Implisitt funksjonsteorem.

$x \in \mathbb{R}^m \quad y \in \mathbb{R} \quad f(x,y)$

$f(x_0, y_0) = 0$ og $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0.$

Da fins funksjon $g : \mathbb{R}^m \to \mathbb{R}$ (g definert for x nær x_0).

s.a. $f(x, g(x)) = 0$

$$\frac{\partial g(x)}{\partial x_i} = - \frac{\frac{\partial f}{\partial x_i}(x, g(x))}{\frac{\partial f}{\partial y}(x, g(x))}$$

$f(x,y) = e^{x+y} + y - 1 \quad x \in \mathbb{R}, y \in \mathbb{R}.$

$f(0,0) = e^0 + 0 - 1 = 0.$

$\frac{\partial f}{\partial y}(x,y) = e^{x+y} + 1 \quad \frac{\partial f}{\partial y}(0,0) = 1 + 1 = 2 \neq 0.$

Da fins g s.a. g definert for x nær 0.

og $f(x, g(x)) = 0$ eller $e^{x+g(x)} + g(x) - 1 = 0.$

$\frac{\partial f}{\partial x} = e^{x+y}$

$$\frac{\partial g}{\partial x}(0) = \frac{-\frac{\partial f}{\partial x}(0,0)}{\frac{\partial f}{\partial y}(0,0)} = -\frac{1}{2}$$

Paraboloide: $x^2 + \frac{y^2}{4} - z^2 - 1 = 0. \qquad z = \pm\sqrt{1 - x^2 - y^2/4}$

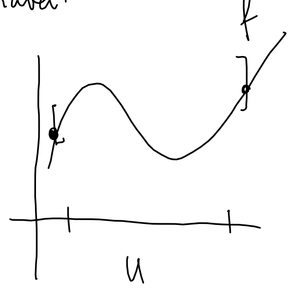
Se på $f(x,y,z) = x^2 + y^2/4 - z^2 - 1 \qquad (x,y,z) = (1,2,1) \quad f(1,2,1) = 0 \quad \frac{\partial f}{\partial z}(1,2,1) = -2(1) = -2 \neq 0.$

Vet at det fins $z = z(x,y)$ s.a. $f(x,y,z(x,y)) = 0.$

$$\frac{\partial z}{\partial x}(1,2,1) = -\frac{\frac{\partial f}{\partial x}(1,2,1)}{\frac{\partial f}{\partial y}(1,2,1)} = -\frac{2}{-2} = 1 \quad , \quad \frac{\partial z}{\partial y}(1,2,1) = -\frac{\frac{\partial f}{\partial y}(1,2,1)}{\frac{\partial f}{\partial z}(1,2,1)} = -\frac{1}{-2} = \frac{1}{2}$$

Max/min til funksjoner av flere variable.

Én variabel:



① Max/min "inne" i U, løser $f'(x)=0$ og ser på $f(x)$.

② Max/min på randen av U. Sjekker på randen.

Flere dimensjoner.

$U \subseteq \mathbb{R}^m$

$f: U \to \mathbb{R}$ kalles <u>begrenset</u> dersom det fins $M < \infty$ s.a. $|f(x)| < M$ for alle $x \in U$.

<u>U</u> kalles begrenset dersom $|x| < M$ for alle $x \in U$.

$c \in U$ c kalles max hvis $f(x) \leq f(c)$ for alle $x \in U$

c kalles min hvis $f(x) \geq f(c)$ for alle $x \in U$.

Teorem:

U begrenset, lukket $U \subseteq \mathbb{R}^m$ $f: U \to \mathbb{R}$, f kontinuerlig.

Da har f max og min i U.

(Det fins $c \in U$ s.a. $f(c) = \max_{x \in U} f(x)$, ...).

Bevis

$M = \sup_{x \in U} f(x)$. ($\sup$ = "minste øvre grense")

M kan være ∞.

Da kan vi finne $x_n \in U$ s.a. $\lim_{n\to\infty} f(x_n) = M$.

$x_n \in U$; lukket & begrenset; $\{x_n\}_{n\geq 1}$ har konvergent delfølge: $x_{n_k} \to c \in U$

$M = \lim_{k\to\infty} f(x_{n_k}) = f\left(\lim_{k\to\infty} x_{n_k}\right) = f(c)$; siden f kont. $f(c) < \infty$.

Lokalt max.

c kalles lokalt max hvis $f(c) \geq f(x)$ for $|x - c| < r$ for en $r > 0$.

c kalles lokalt min hvis $f(c) \leq f(x)$ for alle $x \in B_r(c)$ for $r > 0$.

Hvordan finne max/min?

Lete etter punkter der $f'(x) = 0$.

Hvorfor?

Anta at c er lokalt max. $x \in \mathbb{R}^m$ $x = (x_1, x_2 \dots x_m)$

$c = (c_1, c_2 \dots, c_m)$.

$t \longmapsto f(c_1, c_2, \dots c_{i-1}, t, c_{i+1} \dots c_m) = g(t)$.

$g(t)$ har lokalt max for $t = c_i$

$g'(c_i) = 0$ $\quad g'(t) = \frac{\partial f}{\partial x_i}(c_1, \dots t \dots c_m)$

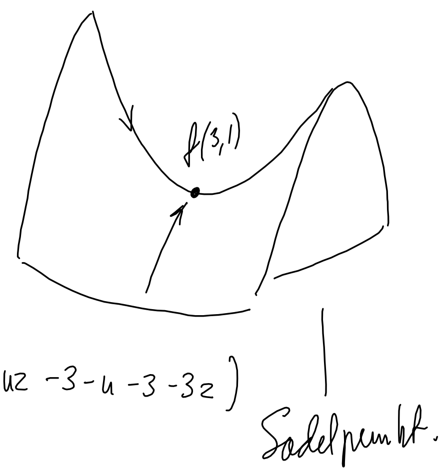
$0 = \frac{\partial f}{\partial x_i}(c_1, \dots c_i \dots c_m)$ Dette gjelder for all $i = 1, \dots, m$

$f'(c) = \nabla f(c) = 0$.

c max/min $\Rightarrow \nabla f(c) = 0$.

$f(x,y) = 3xy - 3x - 9y \qquad U = \mathbb{R}^2$

$$\nabla f(x,y) = \begin{pmatrix} 3y - 3 \\ 3x - 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{matrix} y = 1. \\ x = 3. \end{matrix}$$



Sjekker om max eller min.

$f(3+u, 1+z) = 3(3+u)(1+z) - 3(3+u) - 9(1+z) = 3(3 + u + 3z + uz - 3 - u - 3 - 3z)$

$= 3(-3 + uz) = 3uz - 9 \qquad 9 = f(3,1)$.

$uz > 0$ $f(3+u, 1+z) > f(3,1)$ $\quad uz < 0$ $f(3+u, 1+z) < f(3,1)$.

$f: \mathbb{R}^n \to \mathbb{R}$, $f'(x)$ vektor i $\mathbb{R}^n$: $\mathbb{R}^n \to \mathbb{R}^n$, $f''(x)$ $n \times n$ matrise : $\mathbb{R}^n \to$ $n \times n$ matriser

symmetriske matriser.

Taylor's formel

$f: \mathbb{R}^n \to \mathbb{R}$ 2 ganger kontinuerlig deriverbar

$f(a+y) = f(a) + \nabla f(a) \cdot y + \frac{1}{2}(f''(a+cy)y) \cdot y$. $c \in [0,1]$ tall.

Hesse-matrisen.

$Hf(a) = f''(x) =$ ← matrise med $\frac{\partial^2 f}{\partial x_i \partial x_j}$ i i-te rad $j$te kolonne.

Siden $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$ så er $Hf(a)$ symmetrisk.

Eks $f(x,y,z) = x^2 + yz + e^y$ $f'(x,y,z) = \nabla f(x,y,z) = \begin{pmatrix} 2x \\ z + e^y \\ y \end{pmatrix}$, $f''(x,y,z) = Hf(x,y,z) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & e^y & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Bevis for Taylor:

$g(t) = f(a+ty) \qquad t \in [0,1]. \qquad g : [0,1] \to \mathbb{R}$

$$g(1) = g(0) + \int_0^1 g'(y)\,dy = g(0) - \int_0^1 (1-y)'\, g'(y)\,dy = g(0) - (1-y)g'(y)\Big|_0^1 + \int_0^1 (1-y)g''(y)\,dy$$

$$= g(0) + g'(0) + \int_0^1 (1-y)g''(y)$$

; anta g'' kontinuerlig på $[0,1]$.

da er $m \le g''(y) \le M \quad y \in [0,1]. \quad M = \max g'' \quad m = \min g''$

$$\frac{m}{2} = m\int_0^1 (1-y)\,dy \le \int_0^1 (1-y)g''(y)\,dy \le M\int_0^1 (1-y)\,dy = \frac{M}{2}$$

$$\Downarrow$$

$\frac{1}{2}g''(c)$ for en $c \in [0,1]$.

$$= g(0) + g'(0) + \frac{1}{2}g''(c)$$

$c \in [0,1]$.

$g(t) = f(a+ty)$

$$g'(t) = \frac{d}{dt} f(a+ty) = \frac{d}{dt} f(a_1+ty_1, a_2+ty_2, \ldots, a_n+ty_n)$$

$$= \frac{\partial f}{\partial x_1}(a+ty)\,y_1 + \frac{\partial f}{\partial x_2}(a+ty)\,y_2 + \cdots + \frac{\partial f}{\partial x_n}(a+ty)\,y_n = \nabla f(a+ty)\cdot y$$

$$g''(t) = \frac{d}{dt}\left(\sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(a+ty)\,y_j\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(a+ty)\,y_j\,y_i = \sum_{i=1}^{n}\sum_{j=1}^{n} y_i \frac{\partial^2 f}{\partial x_i \partial x_j}(a+ty)\,y_j = y^T Hf(a+ty)\,y$$

$$g(1) = g(0) + g'(0) + \frac{1}{2}g''(c).$$

$$f(a+ty) = f(a) + \nabla f(a)\cdot y + \frac{1}{2}\underbrace{y^T Hf(a+cy)\,y}_{}$$

Samme som $(Hf(a+cy)y)\cdot y \qquad (f''(a+cy)y)\cdot y$

$$Hf(a+cy) = Hf(a) + \underbrace{(Hf(a+cy) - Hf(a))}_{A(y).}$$; hvis f'' er kontinuerlig, så vil

$Hf(a+cy) - Hf(a) \to 0$ når $|y| \to 0$.

Kan skrive Taylor's formel:

$$\boxed{f(a+y) = f(a) + \nabla f(a)\cdot y + \frac{1}{2}(Hf(a)y)\cdot y + \underbrace{\frac{1}{2}A(y)y\cdot y}_{\mathcal{E}(y).}}$$

Spesiell i et stasjonært punkt. $\nabla f(a) = 0$.

$$f(a+y) = f(a) + \frac{1}{2}(Hf(a)y)\cdot y + \mathcal{E}(y)$$ liten nær 0

$$\lim_{y\to 0} \frac{|\mathcal{E}(y)|}{|y|^2} = 0.$$

fordi:

$$\frac{\mathcal{E}(y)}{|y|^2} \le \frac{(A(y)\cdot y)\cdot y}{|y|^2} \le \frac{|A(y)|\cancel{|y|^2}}{\cancel{|y|^2}}$$

$\le |A(y)| \to 0$ når $y \to 0$.

Siden f'' er kont.

Husk $Hf = f''$ er symmetrisk matrise

Alle egenverdiene til Hf er reelle, det fins en basis av egenvektorer.

Egenverdier $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ $\quad r_1, r_2 \ldots r_n$ ortonormale egenvektorer.

$$(Hf(a)y) \cdot y$$

$$= \sum_{i=1}^{n} c_i \lambda_i r_i \cdot \sum_{j=1}^{n} c_j r_j$$

$$= \sum_{i=1}^{n} \lambda_i c_i^2 \quad \text{siden } r_i \cdot r_j = \begin{cases} 1 & i = j \\ 0 & \text{ellers.} \end{cases}$$

$$y = c_1 r_1 + c_2 r_2 + \ldots + c_n r_n$$

$$Hf(a)y = Hf(a) c_1 r_1 + Hf(a) c_2 r_2 + \ldots + Hf(a) c_n r_n$$

$$= c_1 \lambda_1 r_1 + c_2 \lambda_2 r_2 + \ldots + c_n \lambda_n r_n$$

$$f(y+a) = f(a) + \frac{1}{2} \sum_{i=1}^{n} \lambda_i c_i^2 + \varepsilon(y).$$