SE274 Data Structure

Lecture 8: Search Trees, Part 4

(textbook: Chapter 11, Red-Black Tree)

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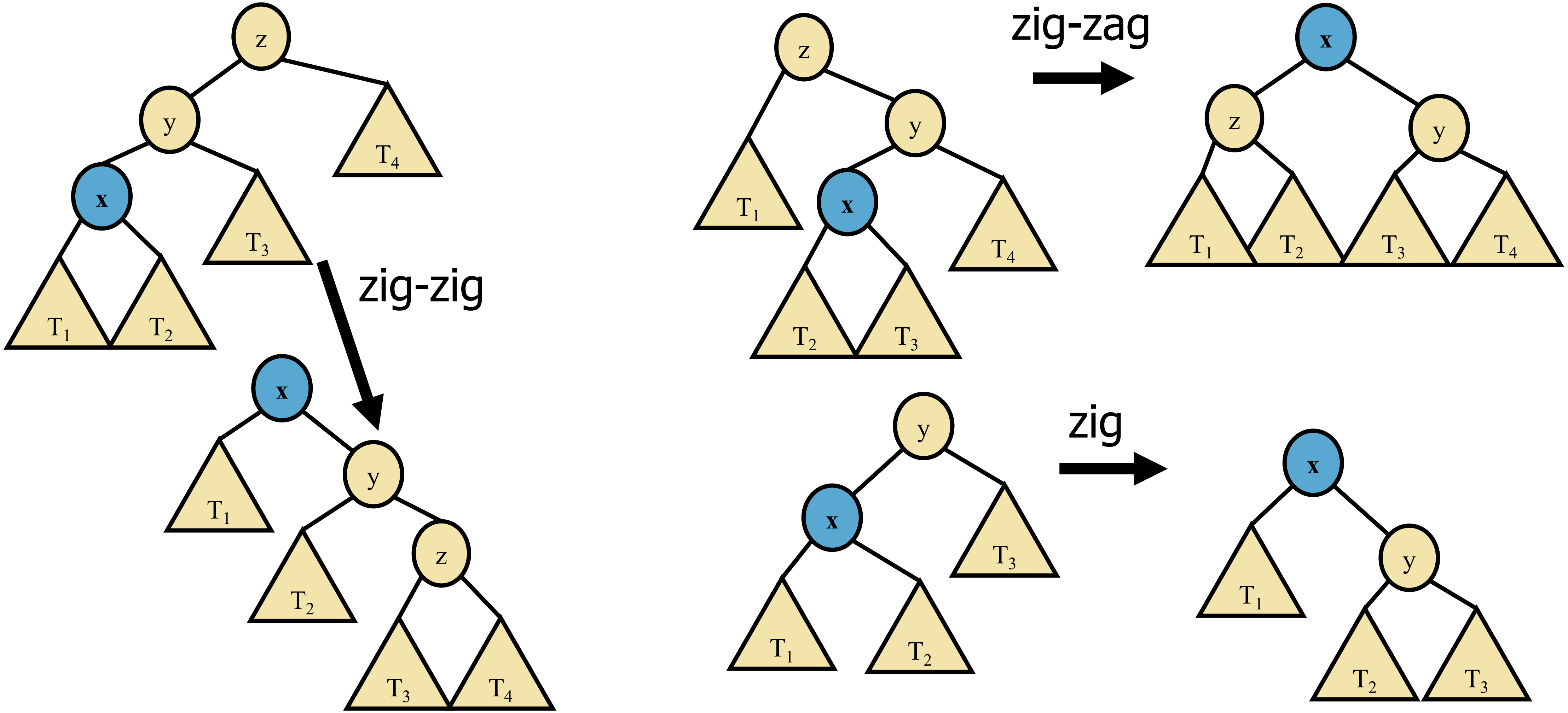
Instructor: Sunjun Kim

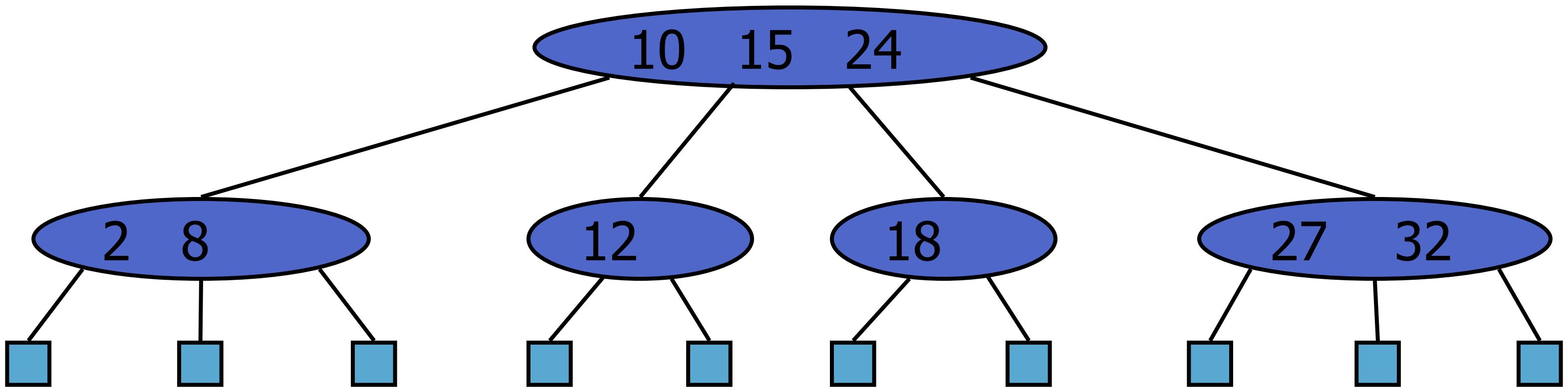
Information&Communication Engineering, DGIST

Recap: Splay tree



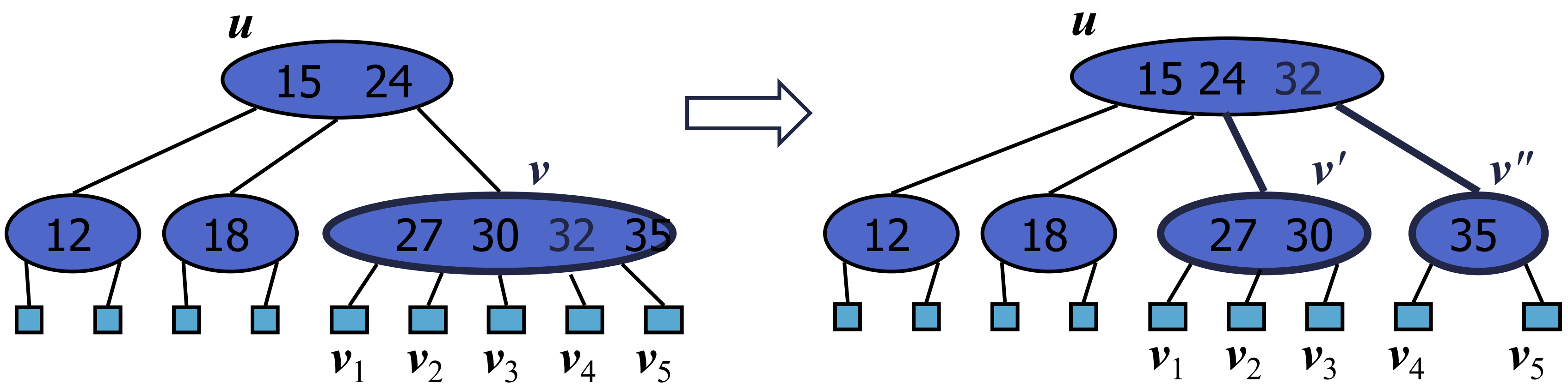
Recap: (2,4) Trees

- A (2,4) tree (also called 2-4 tree or 2-3-4 tree) is a multi-way search with the following properties
  - Node-Size Property: every internal node has at most four children
  - Depth Property: all the external nodes have the same depth
- Depending on the number of children, an internal node of a (2,4) tree is called a 2-node, 3-node or 4-node



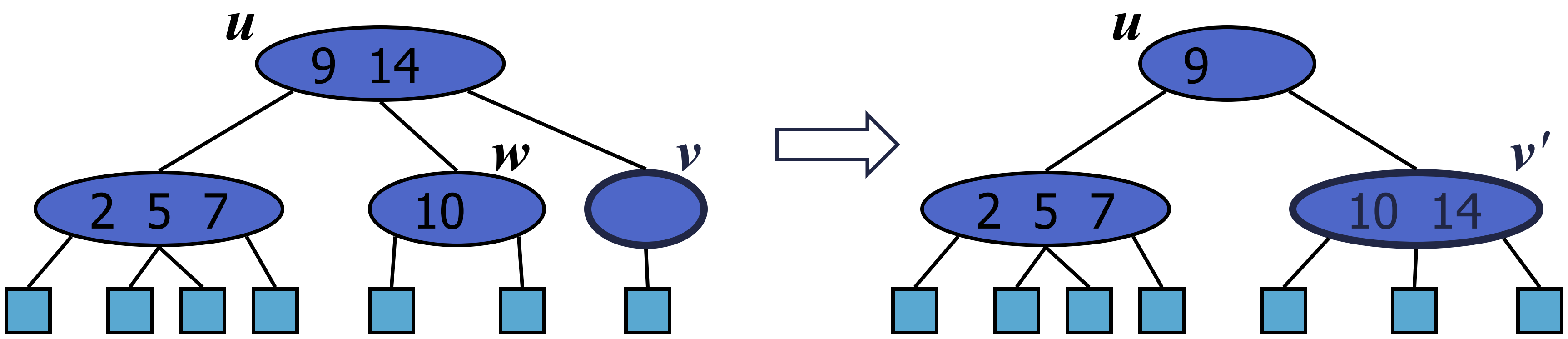
Recap: (2,4) Trees: Overflow and Split

- We handle an overflow at a 5-node $\boldsymbol{v}$ with a split operation:
  - let $v_1 \dots v_5$ be the children of v and $k_1 \dots k_4$ be the keys of v
  - node v is replaced nodes v' and v''
    - v' is a 3-node with keys k_1 k_2 and children v_1 v_2 v_3
    - v'' is a 2-node with key k_4 and children v_4 v_5
  - key k_3 is inserted into the parent u of v (a new root may be created)
- The overflow may propagate to the parent node u



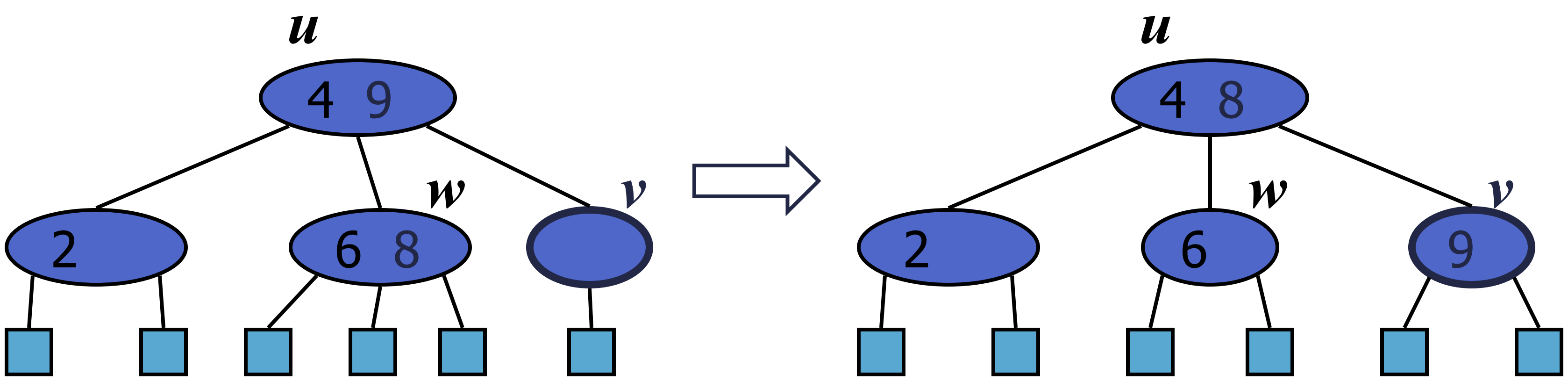
Recap: (2,4) Trees: Underflow and Fusion

- Deleting an entry from a node v may cause an underflow, where node v becomes a 1-node with one child and no keys
- To handle an underflow at node v with parent u, we consider two cases
- Case 1: the adjacent siblings of v are 2-nodes
  - Fusion operation: we merge v with an adjacent sibling w and move an entry from u to the merged node v'
  - After a fusion, the underflow may propagate to the parent u



Recap: (2,4) Trees: Underflow and Transfer

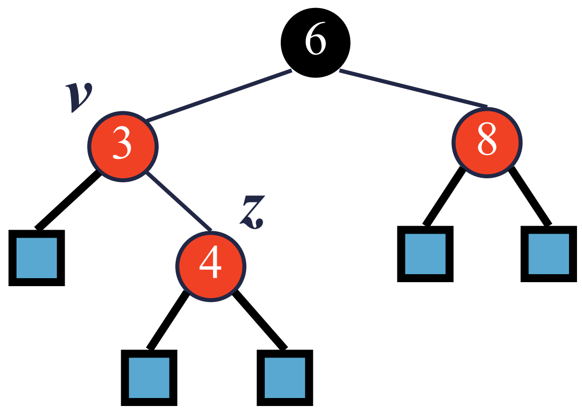
- To handle an underflow at node v with parent u, we consider two cases
- Case 2: an adjacent sibling w of v is a 3-node or a 4-node
  - Transfer operation:
    1. we move a child of w to v
    2. we move an item from u to v
    3. we move an item from w to u
  - After a transfer, no underflow occurs



Comparison of Map Implementations

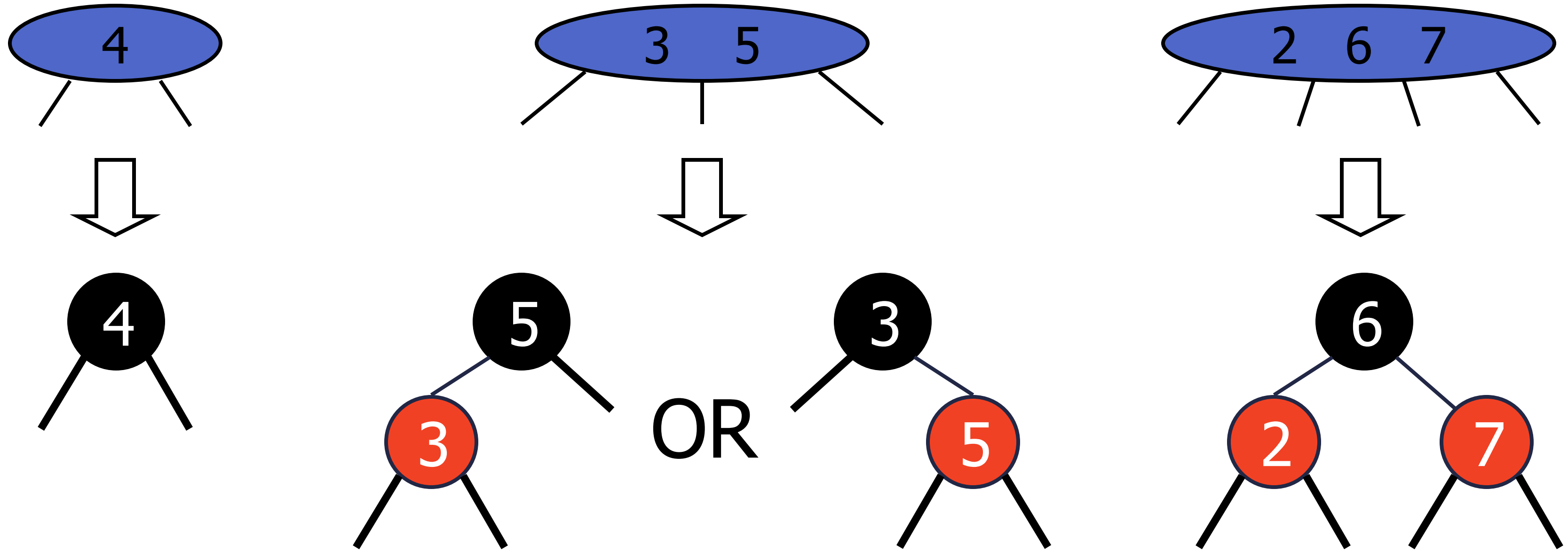
	Search	Insert	Delete	Notes
Hash Table	1 expected	1 expected	1 expected	○ no ordered map methods ○ simple to implement
Skip List	log n high prob.	log n high prob.	log n high prob.	○ randomized insertion ○ simple to implement
AVL and (2,4) Tree	log n worst-case	log n worst-case	log n worst-case	○ complex to implement

Red-Black Trees



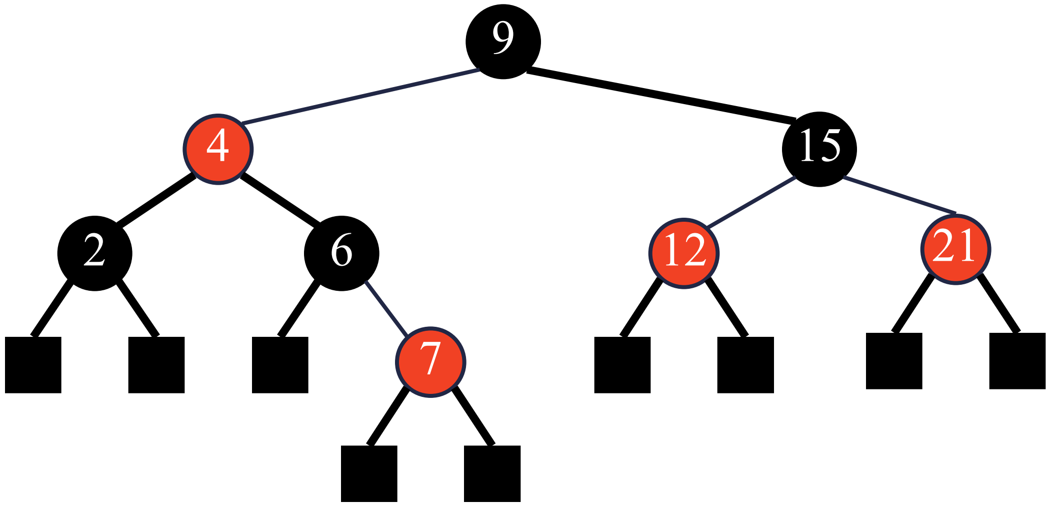
From (2,4) to Red-Black Trees

- A red-black tree is a representation of a (2,4) tree by means of a binary tree whose nodes are colored red or **black**
- In comparison with its associated (2,4) tree, a red-black tree has
  - same logarithmic time performance
  - simpler implementation with a single node type



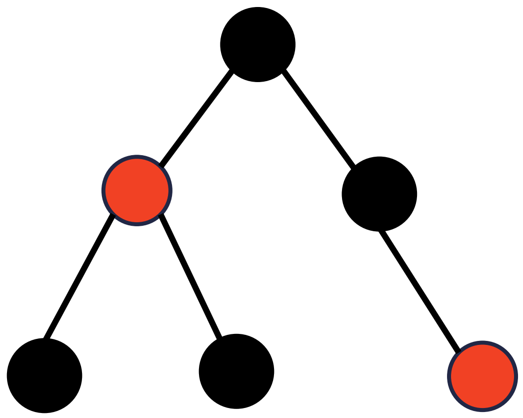
Red-Black Trees

- A red-black tree can also be defined as a binary search tree that satisfies the following properties:
  - Root Property: the root is black
  - External Property: every leaf is black
  - Internal Property: the children of a red node are black
  - Depth Property: all the leaves have the same black depth



Quiz

- Is this a valid red-black tree?



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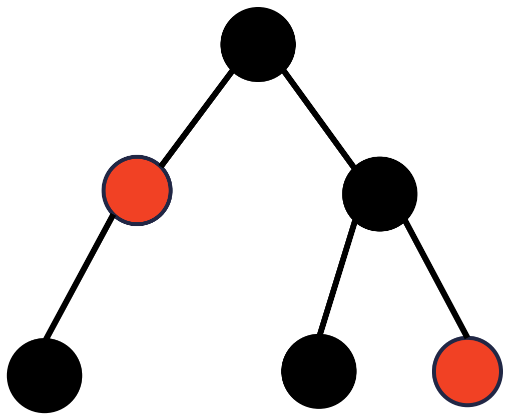

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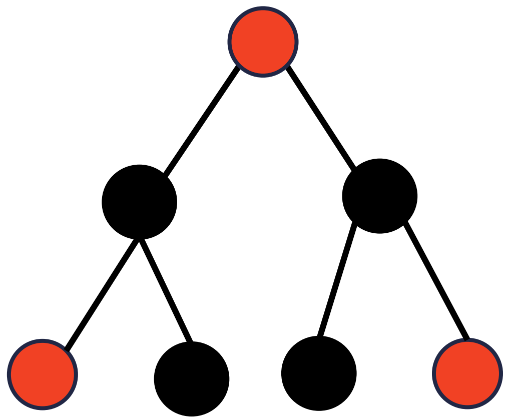

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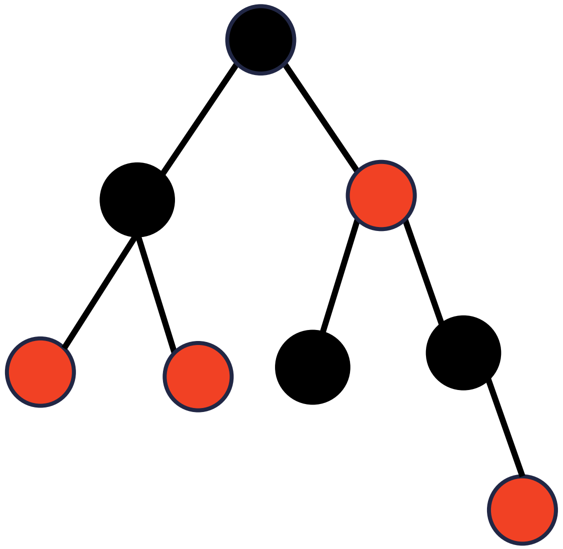

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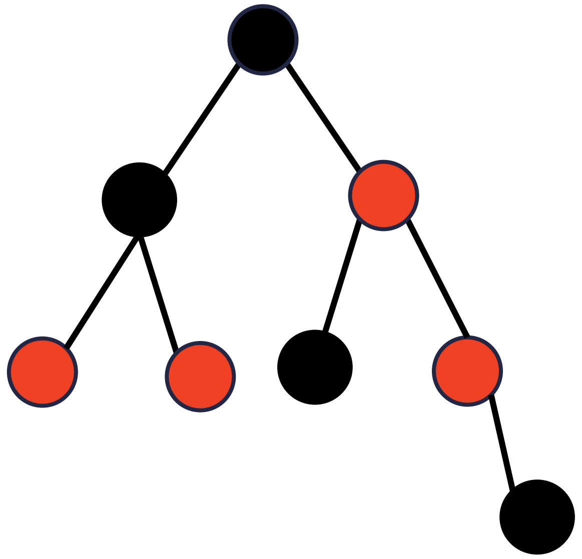

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Height of a Red-Black Tree

- Theorem: A red-black tree storing $\boldsymbol{n}$ items has height $\boldsymbol{O}(\log \boldsymbol{n})$

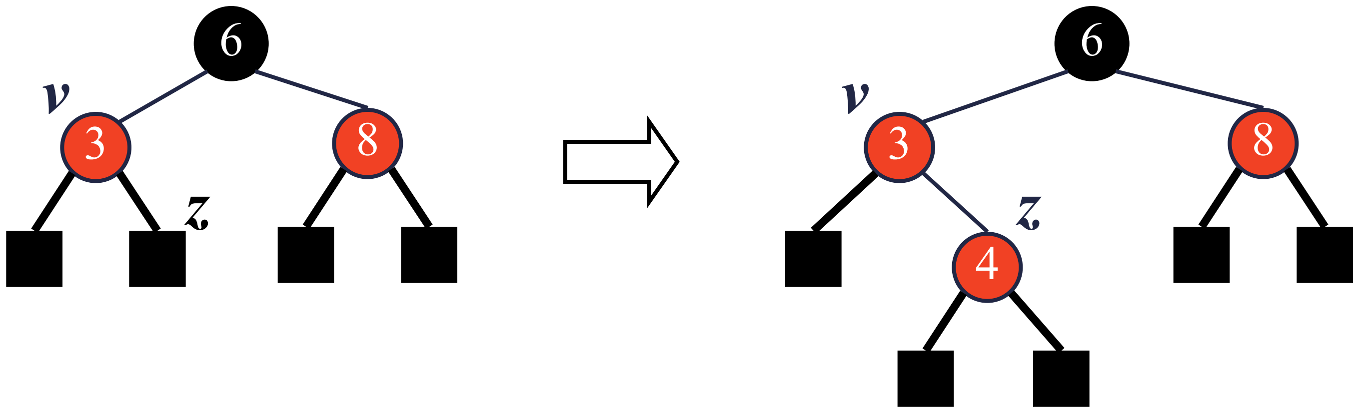
  Proof:

  - The height of a red-black tree is at most twice the height of its associated (2,4) tree, which is $\boldsymbol{O}(\log \boldsymbol{n})$

- The search algorithm for a binary search tree is the same as that for a binary search tree
- By the above theorem, searching in a red-black tree takes $\boldsymbol{O}(\log \boldsymbol{n})$ time

Insertion

- To insert (k, o), we execute the insertion algorithm for binary search trees and color red the newly inserted node z unless it is the root
  - We preserve the root, external, and depth properties
  - If the parent v of z is black, we also preserve the internal property and we are done
  - Else (v is red) we have a double red (i.e., a violation of the internal property), which requires a reorganization of the tree
- Example where the insertion of 4 causes a double red:

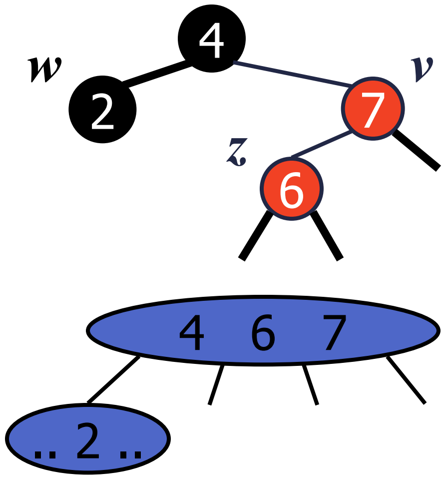


Remedying a Double Red

- Consider a double red with child z and parent v, and let w be the sibling of v

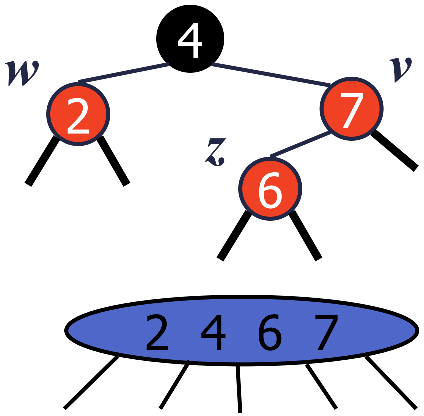
Case 1: w is black

- The double red is an incorrect replacement of a 4-node
- Restructuring: we change the 4-node replacement



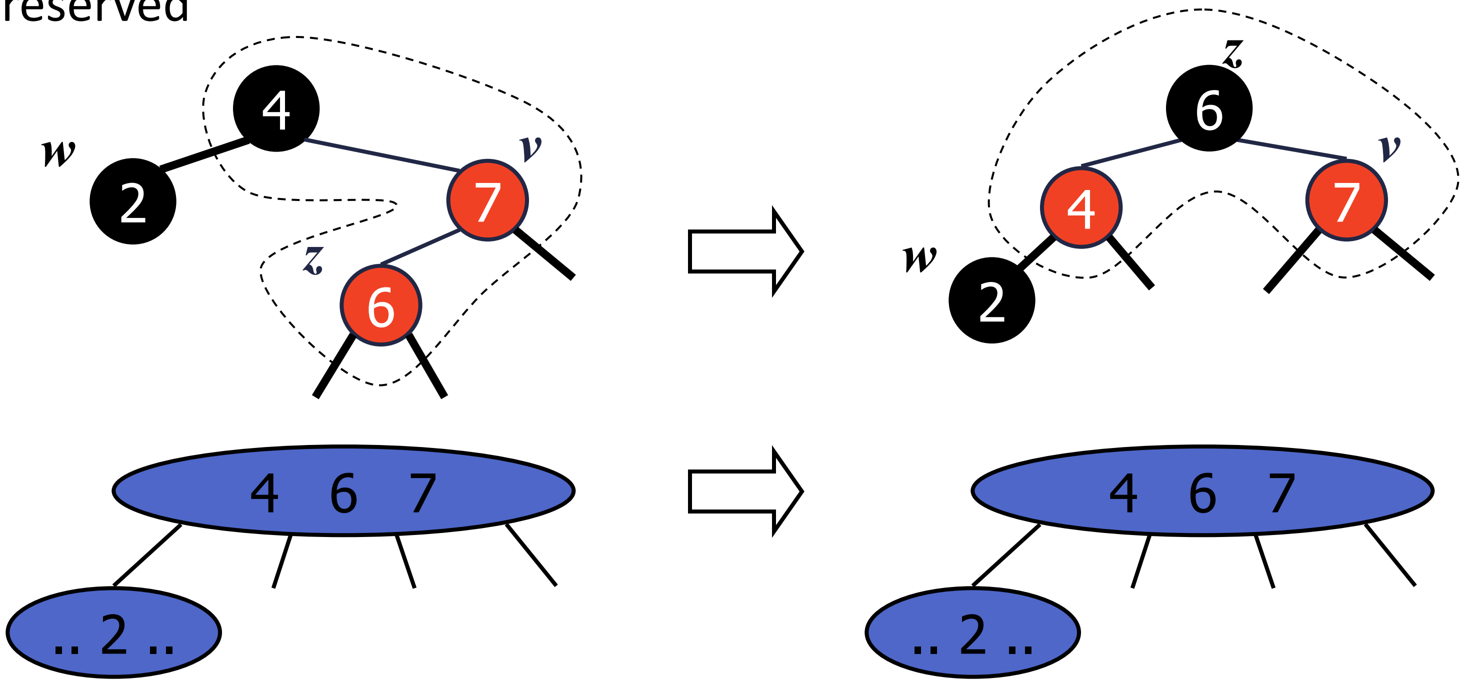
Case 2: w is red

- The double red corresponds to an overflow
- Recoloring: we perform the equivalent of a split



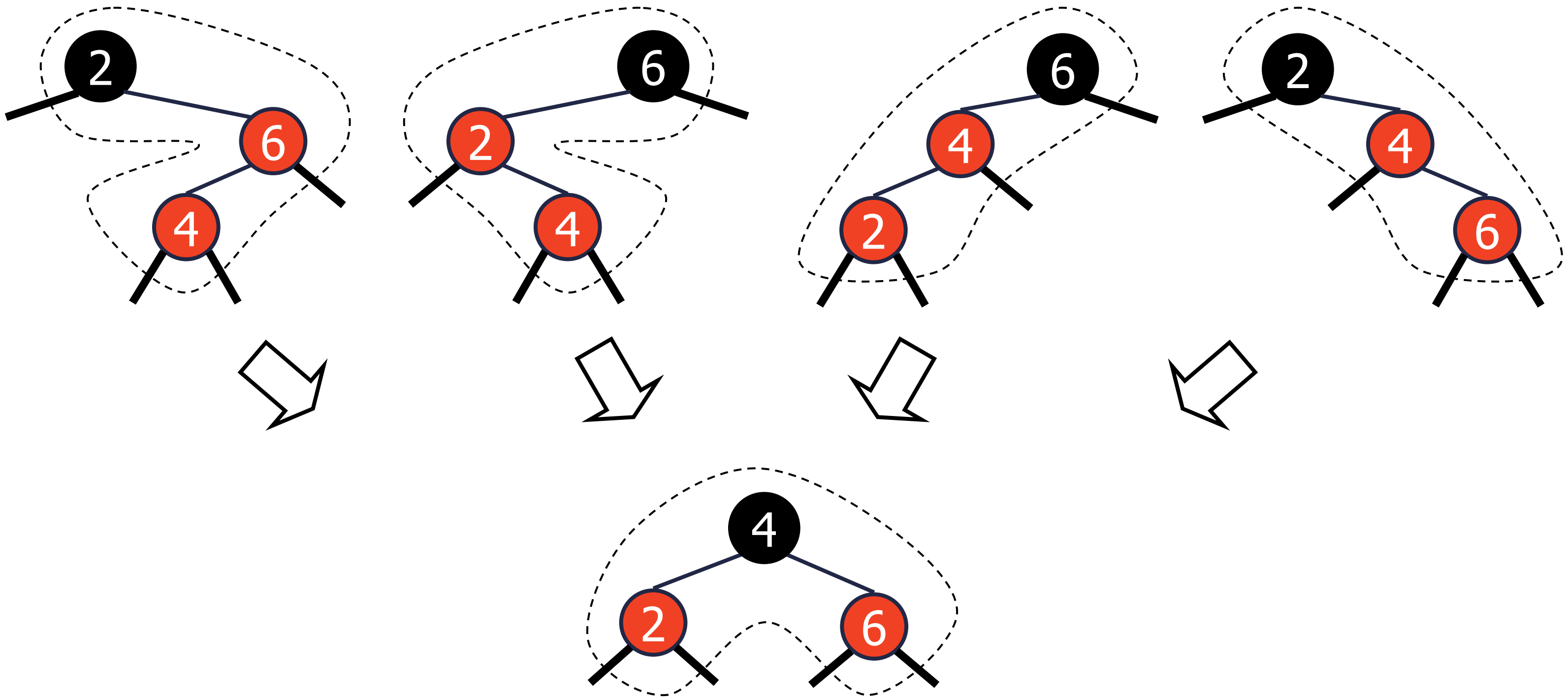
Restructuring

- A restructuring remedies a child-parent double red when the parent red node has a black sibling
- It is equivalent to restoring the correct replacement of a 4-node
- The internal property is restored and the other properties are preserved



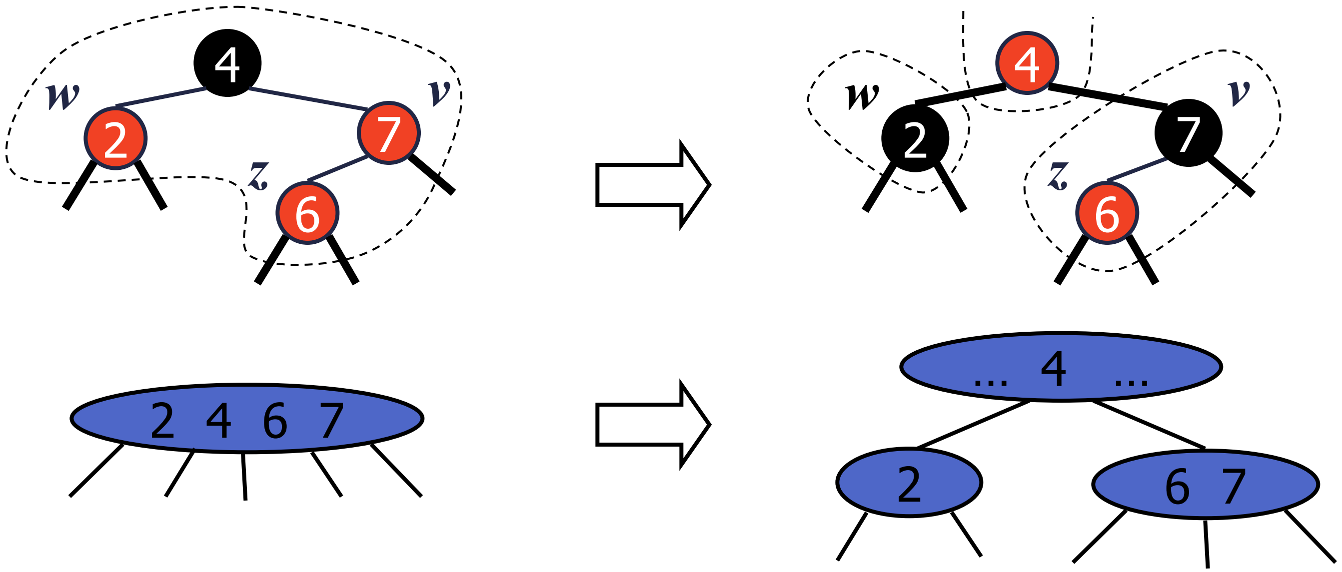
Restructuring (cont.)

- There are four restructuring configurations depending on whether the double red nodes are left or right children

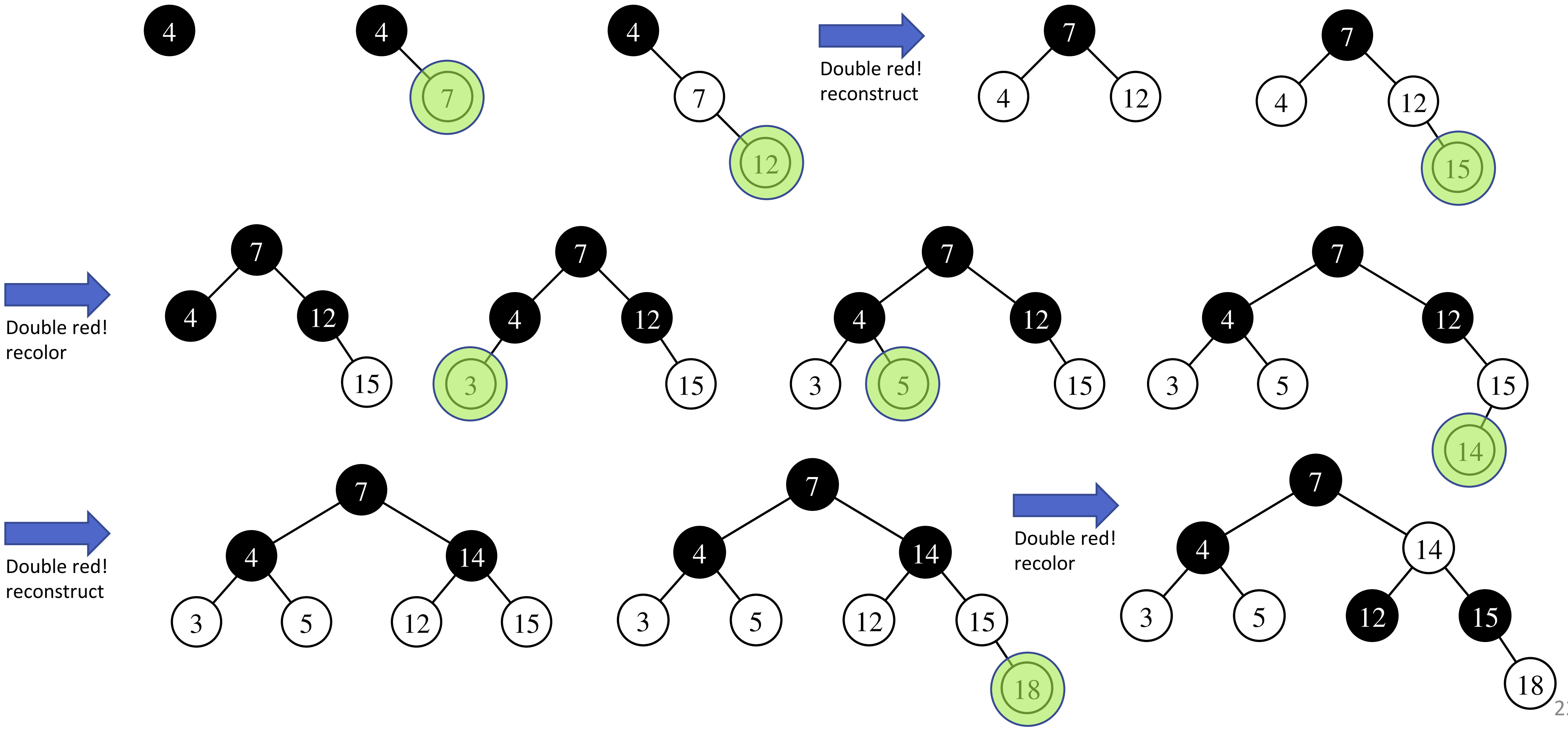


Recoloring

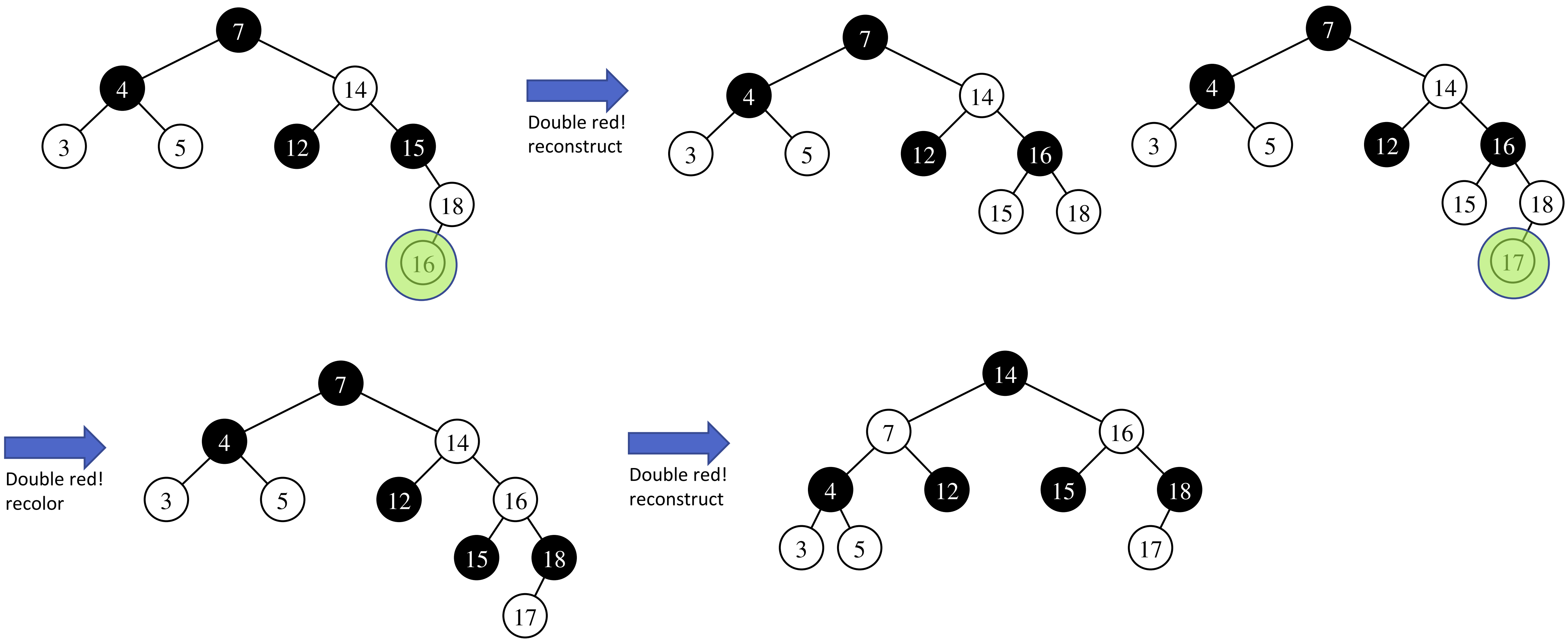
- A recoloring remedies a child-parent double red when the parent red node has a red sibling
- The parent ***v*** and its sibling ***w*** become black and the grandparent ***u*** becomes red, unless it is the root
- It is equivalent to performing a split on a 5-node
- The double red violation may propagate to the grandparent ***u***



Red-Black Insertion: Example



Red-Black Insertion: Example (cont)



Analysis of Insertion

Algorithm ***insert***$(\boldsymbol{k}, \boldsymbol{o})$

1. We search for key $\boldsymbol{k}$ to locate the insertion node $\boldsymbol{z}$

2. We add the new entry $(\boldsymbol{k}, \boldsymbol{o})$ at node $\boldsymbol{z}$ and color $\boldsymbol{z}$ red

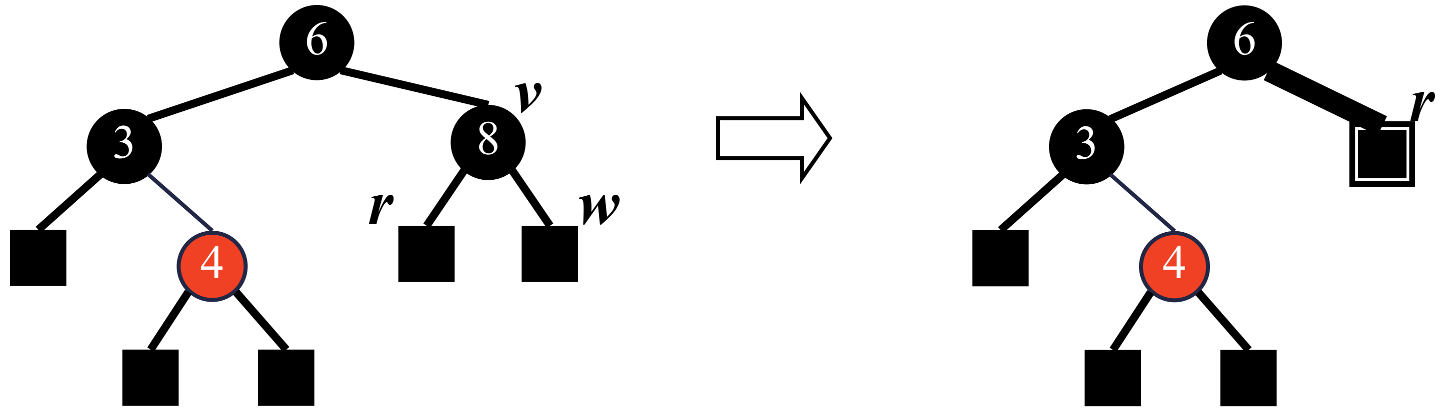
3. **while** ***doubleRed***(z)
   if ***isBlack***(***sibling***(***parent***(z))))
   $z \leftarrow$ ***restructure***(z)
   return
   else { ***sibling***(***parent***(z)) is red }
   $z \leftarrow$ ***recolor***(z)

- Recall that a red-black tree has $O(\log n)$ height
- Step 1 takes $O(\log n)$ time because we visit $O(\log n)$ nodes
- Step 2 takes $O(1)$ time
- Step 3 takes $O(\log n)$ time because we perform
  - $O(\log n)$ recolorings, each taking $O(1)$ time, and
  - at most one restructuring taking $O(1)$ time
- Thus, an insertion in a red-black tree takes $O(\log n)$ time

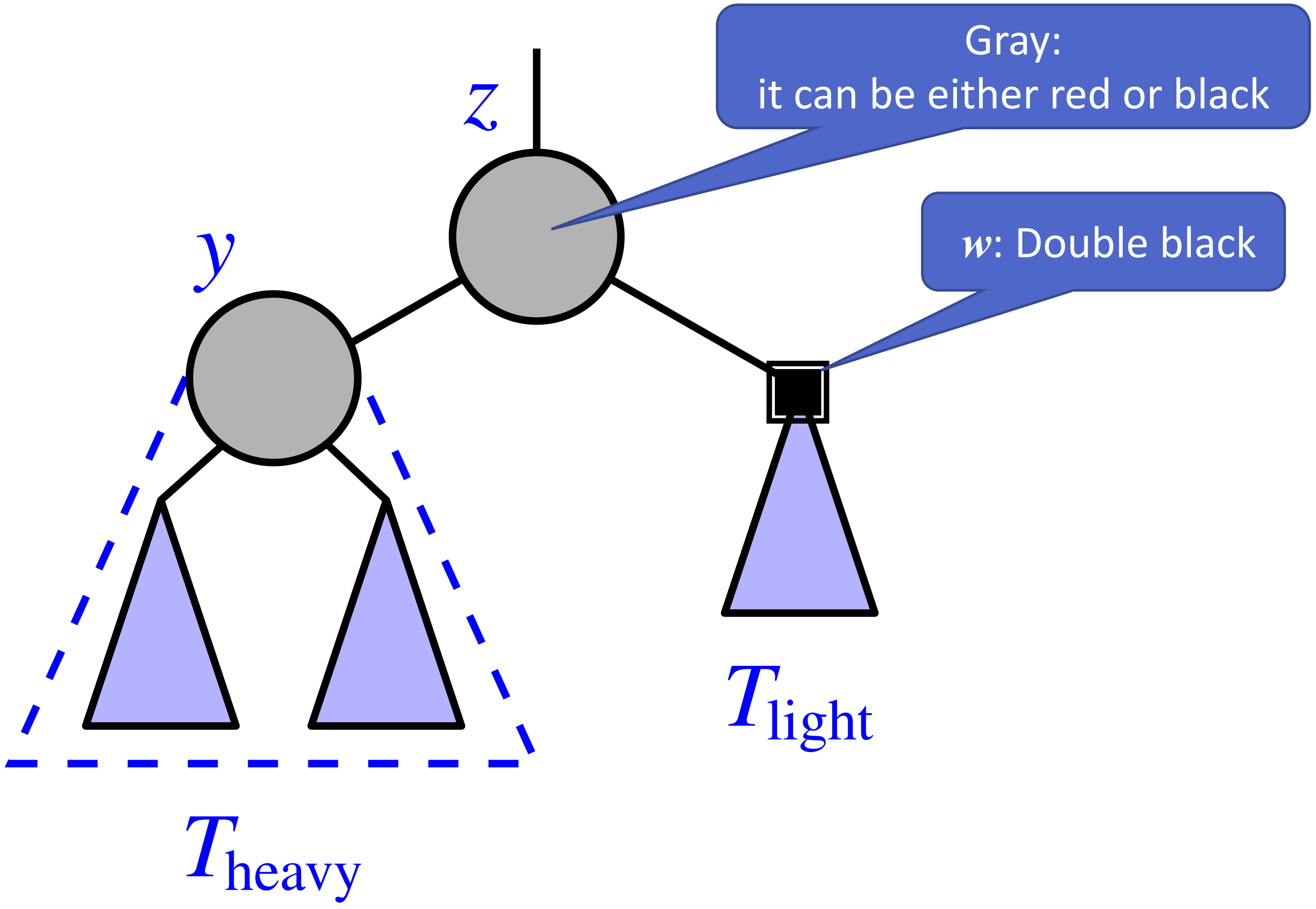
Deletion

- Root Property: the root is black
- External Property: every leaf is black
- Internal Property: the children of a red node are black
- Depth Property: all the leaves have the same black depth

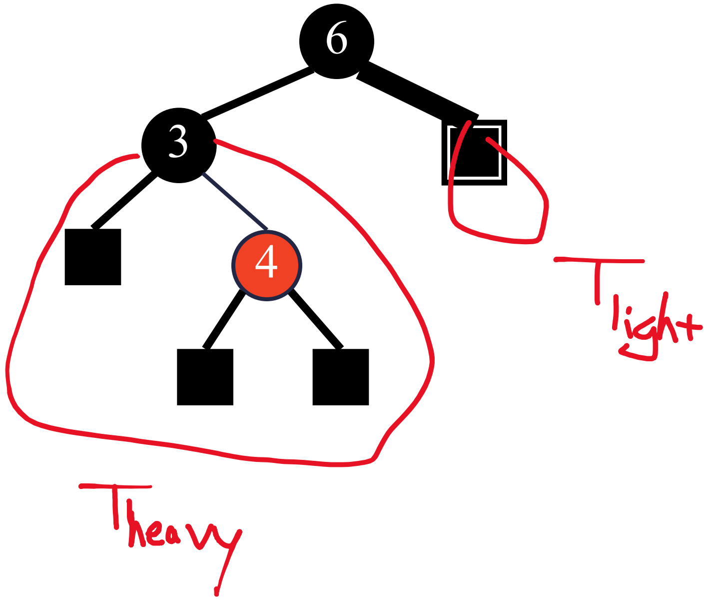
- To perform operation remove(k), we first execute the deletion algorithm for binary search trees
- Let v be the internal node removed, w the external node removed, and r the sibling of w
  - If either v of r was red, we color r black and we are done
  - Else (v and r were both black) we color r ***double black***, which is a violation of the internal property requiring a reorganization of the tree
- Example where the deletion of 8 causes a double black:



T_{heavy} & T_{light}



= has more black depth by 1



Remedying a Double Black

- The algorithm for remedying a double black node $\boldsymbol{w}$ with sibling $\boldsymbol{y}$ considers three cases

  Case 1: $\boldsymbol{y}$ is black and has a red child
  - We perform a restructuring, equivalent to a transfer , and we are done

  Case 2: $\boldsymbol{y}$ is black and its children are both black
  - We perform a recoloring, equivalent to a fusion, which may propagate up the double black violation

  Case 3: $\boldsymbol{y}$ is red
  - We perform an adjustment, equivalent to choosing a different representation of a 3-node
  - After, either Case 1 or Case 2 applies
- Deletion in a red-black tree takes $\boldsymbol{O}(\log \boldsymbol{n})$ time

Case 1: y is black and has a red child

- The algorithm for remedying a double black node w with sibling y considers three cases

  Case 1: y is black and has a red child

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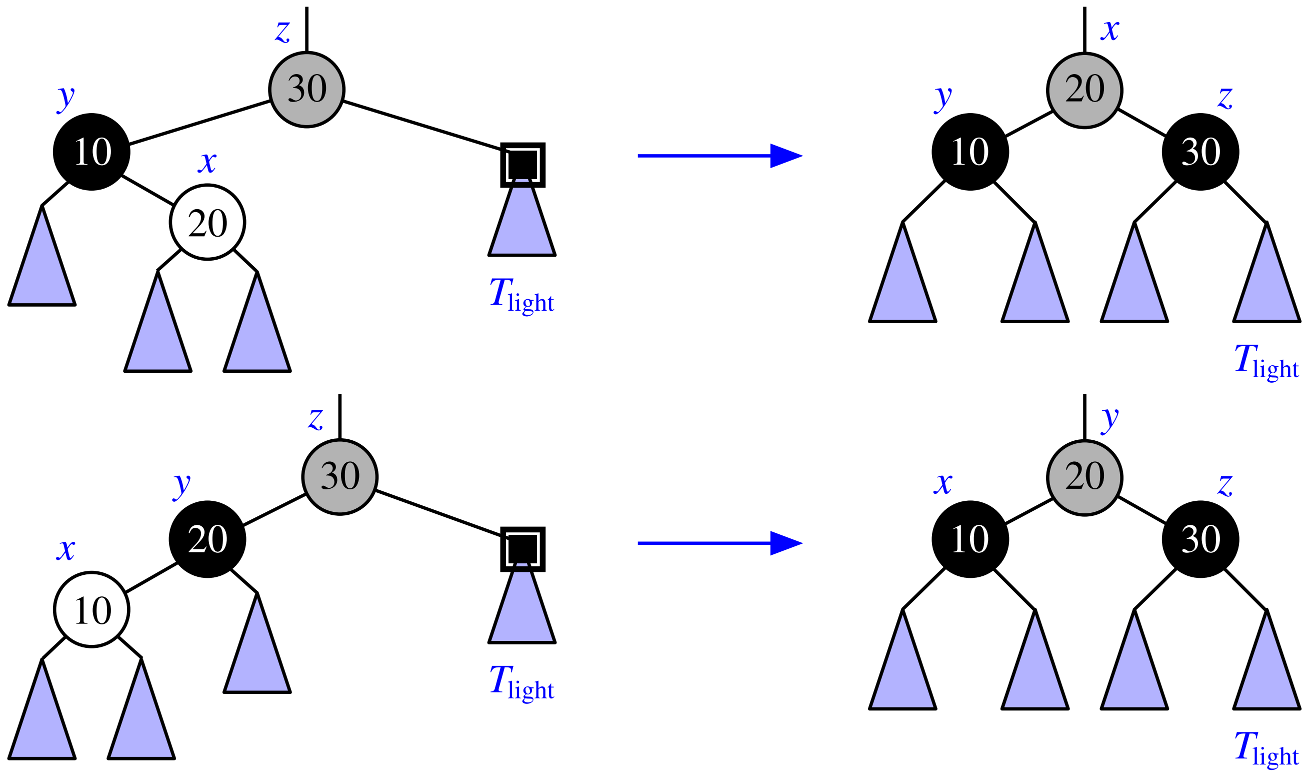

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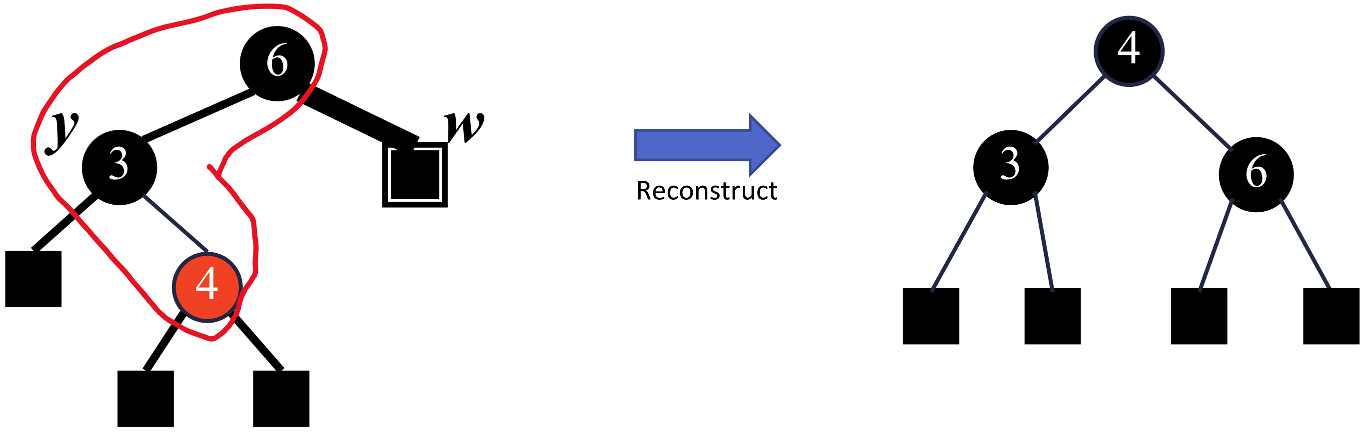

Case 2: y is black and its children are both black

- The algorithm for remedying a double black node w with sibling y considers three cases

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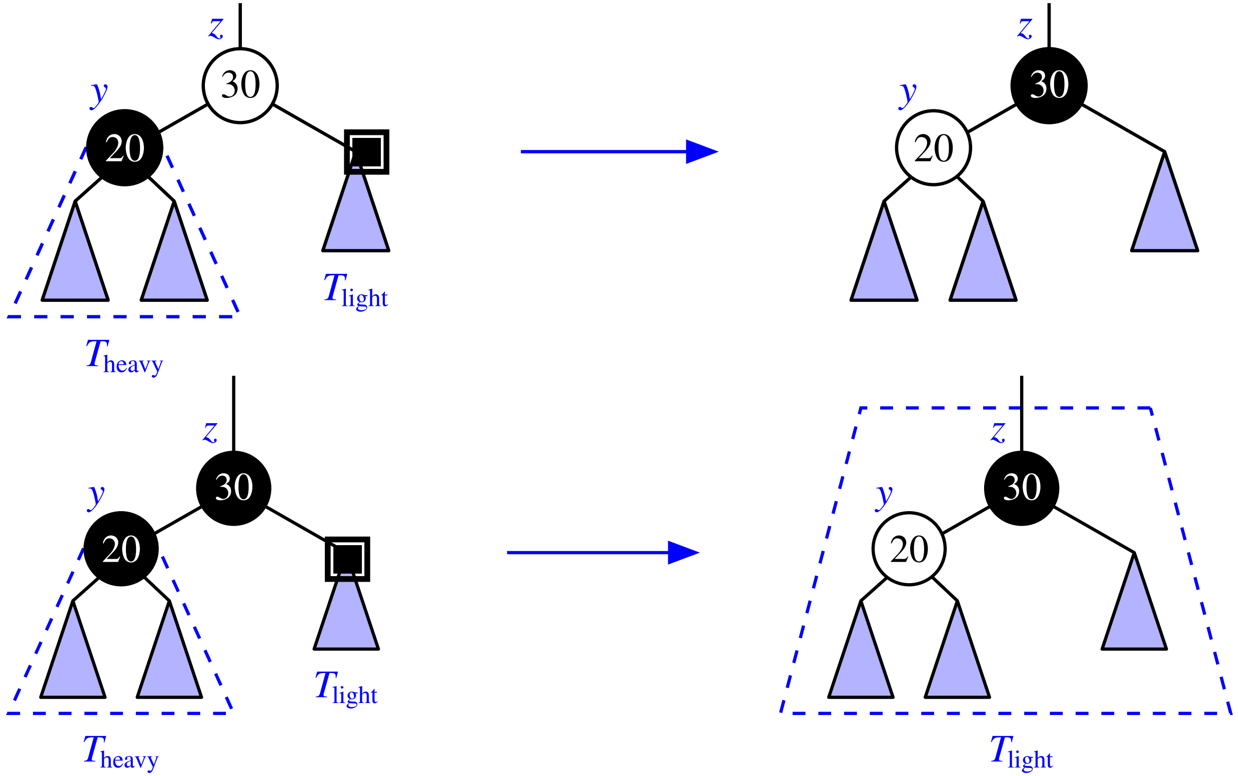

→ (propagate)

Case 2: $\boldsymbol{y}$ is black and its children are both black

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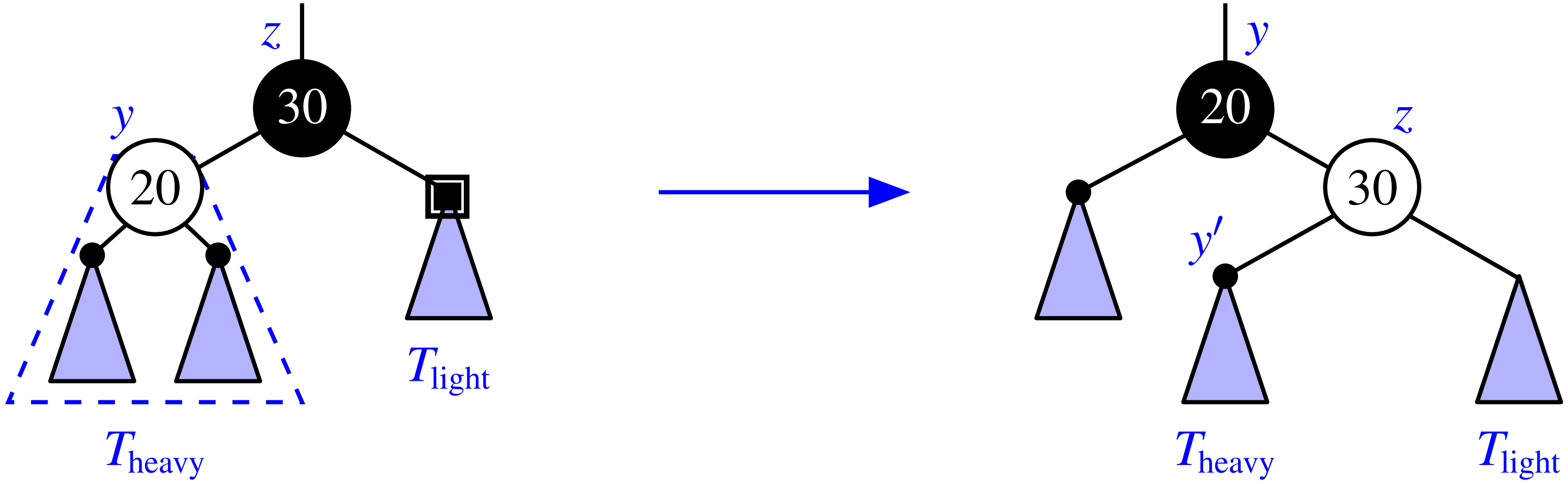
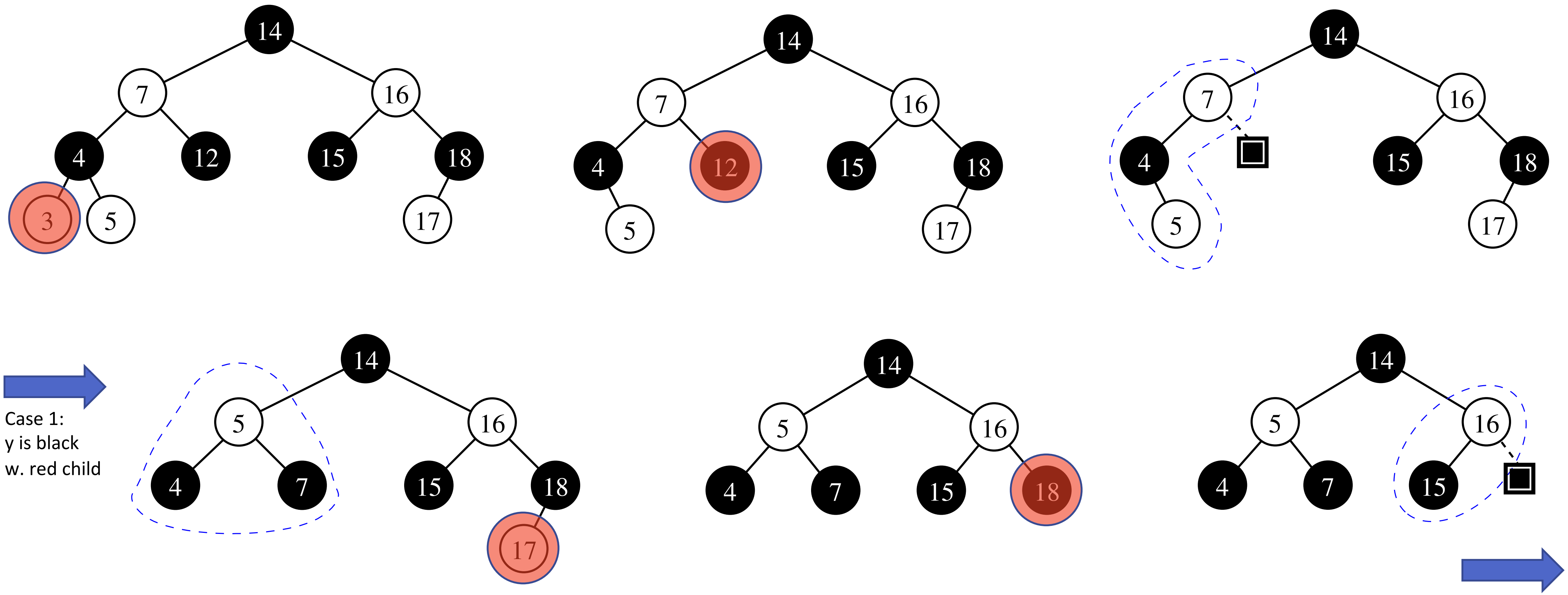


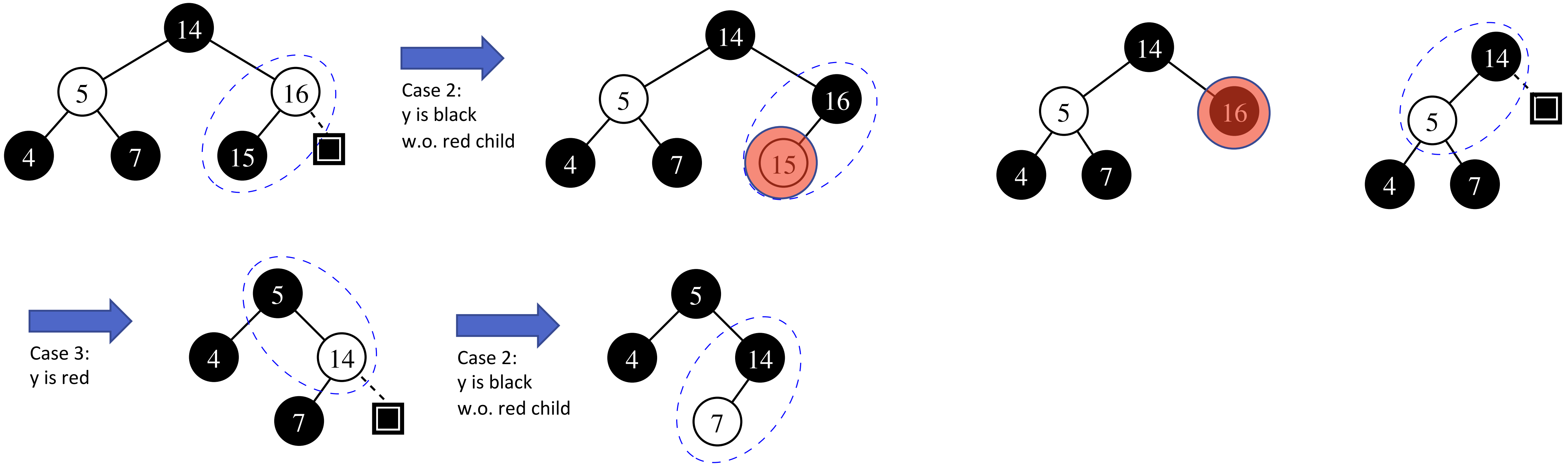
Deletion example



Case 2:
y is black
w.o. red child

- Case 1: y is black and has a red child -> reconstruct
- Case 2: y is black and its children are both black -> recolor
- Case 3: y is red -> arrangement -> Case 1 or 2

Deletion example (cont)



- Case 1: y is black and has a red child -> reconstruct
- Case 2: y is black and its children are both black -> recolor
- Case 3: y is red -> arrangement -> Case 1 or 2

Red-Black Tree Reorganization

Insertion	remedy double red	
Red-black tree action	**(2,4) tree action**	**result**
restructuring	change of 4-node representation	double red removed
recoloring	split	double red removed or propagated up

Deletion	remedy double black	
Red-black tree action	**(2,4) tree action**	**result**
restructuring	transfer	double black removed
recoloring	fusion	double black removed or propagated up
adjustment	change of 3-node representation	restructuring or recoloring follows

Python Implementation

```
class RedBlackTreeMap(TreeMap):
  """Sorted map implementation using a red-black tree."""
  class _Node(TreeMap._Node):
    """Node class for red-black tree maintains bit that denotes color."""
    __slots__ = '_red'      # add additional data member to the Node class

    def __init__(self, element, parent=None, left=None, right=None):
      super().__init__(element, parent, left, right)
      self._red = True        # new node red by default
```

Python Implementation, Part 2

```
#------------------------ positional-based utility methods ------------------------
# we consider a nonexistent child to be trivially black
def _set_red(self, p): p._node._red = True
def _set_black(self, p): p._node._red = False
def _set_color(self, p, make_red): p._node._red = make_red
def _is_red(self, p): return p is not None and p._node._red
def _is_red_leaf(self, p): return self._is_red(p) and self.is_leaf(p)

def _get_red_child(self, p):
  """Return a red child of p (or None if no such child)."""
  for child in (self.left(p), self.right(p)):
    if self._is_red(child):
      return child
  return None

#------------------------ support for insertions ------------------------
def _rebalance_insert(self, p):
  self._resolve_red(p)                      # new node is always red

def _resolve_red(self, p):
  if self.is_root(p):
    self._set_black(p)                      # make root black
  else:
    parent = self.parent(p)
    if self._is_red(parent):                # double red problem
      uncle = self.sibling(parent)
      if not self._is_red(uncle):           # Case 1: misshapen 4-node
        middle = self._restructure(p)       # do trinode restructuring
        self._set_black(middle)             # and then fix colors
        self._set_red(self.left(middle))
        self._set_red(self.right(middle))
      else:                                 # Case 2: overfull 5-node
        grand = self.parent(parent)
        self._set_red(grand)                # grandparent becomes red
        self._set_black(self.left(grand))   # its children become black
        self._set_black(self.right(grand))
        self._resolve_red(grand)            # recur at red grandparent
```

Python Implementation, end

```python
#------------------------ support for deletions ------------------------
def _rebalance_delete(self, p):
  if len(self) == 1:
    self._set_black(self.root())      # special case: ensure that root is black
  elif p is not None:
    n = self.num_children(p)
    if n == 1:                        # deficit exists unless child is a red leaf
      c = next(self.children(p))
      if not self._is_red_leaf(c):
        self._fix_deficit(p, c)
    elif n == 2:                      # removed black node with red child
      if self._is_red_leaf(self.left(p)):
        self._set_black(self.left(p))
      else:
        self._set_black(self.right(p))

def _fix_deficit(self, z, y):
  """Resolve black deficit at z, where y is the root of z's heavier subtree."""
  if not self._is_red(y): # y is black; will apply Case 1 or 2
    x = self._get_red_child(y)
    if x is not None: # Case 1: y is black and has red child x; do "transfer"
      old_color = self._is_red(z)
      middle = self._restructure(x)
      self._set_color(middle, old_color)       # middle gets old color of z
      self._set_black(self.left(middle))       # children become black
      self._set_black(self.right(middle))
    else: # Case 2: y is black, but no red children; recolor as "fusion"
      self._set_red(y)
      if self._is_red(z):
        self._set_black(z)                     # this resolves the problem
      elif not self.is_root(z):
        self._fix_deficit(self.parent(z), self.sibling(z)) # recur upward
  else: # Case 3: y is red; rotate misaligned 3-node and repeat
    self._rotate(y)
    self._set_black(y)
    self._set_red(z)
    if z == self.right(y):
      self._fix_deficit(z, self.left(z))
    else:
      self._fix_deficit(z, self.right(z))
```