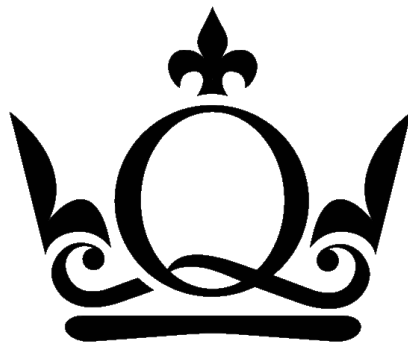


About Negative Temperature

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May 14, 2015



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In this article, it will be briefly given the understanding of negative temperature based on the online material: [Click Me](#). First of all, it should be made clear what 'temperature' really means. By temperature, we mean a parameter that is shared between different systems when they are in equilibrium. And by equilibrium, we usually mean there is no heat flow from one system to the others, if we only focus on the heat interaction between systems for current discussion. Here it should be pointed out that 'no heat flow' does not mean absolute 'no flow' but rather dynamic equilibrium. By far, the 'temperature' means something theoretical, however in daily life we can easily define temperature by 'feeling' how hot the object is. Temperature defined in such way is actually the empirical temperature, which is object specific. In another word, the way we define temperature empirically depends on how the object responds to the 'hotness'. The typical example is the mercury thermometer, for which the expansion of mercury depends how hot the object that it touches is. The advantage of the empirical definition of temperature is it's not an abstract notation at all, e.g. using a mercury thermometer we can actually 'see' the temperature. However the disadvantage is also obvious - it depends on specific object that is selected for measuring the temperature. Also empirical temperature is not suitable for theoretical use to describe how 'temperature' is linked to the property or thermal-process of any systems. Therefore we need theoretical temperature, which, first of all, should definitely be object independent.

Here let's take two systems as an example. When system S_1 and S_2 are at equilibrium, there is no heat flow (again, dynamically) between them. In such case, their temperature should be the same. It is just based on such intuition that the theoretical temperature is defined. From another perspective, if assuming the total energy E of S_1 and S_2 is kept as constant, we have such a state, where S_1 takes up energy of E_1 and S_2 takes up $E_2 = E - E_1$, that the system is at its equilibrium - no heat flow between the two sub-systems. The question is then: how E_1 and E_2 is distributed (while keeping the sum constant - $E = E_1 + E_2 \rightarrow Const$) will make the system at its equilibrium. From the perspective of [microcanonical ensemble](#), the equilibrium state is actually the state with the largest number of microstates. The total number of microstates of the whole system is: $N(E_1, E_2) = N_1(E_1) \times N_2(E_2)$. Then the equilibrium of the system corresponds to the maximization of $N(E_1, E_2)$, which means $N(E_1, E_2)$ is stationary with respect to the small change of E_1 (and/or E_2 , since $E_1 + E_2$ is kept as constant). Here comes the most famous definition in thermodynamics - entropy, which is given by Boltzmann. As a matter of fact, entropy is nothing but taking log of the total number of microstates of the system, to within a multiplicative constant - Boltzmann constant k_B . Therefore for the above system, we have its entropy S as given by:

$$S = k_B \ln N(E_1, E_2) \quad (1)$$

Obviously S is the function of E_1 and E_2 (remember we have $E_2 = E - E_1$, where E is a constant), i.e. $S = S(E_1, E_2)$. From the discussion above, we know that when the system is at its equilibrium, we should have S is stationary with respect to the small change in E_1 and/or E_2 , i.e.

$$dS = dS_1 + dS_2 \quad (2)$$

$$= \left(\frac{dS_1}{dE_1}\right)dE_1 + \left(\frac{dS_2}{dE_2}\right)dE_2 \quad (3)$$

$$= \left(\frac{dS_1}{dE_1}\right)dE_1 - \left(\frac{dS_2}{dE_2}\right)dE_1 \quad (4)$$

$$= \left(\frac{dS_1}{dE_1} - \frac{dS_2}{dE_2}\right)dE_1 = 0 \quad (5)$$

Here there are two things to point out before going on. The first one is that the reason why we could have (2) is because of the basic property of

logrithm function, i.e.

$$\begin{aligned}
 S &= k_B \ln N(E_1, E_2) \\
 &= k_B \ln [N_1(E_1) \times N_2(E_2)] \\
 &= k_B \ln N_1(E_1) + k_B \ln N_2(E_2) \\
 &= S_1 + S_2
 \end{aligned} \tag{6}$$

And the second thing is from (3) to (4), we should recall that $E = E_1 + E_2$ is kept as constant, which will give us:

$$\begin{aligned}
 dE &= dE_1 + dE_2 = 0 \\
 \Rightarrow \\
 dE_2 &= -dE_1
 \end{aligned} \tag{7}$$

Thus we have:

$$\begin{aligned}
 \frac{dS_1}{dE_1} - \frac{dS_2}{dE_2} &= 0 \\
 \Rightarrow \\
 \frac{dS_1}{dE_1} &= \frac{dS_2}{dE_2}
 \end{aligned} \tag{8}$$

From (8) we know that at equilibrium state, the two sub-systems share the quantity defined by dS/dE for each of them. And it is just this quantity that is defined as the theoretical temperature:

$$T = \frac{dS}{dE} \tag{9}$$

Given the definition of theoretical temperature, we know that temperature actually describes how the number of microstates changes as the energy of specific system changes. The most common case is of course the more energy the system possesses, the more number of microstates, which will then give positive temperature value. However in the world of spin, it is possible to have negative temperature. For example, we have N electrons in an external magnetic field. As we all know, each electron has two possible orientations – spin-up (pointing to the magnetic field) and spin-down (pointing opposite to the magnetic field). First of all, if all electrons are spin-up, i.e. all electrons are aligned up pointing to the direction of the magnetic field, the energy of the system is lowest, and the temperature is definitely zero – all electrons at their ground states without any thermal perturbation. Moreover, the entropy is also zero, since the total number of microstates is 1, which gives $S = k_B \ln 1 = 0$.

And now one of the electrons is flipped over to spin-down state, the energy of the system will increase accordingly, since spin-down state is higher in energy. How many states should we have corresponding to such situation? The answer is $C_N^1 = N$, i.e. we can select any one from the N electrons of the system. Thus the entropy is $S = k_B \ln N$. As we move on to flip more electron (e.g. j electrons) over, the number of microstates should be $C_N^j = \frac{N!}{j!(N-j)!}$, and the corresponding entropy is: $S = k_B \ln C_N^j$. Here it should be pointed out that, as we flip more and more electrons over, the energy of the system keeps increasing till all electrons are flipped over. However, this is not the case for entropy. Considering the property of combination function (C_N^j as a function of j , where N is a constant), we know that when j increases from 0 to $N/2$ (or $(N \pm 1)/2$, if N is an odd number), the C_N^j value increases with j , i.e. entropy now increases with energy. That means $T = dS/dE > 0$, and the temperature is positive. However when j crosses over the above boundary, C_N^j will decreases as j increases. This means now the entropy decreases as the energy of the system is increasing, which will

then give *NEGATIVE* temperature since now we have $T = dS/dE < 0$. And this is where the *NEGATIVE* temperature comes from, nothing surprise but rather a property of the spin system.

Furthermore, it should be pointed out that the negative temperature system is 'hotter' than the positive temperature system. What does that mean? Imaging two system touches each other, one of the is with negative temperature, and the other one is with positive temperature. Now if energy flow from the negative-temperature system to the positive-temperature system, what will happen? The answer is: the number of microstates (i.e. the entropy S) of the whole system will increase. Why? For negative-temperature system, when energy was taken away from it, C_N^J will increase since for negative-temperature system entropy increases as the decreasing of energy. And similarly, the accepting of extra energy for the positive-temperature system will also lead to the increase of entropy, since for positive-temperature system entropy increases as the increasing of energy. Therefore, when negative-temperature system and positive-temperature system touches each other, the energy tends to 'flow' from negative to positive system to maximize the entropy (i.e. the total number of microstates) – till the whole system is at its equilibrium. From the above analysis we can see that the negative-temperature system is 'hotter' not because there is more energy in it but rather the energy flowing to the positive-temperature system will lead to a more stable state of the system.

The final thing to point out is that the negative temperature can be only accessed when the spin freedom of particles (e.g. electrons, etc.) are not strongly coupled with other freedoms (e.g angular momentum, etc.), and it was already possible in some nuclear experiments. (for details, see the link: [Click Me](#))