Plenum 26/4-16

4.10: 5a

4.11: 1, 10

5.1: 1de, 4

5.2: 1

4.10:

5.) a) $c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 + c_4\vec{u}_4 = \vec{x}$

$$\begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \vec{x}$$

$$\underbrace{U}_{\text{kjent}}\underbrace{\vec{c}}_{?} = \underbrace{\vec{x}}_{\text{kjent}}$$

Fra MATLAB;

$$\vec{x} = (-1{,}9 - 2{,}4i)\underbrace{\vec{u}_1}_{\text{1. egenvektor}} + (-1{,}9 + 2{,}4i)\vec{u}_2 + 1{,}4\vec{u}_3 + 3{,}9\vec{u}_4$$

4.11:

1) $x_{n+1} = x_n + 3y_n$, $x_0 = 5$, $y_0 = -5$

$y_{n+1} = 2x_n + 2y_n$

Matriseform:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}}^{:=A} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & -3 \\ -2 & \lambda-2 \end{vmatrix} = (\lambda-1)(\lambda-2)-6$$

$$= \lambda^2 - 3\lambda - 4 = 0$$

$$\Downarrow \text{(2. gradsformel)}$$

$$\underline{\lambda_1 = -1} \quad \text{og} \quad \underline{\lambda_2 = 4}$$

$\vec{v}_1$: $A\vec{v}_1 = \lambda_1 \vec{v}_1$

$(\lambda_1 I - A)\vec{v}_1 = 0$

Velger i $y = 2$

$$-I - A = \begin{bmatrix} -2 & -3 \\ -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{v}_1 = \underline{\begin{bmatrix} -3 \\ 2 \end{bmatrix}}$$

$\vec{v}_2$: $$4I - A = \begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{v}_2 = \underline{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

Skriv $(x_0, y_0) = (5, -5)$ som lin. komb. av $\vec{v}_1$ og $\vec{v}_2$:

$$\begin{bmatrix} -3 & 1 & 5 \\ 2 & 1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{3} & -\frac{5}{3} \\ 2 & 1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{3} & -\frac{5}{3} \\ 0 & \frac{5}{3} & -\frac{5}{3} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -\frac{1}{3} & -\frac{5}{3} \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 \\ -5 \end{bmatrix} = -2\begin{bmatrix} -3 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_n \\ y_n \end{bmatrix} = -2(-1)^n \begin{bmatrix} -3 \\ 2 \end{bmatrix} - 4^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \underline{\underline{\begin{bmatrix} 6(-1)^n - 4^n \\ (-4)(-1)^n - 4^n \end{bmatrix}}}$$

Hvorfor?

$$\underline{\begin{bmatrix} x_n \\ y_n \end{bmatrix}} = A \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix}$$

$$= A A \begin{bmatrix} x_{n-2} \\ y_{n-2} \end{bmatrix}$$

$$= A^2 \begin{bmatrix} x_{n-2} \\ y_{n-2} \end{bmatrix}$$

$$= \dots = A^n \begin{bmatrix} x_{n-n} \\ y_{n-n} \end{bmatrix}$$

$$= A^n \underline{\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}}$$

$$= A^n (c_1 \vec{v}_1 + c_2 \vec{v}_2)$$

$$= c_1 (A^n \vec{v}_1) + c_2 (A^n \vec{v}_2)$$

$$= \underline{c_1 \lambda_1^n \vec{v}_1 + c_2 \lambda_2^n \vec{v}_2}$$

10.) $x(t) =$ rov , $y(t) =$ bytte

$x(0) = 500$, $y(0) = 1000$

a) $(\star) \begin{cases} x'(t) = x(t) + y(t) \\ y'(t) = -x(t) + y(t) \end{cases}$

$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

Fra $(\star)$: $\vec{r}\,'(t) = \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}_{:=A} \vec{r}(t)$

Egenverdier:

$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -1 \\ 1 & \lambda - 1 \end{vmatrix} = (\lambda-1)^2 + 1 = \lambda^2 - 2\lambda + 2$

$\lambda = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i = \begin{cases} 1+i & \lambda_1 \\ 1-i & \lambda_2 \end{cases}$

Egenvektorer:

$\lambda_1 I - A = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \sim \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{v}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$y = 1$

Fra Set. 4.10.8: $\vec{v}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

Fra argument s. 387: Sett

$\vec{r}(t) = c_1(t)\vec{v}_1 + c_2(t)\vec{v}_2 \quad (\square)$

$\Downarrow$

$c_1'(t) = \lambda_1 c_1(t)$ og $c_2'(t) = \lambda_2 c_2(t)$

$\Downarrow$

$c_1(t) = C_1 e^{(1+i)t}$ og $c_2(t) = C_2 e^{(1-i)t}$

Startbetingelse:

$\vec{r}(0) = \begin{bmatrix} 500 \\ 1000 \end{bmatrix} \underset{(\square)}{=} C_1\vec{v}_1 + C_2\vec{v}_2 = \begin{bmatrix} -C_1 i + C_2 i \\ C_1 + C_2 \end{bmatrix}$

$\Rightarrow C_1 = 1000 - C_2$ og $-(1000 - C_2)i + C_2 i = 500$

$C_1 = 500 + 250i \Leftarrow C_2 = 500 - 250i$

Fra ($\triangle$):

$$x(t) = -\overbrace{(500+250i)e^{t}(\cos t + i\sin t)}^{c_1(t)}\, i$$

$$+ \underbrace{(500-250i)e^{t}(\cos t - i\sin t)}_{c_2(t)}\, i$$

$$e^{(1+i)t} = e^{t}(\cos t + i\sin t)$$

$$e^{(1-i)t} = e^{t}(\cos(-t) + i\sin(-t)) = e^{t}(\cos t - i\sin t)$$

$$= \dots = e^{t}(1000\sin t + 500\cos t)$$

$$y(t) = (500+250i)e^{t}(\cos t + i\sin t) + (500-250i)e^{t}\cdot(\cos t - i\sin t)$$

$$= e^{t}(1000\cos t - 500\sin t)$$

b)
$$\begin{cases} x'(t) = -2x(t) + 4y(t) \\ y'(t) = x(t) - 2y(t) \end{cases}$$

$$\vec{r}\,'(t) = A\vec{r}(t) \quad \text{der} \quad A = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}$$

Egenvektorer & egenverdier:

$$\det(\lambda I - A) = \begin{vmatrix} \lambda+2 & -4 \\ -1 & \lambda+2 \end{vmatrix} = (\lambda+2)^2 - 4 = \lambda(\lambda+4) = 0$$

$$\Rightarrow \lambda_1 = 0 \quad \text{og} \quad \lambda_2 = -4$$

$$\vec{v}_1: \quad \lambda_1 I - A = -A = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\vec{v}_2: \quad \underbrace{\lambda_2}_{-4} I - A = \begin{bmatrix} -2 & -4 \\ -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$\vec{v}_1$ og $\vec{v}_2$ lin. uavh; så basis!

Fra arg. 387: $\quad \vec{r}(t) = c_1(t)\vec{v}_1 + c_2(t)\vec{v}_2 \quad (\Delta)$

der $\quad c_1(t) = C_1 e^{\lambda_1 t} = C_1 e^{0} = C_1 \quad$ og $\quad c_2(t) = C_2 e^{\lambda_2 t} = C_2 e^{-4t}$

Startbetingelse:

$$\begin{bmatrix}500\\1000\end{bmatrix} = \vec{r}(0) \underset{(\Delta)}{=} C_1 \underbrace{\begin{bmatrix}2\\1\end{bmatrix}}_{\vec{v}_1} + C_2 e^{-4\cdot 0} \underbrace{\begin{bmatrix}-2\\1\end{bmatrix}}_{\vec{v}_2}$$

$$= \begin{bmatrix}2C_1 - 2C_2\\ C_1 + C_2\end{bmatrix}$$

$$\Rightarrow C_1 = 1000 - C_2 \quad \text{og} \quad 2(1000 - C_2) - 2C_2 = 500$$

$$\vdots$$

$$4C_2 = 1500$$

$$C_1 = 625 \quad \Leftarrow \quad C_2 = 375$$

Så fra (Δ): $\vec{r}(t) = \begin{bmatrix}x(t)\\y(t)\end{bmatrix} = \underbrace{625}_{c_1(t)} \begin{bmatrix}2\\1\end{bmatrix} + \underbrace{375e^{-4t}}_{c_2(t)} \begin{bmatrix}-2\\1\end{bmatrix}$

$$= \begin{bmatrix}1250 - 750e^{-4t}\\ 625 + 375e^{-4t}\end{bmatrix}$$

c) Ikke flere byttedyr!

Modell a): $y(t) = 0$? $y(t)$ er 0 når

$$1000\cos t - 500\sin t = 0$$

$$2\cos t = \sin t$$

$$\tan(t) = 2$$

Hvor mange måneder før dette skjer? $\arctan(\tan t) \overset{?}{=} \arctan(2)$

$$t \approx 1{,}1$$

Så etter 1,1 år (ca 13 mnd.) er det ingen byttedyr i modell a).

PASSER BRA!

Modell b): $\lim\limits_{t\to\infty} y(t) \underset{b)}{=} \lim\limits_{t\to\infty} 625 + 375e^{-4t} = 625$

$y(t) = 625 + 375e^{-4t} \neq 0$; Er alltid pos. antall byttedyr i modell b).

PASSER DÅRLIG!