HW4

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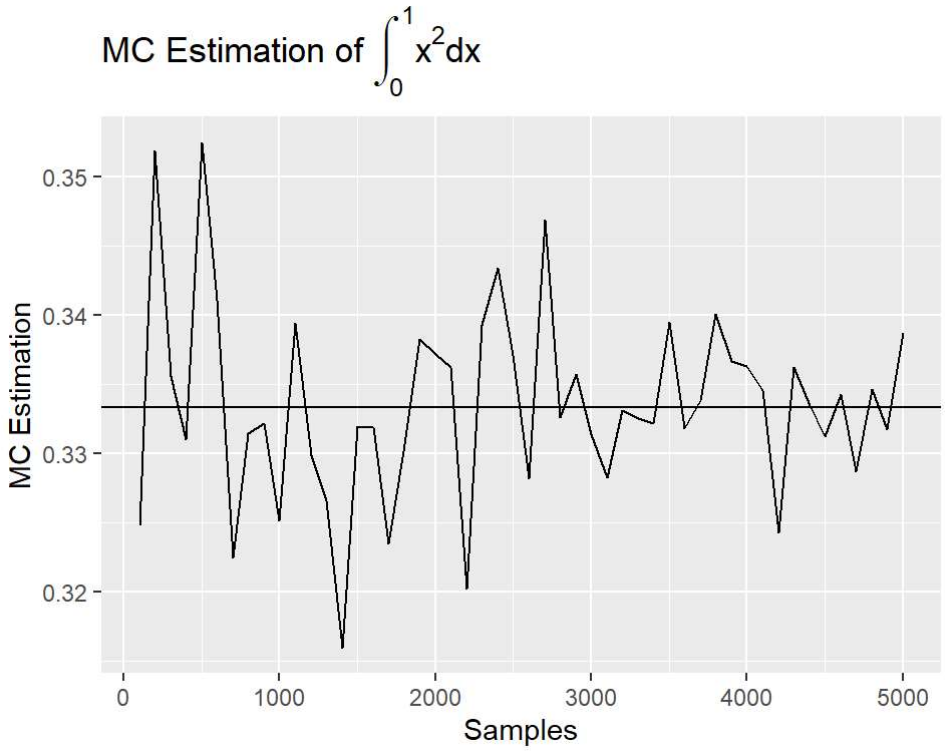
5/19/2021

Problem 1

1a)

$$\int_0^1 x^2 dx$$

We use $f(x) = x^2$ and $p(x) = 1$ uniform over the $[0, 1]$ interval to approximate the integral using the formulation.



The Monte Carlo estimation of $\int_0^1 x^2 dx$ is $\hat{I} = 0.3386958$. The exact evaluation is $I = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$. Differnce is $|I - \hat{I}\ | = 0.0053624$ which is only a factor of $\frac{|I - \hat{I}|}{1/3} = 0.0160873$.

1b)

$\int_{-1}^1 \int_{-1}^1 H(x, y) dx dy$ can be treated as sampling from uniform distribution with support on $[-1, 1] \times [-1, 1]$ square.

The result is 0.7854568, which is $\pi/4$

1c)

$$\int_0^\infty \frac{3}{4}x^4 e^{-x^3/4}dx$$

Notice that $\int_0^\infty \frac{3}{4}x^4 \exp\left(-x\right) \exp(-x^3/4+x)\Gamma(5)\Gamma(5)^{-1}dx$ can be treated as sampling from Gamma distribution with $\alpha = 4$ and $\beta = 1$

The result is 2.2748742.

Problem 2

Here we use importance sampling to approximate

$$I = \frac{1}{\sqrt{2\pi}}\int_1^2 e^{-x^2/2}dx$$

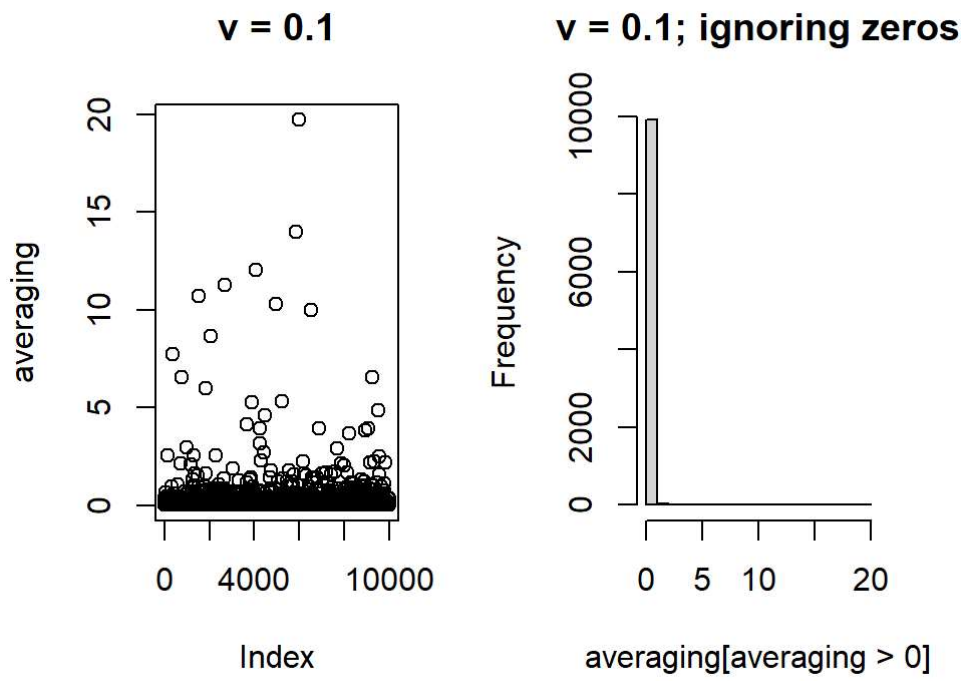
using $f(x) = e^{-x^2/2}/\sqrt{2\pi}$ and $p(x) = \mathbf{1}_{x\in[1,2]}$. Importance sampling is the approximation of

$$\mu = E_p(f(x)) \approx \hat{\mu}_g = \frac{1}{N}\sum_{i=1}^N \frac{f(x_i)p(x_i)}{g(x_i)}$$

where $X_i \sim g(X)$ iid. We rationalize this approximation using the law of large numbers and the fact that

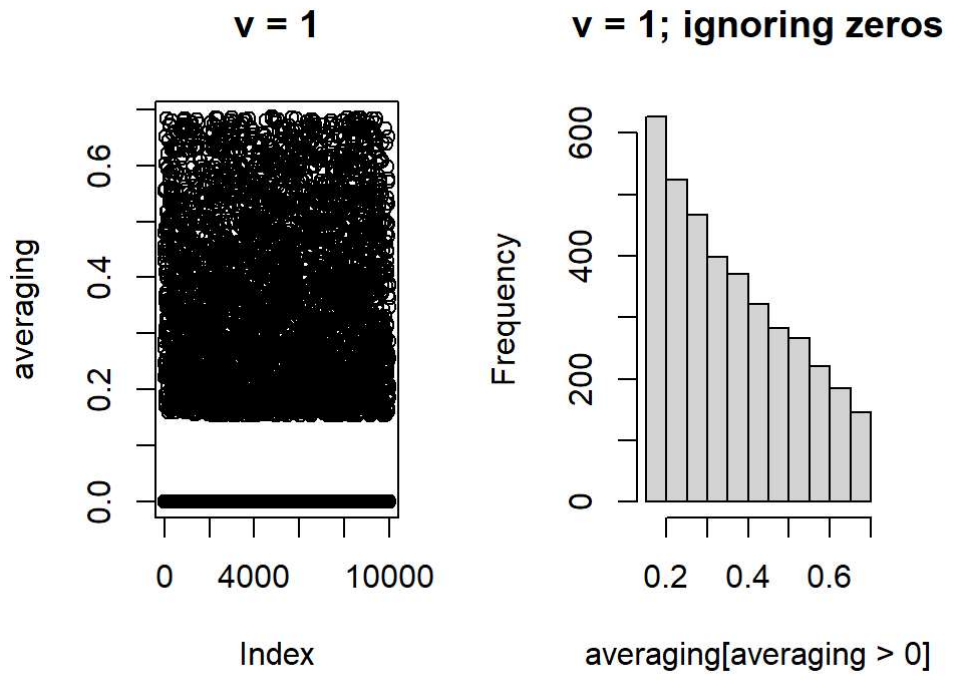
$$\mu = E_p(f(x)) = \int_D f(x)dp(x) = \int_D \frac{f(x)g(x)}{g(x)}dp(x) = \int_D \frac{f(x)p(x)}{g(x)}dg(x) = E_g(\frac{f(x)p(x)}{g(x)})$$

where $\frac{p(x)}{g(x)}$ is the *likelihood ratio*, $g(x)$ is the *importance distribution* given to be $g(x) = N(1.5, \nu)$ for $\nu \in \{0.1, 1, 10\}$, and $p(x)$ is the *nominal distribution*. In this problem, we use $p(x) \sim \mathbf{1}_{[1,2]}$ and $f(x) = e^{-x^2/2}/\sqrt{2\pi}$.



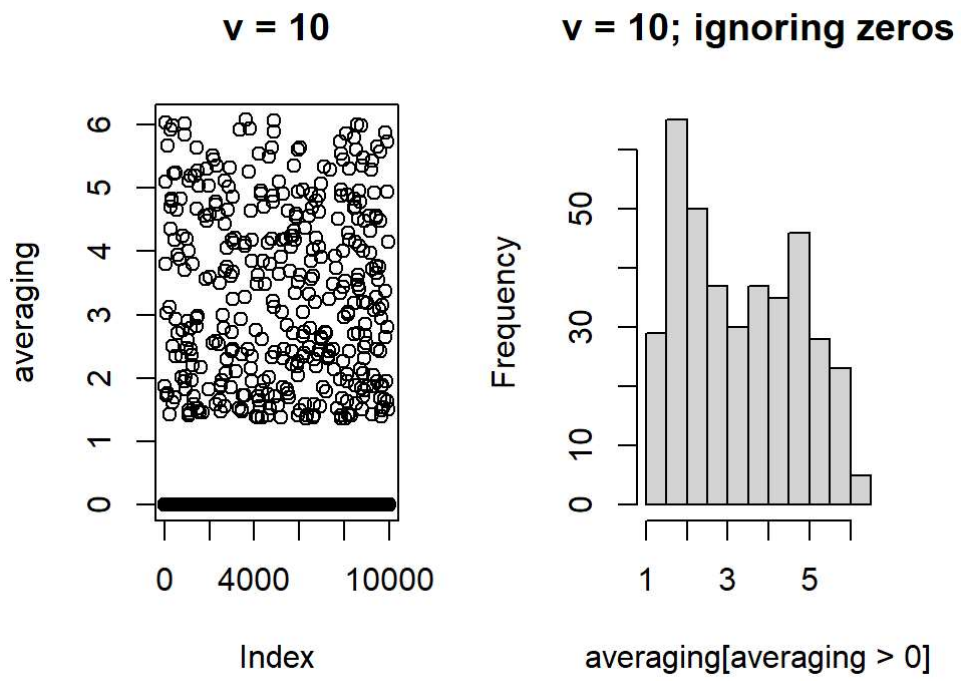
$$\hat{I}_{\nu=0.1} = 0.0945215$$

There are extreme large values.



$$\hat{I}_{\nu=1} = 0.1373354$$

There is no extreme values.



$$\hat{I}_{\nu=10} = 0.1280719$$

There is no extreme values.

The true value of the integral is

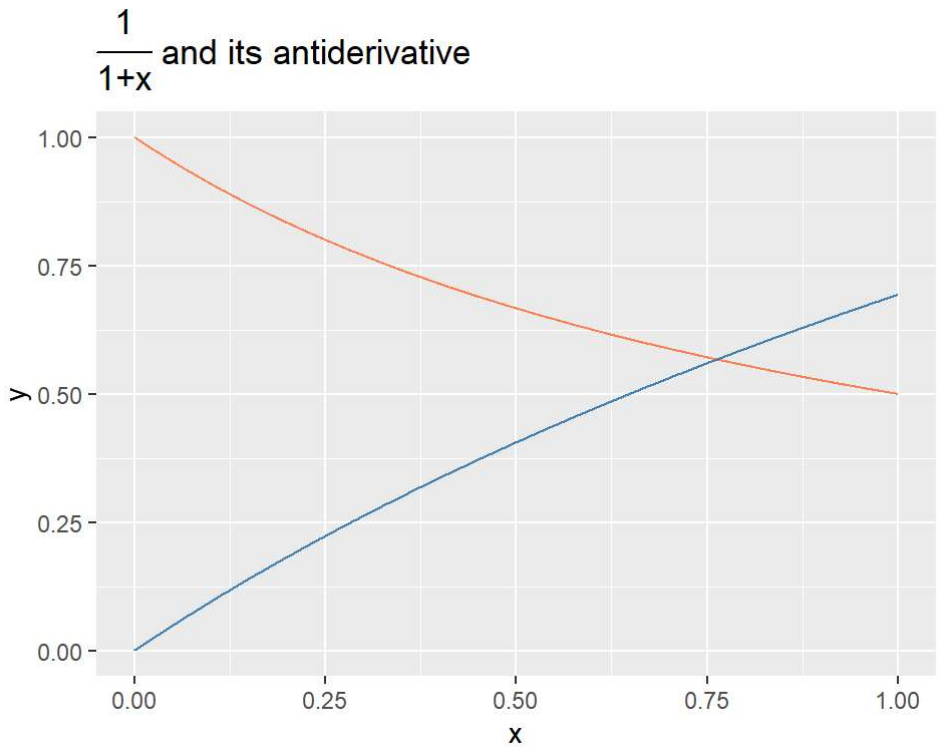
$$\int_1^2 \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \approx 0.135905$$

Problem 3

$$I = \int_0^1 \frac{1}{1+x} dx$$

3a)

We simulate the integral using $h(x) = \frac{1}{1+x}$ and $U_i \sim U[0,1]$ for $i \in \{1, \ldots, n\}$ in the model $I \approx \hat{I}_{MC} = \frac{1}{N} \sum_{i=1}^{N} h(u_i)$ where u_i are the observations of the random variables U_i.



The red line above is the function $h(x) = \frac{1}{1+x}$ and the blue line is $H(x) = \int_0^x h(t)dt$.

The MCMC estimate of the integral is $\hat{I}_{MC} = 0.6937654$ compared with the true integral $I \approx 0.6931472$.

3b)

Using the control variate $c(x) = 1 + x$, we calculate
$\hat{I}_{CV} = \frac{1}{N}\sum_{i=1}^{N} h(U_i) + b(\frac{1}{N}\sum_{i=1}^{N} c(U_i) - E(c(U)))$ where the variance of $\hat{I}_{CV}$ is minimized at
$b = \frac{-cov(h(X),c(X))}{var(c(X))} \approx 0.476696$. Upon calculation, $\hat{I}_{CV}$ is 0.6931796.

3c)

The variance of $\hat{I}_{CV}$ is

$$Var(\hat{I}_{CV}) = \frac{1}{N^2}Var(h(X)) - (\frac{b}{N})^2 Var(c(X)) = Var(\hat{I}_{MC}) - \frac{1}{N^2}\frac{Cov(h(X), c(X))^2}{Var(c(X))}$$

Running the MC and CV estimators 300 times with 1500 samples, $Var(\hat{I}_{MC}) \approx 1.3398899 \times 10^{-5}$ and $Var(\hat{I}_{CV}) \approx 3.9353132 \times 10^{-7}$. Notice that this is an improvement of 1.5320898 orders of magnitude in the variance of the estimator.

3d)

A new estimator for I, $\hat{I}_0 = \log 2$ has 0 variance and 0 bias. Alternatively, we could add another control variate ,for example $\hat{I}_{CV2} = \hat{I}_{CV} + b_2(\frac{1}{N}\sum_{i=1}^{N} c_2(U_i) - E(c_2(U)))$ where $c_2(x) = 1 - \frac{x}{2}$.

Problem 4

4a)

Using the model $y_{ij} = \mu + \alpha_i + e_{ij}$ where e_{ij} are sampled iid from a double exponential distribution, our null hypothesis H_0 is that $\alpha_i = 0$ for all $i \in \{1, \ldots, 3\}$. The alternative hypothesis H_A is that, for at least one $i^\star \in \{1, \ldots, 3\}$, $\alpha_{i^\star} \neq 0$.

To test the null hypothesis H_0 using Monte Carlo methods, we would execute the following algorithm:

```
Algorithm: Monte Carlo test
    input
        Y11...Y1N = Y.1,
        Y21...Y2N = Y.2,
        Y31...Y3N = Y.3,
            Observations for each of three treatments
        M,
            Number of samples for simulated emperical distribution
        K,
            Number of samples for each simulated statistic
        alpha
            Testing significance level
    output
        {0,1}
            0 if fail to reject H_0
            1 if reject H_0

m.hat = mean({Y1...YN})
b.hat = 1/2*sqrt(var(Y1...YN))
d.h0 = doubleExponential.pdf(mean=m.hat, sd=b.hat)

e.cdf = []
for i from 1 to M do:
    x.samp = sample(from=d.h0, n=K)
    x.stat = mean(x.samp)
    e.cdf.append(x.stat)

for i from 1 to 3 do:
    y.i.stat = mean(Y.i)
    if y.i.stat not in quantiles(e.cdf, alpha/2, 1-alpha/2) then:
        return 1
return 0
```

Hence, we generate a null empirical distribution using parameters estimated from the observed data and predicated on the H_0 model and reject the null hypothesis if any of the observed means for any of the treatments fall outside a reasonable simulated empirical quantile.

4b)

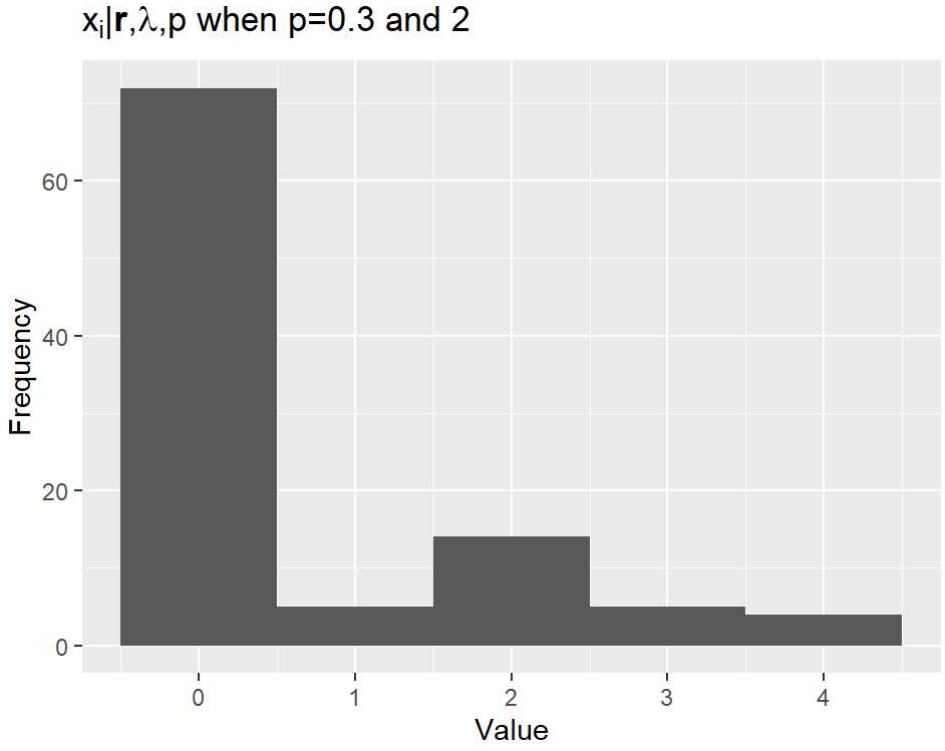
We could use Fisher's 2-sample permutation method applied to each pair of treatments. Note that this scales poorly ($O(n!)$) with the number of treatments.

Problem 5

5a)

We seek to sample $n = 100$ times from $Poisson(\lambda r_i)$ with $r_i \sim Bernoulli(p)$ using $\lambda = 2$ and $p = 0.3$.

The sample generated using $p = 0.3$ and $\lambda = 2$ is below:



5b)

5b).i

$$f(\lambda \mid p, r, x) = \frac{b^a \lambda^{a-1} e^{-b\lambda}}{\Gamma(a)} p^{\sum(r_i)} (1-p)^{n-\sum(r_i)} \prod \frac{r_i^{x_i}}{x_i!} e^{-\lambda \sum r_i} \lambda^{\sum x_i} =$$

$$\lambda^{a+\sum x_i - 1} e^{-(b+\sum r_i)\lambda} \frac{1}{N_{p,r,x}} = \mathbf{\Gamma}_{a+\sum x_i, b+\sum r_i}(\lambda)$$

5b).ii

$$f(\lambda \mid p, r, x) = \frac{b^a \lambda^{a-1} e^{-b\lambda}}{\Gamma(a)} p^{\sum(r_i)} (1-p)^{n-\sum(r_i)} \prod \frac{r_i^{x_i}}{x_i!} e^{-\lambda \sum r_i} \lambda^{\sum x_i} =$$

$$p^{\sum r_i} (1-p)^{n-\sum r_i} \frac{1}{N_{\lambda,r,x}} = \beta_{\sum r_i, n-\sum r_i}(p)$$

where

$$\frac{1}{N_{\lambda,r,x}} = \frac{b^a \lambda^{a-1} e^{-b\lambda}}{\Gamma(a)} \prod \frac{r_i^{x_i}}{x_i!} e^{-\lambda \sum r_i} \lambda^{\sum x_i}$$

5b).iii

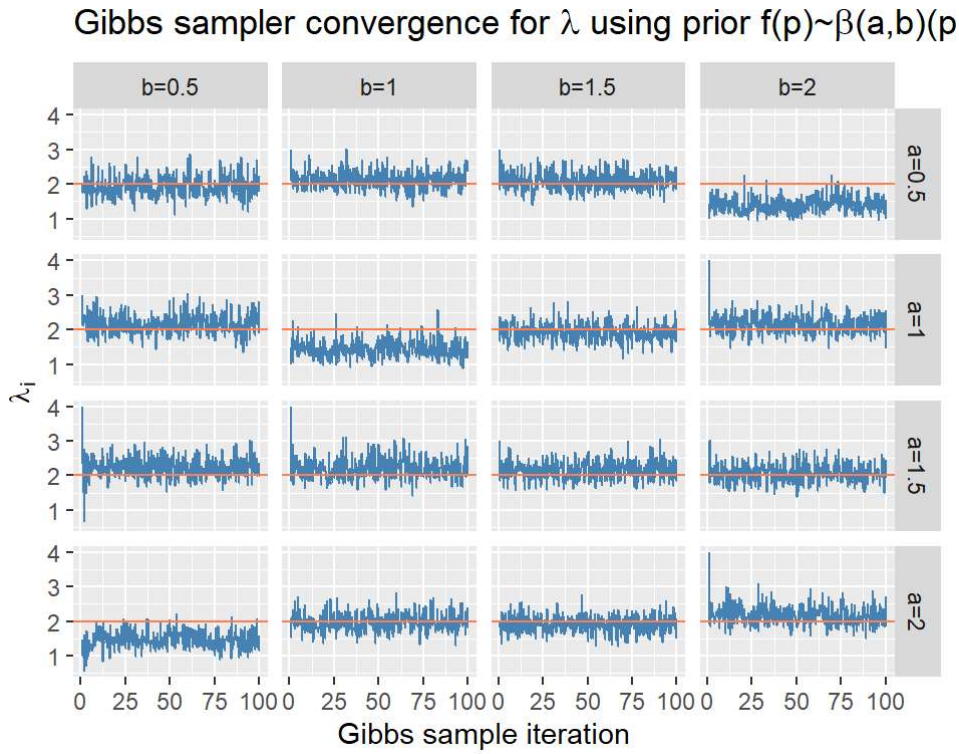
$$f(\lambda \mid p, r, x) = \frac{b^a \lambda^{a-1} e^{-b\lambda}}{\Gamma(a)} p^{\sum(r_i)} (1-p)^{n-\sum(r_i)} \prod \frac{r_i^{x_i}}{x_i!} e^{-\lambda \sum r_i} \lambda^{\sum x_i} =$$

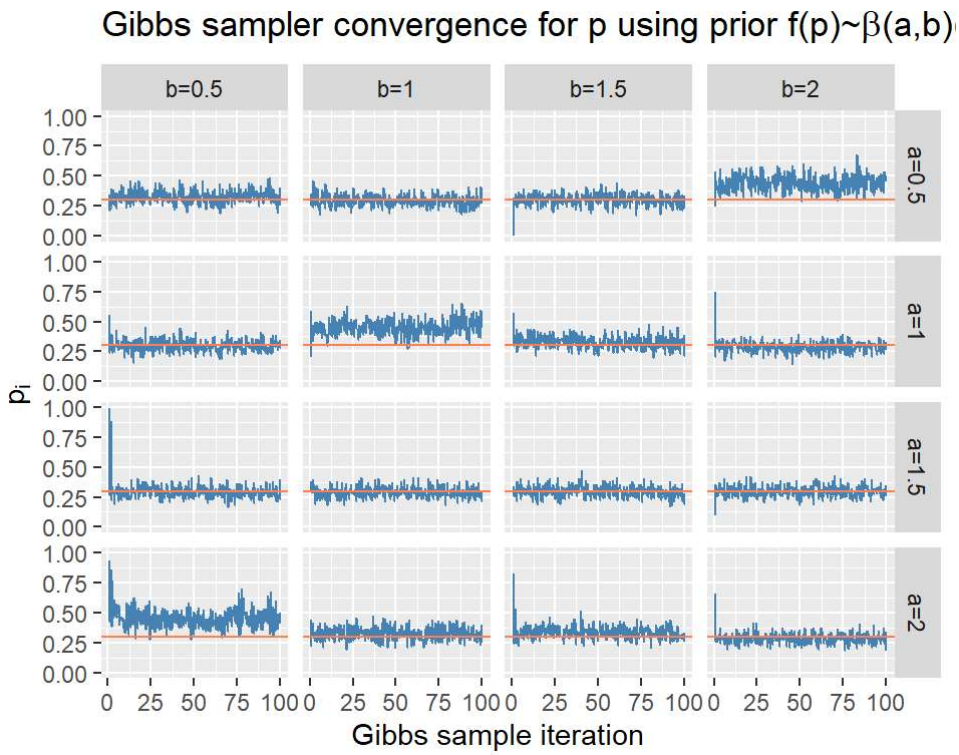
$$(\frac{b^a \lambda^{a-1} e^{-b\lambda}}{\Gamma(a)}) e^{-\lambda r_i} (\lambda r_i)^{\sum x_i} (\prod \frac{1}{x_i!}) p^{r_i} (1-p)^{1-r_i} = \frac{1}{N_{\lambda,x,p}} (e^{-\lambda} \frac{p}{1-p})^{r_i} r_i^{\sum x_i} =$$

$$Bernoulli(\frac{pe^{-\lambda}}{pe^{-\lambda} + (1-p)\mathbf{1}_{x_i=0}})$$

5c)

To sample from the posterior $f(\lambda, \mathbf{r}, p \mid \mathbf{x})$, we condition on the sample generated in part (5a). Then we expect to see high density around the values $p = 0.3$ and $\lambda = 2$ as were used to generate the sample.



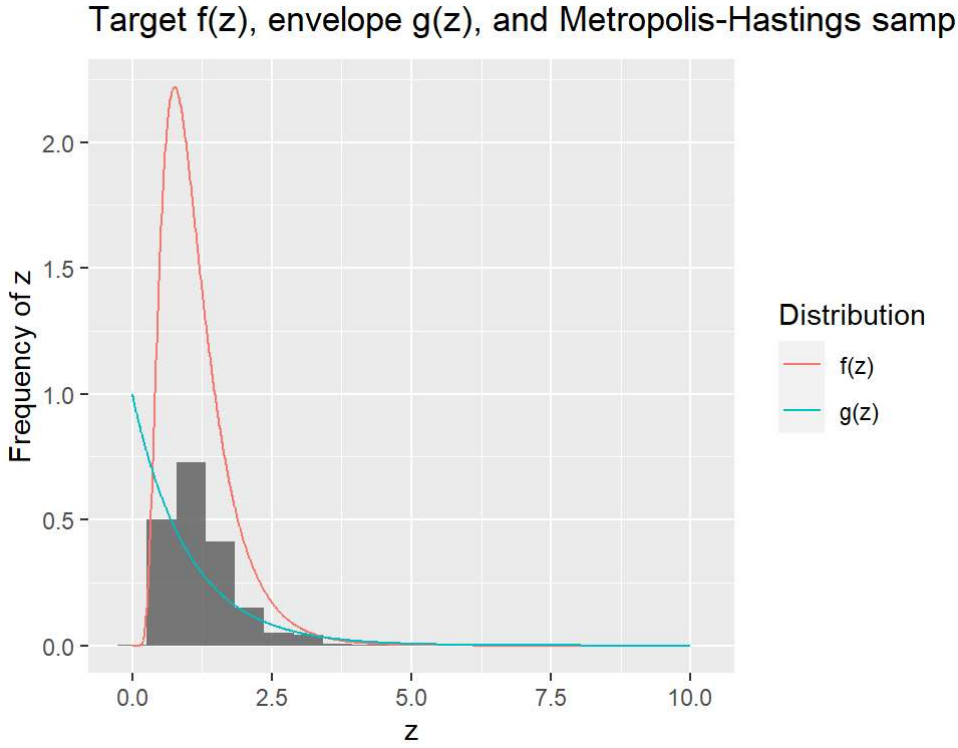


The Gibbs sampler stabilizes very quickly, typically within 15 iterations, though this is not true in general. The choice of prior heavily biases the estimates, as can be concluded by the distance from the stabilized region of the blue chain to the red true value.

Problem 6

$$f_{\theta_1,\theta_2}(z) \propto z^{-3/2} e^{-\theta_1 z - \frac{\theta_2}{z} + 2\sqrt{\theta_1\theta_2} + \log\sqrt{2\theta_2}}$$

We draw 1000 samples by executing the following:



Notice that the sample in grey follows the target distribution in red rather than the sampling envelope in blue even though the target and envelope have a small inner product. Furthermore, we can test the accuracy of this simulation using the following the target distribution:

$$E_f(Z) = \sqrt{\frac{\theta_2}{\theta_1}} = 1.1547005 \quad \text{and} \quad E_f(\frac{1}{Z}) = \sqrt{\frac{\theta_1}{\theta_2}} + \frac{1}{2\theta_2} = 1.1160254$$

In the sample produced by the M-H algorithm above:

$$\frac{1}{N}\sum_{i=1}^{N} z_i = 1.21147 \quad \text{and} \quad \frac{1}{N}\sum_{i=1}^{N} \frac{1}{z_i} = 1.0528308$$

Notice that the test statistics and their true values are pretty close, indicating by another means that the sample is accurately drawn from $f(z)$

Appendix

The following code is used in Question 1 part (a)

```
library(ggplot2)
library(latex2exp)
set.seed(100)

# Define functions

f = function(x){
    return(x^2)
}
p = function(x){
    return(1)
}
f = Vectorize(f)
p = Vectorize(p)

## Parameterize

N = 5000
m = 50
K = 10000
v = 1

## Simulate

pxs = seq(0, 1, length.out=K)
i1s = rep(NA, m)
ns = (1:m)*N/m
for(i in 1:m){
    xs = sample(pxs, ns[i], prob=p(pxs), replace=T)
    i1 = v*mean(f(xs))
    i1s[i] = i1
}

## Plot

df_plt1 = data.frame(x = ns, y = i1s)
plt1 = ggplot(data=df_plt1, aes(x=x, y=y)) +
    geom_line() +
    geom_hline(yintercept = 1/3) +
    labs(
         x="Samples",
         y="MC Estimation",
         title=TeX("MC Estimation of $\\int_0^1 x^2 dx$")
         )
plt1
```

The following code is used in Question 1 part (b)

```
x = runif(10000000, min = -1, max = 1)
y = runif(10000000, min = -1, max = 1)
result = mean(mean(x^2+y^2 < 1))
```

The following code is used in Question 1 part (c)

```
library(reshape)

x = rgamma(10000000, shape = 5, scale = 1)
result = mean(mean(3/4 * exp(-x^3/4+x)))*4*3*2
```

The following code is used in Question 2

```
set.seed(101)
v = 0.1
x = rnorm(10000, mean = 1.5, sd = v)
averaging = 1/sqrt(2*pi) * exp(-x^2/2)*(x<2 & x>1)/dnorm(x, mean = 1.5, sd = v)
par(mfrow = c(1,2))
plot(averaging, main = paste0("v = ",v))
hist(averaging[averaging>0], main = paste0("v = ",v,"; ignoring zeros"))
```

```
set.seed(102)
v = 1
x = rnorm(10000, mean = 1.5, sd = v)
averaging = 1/sqrt(2*pi) * exp(-x^2/2)*(x<2 & x>1)/dnorm(x, mean = 1.5, sd = v)
par(mfrow = c(1,2))
plot(averaging, main = paste0("v = ",v))
hist(averaging[averaging>0], main = paste0("v = ",v,"; ignoring zeros"))
```

```
set.seed(103)
v = 10
x = rnorm(10000, mean = 1.5, sd = v)
averaging = 1/sqrt(2*pi) * exp(-x^2/2)*(x<2 & x>1)/dnorm(x, mean = 1.5, sd = v)
par(mfrow = c(1,2))
plot(averaging, main = paste0("v = ",v))
hist(averaging[averaging>0], main = paste0("v = ",v,"; ignoring zeros"))
```

The following code is used in Question 3 part (a)

```
## Parameterize
N = 1500
us = runif(N, 0, 1)
itru = log(2)
xs = seq(0,1,length.out=N)

## Define functions

h = function(x){
    if(x<0){return(0)}
    else if(x>1){return(0)}
    else{
        return(1/(1+x))
    }
}
h = Vectorize(h)

## Simulate

i1 = mean(h(us))

## Plot h and int(h)

df_3.a = data.frame(x=xs, y=h(xs))
df_3.a.2 = data.frame(x=xs, y=cumsum((1/N)*h(xs)))
plt_3.a = ggplot() +
    geom_line(data=df_3.a, aes(x=x, y=y), color="coral") +
    geom_line(data=df_3.a.2, aes(x=x, y=y), color="steelblue") +
    labs(title=TeX("$\\frac{1}{1+x}$ and its antiderivative"))

plt_3.a
```

The following code is used in Question 3 part (b)

```
cc = function(x){
    if(x<0){return(0)}
    else if(x>1){return(0)}
    else{
        return(1+x)
    }
}
cc = Vectorize(cc)

b = -cov(h(xs), cc(xs)) / var(cc(xs))
i2 = mean(h(us)) + b*(mean(cc(us)) - 1.5)
```

The following code is used in Question 3 part (c)

```
## Parameterize

M = 300

## Simulate

i1s = rep(NA, M)
i2s = rep(NA, M)

for(i in 1:M){
    us = runif(N)
    i1 = mean(h(us))
    i2 = mean(h(us)) + b*(mean(cc(us)) - 1.5)
    i1s[i] = i1
    i2s[i] = i2
}

v1 = var(i1s)
v2 = var(i2s)
```

The following code is used in Question 5 part (a)

```
## Parameterize

n = 100
ltrue = 2
ptrue = 0.3

## Sample from prior

r = runif(n) < ptrue # r Bernoulli(p)

## Sample from posterior

xs = r * rpois(n, ltrue)

## Plot

df = data.frame(x=xs)
plt = ggplot(data=df, aes(x=x)) +
    geom_histogram(bins = max(xs)+1) +
    labs(
         x="Value",
         y="Frequency",
         title=TeX(
            "$x_i | \\textbf{r}, \\lambda, p$ when $p=0.3$ and $\\lambda=2$"
            )
        )
```

The following code is used in Question 5 part (c)

```
## Parameterize

N = 400
A = seq(1/2, 2, by=1/2)
B = A



## Generate data frame

df.main = data.frame(matrix(NA,
                            length(A)*length(B)*N,
                            (5+n)
                          ))
names(df.main) = c("k", "p", "l", "a", "b", paste("r", 1:n, sep=""))

z = 0
for(a in A){for(b in B){

## Generate lambda, r, p randomly

r0 = runif(n) > rbeta(n, a, b)
l0 = ceiling(runif(1)*max(xs))
p0 = rbeta(1, a, b)
r1 = r0
l1 = l0
p1 = p0

Rs = matrix(NA, N, n)
ls = rep(NA, N)
ps = rep(NA, N)
Rs[1,] = r1
ls[1] = l1
ps[1] = p1

## Run the Gibbs sampler

for(i in 2:N){

    r1 = runif(n) < (p1*exp(-l0))/(p1*exp(-l0) + (1-p1)*(xs==0))
    l1 = rgamma(1, shape = a + sum(xs), rate = b + sum(r1))
    p1 = rbeta(1, 1 + sum(r1), n + 1 - sum(r1))



    Rs[i,] = r1
    ls[i] = l1
    ps[i] = p1
}

# Update the main dataframe (k, p, l, a, b, r1, ..., rn)

ixs = (1:N)+z*N
df.main[ixs,][1] = 1:n
```

```
df.main[ixs,][2] = ps   # estimate of p
df.main[ixs,][3] = ls   # estimate of lambda
df.main[ixs,][4] = a
df.main[ixs,][5] = b
df.main[ixs,][6:dim(df.main)[2]] = as.integer(Rs)
z = z + 1

}}

## Plot

plt_lambda = ggplot(data=df.main) +
    geom_line(aes(x=k, y=l), color="steelblue") +
    geom_hline(aes(yintercept=ltrue), color="coral") +
    facet_grid(a ~ b,
            labeller=labeller(
                a = (function (x) paste("a=", x, sep="")),
                b = (function (x) paste("b=", x, sep=""))
            )) +
    labs(x="Gibbs sample iteration",
         y=TeX("$\\lambda_i$"),
         title=TeX(paste(
            "Gibbs sampler convergence for $\\lambda$",
            "using prior $f(p)~\\beta(a,b)(p)$"
            ))
        )

plt_p = ggplot(data=df.main) +
    geom_line(aes(x=k, y=p), color="steelblue") +
    geom_hline(aes(yintercept=ptrue), color="coral") +
    facet_grid(a ~ b,
            labeller=labeller(
                a = (function (x) paste("a=", x, sep="")),
                b = (function (x) paste("b=", x, sep=""))
            )) +
    labs(x="Gibbs sample iteration",
         y=TeX("$p_i$"),
         title=TeX(paste(
            "Gibbs sampler convergence for $p$",
            "using prior $f(p)~\\beta(a,b)(p)$"
            ))
        )


plt_lambda
plt_p
```

The following code is used in Question 6

```
## Parameterize

N = 1000
ran = seq(0, 10, by=0.01)
t1 = 1.5
t2 = 2

## Define functions

f = function(z){



    if(z<=0){return(0)}
    else if(t1<=0){return(0)}
    else if(t2<=0){return(0)}

    # Density
    r = z^(-3/2)*exp(-t1*z-t2/z+2*sqrt(t1*t2)+log(sqrt(2*t2)))
    return(r)
}
f = Vectorize(f)

g = function(z){
    # Envelope density g(z)

    return(dgamma(z, 1, 1))
}
g = Vectorize(g)

r = function(x,y){
    return(min(f(y)/f(x)*g(x)/g(y),1))
}

## Metropolis-Hastings algorithm

samp = rep(NA, N)
envelope = g(ran)
z0 = sample(size=1, x=ran, prob=envelope)
samp[1] = z0

for(i in 2:N){
    z1 = sample(size=1, x=ran, prob=envelope)
    a = r(z0,z1)
    if(a == 1){

        z0 = z1
    }else{

        if(a > runif(1)){

            z0 = z1
        }
```

```
    }
    samp[i] = z0
}

## Plot

dists_df = data.frame(x=ran, f=f(ran), g=g(ran))
plt_dists = ggplot(data=melt(dists_df, id="x")) +
    geom_line(aes(x=x, y=value, color=variable))

plt_methast = ggplot() +
    geom_histogram(
        data=melt(samp),
        aes(x=value, y=..density..),
        alpha=0.8, bins=N/50
    ) +
    geom_line(
        data=melt(dists_df, id="x"),
        aes(x=x, y=value, color=variable)
    ) +
    scale_colour_discrete(
        labels=c("f(z)", "g(z)"),
        name="Distribution"
    ) +
    labs(
        x="z", y="Frequency of z",
        title=TeX(paste(
            "Target $f(z)$,",
            "envelope $g(z)$,",
            "and Metropolis-Hastings sample",
            sep=" "
        ))
    )

plt_methast
```