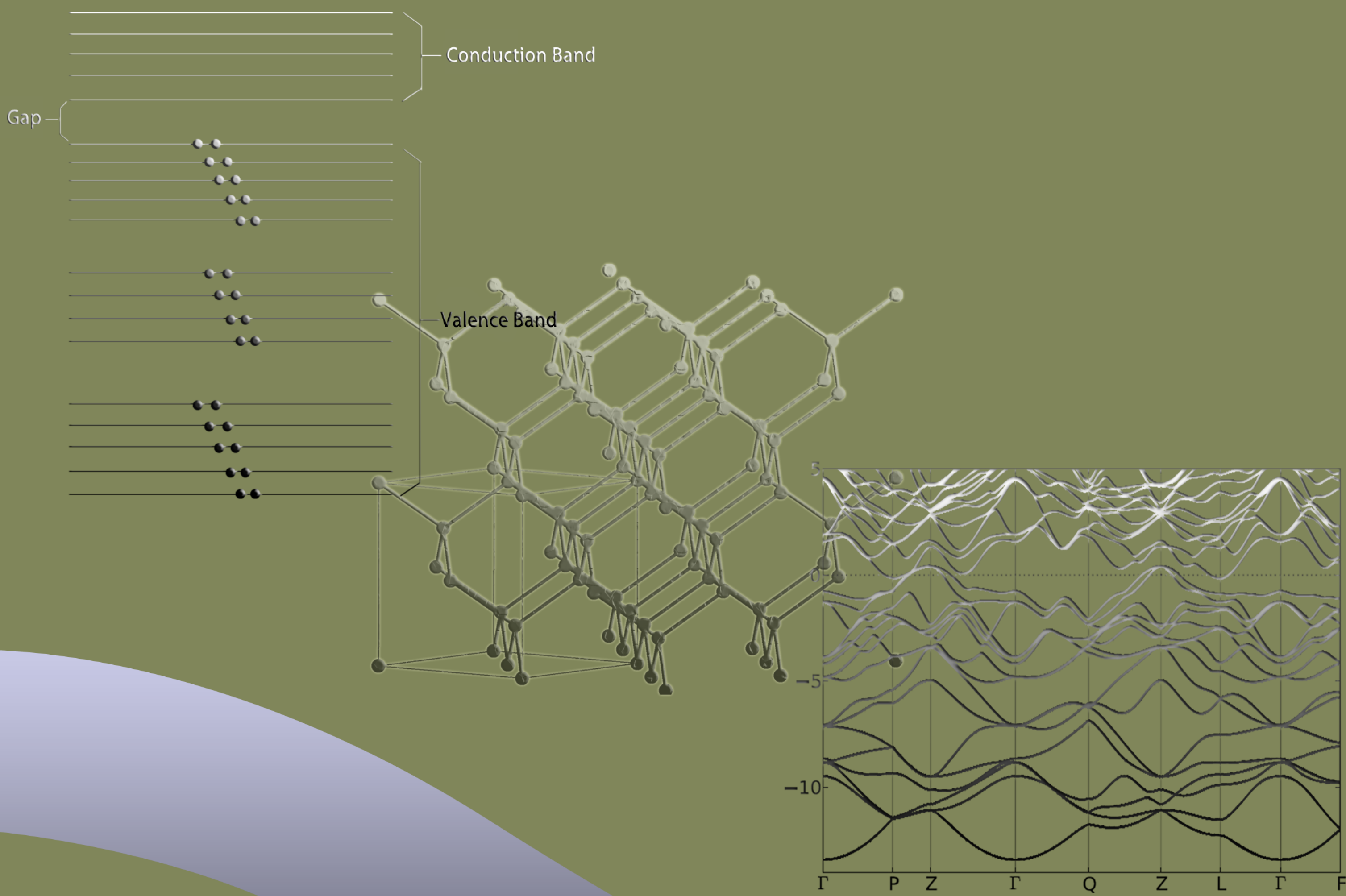
Queen Mary, University of London
School of Physics and Astronomy

General Notes On Mathematics in Physics



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About Inversion

Inversion is an operation acting on the generalized circle, and through the inversion operation, the generalized circle is mapped to another generalized circle. Here the *generalized* circle may mean real circle with certain radius, or it can mean a line, which can be imagined as a circle with infinite radius. As shown in Fig. 1, The inversion of point P with respect to the circle gives another point $P^{'}$, and vice versus. And the following equation should be satisfied for the geometry related to the inversion operation, which actually defines the way how an inversion should be done:

$$OP \times OP^{'} = r^2 \tag{1}$$

where r is the radius of the circle in Fig. 1.

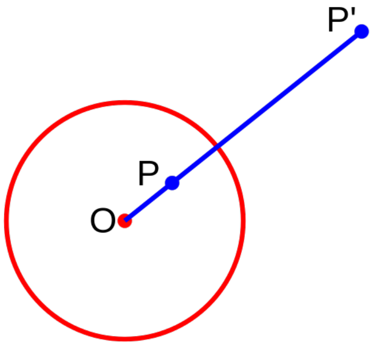

Figure 1 The inversion of a point with respect to a given circle (shown in red). The figure given here is from Wikipedia - see the Wikipedia term of Inversion Geometry.

The inversion of a generalized circle is just the set of inversive points corresponding to each point on the initial circle. Also it is worthy pointing out that the mathematical definition of the inversion operation given in (1) itself gives a way of finding the inversion of a point with respect to a given circle. However, sometimes the geometrical method which does not even need a calculation makes life easier. On the above Wikipedia page, it was also given a geometrical way to find the inversion of a point with respect to a give circle, where the point can be inside or outside the circle (according to the definition of inversion - (1), if the point is on the circle, the inversion point is just the point itself). Fig. 2 illustrates the idea.

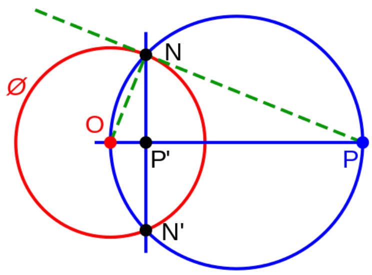

Figure 2 The geometrical way to find the inversion point. The figure given here is from Wikipedia - see the Wikipedia term of Inversion Geometry.

As shown in Fig, 2, to find the inversion of point P with respect to the circle ϕ, first we need to draw the tangential line (through the point of P) of circle ϕ, which gives us the tangential point of N (or N') on the circle. Then through point N or N', we draw a line perpendicular to the line OP, which actually gives the line of NN'. The cross point of line NN' and OP is just what we are going to find – the inversion of point P with respect to the circle ϕ. The reasoning is quite simple: the triangle NOP' and NOP shares the common angle of $\angle NOP$, Also the $\angle ONP$ (in triangle NOP) and $\angle OP'N$ (in triangle NOP') are both right angle. Therefore the triangle NOP' and NOP are similar, which gives the following relation:

$$\frac{ON}{OP'} = \frac{r}{OP'} = \frac{OP}{ON} = \frac{OP}{r}$$
$$\Rightarrow$$
$$r^2 = OP' \times OP \tag{2}$$

That means P' is exactly the inversion of point P with respect to circle ϕ. The above discussion is just some basic understanding of the notion of inversion, as given on Wikipedia, for detailed discussion, refer to the link given above.

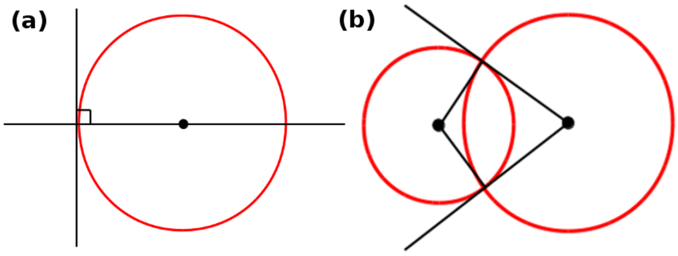

Figure 3 The orthogonality of (a) line and (b) circle with the given circle.

Before ending this session, there are two things to mention. The first one is the inversion is invertible, which means the inversion of inversion leads back to where we start. The second thing is about one of the properties of inversion operation given on Wikipedia – 'A circle (or line) is unchanged by inversion if and only if it is orthogonal to the reference circle at the points of intersection'. And here there may be questions about what it means by 'a circle or a line is orthogonal to the reference circle'. For a line and a circle, 'orthogonal at the intersection point' means the line goes through the center of the circle – the part of the circle at the intersection point can be regarded as a line – the tangential line at the intersection point is perpendicular to the given line. And for the orthogonality of two circles, the situation is quite similar. The illustration for the orthogonality between generalized circles is given in Fig. 3.