Plenum

6.1: 1, 7

6.2: 3

6.3: 4

6.4: (3)

6.1: 1) e) $\iint_R xye^{x^2y}\,dxdy = \int_1^2\int_0^2 xye^{x^2y}\,dxdy$

$$= \int_1^2 \left[\tfrac{1}{2}e^{x^2y}\right]_{x=0}^{2} dy = \int_1^2 \left(\tfrac{1}{2}e^{4y} - \tfrac{1}{2}\right) dy$$

$$= \left[\tfrac{1}{8}e^{4y} - \tfrac{1}{2}y\right]_{y=1}^{2} = \tfrac{1}{8}e^{8} - 1 - \left(\tfrac{1}{8}e^{4} - \tfrac{1}{2}\right)$$

$$= \underline{\underline{\tfrac{1}{8}e^{8} - \tfrac{1}{8}e^{4} - \tfrac{1}{2}}}$$

Regneregel logaritme

f) $\iint_R \ln(xy)\,dxdy = \int_1^e\int_1^e \ln(xy)\,dxdy = \int_1^e\int_1^e \left(\ln(x) + \ln(y)\right)dxdy$

TRIKS!

Delvis int.
$u' = 1$
$v = \ln(x) \Rightarrow u = x$
$v' = \frac{1}{x}$

$$= \int_1^e \left(\left[x\ln(x)\right]_{x=1}^{e} - \int_1^e 1\,dx + \left[x\ln(y)\right]_{x=1}^{e}\right) dy$$

$$= \int_1^e \left(\cancel{e} - 0 - \cancel{e} + 1 + (e-1)\cdot\ln(y)\right) dy$$

$$= \int_1^e \left(1 + (e-1)\ln(y)\right) dy = \left[y\right]_{y=1}^{e} + (e-1)\left[y\ln(y)\right]_{y=1}^{e} - (e-1)\int_1^e 1\,dy$$

$$= e - 1 + (e-1)e - (e-1)^2 = \cancel{e} - 1 + \cancel{e^2} - \cancel{e} - \cancel{e^2} + 2e - 1$$

$$= 2e - 2 = \underline{2(e-1)}$$

g) $$\iint_R \frac{1}{1+x^2y}\,dx\,dy = \int_0^1\int_1^{\sqrt{3}} \frac{1}{1+x^2y}\,dx\,dy = I$$

<u>M</u>: $$\int \frac{1}{1+x^2y}\,dx = \int \frac{1}{\sqrt{y}\,(1+u^2)}\,du = \frac{1}{\sqrt{y}}\arctan(u)+C$$

$$= y^{-\frac{1}{2}}\arctan(\sqrt{y}\,x)+C$$

<u>Substitusjon:</u>
$u = x\sqrt{y}$
$du = \sqrt{y}\,dx$
$\frac{du}{\sqrt{y}} = dx$

$$I = \int_0^1 \left[y^{-\frac{1}{2}} \arctan(x\sqrt{y}) \right]_{x=1}^{\sqrt{3}} dy$$

$$= \int_0^1 \left(y^{-\frac{1}{2}} \left(\arctan(\sqrt{3}\sqrt{y}) - \arctan(\sqrt{y}) \right) \right) dy$$

$$= 2\int_0^1 \left(\arctan(\sqrt{3}u) - \arctan(u) \right) du$$

<u>Substitusjon:</u>
$u = \sqrt{y}$
$du = \frac{1}{2} y^{-\frac{1}{2}} dy$
$2y^{\frac{1}{2}}\,du = dy$

$y = 0 \Rightarrow u = 0$
$y = 1 \Rightarrow u = 1$

$$= 2\left[u\left(\arctan(\sqrt{3}u) - \arctan(u) \right) \right]_{u=0}^{1} - 2\int_0^1 \left(\frac{\sqrt{3}u}{1+3u^2} - \frac{u}{1+u^2} \right) du$$

Delvis <u>int.</u> $v' = 1$
$w = \arctan(\sqrt{3}u) - \arctan(u)$

$$= 2\left(\arctan(\sqrt{3}) - \arctan(1) \right) - 2\left[\frac{\sqrt{3}}{6}\ln(1+3u^2) - \frac{1}{2}\ln(1+u^2) \right]_{u=0}^{1}$$

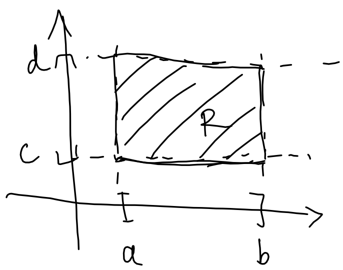
$\arctan(0) = 1$ siden $\tan(0) = \frac{\sin 0}{\cos 0} = 0$

$$= 2\left(\frac{\pi}{3} - \frac{\pi}{4} \right) - 2\,\frac{\sqrt{3}}{6}\ln(4) + 2\,\frac{1}{2}\ln(2)$$

$$= \frac{\pi}{6} + \left(1 - \frac{2\sqrt{3}}{3} \right)\ln(2)$$

ϕ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	ikke def.

7) $f : R \to \mathbb{R}$, kont.



VIS: Fins pkt. $(\bar{x}, \bar{y}) \in R$ s.a.

$$\frac{\iint_R f(x,y)\,dx\,dy}{|R|} = f(\bar{x}, \bar{y})$$

Bevis: La $m := \min_{(x,y)\in R} f(x,y)$ og $M := \max_{(x,y)\in R} f(x,y)$.

Da er:

$$\iint_R f(x,y)\,dx\,dy \le \iint_R M\,dx\,dy = M \iint_R 1\,dx\,dy = M|R|$$

f alltid mindre enn sitt maks.

og

$$\iint_R f(x,y)\,dx\,dy \ge \iint_R m\,dx\,dy = m\iint_R 1\,dx\,dy = m|R|$$

f alltid større enn min.

Så:

$$m|R| \le \iint_R f(x,y)\,dx\,dy \le M|R|$$

$\Downarrow$ $\longrightarrow$ $|R|>0$; $|R|=0$ ingenting

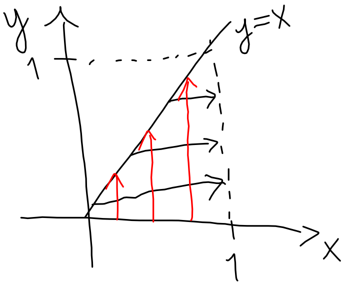
$$m \le \frac{\iint_R f(x,y)\,dx\,dy}{|R|} \le M \qquad (\star)$$

Fra skjæringssetningen vet vi at den kont. funk. $f(x,y)$ tar alle verdier mellom minimum og maksimum. Siden $(\star)$ gir at $\frac{\iint_R f(x,y)\,dx\,dy}{|R|}$ er en slik verdi mellom maks. og min, så må det fins et punkt $(\bar{x}, \bar{y}) \in R$ s.a.

$$f(\bar{x}, \bar{y}) = \frac{\iint_R f(x,y)\,dx\,dy}{|R|}$$

6.2:

3) a) $I = \int_0^1 \int_y^1 e^{x^2}\, dx\, dy$



$x \in [0,1]$ og $\boxed{y \in [0,x]}$

$$I = \int_0^1 \int_0^x e^{x^2}\, dy\, dx = \int_0^1 e^{x^2} [y]_{y=0}^{x}\, dx$$

$$= \int_0^1 e^{x^2} x\, dx = \left[\frac{1}{2} e^{x^2}\right]_{x=0}^{1}$$

$$= \underline{\underline{\frac{1}{2}(e-1)}}$$

b) $\int_0^{\frac{\pi}{2}} \int_x^{\frac{\pi}{2}} \frac{\sin(y)}{y}\, dy\, dx$

$$= \int_0^{\frac{\pi}{2}} \int_0^{y} \frac{\sin(y)}{y}\, dx\, dy$$

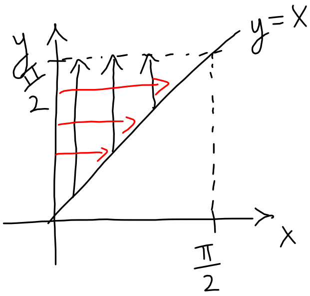
$$= \int_0^{\frac{\pi}{2}} \left[\frac{\sin(y)}{y} x\right]_{x=0}^{y} dy$$

$$= \int_0^{\frac{\pi}{2}} \sin(y)\, dy = [-\cos(y)]_{y=0}^{\frac{\pi}{2}}$$

$$= \underline{\underline{1}}$$

$y \in [0, \frac{\pi}{2}]$

$x \in [0, y]$



Evt: $0 \le x \le \frac{\pi}{2}$

$x \le y \le \frac{\pi}{2}$

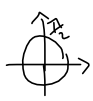
$\Downarrow$

$0 \le x \le y \le \frac{\pi}{2}$

$\Downarrow$

$0 \le y \le \frac{\pi}{2}$

og $0 \le x \le y$



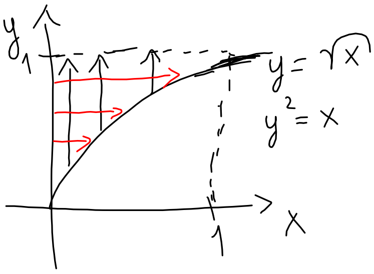
c) $\int_0^1 \int_{\sqrt{x}}^1 e^{\frac{x}{y^2}}\, dy\, dx$

$$= \int_0^1 \int_0^{y^2} e^{\frac{x}{y^2}}\, dx\, dy$$

$$= \int_0^1 \left[y^2 e^{\frac{x}{y^2}}\right]_{x=0}^{y^2} dy$$

$$= \int_0^1 (y^2 e - y^2)\, dy = (e-1)\int_0^1 y^2\, dy = (e-1)\left[\frac{1}{3}y^3\right]_{y=0}^{1}$$

$$= \underline{\frac{e-1}{3}}$$



$y \in [0,1]$

$x \in [0, y^2]$

6.3:

4.) $|A| = \iint_A 1 \, dx\,dy = \iint_A r \, dr \, d\theta = \int_\alpha^\beta \int_0^{r(\theta)} r \, dr \, d\theta$

polarkoord.

$= \int_\alpha^\beta \left[\frac{1}{2} r^2 \right]_{r=0}^{r(\theta)} d\theta = \int_\alpha^\beta \frac{1}{2} \, \underline{\underline{r^2(\theta)}} \, d\theta$

$r(\theta) = \sin(2\theta); \quad \theta \in [0, \frac{\pi}{2}]$

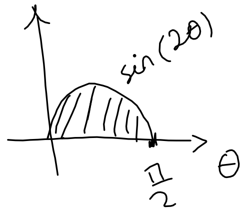
$|A| = \int_0^{\frac{\pi}{2}} \frac{1}{2} \sin^2(2\theta) \, d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{4} \left(1 - \cos(4\theta)\right) d\theta$

TRIG. FORMEL:

$\cos(2x) = 1 - 2\sin^2(x)$

$\sin^2(2\theta) = \frac{1}{2}(1 - \cos(4\theta))$

$= \frac{1}{4} \left[\theta - \frac{1}{4} \sin(4\theta) \right]_{\theta=0}^{\frac{\pi}{2}} = \frac{1}{4} \frac{\pi}{2} = \underline{\underline{\frac{\pi}{8}}}$



$\theta \in [0, \frac{\pi}{2}]$

$2\theta \in [0, \pi]$

6.4: 3) $R = \{(x,y) \mid 0 \le x \le 1,\ x^2 \le y \le 1\}$,

$f(x,y) = xy$

$$(\bar{x}, \bar{y}) = \left(\frac{\iint_R x\, f(x,y)\,dx\,dy}{\iint_R f(x,y)\,dx\,dy}, \frac{\iint_R y\, f(x,y)\,dx\,dy}{\iint_R f(x,y)\,dx\,dy} \right)$$

Mellomregning:

$$\iint_R f(x,y)\,dx\,dy = \int_0^1 \int_{x^2}^1 xy\,dy\,dx = \int_0^1 \left[\frac{1}{2}y^2 x\right]_{y=x^2}^1 dx$$

$$= \int_0^1 \frac{1}{2}x(1-x^4)\,dx = \frac{1}{2}\int_0^1 (x - x^5)\,dx$$

$$= \frac{1}{2}\left[\frac{1}{2}x^2 - \frac{1}{6}x^6\right]_{x=0}^1 = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{6}\right) = \frac{1}{2}\,\frac{3-1}{6} = \underline{\underline{\frac{1}{6}}}$$

$$\iint_R x\, f(x,y)\,dx\,dy = \int_0^1 \int_{x^2}^1 x^2 y\,dy\,dx = \int_0^1 x^2 \left[\frac{1}{2}y^2\right]_{y=x^2}^1 dx$$

$$= \frac{1}{2}\int_0^1 x^2(1-x^4)\,dx = \frac{1}{2}\left[\frac{1}{3}x^3 - \frac{1}{7}x^7\right]_{x=0}^1 dx$$

$$= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{7}\right) = \dots = \underline{\underline{\frac{2}{21}}}$$

$$\iint_R y\, f(x,y)\,dx\,dy = \int_0^1 \int_{x^2}^1 x y^2\,dy\,dx = \int_0^1 \left[x\,\frac{1}{3}y^3\right]_{y=x^2}^1 dx$$

$$= \frac{1}{3}\int_0^1 (x - x^7)\,dx = \frac{1}{3}\left[\frac{1}{2}x^2 - \frac{1}{8}x^8\right]_{x=0}^1$$

$$= \frac{1}{3}\left(\frac{1}{2} - \frac{1}{8}\right) = \dots = \underline{\underline{\frac{1}{8}}}$$

Massemiddelpunktet er:

$$(\bar{x}, \bar{y}) = \left(\frac{\frac{2}{21}}{\frac{1}{6}}, \frac{\frac{1}{8}}{\frac{1}{6}} \right) = \underline{\underline{\left(\frac{4}{7}, \frac{3}{4} \right)}}$$