Poisson Bracket and Commutator

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The defination of Poisson Bracket is:

$$\{A,B\}_P \stackrel{def}{=} \sum_i \left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right) \tag{1}$$

And the definition for commutator:

$$[\hat{A},\hat{B}]_C \stackrel{def}{=} \hat{A}\hat{B} - \hat{B}\hat{A} \tag{2}$$

Both of them have similar algebraic properties:

1. Linearity: $[\alpha_1 A_1 + \alpha_2 A_2, B] = \alpha_1[A_1, B] + \alpha_2[A_2, B]$ and $[A, \beta_2 B_1 + \beta_2 B_2] = \beta_1[A, B_1] + \beta_2[A, B_2]$.
2. Antisymmetry: $[A, B] = -[B, A]$.
3. Leibniz rules: $[AB, C] = A[B, C] + [A, C]B$ and $[A, BC] = B[A, C] + [A, B]C$.
4. Jacobi Identity: [A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.

Later we will see that it is the above algebraic properties that determine the relationship between Poisson Bracket and commutator.
Before going to discuss the relationship between Poisson Bracket and commutator, there is an important aspect (however may be easily ignored) for Poisson Bracket. The definition of Poisson Bracket indicates that quantity A and B are both the function of ps and qs, where p and q are series of coordinates. Most importantly, the number of coordinates p and q should be the same if A and B can be fed into the definition of Poisson Bracket as given by (1). If thinking of the Lagrangian and Hamiltonian mechanics, we can then take qs and ps as the general coordinates and momentum, respectively. Then it is natual that the number of general coordinates qs should be definitely equal to that of general momentum ps, since each 'particle' should have one general coordinate and exactly one value of moment. So generally, the quantity A and B can be written as:

$$A = A(q_1, q_2, \cdots, q_N; p_1, p_2, \cdots, p_N) \tag{3}$$
$$B = B(q_1, q_2, \cdots, q_N; p_1, p_2, \cdots, p_N) \tag{4}$$

Now the relationship between Poisson Bracket and commutator will be discussed following the idea of Poisson Brackets and Commutator Brackets. Starting from calculating the quantity: $[AU, BV]$ using the Leibniz rules as shown above, we have:

$$\begin{aligned}[AU, BV] &= A[U, BV] + [A, BV]U \\ &= AB[U, V] + A[U, B]V + B[A, V]U + [A, B]VU\end{aligned} \tag{5}$$

Then if we use the Leibniz rules in the opposite order, we get a different expression of $[AU, BV]$:

$$\begin{aligned}[AU, BV] &= B[AU, V] + [AU, B]V \\ &= BA[U, V] + B[A, V]U + A[U, B]V + [A, B]UV\end{aligned} \tag{6}$$

Definitely, we should have (5) and (6) identical to each other, since we are calculating exactly the same quantity $[AU, BV]$. That means we can equalize the right hand side of (5) and (6) to give us:

$$AB[U, V] + [A, B]VU = BA[U, V] + [A, B]UV$$

$\Rightarrow$

$$(AB - BA)[U, V] = [A, B](UV - VU)$$

$\Rightarrow$

$$[U, V](UV - VU)^{-1} = (AB - BA)^{-1}[A, B] \tag{7}$$

where we assume that quantity A and B are non-commuting. (If they are commuting, the above discussion – except the last step in (7) – still stands but trivial)
Since the quantity A, B, U and V discussed above are selected artificially, (7) indicates that both sizes should be equal to the same constant. If setting this constant as c, we should have:

$$[A,B] = c(AB - BA) \text{ and } [U,V] = c(UV - VU) \tag{8}$$

This says that for non-commuting quantities, any bracket $[A,B]$ (or $[U,V]$, or any other pair of quantities) with the above algebraic properties 1 through 4, is proportional to the commutator. More importantly, no matter we are talking about quantities pair (A, B) or (U, V), or ANY other pair of quantities, they share exactly the same constant 'c' as shown above in (8), if we are talking about the same bracket for all cases.
Another thing to mention is that if we have A and B as operators, how are we going to feed them into the Poisson Bracket to get the similar expression of the relationship between Possion Bracket and commutator for A and B? Actually, Poisson Bracket should operate on functions but not on operators, so the euqal sign from (5) to (8) is not the commonly spoken 'equal', exactly, maybe we should say equivalent. That means we need to change the operator A and B withing Poisson Bracket to their counterpart of function form. Thus the exact expression for the relatioship between classical Poisson Bracket and quantum commutator for operator A and B (or Poisson Bracket, they are functions) should be:

$$\{A,B\}_P \Leftrightarrow c_{cq}[A,B]_C \tag{9}$$

Then the last question is: what is the constant c_{cq}? In Susskind's book about quantum machenis (Page112-113)[1], the following two equations are given:

$$\frac{d\hat{L}}{dt} = -\frac{i}{\hbar}[\hat{L},\hat{H}]_C \tag{10}$$
$$\frac{dL}{dt} = \{L,H\}_P \tag{11}$$

where the first equation (10) is obtained based on the Schrödinger equation (details see the discussion in Susskind's book – Page109-112). And the derivation for the second equation (11) is given in another book of Susskind about classical mechanics (Page171-172)[2]. Combining (10) and (11), we could easily obtain the following expression:

$$\{L,H\}_P \Leftrightarrow i\hbar[\hat{L},\hat{H}]_C \tag{12}$$

As discussed above, the relatioship between bracket (with the above four properties) and commutator does not depend on the two quantities involved, thus generally we have:

$$\{A,B\}_P \Leftrightarrow i\hbar[\hat{A},\hat{B}]_C \tag{13}$$

which is the relationship between Poisson Bracket and quantum commutator for any two quantities A and B.

References

1 S. Leonard and F. Art, *Quantum Mechanics - The Theoretical Minimum*, Basics Books, A Member of the Perseus Books Group, New York, 2014.

2 S. Leonard and H. George, *Classical Mechanics - The Theoretical Minimum*, Basics Books, A Member of the Perseus Books Group, New York, 2013.