Computational Geometry: Theory and Applications 96 (2021) 101748

Contents lists available at ScienceDirect

Computational Geometry: Theory and Applications

www.elsevier.com/locate/comgeo

Attraction-convexity and normal visibility [☆]

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A R T I C L E   I N F O

Article history:
Received 30 September 2019
Received in revised form 19 January 2021
Accepted 21 January 2021
Available online 2 February 2021

Keywords:
Beacon attraction
Attraction convexity
Convex polygon
Star-shaped polygon
Helly-type characterization

A B S T R A C T

Beacon attraction, or simply *attraction*, is a movement system whereby a point moves in a free space so as to always locally minimize its Euclidean distance to an activated beacon (also a point). This results in the point moving directly towards the beacon when it can, and otherwise sliding along the edge of an obstacle or being stuck (unable to move). When the point can reach the activated beacon by this method, we say that the beacon *attracts* the point. In this paper, we study *attraction-convex* polygons, which are those where every point in the polygon attracts every other point. We find that these polygons are a subclass of *weakly externally visible* polygons, which are those where every point on the boundary is visible from some point arbitrarily distant (or at infinity on the projective plane). We propose a new class of polygons called *normally visible*, and show that this is exactly the class of attraction-convex polygons. This alternative characterization of attraction-convex polygons leads to a simple linear-time attraction-convex polygon recognition algorithm. We also give a Helly-type characterization of inverse-attraction star-shaped polygons.

1. Background

Beacon attraction has appeared in the literature as a model of greedy geographical routing in dense sensor networks. In this application, each node of the network has a location, and each communication packet knows the location of its destination. Nodes having a packet to deliver will forward the packet to their neighbor that is the closest (using Euclidean distance) to the packet's destination [5,6].

Another application, closer to the abstract form, is for a robot that heads towards a “homing signal.” The robot will follow walls it encounters if, in doing so, it gets closer to the source of the signal.

Here we study beacon attraction in the abstract setting, considering only simple singly-connected polygons with interior as our free space. In this setting, the destination point or signal source is called a beacon, and the message or robot is considered to be a point that greedily moves towards the beacon. This results in the point moving directly towards the beacon when it can, and otherwise sliding along an edge or being stuck (unable to move). The point, under this motion, may or may not reach the beacon—if it does reach the beacon, we say that the beacon *attracts* the robot's starting point (see Fig. 1).

Let $P = v_0, v_1, \ldots, v_{n-1}$ be a simple n-vertex polygon. We use the convention that the interior of P lies to the left of the edge directed from v_i to v_{i+1}, i.e. the polygon is described in a counter-clockwise fashion. We follow the convention that all indices are taken modulo n and will assume that no three vertices lie on a line.

☆ Research supported in part by NSERC. An extended abstract of this work appeared in CCCG 2019.
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https://doi.org/10.1016/j.comgeo.2021.101748
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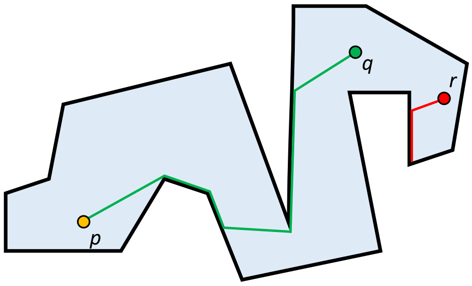

Fig. 1. A beacon at p attracts the point at q but not the point at r. The paths of the attraction are shown. Note that q does not attract p.

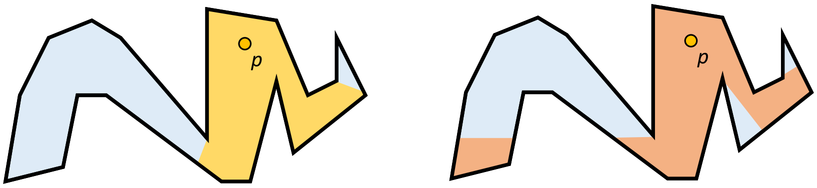

Fig. 2. (a) The attraction region $A(p)$. (b) The inverse attraction region $A^{-1}(p)$. Note that, in this instance, $A^{-1}(p)$ is not connected.

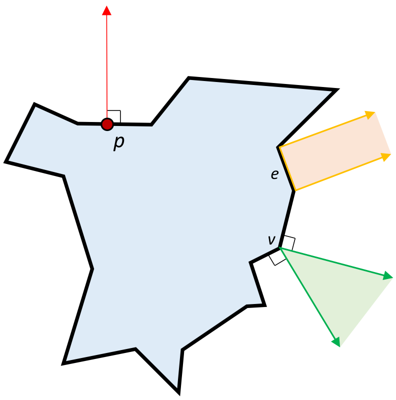

Fig. 3. A normally visible polygon, with the normal from p, the beam from e, and the cone of v.

Given a polygon P, the *attraction region* $A(p)$ of point p is the set of all points in P that p attracts, and the *inverse attraction region* $A^{-1}(p)$ is the set of all points in P that p is attracted to. See Fig. 2 for an illustration of these definitions.

The attraction relation between points has the flavor of a visibility-type relation, with the interesting property of asymmetry: if point p attracts point q, then it does not follow that point q attracts p (as is the case in Fig. 1). In a series of publications, Biro, Gao, Iwerks, Kostitsyna, and Mitchell have studied various visibility-type questions for beacon attraction, such as computing attraction (and inverse-attraction) regions for points, computing attraction kernels, guarding, and routing [2–4].

From a geometric standpoint, visibility, convexity, and star-shapedness are fundamental concepts. For a more detailed overview of these notions, the reader is referred to the book by O'Rourke [8]. Two points x and y in a shape (compact set of points) are called *visible* if the line segment $\overline{xy}$ is contained in the shape. A shape S is called *convex* if $\forall x, y \in S$, x and y are visible. A shape S is called *star-shaped* if $\exists x \in S,\ \forall y \in S$, x and y are visible. The *kernel* of S is the set of all points from which it is star-shaped: $kerS = \{x \in S \mid \forall y \in S,\ x \text{ and } y \text{ are visible}\}$.

These notions are easily extended to other "visibility-type" relations R: a shape S is called *R-convex* if $\forall x, y \in S$, xRy, and it is called *R-star* if $\exists x \in S,\ \forall y \in S,\ xRy$. The *$R$-kernel* of S is $kerS = \{x \in S \mid \forall y \in S,\ xRy\}$.

In this article, we will focus on relations where R is beacon attraction (and inverse beacon attraction). Although Biro *et al.* have studied attraction kernels and their computation, they did not produce characterizations of polygons that are attraction-star-shaped or attraction-convex. We address these issues.

To this end, we introduce a new notion of exterior visibility of a polygon. We call a polygon *normally visible* if the normal (exterior perpendicular) to each point on the boundary hits no point of the interior of the polygon. This is a specialization of weak exterior visibility: weak exterior visibility [9] only requires a ray from each boundary point that does not intersect the interior, whereas normal visibility constrains those rays to be perpendicular to the polygon boundary. Fig. 3 shows a normally-visible polygon and the nonintersecting perpendicular ray for point p.

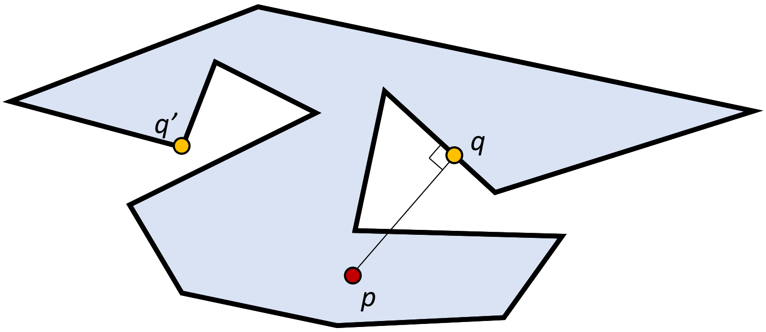

Fig. 4. The dead points q and q' of a beacon at point p.

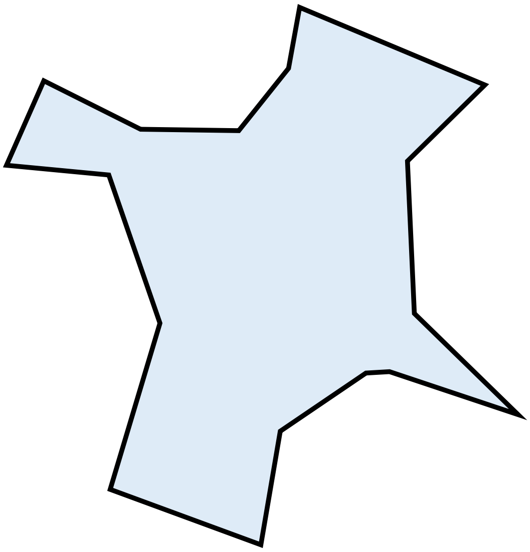

Fig. 5. An attraction-convex polygon.

Another way to conceptualize normally visible polygons is to imagine an ant crawling counterclockwise around the boundary of the polygon. This ant has an attached laser that points directly to its right. If the ant can crawl all the way around the polygon with the laser never hitting the polygon interior, then the polygon is normally visible.

To simplify thinking about normal visibility, it is useful to consider the *beam* $B(e)$ of each edge e, and the *cone* $C(v)$ of each vertex v. The beam of e is the union of all normals from points on e, and the cone of v is the normals that are swept out as the crawling ant continuously changes heading from one edge to the other at v. A polygon is normally visible if no beam or cone hits its interior.

We may dispense with cones by a chain of observations. First, for the polygon interior to intersect a cone, the polygon must *cross* (not simply intersect) one of the cone's two bounding rays, as the polygon is a Jordan curve. Next, the bounding rays of the cone are also bounding rays of the beams of the two adjacent edges. Thus the polygon crosses a ray in a beam; that ray will hit an interior point of the polygon. So we conclude that a polygon is normally visible if no beam hits its interior.

2. Convex characterization

Given a polygon P, a point q is called a *dead point* of point p if a beacon at p does not move q. A dead point of p is either a convex vertex or a point q in the interior of an edge such that pq is perpendicular to the edge (see Fig. 4). In either case, pq is exterior to the polygon in the neighborhood of q. When a beacon is activated, all points end up either at the beacon or at a dead point. Recall that polygon P is attraction-convex if for every pair of points $p, q \in P$, we have that p attracts q. Fig. 5 depicts an attraction-convex polygon. In this section, we show that a polygon is attraction-convex if and only if it is normally visible.

Theorem 1. *A polygon is attraction-convex if and only if it is normally visible.*

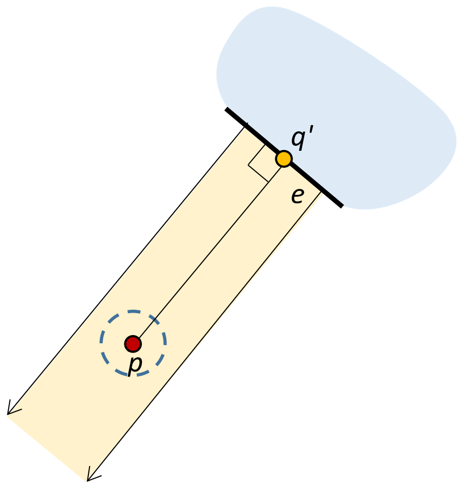

Fig. 6. The yellow region, and thus an ε-ball around p, is contained in the beam of e. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

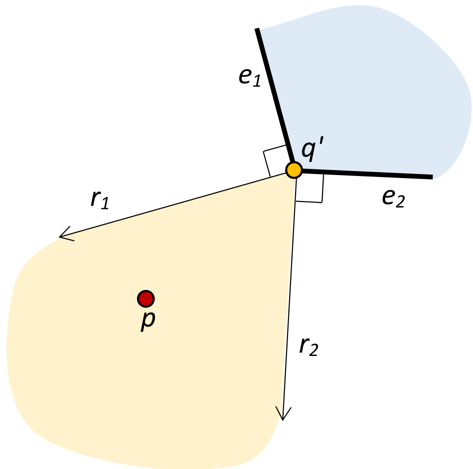

Fig. 7. p lies in the yellow region, which is the cone of q'.

Proof. We first show that if a polygon is attraction-convex, then it is normally visible. Suppose this is not the case. Assume P is an attraction-convex polygon, with q being a point on edge e of P that is not normally visible. The outside ray r starting at q perpendicular to e must encounter some other point q' on the boundary of P. Now we note that, in P, q' does not attract q. (q is a dead point of q'.) This contradicts the assumption that P is attraction-convex.

We now show that if a polygon is normally visible, then it is attraction-convex. We prove the contrapositive: If a polygon P is not attraction-convex, then it is not normally visible.

Since P is not attraction-convex, there are points p and q in P such that p does not attract q. Acting under attraction from p, the point q will arrive at a dead point q' of q, and become stuck there.

If the dead point is on the interior of an edge e of P, then the outside ray r from q' perpendicular to e hits p. The beam $B(e)$ extends along e to both sides of q', which implies that for ε small enough, $B(e)$ contains an ε-neighborhood of p (see Fig. 6). Since any ε-neighborhood of p contains interior points of P, the beam $B(e)$ contains interior points of P, implying that P is not normally visible.

If the dead point q' is a convex vertex of P, then let e_1 and e_2 be the edges of P incident on q'. Let r_1 be the outside ray from q' perpendicular to e_1, and r_2 be the outside ray from q' perpendicular to e_2. In order for q' to be a dead point, p must lie in the closed convex cone from q' bounded by r_1 and r_2 (see Fig. 7).

However, p must be connected to q' by a path interior to P except possibly at p and q'. This path cannot reach q' directly from the cone, as the polygon is simple. Since the path cannot go to infinity, as P does not, it must cross either r_1 or r_2. So a beam of P contains a point of the interior p, and P is not normally visible.

As these are the only two possibilities for the dead point q', the lemma is proved. □

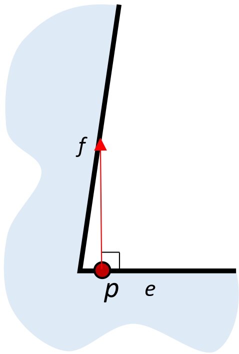

Fig. 8. A polygon having an acute exterior angle cannot be normally visible. The ray from p intersects f.

In the next section, we provide an optimal algorithm for determining whether a polygon is attraction-convex. The following is a useful property from an algorithmic standpoint.

Observation 1. *In an attraction-convex polygon, every interior angle is at most* $3\pi/2$*.*

Proof. Suppose the interior angle at some vertex v were more than $3\pi/2$; then the exterior angle at v would be less than $\pi/2$ (i.e. acute). Let e and f be the edges on v. Choosing p on e sufficiently near v, the perpendicular from p will intersect first f and then directly thereafter the interior of P (see Fig. 8). Thus the polygon is not normally visible. □

3. Algorithm for recognizing attraction-convex polygons

Before outlining the algorithm to recognize an attraction-convex polygon, we first introduce some terminology. Let $P = v_0, v_1, \ldots, v_{n-1}$ be the vertices of a simple polygon ordered in counterclockwise order. All indices are manipulated modulo n. Let $CH(P)$ be the convex hull of polygon P. Let $v_p v_q$ be an edge of the convex hull. If q is not $p+1$, i.e. if v_q is not the next counterclockwise vertex on the boundary of P, we call edge $v_p v_q$ a *pocket lid* and the polygonal chain from $v_p, \ldots, v_q$ on the boundary of P as the *pocket chain*. Together, a pocket lid and the corresponding pocket chain form the boundary of a simple polygon called a *pocket polygon*.

Recall that a polygon is normally visible if for every edge e, $B(e)$ does not intersect the polygon boundary. This means that for every edge e on a pocket chain, $B(e)$ intersects only the pocket lid. In fact, if pocket chain edge $e = ab$, then $B(e) \cap CH(P)$ results in a segment $a'b'$ that is a subsegment of the pocket lid (where a' is the intersection of the normal ray originating from a with the pocket lid and b' is the intersection of the normal ray originating from b with the pocket lid).

A *terrain polygon* is a monotone polygon Q that has a distinguished edge e such that (1) Q is monotone with respect to the direction of e, and (2) e is one of the two monotone chains of Q in this direction. We call a pocket chain $\Pi = v_p, \ldots, v_q$ *a terrain with respect to its lid* $v_p v_q$ if the corresponding pocket polygon is a terrain polygon with distinguished edge $v_p v_q$. Equivalently, Π is a terrain with respect to $v_p v_q$ provided that for every vertex $v_k \in \Pi$, the line perpendicular to $v_p v_q$ going through v_k, denoted by $\ell(v_k)$, intersects Π only at v_k and intersects the lid $v_p v_q$ at a point v'_k. Note that v'_k is the orthogonal projection of v_k onto the pocket lid. Moreover, the sequence $v'_p, v'_{p+1}, \ldots, v'_{q-1}, v'_q$ is a sorted sequence.

Lemma 1. *If a simple polygon* P *is normally visible, then every pocket chain of* P *is a terrain with respect to its pocket lid.*

Proof. For sake of a contradiction, suppose that P is normally visible and that there is a pocket chain $\Pi = v_p, \ldots, v_q$ that is not a terrain with respect to its pocket lid $v_p v_q$. Let Q be the corresponding pocket polygon. If $e \in \Pi$, then the interior of the beam $B(e)$, in a neighborhood of e, is inside Q. Since the beam is infinite, it must leave Q somewhere. Since P is normally visible, the beam cannot leave Q via any edge in Π. Therefore the beam must leave Q via $v_p v_q$. This implies that the angle between the normal of the convex hull edge $v_p v_q$ and the normal of (any) edge e of Π is strictly less than $\pi/2$.

Without loss of generality, assume that $v_p v_q$ is on the X-axis with $v_p = v'_p$ at the origin, and the polygon above. If Π were a terrain with respect to $v_p v_q$, then $v'_p, \ldots, v'_q$ would be a non-decreasing sequence. (Here we use v' as a stand-in for the x-coordinate of v.) Since Π is not a terrain with respect to $v_p v_q$, there must exist a vertex v_k, with $k \in p+1, \ldots, q-1\}$, such that v'_{k-1} and v'_{k+1} are both greater than v'_k or both are less than v'_k in the sequence.

Without loss of generality, assume that both v'_{k-1} and v'_{k+1} are greater than or equal to v'_k. Note that if v'_{k-1} is equal to v'_k, then $v'_{k+1} > v'_k$ and vice versa since $v'_{k-1} = v'_k = v'_{k+1}$ implies that the three vertices are collinear which violates our definition of simple polygon.

```
   // Maintain a stack S of vertices. Initially, S is empty.
   // Let k = |S| and let c_i refer to the i^th element in S.

   ccwScan()
1      push(v_0) // sets k = 1
2      for i ← 1 to n
3          while k ≠ 1 and c_{k-1} c_k v_i is a right turn do
4              pop() // decrements k
5
6          // Now k = 1 or c_{k-1} c_k v_i is straight or left turn
7          push(v_i) // increments k
8          if c_{k-1} c_k v_{i+1} is a right turn and ∠c_{k-1} c_k v_{i+1} < π/2 then
9              return false
10         if c_{k-1} c_k v_{i+1} is a left turn and v_{i-1} v_i v_{i+1} is a right turn then
11             return false
12     return true
```

Fig. 9. The counterclockwise scan of the algorithm.

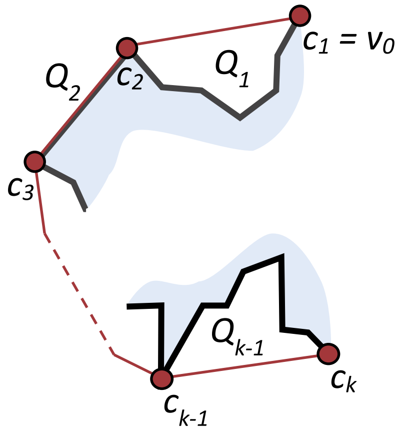

Fig. 10. The spiral chain $c_1, c_2, \ldots c_k$ maintained in the stack and the polygons $Q_1, Q_2, \ldots Q_{k-1}$. Note that Q_2 here is degenerate.

Given this property, we have that either the angle between the normal of $v_{k-1}v_k$ and the normal of $v_p v_q$ is at least $\pi/2$ or the angle between the normal of $v_k v_{k+1}$ and the normal of $v_p v_q$ is at least $\pi/2$, which is a contradiction since these two angles must both be strictly less than $\pi/2$ by Observation 1. □

Our algorithm for recognizing whether a polygon P is attraction-convex consists of three major steps. The first step is to compute the convex hull of P. The second step is to verify that every pocket chain is a terrain with respect to its lid. The third step consists of running a specialized scan twice: once counterclockwise and once clockwise. If either scan returns **false** then the polygon is not attraction-convex; if both return **true** then the polygon is attraction-convex.

The algorithm for the counterclockwise scan is shown in Fig. 9; the clockwise scan is symmetric. Here, v_0 is the leftmost top vertex of the polygon and vertices are numbered sequentially in counterclockwise order.

At the end of each **for** loop iteration, the stack can be viewed as a left-turning spiral chain starting at v_0 and ending at v_i; this chain does not cross the polygonal chain seen so far $(v_0, v_1, \ldots v_i)$. The noncrossing property implies that each edge $c_{j-1}c_j$ of the spiral chain is an edge of a (possibly degenerate) closed polygon Q_j whose boundary consists of that edge and the portion of the boundary of the input polygon from c_j to c_{j+1} (see Fig. 10).

Theorem 2. *The algorithm described above correctly determines if an input polygon is attraction-convex in $O(n)$ time.*

Proof. The first step can be computed in linear time with Melkman's linear time convex hull algorithm [7]. The second step can also be computed in linear time by simply verifying that for every pocket chain $v_p, \ldots, v_q$, the sequence $v'_p, \ldots, v'_q$ appears in the same order. If some pocket chain is not a terrain with respect to its lid, then we stop, and Lemma 1 guarantees us that **false** is the correct answer.

So suppose the algorithm proceeds to the third step and then reports **false**. Without loss of generality, assume that it is the counterclockwise scan that reports **false**. If this was reported from line 9 of the algorithm, then some angle $c_{k-1}c_k v_{i+1}$ is right-turning and acute. Since we have just pushed v_i in line 7, $c_k = v_i$. Similar to the proof of Observation 1, we take a point p sufficiently close to $c_k = v_i$ on $v_i v_{i+1}$, and the normal ray r from p will intersect $c_{k-1}c_k$. If Q_{k-1} is degenerate, then this is an intersection q with the polygon boundary. If it is not degenerate, r enters Q_{k-1} and must exit Q_{k-1} at some

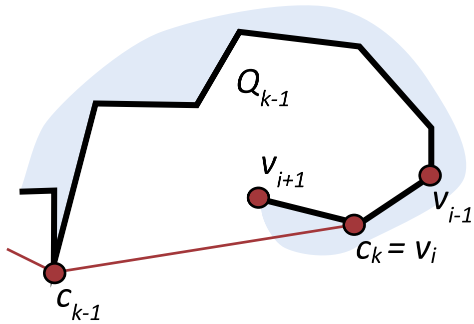

Fig. 11. When $c_{k-1}c_k v_{i+1}$ is a left turn and $v_{i-1}v_i v_{i+1}$ is a right turn, $v_i v_{i+1}$ lies in Q_{k-1}.

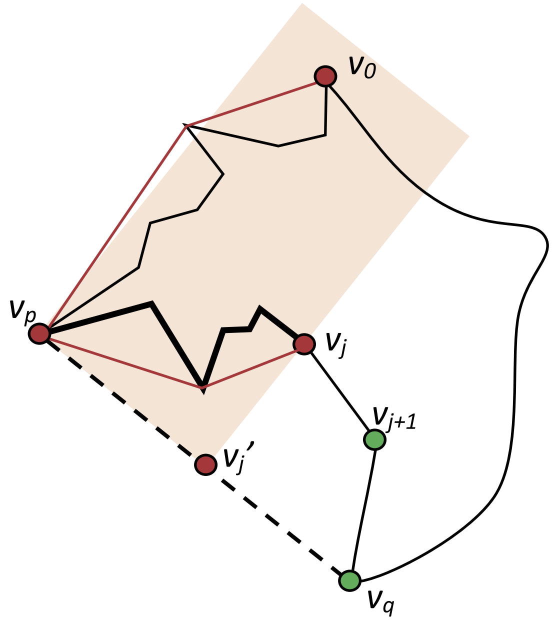

Fig. 12. The subchain $v_p, v_{p+1}, \ldots, v_j$ must lie in the shaded region.

point q not on $c_{k-1}c_k$. The point q is an intersection with the polygon boundary. In either case, the ray r enters the interior of the polygon directly after it hits the point q, so this polygon is not normally visible. (Or attraction-convex, by Theorem 1.)

If instead the **false** was reported in line 11, we will have a similar situation. If $c_{k-1}c_k v_{i+1}$ is a left turn and $v_{i-1}v_i v_{i+1}$ is a right turn, then the edge $v_i v_{i+1}$ lies in Q_{k-1} with polygon interior on the side closest to $c_{k-1}c_k$ (see Fig. 11). If we let p be a point on the relative interior of $v_i v_{i+1}$, and r be the normal ray from p, then the ray r will leave Q_{k-1} at some point other than those on $c_{k-1}c_k$. Immediately after exiting Q_{k-1}, the ray r encounters a point in the interior of P. Thus the polygon is not normally visible.

In either case, then, the algorithm is correct when it reports **false**. Now suppose the algorithm reports **true** from both scans. Consider an arbitrary edge $v_j v_{j+1}$ of the polygon. If $v_j v_{j+1}$ is a convex hull edge, then its beam does not intersect the polygon interior. Otherwise, $v_j v_{j+1}$ is an edge in a pocket chain with some pocket lid $v_p v_q$. During the counterclockwise scan, after v_j was pushed on the stack, the stack was a left-spiral chain from v_0 to v_j, and the angle $c_{k-1}c_k v_{j+1}$ was at least $\pi/2$. The convex hull vertex v_p is somewhere on the left-spiral chain. Because the pocket chain is a terrain with respect to $v_p v_q$, the vertices of the sub chain $v_p, v_{p+1}, \ldots v_{j-1}, v_j$ all lie to the right of $v_j v'_j$ where v'_j is the projection of v_j onto $v_p v_q$ (see Fig. 12). In particular, this means that this chain, and thus the left-spiral chain from v_0, cannot wind around $v_j v_{j+1}$. A similar argument using the clockwise scan establishes that the right-spiral chain from v_0 reaches v_{j+1} without winding around $v_j v_{j+1}$.

Refer now to Fig. 13. Since the angle $c_{k-1}c_k v_{j+1}$ was at least $\pi/2$, the left-spiral chain from the counterclockwise scan may graze the boundary but not the interior of the beam $B(v_j v_{j+1})$. Furthermore, as the beam gets farther from $v_j v_{j+1}$, the left-spiral chain diverges from the beam, and as it doesn't wind, it cannot later intersect the beam. Similarly, the right-spiral chain from the clockwise scan may graze but not intersect the interior of the beam. The polygon P is contained entirely by the polygon consisting of the left-spiral chain from v_0 to v_j, the edge $v_j v_{j+1}$, and the right-spiral chain from v_0 to v_{j+1}

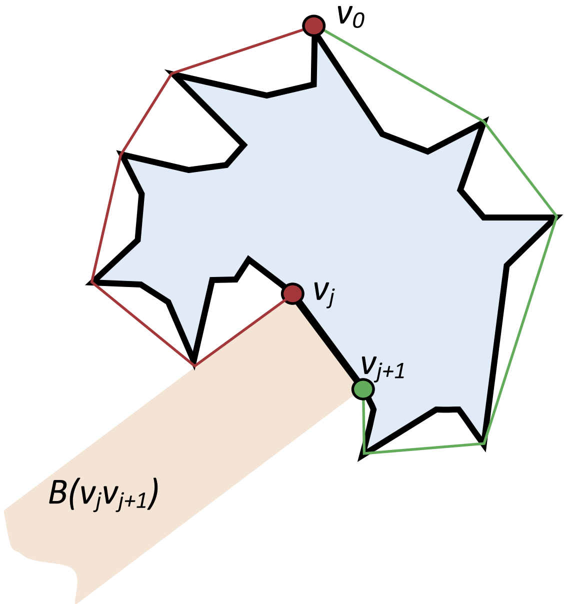

Fig. 13. The left-spiraling chain from v_0 to v_j, the edge v_jv_{j+1}, and the right-spiraling chain from v_0 to v_{j+1} enclose P and can touch only the boundary of $B(v_jv_{j+1})$.

(backwards). Thus the polygon P may intersect the boundary but not the interior of $B(v_jv_{j+1})$. Finally, since both P and the beam are closed, this means that the beam (including its boundary) does not intersect the interior of P.

Since we started with an arbitrary edge v_jv_{j+1}, and (given that both scans return **true**) established that the beam of that edge does not intersect the polygon interior, this means that all beams are free from such intersections and the polygon is normally visible.

Thus the algorithm is correct when it returns **true**, and correct overall. During each of the scans of step three, each vertex of the polygon is pushed onto the stack once, and popped at most once, giving linear stack manipulation. The condition of line 3 is checked once for each pop (linear time) and once for each time through the **for** loop (also linear time). The other conditions are checked once per iteration of the **for** loop, giving linear time for these, and linear time overall. □

4. Geodesic convexity and inverse-attraction-star shapes

4.1. Geodesic convexity

A set $A \subseteq P$ is called geodesically convex relative to P if, for all points p, q in A, the shortest path (geodesic) $\Pi(p,q)$ between p and q in P is contained in A. We will drop the phrase "relative to P" when the set P is understood.

Geodesic convexity is transitive.

Lemma 2. *If R is geodesically convex relative to Q, and Q is geodesically convex relative to P, then R is geodesically convex relative to P.*

Proof. Consider two points p, q in R. These points are also in Q and P. Since Q is geodesically convex relative to P, the shortest path $\Pi(p,q)$ from p to q in P is also the shortest path from p to q in Q. Since R is geodesically convex relative to Q, $\Pi(p,q)$ is also the shortest path from p to q in R. □

Lemma 3. *Let P_1 and P_2 be the two (closed) subpolygons of P formed when P is cut by a chord C. Then P_1 and P_2 are geodesically convex with respect to P.*

Proof. We prove this for P_1. Let x, y be two points in P_1. We claim the shortest path $\Pi(p,q)$ from p to q in P stays in P_1. If not, then there is a section of $\Pi(p,q)$ in P_2 that enters P_2 from some point r_1 of C, and exits P_2 to some point r_2 of C. Replacing this section of $\Pi(p,q)$ with the straight line segment from r_1 to r_2 results in a shorter path, which is a contradiction. □

Lemma 4. *Let P' be a subpolygon of P produced by repeating the operation of cutting off a section of P with a chord. Then P' is geodesically convex with respect to P.*

Proof. This follows directly from Lemmas 3 and 2. □

An *et al.* [1] established that geodesically-convex sets in $\mathbf{R}^2$ have Helly number 3:

Theorem 3 *(Geodesic-convexity Helly Theorem). Let $\mathcal{C}$ be a collection of sets all geodesically convex with respect to some compact base set S in $\mathbf{R}^2$. If every triple of sets in $\mathcal{C}$ has an intersection point, then all sets in $\mathcal{C}$ do.*

4.2. Inverse-attraction star-shaped polygons

Recall that given a polygon P, the *attraction region* $A(p)$ of point $p \in P$ is the set of all points in P that p attracts. Biro [2] showed that this region is itself a polygon. As such, $A(p)$ is sometimes referred to as an attraction polygon.

Lemma 5. *Attraction polygons are geodesically convex.*

Proof. As shown by Biro [2, Thm. 3.2.11], the attraction polygon of a point p in polygon P can be constructed by cutting several regions off of P, each region bounded by a chord (a "split edge"). The result follows from Lemma 4. □

Recall, as defined in Section 1, that a polygon P is inverse-attraction star-shaped provided there exists a point $x \in P$ such that for all points $y \in P$, y attracts x. We provide a Helly-type characterization of inverse-attraction star-shaped polygons.

Theorem 4. *A polygon P is inverse-attraction star-shaped if and only if for every triple of points p, q, and r in P, $A(p) \cap A(q) \cap A(r) \neq \emptyset$.*

Proof. P is inverse-attraction star-shaped if there exists a point k, called a kernel point, that inverse-attracts all points in P. This is the same as saying that k is in $A(p)$ for every p in P.

If P is inverse-attraction star-shaped with kernel point k, then for every triple of points p, q, r in P, k is in $A(p) \cap A(q) \cap A(r)$.

On the other hand, if for every triple of points p, q, r in P, $A(p) \cap A(q) \cap A(r) \neq \emptyset$, then the attraction polygons triple-wise intersect. Since attraction polygons are geodesically convex, we may then invoke the Geodesic-convexity Helly Theorem to obtain a point k that is in the intersection of all of the attraction polygons. □

We can simplify the conditions on the previous theorem to consider only triples of vertices rather than triples of points, but first we require an additional type of convexity, and a result.

A set $Q \subseteq P$ is said to be convex with respect to P if, for every pair of points p, q in Q, the line segment pq is in P if and only if pq is in Q. Biro [2] showed that inverse-attraction polygons are convex with respect to P.

Theorem 5. *A polygon P is inverse-attraction star-shaped if and only if for every triple of vertices u, v, and w in P, $A(u) \cap A(v) \cap A(w) \neq \emptyset$.*

Proof. If P is inverse attraction star-shaped with kernel point k, then for every triple of vertices u, v, w in P, k is in $A(u) \cap A(v) \cap A(w)$.

Suppose now that for every triple of vertices u, v, w in P, $A(u) \cap A(v) \cap A(w) \neq \emptyset$. By the geodesic-convexity Helly theorem, there is a point k in the intersection of all attraction polygons of vertices. We show that k is also in the attraction polygon of every point in P.

Consider an arbitrary point p of P. The point p resides in some triangle uvw of a triangulation of P. $A^{-1}(k)$ includes u, v, and w. Since u, v, and w are pairwise visible, the entire triangle u, v, w is in $A^{-1}(k)$ by Biro's convexity relative to P. □

It is an open problem to determine if there are equivalents of Theorems 4 and 5 for attraction as opposed to inverse attraction.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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