Eksempel; finn max/min.

$A = \{(x,y) \mid x^2+y^2 \leq 1\}$ $f(x,y) = x^2 - y^3$; Finn max/min for f når $(x,y) \in A$.

① Lete etter ekstrempunkter i det indre av A.

$\nabla f(x,y) = 0$. $\nabla f = (2x, -3y^2) = 0$. $\Rightarrow x=0, y=0$. $\boxed{f(0,0)=0}$

② Lete på randen: Rand gitt ved $x^2+y^2=1$. $g(x,y) = x^2+y^2$. $\nabla g(x,y) = (2x, 2y)$.

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 1 \end{cases} \rightarrow \begin{matrix} 2x = \lambda 2x \\ -3y^2 = \lambda 2y \\ x^2+y^2 = 1 \end{matrix}$$

$x=0$: $y^2 = 1$, $y = \pm 1$, $\boxed{(0,\pm 1)}$

$x \neq 0$: $\lambda = 1$. $-3y^2 = 2y$

$\rightarrow y = 0$: $x = \pm 1$, $\boxed{(\pm 1, 0)}$

$\rightarrow y \neq 0$: $-3y = 2$, $y = -\frac{2}{3}$. $x^2 = 1 - (\frac{2}{3})^2 = \frac{5}{9}$, $x = \pm \frac{\sqrt{5}}{3}$, $\boxed{\left(\pm\frac{\sqrt{5}}{3}, -\frac{2}{3}\right)}$

$f(0,1) = 0^2 - 1^3 = \boxed{-1}$ MIN

$f(0,-1) = 0^2 - (-1)^3 = \boxed{1}$ MAX

$f(1,0) = 1^2 - 0^3 = 1$

$f(-1,0) = 1^2 = 1$

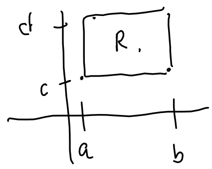
$f\left(+\frac{\sqrt{5}}{3}, -\frac{2}{3}\right) = \frac{5}{9} - \left(-\frac{2}{3}\right)^3 = \frac{5}{9} + \frac{8}{27} = \frac{15}{27} + \frac{8}{27} = \frac{23}{27}$

$f\left(-\frac{\sqrt{5}}{3}, -\frac{2}{3}\right) = \frac{23}{27}$

Integrasjon av funksjoner av flere variable.

i $\mathbb{R}^2$ (områder i $\mathbb{R}^2$).

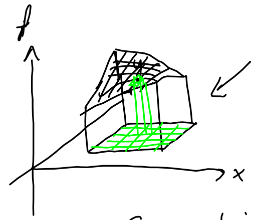
Rektangel $[a,b]\times[c,d]=R$



$f: R \to \mathbb{R}$, kontinuerlig. $f(x,y)>0$.

$x_i = a + i\Delta x \qquad \Delta x = \frac{b-a}{N}$

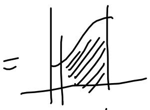
$y_j = c + j\Delta y \qquad \Delta y = \frac{d-c}{N}$



$$V \simeq \sum_{i,j=1}^{N} \underset{\uparrow \atop \text{høyde}}{f(x_i,y_j)}\,\underbrace{\Delta x\,\Delta y}_{\text{grunnflate}} \approx \iint_R f(x,y)\,dx\,dy$$

Dobbeltintegral.

Sammenlign enkeltintegral:

$\int_a^b f(x)\,dx =$ 

Hvordan regne ut?

$$\int_c^d h(y)\,dy$$

$$\iint_R f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$

Eksempel:

$R = [0,1]\times[0,1] \qquad f(x,y) = x^2 + xy + y^2$

$$\iint_R x^2 + xy + y^2 \, dx\,dy = \int_0^1 \left[\int_0^1 x^2 + xy + y^2 \, dx \right] dy = \int_0^1 \left(\left(\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 y + xy^2 \right) \Big|_0^1 \right) dy$$

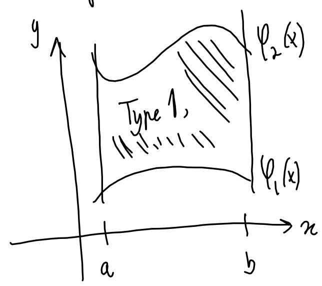
$$= \int_0^1 \tfrac{1}{3} + \tfrac{1}{2}y + y^2 \, dy = \left(\tfrac{1}{3}y + \tfrac{1}{4}y^2 + \tfrac{1}{3}y^3 \right) \Big|_0^1 = \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{3}$$

Teorem

$R = [a,b]\times[c,d]$ f kontinuerlig $f: R \to \mathbb{R}$.

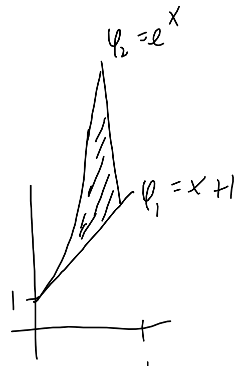
$$\int_a^b \left[\int_c^d f(x,y)\, dy \right] dx = \int_c^d \left[\int_a^b f(x,y)\, dx \right] dy$$

Integraler over andre områder enn rektangler.



$A = \{ (x,y) \mid a \le x \le b, \ \varphi_1(x) \le y \le \varphi_2(x) \}$.

$$\iint_A f(x,y)\, dx\,dy = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\, dy \right] dx$$



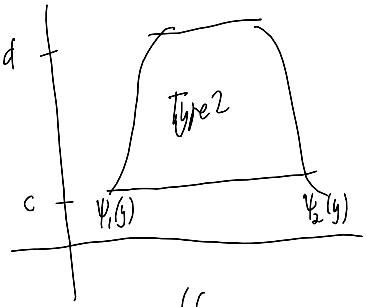
Eksempel: $A = \{ (x,y) \mid 0 \le x \le 1 \quad 1+x \le y \le e^x \}$

$f(x,y) = x + y^2$

$$\iint_A f(x,y)\, dx = \int_0^1 \left[\int_{1+x}^{e^x} x + y^2 \, dy \right] dx = \int_0^1 \left(\left(xy + \tfrac{1}{3}y^3 \right) \Big|_{y=1+x}^{y=e^x} \right) dx$$

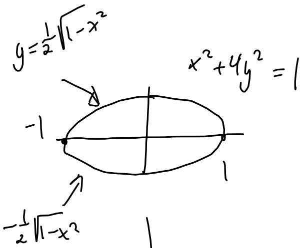
$$= \int_0^1 xe^x - x(1+x) + \tfrac{1}{3}e^{3x} - \tfrac{1}{3}(1+x)^3 \, dx = \ldots$$

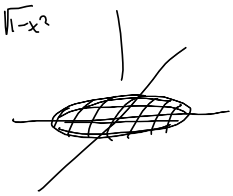
Type 2



$$A = \{(x,y) \mid c \le y \le d,\ \psi_1(y) \le x \le \psi_2(y)\}.$$

$$\iint_A f(x,y)\,dx\,dy = \int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx \right] dy$$





Eksempel

Volum av området $\{(x,y,z) \mid 0 \le z \le 1 - x^2 - 4y^2\}$.

$$V = \int_{-1}^{1} \left[\int_{-\frac{1}{2}\sqrt{1-x^2}}^{\frac{1}{2}\sqrt{1-x^2}} (1 - x^2 - 4y^2)\,dy \right] dx = \ldots\ldots$$