Some notes on second-order ordinary differential equation (ODE)

Yuanpeng Zhang

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School of Physics and Astronomy

Queen Mary, University of London

Starting with the homogeneous second-order ODE, the general solution form should be:

$$y = c_1 y_1(x) + c_2 y_2(x) \tag{1}$$

where c_1 and c_2 are undetermined constants. This is easy to understand, since for any second order ODE, we definitely need two initial conditions to determine the specific solution. For example, if we are given the acceleration of an object, then it is necessary to know the initial position and velicity (or, the other two combined initial conditions), in order to know exactly where the object will be in next moment. And for inhomogeneous second-order ODE, the general solution is in the following form:

$$y = c_s + c_1 y_1(x) + c_2 y_2(x) \tag{2}$$

where c_s is any special solution to the inhomogeneous equation.
Generally, it is the most difficult step to obtain the first general solution (e.g. y_1) for the homogeneous second-order ODE. Once y_1 is obtained, then we can follow the order-reduction method, which will be discussed later in the main text, to find the second general solution (i.e. y_2). Having the general solution for homogeneous equation, then we can use the constant-variation method (i.e. changing the constant c_1 and c_2 in the general solution to undetermined functions) to find the general solution for corresponding inhomogeneous equation. This will also be discussed later in the main text.
One of the mothods to obtain the first general solution for homogeneous second-order ODE is through Laplace transformation, which can transform the ODE to its linear form. For the following discussion, the 3rd order Laguerre equation will be taken as an example:

$$x\frac{d^2y(x)}{dx^2} + 3y(x) + (1-x)\frac{dy(x)}{dx} = 0 \tag{3}$$

and here is the link containing the detailed steps of deriving the first general solution for 3rd order Laguerre equation: Click Me. Following the steps given in the above link, we can finally obtain the first general solution for equation-(3):

$$y_1 = x^2 - 9x^2 + 18x - 6 \tag{4}$$

Having the first general solution to equation-(3), the next step is then to build up the other general solution based on what we have - (4). The idea is to reduce the order of second-order ODE to the first order, and specific steps will be given as following.
Let's begin with writing the Laguerre's equation (n=3) in the following form:

$$y'' + \frac{1-x}{x}y' + \frac{3}{x} = 0 \tag{5}$$

And (5) here is in the general form of second order ODE:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \tag{6}$$

And assume we have got a special solution y_1 to equation-(6) (for our Laguerre's equation (n=3), we indeed know y_1 as given by (4)). To build up the general solution for equation-(6), we write down a guess for the final solution as:

$$y = uy_1 \tag{7}$$

where $u = u(x)$ is an undetermined function of x. So then we have:

$$\frac{dy}{dx} = y_1\frac{du}{dx} + u\frac{dy_1}{dx} \tag{8}$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}(y_1\frac{du}{dx} + u\frac{dy_1}{dx}) \\ &= y_1\frac{d^2y_1}{dx^2} + 2\frac{du}{dx}\frac{dy_1}{dx} + u\frac{d^2y_1}{dx^2}\end{aligned} \tag{9}$$

By substituting (8) and (9) to equation-(6), we have:

$$y_1\frac{d^2u}{dx^2}+[2\frac{dy_1}{dx}+p(x)y_1]\frac{du}{dx}+[\frac{d^2y_1}{dx^2}+p(x)\frac{dy_1}{dx}+q(x)y_1]u=0 \tag{10}$$

For the third term on the left hand side of (10), since we already assume that y_1 is a special solution to equation-(6), we should have:

$$\frac{d^2y_1}{dx^2}+p(x)\frac{dy_1}{dx}+q(x)y_1=0 \tag{11}$$

Therefore, (10) becomes:

$$y_1\frac{d^2u}{dx^2}+[2\frac{dy_1}{dx}+p(x)y_1]\frac{du}{dx}=0 \tag{12}$$

Then we define another function z(x) as:

$$z=\frac{du}{dx} \tag{13}$$

and equation-(12) becomes:

$$y_1\frac{dz}{dx}+[2\frac{dy_1}{dx}+p(x)y_1]z=0 \tag{14}$$

Rearranging equation-(14), we could get:

$$\begin{aligned}\frac{dz}{z}&=-\frac{dx}{y_1}[2\frac{dy_1}{dx}+p(x)y_1]\\&=-[\frac{2dy_1}{y_1}+p(x)dx]\end{aligned} \tag{15}$$

Integrating (15) on both sides, we have:

$$\begin{aligned}&\int\frac{dz}{z}=-[\int\frac{2y_1}{y_1}+\int p(x)dx]\\ \Rightarrow\quad & lnz=-[2lny_1+\int p(x)dx+c_0],\ c_0 \text{ is constant}\\ \Rightarrow\quad & lnz=ln[y_1^{-2}\cdot e^{\int -p(x)dx}\cdot e^{c_0}]\\ \Rightarrow\quad & z=\frac{c_1}{y_1^2}e^{-\int p(x)dx},\ c_1=e^{c_0}, \text{ which is arbitary constant}\end{aligned} \tag{16}$$

Recalling the definition of function $z(x)$ – equation-(13), we could obtain the expression for function $u(x)$, explicitly as following:

$$\begin{aligned}u&=\int z(x)dx\\&=c_1\int\frac{e^{-\int p(x)dx}}{y_1^2}dx+c_2\end{aligned} \tag{17}$$

Then recalling our guess form for the solution $y(x)$ as given by equation-(7), we should definitely have:

$$\begin{aligned}y&=uy_1\\&=y_1[c_1\int\frac{e^{-\int p(x)dx}}{y_1^2}dx+c_2]\end{aligned} \tag{18}$$

So finally, we successfully build up the general form of the solution for equation-(6), based on the first general solution y_1 which we obtained previously through some specific methods (e.g. the Laplace transformation method we use to for Laguerre's equation). Now it's time to go back to our initial problem – the Laguerre's equation with n=3, which is given at the beginning – equation-(5). By comparing equation-(5) with (6), we could then write down:

$$p(x) = \frac{1-x}{x} \tag{19}$$

$$q(x) = \frac{3}{x} \tag{20}$$

specifically for our Laguerre's equation with n=3. Since we already obtained a special solution y_1 using Laplace transformation method:

$$y_1 = x^3 - 9x^2 + 18x - 6 \tag{21}$$

then we can write down the final GENERAL form of solution for equation-(5), by putting equation-(19) and (21) into equation-(18):

$$\begin{aligned} y = {} & c_2(x^3 - 9x^2 + 18x - 6) \\ & + c_1(x^3 - 9x^2 + 18x - 6) \int \frac{e^{-\int \frac{1-x}{x}dx}}{(x^3 - 9x^2 + 18x - 6)^2} dx \end{aligned} \tag{22}$$

The complex integration in the second term should lead to the so-called confluent hypergeometric function of the first kind, which is discussed in details in the link (click on the blue link to go to the webpage).
Having the two general solutions for homogeneous second-order ODE, then we can use the so-called constant-variation method to build up the general solution of corresponding inhomogeneous ODE. First of all, let's write down the general form of inhomogeneous second-order ODE:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = f(x) \tag{23}$$

Assuming we have alreadu obtained the two general solutions y_1 and y_2 for the homogeneous equation corresponding to (23), then the next step to give the general solution for equation-(23) is to obtain a special solution y_s. And this is just where the constant-variation method comes to help. First of all, assume the form of special solution is as following:

$$y_s = u_1 y_1 + u_2 y_2 \tag{24}$$

where u_1 and u_2 are undetermined functions. Here it should be pointed out that there should be many special solution for equation-(23), and that's why it's special but not general solution. Following (24) we have:

$$y_s' = u_1 y_1' + u_2 y_2' + y_1 u_1' + y_2 u_2' \tag{25}$$

If substituting the solution form (24) and its corresponding first and second derivative to equation-(23), there will be two undetermined functions u_1 and u_2 with only one equation. Theoretically, the corresponding function is not solvable. Thus we have to introduce more conditions, and since we are only going to find a special solution, the selection of extra condition is arbitary. For the convenience of calculation, the following condition is given:

$$y_1 u_1' + y_2 u_2' = 0 \tag{26}$$

Thus we have:

$$y_s' = u_1 y_1' + u_2 y_2' \tag{27}$$

$$y_s'' = u_1' y_1'' + u_2' y_2'' + u_1 y_1' + u_2 y_2' \tag{28}$$

By substituting (27) and (28) into (23), and noticing that y_1 and y_2 are the general solution to the homogeneous equation corresponding to equation-(23), we could then obtain:

$$u_1' y_1' + u_2' y_2' = f(x) \tag{29}$$

Together with the arbitarily introduced extra condition-(26), we have:

$$\begin{cases} y_1u_1'+y_2u_2'=0 \\ y_1'u_1'+y_2'u_2'=f(x) \end{cases} \tag{30}$$

Then we can solve (30) to obtain u_1' and u_2' as:

$$u_1'=\frac{-y_2f(x)}{y_1y_2'-y_2y_1'} \tag{31}$$

$$u_2'=\frac{y_1f(x)}{y_1y_2'-y_2y_1'} \tag{32}$$

It should be noticed that (31) and (32) requires the general solution y_1 and y_2 to be linear independent. Furthermore, we can obtain u_1 and u_2 as:

$$u_1=-\int\frac{y_2f(x)}{y_1y_2'-y_2y_1'}dx \tag{33}$$

$$u_2=\int\frac{y_1f(x)}{y_1y_2'-y_2y_1'}dx \tag{34}$$

Thus finally we find a special solution y_s for equation-(23):

$$y_s=-y_1\int\frac{y_2f(x)}{y_1y_2'-y_2y_1'}dx+y_2\int\frac{y_1f(x)}{y_1y_2'-y_2y_1'}dx \tag{35}$$

Finally, we can write down the general solution for equation-(23):

$$y=c_1y_1+c_2y_2-y_1\int\frac{y_2f(x)}{y_1y_2'-y_2y_1'}dx+y_2\int\frac{y_1f(x)}{y_1y_2'-y_2y_1'}dx \tag{36}$$

It is remarkable that the method discussed above is applicable to both constant and variant coefficient second-order ODE.
For some second-order ODE with specific form, the solving is straightforward without going through the steps discussed above. Firstly, for constant coefficient second-order ODE, if the inhomogeneous term is with the form of $f(x)=(ax+b)e^{\lambda x}$ or $f(x)=(acos\beta x+bsin\beta x)e^{\alpha x}$, the special solution can be directly written down with corresponding undetermined constants. Detailed discuss can be found in the on-line material ($Page_{12-14}$): Click Me.
Also, for some inhomogeneous second-order ODE with specific form, they can be solved through reducing the order to first order ODE, which is easier to handle. The first case is:

$$y''=f(x,y') \tag{37}$$

for which the right-hand side does not explicitly contain y. To solve (37), we can define $y'=p$, then we have:

$$y''=\frac{dp}{dx}=p' \tag{38}$$

Then (37) becomes:

$$p'=f(x,p) \tag{39}$$

which is first order ODE. Assuming the general solution for (39) is $p=\phi(x,c_1)$, then we have:

$$\frac{dy}{dx}=\phi(x,c_1) \tag{40}$$

Thus the general solution for (37) is:

$$y=\int\phi(x,c_1)dx+c_2y_2 \tag{41}$$

The second case is:

$$y''=f(y,y') \tag{42}$$

for which the right-hand side does not contain x, explicitly. The method for solving (42) is still defining $y' = p$, however here we write down the second derivative as:

$$y'' = \frac{dp}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx} = p\frac{dp}{dy} \tag{43}$$

Then (42) becomes:

$$p\frac{dp}{dy} = f(y, y') \tag{44}$$

Again (44) is first order ODE ($y' = p$ as the function of y), which we can solve to obtain the general solution as:

$$y' = p = \frac{dy}{dx} = \phi(y, c_1) \tag{45}$$

By rearranging (45) and integrati, we could obtain:

$$\int \frac{dy}{\phi(y, c_1)} = x + c_2 \tag{46}$$

which is just the general solution for (42).
Here is given the link for several references, on which the discussion in this article is based:

- Methods for solving variant coefficient second-order ODE (in Chinese).
- Reducible second-order ODE (in Chinese).
- About second-order ODE.