}\, dz$$

$$= \pi \ln 2 + 2\pi \left[\arctan z\right]_{z=0}^{1}$$

$$= \underline{\underline{\pi \ln 2 + \frac{\pi^2}{2}}}$$

4.1: Gauss-eliminasjon

4) $x - 2y + 3z = 1$
$-x + y - 2z = 0$
$-3x + 5y - 8z = 2$

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ -1 & 1 & -2 & 0 \\ -3 & 5 & -8 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & -1 & 1 & 1 \\ -3 & 5 & -8 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Siste ligning: $0 = 4$ $\Rightarrow$ Systemet har ingen løsninger.

USANT!