SE274 Data Structure

Lecture 9: Graphs

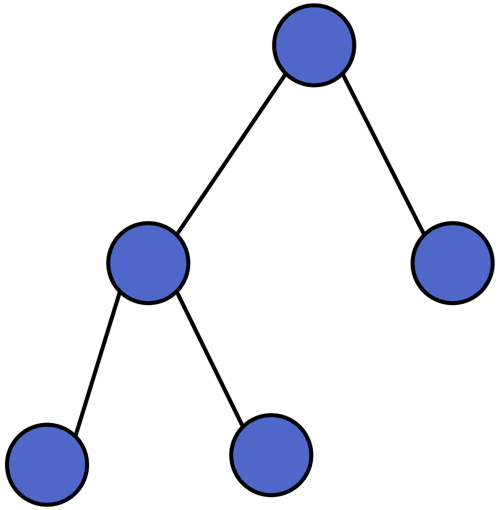
(textbook: Chapter 14)

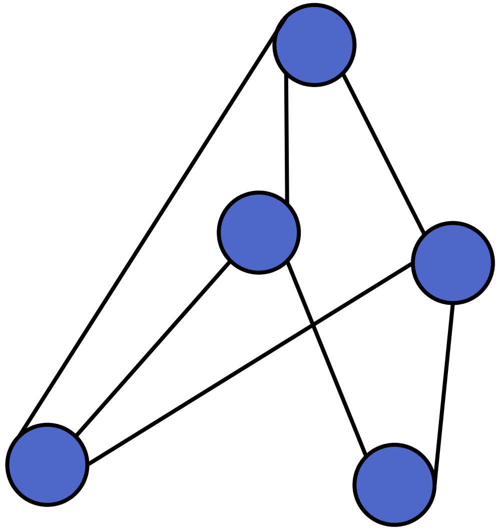
May *13*, 2020

Instructor: Sunjun Kim

Information&Communication Engineering, DGIST

Recap: Graph





Tree $\subset$ Graph

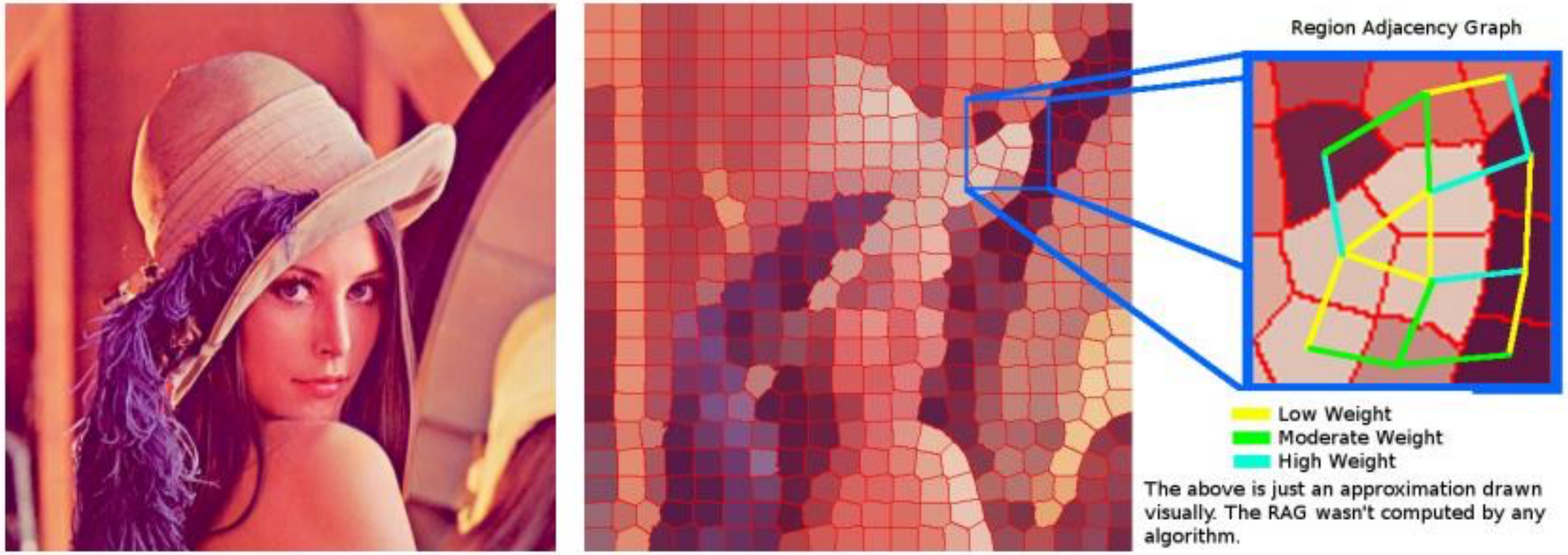
Application of graph

Web, social network, biology, neuroscience, …

코로나19 확산 관계망

http://dj.kbs.co.kr/resources/2020-01-31/

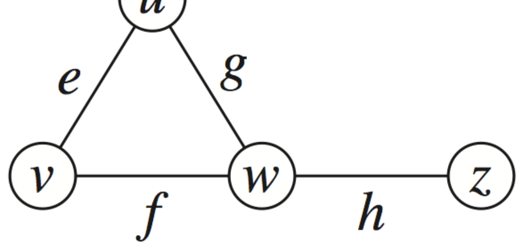
Graphs in Computer Vision

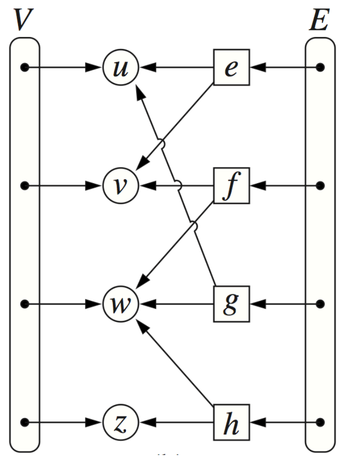


input image regions (here superpixels) RAG

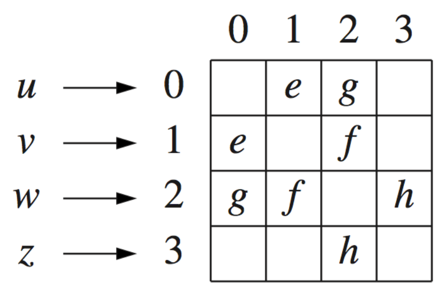
Graph representations

- Graph $G = (V, E)$
  - Directed / Undirected
- Two typical representations
  - Adjacency list, adjacency matrix
- Other representations
  - Compressed sparse row (CSR)
  - Compressed sparse column (CSC)

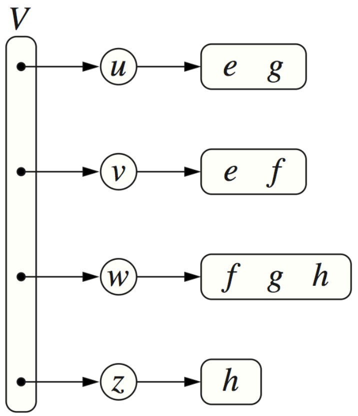




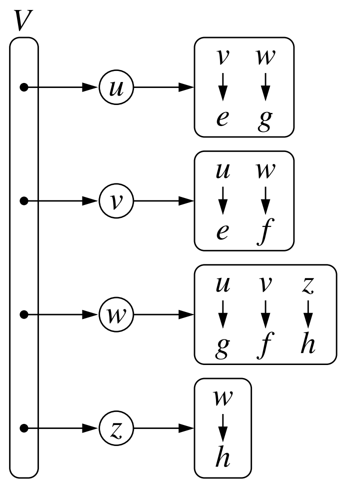
Edge list



Adjacency matrix



Adjacency list



Adjacency map

Graph sparseness

- Most real graphs are spares
- Sparse:$|E| \ll |V|^2$
- Dense: $|E| \approx |V|^2$

- In case of directed graph,

  $\boldsymbol{m} \leq \boldsymbol{n}\,(\boldsymbol{n} - 1)$

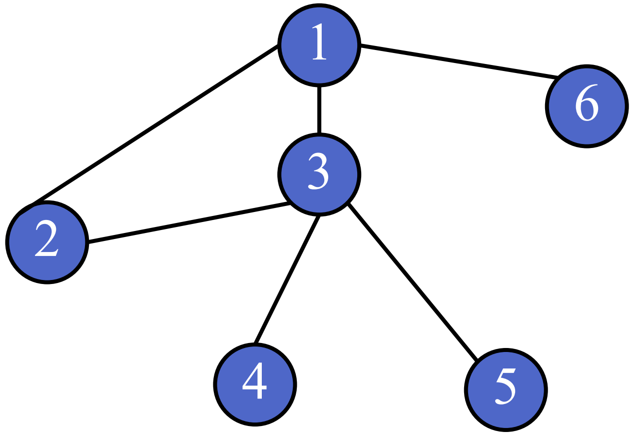
  where $\boldsymbol{m} = |E|$, $\boldsymbol{n}=|V|$

Graph traversals

- ***Graph traversal*** algorithm is a systematic procedure for exploring a graph by examining all of its vertices and edges.
- Good traversal algorithm should be done in $O(|V|+|E|)$

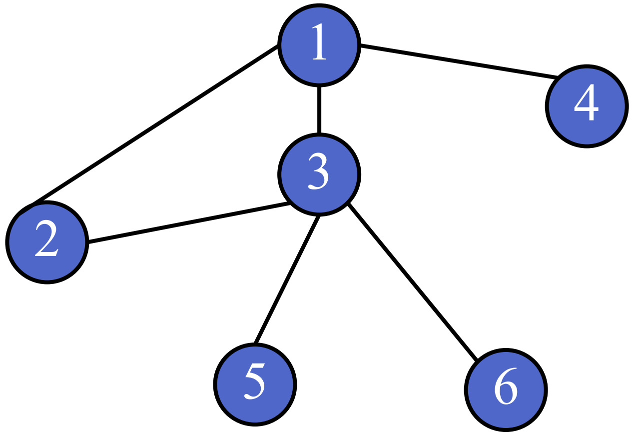
Depth-First Search (DFS)

- Explore a graph until as far as possible, then roll back to explore the next.

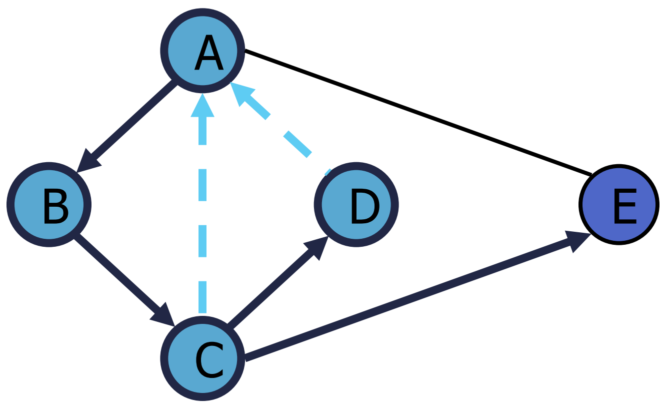


Breadth-First Search (BFS)

- Gradually broaden explored vertices in the same level.

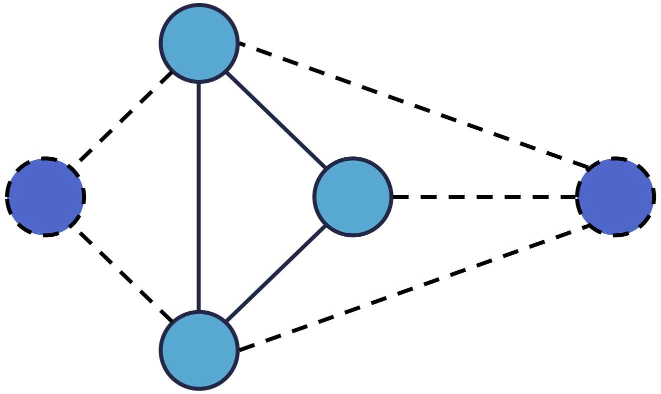


Depth-First Search

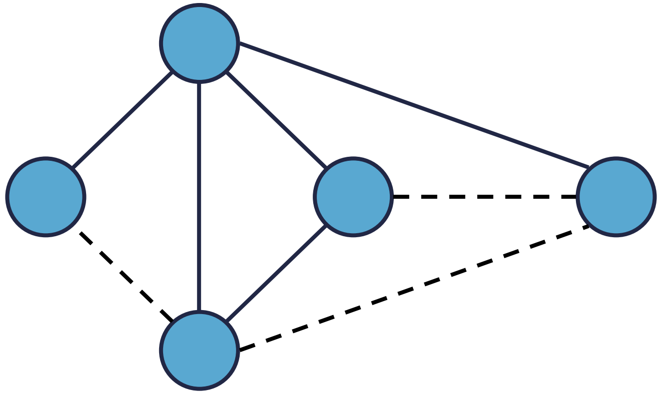


Subgraphs

- A subgraph S of a graph G is a graph such that
  - The vertices of S are a subset of the vertices of G
  - The edges of S are a subset of the edges of G
- A spanning subgraph of G is a subgraph that contains all the vertices of G



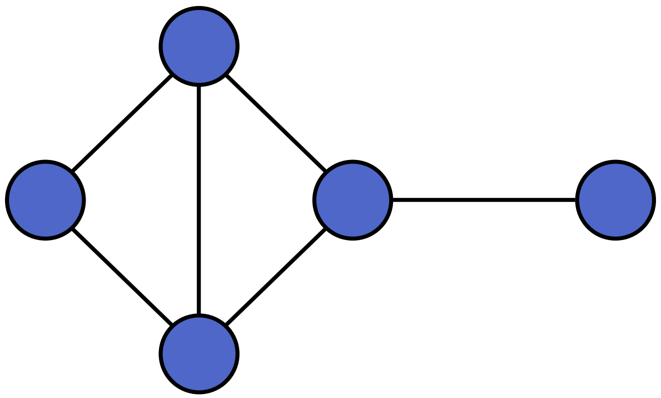
Subgraph



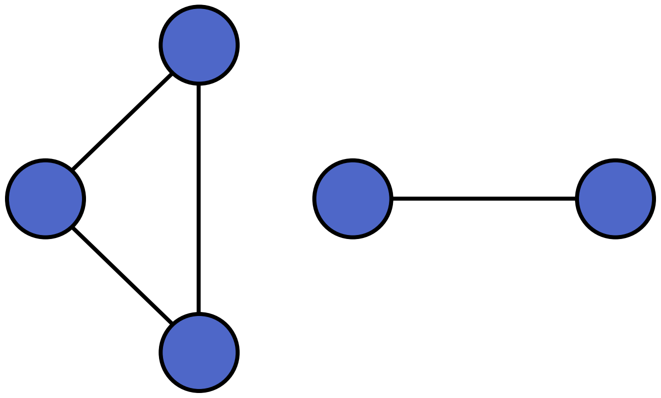
Spanning subgraph

Connectivity

- A graph is connected if there is a path between every pair of vertices
- A connected component of a graph G is a maximal connected subgraph of G



Connected graph

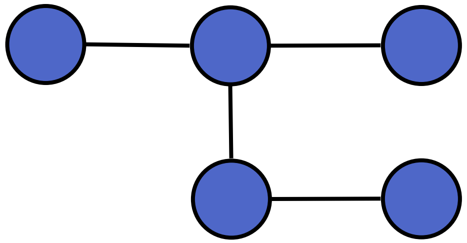


Non connected graph with two connected components

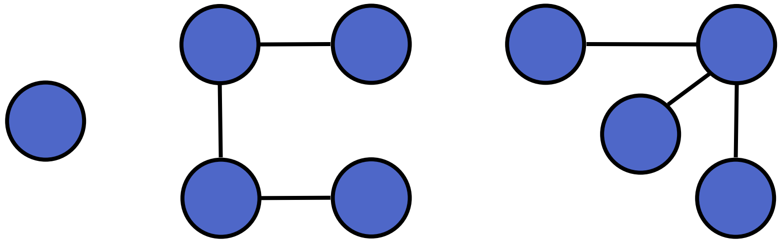
Trees and Forests

- A (free) tree is an undirected graph T such that
  - T is connected
  - T has no cycles

  This definition of tree is different from the one of a rooted tree
- A forest is an undirected graph without cycles
- The connected components of a forest are trees



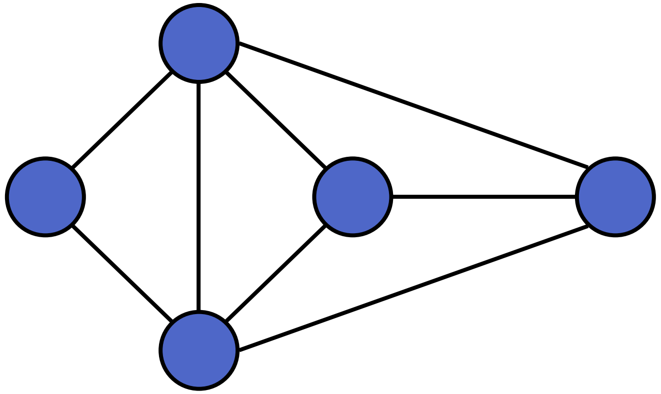
Tree



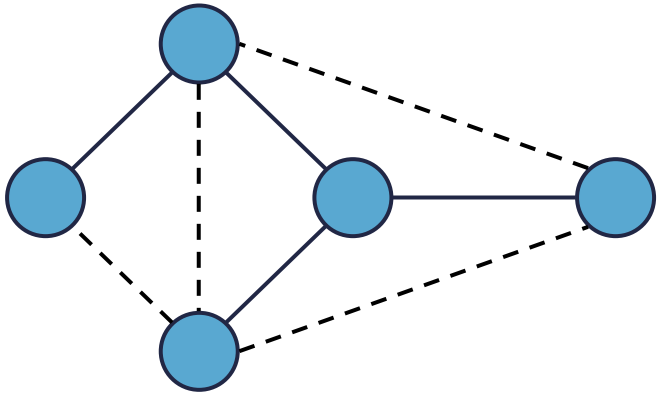
Forest

Spanning Trees and Forests

- A spanning tree of a connected graph is a spanning subgraph that is a tree
- A spanning tree is not unique unless the graph is a tree
- Spanning trees have applications to the design of communication networks
- A spanning forest of a graph is a spanning subgraph that is a forest



Graph



Spanning tree

Depth-First Search

- Depth-first search (DFS) is a general technique for traversing a graph
- A DFS traversal of a graph G
  - Visits all the vertices and edges of G
  - Determines whether G is connected
  - Computes the connected components of G
  - Computes a spanning forest of G
- DFS on a graph with n vertices and m edges takes $O(n + m)$ time
- DFS can be further extended to solve other graph problems
  - Find and report a path between two given vertices
  - Find a cycle in the graph
- Depth-first search is to graphs what Euler tour is to binary trees

DFS Algorithm

- The algorithm uses a mechanism for setting and getting "labels" of vertices and edges

```
Algorithm DFS(G)
   Input graph G
   Output labeling of the edges of G
      as discovery edges and
      back edges
  for all u ∈ G.vertices()
   setLabel(u, UNEXPLORED)
  for all e ∈ G.edges()
   setLabel(e, UNEXPLORED)
  for all v ∈ G.vertices()
   if getLabel(v) = UNEXPLORED
      DFS(G, v)
```

```
Algorithm DFS(G, v)
   Input graph G and a start vertex v of G
   Output labeling of the edges of G
      in the connected component of v
      as discovery edges and back edges
   setLabel(v, VISITED)
   for all e ∈ G.incidentEdges(v)
      if getLabel(e) = UNEXPLORED
         w ← opposite(v,e)
         if getLabel(w) = UNEXPLORED
            setLabel(e, DISCOVERY)
            DFS(G, w)
         else
            setLabel(e, BACK)
```

Example

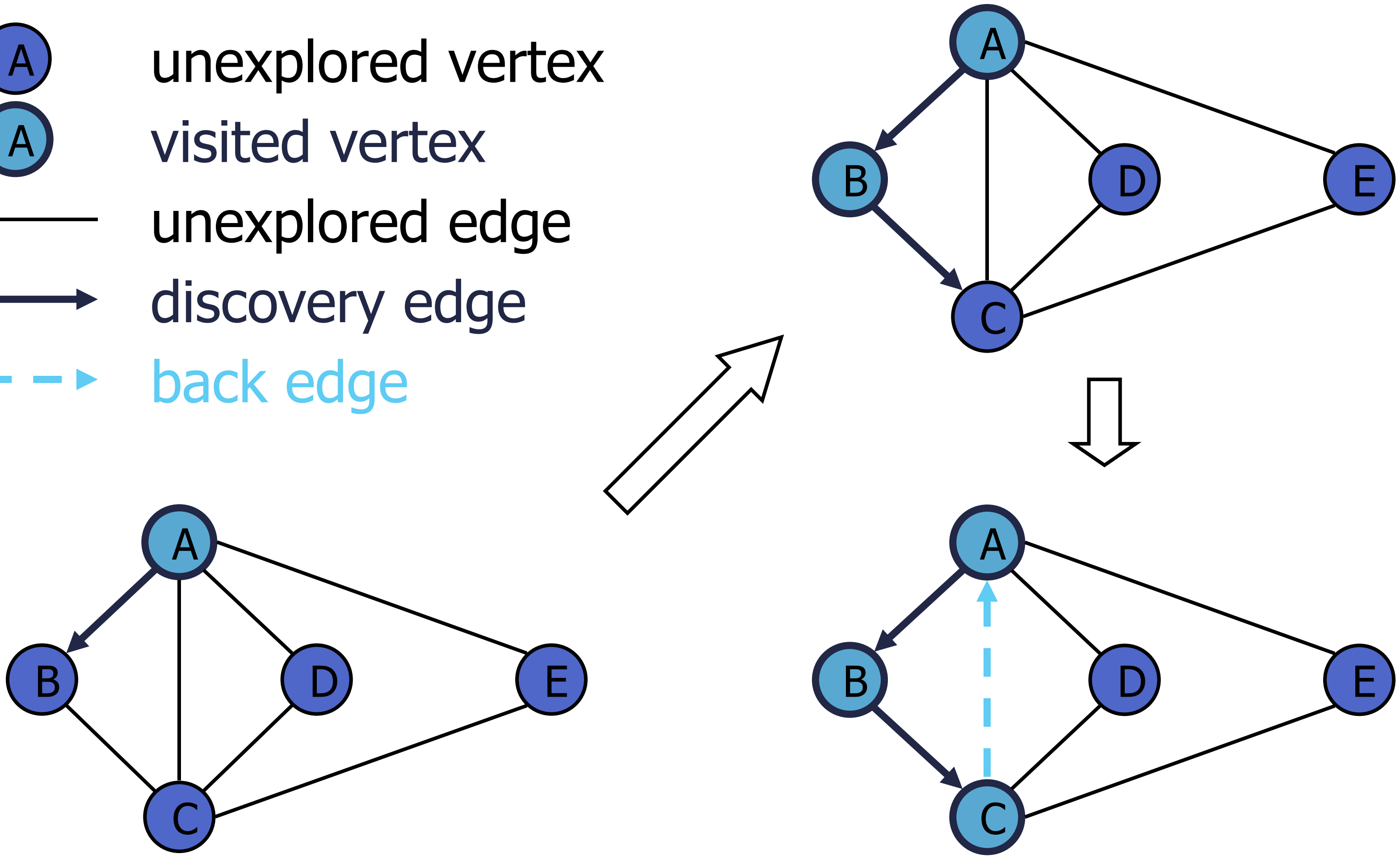
unexplored vertex

visited vertex

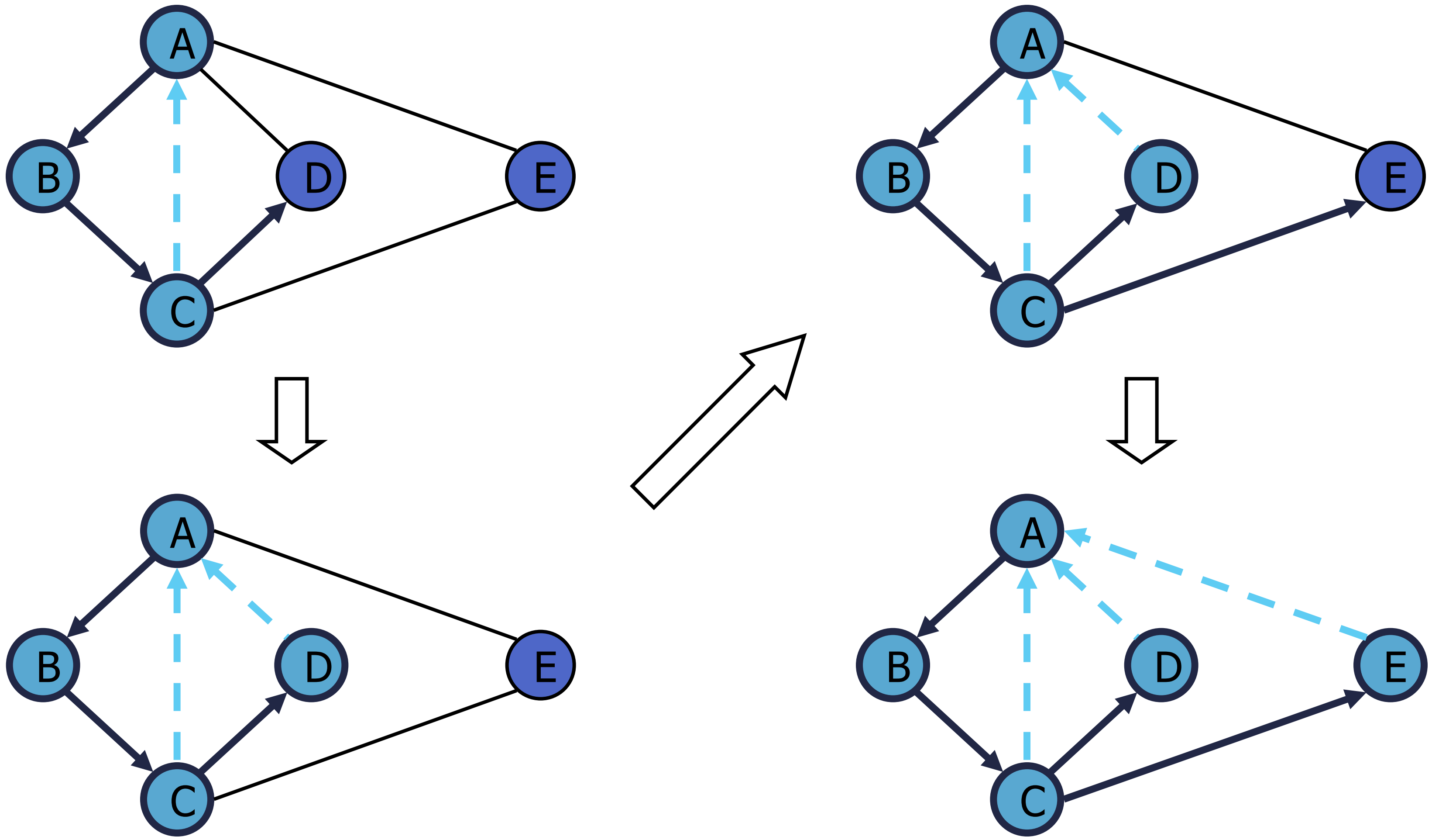
unexplored edge

discovery edge

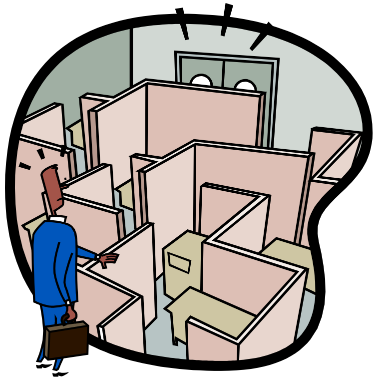
back edge



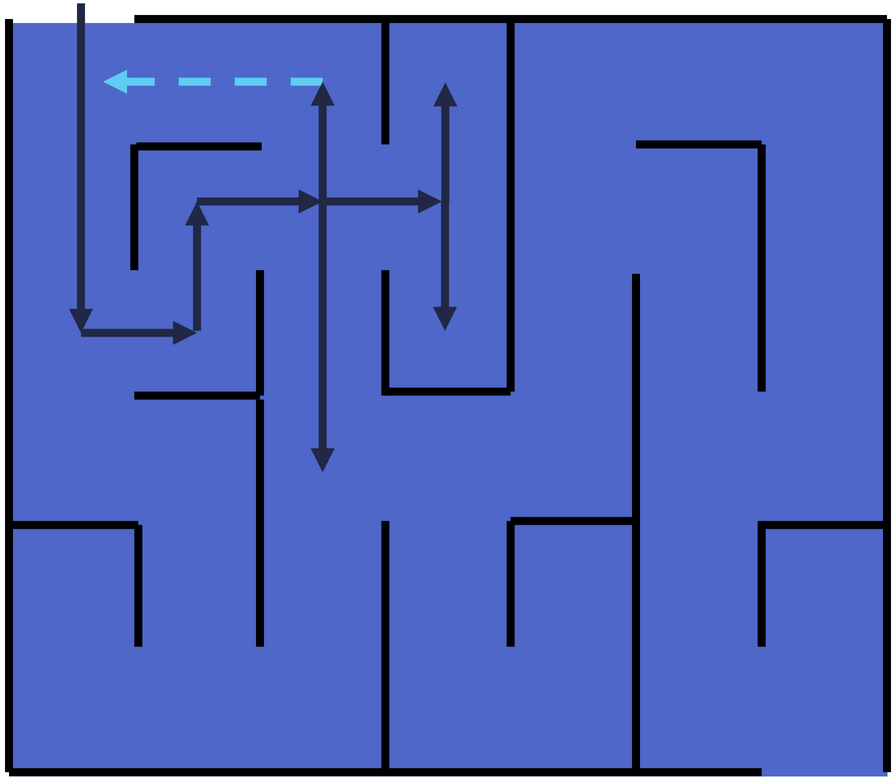
Example (cont.)



DFS and Maze Traversal



- The DFS algorithm is similar to a classic strategy for exploring a maze
  - We mark each intersection, corner and dead end (vertex) visited
  - We mark each corridor (edge) traversed
  - We keep track of the path back to the entrance (start vertex) by means of a rope (recursion stack)



Maze: From the XScreenSaver Collection, 1985
https://www.youtube.com/watch?v=-u4neMXIRA8

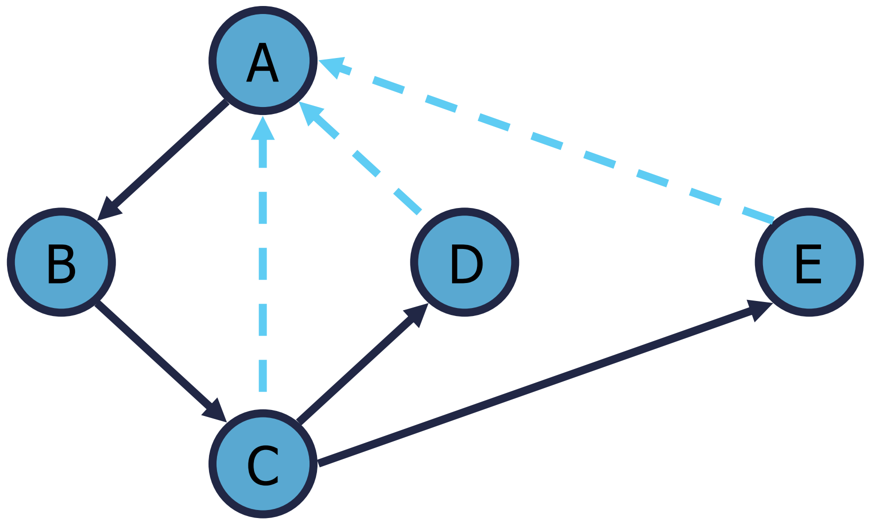
Properties of DFS

Property 1

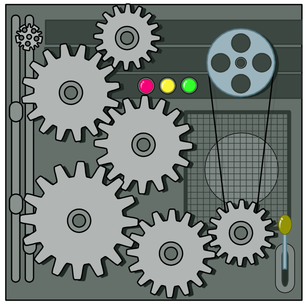
$DFS(G, v)$ visits all the vertices and edges in the connected component of v

Property 2

The discovery edges labeled by $DFS(G, v)$ form a spanning tree of the connected component of v



Analysis of DFS



- Setting/getting a vertex/edge label takes $O(1)$ time
- Each vertex is labeled twice
  - once as UNEXPLORED
  - once as VISITED
- Each edge is labeled twice
  - once as UNEXPLORED
  - once as DISCOVERY or BACK
- Method incidentEdges is called once for each vertex
- DFS runs in $O(n + m)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2m$

Python Implementation

```
def DFS(g, u, discovered):
  """Perform DFS of the undiscovered portion of Graph g starting at Vertex u.

  discovered is a dictionary mapping each vertex to the edge that was used to
  discover it during the DFS. (u should be "discovered" prior to the call.)
  Newly discovered vertices will be added to the dictionary as a result.
  """
  for e in g.incident_edges(u):          # for every outgoing edge from u
    v = e.opposite(u)
    if v not in discovered:              # v is an unvisited vertex
      discovered[v] = e                  # e is the tree edge that discovered v
      DFS(g, v, discovered)              # recursively explore from v
```

Path Finding

- We can specialize the DFS algorithm to find a path between two given vertices u and z using the template method pattern
- We call $DFS(G, u)$ with u as the start vertex
- We use a stack S to keep track of the path between the start vertex and the current vertex
- As soon as destination vertex z is encountered, we return the path as the contents of the stack

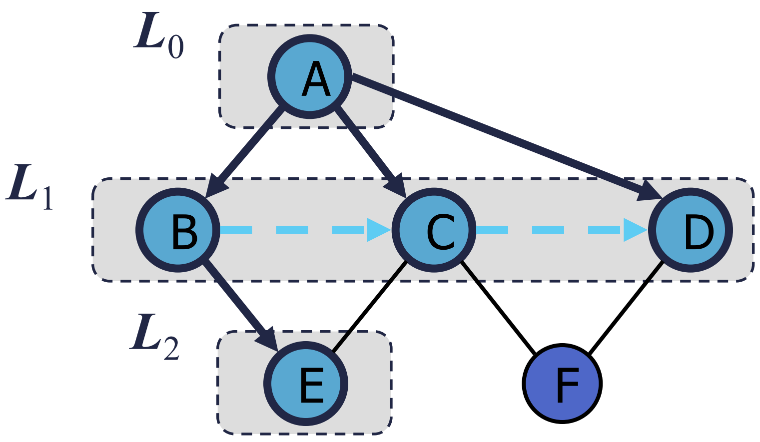
```
Algorithm pathDFS(G, v, z)
  setLabel(v, VISITED)
  S.push(v)
  if v = z
    return S.elements()
  for all e ∈ G.incidentEdges(v)
    if getLabel(e) = UNEXPLORED
      w ← opposite(v,e)
      if getLabel(w) = UNEXPLORED
        setLabel(e, DISCOVERY)
        S.push(e)
        pathDFS(G, w, z)
        S.pop(e)
      else
        setLabel(e, BACK)
  S.pop(v)
```

Cycle Finding

- We can specialize the DFS algorithm to find a simple cycle using the template method pattern
- We use a stack S to keep track of the path between the start vertex and the current vertex
- As soon as a back edge (v, w) is encountered, we return the cycle as the portion of the stack from the top to vertex w

```
Algorithm cycleDFS(G, v, z)
   setLabel(v, VISITED)
   S.push(v)
   for all  e ∈ G.incidentEdges(v)
      if  getLabel(e) = UNEXPLORED
         w ← opposite(v,e)
         S.push(e)
         if getLabel(w) = UNEXPLORED
            setLabel(e, DISCOVERY)
            pathDFS(G, w, z)
            S.pop(e)
         else
            T ← new empty stack
            repeat
               o ← S.pop()
               T.push(o)
            until o = w
            return T.elements()
   S.pop(v)
```

Breadth-First Search



Breadth-First Search

- Breadth-first search (BFS) is a general technique for traversing a graph
- A BFS traversal of a graph G
  - Visits all the vertices and edges of G
  - Determines whether G is connected
  - Computes the connected components of G
  - Computes a spanning forest of G
- BFS on a graph with n vertices and m edges takes $O(n + m)$ time
- BFS can be further extended to solve other graph problems
  - Find and report a path with the minimum number of edges between two given vertices
  - Find a simple cycle, if there is one

BFS Algorithm

- The algorithm uses a mechanism for setting and getting "labels" of vertices and edges

```
Algorithm BFS(G)
   Input graph G
   Output labeling of the edges
      and partition of the
      vertices of G
 for all u ∈ G.vertices()
   setLabel(u, UNEXPLORED)
 for all e ∈ G.edges()
   setLabel(e, UNEXPLORED)
 for all v ∈ G.vertices()
   if getLabel(v) = UNEXPLORED
      BFS(G, v)
```

```
Algorithm BFS(G, s)
  L0 ← new empty sequence
  L0.addLast(s)
  setLabel(s, VISITED)
  i ← 0
  while ¬Li.isEmpty()
    Li+1 ← new empty sequence
    for all v ∈ Li.elements()
      for all e ∈ G.incidentEdges(v)
        if getLabel(e) = UNEXPLORED
          w ← opposite(v,e)
          if getLabel(w) = UNEXPLORED
            setLabel(e, DISCOVERY)
            setLabel(w, VISITED)
            Li+1.addLast(w)
          else
            setLabel(e, CROSS)
    i ← i +1
```

Example

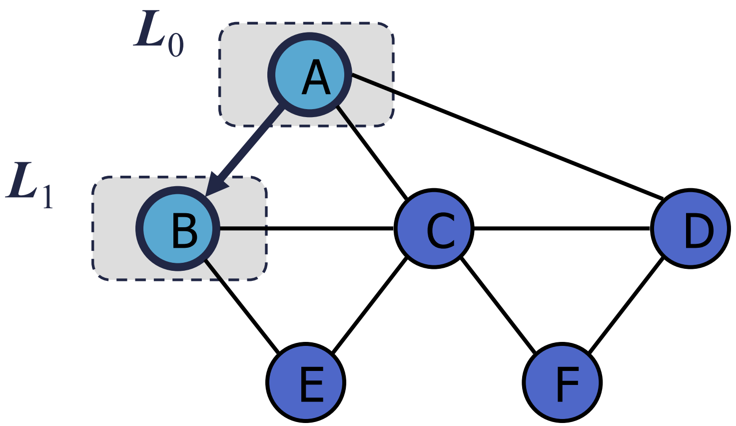
A unexplored vertex

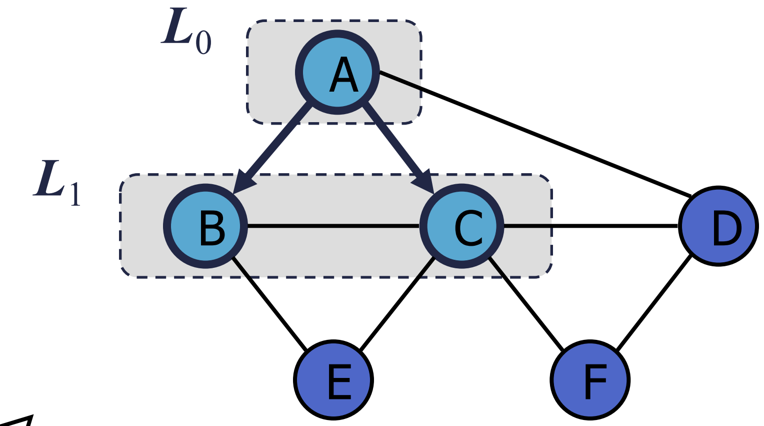
A visited vertex

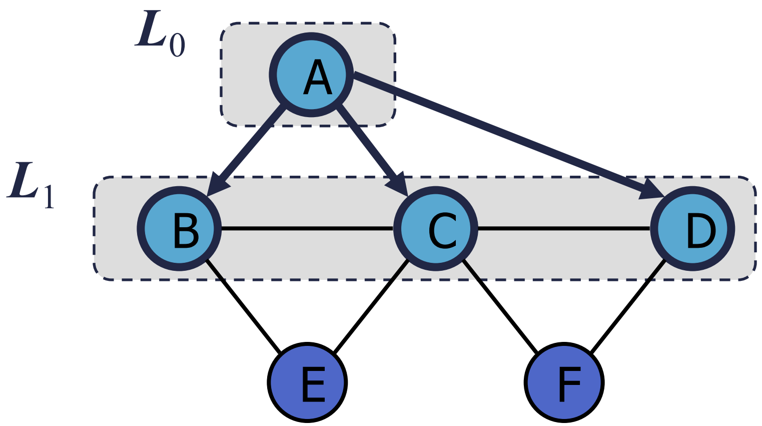
unexplored edge

discovery edge

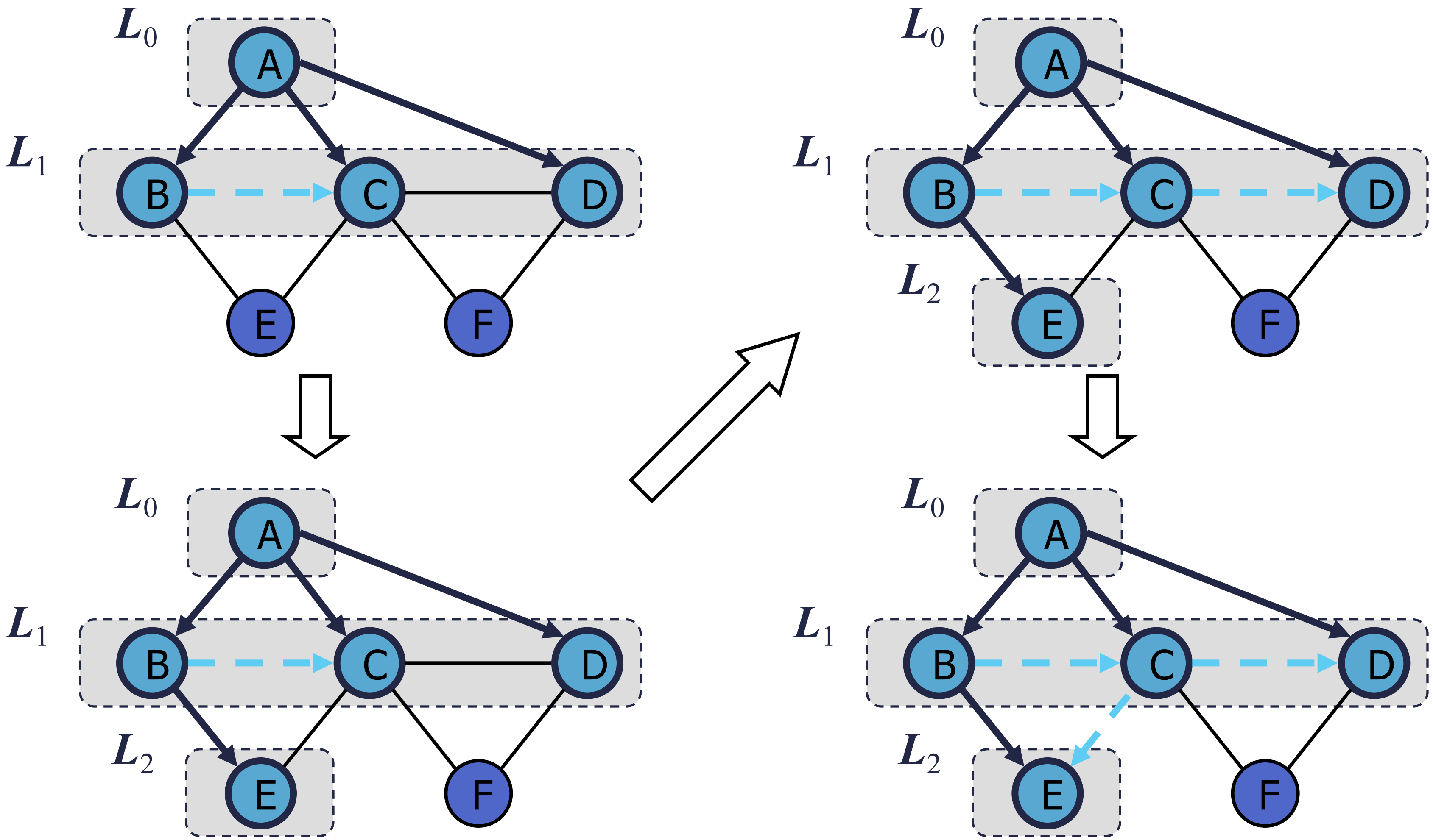
cross edge







Example (cont.)



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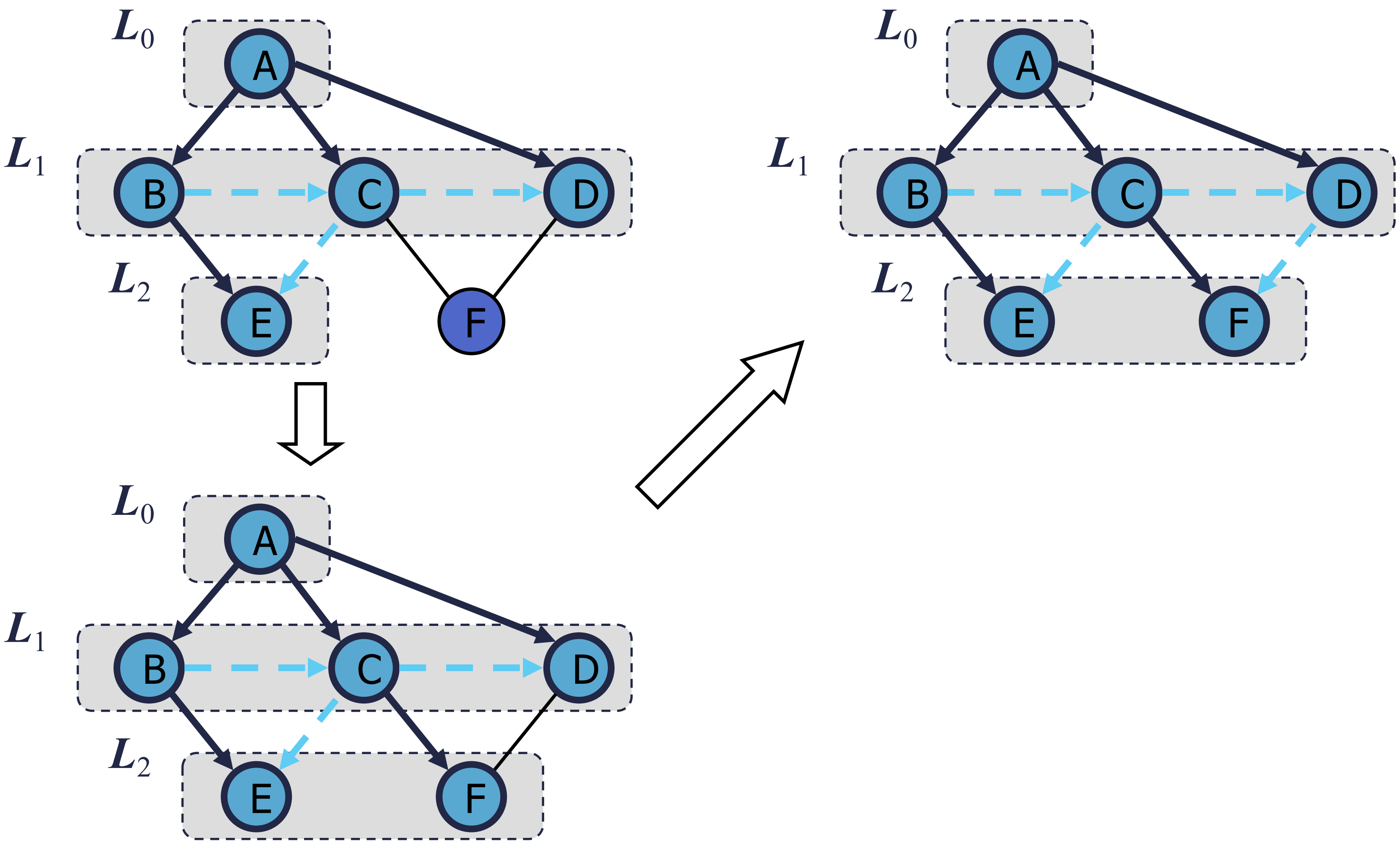


Properties

Notation

G_s: connected component of s

Property 1

$BFS(G, s)$ visits all the vertices and edges of G_s

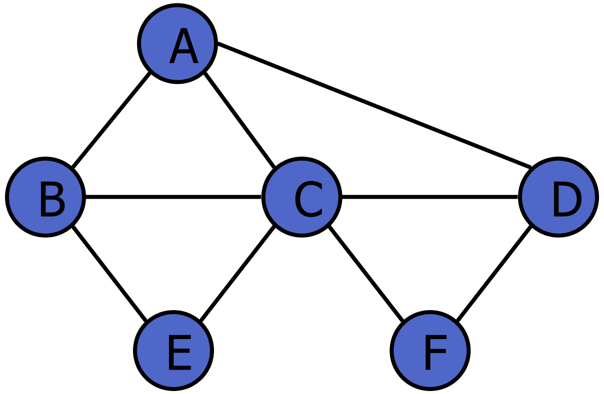
Property 2

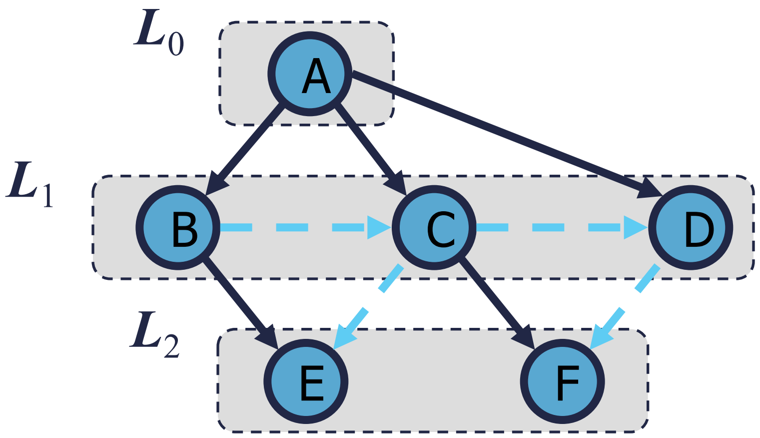
The discovery edges labeled by $BFS(G, s)$ form a spanning tree T_s of G_s

Property 3

For each vertex v in L_i

- The path of T_s from s to v has i edges
- Every path from s to v in G_s has at least i edges





Analysis

- Setting/getting a vertex/edge label takes $O(1)$ time
- Each vertex is labeled twice
  - once as UNEXPLORED
  - once as VISITED
- Each edge is labeled twice
  - once as UNEXPLORED
  - once as DISCOVERY or CROSS
- Each vertex is inserted once into a sequence L_i
- Method incidentEdges is called once for each vertex
- BFS runs in $O(n + m)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2m$

Python Implementation

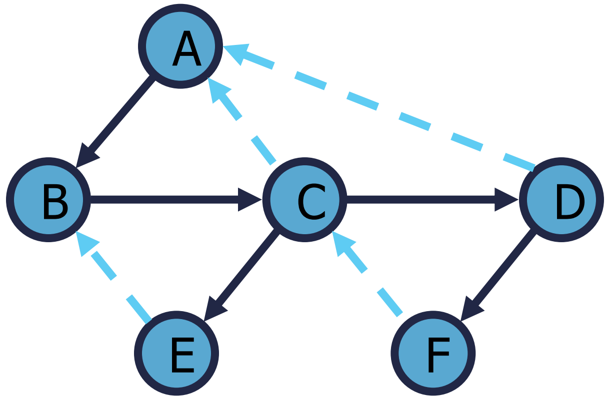
```
def BFS(g, s, discovered):
  """Perform BFS of the undiscovered portion of Graph g starting at Vertex s.

  discovered is a dictionary mapping each vertex to the edge that was used to
  discover it during the BFS (s should be mapped to None prior to the call).
  Newly discovered vertices will be added to the dictionary as a result.
  """
  level = [s]                          # first level includes only s
  while len(level) > 0:
    next_level = [ ]                   # prepare to gather newly found vertices
    for u in level:
      for e in g.incident_edges(u):    # for every outgoing edge from u
        v = e.opposite(u)
        if v not in discovered:        # v is an unvisited vertex
          discovered[v] = e            # e is the tree edge that discovered v
          next_level.append(v)         # v will be further considered in next pass
    level = next_level                 # relabel 'next' level to become current
```

Applications

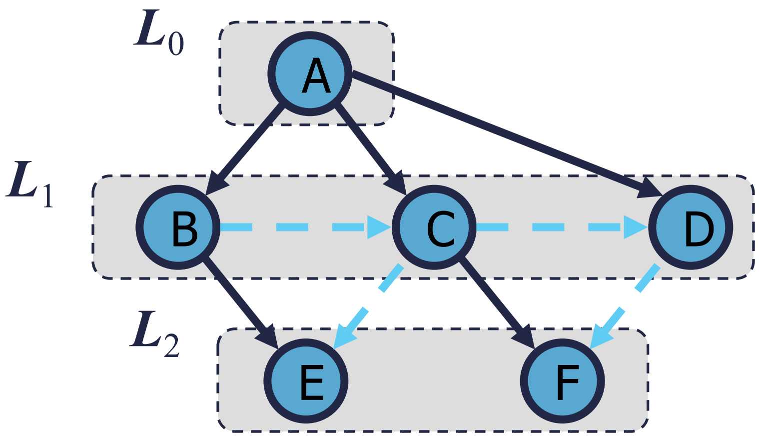
- Using the template method pattern, we can specialize the BFS traversal of a graph G to solve the following problems in $O(n + m)$ time
  - Compute the connected components of G
  - Compute a spanning forest of G
  - Find a simple cycle in G, or report that G is a forest
  - Given two vertices of G, find a path in G between them with the minimum number of edges, or report that no such path exists

DFS vs. BFS

Applications	DFS	BFS
Spanning forest, connected components, paths, cycles	√	√
Shortest paths		√
Biconnected components	√	



DFS



BFS

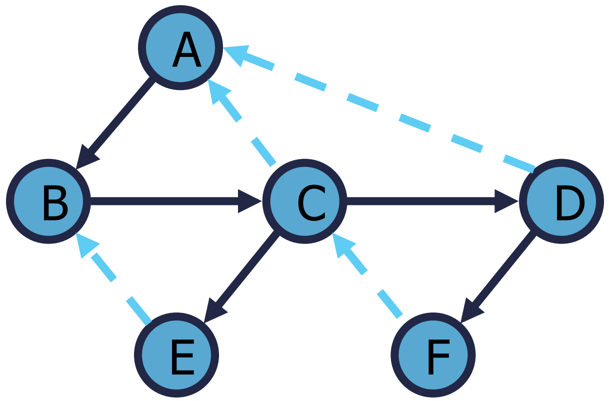
DFS vs. BFS (cont.)

Back edge (v,w)

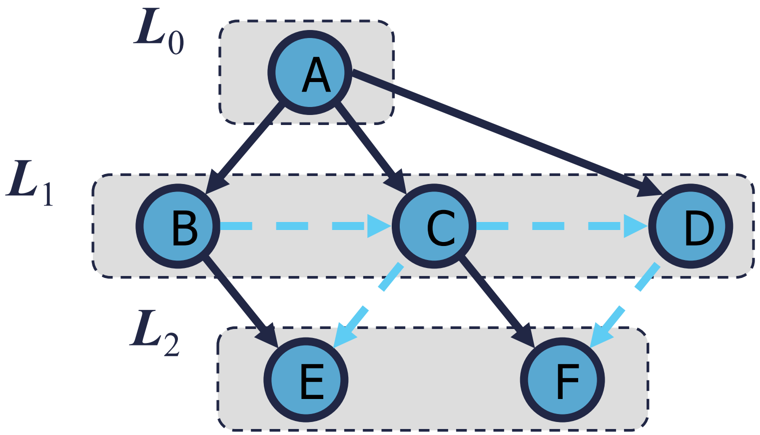
- w is an ancestor of v in the tree of discovery edges

Cross edge (v,w)

- w is in the same level as v or in the next level



DFS



BFS

Formal edge classifications

- In a graph search tree,
  - Back edge: descendeant → ancestor
  - Forward edge: ancestor → descendant
  - Cross edge: all other edges
    (appears in a directed graph)

Example) in a DFS search tree,