SE274 Data Structure

Lecture 6: Maps, Hash Tables, Skip Lists – Part 1

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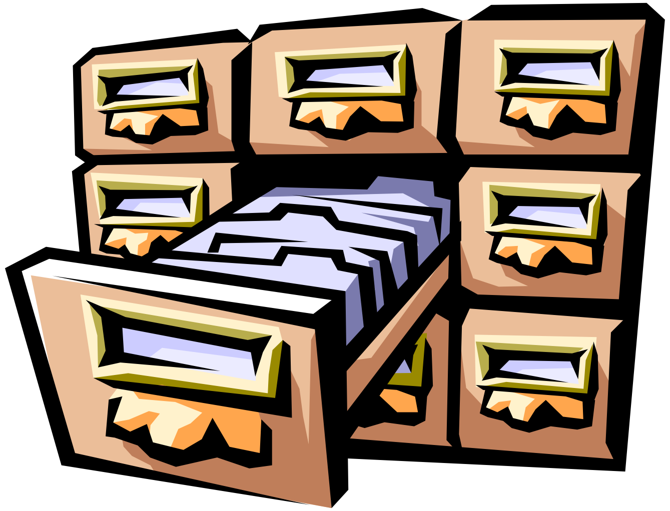
Maps and Dictionaries



Maps

- A **map** is a searchable collection of items that are key-value pairs
- The main operations of a map are for searching, inserting, and deleting items
- Multiple items with the same key are not allowed
- Applications:
  - address book
  - student-record database

Dictionaries

- Python's **dict** class is arguably the most significant data structure in the language.
  - It represents an abstraction known as a **dictionary** in which unique **keys** are mapped to associated **values**.
- Here, we use the term "dictionary" when specifically discussing Python's dict class, and the term "map" when discussing the more general notion of the abstract data type.

The Map ADT (Using **dict** Syntax)

M[k]: Return the value v associated with key k in map M, if one exists; otherwise raise a KeyError. In Python, this is implemented with the special method __getitem__.

M[k] = v: Associate value v with key k in map M, replacing the existing value if the map already contains an item with key equal to k. In Python, this is implemented with the special method __setitem__.

del M[k]: Remove from map M the item with key equal to k; if M has no such item, then raise a KeyError. In Python, this is implemented with the special method __delitem__.

len(M): Return the number of items in map M. In Python, this is implemented with the special method __len__.

iter(M): The default iteration for a map generates a sequence of *keys* in the map. In Python, this is implemented with the special method __iter__, and it allows loops of the form, **for** k **in** M.

More Map Operations

k in M: Return True if the map contains an item with key k. In Python, this is implemented with the special __contains__ method.

M.get(k, d=None): Return M[k] if key k exists in the map; otherwise return default value d. This provides a form to query M[k] without risk of a KeyError.

M.setdefault(k, d): If key k exists in the map, simply return M[k]; if key k does not exist, set M[k] = d and return that value.

M.pop(k, d=None): Remove the item associated with key k from the map and return its associated value v. If key k is not in the map, return default value d (or raise KeyError if parameter d is None).

A Few More Map Operations

M.popitem( ): Remove an arbitrary key-value pair from the map, and return a (k,v) tuple representing the removed pair. If map is empty, raise a KeyError.

M.clear( ): Remove all key-value pairs from the map.

M.keys( ): Return a set-like view of all keys of M.

M.values( ): Return a set-like view of all values of M.

M.items( ): Return a set-like view of (k,v) tuples for all entries of M.

M.update(M2): Assign M[k] = v for every (k,v) pair in map M2.

M == M2: Return True if maps M and M2 have identical key-value associations.

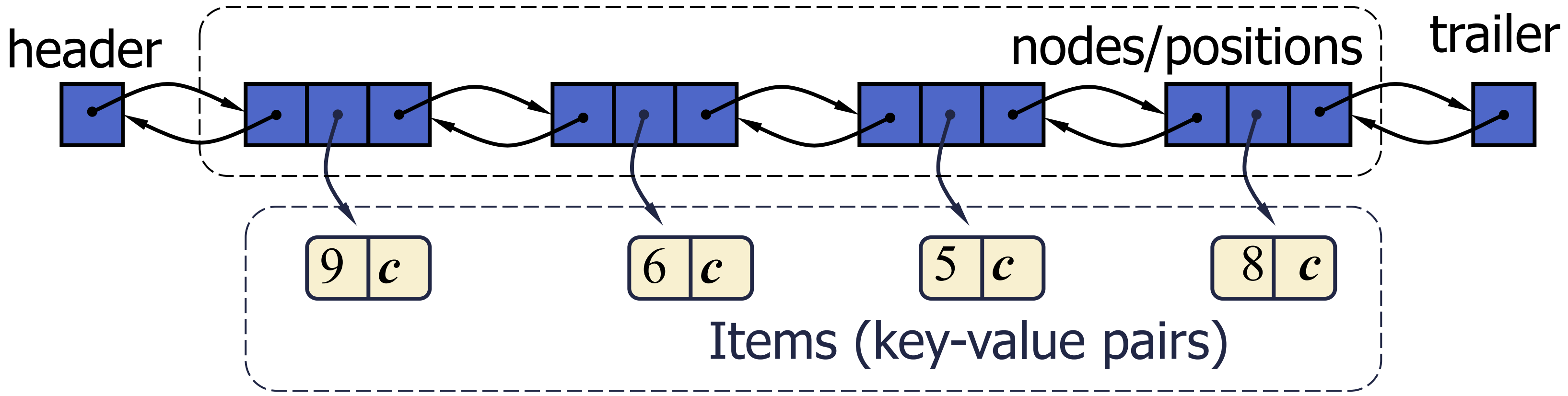
M != M2: Return True if maps M and M2 do not have identical key-value associations.

Example

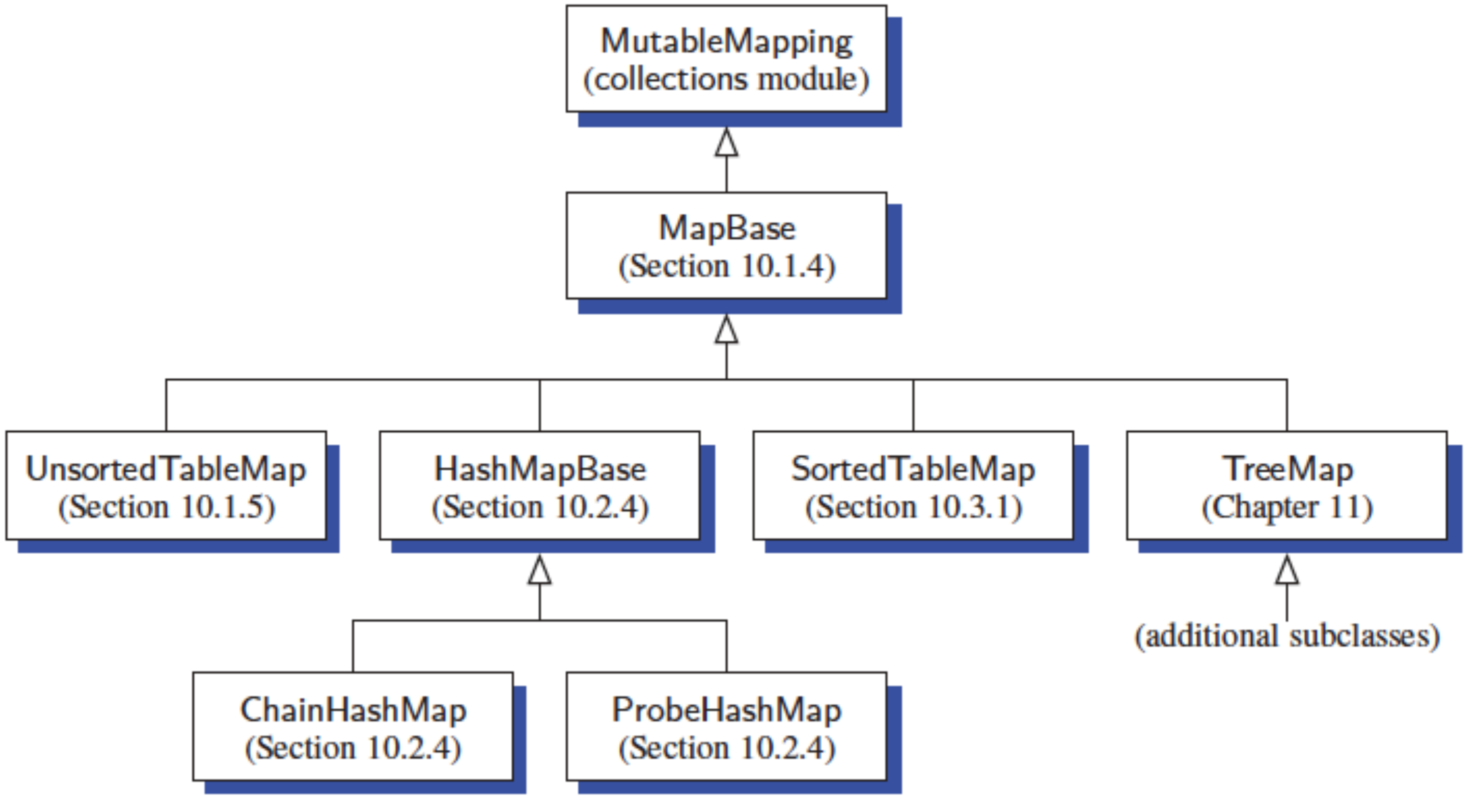
Operation	Return Value	Map
len(M)	0	{ }
M['K'] = 2	–	{'K': 2}
M['B'] = 4	–	{'K': 2, 'B': 4}
M['U'] = 2	–	{'K': 2, 'B': 4, 'U': 2}
M['V'] = 8	–	{'K': 2, 'B': 4, 'U': 2, 'V': 8}
M['K'] = 9	–	{'K': 9, 'B': 4, 'U': 2, 'V': 8}
M['B']	4	{'K': 9, 'B': 4, 'U': 2, 'V': 8}
M['X']	KeyError	{'K': 9, 'B': 4, 'U': 2, 'V': 8}
M.get('F')	None	{'K': 9, 'B': 4, 'U': 2, 'V': 8}
M.get('F', 5)	5	{'K': 9, 'B': 4, 'U': 2, 'V': 8}
M.get('K', 5)	9	{'K': 9, 'B': 4, 'U': 2, 'V': 8}
len(M)	4	{'K': 9, 'B': 4, 'U': 2, 'V': 8}
del M['V']	–	{'K': 9, 'B': 4, 'U': 2}
M.pop('K')	9	{'B': 4, 'U': 2}
M.keys()	'B', 'U'	{'B': 4, 'U': 2}
M.values()	4, 2	{'B': 4, 'U': 2}
M.items()	('B', 4), ('U', 2)	{'B': 4, 'U': 2}
M.setdefault('B', 1)	4	{'B': 4, 'U': 2}
M.setdefault('A', 1)	1	{'A': 1, 'B': 4, 'U': 2}
M.popitem()	('B', 4)	{'A': 1, 'U': 2}

A Simple List-Based Map

- We can efficiently implement a map using an unsorted list
  - We store the items of the map in a list S (based on a doubly-linked list), in arbitrary order



Our MapBase Class



The MapBase Abstract Class

```python
class MapBase(MutableMapping):
  """Our own abstract base class that includes a nonpublic _Item class."""

  #------------------------------- nested _Item class -------------------------------
  class _Item:
    """Lightweight composite to store key-value pairs as map items."""
    __slots__ = '_key', '_value'

    def __init__(self, k, v):
      self._key = k
      self._value = v

    def __eq__(self, other):
      return self._key == other._key      # compare items based on their keys

    def __ne__(self, other):
      return not (self == other)          # opposite of __eq__

    def __lt__(self, other):
      return self._key < other._key       # compare items based on their keys
```

Mapping(Collection)

`__getitem__`, `__iter__`, `__len__`	`__contains__`, `keys`, `items`, `values`, `get`, `__eq__`, and `__ne__`

MutableMapping(Mapping)

`__getitem__`, `__setitem__`, `__delitem__`, `__iter__`, `__len__`	Inherited `Mapping` methods and `pop`, `popitem`, `clear`, `update`, and `setdefault`

An Unsorted List Implementation

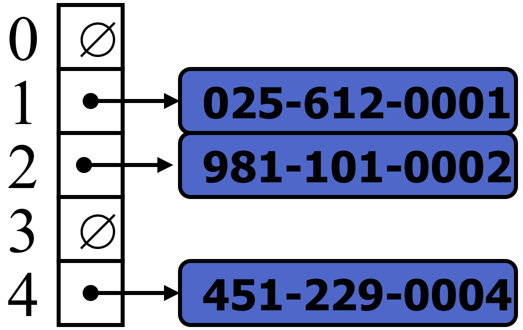
```
class UnsortedTableMap(MapBase):
  """Map implementation using an unordered list."""

  def __init__(self):
    """Create an empty map."""
    self._table = [ ]                              # list of _Item's

  def __getitem__(self, k):
    """Return value associated with key k (raise KeyError if not found)."""
    for item in self._table:
      if k == item._key:
        return item._value
    raise KeyError('Key Error: ' + repr(k))

  def __setitem__(self, k, v):
    """Assign value v to key k, overwriting existing value if present."""
    for item in self._table:
      if k == item._key:                           # Found a match:
        item._value = v                            # reassign value
        return                                     # and quit
    # did not find match for key
    self._table.append(self._Item(k,v))

  def __delitem__(self, k):
    """Remove item associated with key k (raise KeyError if not found)."""
    for j in range(len(self._table)):
      if k == self._table[j]._key:                 # Found a match:
        self._table.pop(j)                         # remove item
        return                                     # and quit
    raise KeyError('Key Error: ' + repr(k))

  def __len__(self):
    """Return number of items in the map."""
    return len(self._table)

  def __iter__(self):
    """Generate iteration of the map's keys."""
    for item in self._table:
      yield item._key                              # yield the KEY
```

Performance of a List-Based Map

- The unsorted list implementation is simple but not efficient.
  - __getitem__, __setitem__, and __delitem__, relies on a for loop to scan the underlying list of items in search of a matching key.

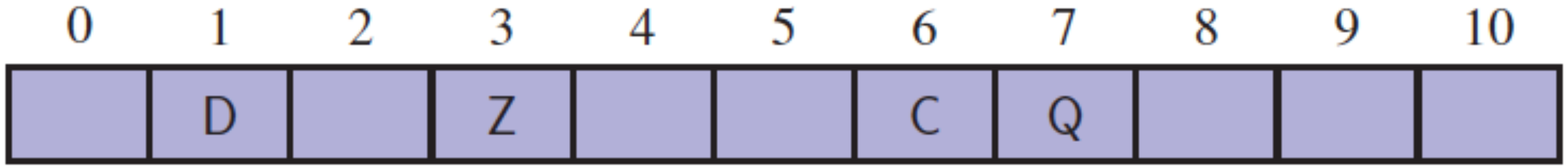
- Performance:
  - Inserting an item takes $O(n)$ time since we first search for the existing keys and then insert the new item.
  - Searching for or removing an item takes $O(n)$ time, since in the worst case (the item is not found) we traverse the entire list to look for an item with the given key.

Hash Tables



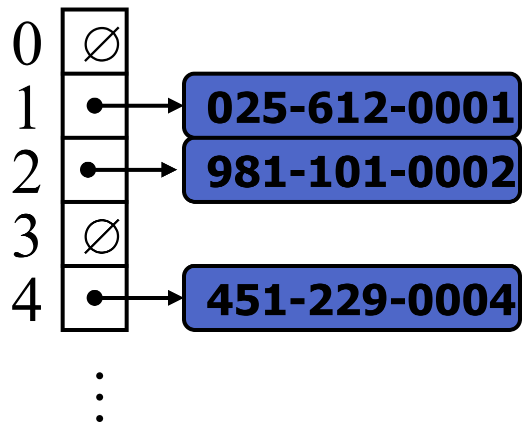
Recall the notion of a Map

- Intuitively, a map M supports the abstraction of using keys as indices with a syntax such as M[k].
- As a mental warm-up, consider a restricted setting in which a map with n items uses keys that are known to be integers in a range from 0 to N – 1, for some N ≥ n.



More General Kinds of Keys

- But what should we do if our keys are not integers in the range from 0 to N – 1?
  - Use a **hash function** to map general keys to corresponding indices in a table.
  - For instance, the last four digits of a Social Security number.

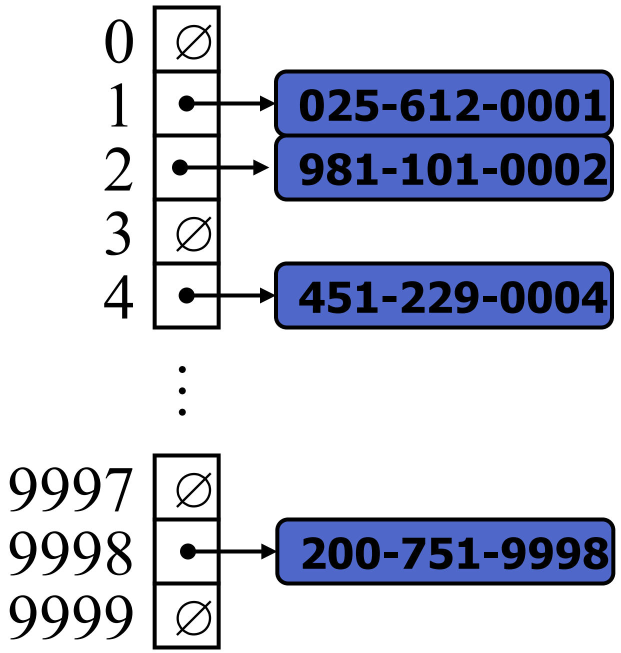


Hash Functions and Hash Tables

- A hash function h maps keys of a given type to integers in a fixed interval $[0, N-1]$
- Example:
  $$h(x) = x \bmod N$$
  is a hash function for integer keys
- The integer $h(x)$ is called the hash value of key x
- A hash table for a given key type consists of
  - Hash function h
  - Array (called table) of size N
- When implementing a map with a hash table, the goal is to store item (k, o) at index $i = h(k)$

SSN Example

- We design a hash table for a map storing entries as (SSN, Name), where SSN (social security number) is a nine-digit positive integer
- Our hash table uses an array of size $N = 10{,}000$ and the hash function
  $h(x) = \text{last four digits of } x$



Hash Functions

- A hash function is usually specified as the composition of two functions:

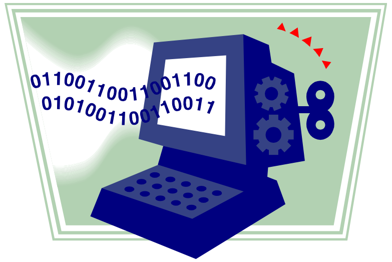
  Hash code:
  h_1: keys $\rightarrow$ integers

  Compression function:
  h_2: integers $\rightarrow [0, N-1]$

- The hash code is applied first, and the compression function is applied next on the result, i.e.,
  $$h(x) = h_2(h_1(x))$$

- The goal of the hash function is to "disperse" the keys in an apparently random way

Hash Codes



- Memory address:
  - We reinterpret the memory address of the key object as an integer.
    - Good in general, except for numeric and string keys
- Integer cast:
  - We reinterpret the bits of the key as an integer
  - Suitable for keys of length less than or equal to the number of bits of the integer

- Component sum:
  - We partition the bits of the key into components of fixed length (e.g., 16 or 32 bits) and we sum the components (ignoring overflows)
  - Suitable for numeric keys of fixed length greater than or equal to the number of bits of the integer type

Hash Codes (cont.)

- Polynomial accumulation:
  - We partition the bits of the key into a sequence of components of fixed length (e.g., 8, 16 or 32 bits)

    $$a_0\, a_1\, \ldots\, a_{n-1}$$

  - We evaluate the polynomial

    $$p(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_{n-1} z^{n-1}$$

    at a fixed value z, ignoring overflows
  - Especially suitable for strings (e.g., the choice $z = 33$ gives at most 6 collisions on a set of 50,000 English words)

- Polynomial $p(z)$ can be evaluated in $O(n)$ time using Horner's rule:
  - The following polynomials are successively computed, each from the previous one in $O(1)$ time

    $$p_0(z) = a_{n-1}$$

    $$p_i(z) = a_{n-i-1} + z p_{i-1}(z)$$

    $$(i = 1, 2, \ldots, n-1)$$

- We have $p(z) = p_{n-1}(z)$

Compression Functions

- Division:
  - $h_2(y) = y \bmod N$
  - The size N of the hash table is usually chosen to be a prime
  - The reason has to do with number theory and is beyond the scope of this course

- Multiply, Add and Divide (MAD):
  - $h_2(y) = [(ay + b) \bmod p] \bmod N$
  - p is a prime number larger than N
  - a and b are nonnegative integers such that

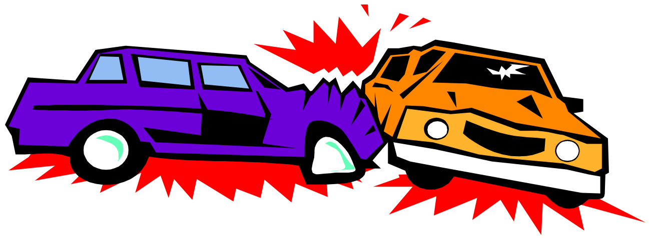
    $a \bmod N \neq 0$

    $a < p$
  - This method distributes the keys in more in uniformly random way.

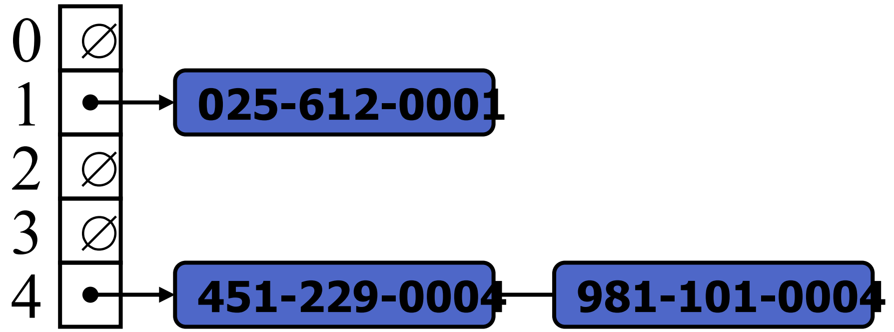
Abstract Hash Table Class

```python
class HashMapBase(MapBase):
  """Abstract base class for map using hash-table with MAD compression."""

  def __init__(self, cap=11, p=109345121):
    """Create an empty hash-table map."""
    self._table = cap * [ None ]
    self._n = 0                                  # number of entries in the map
    self._prime = p                              # prime for MAD compression
    self._scale = 1 + randrange(p-1)             # scale from 1 to p-1 for MAD
    self._shift = randrange(p)                   # shift from 0 to p-1 for MAD

  def _hash_function(self, k):
    return (hash(k)*self._scale + self._shift) % self._prime % len(self._table)

  def __len__(self):
    return self._n

  def __getitem__(self, k):
    j = self._hash_function(k)
    return self._bucket_getitem(j, k)            # may raise KeyError

  def __setitem__(self, k, v):
    j = self._hash_function(k)
    self._bucket_setitem(j, k, v)                # subroutine maintains self._n
    if self._n > len(self._table) // 2:          # keep load factor <= 0.5
      self._resize(2 * len(self._table) - 1)     # number 2^x - 1 is often prime

  def __delitem__(self, k):
    j = self._hash_function(k)
    self._bucket_delitem(j, k)                   # may raise KeyError
    self._n -= 1

  def _resize(self, c):                          # resize bucket array to capacity c
    old = list(self.items())                     # use iteration to record existing items
    self._table = c * [None]                     # then reset table to desired capacity
    self._n = 0                                  # n recomputed during subsequent adds
    for (k,v) in old:
      self[k] = v                                # reinsert old key-value pair
```

Collision Handling



- Collisions occur when different elements are mapped to the same cell
- Separate Chaining: let each cell in the table point to a linked list of entries that map there
- Separate chaining is simple, but requires additional memory outside the table



Map with Separate Chaining

Delegate operations to a list-based map at each cell:

Algorithm get(k):
return A[h(k)].get(k)

Algorithm put(k,v):
t = A[h(k)].put(k,v)
if t = **null then** {k is a new key}
    n = n + 1
return t

Algorithm remove(k):
t = A[h(k)].remove(k)
if t ≠ **null then** {k was found}
    n = n - 1
return t

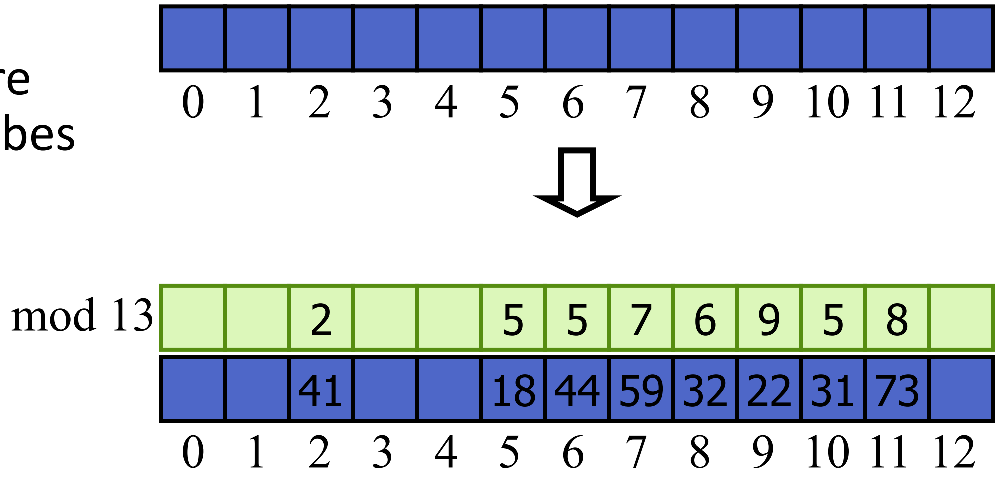
Hash Table with Separate Chaining

```
class ChainHashMap(HashMapBase):
  """Hash map implemented with separate chaining for collision resolution."""

  def _bucket_getitem(self, j, k):
    bucket = self._table[j]
    if bucket is None:
      raise KeyError('Key Error: ' + repr(k))          # no match found
    return bucket[k]                                    # may raise KeyError

  def _bucket_setitem(self, j, k, v):
    if self._table[j] is None:
      self._table[j] = UnsortedTableMap( )      # bucket is new to the table
    oldsize = len(self._table[j])
    self._table[j][k] = v
    if len(self._table[j]) > oldsize:           # key was new to the table
      self._n += 1                              # increase overall map size

  def _bucket_delitem(self, j, k):
    bucket = self._table[j]
    if bucket is None:
      raise KeyError('Key Error: ' + repr(k))          # no match found
    del bucket[k]                                       # may raise KeyError

  def __iter__(self):
    for bucket in self._table:
      if bucket is not None:                            # a nonempty slot
        for key in bucket:
          yield key
```

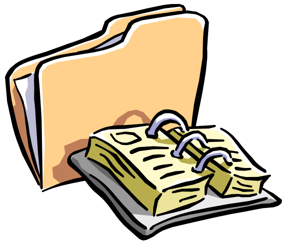
Linear Probing

- Open addressing: the colliding item is placed in a different cell of the table
- Linear probing: handles collisions by placing the colliding item in the next (circularly) available table cell
- Each table cell inspected is referred to as a "probe"
- Colliding items lump together, causing future collisions to cause a longer sequence of probes

- Example:
  - $h(x) = x \bmod 13$
  - Insert keys 18, 41, 22, 44, 59, 32, 31, 73, in this order



Search with Linear Probing



- Consider a hash table A that uses linear probing
- get(k)
  - We start at cell $h(k)$
  - We probe consecutive locations until one of the following occurs
    - An item with key k is found, or
    - An empty cell is found, or
    - N cells have been unsuccessfully probed

Algorithm $get(k)$
    $i \leftarrow h(k)$
    $p \leftarrow 0$
    repeat
        $c \leftarrow A[i]$
        if $c = \varnothing$
            return $null$
        else if $c.getKey\,() = k$
            return $c.getValue()$
        else
            $i \leftarrow (i + 1) \bmod N$
            $p \leftarrow p + 1$
    until $p = N$
    return $null$

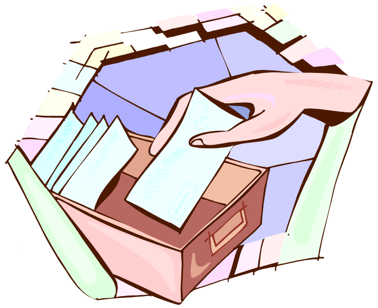
Updates with Linear Probing

- To handle insertions and deletions, we introduce a special object, called $\boldsymbol{AVAILABLE}$, which replaces deleted elements
- remove($\boldsymbol{k}$)
  - We search for an entry with key $\boldsymbol{k}$
  - If such an entry $(\boldsymbol{k}, \boldsymbol{o})$ is found, we replace it with the special item $\boldsymbol{AVAILABLE}$ and we return element $\boldsymbol{o}$
  - Else, we return ***null***
- put($\boldsymbol{k}, \boldsymbol{o}$)
  - We throw an exception if the table is full
  - We start at cell $\boldsymbol{h}(\boldsymbol{k})$
  - We probe consecutive cells until one of the following occurs
    - A cell $\boldsymbol{i}$ is found that is either empty or stores $\boldsymbol{AVAILABLE}$, or
    - $\boldsymbol{N}$ cells have been unsuccessfully probed
  - We store $(\boldsymbol{k}, \boldsymbol{o})$ in cell $\boldsymbol{i}$

Hash Table with Linear Probing

```python
class ProbeHashMap(HashMapBase):
  """Hash map implemented with linear probing for collision resolution."""
  _AVAIL = object( )       # sentinal marks locations of previous deletions

  def _is_available(self, j):
    """Return True if index j is available in table."""
    return self._table[j] is None or self._table[j] is ProbeHashMap._AVAIL

  def _find_slot(self, j, k):
    """Search for key k in bucket at index j.

    Return (success, index) tuple, described as follows:
    If match was found, success is True and index denotes its location.
    If no match found, success is False and index denotes first available slot.
    """
    firstAvail = None
    while True:
      if self._is_available(j):
        if firstAvail is None:
          firstAvail = j                      # mark this as first avail
        if self._table[j] is None:
          return (False, firstAvail)          # search has failed
      elif k == self._table[j]._key:
        return (True, j)                      # found a match
      j = (j + 1) % len(self._table)          # keep looking (cyclically)

  def _bucket_getitem(self, j, k):
    found, s = self._find_slot(j, k)
    if not found:
      raise KeyError('Key Error: ' + repr(k))         # no match found
    return self._table[s]._value

  def _bucket_setitem(self, j, k, v):
    found, s = self._find_slot(j, k)
    if not found:
      self._table[s] = self._Item(k,v)                # insert new item
      self._n += 1                                    # size has increased
    else:
      self._table[s]._value = v                       # overwrite existing

  def _bucket_delitem(self, j, k):
    found, s = self._find_slot(j, k)
    if not found:
      raise KeyError('Key Error: ' + repr(k))         # no match found
    self._table[s] = ProbeHashMap._AVAIL              # mark as vacated

  def __iter__(self):
    for j in range(len(self._table)):                 # scan entire table
      if not self._is_available(j):
        yield self._table[j]._key
```

Double Hashing



- Double hashing uses a secondary hash function $d(k)$ and handles collisions by placing an item in the first available cell of the series
$$(i + jd(k)) \bmod N$$
for $j = 0, 1, \dots, N-1$
- The secondary hash function $d(k)$ cannot have zero values
- The table size N must be a prime to allow probing of all the cells

- Common choice of compression function for the secondary hash function:
$$d_2(k) = q - k \bmod q$$
where
  - $q < N$
  - q is a prime
- The possible values for $d_2(k)$ are
$$1, 2, \dots, q$$

Example of Double Hashing

- Consider a hash table storing integer keys that handles collision with double hashing
  - $N = 13$
  - $h(k) = k \bmod 13$
  - $d(k) = 7 - k \bmod 7$
- Insert keys 18, 41, 22, 44, 59, 32, 31, 73, in this order

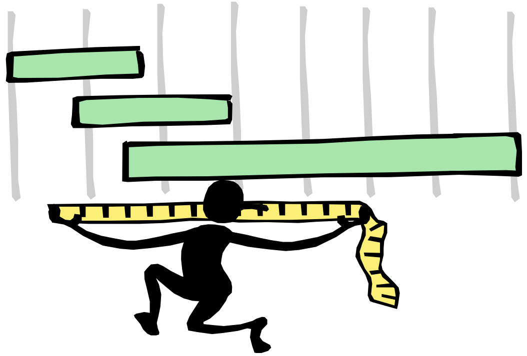
k	$h(k)$	$d(k)$	Probes		
18	5	3	5		
41	2	1	2		
22	9	6	9		
44	5	5	5	10	
59	7	4	7		
32	6	3	6		
31	5	4	5	9	0
73	8	4	8		

0	1	2	3	4	5	6	7	8	9	10	11	12

⇩

31		41			18	32	59	73	22	44		
0	1	2	3	4	5	6	7	8	9	10	11	12

Performance of Hashing



- In the worst case, searches, insertions and removals on a hash table take $O(n)$ time
- The worst case occurs when all the keys inserted into the map collide
- The load factor $\alpha = n/N$ affects the performance of a hash table
- Assuming that the hash values are like random numbers, it can be shown that the expected number of probes for an insertion with open addressing is

  $$1 / (1 - \alpha)$$

- The expected running time of all the dictionary ADT operations in a hash table is $O(1)$
- In practice, hashing is very fast provided the load factor is not close to 100%
- Applications of hash tables:
  - small databases
  - compilers
  - browser caches