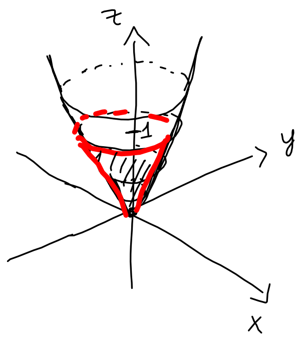
Plenum 8/3-16

6.4: 6, 7, 15

6.5: 2, 8, 10

6.4: 6) $z^2 = x^2 + y^2$, $z \in [0,1]$



Parametrisering: $(x, y, \sqrt{x^2+y^2})$

fordi $z \in [0,1]$

$$\frac{\partial z}{\partial x} = \frac{1}{\cancel{2}\sqrt{x^2+y^2}}\cancel{2}x = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

Areal = (s.595)

$$\iint_A \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\,dy = \iint_A \sqrt{1 + \left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}\, dx\,dy$$

$$= \iint_A \sqrt{1 + \frac{x^2+y^2}{x^2+y^2}}\, dx\,dy = \iint_A \sqrt{2}\, dx\,dy$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^1 r\, dr\, d\theta = \sqrt{2}\, 2\pi \left[\frac{1}{2} r^2\right]_{r=0}^{1}$$

$$= \frac{\sqrt{2}\,\cancel{2}\pi}{\cancel{2}} = \underline{\underline{\sqrt{2}\,\pi}}$$

Hva er A?

$N_c = \{(x,y) \mid \sqrt{x^2+y^2} = c\}$

$x^2 + y^2 = c^2$, sirkel sentrum origo, radius c

Vet: $z \in [0,1] \Rightarrow c \in [0,1]$

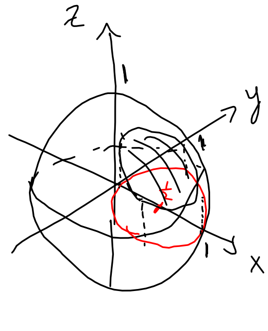
7.) $x^2 + y^2 + z^2 = 1$

$(x - \frac{1}{2})^2 + y^2 \leq \frac{1}{4}$

$(x - \frac{1}{2})^2 + y^2 \leq \left(\frac{1}{2}\right)^2$

sirkel m/ radius $\frac{1}{2}$

sentrum $(\frac{1}{2}, 0)$



Flaten er beskrevet ved: $z = f(x, y) = \sqrt{1 - x^2 - y^2}$

$$\frac{\partial f}{\partial x} = \frac{1}{\cancel{2}\sqrt{1-x^2-y^2}}(-\cancel{2}x) = \frac{-x}{\sqrt{1-x^2-y^2}}$$

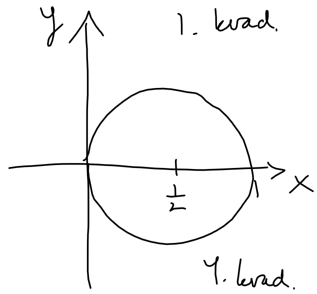
$$\frac{\partial f}{\partial y} = \frac{-y}{\sqrt{1-x^2-y^2}}$$

Formel s. 595:

$$\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} = \sqrt{1 + \frac{x^2}{1-x^2-y^2} + \frac{y^2}{1-x^2-y^2}}$$

$$= \sqrt{\frac{1}{1-x^2-y^2}} = \frac{1}{\sqrt{1-x^2-y^2}}$$

Grenser for integrasjon i polarkoordinater:



Sirkel: $\left(r\cos\theta - \frac{1}{2}\right)^2 + r^2\sin^2\theta \le \frac{1}{4}$

$$\underbrace{r^2\cos^2\theta + r^2\sin^2\theta}_{r^2(\underbrace{\cos^2\theta+\sin^2\theta}_{1})} - r\cos\theta + \cancel{\tfrac{1}{4}} \le \cancel{\tfrac{1}{4}}$$

$$r^2 \le r\cos\theta$$

$$\boxed{r \le \cos\theta}$$

Sirkel er i 1. og 4. kvadrant: $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Flateint: $$\iint_A \frac{1}{\sqrt{1-x^2-y^2}}\,dx\,dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\cos\theta} \frac{1}{\sqrt{1-r^2}}\, r\,dr\,d\theta$$

polar-koordinat

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[-\sqrt{1-r^2} \right]_{r=0}^{\cos\theta} d\theta$$

$u = 1 - r^2$ substitusjon

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 - \sqrt{\underbrace{1-\cos^2\theta}_{\sin^2\theta}}\right) d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - |\sin\theta|)\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\, d\theta - \int_{-\frac{\pi}{2}}^{0} |\sin\theta|\, d\theta - \int_{0}^{\frac{\pi}{2}} |\sin\theta|\, d\theta$$

$$= \pi - \int_{-\frac{\pi}{2}}^{0} (-\sin\theta)\, d\theta - \int_{0}^{\frac{\pi}{2}} \sin\theta\, d\theta$$

$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

$$= \pi + \left[-\cos\theta\right]_{\theta=-\frac{\pi}{2}}^{0} - \left[-\cos\theta\right]_{\theta=0}^{\frac{\pi}{2}}$$

$$= \pi - 1 - 1 = \underline{\underline{\pi - 2}}$$

15.) $x^2 + y^2 + z^2 = 1 \;\to\; z = \sqrt{1-x^2-y^2}$, $f(x,y,z) = 1$

$$\iint_T x \cdot 1\, d\sigma = \iint_A x \left| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right| (x,y)\, dx\, dy$$

$$= \iint_A x \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy$$

s. 595

forrige oppg.

$$= \iint_A \frac{x}{\sqrt{1-x^2-y^2}}\, dx\, dy = \int_0^{2\pi}\int_0^1 \frac{r^2\cos\theta}{\sqrt{1-r^2}}\, dr\, d\theta$$

$$= \int_0^1 \int_0^{2\pi} \cos\theta\, d\theta \frac{r^2}{\sqrt{1-r^2}}\, dr = \int_0^1 \left[\sin\theta\right]_{\theta=0}^{2\pi} \frac{r^2}{\sqrt{1-r^2}}\, dr$$

$$= \int_0^1 0\, dr = \underline{0}$$

$$\iint_T y \cdot 1 \, d\sigma \underset{\text{Tilsv.}}{=} \underline{0}$$

$$\iint_T z \cdot 1 \, dS = \iint_A \frac{\sqrt{1-x^2-y^2}}{\sqrt{1-x^2-y^2}} \, dx\,dy = \iint_A 1 \, dx\,dy$$

$$\underset{\text{polar-koord.}}{=} \int_0^{2\pi}\int_0^1 r \, dr\, d\theta = 2\pi \left[\frac{1}{2} r^2\right]_{r=0}^{1} = \underline{\pi}$$

$u = 1-r^2$ $\quad r=0 \Rightarrow u=1$

$du = -2r\,dr$ $\quad r=1 \Rightarrow u=0$

$$\iint_T 1 \, d\sigma = \int_0^{2\pi}\int_0^1 \frac{r}{\sqrt{1-r^2}} \, dr\, d\theta = 2\pi \left[\sqrt{u}\right]_{u=0}^{1} = \underline{2\pi}$$

Massemiddelpunkt: $(\bar{x}, \bar{y}, \bar{z}) = \underline{\underline{\left(0, 0, \tfrac{1}{2}\right)}}$

6.5: 2) $\vec{r}(t) = (t\sin(t), 2\pi t - t^2), \; t \in [0, 2\pi]$

Fra MATLAB: $\mathcal{C}$ enkel, lukket, stykkvis glatt og pos. orientert $\Rightarrow$ OK å bruke Greens teorem!

Sett: $\underbrace{P(x,y) := 0}_{\frac{\partial P}{\partial y} = 0}$ og $\underbrace{Q(x,y)}_{\frac{\partial Q}{\partial x} = 1} := x$

Areal av område avgr. av kurven $= \iint_R 1 \, dx\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$

$= \int_{\mathcal{C}} P\,dx + Q\,dy = \int_{\mathcal{C}} x\,dy = \int_0^{2\pi} t \sin(t)(2\pi - 2t)\,dt$

Greens teorem

Def P og Q

parametrisering

$y = 2\pi t - t^2$

$dy = (2\pi - 2t)\,dt$

$= 2\pi \int_0^{2\pi} \overset{u}{t} \overset{v'}{\sin(t)}\,dt - 2 \int_0^{2\pi} \overset{u}{t^2} \overset{v'}{\sin(t)}\,dt$

Delvis int.

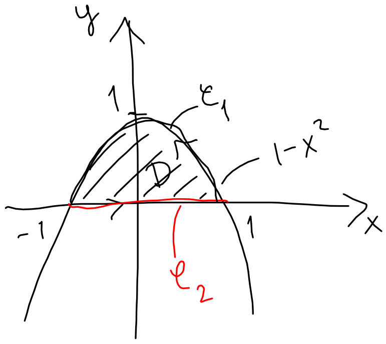
$= 2\pi \left[-t\cos t + \int \cos(t)\,dt \right]_{t=0}^{2\pi} - 2\left[-t^2 \cos t + \int 2t \cos(t)\,dt \right]_{t=0}^{2\pi}$

Delvis integrasjon

$= 2\pi \left[-t\cos t + \int \cos(t)\,dt \right]_0^{2\pi} - 2\left[-t^2\cos t + 2t\sin t + 2\cos(t) \right]_{t=0}^{2\pi}$

$= -4\pi^2 - 2(-4\pi^2 + 2) + 4 = \underline{\underline{4\pi^2}}$

8) $y = 1 - x^2$



a) $I = \int_{\mathcal{C}} -y\,dx + x^2 dy$

$\mathcal{C}_1: (x, 1-x^2), \quad x \in [1, -1]$

$\mathcal{C}_2: (x, 0), \quad x \in [-1, 1]$

$$\int_{\mathcal{C}_1} -y\,dx + x^2 dy = \int_1^{-1} \Big(-(\underbrace{1-x^2}_{y}) + x^2(-2x)\Big)dx$$

$$= \int_1^{-1} (-2x^3 + x^2 - 1)\,dx = \Big[-\frac{1}{2}x^4 + \frac{1}{3}x^3 - x\Big]_{x=1}^{-1}$$

$$= -\frac{1}{3} - \frac{1}{3} + 1 + 1 = \underline{\frac{4}{3}}$$

$$\int_{\mathcal{C}_2} -y\,dx + x^2 dy = \int_{-1}^{1} (0 + x^2 \cdot 0)\,dx = \underline{0}$$

Så: $$\int_{\mathcal{C}} -y\,dx + x^2 dy = \int_{\mathcal{C}_1} -y\,dx + x^2 dy + \int_{\mathcal{C}_2} -y\,dx + x^2 dy$$

$$= \underline{\underline{\frac{4}{3}}}$$

b) Velg: $P(x,y) = -y, \quad Q(x,y) = x^2$

$$\frac{\partial P}{\partial y} = -1, \qquad \frac{\partial Q}{\partial x} = 2x$$

Husk: $y = 1 - x^2$ skjærer x-aksen i -1 og 1.

$$\int_{\mathcal{C}} -y\,dx + x^2 dy = \iint_R \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\,dxdy$$

$\mathcal{C}$ enkel, lukket, stykkevis gl., og orientert mot klokka → Greens!

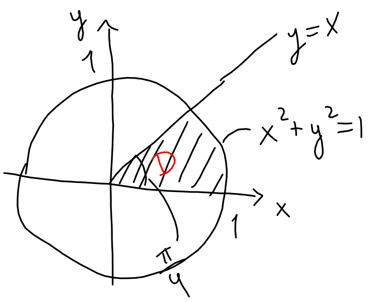
$$= \int_{-1}^{1}\int_0^{1-x^2} (2x+1)\,dy\,dx$$

$$= \int_{-1}^{1} (1-x^2)(2x+1)\,dx = \int_{-1}^{1} (2x - 2x^3 + 1 - x^2)\,dx$$

$$= \left[x^2 - \frac{1}{2}x^4 + x - \frac{1}{3}x^3 \right]_{x=-1}^{1} = 1 + 1 - \frac{1}{3} - \frac{1}{3} = \underline{\underline{\frac{4}{3}}}$$

10.) $D = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1,\ x \geq 0,\ y \geq 0,\ 0 \leq y \leq x \}$

a)



$x = r\cos\theta \quad , r \in [0, 1]$

$y = r\sin\theta \quad , \theta \in [0, \frac{\pi}{4}]$