ON ANNIHILATOR MULTIPLICATION MODULES

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ABSTRACT. An A-module E is an annihilator multiplication module if, for each $e \in E$, there is a finitely generated ideal I of A such that ann(e) = ann(IE). In this paper, we investigate fundamental properties of annihilator multiplication modules and employ them as a framework for characterizing significant classes of rings and modules, including torsion-free modules, multiplication modules, injective modules, and principal ideal von Neumann regular rings. In addition, we establish that, for such modules, the equality $Ass_A(E) = Ass(A)$ holds, thereby providing a precise connection between module-theoretic and ring-theoretic prime structures.

1. Introduction

In this paper, we focus only on commutative rings with a nonzero identity and nonzero unital modules. In particular, A will always represent such a ring, and E will represent such an A-module. The concept of the multiplication module was first introduced and studied by Barnard in [9]. An A-module E is called a multiplication module if every submodule V of E has the form IE for some ideal I of A. Subsequently, El-Bast and Smith, in their influential paper [15], explored further properties and characterizations of multiplication modules. The notion of multiplication modules provides a way to associate a submodule V of E with an ideal $(V:_A E)$ of A, where $(V:_A E) = ann_A(V/E)$. The algebraic properties of V and $(V:_A E)$ are frequently compatible. Many authors have since used multiplication modules as a bridge connecting module theory with ring theory. See, for example, [1], [2], [4], [8], [13], [19]-[22], [26], [27], [30]. A proper submodule V of E is called a prime submodule if for each $a \in A$ and $e \in E$, the condition $ae \in V$ implies $a \in (V :_A E)$ or $e \in V$ [29]. The set of all prime submodules of E is denoted by Spec(E). For any submodule K of E, the variety of K is defined as $V(K) = \{P \in Spec(E) : K \subseteq P\}$. While these varieties generate the Zariski topology in commutative ring theory, they may not always define a topology in general modules. However, if M is a multiplication module, these varieties fulfill the axioms for closed sets of a topology on Spec(M). This is known as the quasi Zariski topology of M [26]. Thus, prime submodules have applications in general topology through the concept of multiplication modules.

Multiplication modules are significant because they encompass several important classes of modules, including simple modules, cyclic modules, and von Neumann regular rings and modules. A ring A (not necessarily commutative) is defined as von Neumann regular if, for each a in A, there exists x in A such that a = axa [32]. Von

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Neumann originally introduced these rings due to their connection with continuous geometry [32]. Their algebraic significance has become increasingly recognized in recent years. In 2018, Jayaram and Tekir extended the concept of von Neumann regular rings to modules by introducing E-von Neumann regular and weak idempotent elements. Specifically, an element a in A is E-von Neumann regular (or weak idempotent) if $aE = a^2E$ (or $a - a^2 \in ann_A(E)$). An A-module E is called a von Neumann regular (vn-regular) module if, for each e in E, $Ae = aE = a^2E$ [17]. Every vn-regular module is also a multiplication module. Furthermore, Jayaram and Tekir demonstrated that a finitely generated A-module E is vn-regular if and only if, for each e in E, there exists a weak idempotent a in A such that Ae = aE[17, Lemma 5]. In functional analysis, Baer *-rings serve as the algebraic counterparts of von Neumann algebras. Every von Neumann algebra is a Baer *-ring because the right annihilator of any subset is generated by a projection. This relationship underscores the structural importance of projections in operator algebras and positions Baer *-rings as an algebraic framework for the study of annihilators and projection lattices. Kaplansky defined a Baer ring as a ring A in which the annihilator $ann(S) = \{a \in A : aS = 0\}$ of any subset S of A is generated by an idempotent b in A [23]. Subsequently, Kist refined this definition by considering S as a singleton, rather than an arbitrary subset. According to Kist [24], a ring A is a Baer ring (also referred to as p.q. Baer or P.P. ring) if, for every a in A, there exists an idempotent b in A such that ann(a) = bA. The present paper adopts Kist's definition of a Baer ring.

Let V be a submodule of E and K a nonempty subset of E. The residual of Vby K is denoted by $(V:_A K) = \{a \in A : aK \subseteq V\}$. When V = (0) and K is a submodule of E, we prefer ann(K) to denote ((0):K). In particular, for each $e \in E$, we use ann(e) instead of ann(Ae). An A-module E is said to be a torsion free if ann(e) = 0 for every $0 \neq e \in E$ [29]. E is said to be a faithful module if ann(E) = 0. It is clear that all torsion-free modules are faithful, and the converse is not true in general. For instance, \mathbb{Z} -module $\mathbb{Z} \oplus \mathbb{Z}_n$, where $n \geq 2$ is a faithful module which is not torsion-free. In a recent study, Jayaram, Tekir, and Koç introduced Baer modules, a new generalization of von Neumann regular modules, and extended various properties of Baer rings to these modules. Recall from [18] that an A-module E is said to be a Baer module if for each $e \in E$ there exists a weak idempotent $a \in A$ such that ann(e)E = aE. Also, they defined an A-module E to be an annihilator multiplication if for each $e \in E$, there exists a finitely generated ideal I of A such that ann(e) = ann(IE). Among the results in their paper, the authors showed that every finitely generated von Neumann regular module is a Baer module [18, Proposition 2.8], and also they proved that every finitely generated Baer module is also an annihilator multiplication module [18, Lemma 2.1]. Furthermore, [18] established that annihilator multiplication modules provide a link between the theory of Baer modules and Baer rings. Specifically, a finitely generated A-module E is a Baer module if and only if E is an annihilator multiplication module and A/ann(E)is a Baer ring [18, Theorem 2.14]. In this article, we aim to investigate additional algebraic properties of annihilator multiplication modules. In Section 2, we analyze the relationships between annihilator multiplication modules and other classical modules, including multiplication modules, von Neumann regular modules, finitely generated Baer modules, torsion-free modules, and simple modules (see Example 1

and Example 2). We also examine the stability of annihilator multiplication modules under direct products, direct sums, quotients, homomorphisms, factor modules, and polynomial modules (see Proposition 2, Proposition 5, Proposition 3, Proposition 4, Corollary 2, Proposition 6). In addition, we study 1-absorbing prime ideals and classical 1-absorbing prime submodules in annihilator multiplication modules (see Proposition 7 and Proposition 8). Section 3 is devoted to the characterization of important classes of rings and modules, such as torsion-free modules, multiplication modules, injective modules, and principal ideal von Neumann regular rings, using annihilator multiplication modules (see Theorem 1, Proposition 9, Proposition 10, Theorem 2). Finally, we prove that $Ass_A(E) = Ass(A)$ in annihilator multiplication modules, where $Ass_A(E)$ and Ass(A) denote the sets of associated primes of E and E, respectively (see Theorem 3).

2. Basic properties of annihilator multiplication modules

This section outlines the fundamental properties of annihilator multiplication modules. The discussion begins with the formal definition of this concept.

Definition 1. An A-module E is said to be an annihilator multiplication module if for each $e \in E$, there exists a finitely generated ideal I of A such that ann(e) = ann(IE).

Example 1. (i) Every multiplication module is an annihilator multiplication module. In particular, every vn-regular module is an annihilator multiplication module.

- $(ii)\ Every\ finitely\ generated\ Baer\ module\ is\ an\ annihilator\ multiplication\ module.$
- (iii) Every torsion-free module is an annihilator multiplication module.
- (iv) Every simple module is an annihilator multiplication module.

Proof. (i): Let E be a multiplication module and choose $e \in E$. Then we have Ae = (Ae : E)E. Then we can write $e = a_1e_1 + a_2e_2 + \cdots + a_ne_n$ for some $e_1, e_2, \ldots, e_n \in (Ae : E)$. Now, put $I = \sum_{i=1}^n Aa_i$. Let $v = be \in Ae$. Then we have $v = be = a_1(be_1) + a_2(be_2) + \cdots + a_n(be_n) \in IE$ which implies that $Ae \subseteq IE \subseteq (Ae : E)E \subseteq Ae$. Thus, we have Ae = IE for some finitely generated ideal I of A, and thus we get ann(e) = ann(IE), that is, E is an annihilator multiplication module. The rest follows from the fact that every von Neumann regular module is a multiplication module.

- (ii): Follows from [18, Lemma 2.11].
- (*iii*): Follows from [18, Example 2.10 (i)].
- (iv): Follows from the fact that every simple module is a multiplication module and by (i).

Example 2. Consider \mathbb{Z} -module $E = \mathbb{Z}_2 \oplus \mathbb{Z}_4$. Then note that E is not a multiplication module, and also not a reduced module since $2^2(\overline{0},\overline{1}) = (\overline{0},\overline{0})$ and $2(\overline{0},\overline{1}) \neq (\overline{0},\overline{0})$. As E is finitely generated, by [18, Proposition 2.7 and Proposition 2.8], E is neither a Baer module nor a von Neumann regular module. On the other hand, E is not a torsion free module and not a simple module. Let $e = (\overline{x},\overline{y}) \in E$. Then it is clear that $ann(e) = 4\mathbb{Z}$, $2\mathbb{Z}$ or \mathbb{Z} . Now we have 3 cases. Case 1: If $ann(e) = 4\mathbb{Z}$, then put $I = \mathbb{Z}$ and note that ann(e) = ann(IE). Case 2: If $ann(e) = 2\mathbb{Z}$, then put $I = 2\mathbb{Z}$ and note that ann(e) = ann(IE) Case 3: If $ann(e) = \mathbb{Z}$, then put $I = 0\mathbb{Z}$ and note that ann(e) = ann(IE). By above all cases, E is an annihilator multiplication module.

SUAT KOC

4

Let E be a multiplication module and V a finitely generated submodule of E. According to [14, Lemma 3.5], there exists a finitely generated ideal I of A such that V = IE. The following result provides an analogous statement for annihilator multiplication modules.

Proposition 1. Let E be an annihilator multiplication A-module. Then,

- (i) If V is a finitely generated submodule of E, there exists a finitely generated ideal I of A such that ann(V) = ann(IE).
- (ii) If V is a submodule of E (not necessarily a finitely generated), then there exists an ideal I of A such that ann(V) = ann(IW).
- *Proof.* (i): Suppose that V is finitely generated. Then we can write $V = Av_1 + Av_2 + \cdots + Av_k$ for some $v_1, v_2, \ldots, v_k \in V$. This implies that $ann(V) = \bigcap_{i=1}^k ann(v_i)$. Since E is an annihilator multiplication module, for each $i = 1, 2, \ldots, k$, there exist finitely generated ideals I_i of A such that $ann(v_i) = ann(I_iE)$. This gives

$$ann(V) = \bigcap_{i=1}^{k} ann(v_i) = \bigcap_{i=1}^{k} ann(I_i E)$$
$$= ann(\left(\sum_{i=1}^{k} I_i\right) E).$$

Since all I_i 's are finitely generated, so is $\sum_{i=1}^k I_i$, which completes the proof.

(ii): A similar argument in the proof of (i) shows that ann(V) = ann(IE) for some ideal I of A.

Let E_i be an A_i -module for each $i=1,2,\ldots,n$, where $n\geq 1$. Assume that $E=E_1\oplus E_2\oplus \cdots \oplus E_n$ and $A=A_1\oplus A_2\oplus \cdots \oplus A_n$. Then E is an A-module with component-wise addition and scalar multiplication.

Proposition 2. Let E_i be an A_i -module for each $i=1,2,\ldots,n$, where $n\geq 1$. Suppose that $E=E_1\oplus E_2\oplus \cdots \oplus E_n$ and $A=A_1\oplus A_2\oplus \cdots \oplus A_n$. Then E is an annihilator multiplication A-module if and only if E_i is an annihilator multiplication A_i -module for each $i=1,2,\ldots,n$.

Proof. (\Rightarrow): Let E be an annihilator multiplication A-module. Now, we will show that E_1 is an annihilator multiplication A_1 -module. Choose $e_1 \in E_1$ and put $e = (e_1, 0, 0, \dots, 0) \in E$. Since E is an annihilator multiplication module, there exists a finitely generated ideal $I = I_1 \oplus I_2 \oplus \cdots \oplus I_n$ of A such that

$$ann(e) = ann(e_1) \oplus A_2 \oplus \cdots \oplus A_n = ann(IE)$$
$$= ann(I_1E_1) \oplus ann(I_2E_2) \oplus \cdots \oplus ann(I_nE_n).$$

Then note that I_1 is a finitely generated ideal of A_1 and $ann(e_1) = ann(I_1E_1)$. Thus, E_1 is an annihilator multiplication A_1 -module. Likewise, E_i is an annihilator multiplication A_i -module for each i = 2, 3, ..., n.

 (\Leftarrow) : Suppose that E_i is an annihilator multiplication A_i -module for each $i=1,2,\ldots,n$. Let $e=(e_1,e_2,\ldots,e_n)\in E$. Since E_i is an annihilator multiplication A_i -module, there exists a finitely generated ideal I_i of A_i such that $ann(e_i)=1$

 $ann(I_iE_i)$. Then we conclude that

$$ann(e) = ann(e_1) \oplus ann(e_2) \oplus \cdots \oplus ann(e_n)$$

= $ann(I_1E_1) \oplus ann(I_2E_2) \oplus \cdots \oplus ann(I_nE_n)$
= $ann(IE)$,

where $I = I_1 \oplus I_2 \oplus \cdots \oplus I_n$ is an ideal of A. Since I_i is a finitely generated ideal of A_i for each $i = 1, 2, \ldots, n$, it follows that I is a finitely generated ideal of A. Hence, E is an annihilator multiplication A-module.

Let E be an A-module and $T \subseteq R$ a multiplicatively closed set. Then we denote the quotient module at T by $T^{-1}E = \left\{ \frac{e}{t} : e \in E, t \in T \right\}$ over the quotient ring $T^{-1}A$ [29].

Proposition 3. Let E be a finitely generated A-module and $T \subseteq A$ a multiplicatively closed set. If E is an annihilator multiplication A-module, then $T^{-1}E$ is an annihilator multiplication $T^{-1}A$ -module.

Proof. Let $\frac{e}{t} \in T^{-1}E$. Since E is an annihilator multiplication A-module, there exists a finitely generated ideal I of A such that ann(e) = ann(IE). This implies that

$$ann_{T^{-1}A}(\frac{e}{t}) = T^{-1}[ann(e)] = T^{-1}[ann(IE)].$$

Since I and E are finitely generated, so is IE. This gives

$$T^{-1}[ann(IE)] = ann_{T^{-1}A} [(T^{-1}I) (T^{-1}E)],$$

and so we have $ann_{T^{-1}A}(\frac{e}{t}) = ann_{T^{-1}A}\left[\left(T^{-1}I\right)\left(T^{-1}E\right)\right]$. Since I is a finitely generated ideal of A, so is $T^{-1}I$. Thus, $T^{-1}E$ is an annihilator multiplication $T^{-1}A$ -module.

In general, the converse of the previous proposition does not hold. The following example illustrates this point.

Example 3. Let p be a prime number and consider \mathbb{Z} -module $E = \mathbb{Z}_p \oplus \mathbb{Q}$. First note that E is not a finitely generated module. Choose $e = (\overline{1},0)$ and note that $ann(e) = p\mathbb{Z}$. Since \mathbb{Z} is a PID, every ideal I has the form $I = n\mathbb{Z}$ for some $n \in \mathbb{Z}$. Then we have two cases. Case 1: If n = 0, then $ann(IE) = \mathbb{Z} \neq p\mathbb{Z}$. Case 2: If $n \neq 0$, then $ann(IE) = (0) \neq p\mathbb{Z}$. This shows that E is not an annihilator multiplication module. Let $T = reg(\mathbb{Z}) = \mathbb{Z} - \{0\}$. Then note that $T^{-1}E$ is a vector space over the field $T^{-1}\mathbb{Z} = \mathbb{Q}$. Since every vector space is torsion free, by Example 1, $T^{-1}E$ is an annihilator multiplication module.

Corollary 1. Let E be a finitely generated annihilator multiplication module. Then for every prime ideal P of A, E_P is an annihilator multiplication A_P -module.

Proposition 4. Let $\phi: E \to E'$ be an A-homomorphism such that ann(E) = ann(E').

- (i) If ϕ is one to one and E' is an annihilator multiplication module, then E is an annihilator multiplication module.
- (ii) If ϕ is surjective, $Ker(\phi)$ is a prime submodule of E and E is an annihilator multiplication module, then E' is an annihilator multiplication module.

- *Proof.* (i): Let $e \in E$. Since E' is an annihilator multiplication module, there exists a finitely generated ideal I of A such that $ann(\phi(e)) = ann(IE')$. Since ϕ is one to one and ann(E') = ann(E), one can easily see that $ann(e) = ann(\phi(e))$ and ann(IE) = ann(IE') which implies that ann(e) = ann(IE). Thus, E is an annihilator multiplication module.
- (ii): Let $e' \in E'$. If e' = 0, then we have ann(e') = A = ann((0)E), we are done. So assume that e' is nonzero. Since ϕ is surjective, we can write $e' = \phi(e)$ for some $e \in E$. As E is an annihilator multiplication module, there exists a finitely generated ideal I of A such that ann(e) = ann(IE). Since ann(E) = ann(E'), we conclude that $ann(IE') = ann(IE) = ann(e) \subseteq ann(\phi(e))$. Now, let $a \in ann(\phi(e))$. Then we have $a\phi(e) = \phi(ae) = 0$ which implies thath $ae \in Ker(\phi)$. Since $Ker(\phi)$ is a prime submodule, we have either $a \in (Ker(\phi) : E)$ or $e \in Ker(\phi)$. Case 1: Let $e \in Ker(\phi) : E$. Then we have $e \in Ker(\phi)$, that is, $e \in Ker(\phi) : E$. Then we have $e \in Ker(\phi) : E$. Case 2: Let $e \in Ker(\phi)$, that is, $e \in Ker(\phi) : E$. Thus, we conclude that $e \in Ker(\phi) : E$ ann $e \in Ker(\phi) : E$ which is a contradiction. Thus, we conclude that $e \in Ker(\phi) : E$ ann $e \in Ker(\phi) : E$ ann $e \in Ker(\phi) : E$ which is, $e \in Ker(\phi) : E$ which is a contradiction. Thus, we conclude that $e \in Ker(\phi) : E$ ann $e \in Ker(\phi) : E$ ann $e \in Ker(\phi) : E$ which completes the proof.

In the previous proposition (ii), the condition " $Ker(\phi)$ is a prime submodule of E" is necessary. See the following example.

Example 4. Consider the \mathbb{Z} -homomorphism $\pi: \mathbb{Z} \oplus \mathbb{Q} \to \mathbb{Z}_p \oplus \mathbb{Q}$ defined by $\pi(e, \frac{a}{b}) = (\overline{e}, \frac{a}{b})$, where p is a prime number. It is clear that $ann(\mathbb{Z} \oplus \mathbb{Q}) = (0) = \mathbb{Z}_p \oplus \mathbb{Q}$. Then note that π is surjective and $Ker(\pi) = p\mathbb{Z} \oplus (0)$ is not a prime submodule of $\mathbb{Z} \oplus \mathbb{Q}$. On the other hand, $\mathbb{Z} \oplus \mathbb{Q}$ is a torsion free module, by Example 1, so is an annihilator multiplication module. However, by Example 2, $\mathbb{Z}_p \oplus \mathbb{Q}$ is not an annihilator multiplication module.

According to [3], a nonzero submodule V of E is defined as an essential submodule, also referred to as a large submodule, if for any submodule K of E, the condition $K \cap V = (0)$ implies K = (0). Furthermore, a submodule V of E is termed a pure submodule if $IE \cap V = IV$ for every ideal I of A [3].

Corollary 2. Let E be an A-module and V a submodule of E. The following statements are satisfied.

- (i) If V is a prime submodule of E such that ann(V) = (V : E) and E is an annihilator multiplication module, then E/V is an annihilator multiplication module.
- (ii) If E is an annihilator multiplication module and ann(E) = ann(V), then V is an annihilator multiplication module.
- (iii) If E is an annihilator multiplication module and V is a pure submodule and essential in E, then V is an annihilator multiplication module.
- *Proof.* (i): Consider the natural epimorphism $\pi: E \to E/V$ defined by $\pi(e) = e + V$ for each $e \in E$. Also note that $V = Ker(\pi)$ is a prime submodule and ann(E) = ann(E/V) = (V:E). Then by Proposition 4 (ii), E/V is an annihilator multiplication module.
- (ii): Consider the injection $i:V\to E$ defined by i(v)=v for each $v\in V$. Then note that i is one to one and ann(E)=ann(V). Then by Proposition 4 (i), V is an annihilator multiplication module.
- (iii): It is enough to show that ann(V) = ann(E). Let $a \in ann(V)$. Then we have aV = (0). Since V is a pure submodule, we conclude that $aV = aE \cap V =$

(0). As V is an essential submodule, we conclude that aE = (0) which implies that $ann(V) \subseteq ann(E)$. Since the reverse inclusion is always hold, we have the equality ann(V) = ann(E).

Proposition 5. Let E be an A-module and $\{E_i\}_{i\in\Delta}$ be a family of submodules of E such that $ann(E_i) = ann(E_j)$ for every $i \neq j$. Then $E = \bigoplus_{i \in \Delta} E_i$ is an annihilator multiplication module if and only if E_i is an annihilator multiplication module for each $i \in \Delta$.

Proof. The if part follows from Corollary 2 (ii). Let E_i be an annihilator multiplication module for each $i \in \Delta$. Choose $e \in E$. Then we can write $e = e_{i_1} + e_{i_2} + \cdots + e_{i_n}$ for some $i_1, i_2, \ldots, i_n \in \Delta$. Let $a \in ann(e)$. Then we conclude that $a(e_{i_1} + e_{i_2} + \cdots + e_{i_n}) = 0$ which implies that $ae_{i_1} = -ae_{i_2} - ae_{i_3} - \cdots - ae_{i_n} \in E_{i_1} \cap \sum_{j=2}^n E_{i_j} = (0)$. Thus we conclude that $a \in ann(e_{i_1})$. Similarly, we have $a \in \bigcap_{j=1}^n ann(e_{i_j})$. Then clearly we obtain $ann(e) = \bigcap_{j=1}^n ann(e_{i_j})$. Since E_{i_j} is an annihilator multiplication module for each $1 \leq j \leq n$, there exist finitely generated ideals I_j such that $ann(e_{i_j}) = ann(I_jE_j)$. On the other hand, we have $ann(I_jE_j) = ann(I_jE)$ since $ann(E_i) = ann(E_j)$ for every $i \neq j$ and E is internal direct sum of the family $\{E_i\}_{i \in \Delta}$. This implies that

$$ann(e) = \bigcap_{j=1}^{n} ann(e_{i_j}) = \bigcap_{j=1}^{n} ann(I_j E_j)$$
$$= \bigcap_{j=1}^{n} ann(I_j E) = ann\left(\left(\sum_{j=1}^{n} I_j\right) E\right).$$

As $\sum_{j=1}^{n} I_j$ is finitely generated, E is an annihilator multiplication module.

The condition that $ann(E_i) = ann(E_j)$ for every $i \neq j$ is a necessary requirement in the preceding proposition. The subsequent example demonstrates the necessity of this condition.

Example 5. Let p be a prime number. Since \mathbb{Q} is a torsion free \mathbb{Z} -module and \mathbb{Z}_p is a multiplication \mathbb{Z} -module, so they are annihilator multiplication \mathbb{Z} -modules. Also, it is clear that $ann(\mathbb{Q}) = (0) \neq ann(\mathbb{Z}_p) = p\mathbb{Z}$. However, \mathbb{Z} -module $E = \mathbb{Q} \oplus \mathbb{Z}_p$ is not an annihilator multiplication module since $ann(\overline{1}) = p\mathbb{Z} \neq ann(IE) = ann(I) = (0)$ for any nonzero ideal I of \mathbb{Z} .

Let E be an A-module and X an indeterminate over A. The notation E[X] refers to the polynomial module over the polynomial ring A[X]. An A-module E is defined as an Armendariz module if, for every $e(X) = e_0 + e_1 X + e_2 X^2 + e_n X^n$ in E[X] and $f(X) = a_0 + a_1 X + a_2 X^2 + a_k X^k$ in A[X] such that f(X)e(X) = 0, the condition holds that $1 \le i \le n$ and $1 \le j \le k$ imply $a_i e_j = 0$ [10]. Prior to examining the stability of the annihilator multiplication property in polynomial modules, the following lemma is required.

Lemma 1. Let M be an Armendariz module. Then,

(i)
$$ann_{A[X]}(e(X)) = \left[\bigcap_{i=0}^{n} ann_{A}(e_{i})\right][X]$$
 for every $e(X) = e_{0} + e_{1}X + \cdots + e_{n}X^{n} \in E[X]$.

8

module.

(ii)
$$ann_{A[X]}(p(X)E[X]) = \left[\bigcap_{i=0}^{k} ann_A(a_iE)\right][X]$$
 for every $p(X) = a_0 + a_1X + \cdots + a_kX^k \in A[X]$.

Proof. (i): Follows from definition of Armendariz module.

(ii): Suppose that $p(X) = a_0 + a_1X + \cdots + a_kX^k \in A[X]$. Let $q(X) = b_0 + b_1X + \cdots + b_tX^t \in A[X]$ such that $q(X) \in ann_{A[X]}(p(X)E[X])$. Then we have q(X)p(X)E[X] = 0 which implies that q(X)(p(X)e) = 0 for each $e \in E$. Since $p(X)e = a_0e + a_1eX + \cdots + a_keX^k$ and E is an Armendariz module, we have $b_ja_ie = a_0e^k + a_1e^k + a_0e^k + a_0e^k$

0 for each $0 \le j \le t$ and $0 \le i \le k$. Then we conclude that $b_j \in \bigcap_{i=0}^k ann_A(a_i E)$

for every $0 \le j \le t$, that is, $ann_{A[X]}(p(X)E[X]) \subseteq \left[\bigcap_{i=0}^k ann_A(a_iE)\right][X]$. For the reverse inclusion, let $q(X) = b_0 + b_1X + \dots + b_tX^t \in \left[\bigcap_{i=0}^k ann_A(a_iE)\right][X]$. Then we

have $b_j \in \bigcap_{i=0}^k ann_A(a_iE)$ which implies that $b_ja_ie = 0$ for every $e \in E$, $0 \le j \le t$ and $0 \le i \le k$. Choose $h(X) = e_0 + e_1X + \dots + e_sX^s \in E[X]$. Then note that $p(X)h(X) = a_0e_0 + (a_1e_0 + a_0e_1)X + \dots + a_ke_sX^{k+s}$. Since $b_ja_ie = 0$ for every $e \in E$, $0 \le j \le t$ and $0 \le i \le k$, we have q(X)p(X)h(X) = 0. This implies that q(X)p(X)E[X] = 0, that is, $q(X) \in ann_{A[X]}(p(X)E[X])$. Thus, we conclude that $\left[\bigcap_{i=0}^k ann_A(a_iE)\right][X] \subseteq ann_{A[X]}(p(X)E[X])$.

Proposition 6. Let E be an Armendariz A-module. Then E is an annihilator multiplication A-module if and only if E[X] is an annihilator multiplication A[X]-

Proof. (\Leftarrow): Suppose that E[X] is an annihilator multiplication A[X]-module. Let $e_0 \in E$ and put $e(X) := e_0$ which is a constant polynomial. Since E[X] is an annihilator multiplication A[X]-module, there exist $p_1(X), p_2(X), \ldots, p_k(X) \in A[X]$ such that

$$ann_{A[X]}(e(X)) = [ann_{A}(e_{0})][X] = ann_{A[X]} \left(\sum_{i=1}^{k} p_{i}(X)E[X] \right)$$
$$= \bigcap_{i=1}^{k} ann_{A[X]}(p_{i}(X)E[X]).$$

Let $p_i(X) = a_{0,i} + a_{1,i}X + a_{2,i}X^2 + \dots + a_{t,i}X^t$. Then by Lemma 1 (ii), we have $ann_{A[X]}(p_i(X)E[X]) = \begin{bmatrix} t \\ j=0 \end{bmatrix} ann_A(a_{j,i}E)$ [X]. This implies that

$$ann_{A[X]}(e(X)) = [ann_{A}(e_{0})][X] = \bigcap_{i=1}^{k} ann_{A[X]}(p_{i}(X)E[X])$$
$$= \left[\bigcap_{i=1}^{k} \bigcap_{j=0}^{t} ann_{A}(a_{j,i}E)\right][X].$$

Thus we obtain,

$$ann_A(e_0) = \bigcap_{i=1}^k \bigcap_{j=0}^t ann_A(a_{j,i}E)$$
$$= ann_A \left(\left(\sum_{\substack{1 \le i \le k \\ 0 \le j \le t}} Aa_{j,i} \right) E \right).$$

Hence, E is an annihilator multiplication A-module since $\sum_{\substack{1 \le i \le k \\ 0 \le i \le t}} Aa_{j,i}$ is a finitely

generated ideal.

 (\Rightarrow) : Let E be an annihilator multiplication A-module and $e(X) = e_0 + e_1 X + e_2 X^2 + \cdots + e_n X^n \in E[X]$. Then by Lemma 1 (i),

$$ann_{A[X]}(e(X)) = \left[\bigcap_{i=0}^{n} ann_{A}(e_{i})\right][X]$$
$$= \left[ann_{A}(Ae_{0} + Ae_{1} + \dots + Ae_{n})\right][X].$$

Since $Ae_0 + Ae_1 + \cdots + Ae_n$ is a finitely generated submodule of E and E is an annihilator multiplication module, by Proposition 1, there exists a finitely generated ideal I of A such that

$$ann_A(Ae_0 + Ae_1 + \dots + Ae_n) = ann_A(IE).$$

This gives

$$ann_{A[X]}(e(X)) = [ann_A(IE)][X] = ann_{A[X]}((IE)[X])$$

= $ann_{A[X]}(I[X]E[X]).$

Since I is finitely generated in A, I[X] is finitely generated in A[X]. Thus, it follows that E[X] is an annihilator multiplication A[X]-module.

According to [11], a proper submodule V of E is defined as a classical prime submodule if, whenever $abe \in V$ for some $a,b \in A$ and $e \in E$, it follows that $ae \in V$ or $be \in V$. Furthermore, V is termed a classical 1-absorbing prime if, for non-units $a,b,c \in A$ and $e \in E$, the condition $abce \in V$ implies that $abe \in V$ or $ce \in V$ [36]. Every classical prime is also a classical 1-absorbing prime; however, the converse does not generally hold. The following example illustrates this distinction.

Example 6. Consider the k[[X]]-module $E = k[[X]]/(X^2)$ where k is a field and X is an indeterminate over k. Let V be the zero submodule of E. Then V is not a classical prime submodule, since $X^2\overline{1} = \overline{0} \in V$ and $X\overline{1} = \overline{X} \notin V$. Let $abc\overline{e} = \overline{0}$ for some nonunits $a,b,c \in k[[X]]$ and $\overline{e} \in M$. Since k[[X]] is a local ring with a unique maximal ideal (X), we have a = Xf, b = Xg and c = Xh for some $f,g,h \in k[[X]]$. This gives $ab\overline{e} = X^2fg\overline{e} = \overline{0} \in V$, and so V is a classical 1-absorbing prime submodule of E.

Recall from [34] that a proper ideal I of A is said to be a 1-absorbing prime ideal if whenever $abc \in I$ for some nonunits $a, b, c \in A$ then $ab \in I$ or $c \in I$.

Proposition 7. Let E be an annihilator multiplication module over which ann(E) is a 1-absorbing prime ideal of A. Then, for every submodule V of E, either ann(V) = ann(E) or ann(V) is a prime ideal of A. In this case, ann(V) is a 1-absorbing

10 SUAT KOC

prime ideal of A. Furthermore, $\{ann(V): V \text{ is a submodule of } E\}$ is totally ordered by inclusion.

Proof. Suppose that E is an annihilator multiplication module and V is a submodule of E such that $ann(V) \neq ann(E)$. Then there exists a proper ideal I of A such that ann(V) = ann(IE). Let $ab \in ann(V)$ for some $a, b \in A$. Then we have abV = aIbE = 0 which implies that $aIb \subseteq ann(E)$. If a or b is unit, then we are done. So assume that a, b are nonunits. Since ann(E) is a 1-absorbing prime ideal, we conclude that $aI \subseteq ann(E)$ or $b \in ann(E)$. Then we have $a \in ann(IE) = ann(V)$ or $b \in ann(E) \subseteq ann(V)$. Thus, ann(V) is a prime ideal of A, so is 1-absorbing prime. Let V, K be two submodules of E such that $ann(V) \neq ann(E)$ and $ann(K) \neq ann(E)$. Then by above, either we have ann(V+K) = ann(E) or ann(V+K) is a prime ideal of A. Now we have two cases. Case 1: Let ann(V+K) = ann(E). Since $ann(V+K) = ann(V) \cap ann(K)$, we have $ann(V) \cap ann(K) = ann(E)$. By assumption, we get ann(K) and ann(V) are prime ideals of A which implies that $\sqrt{ann(E)} = \sqrt{ann(V)} \cap \sqrt{ann(K)} = ann(V) \cap$ ann(K) = ann(E). Since ann(E) is a 1-absorbing prime ideal and semiprime, so is prime ideal. Then we have either $ann(V) \subseteq ann(K)$ or $ann(K) \subseteq ann(V)$. Case **2:** Let $ann(V+K) \neq ann(E)$. In this case, $ann(V+K) = ann(V) \cap ann(K)$ is a prime ideal which implies that either $ann(V) \subseteq ann(K)$ or $ann(K) \subseteq ann(V)$. Hence, $\{ann(V): V \text{ is a submodule of } E\}$ is totally ordered by inclusion.

Proposition 8. Let E be an annihilator multiplication module and $ann(e) \neq ann(E)$ for every $e \in E$. Then the zero submodule of E is a classical prime submodule if and only if it is a classical 1-absorbing prime.

Proof. The if part follows from [36, Proposition 1]. For the only if part, assume that the zero submodule is a classical 1-absorbing prime submodule. Let abe=0 for some $a,b\in A$ and $e\in E$. We may assume that a,b are nonunits of A. Since E is an annihilator multiplication module, there exists a finitely generated ideal I of A such that ann(e)=ann(IE). Since $ann(e)\neq ann(E)$, I is a proper ideal of A. This gives abIE=0. If be=0, then we are done. So, assume that $be\neq 0$. Now, we will show that aIE=0. Choose an arbitrary $z\in E$ and assume that $aIz\neq 0$. Since abIz=0, by [36, Theorem 2], we have bz=0. On the other hand, note that abI(z+e)=0. Since $aIz\neq 0$, again by [36, Theorem 2], we have b(z+e)=0 which implies that be=0 which is a contradiction. Thus, we conclude that aIE=0, that is, $a\in ann(IE)=ann(e)$. Then we conclude that ae=0 completes the proof. \Box

Remark 1. In the previous proposition, condition $ann(e) \neq ann(E)$ is necessary. For example, consider the k[[X]]-module $E = k[[X]]/(X^2)$ where k is a field and X is an indeterminate over k. Then E is clearly a multiplication module, so by Example 1 it is an annihilator multiplication module. However, note that $ann(\overline{1}) = (X^2) = ann(E)$. Also, by previous example, $V = (\overline{0})$ is a classical 1-absorbing prime submodule which is not a classical prime submodule.

3. Characterizations of some special rings/modules

This section examines the characterizations of significant classes of rings and modules using the framework of annihilator multiplication modules.

Theorem 1. Let E be an A-module. Then E is a torsion free module if and only if E is an annihilator multiplication module and faithful module over the domain A.

Proof. (\Rightarrow): If E is a torsion free module, by Example 1 (iii), E is an annihilator multiplication module. Also note that if E is torsion free, then clearly A is a domain and E is a faithful module.

 (\Leftarrow) : Suppose that E is an annihilator multiplication module and faithful module over the domain A. Let $0 \neq e \in E$. Now, we will show that ann(e) = (0). Since E is an annihilator multiplication module, there exists a finitely generated ideal I of A such that ann(e) = ann(IE). As I is a finitely generated ideal, we can write $I = \sum_{i=1}^{n} Ra_i$ for some $a_i \in I$. As E is a faithful module, we have ann(IE) = ann(IE)

write $I = \sum_{i=1}^{n} Ra_i$ for some $a_i \in I$. As E is a faithful module, we have $ann(IE) = ann(I) = \bigcap_{i=1}^{n} ann(a_i) = ann(e)$. Since e is not zero, I can not be a zero ideal. As A is a domain, $ann(a_i) = (0)$ for every $i = 1, 2, \ldots, n$. Then we have ann(e) = (0) which completes the proof.

Consider an A-module E. Given any submodule V of E and a nonempty subset J of A, we define the residual of V by J as $(V:_E J) = \{e \in E : Je \subseteq V\}$. When V is the zero submodule, we use $ann_E(J)$ to represent $((0):_E J)$. As described in [31], E is called a comultiplication module if, for every submodule V of E, there is an ideal I of E such that $V = ann_E(I)$. In fact, E is a comultiplication module precisely when $V = ann_E(ann(V))$. For further exploration of comultiplication modules, see [28] and [35].

Proposition 9. Let E be a comultiplication module. Then E is a multiplication module if and only if E is an annihilator multiplication module.

Proof. (\Rightarrow) : Follows from Example 1.

 (\Leftarrow) : Let E be a comultiplication and annihilator multiplication module. Then by Proposition 1 (ii), for every submodule V of E there exists an ideal I of A such that ann(V) = ann(IE). As E is a comultiplication module, we have $V = ann_E(ann(V)) = ann_E(ann(IE)) = IE$ which completes the proof.

According to [7], an A-module E is defined as a prime module if the zero submodule of E is prime, or equivalently, ann(V) = ann(E) for every nonzero submodule V of E. Additionally, a submodule V of E is defined to be a second submodule if, for each $a \in A$, either aV = (0) or aV = V. In particular, E is called a second module if it is a second submodule of itself [33].

An A-module E is defined as injective if, whenever $f:L\to K$ is an injective homomorphism and $g:L\to E$ is an arbitrary homomorphism, there exists a homomorphism $h:K\to E$ such that fh=g [3]. Furthermore, [33, Theorem 2.3] demonstrates that a prime module E is a second module if and only if it is an injeinjective module over the ring A/ann(E).

Proposition 10. Let E be a second module and annihilator multiplication module. Then E is a prime module and thus every submodule V of E is an annihilator multiplication module. In this case, V is an injective A/ann(E)-module. In particular, if E is a faithful module, then E is an injective A-module. Furthermore, nonzero pure submodules and second submodules of E coincide.

Proof. Suppose that E is a second submodule and annihilator multiplication module. Now, we will show that ann(e) = ann(E) for every $0 \neq e \in E$. Choose $0 \neq e \in E$. As E is an annihilator multiplication module, there exists a finitely generated ideal I of A such that ann(e) = ann(IE). Since I is finitely generated, there exists $a_1, a_2, \ldots, a_n \in I$ such that $I = \sum_{i=1}^n Aa_i$. This implies that $ann(e) = ann(IE) = \bigcap_{i=1}^n ann(a_iE)$. As $e \neq 0$, there exists $t \in \{1, 2, \ldots, n\}$ such that $a_tE \neq 0$. Otherwise, we would have ann(e) = A which implies that e = 0, a contradiction. Since E is a second module, $a_tE = E$ which yields that

$$ann(E) = ann(a_t E) \subseteq ann(e)$$

= $\bigcap_{i=1}^{n} ann(a_i E) \subseteq ann(a_t E) = ann(E)$.

Then we have ann(e) = ann(E). Then it is clear that for every nonzero submodule V of E, we have ann(V) = ann(E). Thus, E is a prime module. Furthermore, by Corollary 2 (ii), V is an annihilator multiplication module every nonzero submodule V of E. Since E is a prime and second module, by [33, Theorem 2.3], E is an injective A/ann(E)-module. For the rest, assume that V is a second submodule of E. Let E be an ideal of E. Since E is a second submodule, either E or E

It is well known that a ring A is von Neumann regular if and only if A is reduced (i.e, a ring without a nonzero nilpotent) and every prime ideal is maximal. Now, we characterize principal ideal von Neumann regular rings in terms of annihilator multiplication modules.

Theorem 2. Let A be a ring. The following statements are equivalent.

- (i) A is a reduced ring and every faithful A-module is an annihilator multiplication module.
 - (ii) A is a principal ideal ring and von Neumann regular ring.
 - (iii) A is a direct product of finitely many fields.

Proof. $(i) \Rightarrow (ii)$: Suppose that A is a reduced ring and every faithful A-module is an annihilator multiplication module. Let I be an ideal of A, and put $E = A \oplus (A/I)$. Note that E is a faithful multiplication module, by assumption, E is an annihilator multiplication module. This implies that $ann((0,\overline{1})) = I = ann(JE) = ann(J)$ for some finitely generated ideal J of A. Then there exist $x_1, x_2, \ldots, x_n \in J$ such that $J = \sum_{i=1}^n Ax_i$. This implies that $I = \bigcap_{i=1}^n ann(x_i)$. Now, take a prime ideal P of A. Then we have $P = \bigcap_{i=1}^n ann(y_i)$ for some $y_1, y_2, \ldots, y_n \in A$. Since P is prime, we conclude that $P = ann(y_i)$ for some $y_i \in A$. As A is a reduced ring,

by [16, Lemma 2.8], P is a minimal prime ideal of A. Thus, every prime ideal of A is maximal, so A is a von Neumann regular ring. Now, we will show that A is a principal ideal ring. Let I be an ideal of A. Then by above, $I = \bigcap_{i=1}^n ann(x_i)$. Since A is a von Neumann regular ring, every principal ideal $(x_i) = (b_i)$ is generated by an idempotent $b_i \in R$. Thus, we conclude that $ann(x_i) = (1-b_i)$, where $1-b_i$ is again an idempotent of A. Since $(b) \cap (c) = (bc)$ for every two idempotents $b, c \in A$, we obtain

$$I = \bigcap_{i=1}^{n} ann(x_i) = \bigcap_{i=1}^{n} (1 - b_i)$$

= $((1 - b_1)(1 - b_2) \cdots (1 - b_n))$

which is a principal ideal. Thus, A is a principal ideal von Neumann regular ring.

 $(ii)\Rightarrow (iii)$: Let A be a principal ideal ring and von Neumann regular ring. Then every prime ideal is maximal and A is reduced. Since A is principal ideal ring, $\sqrt{0}=(0)=Jac(A)$. Since A is Noetherian and zero dimensional, it is Artinian, and so A has only finitely many maximal ideals Q_1,Q_2,\ldots,Q_n . This gives $Q_1\cap Q_2\cap\cdots\cap Q_n=(0)$. Then by Chinese Remainder Theorem, we have $A=A/(0)\cong (A/Q_1)\oplus (A/Q_2)\oplus\cdots\oplus (A/Q_n)$, where A/Q_i is a field.

 $(iii) \Rightarrow (ii)$: It is clear.

 $(ii) \Rightarrow (i)$: Assume that A is a principal ideal ring and von Neumann regular ring. Then A is reduced. Now, choose a faithful R-module E, and $e \in E$. Since A is a principal ideal ring, ann(e) = (a) for some $a \in A$. As A is a von Neumann regular ring, there exists an idempotent $b \in A$ such that (a) = (b) = ann(1-b). As E is a faithful module, we get ann(e) = (b) = ann((1-b)E), where (1-b) is a finitely generated ideal. Thus, E is an annihilator multiplication module. \Box

Let E be an A-module. An ideal P of A is defined as an associated prime of E if P is a prime ideal and P = ann(e) for some $0 \neq e \in E$. The set of all associated primes of E is denoted by $Ass_A(E)$ [25]. In this context, Ass(A) denotes $Ass_A(A)$. Associated prime ideals correspond to the irreducible components underlying a module or ideal. These ideals play a fundamental role in primary decomposition, establish connections between algebraic structures and the geometry of varieties, and are crucial in the study of dimension and depth theory. The following section presents the relationships between $Ass_A(E)$ and Ass(A) in terms of annihilator multiplication modules.

Let E be an A-module. The set of all torsion elements of E is denoted by $T(E) = \{e \in M : ann(e) \neq 0\}$. According to [6], E is classified as a torsion module if T(E) = E; otherwise, E is referred to as a non-torsion module.

Theorem 3. (i) Let E be an annihilator multiplication module faithful module. Then $Ass_A(E) \subseteq Ass(A)$.

(ii) Let E be a non torsion module and annihilator multiplication module. Then $Ass(A) = Ass_A(E)$.

Proof. (i): Let E be an annihilator multiplication module faithful module. Choose $P \in Ass_A(E)$. Then P is a prime ideal and there exists $0 \neq e \in E$ such that P = ann(e). Since E is an annihilator multiplication module, there exists a finitely n

generated ideal I of A such that ann(e) = ann(IE). Then we can write $I = \sum_{i=1}^{n} Aa_i$

for some $a_1, a_2, \ldots, a_n \in I$. This implies that

$$P = ann(e) = ann(IE)$$
$$= \bigcap_{i=1}^{n} ann(a_iE).$$

Since $\bigcap_{i=1}^{n} ann(a_i E) \subseteq P$ and P is a prime ideal, there exists $t \in \{1, 2, \dots, n\}$ such

that $ann(a_tE) \subseteq P = \bigcap_{i=1}^n ann(a_iE) \subseteq ann(a_tE)$, that is, $P = ann(a_tE)$. As E is a faithful module, we have $P = ann(a_tE) = ann(a_t)$. This gives $P \in Ass(A)$, that is, $Ass_A(E) \subseteq Ass(A)$.

(ii) : Since every non torsion module is faithful, by (i), we have $Ass_A(E) \subseteq Ass(A)$. For the reverse inclusion, choose $P \in Ass(A)$. Then there exists $0 \neq a \in A$ such that P = ann(a) is a prime ideal of A. Since E is a non torsion module, there exists $0 \neq e \in E$ such that ann(e) = (0) which implies that P = ann(a) = ann(ae), that is, $P \in Ass_A(E)$. Then we have the equiality $Ass(A) = Ass_A(E)$.

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