

BREAKING THE SYMMETRY IN EXCESS INTERSECTION AND COUNTING SOLUTIONS OF SYSTEMS OF POLYNOMIALS

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ABSTRACT. We revisit the fundamental problem of assigning intersection multiplicities to subsets of solutions of (square) systems of polynomials. Severi [Ann. Mat. Pura Appl. 26 (4), 1947] suggested an intuitive *dynamic* solution to this problem which was later corrected and made rigorous by Lazarsfeld [Compos. Math. 43, 1981]. We consider an *asymmetric* variant of this approach and find an explicit description of the resulting “ordered intersection multiplicity” which opens pathways to step by step solutions to the *affine Bézout problem* of counting isolated solutions to (square) systems of polynomials via “Bernstein-Kushnirenko type” estimates in terms of Newton diagrams. To illustrate our methods we compute the number of common tangent lines to 4 general spheres in the affine 3-space (which is known to be 12 due to Macdonald, Pach, and Theobald [Discrete Comput. Geom. 26 (1), 2001]) via certain ordered intersection multiplicities on the corresponding Grassmannian.

CONTENTS

1. Introduction	2
2. An overview of ordered intersection multiplicity and the main results	3
Part 1. Ordered intersection multiplicity	9
3. Initial observations	9
4. Distinguished components of an ordered intersection	13
5. Combinatorics of intersections of divisors - guessing the formula for the ordered intersection multiplicity	20
6. Proof of the formula for ordered intersection multiplicity	23
Part 2. Computation: estimates and examples	30
7. Local factors of ordered intersection as (usual) intersection multiplicities	30
8. Computation in terms of Newton diagrams	35
9. An example: the number of tangent lines to four generic spheres	46
10. Inductive “solutions” to the affine Bézout problem	52
Part 3. Appendix	55
Appendix A. Miscellaneous algebraic results used in the article	55
Appendix B. A result about completions of local rings at a subvariety	56
Appendix C. Proof of Claim 6.4.2	59
References	67

1. INTRODUCTION

This work is motivated by the *affine Bézout problem* of counting the precise number of isolated solutions of a system f_1, \dots, f_n of n polynomials in n variables over an algebraically closed field \mathbb{k} . A general approach to this problem is to consider appropriate vector spaces of polynomials containing f_i , which corresponds to embedding the f_i in families of base-point free linear systems \mathcal{L}_i (on a “natural” ambient space). In the generic case the given system is “non-degenerate”, and the number of its solutions equals the number of common solutions of generic divisors $G_i \in \mathcal{L}_i$, which is calculated by a version of Bézout’s theorem or Bernstein-Kushnirenko theorem. However, if the system is “degenerate”, then this calculated bound is not sharp, and the intersection of the (effective) divisors F_1, \dots, F_n corresponding to the given polynomials contains “problematic” points (which could be “points at infinity” or non-isolated components). This leads one to the following general problem:

Problem 1.1. Given a subset Z of $\bigcap_i \text{Supp}(F_i)$, count (with appropriate multiplicity) how many of the points in $\bigcap_i \text{Supp}(G_i)$ “move to” Z as the G_i “move to” F_i (where G_i are generic elements of \mathcal{L}_i).

1.2. In this paper we approach Problem 1.1 by considering deformations of $\bigcap_i F_i$ defined locally by $f_1 - t^{r_1}g_1 = \dots = f_n - t^{r_n}g_n = 0$, where the f_i, g_i are local representatives respectively of F_i, G_i , and computing the *ordered intersection multiplicity* of Z , which is the number of points which “move to” generic points of Z as t moves to 0 in the case that each $r_i/r_{i+1} \gg 1$. The point is that the ordered intersection multiplicity admits a fairly explicit and elementary description in terms of sums of products of various “degrees” of Z (the “global” factors) and lengths of certain ideals in the local ring of Z (the “local” factors). In turn this opens a possibility for a step by step solution to the affine Bézout problem via estimates in terms of associated *Newton diagrams*.

1.3. In Section 2 below we provide a detailed introduction to ordered intersection multiplicity and the main results of this article. The subsequent sections are divided into two largely independent parts: in the first part (comprising of Sections 3 to 6) we show that the ordered intersection multiplicity is well defined and find an explicit formula for it; the second part (spanning Sections 7 to 10) provides various expressions of the ordered intersection multiplicity in terms of usual intersection multiplicities of divisors, and resulting estimates in terms of Newton diagrams. Note that for applications to the affine Bézout problem one does not need the dynamic interpretation of ordered intersection multiplicity - it suffices to read only the proof of Theorem 5.7 in Section 5, and then Sections 7 to 10 in the second part.

1.4. The big picture in the context of the affine Bézout problem. A problem associated to, and deeply intertwined with, the affine Bézout problem is that of computing the multiplicity of a (proper) isolated intersection at a nonsingular point. In the “non-degenerate” case this multiplicity can be estimated in terms of associated Newton diagrams, and in the “degenerate” case this multiplicity is equal to the non-degenerate multiplicity plus the multiplicity of certain excess intersections on a blow up at the point (see e.g. [Mon21, Chapter IX]).

1.4.1. In turn, the “ordered intersection multiplicities” can be expressed in terms of the (usual) intersection multiplicities of divisors at isolated points of intersections on a birational modification of the given variety (this modification can be described in terms of the given F_i and “generic” G_i from Problem 1.1 - see Section 10.1). In a sense this provides an inductive “solution” to the affine Bézout problem: start with the non-degenerate bound; if there are excess intersections, then express the corresponding ordered multiplicities in terms of the usual intersection multiplicities of divisors at isolated points of intersection on an appropriate modification; then compute the “non-degenerate portion” of

the latter multiplicities in terms of Newton diagrams and express the “degenerate portion” in terms of excess intersections, and then repeat. We are however not sure about the practicality or complexity of this “solution.”

1.4.2. There are similar inductive routes to the affine Bézout problem when $\bigcup_i \text{Supp}(F_i)$ is the support of a strict normal crossing divisor, i.e. when resolution of singularities is available (e.g. in characteristic zero), or when certain curves are locally set-theoretic complete intersections (which is true in positive characteristics by a result of Cowsik and Nori [CN78]) - see Sections 10.2 and 10.3.

1.4.3. In general it is unknown if every curve is a local set-theoretic complete intersection. Short of the birational “modifications” mentioned in Paragraphs 1.4.1 and 1.4.2, we do not know of any general method to obtain combinatorial or convex geometric estimates of the ordered intersection multiplicity if the associated curves are not set-theoretic complete intersection near the corresponding “centers”. Sometimes it is possible to give such an estimate when a “punctured” germ of the curve is a complete intersection in the complement of a union of “coordinate subspaces” (see Section 8.3); in order to extend this approach to a general method one would need to get a handle on the *tropicalization* of a curve C near a point P , which seems to be a hard problem if C is not a (set-theoretic) complete intersection near P .

2. AN OVERVIEW OF ORDERED INTERSECTION MULTIPLICITY AND THE MAIN RESULTS

2.1. Ordered intersection multiplicity. Consider the set up of Problem 1.1. More precisely, let X be a variety of dimension n , and F_i be an effective Cartier divisor on X which is an element of a base-point free linear system \mathcal{L}_i , $i = 1, \dots, n$. For each choice of $G_i \in \mathcal{L}_i$, $i = 1, \dots, n$, we denote by $C^* = C^*(G_1, \dots, G_n; r_1, \dots, r_n)$ the subscheme of $X \times (\mathbb{k} \setminus \{0\})$ which is defined locally as $V(f_1 - t^{r_1}g_1, \dots, f_n - t^{r_n}g_n)$, where f_i, g_i are local representatives of respectively F_i, G_i . If the G_i are generic, then for almost all $a \in \mathbb{k}$, the number of points on C^* at $t = a$, counted with appropriate multiplicities, is precisely the intersection number $(\mathcal{L}_1, \dots, \mathcal{L}_n)$. Consequently, C^* can be viewed as a moving collection of $(\mathcal{L}_1, \dots, \mathcal{L}_n)$ points, and it seems reasonable, as Severi [Sev47] suggested, to count how many of these points approach “generic” points of a given subvariety $Z \subseteq X$, and take it as a measure of the intersection multiplicity of F_1, \dots, F_n along Z ; we informally denote this number as $(F_1, \dots, F_n)_{\vec{r}}^Z$ (where $\vec{r} := (r_1, \dots, r_n)$). In the case that X is nonsingular, Lazarsfeld [Laz81] rigorously developed this notion for $\vec{r} = (1, \dots, 1)$ following (and correcting) the idea of Severi [Sev47]¹. The point of departure for this article is the observation that $(F_1, \dots, F_n)_{\vec{r}}^Z$ admits an explicit description in the extreme limit that $r_1 \gg r_2 \gg \dots \gg r_n > 0$; motivated by this somewhat of a miracle we consider the *ordered intersection multiplicity*

$$(F_1, \dots, F_n)_Z^{\text{ord}} := \lim_{\min\{\frac{r_1}{r_2}, \dots, \frac{r_{n-1}}{r_n}\} \rightarrow \infty} (F_1, \dots, F_n)_Z^{\vec{r}}$$

We show that $(F_1, \dots, F_n)_Z^{\text{ord}}$ is well defined and provide explicit descriptions and estimates of this number. Unlike the symmetric version $(F_1, \dots, F_n)_Z^{(1, \dots, 1)}$ considered by Severi and Lazarsfeld, the ordered intersection multiplicity in general depends on the ordering of F_1, \dots, F_n , and corresponds to a “step-by-step expansion of the intersection product (F_1, \dots, F_n) from left to right” (Theorem 5.7). However, in the case that Z is an *isolated* point of $V := \bigcap_i \text{Supp}(F_i)$, the ordered intersection

¹ Severi [Sev47] suggested taking the *minimum* possible number of points approaching generic points of Z . Lazarsfeld [Laz81] showed that one should instead take the number of points in a *generic* such deformation.

multiplicity is independent of the ordering, and equals the usual intersection multiplicity of the F_i at Z . It follows that

$$(1) \quad (F_1, \dots, F_n)_X^{iso} = (\mathcal{L}_1, \dots, \mathcal{L}_n) - \sum_{Z \subseteq V_+} (F_1, \dots, F_n)_Z^{ord}$$

where $(F_1, \dots, F_n)_X^{iso}$ is the sum of intersection multiplicities of F_1, \dots, F_n over all isolated points of V , and V_+ is the union of the positive dimensional irreducible components of V . Identity (1) opens the passage to applications of this ordered intersection multiplicity to the affine Bézout problem.

Example 2.2. We start with a classical example, considered e.g. in [Laz81]. Let $F_1 = V(x_0^2 x_1)$, $F_2 = V(x_0 x_1^2) \subseteq \mathbb{P}^2$, and take $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{O}(3)$, the linear system on \mathbb{P}^2 cut out by the cubics. Take $U := \mathbb{P}^2 \setminus V(x_2)$, and affine coordinates $(u_0, u_1) := (x_0/x_2, x_1/x_2)$ on U . Then $F_i = V(f_i)$ on U , where $f_1 := u_0^2 u_1$ and $f_2 := u_0 u_1^2$. Let $\phi_i := f_i - t^{r_i} g_i$, where g_1, g_2 are cubic polynomials in (u_0, u_1) . First we consider the case that $r_1 = r_2 = 1$. Then

$$u_1 \phi_1 - u_0 \phi_2 = t(u_0 g_2 - u_1 g_1)$$

Write C for the (scheme-theoretic) closure of C^* . If g_1, g_2 are generic, then $C = V(\phi_1, \phi_2)$ is defined near $U \times \{0\}$ by

$$\begin{aligned} f_1 - t g_1 &= u_0^2 u_1 - t a_1 + \dots, \text{ and} \\ u_0 g_2 - u_1 g_1 &= u_0 a_2 - u_1 a_1 + \dots \end{aligned}$$

where the a_i are the constant terms of g_i (and “ \dots ” denotes terms with higher order in t). In the generic case both a_i are nonzero, which implies that C is nonsingular at the origin $O := (0, 0, 0)$, and both u_0 and u_1 are parameters of C at O . It follows that

$$(F_1, F_2)_{V(x_0, x_1)}^{(1,1)} = \text{ord}_{(0,0,0)}(t|_C) = \text{ord}_{(0,0,0)}(u_0^2 u_1|_C) = 3$$

Now we consider $Z = V(x_0)$. Then $Z \cap U = V(u_0)$, and the intersection of $C \cap \{t = 0\}$ and $(Z \cap U) \times \{0\}$ is $V(f_1 - t g_1, u_0 g_2 - u_1 g_1, t, u_0) = V(f_1, u_1 g_1, t, u_0)$, which set theoretically consists of the origin O and $V(g_1, u_0) \times \{0\}$. In the generic case the latter set consists of three generic points on $Z \times \{0\}$ distinct from O , and it is not hard to see that at each of these points C is nonsingular and t is a parameter. Consequently,

$$\begin{aligned} (F_1, F_2)_{V(x_0)}^{(1,1)} &= 3, \text{ and similarly,} \\ (F_1, F_2)_{V(x_1)}^{(1,1)} &= 3 \end{aligned}$$

Since $(F_1, F_2) = 9$, it follows that

$$(F_1, F_2) = (F_1, F_2)_{V(x_0, x_1)}^{(1,1)} + (F_1, F_2)_{V(x_0)}^{(1,1)} + (F_1, F_2)_{V(x_1)}^{(1,1)}$$

is a complete decomposition of (F_1, F_2) , and $(F_1, F_2)_Z^{(1,1)} = 0$ for all Z different from $V(x_0), V(x_1)$ or $V(x_0, x_1)$.

Example 2.3. We continue with the same set up as Example 2.2. If $r_1 \gg r_2$, then we claim that $g_2(z) = 0$ for each $z \in (C \cap \{t = 0\}) \cap (U \times \{0\})$. Indeed, $f_2 = u_0 u_1^2 \in \sqrt{u_0^2 u_1 \mathcal{O}_{z,C}} = \sqrt{f_1 \mathcal{O}_{z,C}}$. Consequently, there is $k \geq 1$ such that $f_2^k = f_1 h$ for some $h \in \mathcal{O}_{z,C}$ (it is clear that $k = 2$ works for this example), and

$$t^{kr_2} (g_2)^k = (f_2)^k = f_1 h = h t^{r_1} g_1 \in \mathcal{O}_{z,C}$$

Consequently, if $r_1 > kr_2$, then $(g_2)^k$ vanishes at z , as claimed. Since a generic g_2 does *not* vanish at the origin, it follows that

$$(F_1, F_2)_{V(x_0, x_1)}^{\vec{r}} = 0$$

Since a generic g_2 vanishes at 3 generic points on $V(x_0) \cap U = V(u_0)$, and since $\text{ord}_{u_0}(f_1) = 2$, it can be shown that (see e.g. assertion (2a) of Proposition 7.4)

$$(F_1, F_2)_{V(x_0)}^{\vec{r}} = 2 \times 3 = 6.$$

Similarly, since $\text{ord}_{u_1}(f_1) = 1$, we have

$$(F_1, F_2)_{V(x_1)}^{\vec{r}} = 1 \times 3 = 3.$$

Since $(F_1, F_2) = 9 = 6 + 3$, it follows that if $r_1 \gg r_2$, then

$$(F_1, F_2)_Z^{\vec{r}} = \begin{cases} 6 & \text{if } Z = V(x_0), \\ 3 & \text{if } Z = V(x_1), \\ 0 & \text{otherwise.} \end{cases}$$

If $r_2 \gg r_1$, then it similarly follows that

$$(F_1, F_2)_Z^{\vec{r}} = \begin{cases} 3 & \text{if } Z = V(x_0), \\ 6 & \text{if } Z = V(x_1), \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.4 (Cf. Example 7.7). Let $F_2 = V(x_0 x_1^2)$ be as above and $F'_1 = V(x_0^2)$, i.e. F'_1 is a “component” of F_1 from above. The same arguments as the previous example show that if $r_1 \gg r_2$, then

$$(F'_1, F_2)_Z^{\vec{r}} = \begin{cases} 6 & \text{if } Z = V(x_0), \\ 0 & \text{otherwise.} \end{cases}$$

whereas, if $r_2 \gg r_1$, then

$$(F'_1, F_2)_Z^{\vec{r}} = \begin{cases} 2 & \text{if } Z = V(x_0), \\ 4 & \text{if } Z = V(x_0, x_1), \\ 0 & \text{otherwise.} \end{cases}$$

2.5. We now describe the main results. Recall the basic set up: we are given fixed elements F_i of base-point free linear systems \mathcal{L}_i , $i = 1, \dots, n$, on a variety X of dimension n . Note that we allow F_i to be identically zero on X . For generic $G_i \in \mathcal{L}_i$, the subscheme of $X \times \mathbb{k}$ locally defined by $V(f_1 - t^{r_1} g_1, \dots, f_n - t^{r_n} g_n)$, where f_i, g_i are local representatives respectively of F_i and G_i , restricts to a (possibly non-reduced) curve $C^*(G_1, \dots, G_n; r_1, \dots, r_n)$ on $X \times (\mathbb{k} \setminus S)$, where S is a (possibly empty) finite subset of \mathbb{k} ; we write $C(G_1, \dots, G_n; r_1, \dots, r_n)$, or in short, $C(\vec{G}; \vec{r})$, or simply C , for the scheme-theoretic closure of this curve in $X \times \mathbb{k}$.

Theorem 2.6 (Ordered intersection multiplicity - definition). *Let Z be an irreducible subvariety of X . Then there is $\nu \in \mathbb{R}$ and a proper closed subset \tilde{Z} of Z such that one (and only one) of the following holds:*

- (1) *for each \vec{r} such that $r_i/r_{i+1} > \nu$ for each $i = 1, \dots, n-1$, there is a nonempty Zariski open subset \mathcal{L}^* of global sections of $\prod_i \mathcal{L}_i$ such that C does not intersect $(Z \setminus \tilde{Z}) \times \{0\}$ for any $(G_1, \dots, G_n) \in \mathcal{L}^*$,*

- (2) or there is a positive integer N_Z depending only on Z such that for each \vec{r} such that $r_i/r_{i+1} > \nu$ for each $i = 1, \dots, n-1$, and each nonempty open subset Z^* of Z , there is a nonempty open subset \mathcal{L}^* of $\prod_i \mathcal{L}_i$ such that
- (a) $C \cap ((Z^* \setminus \check{Z}) \times \{0\}) \neq \emptyset$ for each $(G_1, \dots, G_n) \in \mathcal{L}^*$, and
 - (b) the sum of $\text{ord}_B(t)$ over the branches² B of C centered at $(Z^* \setminus \check{Z}) \times \{0\}$ is precisely N_Z .

We define $(F_1, \dots, F_n)_Z^{\text{ord}}$ to be zero in the first case, and equal to N_Z in the second case. There are finitely many irreducible subvarieties Z of X such that $(F_1, \dots, F_n)_Z^{\text{ord}}$ is nonzero. If X is complete, then

$$(\mathcal{L}_1, \dots, \mathcal{L}_n) = \sum_Z (F_1, \dots, F_n)_Z^{\text{ord}}$$

The subset \check{Z} can be described as follows: \check{Z} is the intersection of Z with the union of all irreducible subvarieties $Z' \neq Z$ of X such that $\text{codim}(Z') \geq \text{codim}(Z)$ and $(F_1, \dots, F_n)_{Z'}^{\text{ord}} \neq 0$.

The subvarieties Z for which $(F_1, \dots, F_n)_Z^{\text{ord}}$ is nonzero can be explicitly described in terms of intersections of irreducible components of $\text{Supp}(F_i)$. We found this surprising since descriptions of “distinguished varieties” tend to require passing to a “modification” of the original space (normal cone in Fulton-MacPherson theory [Ful98], and a transcendental extension in Stückrad-Vogel theory [Vog84]).

Theorem 2.7. *Let i^* be the smallest index such that F_{i^*} is not identically zero. For each collection of indices $i^* = i_1 < \dots < i_\rho \leq n$, where $1 \leq \rho \leq n$, construct subvarieties $Z_{i_1, \dots, i_\rho}^{(j)}$, $0 \leq j \leq \rho$, as follows:*

- (a) $Z_{i_1, \dots, i_\rho}^{(0)} = X$,
- (b) for $1 \leq j \leq \rho-1$, $Z_{i_1, \dots, i_\rho}^{(j)}$ is the union of all irreducible components of $Z_{i_1, \dots, i_\rho}^{(j-1)} \cap \text{Supp}(F_{i_j})$ which are not contained in $\bigcup_{j' > j} \text{Supp}(F_{i_{j'}})$, but are contained in $\text{Supp}(F_{i_i})$ for each i , $i_j < i < i_{j+1}$,
- (c) $Z_{i_1, \dots, i_\rho}^{(\rho)}$ is the union of all irreducible components of $Z_{i_1, \dots, i_\rho}^{(\rho-1)} \cap \text{Supp}(F_{i_\rho})$ which are contained in $\bigcap_{i=1}^n \text{Supp}(F_i)$.

Then $(F_1, \dots, F_n)_Z^{\text{ord}} > 0$ only if there are $i^* = i_1 < \dots < i_\rho \leq n$, where $\rho := \text{codim}(Z)$, such that Z is an irreducible component of some $Z_{i_1, \dots, i_\rho}^{(\rho)}$.

Definition 2.8. By an (i_1, \dots, i_ρ) -type distinguished component for (the ordered intersection of) F_1, \dots, F_n , we mean an irreducible component of some $Z_{i_1, \dots, i_\rho}^{(\rho)}$ from Theorem 2.7. Note that it is possible for a distinguished component to have zero ordered intersection multiplicity - see Corollary 2.12 below.

Remark 2.9. If all F_i are identically zero, then the only distinguished component for F_1, \dots, F_n is X , and the number prescribed by Theorem 2.11 below for $(F_1, \dots, F_n)_X^{\text{ord}}$ is precisely the intersection number $(\mathcal{L}_1, \dots, \mathcal{L}_n)$.

Remark 2.10. Every irreducible component Z of $\bigcap_i \text{Supp}(F_i)$ is a distinguished component for F_1, \dots, F_n . Indeed, if $\rho := \text{codim}(Z) = 1$, then $i_1 = i^*$ works (where i^* is as in Theorem 2.7).

² Let C be a curve and $\pi : C' \rightarrow C$ be a desingularization of C . A branch B of C is the germ of a point z in C' . The center of B is $y := \pi(z) \in C$; we also say that B is a branch of C at y . The order at B of a regular function f on C , denoted $\text{ord}_B(f)$, is the order of vanishing of f at z .

Otherwise if $\rho > 1$, then set $i_1 = i^*$ and choose an irreducible component H_1 of $V(f_{i^*})$ containing Z , and let i_2 be the smallest index such that H_1 is not contained in any irreducible component of $V(f_{i_2})$. If $\rho = 2$, then stop. Otherwise pick an irreducible component H_2 of $H_1 \cap V(f_{i_2})$ containing Z , and pick the smallest i_3 such that H_2 is not contained in $V(f_{i_3})$. It is straightforward to check that this process can be continued up to ρ steps, and at the end we have i_1, \dots, i_ρ as claimed.

For the distinguished components for (the ordered intersection of) F_1, \dots, F_n , we can give an explicit expression of $(F_1, \dots, F_n)_{\mathbb{Z}}^{\text{ord}}$ in terms of the length of certain local rings.

Theorem 2.11. *Let Z be an (i_1, \dots, i_ρ) -type distinguished component for F_1, \dots, F_n , where $\rho := \text{codim}(Z)$. Fix an open affine neighborhood U of X such that $U \cap Z \neq \emptyset$ and each F_i is defined by a regular function f_i on U which is either identically zero (in the case that F_i is identically zero on X), or a non zero-divisor in $\mathbb{k}[U]$. Consider ideals $I_{i_1, \dots, i_\rho}^{(j)}$, $0 \leq j \leq \rho$, of $\mathbb{k}[U]$ defined as follows:*

- (a) $I_{i_1, \dots, i_\rho}^{(0)}$ is the zero ideal,
- (b) $I_{i_1, \dots, i_\rho}^{(1)}$ is the principal ideal generated by f_{i_1} ,
- (c) for $2 \leq j \leq \rho$, $I_{i_1, \dots, i_\rho}^{(j)}$ is the ideal generated by f_{i_j} and the intersection of all primary components \mathfrak{q} of $I_{i_1, \dots, i_\rho}^{(j-1)}$ such that the corresponding prime ideal $\sqrt{\mathfrak{q}}$ contains f_i for each $i < i_j$, but does not contain f_{i_j} .

Then

$$(F_1, \dots, F_n)_{\mathbb{Z}}^{\text{ord}} = \sum_{i_1, \dots, i_\rho} \text{len}(\mathcal{O}_{Z, X} / I_{i_1, \dots, i_\rho}^{(\rho)} \mathcal{O}_{Z, X}) \deg_{i'_1, \dots, i'_{n-\rho}}(Z)$$

where the sum is over all i_1, \dots, i_ρ such that Z is an (i_1, \dots, i_ρ) -type distinguished component for F_1, \dots, F_n , and $i'_1, \dots, i'_{n-\rho}$ are the elements of $\{1, \dots, n\} \setminus \{i_1, \dots, i_\rho\}$, and

$$\deg_{i'_1, \dots, i'_{n-\rho}}(Z) := (\mathcal{L}_{i'_1}|_Z, \dots, \mathcal{L}_{i'_{n-\rho}}|_Z)$$

Corollary 2.12. *Let Z be an irreducible subvariety of X . Then $(F_1, \dots, F_n)_{\mathbb{Z}}^{\text{ord}} > 0$ if and only if there are $i^* = i_1 < \dots < i_\rho \leq n$, where $\rho := \text{codim}(Z)$ and i^* is the smallest index such that F_{i^*} is not identically zero, such that*

- (1) Z is an (i_1, \dots, i_ρ) -type distinguished component for F_1, \dots, F_n , and
- (2) $\deg_{i'_1, \dots, i'_{n-\rho}}(Z) > 0$,

where $i'_1, \dots, i'_{n-\rho}$ are the elements of $\{1, \dots, n\} \setminus \{i_1, \dots, i_\rho\}$. □

Remark 2.13. $\deg_{i'_1, \dots, i'_{n-\rho}}(Z)$ is the number (counted with appropriate multiplicities) of points in $Z \cap \bigcap_j \text{Supp}(G_{i'_j})$ for generic $G_{i'_j} \in \mathcal{L}_{i'_j}$. It is analogous to the usual degree of Z , and can be considered a “global factor” since it depends on the intersection theory of X and the line bundles \mathcal{L}_i . The term $\text{len}(\mathcal{O}_{Z, X} / I_{i_1, \dots, i_\rho}^{(\rho)} \mathcal{O}_{Z, X})$ is the “local factor” since it depends on how the F_{i_j} vanish along Z .

Remark 2.14. Let Z be a distinguished component for F_1, \dots, F_n . If Z is a point, then the “global factor” is 1 (since $\rho = n$, and therefore $\{i'_1, \dots, i'_{n-\rho}\}$ is the empty set), and if in addition Z is an isolated point of $\bigcap_i \text{Supp}(F_i)$, then the “local factor” $\text{len}(\mathcal{O}_{Z, X} / I_{i_1, \dots, i_\rho}^{(\rho)} \mathcal{O}_{Z, X})$ is precisely the intersection multiplicity of F_1, \dots, F_n at Z (this follows e.g. since intersections of divisors are well defined as cycle classes on the support of the intersection [Ful98, Theorem 2.4]). On the other extreme, if $Z = X$ (which is only possible if all the F_i are identically zero), then as stated in Remark 2.9, the local factor is 1, and the global factor is $(\mathcal{L}_1, \dots, \mathcal{L}_n)$.

Remark 2.15. The ordered multiplicity is additive in the “first”, i.e. the “left most” component. However, it is in general *not* additive in any other component. Indeed, let $F_1 = V(x_0^2 x_1)$, $F_2 = V(x_0 x_1^2)$, $F'_1 = V(x_0^2)$ be the divisors on $X := \mathbb{P}^2$ from Examples 2.2 to 2.4. It follows from the computations in Examples 2.3 and 2.4 that

$$(F_2, F_1)_{\mathbb{P}^2}^{\text{ord}} = \begin{cases} 3 & \text{if } Z = V(x_0), \\ 6 & \text{if } Z = V(x_1), \\ 0 & \text{otherwise.} \end{cases}$$

$$(F_2, F'_1)_{\mathbb{P}^2}^{\text{ord}} = \begin{cases} 2 & \text{if } Z = V(x_0), \\ 4 & \text{if } Z = V(x_0, x_1), \\ 0 & \text{otherwise.} \end{cases}$$

Write $F''_1 := V(x_1)$, so that $F_1 = F'_1 + F''_1$. It can be shown similarly (cf. Example 7.7) that

$$(F_2, F''_1)_{\mathbb{P}^2}^{\text{ord}} = \begin{cases} 2 & \text{if } Z = V(x_1), \\ 1 & \text{if } Z = V(x_0, x_1), \\ 0 & \text{otherwise.} \end{cases}$$

In particular, it follows that in general $(F_2, F_1)_{\mathbb{P}^2}^{\text{ord}} \neq (F_2, F'_1)_{\mathbb{P}^2}^{\text{ord}} + (F_2, F''_1)_{\mathbb{P}^2}^{\text{ord}}$, even though $(F_2, F_1) = (F_2, F'_1) + (F_2, F''_1)$.

2.16. Theorems 2.6, 2.7 and 2.11 are proved in Part 1 of this article (i.e. Sections 3 to 6). In Part 2 we explicitly compute the ordered intersection multiplicity in some special cases and present examples to illustrate applications to the affine Bézout problem of counting isolated solutions of systems of polynomials. In particular, we consider some criteria under which the following holds:

(property I) the “local factor” $\text{len}(\mathcal{O}_{Z,X}/I_{i_1, \dots, i_\rho}^{(\rho)} \mathcal{O}_{Z,X})$ can be represented as an intersection multiplicity of hypersurfaces on a nonsingular variety at a proper isolated point of intersection.

If X is nonsingular, then (property I) holds e.g. when $\text{codim}(Z) \leq 2$ or $\text{codim}(Z) = n$, or when Z is an isolated component of $\bigcap_{j=1}^{\rho} \text{Supp}(F_{i_j})$ (Proposition 7.4). In case (property I) holds, one can estimate the intersection multiplicity in terms of corresponding Newton diagrams and can also explicitly identify when this estimate is not exact using the results from [Mon21, Chapter IX] - this is described in Sections 8.1 and 8.2. In Section 8.3 we describe a scenario under which a similar convex geometric estimate for the local factor is available even when (property I) does not hold. To illustrate an application of all these estimates to the affine Bézout problem, in Section 9 we compute the number of common tangent lines to 4 general spheres in \mathbb{A}^3 by setting it up as a counting problem on the Grassmannian of lines in \mathbb{P}^3 (or equivalently, planes in \mathbb{A}^4), and estimating the ordered intersection multiplicity of corresponding excess intersections³. Finally, Section 10 presents a few inductive “solutions” to the affine Bézout problem mentioned in Section 1.4.

³ This problem was solved by Macdonald, Pach, and Theobald [Discrete Comput. Geom. 26 (1), 2001] via modelling it as a different counting problem without any excess intersections. Our exposition was inspired by a paragraph by Sottile and Theobald in [ST02] which mentions a similar explanation in terms of usual excess intersection multiplicities in a manuscript by Aluffi and Fulton.

Part 1. Ordered intersection multiplicity

3. INITIAL OBSERVATIONS

3.1. In this section we make a few elementary observations which are the basis of all our results on ordered intersection multiplicity. We work locally on an irreducible affine variety X with dimension n and fixed regular functions f_1, \dots, f_η (in this section η can be an arbitrary positive integer, i.e. does not have to equal n). Given $g_i \in \mathbb{k}[X]$ and positive integers r_i , $1 \leq i \leq \eta$, let

$$C^*(g_1, \dots, g_\eta; r_1, \dots, r_\eta) := V(f_1 - t^{r_1}g_1, \dots, f_\eta - t^{r_\eta}g_\eta) \cap (X \times (\mathbb{k} \setminus \{0\}))$$

and let $C(g_1, \dots, g_\eta; r_1, \dots, r_\eta)$ be the closure of $C^*(g_1, \dots, g_\eta; r_1, \dots, r_\eta)$ in $X \times \mathbb{k}$. Note that in this section and the next we treat these objects only as algebraic *varieties*, not schemes (in Section 6 we would treat these as schemes), and often denote them simply by C^* and C if the g_i and r_j are clear from the context. Given a subvariety Z of X , we will study under what conditions C intersect generic points of Z at $t = 0$ for generic g_i in the case that $r_i/r_{i+1} \gg 1$ for each i . We start with a simple observation.

Lemma 3.2. *Let B be a branch of a curve on $C^*(g_1, \dots, g_\eta; r_1, \dots, r_\eta)$ centered at $(z, 0)$ for some $z \in X$. Then*

$$\text{ord}_B(f_i) \geq r_i \text{ord}_B(t)$$

for each i ; moreover $\text{ord}_B(f_i) = r_i \text{ord}_B(t)$ if and only if $g_i(z) \neq 0$. (See Footnote 2 on page 6 for the definition of (order of a function at) a branch of a curve.) \square

3.3. For $\nu \in \mathbb{R}$, we write (condition $r./r._+ > \nu$) as a shorthand for the following condition:

$$(\text{condition } r./r._+ > \nu) \quad r_i > 0 \text{ and } r_i/r_{i+1} > \nu, \ 1 \leq i \leq \eta - 1, \text{ and } r_\eta > \nu$$

We say that “a certain property P is true under (condition $r./r._+ >> 1$)” to mean the following:

$$(\text{condition } r./r._+ >> 1) \quad \begin{array}{l} \text{there is a real number } \nu \text{ such that property } P \text{ holds} \\ \text{whenever (condition } r./r._+ > \nu) \text{ holds.} \end{array}$$

Proposition 3.4. *Let Z be an irreducible subvariety of X such that $\prod_{i=1}^s f_{k'_i} \in \sqrt{\langle f_1, \dots, f_k \rangle \mathcal{O}_{Z,X}}$ for $k'_1, \dots, k'_s > k$. Then there is a nonempty open subset Z^* of Z such that under (condition $r./r._+ >> 1$), for all $z \in Z^*$, if $(z, 0) \in C$, then $g_{k'_i}(z) = 0$ for some $i = 1, \dots, s$.*

Proof. Assume $(\prod_{i=1}^s f_{k'_i})^m = \sum_{j=1}^k h_j f_j$ with $h_j \in \mathcal{O}_{Z,X}$. Let U be an open subset of X such that $U \cap Z \neq \emptyset$ and each h_j is a regular function on U . Choose r_1, \dots, r_η such that

$$r_j > sm \max\{r_{k'_1}, \dots, r_{k'_s}\}$$

for each $j = 1, \dots, k$. Let B be a branch (of a curve) on $C^* = C^*(g_1, \dots, g_\eta; r_1, \dots, r_\eta)$ centered at $(z, 0)$ where $g_i \in L_i$, $i = 1, \dots, \eta$. Pick i' such that $\text{ord}_B(f_{k'_{i'}}) = \max_{i=1}^s \text{ord}_B(f_{k'_i})$. Then

$$\begin{aligned} \text{ord}_B(f_{k'_{i'}}) &\geq \frac{1}{s} \sum_{i=1}^s \text{ord}_B(f_{k'_i}) = \frac{1}{sm} \text{ord}_B((\prod_{i=1}^s f_{k'_i})^m) \\ &\geq \frac{1}{sm} \min_{j=1}^k \text{ord}_B(f_j) \geq \frac{1}{sm} \min_{j=1}^k r_j \text{ord}_B(t) > r_{k'_{i'}} \text{ord}_B(t) \end{aligned}$$

It follows that $g_{k'_{i'}}(z) = 0$ (Lemma 3.2), as required. \square

3.5. Fix a subvariety Z of X and $1 \leq i_1 < \dots < i_\rho \leq \eta$. Proposition 3.4 suggests that it may be fruitful to understand when there is a nonempty open subset Z^* of Z such that under (condition $r./r._+ >> 1$) there is $z \in Z^*$ which satisfies the following property:

$$(\emptyset) \quad (z, 0) \in C, \text{ and } g_{i_j}(z) \neq 0 \text{ for all } j = 1, \dots, \rho.$$

We say that a branch B on C satisfies (\emptyset) if the center of B is of the form $(z, 0)$ such that z satisfies (\emptyset) . We write $(\emptyset(\nu))$ to denote the combination of (condition $r./r._+ > \nu$) and (\emptyset) , i.e. z (respectively, B) and r_1, \dots, r_η satisfy $(\emptyset(\nu))$ if and only if

$$(\emptyset(\nu)) \quad r_1, \dots, r_\eta \text{ satisfy (condition } r./r._+ > \nu), \text{ and } z \text{ (respectively, } B) \text{ satisfies } (\emptyset).$$

Remark 3.6. Assume B is a branch on C which satisfies (\emptyset) . Then Lemma 3.2 implies that

$$\text{ord}_B(f_{i_j}) = \text{ord}_B(g_{i_j} t^{r_{i_j}}) = r_{i_j} \text{ord}_B(t), \quad j = 1, \dots, \rho.$$

In particular, if i^* is the smallest index such that f_{i^*} is not identically zero, then $i_j \geq i^*$ for each j .

Proposition 3.7. Let Z be an irreducible subvariety of X .

- (1) If each f_i is identically zero, or $Z = X$, or $Z \not\subseteq V(f_1, \dots, f_\eta)$, then there is a nonempty open subset Z^* of Z such that no $z \in Z^*$ satisfies (\emptyset) for any choice of $g_i, r_j, \rho, i_1, \dots, i_\rho$ such that $1 \leq \rho \leq \eta$ and $1 \leq i_1 < \dots < i_\rho \leq \eta$.
- (2) Otherwise, let i^* be as in Remark 3.6 the smallest index such that f_{i^*} is not identically zero. Then there is $\nu \in \mathbb{R}$ and a nonempty open subset Z^* of Z such that if there are $z \in Z^*$ and r_1, \dots, r_η which satisfy $(\emptyset(\nu))$ for some g_1, \dots, g_η , then the number of distinct j such that $i_j > i^*$ is at most $\text{codim}(Z) - 1$.

Proof. If each f_i is identically zero, then the first assertion holds with $Z^* := Z$ (Lemma 3.2). So assume there is i such that f_i is not identically zero, or in other words, $i^* \leq \eta$. If $Z = X$ (respectively, $Z \not\subseteq V(f_1, \dots, f_\eta)$), then again the first assertion holds with $Z^* := X \setminus V(f_{i^*})$ (respectively, $Z^* := Z \setminus V(f_1, \dots, f_\eta)$). So we may assume

$$Z \subseteq V(f_1, \dots, f_\eta) \subsetneq X$$

3.7.1. Now we assume the proposition is not true. Then, since there are only finitely many possibilities for the collections of i_j , there is a fixed collection $i^* < i_1 < \dots < i_\rho \leq \eta$ such that

- (i) $\rho = \text{codim}(Z) \geq 1$, and
- (ii) for all $\nu \in \mathbb{R}$ and all nonempty open subsets Z^* of Z , there are $z \in Z^*$ and g_i, r_j which satisfy $(\emptyset(\nu))$.

3.7.2. We claim that at least one irreducible component of $V(f_{i^*})$ containing Z is not contained in $\bigcup_{j=1}^\rho V(f_{i_j})$. Indeed, otherwise $\prod_{j=1}^\rho f_{i_j} \in \sqrt{f_{i^*} \mathcal{O}_{Z,X}}$, and we get a contradiction to property (ii) due to Proposition 3.4. Choose an open subset $U_1 \subseteq X$ intersecting Z and a regular function h_1 on U_1 such that

- h_1 identically vanishes on each irreducible component of $V(f_{i^*})$ which contains Z but is not contained in $V(\prod_{j=1}^\rho f_{i_j})$,
- h_1 does not vanish identically on any irreducible component of $V(\prod_{j=1}^\rho f_{i_j})$.

Since $h_1 \prod_{j=1}^\rho f_{i_j}$ identically vanishes on each irreducible component of $V(f_{i^*})$ containing Z , there is an open subset $U_1^* \subseteq U_1$ intersecting Z and $m_1 \geq 1$ such that

$$(h_1 \prod_{j=1}^\rho f_{i_j})^{m_1} \in f_{i^*} \mathbb{k}[U_1^*]$$

Now, if B is a branch on C centered at $U_1^* \times \{0\}$ that satisfies $(\emptyset(\nu))$ for some $\nu \geq 1$, then Remark 3.6 implies that

$$\text{ord}_B(f_{i_1}) = r_{i_1} \text{ord}_B(t) > r_{i_j} \text{ord}_B(t) = \text{ord}_B(f_{i_j}), \quad j = 2, \dots, \rho,$$

so that

$$m_1(\text{ord}_B(h_1) + \rho r_{i_1} \text{ord}_B(t)) \geq m_1(\text{ord}_B(h_1) + \sum_{j=1}^{\rho} \text{ord}_B(f_{i_j})) \geq \text{ord}_B(f_{i^*}) \geq r_{i^*} \text{ord}_B(t)$$

It follows that

$$\text{ord}_B(h_1) > \frac{r_{i^*} - m_1 \rho r_{i_1}}{m_1} \text{ord}_B(t)$$

Since $i^* < i_1$, it is then clear that we can find $\nu_1 \geq 1$ and $\mu_1 > 0$ such that if B is a branch on C centered at $U_1^* \times \{0\}$ that satisfies $(\emptyset(\nu))$ for some $\nu \geq \nu_1$, then

$$\text{ord}_B(h_1) > \frac{r_{i^*}}{\mu_1} \text{ord}_B(t) > r_{i_1} \text{ord}_B(t) = \text{ord}_B(f_{i_1})$$

If $\rho = 1$, then we stop. Otherwise we consider $V(h_1) \cap V(f_{i_1})$, which by construction of h_1 , has pure codimension 2 in X .

Claim 3.7.3. *At least one irreducible component of $V(h_1) \cap V(f_{i_1})$ containing Z is not contained in $\bigcup_{j=2}^{\rho} V(f_{i_j})$.*

Proof. Indeed, otherwise $\prod_{j=2}^{\rho} f_{i_j} \in \sqrt{\langle h_1, f_{i_1} \rangle \mathcal{O}_{Z,X}}$. Then there is $U_2' \subseteq U_1^*$ and $m_2' \geq 1$ such that

$$\left(\prod_{j=2}^{\rho} f_{i_j} \right)^{m_2'} \in \langle h_1, f_{i_1} \rangle \mathbb{k}[U_2]$$

If B is a branch on C centered at $U_2' \times \{0\}$ which satisfies $(\emptyset(\nu))$ for some $\nu \geq \nu_1$ as in the end of Paragraph 3.7.2, then

$$\begin{aligned} m_2'(\rho - 1)r_{i_2} \text{ord}_B(t) &= m_2'(\rho - 1) \text{ord}_B(f_{i_2}) \\ &\geq m_2' \sum_{j=2}^{\rho} \text{ord}_B(f_{i_j}) \\ &\geq \min\{\text{ord}_B(h_1), \text{ord}_B(f_{i_1})\} \\ &\geq \text{ord}_B(f_{i_1}) \\ &= r_{i_1} \text{ord}_B(t) \end{aligned}$$

which implies

$$r_{i_2} \geq \frac{r_{i_1}}{m_2'(\rho - 1)}$$

However, this is impossible if we take $\nu > m_2'(\rho - 1)$ (it is permissible to take arbitrarily large ν due to assumption (ii) in Paragraph 3.7.1). This contradiction proves the claim. \square

3.7.4. Choose an open subset $U_2 \subseteq U_1^*$ intersecting Z and a regular function h_2 on U_2 such that

- h_2 identically vanishes on each irreducible component of $V(h_1, f_{i_1})$ which contains Z but is not contained in $V(\prod_{j=2}^{\rho} f_{i_j})$,
- h_2 does not vanish identically on any irreducible component of $V(h_1 \prod_{j=2}^{\rho} f_{i_j})$.

Since $h_2 \prod_{j=2}^{\rho} f_{i_j}$ identically vanishes on each irreducible component of $V(h_1, f_{i_1})$ containing Z , there is an open subset $U_2^* \subseteq U_2$ intersecting Z and $m_2 \geq 1$ such that

$$(h_2 \prod_{j=2}^{\rho} f_{i_j})^{m_2} \in \langle h_1, f_{i_1} \rangle \mathbb{k}[U_2^*]$$

It follows as in the proof of Claim **3.7.3** that if B is a branch on C centered at $U_2^* \times \{0\}$ which satisfies $(\emptyset(\nu))$ for some $\nu \geq \nu_1$, then

$$m_2(\text{ord}_B(h_2) + (\rho - 1)r_{i_2} \text{ord}_B(t)) \geq r_{i_1} \text{ord}_B(t)$$

so that

$$\text{ord}_B(h_2) \geq \frac{r_{i_1} - m_2(\rho - 1)r_{i_2}}{m_2} \text{ord}_B(t)$$

Consequently we can choose $\nu_2 \geq \nu_1$ and a positive number μ_2 such that if B is a branch on C centered at $U_2^* \times \{0\}$ which satisfies $(\emptyset(\nu))$ for some $\nu \geq \nu_2$, then

$$\text{ord}_B(h_2) > \frac{r_{i_1}}{\mu_2} \text{ord}_B(t) > r_{i_2} \text{ord}_B(t) = \text{ord}_B(f_{i_2})$$

3.7.5. If $\rho = 2$, then we stop. Otherwise we proceed as above, and after ρ steps we end up with an open subset U_ρ^* of X such that $U_\rho^* \cap Z \neq \emptyset$, and regular functions h_1, \dots, h_ρ on U_ρ^* , and positive real numbers $\nu_1, \dots, \nu_\rho, \mu_1, \dots, \mu_\rho$ such that for each k , $1 \leq k \leq \rho$,

- (1) $Y_k := V(h_1, h_2, \dots, h_k) \cap U_\rho^*$ is a pure codimension k (possibly reducible) subvariety of X containing Z , and
- (2) if B is a branch on C centered at $U_\rho^* \times \{0\}$ which, together with $r_1, \dots, r_\eta \in \mathbb{R}$, satisfies $(\emptyset(\nu))$ for some $\nu \geq \nu_k$, then

$$\text{ord}_B(h_k) > \frac{1}{\mu_k} r_{i_{k-1}} \text{ord}_B(t)$$

where we set $i_0 := i^*$ for consistency.

Now since $Y_\rho \supseteq Z$ has pure codimension $\rho = \text{codim}(Z)$, there is an open subset U^* of U_ρ^* such that $Y_\rho \cap U^* = Z \cap U^* \neq \emptyset$. Since $Z \subseteq V(f_1, \dots, f_\eta) \subseteq V(f_{i_\rho})$, it follows that $(f_{i_\rho})^{m_\rho} \in \langle h_1, \dots, h_\rho \rangle \mathbb{k}[U^*]$ for some $m_\rho \geq 1$. By assumption (ii) from Paragraph **3.7.1**, there are $z \in U^* \cap Z$ and g_i, r_j which satisfy $(\emptyset(\nu))$ for some $\nu > \max\{\nu_1, \dots, \nu_\rho, m_\rho \mu_1, \dots, m_\rho \mu_\rho\}$. Then for each branch B of C centered at $(z, 0)$,

$$\text{ord}_B(f_{i_\rho}) \geq \min_{k=1}^{\rho} \frac{\text{ord}_B(h_k)}{m_\rho} \geq \min_{k=1}^{\rho} \frac{r_{i_{k-1}}}{m_\rho \mu_k} \text{ord}_B(t) > r_{i_\rho} \text{ord}_B(t)$$

which contradicts Remark **3.6** and completes the proof of Proposition **3.7**. \square

4. DISTINGUISHED COMPONENTS OF AN ORDERED INTERSECTION

In this section we determine the “distinguished components” of ordered intersection. The main result of this section is Theorem 4.8 which shows that the property of “having zero (or nonzero) ordered intersection multiplicity” is well defined, and there can be at most finitely many irreducible subvarieties with nonzero ordered intersection multiplicity, and all of them must be of the form described in Theorem 2.7. The arguments used in this section are elementary (but involved) except for one place in the proof of Lemma 4.6 where de Jong’s theorem on alteration [dJ96, Theorem 4.1] is used⁴.

4.1. Set up. We start with a set up which is a bit more general than that of Problem 1.1. We work locally on an irreducible affine variety X of dimension n with fixed regular functions f_1, \dots, f_n and subsets L_1, \dots, L_n of $\mathbb{k}[X]$ such that for each i ,

- (1) L_i is an irreducible constructible set of dimension ≥ 1 , and
- (2) L_i is base-point free, i.e. for each $x \in X$, there is $g \in L_i$ such that $g(x) \neq 0$.

Given $g_i \in L_i$ and positive integers r_i , $1 \leq i \leq n$, we consider $C = C(g_1, \dots, g_n; r_1, \dots, r_n)$ as in Section 3. Building on Propositions 3.4 and 3.7, the following result shows how the ordering of f_1, \dots, f_n dictates the subvarieties Z of X for which $C = C(g_1, \dots, g_n; r_1, \dots, r_n)$ may intersect $Z \times \{0\}$ under (condition $r./r._+ >> 1$) in the case that $\eta = n$.

Theorem 4.2. *Assume we are in the set up of Section 4.1. Let Z be a subvariety of X with $\rho := \text{codim}(Z) > 0$. Then there are $\nu^* \in \mathbb{R}$, a nonempty Zariski open subset Z^* of Z , and a nonempty Zariski open subset L^* of $\prod_i L_i$ such that for each $(g_1, \dots, g_n) \in L^*$, one (and only one) of the following holds for every $\nu \geq \nu^*$:*

- (1) *either C does not intersect $Z^* \times \{0\}$ under (condition $r./r._+ > \nu$),*
- (2) *or there are $z \in Z^*$ and $r_1, \dots, r_n \in \mathbb{R}$ such that (condition $r./r._+ > \nu$) holds and $(z, 0) \in C$. In this case there are ρ choices for i , say $1 \leq i_1 < \dots < i_\rho \leq n$, such that*
 - (a) *$i_1 = i^*$, where i^* , as in Remark 3.6, is the smallest index such that f_{i^*} is not identically zero, and*
 - (b) *$g_i(z) \neq 0$ if and only if $i = i_j$ for some $j = 1, \dots, \rho$.*

Remark 4.2.1. Note that Theorem 4.2 leaves open the possibility that some $(g_1, \dots, g_n) \in L^*$ satisfies the first assertion, whereas some other $(g_1, \dots, g_n) \in L^*$ satisfies the second. Moreover, it does not give any control on whether any of these options holds “uniformly”. Theorem 4.8 below provides a uniform version.

Remark 4.2.2. It is possible (when $1 < \rho < n$) that the same Z satisfies the second assertion of Theorem 4.2 for more than one choices of i_1, \dots, i_ρ (see e.g. identity (13) in Example 7.9).

4.2.3. Proof of Theorem 4.2. Let L' be the subset of $\prod_i L_i$ consisting of all (g_1, \dots, g_n) such that $Z \cap V(g_{i'_1}, \dots, g_{i'_{n-\rho+1}}) = \emptyset$ for all choices of $1 \leq i'_1 < \dots < i'_{n-\rho+1} \leq n$. Then L' is a full-dimensional constructible subset of $\prod_i L_i$, and therefore contains a nonempty Zariski open subset L^* of $\prod_i L_i$. Now due to Proposition 3.7 there is $\nu^* \in \mathbb{R}$ and a nonempty open subset Z^* of Z such that the following holds for all $\nu \geq \nu^*$: if $z \in Z^*$ and $r_1, \dots, r_n \in \mathbb{R}$ are such that $(z, 0) \in C$ and (condition $r./r._+ > \nu$) holds, then there are at most ρ choices for k such that $g_k(z) \neq 0$. Consequently, for each $(g_1, \dots, g_n) \in L^*$ and $\nu \geq \nu^*$ there are two possible scenarios:

- (i) *either C does not intersect $Z^* \times \{0\}$ under (condition $r./r._+ > \nu$),*
- (ii) *or there are $z \in Z^*$ and $r_1, \dots, r_n \in \mathbb{R}$ such that (condition $r./r._+ > \nu$) holds and $(z, 0) \in C$.*

⁴ We believe the results of this article could have been proved without using de Jong’s theorem on alteration; however, that would have made the arguments much more involved.

The first scenario is precisely the first possibility stated in Theorem 4.2, and under the second scenario, the second assertion of Theorem 4.2 holds due to Proposition 3.7 and the construction of L^* . \square

4.3. In light of Theorem 4.2, from now on we consider conditions (\emptyset) and $(\emptyset(\nu))$ only in the special case $i_1 = i^*$. Before proceeding further, we explicitly state a somewhat technical, but straightforward, implication of the condition $(\emptyset(\nu))$ that was implicitly used several times in the proof of Proposition 3.7. Fix $i^* = i_1 < i_2 < \dots < i_\rho \leq n$, a nonempty open subset Z^* of Z and a collection \mathcal{H}^* of pairs (h, i) such that $1 \leq i \leq n$, and h is a regular function on an open subset of X containing Z^* . We say that \mathcal{H}^*, Z^* satisfy $(\emptyset(\mu^*, \nu^*))$ for $\mu^*, \nu^* \in \mathbb{R}$ if the following holds:

there are $\nu \geq \nu^*, r_1, \dots, r_n \in \mathbb{R}$ and a branch B of $C = C(g_1, \dots, g_n; r_1, \dots, r_n)$ centered at $Z^* \times \{0\}$ such that $(\emptyset(\nu))$ holds, and in addition, $\text{ord}_B(h) > \mu^* \text{ord}_B(f_i) \geq \mu^* r_i \text{ord}_B(t)$ for each $(h, i) \in \mathcal{H}^*$.

Lemma 4.4. Fix $i^* = i_1 < i_2 < \dots < i_\rho \leq n$ and an open subset U^* of X such that $U^* \cap Z \neq \emptyset$. Let \mathcal{H} be a collection of pairs (h, i) of $h \in \mathbb{k}[U^*]$ and $1 \leq i \leq n$ such that

- (i) property $(\emptyset(\mu^*, \nu^*))$ holds with $\mathcal{H}^* = \mathcal{H}$ and $Z^* = U^* \cap Z$ for each $\mu^*, \nu^* \in \mathbb{R}$;
- (ii) there is $j, 1 \leq j \leq \rho - 1$, such that $i \leq i_{j+1}$ for each $(h, i) \in \mathcal{H}$.

Let $Y := V(h : (h, i) \in \mathcal{H})$. Then

- (1) $Y \cap U^* \not\subseteq \bigcup_{j' > j} V(f_{i_{j'}})$.
- (2) If $h^* \in \mathbb{k}[U^*]$ is such that $h^* \prod_{j' > j} f_{i_{j'}}$ vanishes on $Y \cap U^*$, then for each $\mu^*, \nu^* \in \mathbb{R}$, then $\mathcal{H}^* = \mathcal{H} \cup \{(h^*, i_{j+1})\}$ also satisfies $(\emptyset(\mu^*, \nu^*))$ with $Z^* = U^* \cap Z$.

Proof. First we prove assertion (2). By assumption (and Hilbert's Nullstellensatz) there is $m \geq 1$ such that $(h^* \prod_{j' > j} f_{i_{j'}})^m$ is in the ideal of $\mathbb{k}[U^*]$ generated by all h such that $(h, i) \in \mathcal{H}$ for some i . If B is any branch B of C centered at $U^* \times \{0\}$ which satisfies $(\emptyset(\nu))$ for some $\nu \geq 1$, then

$$\text{ord}_B(h^*) \geq \frac{1}{m} \left(\min_{(h, i) \in \mathcal{H}} \text{ord}_B(h) - m \sum_{j' > j} \text{ord}_B(f_{i_{j'}}) \right) \geq \frac{1}{m} \left(\min_{(h, i) \in \mathcal{H}} \text{ord}_B(h) - m(\rho - j) \text{ord}_B(f_{i_{j+1}}) \right)$$

Now, given μ, ν , choose $r_1, \dots, r_n \in \mathbb{R}$ and a branch B of C centered at $(U^* \cap Z) \times \{0\}$ such that $(\emptyset(\mu^*, \nu^*))$ holds for $\mu^* = m(\rho - j + \mu)$ and $\nu^* = \nu$. Then $\text{ord}_B(h^*) > \mu \text{ord}_B(f_{i_{j+1}})$, as required to complete the proof of assertion (2). Now, if assertion (1) is not true, then $\prod_{j' > j} f_{i_{j'}}$ vanishes on $Y \cap U^*$, and assertion (2) applied with $h^* := 1$ gives a contradiction. This completes the proof. \square

4.5. Our next result shows that any subvariety of X not listed in Theorem 2.7 has zero ordered intersection multiplicity. Let Z be an irreducible subvariety of X . Assume at least one f_i is not identically zero; as in Remark 3.6, let i^* be the smallest such i . Fix $i^* = i_1 < \dots < i_\rho \leq n$, where $\rho := \text{codim}(Z)$ and define subvarieties $\Phi_Z^{(0)}, \dots, \Phi_Z^{(\rho)}$ of X as follows:

- (a) $\Phi_Z^{(0)} := X$,
- (b) for each $j = 1, \dots, \rho - 1$, $\Phi_Z^{(j)}$ is the union of all irreducible components of $\Phi_Z^{(j-1)} \cap V(f_{i_j})$ containing Z which are not contained in $\bigcup_{j'=j+1}^\rho V(f_{i_{j'}})$, but are contained in $V(f_i)$ for each $i, i_j < i < i_{j+1}$.
- (c) for $j = \rho$,

$$\Phi_Z^{(\rho)} = \begin{cases} Z & \text{if } \Phi_Z^{(\rho-1)} \neq \emptyset \text{ and } Z \subseteq V(f_1, \dots, f_n), \\ \emptyset & \text{otherwise.} \end{cases}$$

Lemma 4.6. *If $\Phi_Z^{(j)} = \emptyset$ for some $j = 1, \dots, \rho$, then there is $\nu \in \mathbb{R}$, a nonempty open subset Z^* of Z , and a nonempty Zariski open subset L^* of $\prod_i L_i$ such that no branch B of $C = C(g_1, \dots, g_n; r_1, \dots, r_n)$ centered at $Z^* \times \{0\}$ satisfies $(\emptyset(\nu))$ for any $(g_1, \dots, g_n) \in L^*$. In other words, for each $(g_1, \dots, g_n) \in L^*$, and each choice of positive r_1, \dots, r_n such that $r_i/r_{i+1} > \nu$, $1 \leq i \leq n-1$, and each $z \in Z^*$ such that $(z, 0) \in C$, there is j , $1 \leq j \leq \rho$, such that $g_{i_j}(z) = 0$.*

Proof. We prove the lemma in several steps. First we reduce it to the case that the union of the set of zeroes of the f_i is the support of a *strict normal crossing divisor*⁵ using de Jong's *alterations*⁶[dJ96, Theorem 4.1].

Proposition 4.6.1. *Let $\pi : X' \rightarrow X$ be a surjective morphism between varieties of the same dimension n . Assume Lemma 4.6 holds for the ordered intersection of $V(\pi^*(f_i))$, $i = 1, \dots, n$, on X' . Then Lemma 4.6 holds for the ordered intersection of $V(f_i)$, $i = 1, \dots, n$, on X .*

Proof. It suffices to consider the case that $Z \subseteq V(f_1, \dots, f_n)$, and $\Phi_Z^{(j)}$, as defined in Lemma 4.6, is empty for some $j = 1, \dots, \rho-1$, where $\rho := \text{codim}(Z)$. Due to Theorem 4.2 then it suffices to show that there is no subvariety Z' of $\pi^{-1}(Z)$ such that

- (i) the codimension of Z' in X' equals the codimension ρ of Z in X ,
- (ii) $\pi(Z') = Z$,
- (iii) $\Phi_{Z'}^{(j)}$, as defined in Lemma 4.6, are nonempty for each $j = 1, \dots, \rho$.

Assume to the contrary that there is a subvariety Z' of $\pi^{-1}(Z)$ satisfying the above properties. Note the following (obvious) observation:

- (iv) given a subset W' of X' , $W' \subseteq V(\pi^*(f_i))$ if and only if $\pi(W') \subseteq V(f_i)$. In particular, if W' is irreducible and $W' \not\subseteq V(\pi^*(f_i))$, and $\dim(W') > \dim(\pi(W'))$, then
 - (a) either $W' \cap V(\pi^*(f_i)) = \emptyset$,
 - (b) or $\dim(\pi(W') \cap V(f_i)) = \dim(\pi(W')) - 1 < \dim(W') - 1 = \dim(W' \cap V(\pi^*(f_i)))$.

It follows from the definition of $\Phi^{(\cdot)}$ and applications of the preceding observation that for each $j = 1, \dots, \rho$,

- (v) $\pi(\Phi_{Z'}^{(j)})$ is the union of $\Phi_Z^{(j)}$ and a (possibly empty) subvariety W_j of codimension $> j$ in X .

Since $\Phi_Z^{(\rho-1)} = \emptyset$ by assumption, it follows that $\pi(\Phi_{Z'}^{(\rho-1)}) = Z \subseteq V(f_{i_\rho})$. However, then observation (iv) implies that $\Phi_{Z'}^{(\rho-1)} \subseteq V(\pi^*(f_{i_\rho}))$. This contradicts the definition of $\Phi^{(\rho-1)}$, and therefore, proves Proposition 4.6.1. \square

4.6.2. We go back to the proof of Lemma 4.6. It suffices to consider the case that

- (i) $Z \subseteq V(f_1, \dots, f_n)$, for otherwise Lemma 4.6 is obvious with $Z^* := Z \setminus V(f_1, \dots, f_n)$,
- (ii) $\rho := \text{codim}(Z) > 1$, since Lemma 4.6 is clearly true if $\rho \leq 1$,
- (iii) and $\bigcup_i V(f_i)$ with the reduced scheme structure is a strict normal crossing divisor (due to Proposition 4.6.1).

We will prove the contrapositive of Lemma 4.6. In particular, we assume that

⁵ An effective Cartier divisor D on a nonsingular variety X is a *strict normal crossing divisor* if near each $x \in \text{Supp}(D)$, D can be expressed as a sum of divisors of regular functions which collectively form a part of a regular system of parameters in $\mathcal{O}_{x,X}$.

⁶ We use the following weak version of de Jong's theorem on "alterations" [dJ96, Theorem 4.1]: Let Z be a proper closed subset of a variety X . There is a surjective generically finite morphism $\pi : X' \rightarrow X$ such that X' is nonsingular, and $\pi^{-1}(Z)$ is the support of a strict normal crossing divisor on X' .

- (iv) for each $\nu \in \mathbb{R}$, each nonempty open subset Z^* of Z , and each nonempty Zariski open subset L^* of $\prod_i L_i$, there are $r_1, \dots, r_n \in \mathbb{R}$, $(g_1, \dots, g_n) \in L^*$, and a branch B of $C = C(g_1, \dots, g_n; r_1, \dots, r_n)$ centered at $Z^* \times \{0\}$ such that $(\emptyset(\nu))$ holds.

Under these assumptions, we will show that $\Phi_Z^{(j)}$ is nonempty for each $j = 1, \dots, \rho$.

4.6.3. By assumption (iii), there is an open subset U of X with local system of coordinates (x_1, \dots, x_n) such that $U \cap Z \neq \emptyset$ and each f_i which is not identically zero is an invertible function on U multiplied by a monomial in the x_i . Define subvarieties $\Phi_Z^{(j)}$ of X containing $\Phi_Z^{(j)}$, $j = 0, \dots, \rho$, as follows:

- (a) $\Phi_Z^{(0)} := X$,
- (b) for each $j = 1, \dots, \rho$, $\Phi_Z^{(j)}$ is the union of all irreducible components of $\Phi_Z^{(j-1)} \cap V(f_{i_j})$ containing Z which are not contained in $\bigcup_{j'=j+1}^\rho V(f_{i_{j'}})$.

Claim 4.6.4. *Under the assumptions of Paragraph 4.6.2, $\Phi_Z^{(j)} \neq \emptyset$, $j = 0, \dots, \rho$. There are indices k_1, \dots, k_ρ and a nonempty open subset U' of U such that*

- (1) $U' \cap Z \neq \emptyset$,
- (2) for each j ,
 - (a) x_{k_j} divides f_{i_j} , but does not divide $f_{i_{j'}}$ for any $j' > j$,
 - (b) $\Phi_Z^{(j)} \cap U' \supseteq V(x_{k_1}, \dots, x_{k_j}) \cap U' \neq \emptyset$,
- (3) In particular, $\Phi_Z^{(\rho)} \cap U' = V(x_{k_1}, \dots, x_{k_\rho}) \cap U' \neq \emptyset$.

Proof. Write $f_{i_1} = u_{i_1} \prod_k x_k^{\alpha_k}$, where u_{i_1} is invertible on U . Let f'_{i_1} be the product of all $x_k^{\alpha_k}$ such that x_k does not divide f_{i_j} for any $j > 1$. Due to assumption (iv) of Paragraph 4.6.2, we may apply Lemma 4.4 with $j = 1$, $\mathcal{H} = \{(f_{i_1}, i_2)\}$ and $h^* = f'_{i_1}$ to obtain that property $(\emptyset(\mu^*, \nu^*))$ is satisfied for all μ^*, ν^* with $\mathcal{H}^* = (f'_{i_1}, i_2)$ and $Z^* = Z \cap U$; in particular, $V(f'_{i_1}) \cap U \neq \emptyset$. Now assume we have inductively defined monomials $f'_{i_1}, \dots, f'_{i_j}$ in the x_i such that

- (i) $V(f'_{i_1}, \dots, f'_{i_j}) \cap U \neq \emptyset$,
- (ii) each f'_{i_k} divides f_{i_k} , and is relatively prime to $\prod_{j' > k} f_{i_{j'}}$,
- (iii) property $(\emptyset(\mu^*, \nu^*))$ is satisfied for all μ^*, ν^* with $\mathcal{H}^* = \{(f'_{i_1}, i_2), \dots, (f'_{i_j}, i_{j+1})\}$ and $Z^* = Z \cap U$.

If $j = \rho - 1$, then we define $f'_{i_{j+1}} := f_{i_\rho}$, and stop. Otherwise, write $f_{i_{j+1}} = u_{i_{j+1}} \prod_k x_k^{\alpha_k}$, where $u_{i_{j+1}}$ is invertible on U , and define $f'_{i_{j+1}}$ to be the product of all $x_k^{\alpha_k}$ such that x_k does not divide $f_{i_{j'}}$ for any $j' > j + 1$. It is then straightforward to see by an application of Lemma 4.4 for $j + 1$ with $\mathcal{H} = \{(f'_{i_1}, i_2), \dots, (f'_{i_j}, i_{j+1})\}$ and $h^* = f'_{i_{j+1}}$ that the process can be continued to the next step with $f'_{i_{j+1}}$. This implies in particular that $V(f'_{i_1}, \dots, f'_{i_\rho})$ is a nonempty union of “coordinate subspaces” of codimension ρ , i.e. varieties of the form $x_{k_1} = \dots = x_{k_\rho} = 0$ for $1 \leq k_1 < \dots < k_\rho \leq n$. Since (x_1, \dots, x_n) is a local system of coordinates on U , and Z is irreducible of codimension ρ , there is an open subset U' of U such that $U' \cap Z$ is nonempty and equals the intersection with U' of precisely one such coordinate subspace. All assertions of the claim follow from this observation. \square

Consider k_1, \dots, k_ρ from Claim 4.6.4. Pick j , $1 \leq j \leq \rho - 1$, and i such that $i_j < i < i_{j+1}$.

Claim 4.6.5. *There is $j' \leq j$ such that $x_{k_{j'}}$ divides f_i .*

Proof. Write $f_i = u_i \prod_k x_k^{\alpha_k}$ where u_i is invertible on U . Assume the claim does not hold, so that $\alpha_{k_{j'}} = 0$ for all $j' \leq j$. Since (x_1, \dots, x_n) form a local system of coordinates everywhere on U , it

follows that $Z^* := (Z \cap U) \setminus V(x_k : k \neq k_{j'} \text{ for any } j')$ is nonempty. If r_1, \dots, r_n and B satisfy property (iv) of Paragraph 4.6.2 for Z^* with $\nu \geq 1$, then

$$\text{ord}_B(f_i) = \sum_{j' > j} \alpha_{k_{j'}} \text{ord}_B(x_{k_{j'}}) \leq \sum_{j' > j} \alpha_{k_{j'}} \text{ord}_B(f_{i_{j'}}) \leq (\rho - j) \left(\max_{j'=j+1}^{\rho} \alpha_{k_{j'}} \right) r_{i_{j+1}} \text{ord}_B(t)$$

However, then $\text{ord}_B(f_i) < r_i \text{ord}_B(t)$ for $\nu \gg 1$, contradicting property (condition $r./r._+ > \nu$). This proves the claim. \square

If U' is as in Claim 4.6.4, then $\Phi_Z^{(j)} \cap U' \supseteq V(x_{k_1}, \dots, x_{k_j}) \cap U' \neq \emptyset$ for each $j = 1, \dots, \rho$ (Claims 4.6.4 and 4.6.5). This completes the proof of Lemma 4.6. \square

4.7. We now show how Lemma 4.6 implies that the “ordered intersection multiplicity” can be nonzero for at most finitely many subvarieties of X . As in Remark 3.6, let i^* be the smallest index such that f_{i^*} is not identically zero. For each $i^* = i_1 < \dots < i_\rho \leq n$, where $1 \leq \rho \leq n := \dim(X)$, let $Z_{i_1, \dots, i_\rho}^{(j)}$, $0 \leq j \leq \rho$, be as in Theorem 2.7, i.e.

- (a) $Z_{i_1, \dots, i_\rho}^{(0)} = X$,
- (b) for $1 \leq j \leq \rho - 1$, $Z_{i_1, \dots, i_\rho}^{(j)}$ is the union of all irreducible components of $Z_{i_1, \dots, i_\rho}^{(j-1)} \cap \text{Supp}(F_{i_j})$ which are not contained in $\bigcup_{j' > j} \text{Supp}(F_{i_{j'}})$, but are contained in $\text{Supp}(F_{i_j})$ for each $i, i_j < i < i_{j+1}$,
- (c) $Z_{i_1, \dots, i_\rho}^{(\rho)}$ is the union of all irreducible components of $Z_{i_1, \dots, i_\rho}^{(\rho-1)} \cap \text{Supp}(F_{i_\rho})$ which are contained in $\bigcap_{i=1}^n \text{Supp}(F_i)$.

We write $\mathcal{Z}^{(\rho)}$ for the union of $Z_{i_1, \dots, i_\rho}^{(\rho)}$ over all choices of i_1, \dots, i_ρ .

Theorem 4.8. Assume we are in the set up of Section 4.1.

- (1) For each ρ and each irreducible component $\mathcal{Z}_k^{(\rho)}$ of $\mathcal{Z}^{(\rho)}$, choose a nonempty open subset $\mathcal{Z}_k^{(\rho,*)}$ of $\mathcal{Z}_k^{(\rho)}$. Then there is $\nu \in \mathbb{R}$, and a nonempty open subset L^* of $\prod_i L_i$ such that under (condition $r./r._+ > \nu$) for each $(g_1, \dots, g_n) \in L^*$ and each point $(z, 0)$ on C at $t = 0$:
 - (a) $z \in \bigcup_{\rho, k} \mathcal{Z}_k^{(\rho,*)}$,
 - (b) if $z \in \mathcal{Z}_k^{(\rho,*)}$, then there are $i^* = i_1 < \dots < i_\rho \leq n$, such that $\mathcal{Z}_k^{(\rho)}$ is an irreducible component of $Z_{i_1, \dots, i_\rho}^{(\rho)}$, and in addition, $g_i(z) \neq 0$ if and only if $i = i_j$ for some j .
- (2) In particular, given an irreducible subvariety Z of X , if Z is not an irreducible component of any $\mathcal{Z}^{(\rho)}$, then there is $\nu \in \mathbb{R}$, a nonempty open subset Z^* of Z and a nonempty open subset L^* of $\prod_i L_i$ such that under (condition $r./r._+ > \nu$), $C \cap (Z^* \times \{0\}) = \emptyset$ for each $(g_1, \dots, g_n) \in L^*$.

Remark 4.8.1. In the case that Z satisfies (respectively, does not satisfy) the conclusion of assertion (2) of Theorem 4.8, we say that the “ordered intersection multiplicity of f_1, \dots, f_n at Z is zero (respectively, nonzero).”

4.8.2. Proof of assertion (2) of Theorem 4.8. The second assertion of Theorem 4.8 follows from the first. Indeed, given Z as in assertion (2), apply assertion (1) with

$$\mathcal{Z}_k^{(\rho,*)} := \begin{cases} \mathcal{Z}_k^{(\rho)} \setminus Z & \text{if } \mathcal{Z}_k^{(\rho)} \supsetneq Z, \\ \mathcal{Z}_k^{(\rho)} & \text{otherwise.} \end{cases}$$

Then assertion (2) holds with Z^* being the complement in Z of the union of all $\mathcal{Z}_k^{(\rho)}$ such that $\mathcal{Z}_k^{(\rho)} \not\supseteq Z$.

4.8.3. Proof of assertion (1a) of Theorem 4.8 - step 1. Starting with $Z'_0 := V(f_1, \dots, f_n)$, inductively define Z'_i , $i \geq 1$, as follows: for every irreducible component $Z'_{i-1;j}$ of Z'_{i-1} , let

$$Z''_{i-1;j} := \begin{cases} Z'_{i-1;j} \setminus \mathcal{Z}_k^{(\rho,*)} & \text{if } Z'_{i-1;j} = \mathcal{Z}_k^{(\rho)}, \\ Z'_{i-1;j} & \text{if there are no } \rho, k \text{ such that } Z'_{i-1;j} = \mathcal{Z}_k^{(\rho)}. \end{cases}$$

Then define Z'_i to be the union of $Z''_{i-1;j}$ over all j . This process can be repeated only finitely many times, and at the end we arrive at Z'_{j_1} such that

- (i) no irreducible component of Z'_{j_1} is an irreducible component of some $\mathcal{Z}^{(\rho)}$, and
- (ii) $V(f_1, \dots, f_n) = Z'_{j_1} \cup \bigcup_{\rho, k} \mathcal{Z}_k^{(\rho,*)}$.

Now for each irreducible component $Z'_{j_1;k}$ of Z'_{j_1} , apply Lemma 4.6 with $Z = Z'_{j_1;k}$ and every possible choice of i_1, \dots, i_ρ , where $\rho := \text{codim}(Z'_{j_1;k})$. Due to property (i) this yields $\nu'_{j_1;k} \in \mathbb{R}$ and nonempty open subsets $Z'^{*}_{j_1;k}$ of $Z'_{j_1;k}$ and $L'^{*}_{j_1;k}$ of $\prod_i L_i$ such that for each such choice of i_1, \dots, i_ρ , each $(g_1, \dots, g_n) \in L'^{*}_{j_1;k}$ and each r_1, \dots, r_n such that (condition $r./r._+ > \nu$) holds with $\nu = \nu'_{j_1;k}$, no $z \in Z'^{*}_{j_1;k}$ satisfies (\emptyset). It then follows (due to Theorem 4.2) that there are $\nu''_{j_1;k} \geq \nu'_{j_1;k}$ and nonempty open subsets $Z''^{*}_{j_1;k}$ of $Z'^{*}_{j_1;k}$ and $L''^{*}_{j_1;k}$ of $L'^{*}_{j_1;k}$ such that $(Z''^{*}_{j_1;k} \times \{0\}) \cap C = \emptyset$ under (condition $r./r._+ > \nu$) with $\nu = \nu''_{j_1;k}$ for each $(g_1, \dots, g_n) \in L''^{*}_{j_1;k}$. Define

$$Z''^{1,*} := \bigcup_k Z''^{*}_{j_1;k}$$

$$Z'^2_0 := \bigcup_k (Z'_{j_1;k} \setminus Z''^{*}_{j_1;k})$$

4.8.4. Proof of assertion (1a) of Theorem 4.8 - step 2. Now, as in Paragraph 4.8.3, for $i \geq 1$, inductively define Z'^2_i from Z'^2_{i-1} by taking each irreducible component $Z'^2_{i-1;j}$ of Z'^2_{i-1} which equals $\mathcal{Z}_k^{(\rho)}$ for some ρ, k , and replacing it by $Z'^2_{i-1;j} \setminus \mathcal{Z}_k^{(\rho,*)}$. This process stops as above with $Z'^2_{j_2}$ such that

- (i) no irreducible component of $Z'^2_{j_2}$ is an irreducible component of any $\mathcal{Z}^{(\rho)}$, and
- (ii) $V(f_1, \dots, f_n) = Z'^2_{j_2} \cup Z'^{1,*} \cup \bigcup_{\rho, k} \mathcal{Z}_k^{(\rho,*)}$.

As in Paragraph 4.8.3, we apply Lemma 4.6 and Theorem 4.2 to each irreducible component $Z'^2_{j_2;k}$ of $Z'^2_{j_2}$ to obtain a nonempty open subset $Z''^{2,*}_{j_2;k}$ of $Z'^2_{j_2;k}$ and $L''^{2,*}_{j_2;k}$ of $\prod_i L_i$ such that under (condition $r./r._+ > \nu$) for some $\nu \gg 1$, $(Z''^{2,*}_{j_2;k} \times \{0\}) \cap C = \emptyset$ for each $(g_1, \dots, g_n) \in L''^{2,*}_{j_2;k}$. Define

$$Z''^{2,*} := \bigcup_k Z''^{2,*}_{j_2;k}$$

$$Z'^3_0 := \bigcup_k (Z'^2_{j_2;k} \setminus Z''^{2,*}_{j_2;k})$$

and continue as above. Since $\dim(Z'^{i+1}_0) < \dim(Z'^i_0)$ for each i , this process stops after finitely many steps and we end up with a decomposition

$$V(f_1, \dots, f_n) = Z''^{1,*} \cup \dots \cup Z''^{j,*} \cup \bigcup_{\rho, k} \mathcal{Z}_k^{(\rho,*)}$$

and an open subset $L^* := \bigcap_{i,k} L_{ji;k}''^{i,*}$ of $\prod_i L_i$ such that under (condition $r./r._+ > \nu$) for a sufficiently large ν ,

$$\left(\bigcup_i Z''^{i,*} \times \{0\} \right) \cap C = \emptyset$$

for each $(g_1, \dots, g_n) \in L^*$. This immediately implies assertion (1a) of Theorem 4.8.

4.8.5. Proof of assertion (1b) of Theorem 4.8. For each $\mathcal{Z}_k^{(\rho)}$, consider the collection $\mathcal{I}_k^{(\rho)}$ of all $i^* = i_1 < \dots < i_\rho \leq n$ such that

- either $Z_{i_1, \dots, i_\rho}^{(\rho)} = \emptyset$, or
- $Z_{i_1, \dots, i_\rho}^{(\rho)} \neq \emptyset$, but $\mathcal{Z}_k^{(\rho)}$ is *not* an irreducible component of $Z_{i_1, \dots, i_\rho}^{(\rho)}$.

For each such choice of i_1, \dots, i_ρ , due to Lemma 4.6 there is a nonempty open subset $\mathcal{Z}_k'^{(\rho)}(i_1, \dots, i_\rho)$ of $\mathcal{Z}_k^{(\rho)}$, and a nonempty open subset $L_k'^{(\rho,*)}(i_1, \dots, i_\rho)$ of $\prod_i L_i$ such that the following holds under (condition $r./r._+ > \nu$) for sufficiently large ν :

- (i) for each $(g_1, \dots, g_n) \in L_k'^{(\rho,*)}(i_1, \dots, i_\rho)$, if $z \in \mathcal{Z}_k'^{(\rho)}(i_1, \dots, i_\rho)$ is such that $(z, 0) \in C$, then there is j such that $g_{i_j}(z) = 0$.

Moreover, due to Theorem 4.2 there is a nonempty open subset $\mathcal{Z}_k'^{(\rho,*)}$ of $\mathcal{Z}_k^{(\rho)}$, and a nonempty open subset $L_k'^{(\rho,*)}$ of $\prod_i L_i$ such that the following holds under (condition $r./r._+ > \nu$) for sufficiently large ν :

- (ii) for each $(g_1, \dots, g_n) \in L_k'^{(\rho,*)}$, if $z \in \mathcal{Z}_k'^{(\rho,*)}$ is such that $(z, 0) \in C$, then there are $i^* = i_1 < \dots < i_\rho \leq n$ such that $g_i(z) \neq 0$ if and only if $i = i_j$ for some j .

Now let

$$\mathcal{Z}_k''^{(\rho,*)} := \mathcal{Z}_k^{(\rho,*)} \cap \mathcal{Z}_k'^{(\rho,*)} \cap \bigcap_{(i_1, \dots, i_\rho) \in \mathcal{I}_k^{(\rho)}} \mathcal{Z}_k'^{(\rho,*)}(i_1, \dots, i_\rho)$$

where \mathcal{Z}_k^* are the open subsets given in assertion (1) of Theorem 4.8. Applying the arguments of the proof of assertion (1a) with $\mathcal{Z}_k''^{(\rho,*)}$ instead of $\mathcal{Z}_k^{(\rho,*)}$, we obtain an open subset L^* of $\prod_i L_i$ such that for sufficiently large ν the following holds under (condition $r./r._+ > \nu$):

- (iii) for each $(g_1, \dots, g_n) \in L^*$, if $(z, 0) \in C$ at $t = 0$, then $z \in \bigcup_{\rho, k} \mathcal{Z}_k''^{(\rho,*)}$.

Let

$$L''^* := L^* \cap \bigcap_{\rho, k} L_k'^{(\rho,*)} \cap \bigcap_{\rho, k, i_1, \dots, i_\rho} L_k'^{(\rho,*)}(i_1, \dots, i_\rho)$$

where the ρ, k vary over all irreducible components $\mathcal{Z}_k^{(\rho)}$ of all $\mathcal{Z}^{(\rho)}$, and (i_1, \dots, i_ρ) vary over $\mathcal{I}_k^{(\rho)}$. Then property (iii) implies that the following holds under (condition $r./r._+ > \nu$) for sufficiently large ν : for each $(g_1, \dots, g_n) \in L''^*$, if $(z, 0) \in C$ at $t = 0$, then $z \in \bigcup_k \mathcal{Z}_k''^{(\rho,*)}$. Moreover, for each such $z \in \mathcal{Z}_k''^{(\rho,*)}$, property (ii) implies that there are $i^* = i_1 < \dots < i_\rho \leq n$ such that $g_i(z) \neq 0$ if and only if $i = i_j$ for some j . However, then Z must be an irreducible component of $Z_{i_1, \dots, i_\rho}^{(\rho)}$ due to property (i). This completes the proof of assertion (1b), and consequently, the proof of Theorem 4.8. \square

5. COMBINATORICS OF INTERSECTIONS OF DIVISORS - GUESSING THE FORMULA FOR THE ORDERED INTERSECTION MULTIPLICITY

5.1. Consider effective Cartier divisors F_1, \dots, F_n on a variety X of pure dimension n . We allow for the scenario that $\bigcap \text{Supp}(F_i)$ has nonzero dimension, and try to express the intersection product (F_1, \dots, F_n) as a sum of contributions from the “distinguished components”, i.e. the irreducible components of $Z_{i_1, \dots, i_\rho}^{(\rho)}$ from Theorem 2.7. Recall from the definition of the $Z_{i_1, \dots, i_\rho}^{(\rho)}$ that for all such components, i_1 takes a fixed value, which we denoted as i^* . Moreover, $Z_{i_1, \dots, i_\rho}^{(\rho)}$ is constructed via an inductive step starting from irreducible components of $\text{Supp}(F_{i_1})$ which are contained in $\text{Supp}(F_{i_2})$, but *not* contained in $\text{Supp}(F_i)$ for any $i, i_1 < i < i_2$; in other words, for each such component Z of $\text{Supp}(F_{i_1})$, i_2 is the *smallest* value of i such that $Z \not\subseteq \text{Supp}(F_i)$.

5.2. Motivated by the observations in the preceding paragraph, we do the following:

- (i) Fix $i_1 := i^*, 1 \leq i^* \leq n$. Denote the irreducible components of F_{i_1} as $F_{i_1;j}$, so that as a Weil divisor F_{i_1} is of the form $\sum_j m_{i_1;j} F_{i_1;j}$.
- (ii) For each j we consider the smallest index $k_j > i_1$ such that $F_{i_1;j} \not\subseteq \text{Supp}(F_{k_j})$

Then

$$\begin{aligned} (F_1, \dots, F_n) &= (F_{i_1}, \dots, F_n, F_1, \dots, F_{i_1-1}) = \left(\sum_j m_{i_1;j} F_{i_1;j}, F_2, \dots, F_n, F_1, \dots, F_{i_1-1} \right) \\ &= \sum_j m_{i_1;j} (F_{i_1;j}, F_{k_j}, F_{k_j+1}, \dots, F_n, F_1, \dots, F_{i_1-1}, F_2, \dots, F_{k_j-1}) \end{aligned}$$

In line with the construction of the $Z_{i_1, \dots, i_\rho}^{(j)}$ from Theorem 2.7, we write $i_2 = k_j$, and rearrange the terms by fixing first a value of i_2 and summing over all j such that $k_j = i_2$ to obtain

$$\begin{aligned} (F_1, \dots, F_n) &= \sum_{i_2} \sum_{k_j=i_2} m_{i_1;j} (F_{i_1;j}, F_{i_2}, F_{i_2+1}, \dots, F_n, F_1, \dots, F_{i_1-1}, F_{i_1+1}, \dots, F_{i_2-1}) \\ &= \sum_{i_2} (F_{i_1;\Sigma i_2}, F_{i_2}, F_{i_2+1}, \dots, F_n, F_1, \dots, F_{i_1-1}, F_{i_1+1}, \dots, F_{i_2-1}) \end{aligned}$$

where $F_{i_1;\Sigma i_2}$ is the Weil divisor $\sum_{j:k_j=i_2} m_{i_1;j} F_{i_1;j}$.

5.3. Note that each irreducible component of $\text{Supp}(F_{i_1;\Sigma i_2}) \cap \text{Supp}(F_{i_2})$ has codimension two, so that the intersection

$$F_{i_1,i_2} := F_{i_1;\Sigma i_2} \cap F_{i_2}$$

can be represented as a codimension two cycle $\sum_j m_{i_1,i_2;j} F_{i_1,i_2;j}$ consisting of linear combination of irreducible components of $\text{Supp}(F_{i_1;\Sigma i_2}) \cap \text{Supp}(F_{i_2})$. Then

$$\begin{aligned} (F_1, \dots, F_n) &= \sum_{i_2} (F_{i_1,i_2}, F_{i_2+1}, \dots, F_n, F_1, \dots, F_{i_1-1}, F_{i_1+1}, \dots, F_{i_2-1}) \\ &= \sum_{i_2, j} m_{i_1,i_2;j} (F_{i_1,i_2;j}, F_{i_2+1}, \dots, F_n, F_1, \dots, F_{i_1-1}, F_{i_1+1}, \dots, F_{i_2-1}) \end{aligned}$$

We now apply the same procedure as in Section 5.2: for each j , write

$$\begin{aligned} (F_{i_1,i_2;j}, F_{i_2+1}, \dots, F_n, F_1, \dots, F_{i_1-1}, F_{i_1+1}, \dots, F_{i_2-1}) \\ = (F_{i_1,i_2;j}, F_{k'_j}, F_{k'_j+1}, \dots, F_n, F_1, \dots, F_{i_1-1}, F_{i_1+1}, \dots, F_{i_2-1}, F_{i_2+1}, \dots, F_{k'_j-1}) \end{aligned}$$

where k'_j is the smallest index such that $F_{i_1, i_2; j} \not\subseteq \text{Supp}(F_{k'_j})$. Then fix first a value of i_3 and sum over all j such that $k'_j = i_3$ to obtain

$$\begin{aligned} (F_1, \dots, F_n) &= \sum_{i_2, i_3} (F_{i_1, i_2; \Sigma i_3}, F_{i_3}, F_{i_3+1}, \dots, F_n, F_1, \dots, F_{i_1-1}, F_{i_1+1}, \dots, F_{i_2-1}, F_{i_2+1}, \dots, F_{i_3-1}) \\ &= \sum_{i_2, i_3} (F_{i_1, i_2, i_3}, F_{i_3+1}, \dots, F_n, F_1, \dots, F_{i_1-1}, F_{i_1+1}, \dots, F_{i_2-1}, F_{i_2+1}, \dots, F_{i_3-1}) \end{aligned}$$

where

$$\begin{aligned} F_{i_1, i_2; \Sigma i_3} &:= \sum_{k'_j = i_3} m_{i_1, i_2; j} F_{i_1, i_2; j}, \text{ and} \\ F_{i_1, i_2, i_3} &:= F_{i_1, i_2; \Sigma i_3} \cap F_{i_3} \end{aligned}$$

5.4. Repeating this process as many times as possible we obtain

$$\begin{aligned} (F_1, \dots, F_n) &= \sum_{\rho=1}^n \sum_{i^* = i_1 < \dots < i_\rho \leq n} (F_{i_1, \dots, i_\rho}, F_{i'_1}, \dots, F_{i'_{n-\rho}}) \\ &= \sum_{\rho=1}^n \sum_{i^* = i_1 < \dots < i_\rho \leq n} \sum_j m_{i_1, \dots, i_\rho; j} (F_{i_1, \dots, i_\rho; j}, F_{i'_1}, \dots, F_{i'_{n-\rho}}) \end{aligned}$$

where $1 \leq i'_1 < \dots < i'_{n-\rho} \leq n$ are the elements of $\{1, \dots, n\} \setminus \{i_1, \dots, i_\rho\}$. It follows from the construction of the $F_{i_1, \dots, i_\rho; j}$ that for each j ,

- (i) $F_{i_1, \dots, i_\rho; j} \subseteq \bigcap_{j'=1}^{n-\rho} \text{Supp}(F_{i'_{j'}})$ for each j , and
- (ii) $\text{codim}(F_{i_1, \dots, i_\rho; j}) = \rho$.

Due to observation (ii), we can rearrange the terms of the above sum to obtain

$$(F_1, \dots, F_n) = \sum_Z \sum_{F_{i_1, \dots, i_\rho; j} = Z} m_{i_1, \dots, i_\rho; j} \deg_{i'_1, \dots, i'_{n-\rho}}(Z)$$

where $\deg_{i'_1, \dots, i'_{n-\rho}}(Z) := (\mathcal{L}_{i'_1}|_Z, \dots, \mathcal{L}_{i'_{n-\rho}}|_Z)$, the outer sum is over all irreducible subvarieties Z of X , and the inner sum is over all $\rho, i_1, \dots, i_\rho, j$ such that $\rho = \text{codim}(Z)$ and $Z = F_{i_1, \dots, i_\rho; j}$.

5.5. We claim that $F_{i_1, \dots, i_\rho; j}$ from Section 5.4 are precisely the irreducible components of $Z_{i_1, \dots, i_\rho}^{(\rho)}$ from Theorem 2.7. Indeed, recall the definition of $Z_{i_1, \dots, i_\rho}^{(j)}$:

- (a) $Z_{i_1, \dots, i_\rho}^{(0)} = X$,
- (b) for $1 \leq j \leq \rho - 1$, $Z_{i_1, \dots, i_\rho}^{(j)}$ is the union of all irreducible components of $Z_{i_1, \dots, i_\rho}^{(j-1)} \cap \text{Supp}(F_{i_j})$ which are not contained in $\bigcup_{j' > j} \text{Supp}(F_{i_{j'}})$, but are contained in $\text{Supp}(F_{i_i})$ for each $i, i_j < i < i_{j+1}$,
- (c) $Z_{i_1, \dots, i_\rho}^{(\rho)}$ is the union of all irreducible components of $Z_{i_1, \dots, i_\rho}^{(\rho-1)} \cap \text{Supp}(F_{i_\rho})$ which are contained in $\bigcap_{i=1}^n \text{Supp}(F_i)$.

It is clear from the definition that $\text{Supp}(F_{i_1; \Sigma i_2})$ is the union of $Z_{i_1, \dots, i_\rho}^{(1)}$ and a (possibly empty) collection of irreducible components of $\text{Supp}(F_{i_1})$ which are contained in $\bigcup_{j'=2}^\rho \text{Supp}(F_{i_{j'}})$. By an easy

induction then it follows that $\text{Supp}(F_{i_1, \dots, i_{k-1}; \Sigma i_k})$ is the union of $Z_{i_1, \dots, i_\rho}^{(k-1)}$ and a (possibly empty) collection of irreducible components which are contained in $\bigcup_{j'=k+1}^\rho \text{Supp}(F_{i_{j'}})$. However, this means that $\text{Supp}(F_{i_1, \dots, i_{\rho-1}; \Sigma i_\rho})$ is precisely $Z_{i_1, \dots, i_\rho}^{(\rho-1)}$. This in turn implies (due to observation (i) from Section 5.4) that $Z_{i_1, \dots, i_\rho}^{(\rho)}$ is precisely the union of the $F_{i_1, \dots, i_\rho; j}$, as claimed.

5.6. Now we compute the numbers $m_{i_1, \dots, i_\rho; j}$ from Section 5.4. Take an affine open subset U of X such that U intersects $F_{i_1, \dots, i_\rho; j}$, and each F_i is the divisor of some $f_i \in \mathbb{k}[U]$. Recall the definition of $I_{i_1, \dots, i_\rho}^{(k)}$ from Theorem 2.11:

- (a) $I_{i_1, \dots, i_\rho}^{(1)}$ is the principal ideal generated by f_{i_1} ,
- (b) for $2 \leq k \leq \rho$, $I_{i_1, \dots, i_\rho}^{(k)}$ is the ideal generated by f_{i_k} and the intersection of all primary components \mathfrak{q} of $I_{i_1, \dots, i_\rho}^{(k-1)}$ such that the corresponding prime ideal $\sqrt{\mathfrak{q}}$ contains f_i for each $i < i_j$, but does *not* contain f_{i_k} .

It is straightforward to check from the definitions and Proposition A.2 that for each $k \leq \rho$, the subscheme of U defined by $I_{i_1, \dots, i_\rho}^{(k)}$ is precisely the cycle F_{i_1, \dots, i_k} constructed via the inductive procedure in Sections 5.2 to 5.4. In particular, $m_{i_1, \dots, i_\rho; j}$ is precisely the multiplicity of $Z = F_{i_1, \dots, i_\rho; j}$ in $V(I_{i_1, \dots, i_\rho}^{(\rho)})$. Consequently,

$$(2) \quad m_{i_1, \dots, i_\rho; j} = \text{len}_{\mathcal{O}_{Z, V(I_{i_1, \dots, i_\rho}^{(\rho)})}}(\mathcal{O}_{Z, V(I_{i_1, \dots, i_\rho}^{(\rho)})}) = \text{len}(\mathcal{O}_{Z, X} / I_{i_1, \dots, i_\rho}^{(\rho)} \mathcal{O}_{Z, X})$$

We summarize the observations from Sections 5.4 to 5.6 in the following theorem:

Theorem 5.7. *Let F_1, \dots, F_n be effective Cartier divisors on a variety X of pure dimension n . Fix i^* , $1 \leq i^* \leq n$. Then*

$$(F_1, \dots, F_n) = \sum_Z \sum_{i_1, \dots, i_\rho} \text{len}(\mathcal{O}_{Z, X} / I_{i_1, \dots, i_\rho}^{(\rho)} \mathcal{O}_{Z, X}) \deg_{i'_1, \dots, i'_{n-\rho}}(Z)$$

where the outer sum is over all irreducible subvarieties Z of X , the inner sum is over all i_1, \dots, i_ρ where $\rho := \text{codim}(Z)$ and $i^* = i_1 < \dots < i_\rho \leq n$ are such that Z is an irreducible component of $Z_{i_1, \dots, i_\rho}^{(\rho)}$ from Section 5.5, the $I_{i_1, \dots, i_\rho}^{(\rho)}$ are as in Section 5.6, and finally, $1 \leq i'_1 < \dots < i'_{n-\rho} \leq n$ are the elements of $\{1, \dots, n\} \setminus \{i_1, \dots, i_\rho\}$, and $\deg_{i'_1, \dots, i'_{n-\rho}}(Z)$ is the number (counted with appropriate multiplicities) of points in the intersection of Z and generic Cartier divisors linearly equivalent to $F_{i'_j}$, $j = 1, \dots, n - \rho$. \square

5.8. Based on Theorem 5.7 we may guess that the ordered intersection multiplicity of F_1, \dots, F_n along an irreducible subvariety Z equals

$$\sum_{i_1, \dots, i_\rho} \text{len}(\mathcal{O}_{Z, X} / I_{i_1, \dots, i_\rho}^{(\rho)} \mathcal{O}_{Z, X}) \deg_{i'_1, \dots, i'_{n-\rho}}(Z)$$

where $\rho := \text{codim}(Z)$, and the sum is over all i_1, \dots, i_ρ such that Z is an irreducible component of $Z_{i_1, \dots, i_\rho}^{(\rho)}$. In the next section we show that this guess is indeed correct.

6. PROOF OF THE FORMULA FOR ORDERED INTERSECTION MULTIPLICITY

The main result of this section is the computation of the formula of the ordered intersection multiplicity (Theorem 6.4). All our main results on ordered intersection multiplicity, namely Theorems 2.6, 2.7 and 2.11, are straightforward consequences of Theorem 6.4 combined with Theorems 4.8 and 5.7.

6.1. We continue with the set up from Section 4.1. In addition we assume that for each i ,

- (1) L_i is a base-point free (finite dimensional) vector subspace of $\mathbb{k}[X]$ containing f_i .

By \bar{X} we denote a compactification⁷ of X such that

- (2) there are linear systems \mathcal{L}_i on \bar{X} such that $L_i = \mathcal{L}_i|_X$, $i = 1, \dots, n$; in particular, for each i , either $f_i \equiv 0$, or there is a Cartier divisor $F_i \in \mathcal{L}_i$ such that f_i is a local representative of F_i on X .

(For property (2) one can e.g. take an arbitrary completion \bar{X}_0 of X , and then define \bar{X} to be the closure in $\bar{X}_0 \times \prod_{i=1}^n \mathbb{P}^{n_i}$ where $d_i := \dim(L_i)$, of the graph of the morphism $x \mapsto (x, [1 : g_{1,1}(x) : \dots : g_{1,d_1}(x)], \dots, [1 : g_{n,1}(x) : \dots : g_{n,d_n}(x)])$, where $g_{i,1}, \dots, g_{i,d_i}$ form a basis of L_i .)

6.2. Consider subvarieties $Z_{i_1, \dots, i_{\rho'}}^{(\rho')}$ defined in the same way as in Theorem 2.7, but with \bar{X} in place of X , and as usual, let $\mathcal{Z}^{(\rho')}$ be the union of $Z_{i_1, \dots, i_{\rho'}}^{(\rho')}$ over all $i_1, \dots, i_{\rho'}$, and let $\mathcal{Z}_k^{(\rho')}$, $k = 1, 2, \dots$, be the irreducible components of $\mathcal{Z}^{(\rho')}$. Pick nonempty Zariski open subsets $\mathcal{Z}_k^{(\rho',*)}$ of $\mathcal{Z}_k^{(\rho')}$ such that

- (3) $\mathcal{Z}_{k_1}^{(\rho'_1,*)} \cap \mathcal{Z}_{k_2}^{(\rho'_2,*)} = \emptyset$ unless $\rho'_1 = \rho'_2$ and $k_1 = k_2$.

Due to Theorem 4.8 we can pick $\nu_1^* \in \mathbb{R}$ and a nonempty Zariski open subset L_1^* of $\prod_i L_i$ such that if (condition $r./r.+ > \nu$) holds with $\nu \geq \nu_1^*$, then for each $(g_1, \dots, g_n) \in L_1^*$ and each point $(z, 0)$ on $C = C(g_1, \dots, g_n; r_1, \dots, r_n)$ at $t = 0$, there are unique integers ρ_z , k_z , and $i^* = i_1(z) < \dots < i_{\rho_z}(z) \leq n$, and $h_{\rho_z, k_z, i} \in \mathbb{k}[X]$, $i = 1, \dots, n$, such that

- (4) $z \in \mathcal{Z}_{k_z}^{(\rho_z,*)}$,
(5) $\mathcal{Z}_{k_z}^{(\rho_z)}$ is an irreducible component of $Z_{i_1(z), \dots, i_{\rho_z}(z)}^{(\rho_z)}$,
(6) there is a neighborhood of $\mathcal{Z}_{k_z}^{(\rho_z,*)}$ in \bar{X} such that for each $i = 1, \dots, n$,
(a) F_i is locally represented by $f_i/h_{\rho_z, k_z, i}$,
(b) $h/h_{\rho_z, k_z, i}$ is a regular function for each $h \in L_i$,
(7) on a punctured neighborhood of $(z, 0)$ in $\bar{X} \times \mathbb{k}$, C is defined by $f_i/h_{\rho_z, k_z, i} - t^{r_i} g_i/h_{\rho_z, k_z, i} = 0$, $i = 1, \dots, n$,
(8) $(g_i/h_{\rho_z, k_z, i})(z) \neq 0$ if and only if $i = i_j(z)$ for some j .

Due to property (2) (from Section 6.1) we may assume that

- (9) if $\mathcal{Z}_k^{(\rho')}$ is the closure of a subvariety of X , then $\mathcal{Z}_k^{(\rho',*)} \subseteq X \cap Z$ and $h_{\rho', k, i} = 1$, $i = 1, \dots, n$.

Restricting L_1^* to an appropriate open subset if necessary, we may further ensure that

- (10) for each $\mathcal{Z}_k^{(\rho',*)}$ which is an irreducible component of $Z_{i_1, \dots, i_{\rho'}}^{(\rho')}$,

$$|\mathcal{Z}_k^{(\rho',*)} \cap V(\frac{g_{i'_1}}{h_{\rho', k, i'_1}}, \dots, \frac{g_{i'_{n-\rho'}}}{h_{\rho', k, i'_{n-\rho'}}})| < \infty$$

⁷ i.e. a complete variety containing X as a dense subset.

where $i'_1, \dots, i'_{n-\rho'}$ are the elements in the complement of $\{1, \dots, n\} \setminus \{i_1, \dots, i_{\rho'}\}$. Define

$$\mathcal{Z}'_0 := \bigcup_{\rho', k} \left(\mathcal{Z}_k^{(\rho', *)} \cap V\left(\frac{g_{i'_1}}{h_{\rho', k, i'_1}}, \dots, \frac{g_{i'_{n-\rho'}}}{h_{\rho', k, i'_{n-\rho}}}\right) \right)$$

6.3. Given $\vec{g} = (g_1, \dots, g_n) \in \prod_i L_i$, we write below $C(\vec{g})$ for C to explicitly mention the dependence on \vec{g} . For $\vec{a} = (a_1, \dots, a_n) \in \mathbb{k}^n$, we write $\vec{a}\vec{g} := (a_1 g_1, \dots, a_n g_n)$. We now introduce a birational modification \tilde{X} of $\bar{X} \times \mathbb{k}$ on which the image of $C(\vec{a}\vec{g})$ can be explicitly described for generic $\vec{a} \in \mathbb{k}^n$. Indeed, let $W := \mathbb{P}^n(r_1, \dots, r_n, 1)$ be the n -dimensional weighted projective space with weights $(r_1, \dots, r_n, 1)$, and \tilde{X} be the closure in $\bar{X} \times \mathbb{k} \times W$ of the graph of the map

$$\bar{X} \times \mathbb{k} \ni (x, t) \mapsto \left[\frac{f_1}{g_1}(x) : \dots : \frac{f_n}{g_n}(x) : t \right] \in W$$

Given $\vec{a} \in (\mathbb{k} \setminus \{0\})^n$, let $\tilde{C}(\vec{a}\vec{g})$ be the “scheme-theoretic strict transform”⁸ of $C(\vec{a}\vec{g})$ on \tilde{X} . If $(z, 0) \in C(\vec{a}\vec{g})$ and $\tilde{z} \in \tilde{C}(\vec{a}\vec{g})$ is a pre-image of $(z, 0)$, then it is straightforward to see that $\tilde{z} \in \tilde{U}_{n+1} := \pi_W^{-1}(W \setminus V(w_{n+1}))$, where $\pi_W : \tilde{X} \rightarrow W$ is the natural projection and $[w_1 : \dots : w_{n+1}]$ are (weighted) homogeneous coordinates on W . Note that $u_i := w_i/(w_{n+1})^{r_i}$, $i = 1, \dots, n$, are well-defined regular functions on \tilde{U}_{n+1} .

Claim 6.3.1. *Consider the set up in Sections 6.1 and 6.2. Pick $\nu \geq \nu_1^*$ and r_1, \dots, r_n satisfying (condition $r./r. + > \nu$), and $\vec{g} \in L_1^*$ (where ν_1^* and L_1^* are from Section 6.2). There is a nonempty Zariski open subset \mathcal{A}^* of $(\mathbb{k} \setminus \{0\})^n$ such that for each $\vec{a} = (a_1, \dots, a_n) \in \mathcal{A}^*$,*

- (1) $C(\vec{a}\vec{g}) \cap (\bar{X} \times \{0\})$ is set-theoretically the same as $\mathcal{Z}'_0 \times \{0\}$ (where \mathcal{Z}'_0 is from property (10) of Section 6.2),
- (2) $\tilde{C}(\vec{a}\vec{g})$ is scheme-theoretically equal to $V(u_1 - a_1, \dots, u_n - a_n)$ near $\pi^{-1}(X \times \{0\}) \cap \tilde{U}_{n+1}$ (where $\pi : \tilde{X} \rightarrow X \times \mathbb{k}$ is the projection map).

Let $\tilde{\mathcal{E}}_0$ be the (finite) collection of n -dimensional subvarieties E of $\pi^{-1}(X \times \{0\})$ such that $E \cap \tilde{U}_{n+1} \neq \emptyset$, and $\pi_W|_E : E \rightarrow W$ is generically finite. Then

- (3) for each $\vec{a} = (a_1, \dots, a_n) \in \mathcal{A}^*$ and for each $z \in \mathcal{Z}'_0$,

$$\text{len}(\mathcal{O}_{(z,0), C(\vec{a}\vec{g})} / t \mathcal{O}_{(z,0), C(\vec{a}\vec{g})}) = \sum_{E \in \tilde{\mathcal{E}}_0} \text{ord}_E(t) \deg(\pi_W|_E)$$

Proof. Since L_1^* is open (and nonempty), there is a nonempty open subset \mathcal{A}_1^* of $(\mathbb{k} \setminus \{0\})^n$ such that $\vec{a}\vec{g} \in L_1^*$ for all $\vec{a} \in \mathcal{A}_1^*$. Assertion (1) follows (with $\mathcal{A}^* = \mathcal{A}_1^*$) from properties (8) and (10) of L_1^* from Section 6.2. Since $\pi_W|_{\tilde{U}_{n+1}}$ is defined by (u_1, \dots, u_n) , by definition of $\tilde{\mathcal{E}}_0$ there is a nonempty open subset \mathcal{A}^* of \mathcal{A}_1^* such that for each $\vec{a} \in \mathcal{A}^*$, the subscheme $\tilde{C}_{\vec{a}} := V(u_1 - a_1, \dots, u_n - a_n)$ of \tilde{U}_{n+1}

- (i) intersects each $E \in \tilde{\mathcal{E}}_0$ at finitely many points,
- (ii) is purely one dimensional near every such point,
- (iii) does *not* intersect any points in $E \cap E'$ for any $E \neq E' \in \tilde{\mathcal{E}}_0$,
- (iv) does *not* intersect any irreducible component of $\pi^{-1}(X \times \{0\})$ other than those in $\tilde{\mathcal{E}}_0$.

⁸ The scheme-structure on $\tilde{C}(\vec{a}\vec{g})$ is given as follows: near all but finitely many points of $C(\vec{a}\vec{g})$, X is locally isomorphic to \tilde{X} , so that the scheme-structure on $C(\vec{a}\vec{g})$ can be uniquely transferred to $\tilde{C}(\vec{a}\vec{g})$. Over the remaining points the scheme-structure can be extended by taking the scheme-theoretic closure.

For each $\vec{a} \in \mathcal{A}^*$ and each $z \in X$, it follows that $\tilde{C}(\vec{a}\vec{g}) = \tilde{C}_{\vec{a}}$ near $\pi^{-1}(z, 0) \cap \tilde{U}_{n+1}$, and

$$\begin{aligned} \text{len}(\mathcal{O}_{(z,0),C(\vec{a}\vec{g})}/t\mathcal{O}_{(z,0),C(\vec{a}\vec{g})}) &= \sum_{\tilde{z} \in \pi^{-1}(z,0)} \text{len}(\mathcal{O}_{\tilde{z},\tilde{C}(\vec{a}\vec{g})}/t\mathcal{O}_{\tilde{z},\tilde{C}(\vec{a}\vec{g})}) \\ &= \sum_{E \in \tilde{\mathcal{E}}_0} \text{ord}_E(t) \deg((\pi_W)|_E) \end{aligned}$$

which completes the proof of Claim **6.3.1**. \square

6.3.2. Let $\nu, r_1, \dots, r_n, \vec{g}, \mathcal{A}^*$ be as in Claim **6.3.1**. For a positive integer m , define

$$\begin{aligned} \mathcal{A}_m^* &:= \{\vec{a} \in \mathcal{A}^* : \vec{\mu}\vec{a} := (\mu_1 a_1, \dots, \mu_n a_n) \in \mathcal{A}^* \text{ for all } \vec{\mu} := (\mu_1, \dots, \mu_n) \\ &\quad \text{such that } \mu_1^m = \dots = \mu_n^m = 1\} \end{aligned}$$

Then \mathcal{A}_m^* is a nonempty Zariski open subset of \mathcal{A}^* . Pick $\vec{a} \in \mathcal{A}_m^*$ and $z \in \bar{X}$ such that $(z, 0) \in C(\vec{a}\vec{g})$. For convenience write $\rho' = \rho_z$, $k = k_z$ and $i_j = i_j(z)$, $j = 1, \dots, \rho'$ (where $\rho_z, k_z, i_j(z)$ are as in Section **6.2**). Let $i'_1 < \dots < i'_{n-\rho'}$ be the elements of $\{1, \dots, n\} \setminus \{i_1, \dots, i_{\rho'}\}$.

Corollary 6.3.3. Let $C_{m,i_1,\dots,i_{\rho'}}(\vec{a}\vec{g}; \vec{r})$ (or in short, $C_{m,i_1,\dots,i_{\rho'}}(\vec{a}\vec{g})$) be the scheme-theoretic closure in $X \times \mathbb{k}$ of the subscheme of $X \times (\mathbb{k} \setminus \{0\})$ defined by the following equations:

$$\begin{aligned} f_{i_j} - a_{i_j} t^{r_{i_j}} g_{i_j} &= 0, & j &= 1, \dots, \rho', \\ (f_{i'_j})^m - (a_{i'_j})^m t^{mr_{i'_j}} (g_{i'_j})^m &= 0, & j &= 1, \dots, n - \rho', \end{aligned}$$

where $i'_1 < \dots < i'_{n-\rho'}$ are the elements of $\{1, \dots, n\} \setminus \{i_1, \dots, i_{\rho'}\}$. Then

$$\text{len}(\mathcal{O}_{(z,0),C(\vec{a}\vec{g})}/t\mathcal{O}_{(z,0),C(\vec{a}\vec{g})}) = \frac{1}{m^{n-\rho'}} \text{len}(\mathcal{O}_{(z,0),C_{m,i_1,\dots,i_{\rho'}}(\vec{a}\vec{g})}/t\mathcal{O}_{(z,0),C_{m,i_1,\dots,i_{\rho'}}(\vec{a}\vec{g})})$$

Proof. Let $\mathcal{M}_{i_1,\dots,i_{\rho'}}$ be the set of all $(\mu_1, \dots, \mu_n) \in (\mathbb{k} \setminus \{0\})^n$ such that $\mu_1^m = \dots = \mu_n^m = 1$, and in addition, $\mu_{i_j} = 1$, $j = 1, \dots, \rho'$. Since $|\mathcal{M}_{i_1,\dots,i_{\rho'}}| = m^{n-\rho'}$, Claim **6.3.1** implies that

$$\text{len}(\mathcal{O}_{(z,0),C(\vec{a}\vec{g})}/t\mathcal{O}_{(z,0),C(\vec{a}\vec{g})}) = \frac{1}{m^{n-\rho'}} \sum_{\vec{\mu} \in \mathcal{M}_{i_1,\dots,i_{\rho'}}} \text{len}(\mathcal{O}_{(z,0),C(\vec{\mu}\vec{a}\vec{g})}/t\mathcal{O}_{(z,0),C(\vec{\mu}\vec{a}\vec{g})})$$

The right hand side of the identity above equals $\frac{1}{m^{n-\rho'}} \text{len}(\mathcal{O}_{(z,0),C_{m,i_1,\dots,i_{\rho'}}(\vec{a}\vec{g})}/t\mathcal{O}_{(z,0),C_{m,i_1,\dots,i_{\rho'}}(\vec{a}\vec{g})})$ due to the additivity of the “order” function of a non zero-divisor over branches of a curve (see e.g. [Mon21, Theorem IV.24]). \square

Proposition 6.3.4. Let $Z', i_1, \dots, i_{\rho'}$ be such that Z' is an irreducible component of $Z_{i_1,\dots,i_{\rho'}}^{(\rho')}$, and let r_1, \dots, r_n be as in Claim **6.3.1**. Then there is a nonempty open subset L^* of $\prod_i L_i$ such that the following sum is constant for each $g \in L^*$:

$$\sum_{\substack{z \in Z' \\ Z_{k_z}^{(\rho_z)} = Z' \\ (i_1(z), \dots, i_{\rho_z}(z)) = (i_1, \dots, i_{\rho'})}} \text{len}(\mathcal{O}_{(z,0),C(\vec{g})}/t\mathcal{O}_{(z,0),C(\vec{g})})$$

Proof. Let ν, r_1, \dots, r_n and $W := \mathbb{P}^n(r_1, \dots, r_n, 1)$ be as in Claim **6.3.1**, and let Y be the closure in $\prod_i L_i \times \bar{X} \times \mathbb{k} \times W$ of the graph of the map

$$\prod_i L_i \times \bar{X} \times \mathbb{k} \ni (\vec{g}, x, t) \mapsto [\frac{f_1}{g_1}(x) : \dots : \frac{f_n}{g_n}(x) : t] \in W$$

Fix $i_1, \dots, i_{\rho'}$ such that Z' is an irreducible component of $Z_{i_1, \dots, i_{\rho'}}^{(\rho')}$, and consider

$$Y_0 := \{(\vec{g}, x, t) \in Y : x \in Z, g_{i_1'}(x) = \dots = g_{i_{n-\rho'}'}(x) = t = 0\},$$

where the i_j' are the elements of $\{1, \dots, n\} \setminus \{i_1, \dots, i_{\rho'}\}$. Note that Y_0 is a (possibly reducible) closed subvariety of Y . Let $[w_1 : \dots : w_{n+1}]$ be (weighted) homogeneous coordinates on W , and $u_i := w_i / (w_{n+1})^{r_i}$, $i = 1, \dots, n$, be the coordinates on $U_{n+1} := W \setminus V(w_{n+1})$. Let

$$\tilde{L}_0 := Y_0 \cap V(u_1 - 1, \dots, u_n - 1) \cap \pi_W^{-1}(U_{n+1})$$

where $\pi_W : Y \rightarrow W$ is the natural projection. Let $\pi_L : Y \rightarrow \prod_i L_i$ is the natural projection. If $\pi_L|_{\tilde{L}_0}$ is *not* dominant, then Proposition 6.3.4 holds with the required sum equal to zero. So assume $\pi_L|_{\tilde{L}_0}$ is dominant. Then the arguments from the proof of Claim 6.3.1 imply that

- (i) there is a nonempty open subset L_0^* of $\prod_i L_i$ over which each fiber of $\pi_L|_{\tilde{L}_0}$ is zero dimensional, and each fiber of $\pi_L|_{Y_0}$ has pure dimension n .

In particular, Y_0 has codimension one in Y , and consequently, some irreducible components of Y_0 are also irreducible components of $V(t) \subseteq Y$. The observations from the proof of Claim 6.3.1 then imply that

- (ii) there is a nonempty open subset L_1^* of L_0^* such that on $\tilde{U}_{n+1}^1 := \pi_W^{-1}(U_{n+1}) \cap \pi_L^{-1}(L_1^*)$, we can consider \tilde{L}_0 as the (restriction of the) closed subscheme $V(t, u_1 - 1, \dots, u_n - 1)$.
- (iii) Denote the irreducible components of $\tilde{L}_0 \cap \tilde{U}_{n+1}^1$ by $\tilde{L}_{0,i}$. Then for each \vec{g} in a nonempty open subset L_2^* of L_1^* , the “scheme-theoretic strict transform” (see Footnote 8 on Page 24) $\tilde{C}(\vec{g})$ of $C(\vec{g})$ on Y is the same as the subscheme $V(u_1 - 1, \dots, u_n - 1)$ near $\{\vec{g}\} \times Z \times \{0\}$, and

$$\sum_{\substack{z \in Z' \\ \mathcal{Z}_{k_z}^{(\rho_z)} = Z' \\ (i_1(z), \dots, i_{\rho_z}(z)) = (i_1, \dots, i_{\rho'})}} \text{len}(\mathcal{O}_{(z,0),C(\vec{g})}/t\mathcal{O}_{(z,0),C(\vec{g})}) = \sum_i \text{len}(\mathcal{O}_{\tilde{L}_{0,i},\tilde{L}_0}) \deg(\pi_L|_{\tilde{L}_{0,i}})$$

Since the right hand side does *not* depend on $\vec{g} \in L_2^*$, this completes the proof of Proposition 6.3.4 with $L^* = L_2^*$. \square

The following theorem is the main result of this section. It confirms that the formula for the ordered intersection multiplicity guessed in Section 5.8 is the correct one. For this result X is an n -dimensional affine variety, $f_1, \dots, f_n \in \mathbb{k}[X]$, and L_i are base-point free (finite dimensional) vector subspace of $\mathbb{k}[X]$ containing f_i , $i = 1, \dots, n$. Let Z be a distinguished component for (the ordered intersection of the hypersurfaces defined by) $V(f_1), \dots, V(f_n)$ (Definition 2.8), and \check{Z} be the intersection of Z with the union of all other distinguished components Z' with $\text{codim}(Z') \geq \text{codim}(Z)$.

Theorem 6.4. *There is $\nu \in \mathbb{R}$ with the following property: for each $\vec{r} = (r_1, \dots, r_n)$ satisfying (condition $r./r._+ > \nu$) and each nonempty open subset Z^* of Z , there is a nonempty open subset L^* of $\prod_i L_i$ such that for each $\vec{g} = (g_1, \dots, g_n) \in L^*$,*

$$\sum_{\substack{\text{branch } B \text{ of } C(\vec{g};\vec{r}) \\ \text{center}(B) \in (Z^* \setminus \check{Z}) \times \{0\}}} \text{ord}_B(t) = \sum_{i_1, \dots, i_{\rho}} \text{len}(\mathcal{O}_{Z,X}/I_{i_1, \dots, i_{\rho}}^{(\rho)} \mathcal{O}_{Z,X}) \deg_{i_1', \dots, i_{n-\rho}'}(Z)$$

where the i_j' on the right hand side are the elements of $\{1, \dots, n\} \setminus \{i_1, \dots, i_{\rho}\}$, and the sum is over all i_1, \dots, i_{ρ} such that Z is an irreducible component of $Z_{i_1, \dots, i_{\rho}}^{(\rho)}$.

Proof. The proof of Theorem 6.4 straddles from Paragraphs 6.4.1 to 6.4.9 below.

6.4.1. Fix i_1, \dots, i_ρ such that Z is an irreducible component of $Z_{i_1, \dots, i_\rho}^{(\rho)}$. For each $j = 1, \dots, \rho - 1$,

- let $I_{i_1, \dots, i_\rho}^{(j)}$ be the intersection of all primary components \mathfrak{q} of $I_{i_1, \dots, i_\rho}^{(j)}$ which are used in the construction of $I_{i_1, \dots, i_\rho}^{(j+1)}$, i.e. $f_{i_{j+1}} \notin \sqrt{\mathfrak{q}}$, and $f_i \in \sqrt{\mathfrak{q}}$ for each $i < i_j$;
- let $I_{i_1, \dots, i_\rho}^{\prime\prime(j)}$ be the intersection of all other primary components \mathfrak{q} of $I_{i_1, \dots, i_\rho}^{(j)}$.

It is then clear that for each $j = 1, \dots, \rho - 1$,

$$\begin{aligned} I_{i_1, \dots, i_\rho}^{(j)} &= I_{i_1, \dots, i_\rho}^{\prime(j)} \cap I_{i_1, \dots, i_\rho}^{\prime\prime(j)}, \text{ and} \\ I_{i_1, \dots, i_\rho}^{(j+1)} &= I_{i_1, \dots, i_\rho}^{\prime(j)} + f_{i_{j+1}} \mathbb{k}[X] \end{aligned}$$

We will prove the following result in Appendix C below.

Claim 6.4.2. *There is a nonempty open subset Z_0^* of Z and $\nu_0^* \in \mathbb{R}$ such that if $z \in Z_0^*$ and $g_1, \dots, g_n, r_1, \dots, r_n$ satisfy $(\emptyset(\nu))$ with $\nu \geq \nu_0^*$, then*

- (1) *for each j , $1 \leq j \leq \rho$,*

$$I_{i_1, \dots, i_\rho}^{(j)} \hat{\mathcal{O}}_{(z,0),C} \subseteq t^{r_{i_j}} \hat{\mathcal{O}}_{(z,0),C}$$

In addition, Z_0^ does not change (but ν_0^* possibly changes) if the f_i are replaced by $(f_i)^{m_i}$ for $m_i \geq 1$.*

Pick $i' \in \{1, \dots, n\} \setminus \{i_1, \dots, i_\rho\}$. Recall that $i_1 = i^$, where i^* is the smallest index such that F_{i^*} is not identically zero.*

- (2) *If $i' < i^*$, then $g_{i'}|_C \equiv 0 \in t\hat{\mathcal{O}}_{(z,0),C}$.*
- (3) *Assume $i' > i^*$. Pick the largest j such that $i_j < i'$. Then $f_{i'} \in \sqrt{I_{i_1, \dots, i_\rho}^{(j)}}$. If $f_{i'} \in I_{i_1, \dots, i_\rho}^{\prime(j)}$, then $g_{i'} \in t\hat{\mathcal{O}}_{(z,0),C}$.*

6.4.3. We now apply the constructions of Sections 6.1 and 6.2. In particular, we assume \bar{X} is a compactification of X , and for each $i = 1, \dots, n$, F_i is either identically zero (in the case that f_i is identically zero), or a Cartier divisor on \bar{X} extending $V(f_i) \subseteq X$. Let $\mathcal{Z}_k^{(\rho',*)}$ be defined as in Section 6.2 with the following additional constraint:

- (i) $\mathcal{Z}_k^{(\rho',*)} \subseteq Z^*$, where ρ^*, k^* are the unique indices such that $\mathcal{Z}_{k^*}^{(\rho^*)}$ is the closure of Z in \bar{X} (recall that Z^* is the nonempty open subset of Z from the statement of Theorem 6.4).

For each ρ', k , recall that F_i is locally represented near $\mathcal{Z}_k^{(\rho',*)}$ by $f_i/h_{\rho',k,i}$ (Property (6) of Section 6.2). Consider the ideals $I_{\rho',k;i_1, \dots, i_{\rho'}}^{(j)}$ obtained by applying the construction of $I_{i_1, \dots, i_{\rho'}}^{(j)}$ from Theorem 2.11 with $f_i/h_{\rho',k,i}$ in place of f_i , $i = 1, \dots, n$. Replacing the $\mathcal{Z}_k^{(\rho',*)}$ by appropriate open subsets if necessary, we may ensure that for each ρ', k ,

- (ii) $\mathcal{Z}_k^{(\rho',*)}$ does not intersect any irreducible component of $\bigcup_{i_1, \dots, i_{\rho'}} V(I_{i_1, \dots, i_{\rho'}}^{(\rho')})$ other than $\mathcal{Z}_k^{(\rho')}$ itself.

6.4.4. Analogously, define $I_{\rho',k;i_1, \dots, i_{\rho'}}^{\prime(j)}$ as in Paragraph 6.4.1, and pick $M \geq 1$ such that $(f_{i'}/h_{\rho',k,i'})^M \in I_{\rho',k;i_1, \dots, i_{\rho'}}^{\prime(j)}$ for each $k, \rho', i_1, \dots, i_{\rho'}$, and each j, l such that

- either $i_j < i' < i_{j+1}$, or
- $j = \rho$ and $i' > i_{\rho'}$.

For each $k, \rho', i_1, \dots, i_{\rho'}$, and $m \geq M$, define $C_{m, i_1, \dots, i_{\rho'}}(\vec{g}; \vec{r})$ as in Corollary 6.3.3. Claim 6.4.2 implies that there are nonempty open subset $\mathcal{Z}_{k,0}^{(\rho',*)} \subseteq \mathcal{Z}_k^{(\rho',*)}$ and $\nu_m^* \in \mathbb{R}$ such that for each $k, \rho', i_1, \dots, i_{\rho'}$, if (condition $r./r.+ > \nu$) holds with $\nu \geq \nu_m^*$, and $z \in \mathcal{Z}_{k,0}^{(\rho',*)}$ is such that $(z, 0) \in C_{m, i_1, \dots, i_{\rho'}}(\vec{g}; \vec{r})$ and $(g_{i_j}/h_{\rho', k, i_j})(z) \neq 0$, $1 \leq j \leq \rho'$, then

- (i) $I_{\rho', k; i_1, \dots, i_{\rho'}}^{(\rho')} \hat{\mathcal{O}}_{(z,0), C_{m, i_1, \dots, i_{\rho'}}(\vec{g}; \vec{r})} \subseteq t\hat{\mathcal{O}}_{(z,0), C_{m, i_1, \dots, i_{\rho'}}(\vec{g}; \vec{r})}$,
- (ii) $(g_{i'_j}/h_{\rho', k, i'_j})^m \in t\hat{\mathcal{O}}_{(z,0), C_{m, i_1, \dots, i_{\rho'}}(\vec{g}; \vec{r})}$ for each $j = 1, \dots, n - \rho'$, where $i'_1 < \dots < i'_{n-\rho'}$ are the elements of $\{1, \dots, n\} \setminus \{i_1, \dots, i_{\rho'}\}$.

6.4.5. Now run the constructions of Section 6.2 with all $\mathcal{Z}_k^{(\rho',*)}$ replaced by $\mathcal{Z}_{k,0}^{(\rho',*)}$ from Paragraph 6.4.4. Consider resulting $\nu_1^* \in \mathbb{R}$ and $L_1^* \subseteq \prod_i L_i$ from Section 6.2. Pick $\nu \geq \nu_1^*$ and $\vec{r} = (r_1, \dots, r_n)$ satisfying (condition $r./r.+ > \nu$). Given $\vec{g} = (g_1, \dots, g_n) \in L_1^*$ and $m \geq M$, consider the nonempty open subset $\mathcal{A}_{m, \vec{r}}^*(\vec{g})$ of $(\mathbb{k} \setminus \{0\})^n$ constructed in Paragraph 6.3.2 (here we write $\mathcal{A}_{m, \vec{r}}^*(\vec{g})$ instead of \mathcal{A}_m^* in order to explicitly denote the dependence on \vec{r}, \vec{g}). For each $\vec{a} = (a_1, \dots, a_n) \in \mathcal{A}_{m, \vec{r}}^*(\vec{g})$, consider $C(\vec{a}\vec{g}; \vec{r})$ where $\vec{a}\vec{g} = (a_1 g_1, \dots, a_n g_n) \in L_1^*$. Let \mathcal{S} be the collection of all the $\mathcal{Z}_k^{(\rho')}$ over all ρ', k . Then

$$\begin{aligned} (\mathcal{L}_1, \dots, \mathcal{L}_n) &= \sum_{z \in \bar{X}} \text{len}(\mathcal{O}_{(z,0), C(\vec{a}\vec{g}; \vec{r})} / t\mathcal{O}_{(z,0), C(\vec{a}\vec{g}; \vec{r})}) \\ &= \sum_{Z' \in \mathcal{S}} \sum_{\substack{i_1, \dots, i_{\rho'} \\ \rho' = \text{codim}(Z') \\ Z' \subseteq Z_{i_1, \dots, i_{\rho'}}^{(\rho')} \\ (i_1(z), \dots, i_{\rho'}(z)) = (i_1, \dots, i_{\rho'})}} \sum_{\substack{z \in Z' \\ \mathcal{Z}_{k_z}^{(\rho_z)} = Z'}} \text{len}(\mathcal{O}_{(z,0), C(\vec{a}\vec{g}; \vec{r})} / t\mathcal{O}_{(z,0), C(\vec{a}\vec{g}; \vec{r})}) \end{aligned}$$

6.4.6. For each summand of the (innermost) sum in the above identity, Corollary 6.3.3 implies that

$$\text{len}(\mathcal{O}_{(z,0), C(\vec{a}\vec{g}; \vec{r})} / t\mathcal{O}_{(z,0), C(\vec{a}\vec{g}; \vec{r})}) = \frac{1}{m^{n-\rho'}} \text{len}(\mathcal{O}_{(z,0), C_{m, i_1, \dots, i_{\rho'}}(\vec{a}\vec{g}; \vec{r})} / t\mathcal{O}_{(z,0), C_{m, i_1, \dots, i_{\rho'}}(\vec{a}\vec{g}; \vec{r})})$$

Now assume ν (from Paragraph 6.4.5) is greater than or equal to ν_m^* from Paragraph 6.4.4. Then writing $C_* := C_{m, i_1, \dots, i_{\rho'}}(\vec{a}\vec{g}; \vec{r})$ as a shorthand, we have

$$\begin{aligned} \text{len}(\mathcal{O}_{(z,0), C_*} / t\mathcal{O}_{(z,0), C_*}) &= \text{len}(\hat{\mathcal{O}}_{(z,0), C_*} / t\hat{\mathcal{O}}_{(z,0), C_*}) \\ &= \text{len}(\hat{\mathcal{O}}_{(z,0), \bar{X} \times \mathbb{k}} / \langle I(C_*) \rangle \hat{\mathcal{O}}_{(z,0), \bar{X} \times \mathbb{k}}) \\ &= \text{len}(\hat{\mathcal{O}}_{(z,0), \bar{X} \times \mathbb{k}} / \langle I(C_*), t, I_{\rho', k; i_1, \dots, i_{\rho'}}^{(\rho')}(\frac{g_{i'_1}}{h_{\rho', k, i'_1}})^m, \dots, (\frac{g_{i'_{n-\rho'}}}{h_{\rho', k, i'_{n-\rho'}}})^m \rangle \hat{\mathcal{O}}_{(z,0), \bar{X} \times \mathbb{k}}) \\ &\quad \text{(due to properties (i) and (ii) from Paragraph 6.4.4)} \\ &\leq \text{len}(\hat{\mathcal{O}}_{(z,0), \bar{X} \times \mathbb{k}} / \langle t, I_{\rho', k; i_1, \dots, i_{\rho'}}^{(\rho')}(\frac{g_{i'_1}}{h_{\rho', k, i'_1}})^m, \dots, (\frac{g_{i'_{n-\rho'}}}{h_{\rho', k, i'_{n-\rho'}}})^m \rangle \hat{\mathcal{O}}_{(z,0), \bar{X} \times \mathbb{k}}) \\ &= \text{len}(\hat{\mathcal{O}}_{z, \bar{X}} / \langle I_{\rho', k; i_1, \dots, i_{\rho'}}^{(\rho')}(\frac{g_{i'_1}}{h_{\rho', k, i'_1}})^m, \dots, (\frac{g_{i'_{n-\rho'}}}{h_{\rho', k, i'_{n-\rho'}}})^m \rangle \hat{\mathcal{O}}_{z, \bar{X}}) \end{aligned}$$

Claim 6.4.7. For each $m \geq M$, fix $\nu_m \geq \max\{\nu_m^*, \nu_1^*\}$ (where ν_1^* is from Paragraph 6.4.5) and \vec{r}_m such that (condition $r./r.+ > \nu$) holds with $\vec{r} = \vec{r}_m$ and $\nu = \nu_m$. If $\mathcal{A}_{\vec{g}}^* := \bigcap_{m \geq M} \mathcal{A}_{m, \vec{r}_m}^*(\vec{g})$ is

nonempty, then for all sufficiently large m ,

$$\text{len}(\mathcal{O}_{(z,0)}, C(\vec{a}\vec{g}; \vec{r}_m) / t\mathcal{O}_{(z,0), C(\vec{a}\vec{g}; \vec{r}_m)}) \leq \text{len}(\mathcal{O}_{Z', \bar{X}} / I_{\rho', k; i_1, \dots, i_{\rho'}}^{(\rho')} \mathcal{O}_{Z', \bar{X}}) \left(\frac{g_{i'_1}}{h_{\rho', k, i'_1}}|_{Z'}, \dots, \frac{g_{i'_{n-\rho'}}}{h_{\rho', k, i'_{n-\rho'}}}|_{Z'} \right)_z$$

for each $\vec{a} \in \mathcal{A}_{\vec{g}}^*$ and $z \in Z' \cap V(\frac{g_{i'_1}}{h_{\rho', k, i'_1}}, \dots, \frac{g_{i'_{n-\rho'}}}{h_{\rho', k, i'_{n-\rho'}}})$, where $(\cdot, \dots, \cdot)_z$ denotes the intersection multiplicity at z (of Cartier divisors on Z').

Proof. Indeed, it follows from Paragraph 6.4.6 that

$$\text{len}(\mathcal{O}_{(z,0)}, C(\vec{a}\vec{g}; \vec{r}_m) / t\mathcal{O}_{(z,0), C(\vec{a}\vec{g}; \vec{r}_m)}) \leq \frac{1}{m^{n-\rho}} \text{len}(\mathcal{O}_{z, \bar{X}} / \langle I_{\rho', k; i_1, \dots, i_{\rho'}}^{(\rho')}, (\frac{g_{i'_1}}{h_{\rho', k, i'_1}})^m, \dots, (\frac{g_{i'_{n-\rho'}}}{h_{\rho', k, i'_{n-\rho'}}})^m \rangle \mathcal{O}_{z, \bar{X}})$$

The limit formula of Lech [Nor68, Theorem 7.5.10] then implies that the limit of the right hand side as $m \rightarrow \infty$ is $e_{R_z}(\frac{g_{i'_1}}{h_{\rho', k, i'_1}}, \dots, \frac{g_{i'_{n-\rho'}}}{h_{\rho', k, i'_{n-\rho'}}} | R_z)$, where $R_z := \mathcal{O}_{z, \bar{X}} / I_{\rho', k; i_1, \dots, i_{\rho'}}^{(\rho')} \mathcal{O}_{z, \bar{X}}$, and $e(\cdot | \cdot)$ is the “multiplicity” symbol. Since the left hand side is always an integer, it follows that for all $m \gg 1$,

$$\text{len}(\mathcal{O}_{(z,0)}, C(\vec{a}\vec{g}; \vec{r}_m) / t\mathcal{O}_{(z,0), C(\vec{a}\vec{g}; \vec{r}_m)}) \leq e_{R_z}(\frac{g_{i'_1}}{h_{\rho', k, i'_1}}, \dots, \frac{g_{i'_{n-\rho'}}}{h_{\rho', k, i'_{n-\rho'}}} | R_z)$$

for each $z \in Z' \cap V(\frac{g_{i'_1}}{h_{\rho', k, i'_1}}, \dots, \frac{g_{i'_{n-\rho'}}}{h_{\rho', k, i'_{n-\rho'}}})$ (recall that the latter set is finite due to condition (10) of Section 6.2). It follows from the definitions that in this case the multiplicity symbol is same as the intersection multiplicity at z of the Cartier divisor defined by $g_{i'_j} / h_{\rho', k, i'_j}$, $j = 1, \dots, n - \rho'$, on the subscheme \tilde{Z}' of \bar{X} defined by $I_{\rho', k; i_1, \dots, i_{\rho'}}^{(\rho')}$. Since near each $z \in Z' \cap V(\frac{g_{i'_1}}{h_{\rho', k, i'_1}}, \dots, \frac{g_{i'_{n-\rho'}}}{h_{\rho', k, i'_{n-\rho'}}})$, the only irreducible component of \tilde{Z}' is Z' itself (property (ii) of Paragraph 6.4.3), it follows that

$$\begin{aligned} e_{R_z}(\frac{g_{i'_1}}{h_{\rho', k, i'_1}}, \dots, \frac{g_{i'_{n-\rho'}}}{h_{\rho', k, i'_{n-\rho'}}} | R_z) &= (\frac{g_{i'_1}}{h_{\rho', k, i'_1}}|_{\tilde{Z}'}, \dots, \frac{g_{i'_{n-\rho'}}}{h_{\rho', k, i'_{n-\rho'}}}|_{\tilde{Z}'})_z \\ &= \text{len}(\mathcal{O}_{Z', \bar{X}} / I_{\rho', k; i_1, \dots, i_{\rho'}}^{(\rho')} \mathcal{O}_{Z', \bar{X}}) \left(\frac{g_{i'_1}}{h_{\rho', k, i'_1}}|_{Z'}, \dots, \frac{g_{i'_{n-\rho'}}}{h_{\rho', k, i'_{n-\rho'}}}|_{Z'} \right)_z \end{aligned}$$

This completes the proof of the claim. \square

Proposition 6.4.8. Fix $i_1, \dots, i_{\rho'}$ such that $Z' = \mathcal{Z}_k^{(\rho')}$ is an irreducible component of $\mathcal{Z}_{i_1, \dots, i_{\rho'}}^{(\rho')}$. Then there is ν^{*} with the following property: for all $\nu \geq \nu^{*}$ and \vec{r} satisfying (condition $r./r._+ > \nu$), there is a nonempty open subset L^{*} of $\prod_i L_i$ such that for each $\vec{g} = (g_1, \dots, g_n) \in L^{*}$,

$$\sum_{\substack{z \in Z' \\ \mathcal{Z}_{k_z}^{(\rho_z)} = Z' \\ (i_1(z), \dots, i_{\rho_z}(z)) = (i_1, \dots, i_{\rho'})}} \text{len}(\mathcal{O}_{(z,0), C(\vec{g}; \vec{r})} / t\mathcal{O}_{(z,0), C(\vec{g}; \vec{r})}) \leq \text{len}(\mathcal{O}_{Z', \bar{X}} / I_{\rho', k; i_1, \dots, i_{\rho'}}^{(\rho')} \mathcal{O}_{Z', \bar{X}}) \deg_{i'_1, \dots, i'_{n-\rho'}}(Z)$$

Proof. We may assume without loss of generality that \mathbb{k} is uncountable⁹. Proposition 6.3.4 implies that for each choice of sufficiently large ν and r_1, \dots, r_n satisfying (condition $r./r._+ > \nu$), the sum on the left hand side of the above relation is a constant $e_{\nu, \vec{r}}$ for all \vec{g} in a nonempty open subset $L_{\nu, \vec{r}}^*$

⁹ Since one can proceed after embedding \mathbb{k} into the algebraic closure of power series in a new variable.

of $\prod_i L_i$. It suffices to show that $e_{\nu, \vec{r}} \leq \lambda := \text{len}(\mathcal{O}_{Z', \bar{X}} / I_{\rho', k; i_1, \dots, i_{\rho'}}^{(\rho')}) \deg_{i'_1, \dots, i'_{n-\rho'}}(Z)$ if ν is sufficiently large. Indeed, otherwise for each m , there are ν_m, \vec{r}_m as in Claim 6.4.7 such that $e_{\nu_m, \vec{r}_m} > \lambda$. Since \mathbb{k} is uncountable, there is $\vec{g} \in L_1^* \cap \bigcap_m L_{\nu_m, \vec{r}_m}^*$ (where L_1^* is from Paragraph 6.4.5). Now for each m , let $\mathcal{A}_{m, \vec{r}_m}^*(\vec{g})$ be the nonempty open subset of $(\mathbb{k} \setminus \{0\})^n$ consisting of all $\vec{a} \in \mathcal{A}_{m, \vec{r}_m}^*(\vec{g})$ such that $\vec{a}\vec{g} \in L_{\nu_m, \vec{r}_m}^*$. The uncountability of \mathbb{k} again implies that there is $\vec{a} \in \bigcap_m \mathcal{A}_{m, \vec{r}_m}^*(\vec{g})$, which yields a contradiction via Claim 6.4.7. \square

6.4.9. Combining the observations from Paragraph 6.4.5, Proposition 6.4.8 and Theorem 5.7 we obtain that for each \vec{r} satisfying (condition $r./r. + > \nu$) with sufficiently large ν , there is a nonempty open subset L^* of $\prod_i L_i$ such that for all $\vec{g} \in L^*$,

$$\begin{aligned} (\mathcal{L}_1, \dots, \mathcal{L}_n) &= \sum_{Z' \in \mathcal{S}} \sum_{\substack{i_1, \dots, i_{\rho'} \\ \rho' = \text{codim}(Z') \\ Z' \subseteq Z_{i_1, \dots, i_{\rho'}}^{(\rho')} \\ (i_1(z), \dots, i_{\rho_z}(z)) = (i_1, \dots, i_{\rho'})}} \sum_{\substack{z \in Z' \\ \mathcal{Z}_{kz}^{(\rho_z)} = Z'}} \text{len}(\mathcal{O}_{(z,0), C(\vec{g}; \vec{r})} / t\mathcal{O}_{(z,0), C(\vec{g}; \vec{r})}) \\ &\leq \sum_{Z' \in \mathcal{S}} \sum_{\substack{i_1, \dots, i_{\rho'} \\ \rho' = \text{codim}(Z') \\ Z' \subseteq Z_{i_1, \dots, i_{\rho'}}^{(\rho')} \\ (i_1(z), \dots, i_{\rho_z}(z)) = (i_1, \dots, i_{\rho'})}} \text{len}(\mathcal{O}_{Z', \bar{X}} / I_{\rho', k; i_1, \dots, i_{\rho'}}^{(\rho')}) \deg_{i'_1, \dots, i'_{n-\rho'}}(Z') \\ &= (\mathcal{L}_1, \dots, \mathcal{L}_n) \end{aligned}$$

Then the inequality in the middle must be an equality, and for each $Z', i_1, \dots, i_{\rho'}$ and each $\vec{g} \in L^*$ we must have

$$\sum_{\substack{z \in Z' \\ \mathcal{Z}_{kz}^{(\rho_z)} = Z' \\ (i_1(z), \dots, i_{\rho_z}(z)) = (i_1, \dots, i_{\rho'})}} \text{len}(\mathcal{O}_{(z,0), C(\vec{g}; \vec{r})} / t\mathcal{O}_{(z,0), C(\vec{g}; \vec{r})}) = \text{len}(\mathcal{O}_{Z', \bar{X}} / I_{\rho', k; i_1, \dots, i_{\rho'}}^{(\rho')}) \deg_{i'_1, \dots, i'_{n-\rho'}}(Z')$$

Theorem 6.4 now follows from taking Z' equal to the closure \bar{Z} of Z in \bar{X} . \square

Part 2. Computation: estimates and examples

7. LOCAL FACTORS OF ORDERED INTERSECTION AS (USUAL) INTERSECTION MULTIPLICITIES

7.1. Recall the basic set up: we are given base-point free linear systems $\mathcal{L}_1, \dots, \mathcal{L}_n$ on a purely n -dimensional variety X , and for each i , we have a fixed element $F_i \in \mathcal{L}_i$, which is either an effective (Cartier) divisor, or the zero element in \mathcal{L}_i . Theorem 5.7 shows that¹⁰

$$(3) \quad (\mathcal{L}_1, \dots, \mathcal{L}_n) = \sum_Z (F_1, \dots, F_n)_Z^{\text{ord}}$$

¹⁰ In Theorem 5.7 no F_i is allowed to be identically zero. If say k of the F_i are identically zero, then one obtains identities (3) and (4) as follows: first reorder the F_i such that all the identically zero elements appear first, and then appear the remaining ones, but with the *ordering among the non identically-zero elements unchanged*. Then replace the identically zero elements by generic elements of the corresponding \mathcal{L}_i , and apply Theorem 5.7 with $i^* = k + 1$.

where $(F_1, \dots, F_n)_Z^{\text{ord}}$ is the “ordered intersection multiplicity” of F_1, \dots, F_n along Z , which is zero unless Z is a “distinguished component” of the ordered intersection of F_1, \dots, F_n , in which case

$$(4) \quad (F_1, \dots, F_n)_Z^{\text{ord}} = \sum_{i_1, \dots, i_\rho} \text{len}(\mathcal{O}_{Z,X}/I_{i_1, \dots, i_\rho}^{(\rho)} \mathcal{O}_{Z,X}) \deg_{i'_1, \dots, i'_{n-\rho}}(Z)$$

where $\rho := \text{codim}(Z)$, and the sum is over all i_1, \dots, i_ρ are such that Z is an irreducible component of $Z_{i_1, \dots, i_\rho}^{(\rho)}$ from Theorem 2.7, and $I_{i_1, \dots, i_\rho}^{(\rho)}$ are as in Theorem 2.11. In this section we describe a few cases where the “local factors” $\text{len}(\mathcal{O}_{Z,X}/I_{i_1, \dots, i_\rho}^{(\rho)} \mathcal{O}_{Z,X})$ can be interpreted as “intersection multiplicities of regular sequences” at nonsingular points, and work out basic examples in dimension ≤ 3 .

7.2. First we recall the relevant definitions:

- Let h_1, \dots, h_k be regular functions on an open subset U of X , and W be an irreducible component of $V(h_1, \dots, h_k)$. If $\text{codim}(W) = k$, then we say that h_1, \dots, h_k form a *regular sequence near W* ; if in addition W is not contained in the set $\text{Sing}(X)$ of singular points of X , then the *intersection multiplicity of h_1, \dots, h_k along W* is defined as:

$$(h_1, \dots, h_k)_W := \text{len}(\mathcal{O}_{W,X}/\langle h_1, \dots, h_k \rangle \mathcal{O}_{W,X})$$

- If D_1, \dots, D_k are effective Cartier divisors on X and W is an irreducible component of $\bigcap_{i=1}^k \text{Supp}(D_i)$ such that $\text{codim}(W) = k$ and $W \not\subseteq \text{Sing}(X)$, then the intersection multiplicity of D_1, \dots, D_k along W is:

$$(D_1, \dots, D_k)_W := (h_1, \dots, h_k)_{W \cap U}$$

where h_i are local representatives of D_i on an open subset U of X such that $U \cap W \neq \emptyset$.

7.3. We continue to work in the set up of Section 7.1. By a *distinguished component* of (the ordered intersection of) F_1, \dots, F_n we mean an irreducible component of some $Z_{i_1, \dots, i_\rho}^{(\rho)}$ from Theorem 2.7. As in Section 4.7 we write $\mathcal{Z}^{(\rho)}$ for the union of all distinguished components of codimension ρ .

Proposition 7.4. *Let Z be a distinguished component of F_1, \dots, F_n such that $Z \not\subseteq \text{Sing}(X)$. Let i^* be the smallest index such that F_{i^*} is not identically zero, and $\rho := \text{codim}(Z)$.*

- (1) *If Z is an irreducible component of $\bigcap_{j=1}^\rho \text{Supp}(F_{i_j})$, then*

$$\text{len}(\mathcal{O}_{Z,X}/I_{i_1, \dots, i_\rho}^{(\rho)} \mathcal{O}_{Z,X}) = (F_{i_1}, \dots, F_{i_\rho})_Z$$

- (2) *In particular,*

- (a) *If $\rho = 1$, then*

$$\begin{aligned} \text{len}(\mathcal{O}_{Z,X}/I_{i^*}^{(1)} \mathcal{O}_{Z,X}) &= (F_{i^*})_Z \\ (F_1, \dots, F_n)_Z^{\text{ord}} &= (F_{i^*})_Z \deg_{1, \dots, i^*, \dots, n}(Z) \end{aligned}$$

- (b) *If $\rho = n$ and Z is an isolated point in $\bigcap_{i=1}^n \text{Supp}(F_i)$, then*

$$(F_1, \dots, F_n)_Z^{\text{ord}} = \text{len}(\mathcal{O}_{Z,X}/I_{1, \dots, n}^{(n)} \mathcal{O}_{Z,X}) = (F_1, \dots, F_n)_Z$$

- (3) *If $\rho = 2$, then pick an open affine subset U of X such that $U \cap Z \neq \emptyset$, and each F_i is defined on U by some $f_i \in \mathbb{k}[U]$. Since $\mathcal{O}_{Z,X}$ is a UFD¹¹, it contains (up to multiplication by units) a unique element f_{i^*, i_2} which is the least common multiple of all $f \in \mathcal{O}_{Z,X}$ such that*

- (a) *f divides f_{i^*} ,*

¹¹ Due to Auslander–Buchsbaum theorem.

- (b) each irreducible factor of f divides each f_i , $i < i_2$, and
(c) no irreducible factor of f divides f_{i_2} .

Then

$$\begin{aligned} \text{len}(\mathcal{O}_{Z,X}/I_{i^*,i_2}^{(2)} \mathcal{O}_{Z,X}) &= (f_{i^*,i_2}, f_{i_2})_Z \\ (F_1, \dots, F_n)_Z^{\text{ord}} &= \sum_{i_2} (f_{i^*,i_2}, f_{i_2})_Z \deg_{i'_1, \dots, i'_{n-2}}(Z) \end{aligned}$$

where the sum is over all i_2 such that Z is an irreducible component of $Z_{i^*,i_2}^{(\rho)}$, and i'_1, \dots, i'_{n-2} are the elements of $\{1, \dots, n\} \setminus \{i^*, i_2\}$.

Proof. It follows in a straightforward way from the definitions. \square

7.5. For the next result, we pick i_1, \dots, i_ρ such that Z is an irreducible component of $Z_{i_1, \dots, i_\rho}^{(\rho)}$. In Theorem 2.11 we constructed ideals $I_{i_1, \dots, i_\rho}^{(j)}$ on an affine neighborhood U in X of some point in Z . Recall that $V(I_{i_1, \dots, i_\rho}^{(j)})$ has pure codimension j . In particular, if $i'_1, \dots, i'_{n-\rho}$ are the elements of $\{1, \dots, n\} \setminus \{i_1, \dots, i_\rho\}$, and $g_{i'_k}$ are local representatives on U of generic elements in $\mathcal{L}_{i'_k}$, then $V(I_{i_1, \dots, i_\rho}^{(\rho-1)}(g_{i'_1}, \dots, g_{i'_{n-\rho}}))$ has pure dimension one (or is empty); to simplify the notation we write $\vec{i} := (i_1, \dots, i_\rho)$, $\vec{g} := (g_{i'_1}, \dots, g_{i'_{n-\rho}})$ and $C_{\vec{i}, \vec{g}} := V(I_{i_1, \dots, i_\rho}^{(\rho-1)}(g_{i'_1}, \dots, g_{i'_{n-\rho}}))$.

Proposition 7.6. *With the above notations,*

$$\text{len}(\mathcal{O}_{Z,X}/I_{i_1, \dots, i_\rho}^{(\rho)} \mathcal{O}_{Z,X}) \deg_{i'_1, \dots, i'_{n-\rho}}(Z) = \sum_{P \in Z \cap C_{\vec{i}, \vec{g}}} \text{ord}_P(f_{i_\rho} |_{C_{\vec{i}, \vec{g}}})$$

where $\text{ord}_P(\cdot)$ is either zero (if $C_{\vec{i}, \vec{g}} = \emptyset$), or the order of vanishing¹² of $f_{i_\rho} |_{C_{\vec{i}, \vec{g}}}$ at P . Consequently,

$$(F_1, \dots, F_n)_Z^{\text{ord}} = \sum_{i_1, \dots, i_\rho} \sum_{P \in Z \cap C_{\vec{i}, \vec{g}}} \text{ord}_P(f_{i_\rho} |_{C_{\vec{i}, \vec{g}}})$$

where the sum is over all $i_1 = i^* < i_2 < \dots < i_\rho$ such that Z is an irreducible component of $Z_{i_1, \dots, i_\rho}^{(\rho)}$.

Proof. This follows from identity (2) on page 22 and Theorem 2.11. \square

Example 7.7. Let $F_i = \sum_j m_{i,j} C_j + \sum_k m'_{i,k} C'_{i,k}$, $i = 1, 2$, be curves (i.e. purely one dimensional schemes) on a surface X , where $m_{i,j}, m'_{i,k}$ are positive integers, and $C_j, C'_{i,k}$ are pairwise distinct irreducible curves; in particular, C_j are precisely the irreducible components common to both F_1 and F_2 . Then

$$\begin{aligned} \bigcap_i \text{Supp}(F_i) &= \bigcup_j C_j \cup \bigcup_{k_1, k_2} (C'_{1,k_1} \cap C'_{2,k_2}) \\ (5) \quad \mathcal{Z}^{(1)} &= \bigcup_j C_j \\ \mathcal{Z}^{(2)} &= \bigcup_k (C'_{1,k} \cap \text{Supp}(F_2)) \end{aligned}$$

¹² If w is a (closed) point of a purely one dimensional scheme W and $f \in \mathcal{O}_{w,W}$, then $\text{ord}_w(f)$ is the length of $\mathcal{O}_{w,W}/f\mathcal{O}_{w,W}$ as a module over $\mathcal{O}_{w,W}$ [Ful98, Section 1.2].

For each j ,

$$(F_1, F_2)_{C_j}^{\text{ord}} = m_{1,j}(C_j, F_2) = m_{1,j} \deg_2(C_j)$$

If $C_j \not\subseteq \text{Sing}(X)$, then F_1 can be represented by some f_1 near a generic (nonsingular) point of C_j , and $m_{1,j} = (f_1)_{C_j}$ is the order of vanishing of f_1 along C_j , so that

$$(6) \quad (F_1, F_2)_{C_j}^{\text{ord}} = (f_1)_{C_j} \deg_2(C_j)$$

Similarly, if $P \in \mathcal{Z}^{(2)}$ is a nonsingular point of X , then

$$(7) \quad (F_1, F_2)_P^{\text{ord}} = (f'_1, f_2)_P$$

where f'_1, f_2 are local equations of respectively $\sum_k m'_{1,k} C'_{1,k}$ and F_2 near P .

Example 7.8. More generally, assume F_1, \dots, F_{n-1} intersect properly, and the scheme theoretic intersection of F_1, \dots, F_{n-1} is $\sum_j m_j C_j + \sum_k m'_k C'_k$, where the C_j, C'_k are irreducible (reduced) curves such that $\text{Supp}(F_n)$ contains each C_j , and does *not* contain any C'_k . Then

$$\mathcal{Z}^{(\rho)} = \begin{cases} \emptyset & \text{if } \rho < n-1, \\ \bigcup_j C_j & \text{if } \rho = n-1, \\ \bigcup_k (C'_k \cap \text{Supp}(F_n)) & \text{if } \rho = n \end{cases}$$

$$(F_1, \dots, F_n)_Z^{\text{ord}} = \begin{cases} m_j \deg_n(C_j) & \text{if } Z = C_j, \\ m'_k \text{ord}_P(F_n|_{C'_k}) & \text{if } Z = P \in C'_k, \\ 0 & \text{otherwise} \end{cases}$$

Example 7.9. Let X be a variety of pure dimension 3, and $F_i = \sum_j m_{i,j} H_j + F'_i$, where $m_{i,j}$ are positive integers, H_j are irreducible hypersurfaces which are not contained in $\bigcup_i \text{Supp}(F'_i)$, and the F'_i are hypersurfaces with no irreducible components common to all F'_i . We will describe the distinguished components of the ordered intersection of the F_i and the corresponding ordered intersection multiplicity.

7.9.1. Since H_j are codimension one irreducible components of each F_i , it follows that

$$(8) \quad \mathcal{Z}^{(1)} = \bigcup_j \text{Supp}(H_j)$$

Since

$$(9) \quad (F_1, F_2, F_3) = \left(\sum_j m_{1,j} H_j + F'_1, F_2, F_3 \right) = \sum_j m_{1,j} (H_j, F_2, F_3) + (F'_1, F_2, F_3)$$

it follows that

$$(F_1, F_2, F_3)_{H_j}^{\text{ord}} = m_{1,j}(H_j, F_2, F_3) = m_{1,j} \deg_{2,3}(H_j)$$

If $H_j \not\subseteq \text{Sing}(X)$, then

$$(10) \quad (F_1, F_2, F_3)_{H_j}^{\text{ord}} = (f_1)_{H_j} \deg_{2,3}(H_j)$$

where f_1 is a local equation of F_1 near a generic (nonsingular) point of H_j .

7.9.2. Now we consider (F'_1, F_2, F_3) , the second summand of the right hand side of (9). Write $F'_1 = F'_{1,2} + F''_1$, where $F'_{1,2}$ is the sum of the irreducible components of F'_1 (with appropriate multiplicity) which are common to F_2 , and F''_1 does not have any irreducible component in common with F'_2 . It follows that

$$(F'_1, F_2, F_3) = (F'_{1,2} + F''_1, F_2, F_3) = (F'_{1,2}, F_3, F_2) + (F''_1, F_2, F_3)$$

where the intersection of the first two components in each triplet has codimension two. Write the scheme theoretic intersections

$$\begin{aligned} F'_{1,2} \cap F_3 &= \sum_k m'_k C'_k, \\ F''_1 \cap F_2 &= \sum_i m''_{1,i} C''_{1,i} + \sum_j m''_{2,j} C''_{2,j} \end{aligned}$$

where $C''_{1,i}$ are irreducible curves which are contained in $\text{Supp}(F_3)$, and $C''_{2,j}$ are irreducible curves which are not contained in $\text{Supp}(F_3)$. Then

$$(11) \quad (F'_1, F_2, F_3) = \sum_k m'_k (C'_k, F_2) + \sum_i m''_{1,i} (C''_{1,i}, F_3) + \sum_j m''_{2,j} (C''_{2,j}, F_3)$$

It follows that

$$(12) \quad \mathcal{Z}^{(2)} = \bigcup_k C'_k \cup \bigcup_i C''_{1,i}$$

and if C is one of the C'_k or $C''_{1,i}$, then

$$(13) \quad (F_1, F_2, F_3)_{\mathcal{C}}^{\text{ord}} = m'_k \deg_2(C) + m''_{1,i} \deg_3(C)$$

where k and i are such that $C'_k = C$ and $C''_{1,i} = C$ (if there is no such k or i , then we accordingly set $m'_k = 0$ or $m''_{1,i} = 0$). If $C \not\subseteq \text{Sing}(X)$, then this multiplicity can also be expressed as:

$$(14) \quad (F_1, F_2, F_3)_{\mathcal{C}}^{\text{ord}} = (F'_{1,2}, F_3)_C \deg_2(C) + (F''_1, F_2)_C \deg_3(C)$$

which is in line with assertion (3) of Proposition 7.4.

7.9.3. Finally, from the last sum in the right hand side of (11) we deduce

$$(15) \quad \mathcal{Z}^{(3)} = \bigcup_j C''_{2,j} \cap \text{Supp}(F_3)$$

and if $P \in \mathcal{Z}^{(3)}$, then

$$(16) \quad (F_1, F_2, F_3)_P^{\text{ord}} = \sum_{C''_{2,j} \ni P} m''_{2,j} \text{ord}_P(f_{3,P}|_{C''_{2,j}})$$

where $f_{3,P}$ is a local equation of F_3 near P . If $C''_2 := \sum_{C''_{2,j} \ni P} m''_{2,j} C''_{2,j}$ is a local complete intersection near P defined by say f''_1, f''_2 , then of course $(F_1, F_2, F_3)_P^{\text{ord}}$ would equal the intersection multiplicity of $f''_1, f''_2, f_{3,P}$ at P ; however, C''_2 may *not* be a complete intersection at P even when X is nonsingular at P . It follows that unlike codimension ≤ 2 cases, at a codimension three distinguished component, even if it is not contained in $\text{Sing}(X)$, the ordered intersection multiplicity can *not* in general be represented as a (sum of) product(s) of degree times a usual intersection multiplicity.

Example 7.10. As an example of the scenario described at the end of the preceding example, consider

$$\begin{aligned} f_1 &= x^3 - yz \\ f_2 &= y^2 - xz \\ f_3 &= x^2 + xz^2 + y^3 + yz^3 \end{aligned}$$

on \mathbb{k}^3 , and let F_i be the closure of $V(f_i)$ on \mathbb{P}^3 . Scheme theoretically,

$$(17) \quad F_1 \cap F_2 = C + Z$$

where $Z := V(x, y)$ is the “ z -axis”, and C is the (closure of the) monomial curve $\{(t^3, t^4, t^5) : t \in \mathbb{k}\}$. It follows that

$$\bigcap_{i=1}^3 \text{Supp}(F_i) = Z \cup S$$

for a set S of isolated points. We will compute the number $(F_1, F_2, F_3)_{\mathbb{P}^3}^{iso}$ (counted with appropriate multiplicities) of isolated points in $\bigcap_i \text{Supp}(F_i)$. First note that $\mathcal{Z}^{(2)} = \emptyset$, $\mathcal{Z}^{(1)} = Z$, and consequently, it follows from identity (1) on page 4 that

$$(18) \quad (F_1, F_2, F_3)_{\mathbb{P}^3}^{iso} = (F_1, F_2, F_3) - (F_1, F_2, F_3)_Z^{\text{ord}} - \sum_{P \in Z} (F_1, F_2, F_3)_P^{\text{ord}}$$

Identities (14) and (17) imply that

$$(F_1, F_2, F_3)_Z^{\text{ord}} = (F_1, F_2)_Z \deg_3(Z) = 1 \times 4 = 4$$

It follows from (15) that the codimension three distinguished components of the ordered intersection of the F_i are simply the points in $C \cap \text{Supp}(F_3)$. Consequently, the only nonzero contributions to the last sum of the right hand side of (18) come from the points in $Z \cap C$. There are two such points in \mathbb{P}^3 , namely the origin P_0 in \mathbb{k}^3 , and the point P_∞ at infinity on the z -axis. Now, it is well known that C is *not* a complete intersection near P_0 (see e.g. [Har77, Exercise 1.11]), so that $(F_1, F_2, F_3)_{P_0}^{\text{ord}}$, as expressed in identity (16), can *not* be readily expressed as the usual intersection multiplicity of polynomials at P_0 (in Proposition 8.3.5 below we provide an estimate which is sometimes useful in these scenarios). In any event, for the present example, we can compute via the explicit parametrization $\phi : t \mapsto (t^3, t^4, t^5)$ of C near P_0 that

$$(F_1, F_2, F_3)_{P_0}^{\text{ord}} = (F_1, F_2)_C \text{ord}_t(\phi^*(f_3)) = 1 \times 6 = 6$$

Similarly, one can compute that

$$(F_1, F_2, F_3)_{P_\infty}^{\text{ord}} = 1$$

It then follows from Bézout’s theorem and (18) that

$$(F_1, F_2, F_3)_{\mathbb{P}^3}^{iso} = 3 \times 2 \times 4 - 4 - 6 - 1 = 13$$

8. COMPUTATION IN TERMS OF NEWTON DIAGRAMS

We continue to work in the set up of Section 7.1. Proposition 7.4 shows that the “local factors” of ordered intersection multiplicity at a subvariety Z of X can sometimes be represented in terms of intersection multiplicities of irreducible components of complete intersection. In this section we use the results of [Mon21] to estimate this intersection multiplicity in terms of related “Newton diagrams” (provided Z contains nonsingular points of X), and give a few examples of how this can be applied to the affine Bézout problem of counting isolated points of intersections.

8.1. (Usual) intersection multiplicity at a nonsingular point. In this section we recall basic results [Mon21, Theorems IX.1, IX.8] on the computation of intersection multiplicity of n hypersurfaces at a nonsingular point on an n -dimensional variety and some relevant notation that we use in subsequent sections.

8.1.1. The n -dimensional *mixed volume* is the unique symmetric multiadditive functional MV on n -tuples of polytopes in \mathbb{R}^n such that $MV(\mathcal{P}, \dots, \mathcal{P}) = n! \text{Vol}_n(\mathcal{P})$ for each polytope \mathcal{P} , where Vol_n is the n -dimensional Euclidean volume.

8.1.2. Let \mathcal{A} be a (possibly infinite) subset of $(\mathbb{Z}_{\geq 0})^n$. The convex hull of $\mathcal{A} + (\mathbb{R}_{\geq 0})^n$ in \mathbb{R}^n is a convex polyhedron; the *Newton diagram* $\text{ND}(\mathcal{A})$ of \mathcal{A} is the union of the compact faces of this polyhedron. Given variables x_1, \dots, x_n , the *support* of a power series $f = \sum_{\alpha} c_{\alpha} x^{\alpha}$ in (x_1, \dots, x_n) , denoted $\text{Supp}(f)$, is the set of all α such that $c_{\alpha} \neq 0$. The Newton diagram of f , denoted $\text{ND}(f)$, is the Newton diagram of $\text{Supp}(f)$.

8.1.3. We write $[n] := \{1, \dots, n\}$. If $I \subseteq [n]$ and k is a field, we write k^I for the “ $|I|$ -dimensional coordinate subspace” $\{(x_1, \dots, x_n) \in k^n : x_i = 0 \text{ if } i \notin I\}$ of k^n . By $\pi_I : k^n \rightarrow k^I$ we denote the natural projection in the coordinates indexed by I .

8.1.4. Let ν be a weighted order on $\mathbb{k}[x_1, \dots, x_n]$ corresponding to weights ν_j for $x_j, j = 1, \dots, n$. We identify ν with the element in $(\mathbb{R}^n)^*$ with coordinates (ν_1, \dots, ν_n) with respect to the dual basis. For $\mathcal{S} \subseteq \mathbb{R}^n$, we write $\text{In}_{\nu}(\mathcal{S}) := \{\alpha \in \mathcal{S} : \langle \nu, \alpha \rangle = \inf_{\mathcal{S}}(\nu)\}$. The “initial form” of $f = \sum_{\alpha} c_{\alpha} x^{\alpha}$ with respect to ν is $\text{In}_{\nu}(f) := \sum_{\alpha \in \text{In}_{\nu}(\text{Supp}(f))} c_{\alpha} x^{\alpha}$. More generally, if $\text{Supp}(f) \subseteq \mathcal{S}$, we write

$$\text{In}_{\mathcal{S}, \nu}(f) := \sum_{\alpha \in \text{In}_{\nu}(\mathcal{S})} c_{\alpha} x^{\alpha} = \begin{cases} \text{In}_{\nu}(f) & \text{if } \text{Supp}(f) \cap \text{In}_{\nu}(\mathcal{S}) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

We say that ν is *centered at the origin* if each ν_i is positive and that ν is *primitive* if it is nonzero and the greatest common divisor of ν_1, \dots, ν_n is 1. If ν is centered at the origin, then it also extends to a weighted order on the ring of power series in (x_1, \dots, x_n) .

8.1.5. Given Newton diagrams $\Gamma_1, \dots, \Gamma_n$ in \mathbb{R}^n , define

$$(\Gamma_1, \dots, \Gamma_n)_0 := \min\{(f_1, \dots, f_n)_0 : \text{Supp}(f_j) \subseteq \Gamma_j + (\mathbb{R}_{\geq 0})^n, j = 1, \dots, n\}$$

We now recall the basic result regarding $(\Gamma_1, \dots, \Gamma_n)_0$. It gives an expression of $(\Gamma_1, \dots, \Gamma_n)_0$ in terms of the following quantity:

$$(\Gamma_1, \dots, \Gamma_n)_0^* := \sum_{\nu \in \mathcal{V}'_0} \min_{\Gamma_1}(\nu) \text{MV}'_{\nu}(\text{In}_{\nu}(\Gamma_2), \dots, \text{In}_{\nu}(\Gamma_n))$$

where \mathcal{V}'_0 is the set of primitive weighted orders centered at the origin, and MV'_{ν} is the $(n-1)$ -dimensional mixed volume of $\psi_{\nu}(\text{In}_{\nu}(\Gamma_j) + \alpha_j), j = 2, \dots, n$, where α_j is an arbitrary element in \mathbb{Z}^n such that $\text{In}_{\nu}(\Gamma_j) + \alpha_j \subseteq \mathbb{R}_{\nu^{\perp}}^n := \{\alpha \in \mathbb{R}^n : \langle \nu, \alpha \rangle = 0\}$, and $\psi_{\nu} : \mathbb{R}_{\nu^{\perp}}^n \cap \mathbb{Z}^n \rightarrow \mathbb{Z}^{n-1}$ is an arbitrary isomorphism of \mathbb{Z} -modules.

Theorem 8.1.6 ([Mon21, Theorems IX.1, IX.8]). *Let $\Gamma := (\Gamma_1, \dots, \Gamma_n)$ be a collection of Newton diagrams in $(\mathbb{Z}_{\geq 0})^n$. For each $I \subseteq [n]$, let $T_{\Gamma}^I := \{j : \Gamma_j \cap \mathbb{R}^I \neq \emptyset\}$ be the set of all indices j such that Γ_j touches \mathbb{R}^I . Define*

$$\mathcal{T}_{\Gamma, 1} := \{I \subseteq [n] : I \neq \emptyset, |T_{\Gamma}^I| = |I|, 1 \in T_{\Gamma}^I\}$$

Then

- (1) If $0 \notin \bigcup_j \Gamma_j$ and there is $I \subset [n]$ such that $|T_\Gamma^I| < |I|$, then $(\Gamma_1, \dots, \Gamma_n)_0 = \infty$.
 (2) Otherwise

$$(\Gamma_1, \dots, \Gamma_n)_0 = \sum_{I \in \mathcal{T}_{\Gamma,1}} (\Gamma_1 \cap \mathbb{R}^I, \Gamma_{j_2} \cap \mathbb{R}^I, \dots, \Gamma_{j_{|I|}} \cap \mathbb{R}^I)_0^* \\ \times (\pi_{[n] \setminus I}(\Gamma_{j'_1}), \dots, \pi_{[n] \setminus I}(\Gamma_{j'_{n-|I|}}))_0$$

where for each $I \in \mathcal{T}_{\Gamma,1}$, $j_1 = 1, j_2, \dots, j_{|I|}$ are elements of T_Γ^I , and $j'_1, \dots, j'_{n-|I|}$ are elements of $[n] \setminus T_\Gamma^I$.

Pick $f_1, \dots, f_n \in \mathbb{k}[[x_1, \dots, x_n]]$ such that $\text{Supp}(f_j) \subseteq \Gamma_j + (\mathbb{R}_{\geq 0})^n$, $j = 1, \dots, n$. Then it is clear that $(f_1, \dots, f_n)_0 \geq (\Gamma_1, \dots, \Gamma_n)_0$.

- (3) In the situation of assertion (2), $(f_1, \dots, f_n)_0 = (\Gamma_1, \dots, \Gamma_n)_0$ if and only if the following “non-degeneracy” condition holds¹³: for each nonempty subset I of $[n]$ and each weighted order ν centered at the origin, there is no common root of $\text{In}_{\Gamma_j \cap \mathbb{R}^I, \nu}(f_j|_{\mathbb{k}^I})$, $j = 1, \dots, n$, on $(\mathbb{k} \setminus \{0\})^n$. \square

Example 8.1.7. Consider

$$\begin{aligned} f_1 &= x^4 + xy + y^2(1 - y + 7yz) \\ f_2 &= 2xz^2 + y(1 - y)(4 + z^2) \\ f_3 &= x^2(5 + 8xz) + y(1 - y + 3z^3) \end{aligned}$$

We will compute the number $(f_1, f_2, f_3)_{\mathbb{k}^3}^{\text{iso}}$ of isolated points in $V(f_1, f_2, f_3)$ (with appropriate multiplicities). For convenience in this example we fix the characteristic of \mathbb{k} to zero (essentially since the Newton diagrams depend on characteristics). Let F_i be the closure of $V(f_i)$ on \mathbb{P}^3 . Then $\bigcap_i \text{Supp}(F_i)$ consists of the “ z -axis” $Z := V(x, y)$, and finitely many points. Since $\dim(Z) = 1$, it follows that $Z^{(2)} = \emptyset$ and $Z^{(1)} = Z$ (where $Z^{(\rho)}$ denotes the union of the codimension ρ distinguished components of the ordered intersection of F_1, F_2, F_3), and consequently, identity (1) on page 4 implies that

$$(F_1, F_2, F_3)_{\mathbb{P}^3}^{\text{iso}} = (F_1, F_2, F_3) - (F_1, F_2, F_3)_Z^{\text{ord}} - \sum_{P \in Z} (F_1, F_2, F_3)_P^{\text{ord}}$$

Note that

$$(f_1, f_2, f_3)_{\mathbb{k}^3}^{\text{iso}} = (F_1, F_2, F_3)_{\mathbb{P}^3}^{\text{iso}} - \sum_{\substack{P \in \mathbb{P}^3 \setminus \mathbb{k}^3 \\ P \text{ isolated in } \bigcap_i \text{Supp}(F_i)}} (F_1, F_2, F_3)_P$$

An examination of the leading forms of the f_i shows that $\bigcap_i \text{Supp}(F_i) \setminus \mathbb{k}^3$ consists of two points: $P_z := [0 : 0 : 1 : 0]$ and $P_y := [0 : 1 : 0 : 0]$ with respect to homogeneous coordinates $[x : y : z : w]$. Since $P_z \in Z$, it follows that

$$\begin{aligned} (f_1, f_2, f_3)_{\mathbb{k}^3}^{\text{iso}} &= (F_1, F_2, F_3) - (F_1, F_2, F_3)_Z^{\text{ord}} - \sum_{P \in Z} (F_1, F_2, F_3)_P^{\text{ord}} - (F_1, F_2, F_3)_{P_y} \\ (19) \quad &= 64 - (F_1, F_2, F_3)_Z^{\text{ord}} - \sum_{P \in Z} (F_1, F_2, F_3)_P^{\text{ord}} - (F_1, F_2, F_3)_{P_y} \end{aligned}$$

¹³ There is a more “efficient” version of the non-degeneracy condition which allows to disregard certain subsets of $[n]$ [Mon21, Theorem IX.9].

since $(F_1, F_2, F_3) = \prod_{i=1}^3 \deg(f_i) = 64$ due to Bézout's theorem. We now compute $(F_1, F_2, F_3)_{P_y}$ using Theorem 8.1.6. In coordinates $(x', z', w') := (x/y, z/y, 1/y)$, the local equations of the F_i at P are:

$$\begin{aligned} f_{1,y} &= x'^4 + x'w'^2 + w'^2 - w' + 7z' \\ f_{2,y} &= 2x'z'^2w' + (w' - 1)(4w'^2 + z'^2) \\ f_{3,y} &= x'^2(5w'^2 + 8x'z') + w'^3 - w'^2 + 3z'^3 \end{aligned}$$

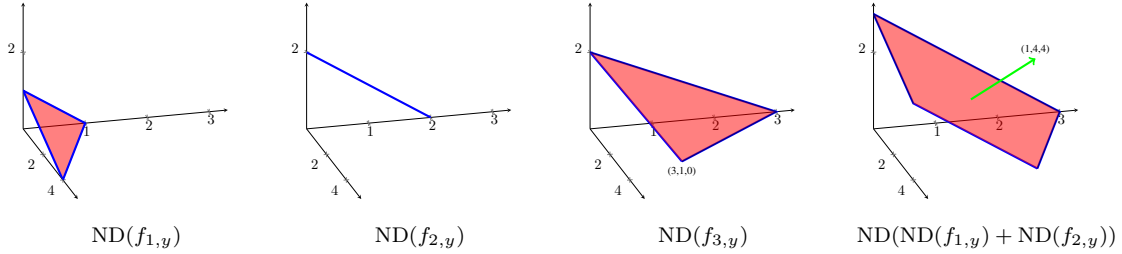


FIGURE 1. Newton diagrams of the $f_{i,y}$

To compute $(f_{1,y}, f_{2,y}, f_{3,y})_{P_y}$ we apply Theorem 8.1.6 with $(\Gamma_1, \Gamma_2, \Gamma_3) := (\text{ND}(f_{3,y}), \text{ND}(f_{2,y}), \text{ND}(f_{1,y}))$ (we could have used any permutation of the $\text{ND}(f_{i,y})$ - this particular choice leads to a relatively shorter computation). Then the set $\mathcal{T}_{\Gamma,1}$ from Theorem 8.1.6 consists only of one I , namely $\{1, 2, 3\}$ (see Figure 1), so that

$$\begin{aligned} (f_{1,y}, f_{2,y}, f_{3,y})_{P_y} &\geq (\text{ND}(f_{3,y}), \text{ND}(f_{2,y}), \text{ND}(f_{1,y}))_0 = (\text{ND}(f_{3,y}), \text{ND}(f_{2,y}), \text{ND}(f_{1,y}))_0^* \\ &= \min_{\text{ND}(f_{3,y})} (\nu) \times \text{MV}'_\nu(\text{In}_\nu(\text{ND}(f_{1,y})), \text{In}_\nu(\text{ND}(f_{2,y}))) \quad (\text{where } \nu := (1, 4, 4)) \\ &= 7 \times 2 = 14 \end{aligned}$$

It is not hard to check that $(f_{3,y}, f_{1,y}, f_{2,y})$ satisfy the non-degeneracy condition from assertion (3) of Theorem 8.1.6, so that

$$(20) \quad (f_{1,y}, f_{2,y}, f_{3,y})_{P_y} = 14$$

We will compute the other terms of identity (19) in Examples 8.2.2 and 8.3.7.

8.2. (possibly non-isolated) complete intersections. Consider an $(n - k)$ -dimensional irreducible component Z of $V(f_1, \dots, f_k)$, where the f_j are regular functions on an n -dimensional affine variety X . Assume in addition that $Z \not\subseteq \text{Sing}(X)$. Under this condition we show that the intersection multiplicity $(f_1, \dots, f_k)_Z$ of f_1, \dots, f_k along Z can be estimated in terms of certain Newton diagrams. Indeed, since $Z \not\subseteq \text{Sing}(X)$, there are “regular coordinates”¹⁴ (x_1, \dots, x_n) on an open subset U of X such that $U \cap Z = V(x_1, \dots, x_k) \cap U \neq \emptyset$. Then there are $f_{j,\alpha} \in \mathbb{k}[U]$ such that for each $a \in U$,

$$f_j = \sum_{\alpha \in (\mathbb{Z}_{\geq 0})^n} f_{j,\alpha} \prod_{i=1}^n (x_i - a_i)^{\alpha_i} \in \mathbb{k}[[x_1 - a_1, \dots, x_n - a_n]]$$

¹⁴ We say (g_1, \dots, g_n) are *regular coordinates* on a nonsingular variety Y if $\dim(Y) = n$ and for each $y \in Y$, $\hat{\mathcal{O}}_{y,Y} \cong \mathbb{k}[[g_1 - g_1(y), \dots, g_n - g_n(y)]]$.

where $a_i := x_i(a)$ (see e.g. [BM97, Lemma 3.5]). Define

$$f_j^{[k]} := \sum_{\alpha \in \mathbb{Z}^{[k]}} f_{j,\alpha} \prod_{i=1}^k x_i^{\alpha_i} \in \mathbb{k}[U][[x_1, \dots, x_k]]$$

(recall from Paragraph 8.1.3 that $\mathbb{Z}^{[k]}$ is the subset of all elements in \mathbb{Z}^n whose last $(n-k)$ -coordinates are identically zero). The *Newton diagram* $\text{ND}(f_j^{[k]})$ with respect to (x_1, \dots, x_k) is the Newton diagram of the set of all α such that $f_{j,\alpha} \neq 0$.

Proposition 8.2.1. *Let $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^k$ be the projection on the first k coordinates. Then*

- (1) $(f_1, \dots, f_k)_Z \geq (\pi(\text{ND}(f_1^{[k]})), \dots, \pi(\text{ND}(f_k^{[k]})))_0$
- (2) *If the right hand side is finite, then the above relation holds with an equality if and only if there is $a \in U \cap Z$ such that the following pair of “non-degeneracy” conditions hold:*
 - (a) *for each $j = 1, \dots, k$, and each vertex α of $\pi(\text{ND}(f_j^{[k]}))$, the coefficient $f_{j,\alpha}$ is nonzero at a , and*
 - (b) *for each nonempty subset I of $[k]$ and each weighted order ν centered at the origin, there is no common root of $\text{In}_\nu(f_j^{[k]}|_{\mathbb{k}^I})$, $j = 1, \dots, k$, on $\{a\} \times (\mathbb{k} \setminus \{0\})^k$.*
- (3) *Moreover, the set of all $a \in U \cap Z$ such that the above non-degeneracy conditions hold is a (possibly empty) Zariski open subset of Z , and every such a (if exists) has an open neighborhood U_a on X such that $V(f_1, \dots, f_k) \cap U_a$ is a complete intersection with only one irreducible component, namely $Z \cap U_a$.*

Proof. Since $(f_1, \dots, f_k)_Z = (f_1, \dots, f_k, x_{k+1} - a_{k+1}, \dots, x_n - a_n)_0$ for generic $a \in U \cap Z$, all statements except for the last one are immediate corollaries of Theorem 8.1.6. The Zariski openness of the set of $a \in U \cap Z$ satisfying the non-degeneracy conditions is a consequence of the Zariski openness of the n -tuples of polynomials which are non-degenerate at the origin [Mon21, Theorem IX.8]. For the local complete intersection property, pick $a \in U \cap Z$ which satisfies condition (a) and in addition, also belongs to another irreducible component Z' of $V(f_1, \dots, f_k)$. Pick an irreducible curve C on Z' such that $a \in C$ and $C \not\subseteq Z$. Let I be the subset of $[k]$ consisting of all i such that $x_i|_C \neq 0$. If B is a branch on C at a , then condition (b) is violated with I and $\nu := (\text{ord}_B(x_i))_{i \in I}$. This completes the proof of the last assertion of Proposition 8.2.1. \square

Example 8.2.2. We continue with the computation in characteristic zero started in Example 8.1.7. Now we compute the ordered intersection multiplicity of F_1, F_2, F_3 along Z , i.e. the term $(F_1, F_2, F_3)_Z^{\text{ord}}$ from the right hand side of identity (19). Since Z is an irreducible component of $V(f_1, f_2)$, assertion 1 of Proposition 7.4 implies that

$$\text{len}(\mathcal{O}_{Z,X}/I_{1,2}^{(\rho)} \mathcal{O}_{Z,X}) = (f_1, f_2)_Z$$



FIGURE 2. “Generic” Newton diagrams of f_1, f_2 with respect to (x, y) -coordinates

Since $Z = V(x, y)$, in the notation of Proposition **8.2.1** we need to consider $f_j^{\{1,2\}}$, $j = 1, 2$, which are simply f_j but where we consider monomials in z as “coefficients”. Figure 2 displays the Newton diagrams of $f_j^{\{1,2\}}$, which are also identical to $\pi(\text{ND}(f_j))$ where π is the projection onto (x, y) -coordinates. Since $\pi(\text{ND}(f_2))$ has only one edge E , and the edge has inner normal $\nu := (1, 1)$, the first assertion of Proposition **8.2.1** and Theorem **8.1.6** imply that

$$(21) \quad (f_1, f_2)_Z \geq (\pi(\text{ND}(f_1)), \pi(\text{ND}(f_2)))_0 = \nu(f_1) \times \text{len}(E) = 2 \times 1 = 2$$

The initial forms of $f_j^{\{1,2\}}$ corresponding to the weighted order $\nu := (1, 1)$ are:

$$\text{In}_\nu(f_1^{\{1,2\}}) = xy + y^2 \quad \text{In}_\nu(f_2^{\{1,2\}}) = 2xz^2 + y(4 + z^2)$$

The second assertion of Proposition **8.2.1** implies that the inequality (21) is satisfied with an equality if and only if there is $z \in \mathbb{k}$ such that following non-degeneracy condition is satisfied:

$$(22) \quad \text{there is no common solution to } \text{In}_\nu(f_1^{\{1,2\}}) = \text{In}_\nu(f_2^{\{1,2\}}) = 0 \text{ with } xyz(4 + z^2) \neq 0.$$

It is clear that the above condition is satisfied for almost all $z \in \mathbb{k}$. It follows that

$$(23) \quad \begin{aligned} (f_1, f_2)_Z &= 2, \text{ and therefore,} \\ (F_1, F_2, F_3)_Z^{\text{ord}} &= (f_1, f_2)_Z \times \deg_3(Z) = 2 \times 4 = 8 \end{aligned}$$

In order to compute $(f_1, f_2, f_3)_{\mathbb{k}^3}^{\text{iso}}$ using (19), it remains to compute $\sum_{P \in Z} (F_1, F_2, F_3)_P^{\text{ord}}$. Write $F_1 \cap F_2 = mZ + C$, where the C is a (possibly non-reduced) curve whose support does not contain Z . Then

$$(24) \quad \sum_{P \in Z} (F_1, F_2, F_3)_P^{\text{ord}} = \sum_{P \in Z} \text{ord}_P(f_3|_C)$$

where $f_{3,P}$ are local equations of F_3 near P (see, e.g. Example **7.9**). Note that the summand on the right hand side is nonzero only for the (finitely many) points P in $Z \cap \text{Supp}(C)$. The third assertion of Proposition **8.2.1** implies that every such P must satisfy one of the following:

- (i) $P = (0, 0, z)$ where z is such that Condition (22) does *not* hold, i.e.
 - (a) $z = 0$,
 - (b) or $4 + z^2 = 0$,
 - (c) or $z(4 + z^2) \neq 0$, but there is a common solution to $\text{In}_{(1,1)}(f_1^{\{1,2\}}) = \text{In}_{(1,1)}(f_2^{\{1,2\}}) = 0$ at z with $x, y, z \in \mathbb{k} \setminus \{0\}$ (this is possible if and only if $z = \pm 2$);
- (ii) or P does *not* belong to the coordinate chart of X with coordinates (x, y, z) , i.e. P is the point $P_\infty := [0 : 0 : 1 : 0]$ with respect to homogeneous coordinates $[x : y : z : w]$.

We will compute $\text{ord}_P(F_3|_C)$ in each of these cases in Example **8.3.7** below.

8.3. (a special type of) non set-theoretic complete intersections. Consider Example **8.2.2** above, and take one of the points, say $P = (0, 0, 0)$, that satisfies one of the two conditions given at the end of the example. We need to compute $\text{ord}_P(f_3|_C)$; at the least we would like to estimate it *from below*. If C were a complete intersection of f_1, f_2 near P , then $\text{ord}_P(f_3|_C)$ would simply be the intersection multiplicity $(f_1, f_2, f_3)_P$ of f_1, f_2, f_3 at P , and we could have used Theorem **8.1.6** to estimate it. However, C is *not* a complete intersection of f_1, f_2 near P . As we have seen, $V(f_1, f_2)$ has another irreducible component Z containing P , so that $\text{ord}_P(f_3|_C) < (f_1, f_2, f_3)_P$, i.e. $(f_1, f_2, f_3)_P$ can *not* be used to estimate $\text{ord}_P(f_3|_C)$ from below (in fact in this particular case $f_3|_Z \equiv 0$, so that $(f_1, f_2, f_3)_P = \infty$). The main result of this section (Proposition **8.3.5**) can sometimes be used to give such an estimate for $\text{ord}_P(f_3|_C)$ in the case that the “punctured” germ of C at P is a complete intersection in the complement of a union of “coordinate subspaces”.

8.3.1. First we state a variant of the non-degeneracy condition from Theorem 8.1.6: we say that $f_1, \dots, f_m \in \mathbb{k}[[x_1, \dots, x_n]]$ are $(*, k)$ -non-degenerate at the origin if they satisfy the following condition:

(ND $_{0}^{*,k}$) for each weighted order ν centered at the origin such that $\dim(\text{In}_\nu(\sum \text{Supp}(f_j))) \leq k$, there is no common root of $\text{In}_\nu(f_j)$, $j = 1, \dots, m$, on $(\mathbb{k} \setminus \{0\})^n$.

We say that f_1, \dots, f_m are $*$ -non-degenerate at the origin if they are $(*, k)$ -non-degenerate at the origin for each $k \geq 0$.

8.3.2. Let z be a nonsingular point on a variety X of dimension n , and (x_1, \dots, x_n) be a system of regular coordinates near z such that each x_i vanishes at z , i.e. z is the “origin” with respect to (x_1, \dots, x_n) . Given a subset I of $[n]$, we write $K^I \subseteq X$ for the set of zeroes of all x_i such that $i \notin I$ (i.e. K^I is the analogue of the “coordinate subspace” \mathbb{k}^I); in particular,

$$K^I = \begin{cases} X & \text{if } I = [n], \\ V(x_1, \dots, x_n) & \text{if } I = \emptyset. \end{cases}$$

Let \mathcal{I} be a given collection of subsets of $[n]$. Consider $f_1, \dots, f_{n-1} \in \mathcal{O}_{z,X}$. We are interested in the irreducible components of $V(f_1, \dots, f_{n-1})$ which contain z , and in addition, are *not* contained in K^I for any $I \in \mathcal{I}$; let $V_{\mathcal{I}}'$ be the union of all such components. Write $\tilde{\mathcal{I}} := \{I \subseteq [n] : I \subseteq I' \text{ for some } I' \in \mathcal{I}\}$, i.e. $\tilde{\mathcal{I}}$ is the “closure” of \mathcal{I} under the operation of taking subsets; note that $\tilde{\mathcal{I}}$ is the largest collection of subsets of $[n]$ such that

$$\bigcup_{I \in \tilde{\mathcal{I}}} K^I = \bigcup_{I \in \mathcal{I}} K^I$$

Proposition 8.3.3. For each $I \subseteq [n]$, let $T^I := \{j : f_j|_{K^I} \not\equiv 0 \text{ near } z\}$ (i.e. T^I is the analogue of T_Γ^I from Theorem 8.1.6). With the set up from the preceding paragraph, assume the following conditions hold for all $I \notin \tilde{\mathcal{I}}$:

- (1) $|T^I| \geq |I| - 1$,
- (2) $f_1|_{K^I}, \dots, f_{n-1}|_{K^I}$ are $(*, |T^I| - 1)$ -non-degenerate at the origin.

where we treat the f_i as power series in (x_1, \dots, x_n) . Then either $V_{\mathcal{I}}' = \emptyset$, or $V_{\mathcal{I}}'$ is a curve (i.e. $V_{\mathcal{I}}'$ is purely one dimensional).

Proof. Assume $V_{\mathcal{I}}' \neq \emptyset$. Pick an irreducible component V' of $V_{\mathcal{I}}'$. First note that $\dim(V') \geq 1$. Let I be the smallest subset of $[n]$ such that $V' \subseteq K^I$. Then $I \notin \tilde{\mathcal{I}}$. Without loss of generality we may assume that $I = \{1, \dots, m\}$. Consider a subset \mathcal{A} of $(\mathbb{Z}_{\geq 0})^m$ such that

- (i) the radical of the ideal $\mathfrak{q}_{\mathcal{A}}$ of \mathcal{O}_{z,K^I} generated by monomials x^α , $\alpha \in \mathcal{A}$, is the maximal ideal of \mathcal{O}_{z,K^I} , and
- (ii) every $m - 1$ dimensional face of $\text{ND}(\sum_{i=1}^{n-1} \text{Supp}(f_i|_{K^I}))$ (if there is one) is also a face of $\text{ND}(\mathcal{A})$.

Consider the blow up \tilde{K}^I of K^I at z with respect to $\mathfrak{q}_{\mathcal{A}}$. The irreducible components of the exceptional divisor E on \tilde{K}^I are toric varieties $X_{\mathcal{Q}}$ corresponding to $m - 1$ dimensional faces \mathcal{Q} of $\text{ND}(\mathcal{A})$, and the proper torus orbits of each such $X_{\mathcal{Q}}$ are toric varieties $X_{\mathcal{Q}'}$ corresponding to proper faces \mathcal{Q}' of \mathcal{Q} . Recall that $|T^I| \geq m - 1$ (property (1) of the f_j). It follows from property (2) of the f_j and property (ii) of \mathcal{A} that

- (iii) if $|T^I| > m - 1$, then the strict transform \tilde{V}' of V' on \tilde{K}^I does *not* intersect $X_{\mathcal{Q}}$ for any face \mathcal{Q} of $\text{ND}(\mathcal{A})$,

- (iv) if $|T^I| = m-1$, then \tilde{V}' does *not* intersect $X_{\mathcal{Q}'}$ for any face \mathcal{Q}' of $\text{ND}(\mathcal{A})$ such that $\dim(\mathcal{Q}') \leq m-2$

(see e.g. [Mon21, Corollary VI.34]). Since case (iii) leads to a contradiction, it follows that $|T^I| = m-1$. Bernstein's theorem then implies that the number of points in $\tilde{V}' \cap X_{\mathcal{Q}}$ (counted with appropriate multiplicity) is $MV'_{\nu}(\text{In}_{\nu}(\text{ND}(f_{j_1})), \dots, \text{In}_{\nu}(\text{ND}(f_{j_{m-1}})))$, where ν is the primitive inner normal to \mathcal{Q} , and $T^I = \{j_1, \dots, j_{m-1}\}$. In particular, it follows that $\tilde{V}' \cap E$ is finite, and therefore, \tilde{V}' is a curve, as required. \square

8.3.4. Let z be a nonsingular point on a variety X of dimension n and C be a curve, i.e. purely one dimensional scheme, containing z . Assume we are interested in estimating $\text{ord}_z(f|_C)$ for some $f \in \mathcal{O}_{z,X}$. Proposition 8.3.5 below provides an estimate under the following assumptions: there are $x_1, \dots, x_n, f_1, \dots, f_{n-1} \in \mathcal{O}_{z,X}$ and a collection \mathcal{I} of subsets of $[n]$ such that

- (a) each x_i vanishes at z , and (x_1, \dots, x_n) form a system of regular coordinates near z ,
- (b) no irreducible component of C containing z is contained in $K^I := V(x_{i'} : i' \notin I)$ for any $I \in \mathcal{I}$,
- (c) f_1, \dots, f_{n-1} generate the ideal of C on $U \setminus \bigcup_{I \in \mathcal{I}} K^I$ for some neighborhood U of z in X (or more generally, the irreducible components of $V(f_1, \dots, f_{n-1})$ which contain z and are not contained in $\bigcup_{I \in \mathcal{I}} K^I$ are precisely the irreducible components of C containing z , and in addition, the multiplicity of each such component in $V(f_1, \dots, f_{n-1})$ is also equal to its multiplicity in C),
- (d) when expressed as power series in (x_1, \dots, x_n) , f_1, \dots, f_{n-1} satisfy conditions (1) and (2) from Proposition 8.3.3 for each $I \notin \bar{\mathcal{I}}$, where $\bar{\mathcal{I}}$ is the closure of \mathcal{I} under the operation of taking subsets (see Paragraph 8.3.2).

Proposition 8.3.5. Let $\mathcal{T} := \{I \subseteq [n] : I \notin \bar{\mathcal{I}}, I \neq \emptyset, |T^I| = |I| - 1\}$, where T^I is as in Proposition 8.3.3. Under the assumptions in Paragraph 8.3.4,

$$(25) \quad \text{ord}_z(f|_C) \geq \sum_{I \in \mathcal{T}} (\text{ND}(f) \cap \mathbb{R}^I, \text{ND}(f_{j_1}) \cap \mathbb{R}^I, \dots, \text{ND}(f_{j_{|I|-1}}) \cap \mathbb{R}^I)_0^* \\ \times (\pi_{[n] \setminus I}(\text{ND}(f'_{j'_1})), \dots, \pi_{[n] \setminus I}(\text{ND}(f'_{j'_{n-|I|}})))_0$$

where for each $I \in \mathcal{T}$, $j_1, \dots, j_{|I|-1}$ are elements of T^I , and $j'_1, \dots, j'_{n-|I|}$ are elements of $[n-1] \setminus T^I$, and $(\cdot, \dots, \cdot)_0^*$ is defined as in Paragraph 8.1.5. Moreover, the above relation is satisfied with an equality if and only if $f_1|_{K^I}, \dots, f_{n-1}|_{K^I}, f|_{K^I}$ are *-non-degenerate at the origin for every $I \in \mathcal{T}$.

Proof. This follows from the arguments of the proof of Proposition 8.3.3 and standard toric geometry arguments (see e.g. [Mon21, Corollary VII.27]). \square

8.3.6. Remarks on the applicability of Proposition 8.3.5.

- (1) The assumptions in Paragraph 8.3.4 in particular implies that on the set-theoretic level there is a decomposition near z of the form $V(f_1, \dots, f_{n-1}) = C \cup W$ with $W \subseteq \bigcup_i V(x_i)$. We do not know if this is possible to achieve for an arbitrary C , since this puts non-trivial constraints on the singularities of W (e.g. the local embedding dimension of each irreducible component of W is $\leq n-1$).

(2) The following example shows that assumption (d) in Paragraph 8.3.4 is necessary: consider

$$f_1 = (x - y + z)(x + y - z)$$

$$f_2 = (x - y - 2z)$$

$$f_3 = x^2 + 3y^2 + 5z^2$$

over a field of characteristic zero or greater than 5. Then $V(f_1, f_2)$ is a curve with two components: C_1 with parametrization $t \mapsto (t, t, 0)$, and C_2 with parametrization $t \mapsto (-3t, t, -2t)$. Let z be the origin, $C = C_2$, and $f = f_3$. Then $\text{ord}_z(f_3|_C) = 2$, but the bound on the right hand side of (25) (applied with $\mathcal{I} = \{1, 2\}$) is

$$\begin{aligned} & (\text{ND}(f_3), \text{ND}(f_1), \text{ND}(f_2))_0^* \\ &= \min_{\text{ND}(f_3)} (\nu) \times \text{MV}'_\nu(\text{In}_\nu(\text{ND}(f_2)), \text{In}_\nu(\text{ND}(f_3))) \quad (\text{where } \nu := (1, 1, 1)) \\ &= 2 \times 2 = 4 > 2 = \text{ord}_z(f_3|_C) \end{aligned}$$

Example 8.3.7. We continue with Examples 8.1.7 and 8.2.2. Recall from the end of Example 8.2.2 that it remains to compute $\text{ord}_P(F_3|_C)$ in four cases, where C is a (possibly non-reduced) curve such that $F_1 \cap F_2 = mZ + C$, and $Z = V(x, y)$ is the z -axis. We will use Proposition 8.3.5 for this computation.

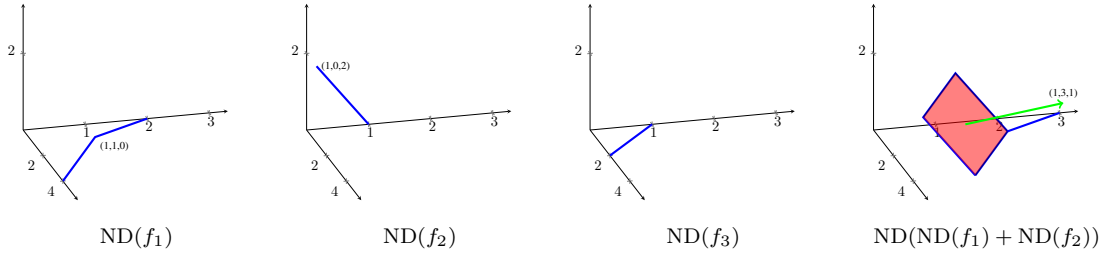


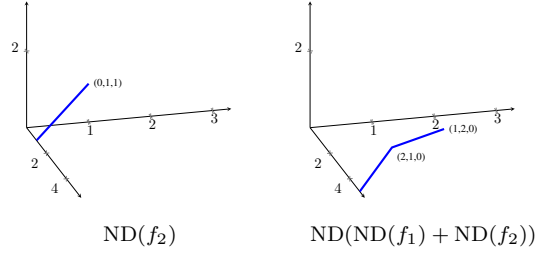
FIGURE 3. Newton diagrams of the f_i

8.3.7.1. Case (i.a). $z = 0$, i.e. $P = P_0 := (0, 0, 0)$. Consider the usual coordinate chart $U \cong \mathbb{k}^3$ with coordinates (x, y, z) . Since $V(f_1, f_2)$ is set-theoretically $Z \cup C$ and $Z \cap U = K^I$ with $I = \{3\}$ (in the notation of Paragraph 8.3.2), we can try to use Proposition 8.3.5 to compute $\text{ord}_{P_0}(f_3|_C)$. Figure 3 shows the Newton diagrams of the f_i with respect to (x, y, z) -coordinates. It is easy to see that f_1, f_2 satisfy condition (d) from Paragraph 8.3.4, so that Proposition 8.3.5 applies. Since in this case the set \mathcal{T} from Proposition 8.3.5 consists of only one set, namely $I := \{1, 2, 3\}$, it follows from inequality (25) (and the last panel of Figure 3) that

$$\begin{aligned} \text{ord}_{P_0}(f_3|_C) &\geq (\text{ND}(f_3), \text{ND}(f_1), \text{ND}(f_2))_0^* \\ &= \min_{\text{ND}(f_3)} (\nu) \times \text{MV}'_\nu(\text{In}_\nu(\text{ND}(f_1)), \text{In}_\nu(\text{ND}(f_2))) \quad (\text{where } \nu := (1, 3, 1)) \\ &= 2 \times 2 = 4 \end{aligned}$$

It is straightforward to check that the f_i satisfy the non-degeneracy condition of Proposition 8.3.5, so that

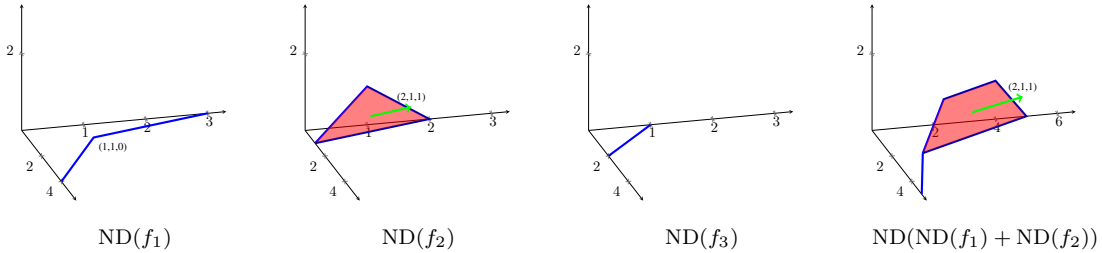
$$(26) \quad \text{ord}_{P_0}(f_3|_C) = 4$$

FIGURE 4. Newton diagrams in (x, y, \tilde{z}) -coordinates

8.3.7.2. Case (i.b). $z^2 + 4 = 0$, i.e. $z = \pm 2\sqrt{-1}$. We start with the case that $z = 2\sqrt{-1}$, i.e. $P = (0, 0, 2\sqrt{-1})$. Consider coordinates (x, y, \tilde{z}) , where $\tilde{z} := z - 2\sqrt{-1}$, on $U := \mathbb{k}^3$. The Newton diagram of f_1 and f_3 with respect to these coordinates are the same as the ones with respect to (x, y, z) -coordinates (the first and the third panels of Figure 3). Figure 4 shows the Newton diagrams of f_2 and of the sum of the Newton diagrams of f_1 and f_2 . As in the previous case, it is easy to see that Proposition 8.3.5 applies with $\mathcal{I} := \{\{3\}\}$, and its non-degeneracy condition is also satisfied. Moreover, since the Newton diagram of $\text{ND}(f_1) + \text{ND}(f_2)$ is one-dimensional, it follows that the right hand side of the inequality (25) is zero, so that $\text{ord}_P(f_3|_C) = 0$. Applying the same arguments for $P = (0, 0, -\sqrt{-1})$, it follows that

$$(27) \quad \text{ord}_{(0,0,\pm 2\sqrt{-1})}(f_3|_C) = 0$$

In other words, C does *not* intersect the z -axis at $(0, 0, \pm 2\sqrt{-1})$.

FIGURE 5. Newton diagrams in (x', y, z') -coordinates

8.3.7.3. Case (i.c). $z = \pm 2$. First we treat the case $z = 2$, i.e. $P = (0, 0, 2)$. With respect to the coordinates (x, y, z') , where $z' := z - 2$, we have:

$$\begin{aligned} f_1 &= x^4 + xy + y^2(1 + 13y + 7yz') \\ f_2 &= 8(x + y) + 2xz'(4 + z'^2) - 8y^2 + yz'(1 - y)(4 + z') \\ f_3 &= x^2(5 + 16x + 8xz') + y(25 - y + 3(12z' + 6z'^2 + z'^3)) \end{aligned}$$

Note that f_1, f_2 violate condition (d) from Paragraph 8.3.4 since with $\nu = (1, 1, 1)$, the initial forms of f_1 and f_2 are respectively $xy + y^2$ and $8(x + y)$, which have common solutions with nonzero

coordinates. However, if we change coordinates to (x', y, z') with $x' := x + y$, then

$$f_1 = x'^4 + x'y + 13y^3 + \dots$$

$$f_2 = 8x' - 4yz' - 8y^2 + \dots$$

$$f_3 = 5x'^2 + 25y + \dots$$

Then it is easy to check that with respect to these coordinates f_1, f_2, C satisfy the conditions of Paragraph 8.3.4 with $\mathcal{I} := \{\{3\}\}$, and in addition, the non-degeneracy condition of Proposition 8.3.5 is also satisfied. It follows that

$$\begin{aligned} \text{ord}_P(f_3|_C) &= (\text{ND}(f_3), \text{ND}(f_1), \text{ND}(f_2))_0^* \\ &= \min_{\text{ND}(f_3)} (\nu) \times \text{MV}'_\nu(\text{In}_\nu(\text{ND}(f_1)), \text{In}_\nu(\text{ND}(f_2))) \quad (\text{where } \nu := (2, 1, 1)) \\ &= 1 \times 1 = 1 \end{aligned}$$

Similarly $\text{ord}_{(0,0,-2)}(f_3|_C) = 1$ as well. It follows that

$$(28) \quad \text{ord}_{(0,0,2)}(f_3|_C) + \text{ord}_{(0,0,-2)}(f_3|_C) = 2$$

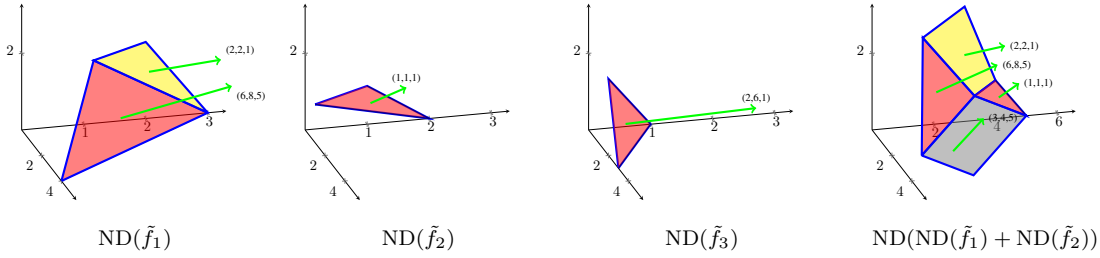


FIGURE 6. Newton diagrams in $(\tilde{x}, \tilde{y}, \tilde{w})$ -coordinates

8.3.7.4. Case (ii). It remains to consider the last case, i.e. $P = [0 : 0 : 1 : 0]$ with respect to homogeneous coordinates $[x : y : z : w]$. With respect to coordinates $(\tilde{x}, \tilde{y}, \tilde{w}) := (x/z, y/z, 1/z)$, P is the origin, and the f_i take the following form:

$$\tilde{f}_1 := \tilde{x}^4 + \tilde{x}\tilde{y}\tilde{w}^2 + \tilde{y}^2(\tilde{w}^2 - \tilde{y}\tilde{w} + 7\tilde{y})$$

$$\tilde{f}_2 := 2\tilde{x}\tilde{w} + \tilde{y}(\tilde{w} - \tilde{y})(4\tilde{w}^2 + 1)$$

$$\tilde{f}_3 := \tilde{x}^2(5\tilde{w}^2 + 8\tilde{x}) + \tilde{y}(\tilde{w}^3 - \tilde{y}\tilde{w}^2 + 3)$$

It is easy to check that the conditions of Paragraph 8.3.4 and the non-degeneracy condition of Proposition 8.3.5 are satisfied by $\tilde{f}_1, \tilde{f}_2, C$ with $\mathcal{I} := \{\{3\}\}$. It follows that

$$\begin{aligned} \text{ord}_P(\tilde{f}_3|_C) &= (\text{ND}(\tilde{f}_3), \text{ND}(\tilde{f}_1), \text{ND}(\tilde{f}_2))_0^* \\ &= \sum_{i=1}^4 \min_{\text{ND}(\tilde{f}_3)} (\nu_i) \times \text{MV}'_{\nu_i}(\text{In}_{\nu_i}(\text{ND}(\tilde{f}_1)), \text{In}_{\nu_i}(\text{ND}(\tilde{f}_2))) \end{aligned}$$

where ν_i vary over the (inner) normals to the four facets of the Newton diagram of the sum of $\text{ND}(\tilde{f}_1)$ and \tilde{f}_2 (see the last panel of Figure 6). Since the initial term of \tilde{f}_2 (respectively, \tilde{f}_1) with

respect to weights $(6, 8, 5)$ (respectively, $(1, 1, 1)$) is a singleton, it follows that $MV'_{\nu_i}(\text{In}_{\nu_i}(\text{ND}(\tilde{f}_1)), \text{In}_{\nu_i}(\text{ND}(\tilde{f}_2))) = 0$ for $\nu_i = (6, 8, 5)$ or $(1, 1, 1)$. Consequently,

$$\begin{aligned} \text{ord}_P(\tilde{f}_3|_C) &= \min_{\text{ND}(\tilde{f}_3)} ((2, 2, 1)) \times MV'_{(2,2,1)}(\text{In}_{(2,2,1)}(\text{ND}(\tilde{f}_1)), \text{In}_{(2,2,1)}(\text{ND}(\tilde{f}_2))) \\ &\quad + \min_{\text{ND}(\tilde{f}_3)} ((3, 4, 5)) \times MV'_{(3,4,5)}(\text{In}_{(3,4,5)}(\text{ND}(\tilde{f}_1)), \text{In}_{(3,4,5)}(\text{ND}(\tilde{f}_2))) \\ &= 2 \times 1 + 4 \times 1 = 6 \end{aligned}$$

It then follows from identities (19), (20), (23), (26), (27), (28) that

$$(29) \quad (f_1, f_2, f_3)_{\mathbb{k}^3}^{iso} = 64 - 8 - 4 - 0 - 2 - 6 - 14 = 30$$

9. AN EXAMPLE: THE NUMBER OF TANGENT LINES TO FOUR GENERIC SPHERES

9.1. In this section we apply the results of Sections 7 and 8 to compute the number of tangent lines to four generic spheres in \mathbb{k}^3 in the case that $\text{char}(\mathbb{k}) = 0$. This number was determined to be 12 by Macdonald, Pach and Theobald [MPT01] by expressing the governing equations in a cleverly chosen system of coordinates so that a direct application of Bézout's theorem solves the problem. On the other hand, a natural setting of this counting problem is the Grassmannian of lines in \mathbb{P}^3 (or equivalently, planes in \mathbb{k}^4). Formulating the problem in corresponding Plücker coordinates gives 5 quadric equations in \mathbb{P}^5 whose common zeroes include a one-dimensional component at infinity. We explain the difference of $2^5 - 12 = 20$ between the Bézout bound and the correct number as the sum of ordered intersection multiplicities corresponding to the excess intersection. This section was inspired by a paragraph by Sottile and Theobald (in [ST02] which solved the tangent line problem in higher dimensions) which mentions a similar explanation in terms of usual excess intersection multiplicities in a manuscript by Aluffi and Fulton.

9.2. Consider the open subset U of the space of lines in \mathbb{k}^3 parameterized by

$$(u_1, u_2, v_1, v_2) \in \mathbb{k}^4 \mapsto L_{u_1, u_2, v_1, v_2} := \{(u_1 + tv_1, u_2 + tv_2, t) : t \in \mathbb{k}\}$$

The points of intersection of L_{u_1, u_2, v_1, v_2} and the sphere in \mathbb{k}^3 with center (a_1, a_2, a_3) and radius r_1 correspond to solutions (in t) of the following equation:

$$(u_1 + tv_1 - a_1)^2 + (u_2 + tv_2 - a_2)^2 + (t - a_3)^2 = r_1^2$$

Consequently, L_{u_1, u_2, v_1, v_2} is tangent to the sphere if and only if the discriminant of the above equation is zero, i.e.

$$(v_1(u_1 - a_1) + v_2(u_2 - a_2) - a_3)^2 - (v_1^2 + v_2^2 + 1)((u_1 - a_1)^2 + (u_2 - a_2)^2 + a_3^2 - r_1^2) = 0$$

which simplifies to

$$((u_2v_1 - u_1v_2) - (a_2v_1 - a_1v_2))^2 + (u_1 + a_3v_1 - a_1)^2 + (u_2 + a_3v_2 - a_2)^2 - r_1^2(v_1^2 + v_2^2 + 1) = 0$$

It turns out that (up to changing some signs) the Plücker embedding of the Grassmannian of lines in \mathbb{P}^3 is determined by (u_1, u_2, v_1, v_2, w) , where $w := u_2v_1 - u_1v_2$. Consequently, in the Plücker

coordinates, the lines tangent to four spheres are precisely the points in $V(f_0, \dots, f_4) \subseteq \mathbb{k}^5$, where

$$\begin{aligned} f_0 &:= w - u_2 v_1 + u_1 v_2 \\ f_1 &:= (w - (a_2 v_1 - a_1 v_2))^2 + (u_1 + a_3 v_1 - a_1)^2 + (u_2 + a_3 v_2 - a_2)^2 - r_1^2(v_1^2 + v_2^2 + 1) \\ f_2 &:= (w - (b_2 v_1 - b_1 v_2))^2 + (u_1 + b_3 v_1 - b_1)^2 + (u_2 + b_3 v_2 - b_2)^2 - r_2^2(v_1^2 + v_2^2 + 1) \\ f_3 &:= (w - (c_2 v_1 - c_1 v_2))^2 + (u_1 + c_3 v_1 - c_1)^2 + (u_2 + c_3 v_2 - c_2)^2 - r_3^2(v_1^2 + v_2^2 + 1) \\ f_4 &:= (w - (d_2 v_1 - d_1 v_2))^2 + (u_1 + d_3 v_1 - d_1)^2 + (u_2 + d_3 v_2 - d_2)^2 - r_4^2(v_1^2 + v_2^2 + 1) \end{aligned}$$

where the r_1, \dots, r_4 are the radii and $(a_1, a_2, a_3), \dots, (d_1, d_2, d_3)$ are the centers of the spheres. Since f_0 and f_1 does not have any irreducible components in common, the codimension of $V(f_0, f_1)$ is 2 in \mathbb{k}^5 . Then for generic choices e.g. of b_1, c_1, d_1 , it follows that $V(f_0, \dots, f_4)$ only has isolated points in \mathbb{k}^5 . We will show that the number (counted with appropriate multiplicities) of these points is 12. Note that the Bézout bound for the number of solutions of the f_i in \mathbb{P}^5 is $2^5 = 32$.

9.3. Choosing appropriate coordinates on \mathbb{k}^3 , without loss of generality we may assume $a_1 = a_2 = a_3 = b_1 = b_2 = 0$ and $r_1 = 1$. It follows that

$$\begin{aligned} f_1 &= w^2 + u_1^2 + u_2^2 - (v_1^2 + v_2^2 + 1) \\ f_2 &= w^2 + (u_1 + b_3 v_1)^2 + (u_2 + b_3 v_2)^2 - r_2^2(v_1^2 + v_2^2 + 1) \end{aligned}$$

9.4. Let F_i be the divisor on \mathbb{P}^5 corresponding to f_i . To determine the intersection “at infinity” on \mathbb{P}^5 of the F_i , we consider their leading forms:

$$\begin{aligned} \text{Ld}(f_0) &= u_2 v_1 - u_1 v_2 \\ \text{Ld}(f_1) &= w^2 + u_1^2 + u_2^2 - v_1^2 - v_2^2 \\ \text{Ld}(f_2) &= w^2 + (u_1 + b_3 v_1)^2 + (u_2 + b_3 v_2)^2 - r_2^2(v_1^2 + v_2^2) \\ \text{Ld}(f_3) &= (w - (c_2 v_1 - c_1 v_2))^2 + (u_1 + c_3 v_1)^2 + (u_2 + c_3 v_2)^2 - r_3^2(v_1^2 + v_2^2) \\ \text{Ld}(f_4) &= (w - (d_2 v_1 - d_1 v_2))^2 + (u_1 + d_3 v_1)^2 + (u_2 + d_3 v_2)^2 - r_4^2(v_1^2 + v_2^2) \end{aligned}$$

$V(\text{Ld}(f_0), \text{Ld}(f_1))$ has codimension two in the hyperplane at infinity on \mathbb{P}^5 . For generic choices of the b_3, c_i, d_j it follows that the common zero set of the $\text{Ld}(f_i)$ is empty if either v_1 or v_2 is nonzero, and consequently,

$$\bigcap_i \text{Supp}(F_i) \cap (\mathbb{P}^5 \setminus \mathbb{k}^5) = Z := V(z, v_1, v_2, w^2 + u_1^2 + u_2^2)$$

where z is the “homogenizing variable” which forms a system of homogeneous coordinates $[u_1 : u_2 : v_1 : v_2 : w : z]$ on \mathbb{P}^5 ; note that Z is a one dimensional “imaginary circle at infinity”. It then follows from identity (1) on page 4 and Bézout’s theorem that

$$\begin{aligned} (f_0, \dots, f_4)_{\mathbb{k}^5}^{\text{iso}} &= (F_0, \dots, F_4) - (F_0, \dots, F_4)_Z^{\text{ord}} - \sum_{P \in Z} (F_0, \dots, F_4)_P^{\text{ord}} \\ (30) \quad &= 32 - (F_0, \dots, F_4)_Z^{\text{ord}} - \sum_{P \in Z} (F_0, \dots, F_4)_P^{\text{ord}} \end{aligned}$$

9.5. Since Z is an irreducible component of $\bigcap_{j=0}^3 \text{Supp}(F_j)$, assertion (1) of Proposition 7.4 implies that

$$(31) \quad \begin{aligned} (F_0, \dots, F_4)_{\tilde{Z}}^{\text{ord}} &= (F_0, \dots, F_3)_Z \deg_4(Z) = (F_0, \dots, F_3)_Z \deg(Z) \deg(F_4) \\ &= 4(F_0, \dots, F_3)_Z \end{aligned}$$

To compute $(F_0, \dots, F_3)_Z$, we work on the coordinate chart $U_w := \mathbb{P}^5 \setminus V(w)$. With the coordinates $(\tilde{u}_1, \tilde{u}_2, \tilde{v}_1, \tilde{v}_2, \tilde{z}) := (u_1/w, u_2/w, v_1/w, v_2/w, 1/w)$ on U_w , the polynomials defining $F_i \cap U_w$ are:

$$\begin{aligned} \tilde{f}_0 &:= \tilde{z} - \tilde{u}_2 \tilde{v}_1 + \tilde{u}_1 \tilde{v}_2 \\ \tilde{f}_1 &:= 1 + \tilde{u}_1^2 + \tilde{u}_2^2 - (\tilde{v}_1^2 + \tilde{v}_2^2 + \tilde{z}^2) \\ \tilde{f}_2 &:= 1 + (\tilde{u}_1 + b_3 \tilde{v}_1)^2 + (\tilde{u}_2 + b_3 \tilde{v}_2)^2 - r_2^2(\tilde{v}_1^2 + \tilde{v}_2^2 + \tilde{z}^2) \\ \tilde{f}_3 &:= (1 - (c_2 \tilde{v}_1 - c_1 \tilde{v}_2))^2 + (\tilde{u}_1 + c_3 \tilde{v}_1 - c_1 \tilde{z})^2 + (\tilde{u}_2 + c_3 \tilde{v}_2 - c_2 \tilde{z})^2 - r_3^2(\tilde{v}_1^2 + \tilde{v}_2^2 + \tilde{z}^2) \\ \tilde{f}_4 &:= (1 - (d_2 \tilde{v}_1 - d_1 \tilde{v}_2))^2 + (\tilde{u}_1 + d_3 \tilde{v}_1 - d_1 \tilde{z})^2 + (\tilde{u}_2 + d_3 \tilde{v}_2 - d_2 \tilde{z})^2 - r_4^2(\tilde{v}_1^2 + \tilde{v}_2^2 + \tilde{z}^2) \end{aligned}$$

Note that $\tilde{S} := V(\tilde{f}_0, \tilde{f}_1)$ is a nonsingular threefold near each point of $Z \cap U_w$, so that $(F_0, \dots, F_3)_Z$ is the intersection multiplicity along Z of the (restrictions of the) divisors of \tilde{f}_2 and \tilde{f}_3 on \tilde{S} ; in other words,

$$(32) \quad (F_0, \dots, F_3)_Z = (\tilde{f}_2|_{\tilde{S}}, \tilde{f}_3|_{\tilde{S}})_Z$$

9.5.1. To compute the divisor of \tilde{f}_2 on \tilde{S} note that

$$\begin{aligned} \tilde{f}_2 &= 1 + \tilde{u}_1^2 + \tilde{u}_2^2 + 2b_3(\tilde{u}_1 \tilde{v}_1 + \tilde{u}_2 \tilde{v}_2) - r_2^2(\tilde{v}_1^2 + \tilde{v}_2^2 + \tilde{z}^2) \\ &\equiv 2b_3(\tilde{u}_1 \tilde{v}_1 + \tilde{u}_2 \tilde{v}_2) + (1 - r_2^2)(\tilde{v}_1^2 + \tilde{v}_2^2 + \tilde{z}^2) \pmod{\tilde{f}_1} \end{aligned}$$

Now

$$\begin{aligned} \tilde{v}_1^2 + \tilde{v}_2^2 + \tilde{z}^2 &\equiv \tilde{v}_1^2 + \tilde{v}_2^2 + (\tilde{u}_2 \tilde{v}_1 - \tilde{u}_1 \tilde{v}_2)^2 \pmod{\tilde{f}_0} \\ &= \tilde{v}_1^2(1 + \tilde{u}_2^2) + \tilde{v}_2^2(1 + \tilde{u}_1^2) - 2\tilde{u}_1 \tilde{u}_2 \tilde{v}_1 \tilde{v}_2 \\ &= \tilde{v}_1^2(1 + \tilde{u}_1^2 + \tilde{u}_2^2) + \tilde{v}_2^2(1 + \tilde{u}_1^2 + \tilde{u}_2^2) - (\tilde{u}_1^2 \tilde{v}_1^2 + \tilde{u}_2^2 \tilde{v}_2^2 + 2\tilde{u}_1 \tilde{u}_2 \tilde{v}_1 \tilde{v}_2) \\ &= (1 + \tilde{u}_1^2 + \tilde{u}_2^2)(\tilde{v}_1^2 + \tilde{v}_2^2) - (\tilde{u}_1 \tilde{v}_1 + \tilde{u}_2 \tilde{v}_2)^2 \\ &\equiv (\tilde{v}_1^2 + \tilde{v}_2^2 + \tilde{z}^2)(\tilde{v}_1^2 + \tilde{v}_2^2) - (\tilde{u}_1 \tilde{v}_1 + \tilde{u}_2 \tilde{v}_2)^2 \pmod{\tilde{f}_1} \end{aligned}$$

so that, modulo \tilde{f}_0 and \tilde{f}_1 ,

$$\tilde{v}_1^2 + \tilde{v}_2^2 + \tilde{z}^2 \equiv -\frac{(\tilde{u}_1 \tilde{v}_1 + \tilde{u}_2 \tilde{v}_2)^2}{1 - \tilde{v}_1^2 - \tilde{v}_2^2}$$

near $Z \cap U_w$. Consequently,

$$(33) \quad \tilde{f}_2 \equiv \tilde{v}_0(2b_3 - \frac{1 - r_2^2}{1 - \tilde{v}_1^2 - \tilde{v}_2^2} \tilde{v}_0)$$

modulo \tilde{f}_0 and \tilde{f}_1 near $Z \cap U_w$, where $\tilde{v}_0 := (\tilde{u}_1 \tilde{v}_1 + \tilde{u}_2 \tilde{v}_2)$. In particular, when $b_3 \neq 0$, the divisor of \tilde{f}_2 on \tilde{S} is defined by \tilde{v}_0 near all points of $Z \cap U_w$.

9.5.2. Now we compute the divisor of \tilde{f}_3 on \tilde{S} . Since $(\tilde{v}_0, \tilde{v}_1)$ have linearly independent differentials whenever $\tilde{u}_2 \neq 0$, we may mod out by the relation $\tilde{v}_2 = (\tilde{v}_0 - \tilde{u}_1\tilde{v}_1)/\tilde{u}_2$ as well as by f_0 and f_1 . This yields, after a long computation, that

$$(34) \quad \tilde{f}_3 \equiv \frac{\tilde{h}_{3,0}}{(1 - \tilde{v}_1^2 - \tilde{v}_2^2)\tilde{u}_2^2} \tilde{v}_0 + \frac{1}{\tilde{u}_2^2} ((c_1\tilde{u}_1 + c_2\tilde{u}_2)^2 + (c_3\tilde{u}_2 + c_1)^2 + (c_3\tilde{u}_1 - c_2)^2) \tilde{v}_1^2$$

where $\tilde{h}_{3,0}$ is a polynomial in all $\tilde{u}_i, \tilde{v}_j, c_k$ and r_3 .

9.5.3. It follows that at each point P of $Z \cap U_w \cap \{\tilde{u}_2 \neq 0\}$, the ideal generated by \tilde{f}_2, \tilde{f}_3 in $\mathcal{O}_{P,\tilde{S}}$ is the same as the ideal generated by \tilde{v}_0 and $\tilde{h}_{3,1}\tilde{v}_1^2$, where

$$\tilde{h}_{3,1} := (c_1\tilde{u}_1 + c_2\tilde{u}_2)^2 + (c_3\tilde{u}_2 + c_1)^2 + (c_3\tilde{u}_1 - c_2)^2$$

Since $\tilde{h}_{3,1} \neq 0$ near generic points of Z , it follows from identities (33) and (34) that

$$(\tilde{f}_2|_{\tilde{S}}, \tilde{f}_3|_{\tilde{S}})_Z = (\tilde{v}_0|_{\tilde{S}}, \tilde{v}_1^2|_{\tilde{S}})_Z = 2$$

as long as $b_3 \neq 0$ (since \tilde{v}_0, \tilde{v}_1 generate the ideal of Z in $\mathcal{O}_{P,\tilde{S}}$ for generic $P \in \tilde{S} \cap Z$, the above equality can be regarded as a (rather trivial) application of Proposition 8.2.1). Identities (31) and (32) then imply that

$$(35) \quad (F_0, \dots, F_4)_Z^{\text{ord}} = 4 \times 2 = 8$$

9.6. It remains to compute the sum $\sum_{P \in Z} (F_0, \dots, F_4)_P^{\text{ord}}$ from the right hand side of (30). Write the scheme theoretic intersection of F_0, \dots, F_3 as $mZ + C$. Then

$$(36) \quad \sum_{P \in Z} (F_0, \dots, F_4)_P^{\text{ord}} = \sum_{P \in Z} \text{ord}_P(F_4|_C)$$

(see e.g. Example 7.8).

9.6.1. First we consider the case that $P \in U_{w,2} := U_w \cap \{\tilde{u}_2 \neq 0\}$. The arguments from Section 9.5 imply that

- if $b_3 \neq 0$, then $C \cap U_{w,2}$ is the scheme theoretic intersection of \tilde{S} and $V(\tilde{v}_0, \tilde{h}_{3,1})$, where $\tilde{h}_{3,1} := (c_1\tilde{u}_1 + c_2\tilde{u}_2)^2 + (c_3\tilde{u}_2 + c_1)^2 + (c_3\tilde{u}_1 - c_2)^2$, and
- the restriction of \tilde{f}_4 to $C \cap U_{w,2}$ equals (the restriction of) $\tilde{h}_{4,1}\tilde{v}_1^2/\tilde{u}_2^2$, where $\tilde{h}_{4,1} := (d_1\tilde{u}_1 + d_2\tilde{u}_2)^2 + (d_3\tilde{u}_2 + d_1)^2 + (d_3\tilde{u}_1 - d_2)^2$.

Since $U_{w,2} \cap Z \cap C = U_{w,2} \cap V(\tilde{z}, \tilde{v}_1, \tilde{v}_2, \tilde{u}_1^2 + \tilde{u}_2^2 + 1, \tilde{h}_{3,1})$, for generic c_1, c_2, c_3 , there are precisely $2 \times 2 = 4$ points in $U_{w,2} \cap Z \cap C$ due to Bézout's theorem. If d_1, d_2, d_3 are generic, then $\tilde{h}_{4,1}$ is nonzero at each of these points, so that

$$(37) \quad \begin{aligned} \sum_{P \in Z \cap U_{w,2}} \text{ord}_P(F_4|_C) &= \sum_{P \in Z \cap C \cap U_{w,2}} \text{ord}_P((\tilde{h}_{4,1}\tilde{v}_1^2)|_C) \\ &= \sum_{P \in Z \cap C \cap U_{w,2}} \text{ord}_P(\tilde{v}_1^2|_C) = 4 \times 2 = 8 \end{aligned}$$

9.6.2. It remains to consider the points in $Z \setminus U_{w,2} = V(z, v_1, v_2, u_2, u_1^2 + w^2) \cup V(z, v_1, v_2, w, u_1^2 + u_2^2)$. First we consider the case that $P \in V(z, v_1, v_2, u_2, u_1^2 + w^2)$. Running the same computations from Paragraph 9.5.2 with the substitution $\tilde{v}_1 = (\tilde{v}_0 - \tilde{u}_2 \tilde{v}_2)/\tilde{u}_1$ (instead of the substitution of \tilde{v}_2) shows that near Z on $\tilde{S} \cap \{\tilde{u}_1 \neq 0\}$,

$$\tilde{f}_3 \equiv \frac{\tilde{h}_{3,1}}{\tilde{u}_1^2} \tilde{v}_1^2$$

modulo (the ideal generated by) \tilde{v}_0 . If c_1, c_2, c_3 are generic, then $\tilde{h}_{3,1}$ is nonzero when $\tilde{u}_2 = 0$. It follows that for each $P \in V(z, v_1, v_2, u_2, u_1^2 + w^2)$, Z is the *only* irreducible component of $\bigcap_{i=0}^3 F_i$ near P ; in particular,

$$(38) \quad \sum_{P \in V(z, v_1, v_2, u_2, u_1^2 + w^2)} \text{ord}_P(F_4|_C) = 0$$

9.6.3. Now we treat the case that $P \in V(z, v_1, v_2, w, u_1^2 + u_2^2)$. Consider the chart $U_2 := \mathbb{P}^5 \setminus V(u_2)$ with coordinates $(u'_1, v'_1, v'_2, w', z') := (u_1/u_2, v_1/u_2, v_2/u_2, w/u_2, 1/u_2)$. The polynomials defining $F_i|_{U_2}$ are:

$$\begin{aligned} f'_0 &:= z'w' - v'_1 + u'_1v'_2 \\ f'_1 &:= w'^2 + u_1'^2 + 1 - (v_1'^2 + v_2'^2 + z'^2) \\ f'_2 &:= w'^2 + (u'_1 + b_3v'_1)^2 + (1 + b_3v'_2)^2 - r_2^2(v_1'^2 + v_2'^2 + z'^2) \\ f'_3 &:= (w' - (c_2v'_1 - c_1v'_2))^2 + (u'_1 + c_3v'_1 - c_1z')^2 + (1 + c_3v'_2 - c_2z')^2 - r_3^2(v_1'^2 + v_2'^2 + z'^2) \\ f'_4 &:= (w' - (d_2v'_1 - d_1v'_2))^2 + (u'_1 + d_3v'_1 - d_1z')^2 + (1 + d_3v'_2 - d_2z')^2 - r_4^2(v_1'^2 + v_2'^2 + z'^2) \end{aligned}$$

Let $S' := V(f'_0, f'_1) \subseteq U_2$. Computations similar to those in Paragraph 9.5.1 show that the following hold in $\hat{\mathcal{O}}_{P,S'} \cong \mathbb{k}[[v'_2, w', z']]$:

$$\begin{aligned} f'_2 &= (1 - r_2^2)z'^2 + 2b_3u'_1w'z' - 2b_3v'_2w'^2 + \cdots \\ f'_3 &= -2(c_2 + c_1u'_1)z' - 2(c_2u'_1 - c_1)v'_2w' + (c_2u'_1 - c_1)^2v'_2{}^2 + \cdots \end{aligned}$$

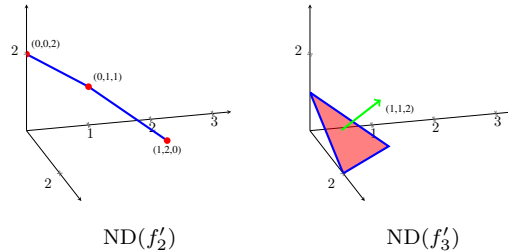


FIGURE 7. Newton diagrams of f'_2 and f'_3 in (v'_2, w', z') -coordinates

Since near P on a set-theoretic level $V(f'_2, f'_3) \cap S'$ is the union of C and $Z \cap S' \cong V(v'_2, z')$, we may try to use Proposition 8.3.5 to estimate $\text{ord}_P(f'_4|_C)$. However, it is straightforward to check that

for weighted orders with weights $\nu := (1, \omega, 1 + \omega)$ with respect to (v'_2, w', z') -coordinates, where $0 < \omega < 1$, the initial forms of f'_2 and f'_3 at P are:

$$\text{In}_\nu(f'_2) = 2b_3w'(u'_1z' - v'_2w') = 2b_3u'_1w'(z' + u'_1v'_2w')$$

$$\text{In}_\nu(f'_3) = -2((c_2 + c_1u'_1)z' + (c_2u'_1 - c_1)v'_2w') = -2(c_2 + c_1u'_1)(z' + u'_1v'_2w')$$

(since $u_1'^2 = -1$ at P) which are *proportional*, and consequently, have common solutions in $(\mathbb{k} \setminus \{0\})^3$, e.g. $(v'_2, w', z') = (1, 1, -u'_1)$. Consequently, (f'_2, f'_3) violates condition (d) of Paragraph 8.3.4, and Proposition 8.3.5 does *not* apply. However, if we let $z'' := z' + u'_1v'_2w'$ and expand f'_2, f'_3 in (v'_2, w', z'') -coordinates, then

$$f'_2 = (1 - r_2^2)z''^2 + 2b_3u'_1w'z'' + \dots$$

$$f'_3 = -2(c_2 + c_1u'_1)z'' + (c_2u'_1 - c_1)^2v_2'^2 + \dots$$

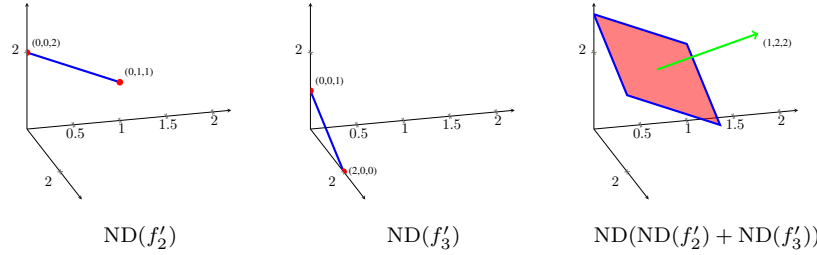


FIGURE 8. Newton diagrams in (v'_2, w', z'') -coordinates

It is straightforward to check that if b_3, c_1, c_2 are generic, then (f'_2, f'_3, f'_4) satisfy all assumptions and non-degeneracy conditions of Proposition 8.3.5, so that

$$\begin{aligned} \text{ord}_P(f'_4|_C) &= (\text{ND}(f'_2), \text{ND}(f'_3), \text{ND}(f'_4))_0^* \\ &= \min_{\text{ND}(f'_4)} (\nu) \times \text{MV}'_\nu(\text{In}_\nu(\text{ND}(f'_2)), \text{In}_\nu(\text{ND}(f'_4))) \quad (\text{where } \nu := (1, 2, 2)) \\ &= 2 \times 1 = 2 \end{aligned}$$

Since there are two points in $V(z, v_1, v_2, w, u_1^2 + u_2^2)$, it follows that

$$(39) \quad \sum_{P \in V(z, v_1, v_2, w, u_1^2 + u_2^2)} \text{ord}_P(F_4|_C) = 2 \times 2 = 4$$

9.7. Combining equations (30), (35), (36), (37), (38), (39) we obtain

$$(f_0, \dots, f_4)_{\mathbb{k}^5}^{iso} = 32 - 8 - 8 - 0 - 4 = 12$$

when b_3, c_i, d_j are generic. This completes the computation of the number of tangent lines to four generic spheres in \mathbb{k}^3 (when $\text{char}(\mathbb{k}) = 0$). Note that the above arguments did *not* require any r_i to be generic, i.e. we showed that the bound of 12 tangent lines can be achieved even if the radii of the spheres are fixed (and nonzero).

10. INDUCTIVE “SOLUTIONS” TO THE AFFINE BÉZOUT PROBLEM

In Sections 7 to 9 we described a few scenarios in which ordered intersection multiplicity can be interpreted in terms of intersection multiplicities of regular sequences. However, as pointed out in Paragraph 8.3.6, those techniques are *not* always applicable. In this section we present some other interpretations in terms of intersection multiplicities of regular sequences which, in theory, are more generally available, but seem rather non-constructive (those from Sections 10.1 and 10.2) or complicated to obtain (the interpretation from Section 10.3). Each of these interpretations, say interpretation \mathcal{I} , gives rise to a (similarly non-constructive or complicated) algorithm for counting isolated solutions of (square) systems of polynomials as follows:

- (1) start with an *upper* bound which is exact in a generic scenario (e.g. Bézout’s bound or some other Bernstein-Kushnirenko type estimate [Mon21, Theorems X.4 and X.7] when the ambient space is \mathbb{k}^n or \mathbb{P}^n); if the system is *non-degenerate*, i.e. there are no “extra intersections” (e.g. “intersections at infinity” or non-isolated points in the intersection), then this bound gives the correct result;
- (2) otherwise express the corresponding ordered multiplicities in terms of usual intersection multiplicities of regular sequences using interpretation \mathcal{I} ;
- (3) compute Bernstein-Kushnirenko type *lower* bounds for the latter intersection multiplicity in terms of associated Newton diagrams (using Theorem 8.1.6 and Proposition 8.2.1); if the system is non-degenerate, then these estimates are exact;
- (4) otherwise there are excess intersections in corresponding toric blow-ups at the origin, and repeat the process starting from step (2) to estimate the corresponding ordered intersection multiplicities.

10.1. An “asymptotic” interpretation. Consider the set up from Section 6.1. Given $\vec{g} = (g_1, \dots, g_n) \in \prod_i L_i$ and $\vec{r} = (r_1, \dots, r_n) \in \mathbb{R}^n$, write $\tilde{X}(\vec{g}; \vec{r})$ for the “weighted blow up” of $\bar{X} \times \mathbb{k}$ considered in Section 6.3, i.e. the closure in $\bar{X} \times \mathbb{k} \times \mathbb{P}^n(r_1, \dots, r_n, 1)$ of the graph of the map

$$\bar{X} \times \mathbb{k} \ni (x, t) \mapsto \left[\frac{f_1}{g_1}(x) : \dots : \frac{f_n}{g_n}(x) : t \right] \in \mathbb{P}^n(r_1, \dots, r_n, 1)$$

As in Section 6.3 denote the (weighted) homogeneous coordinates on $\mathbb{P}^n(r_1, \dots, r_n, 1)$ by $[w_1 : \dots : w_{n+1}]$, and let $u_i := w_i / (w_{n+1})^{r_i}$, $i = 1, \dots, n$, be the standard affine coordinates on $\tilde{U}_{n+1} := \tilde{X}(\vec{g}; \vec{r}) \setminus V(w_{n+1})$. The following result shows that under (condition $r./r._+ >> 1$) the ordered intersection multiplicity can be interpreted as a sum of usual intersection multiplicities at nonsingular points on the normalization of \tilde{U}_{n+1} .

Proposition 10.1.1. *Let X be an n -dimensional affine variety, $f_1, \dots, f_n \in \mathbb{k}[X]$, and for each $i = 1, \dots, n$, let L_i be a base-point free (finite dimensional) vector subspace of $\mathbb{k}[X]$ containing f_i . Let Z be a distinguished component for (the ordered intersection of the hypersurfaces defined by) $V(f_1), \dots, V(f_n)$ (Definition 2.8), and \check{Z} be the intersection of Z with the union of all other distinguished components Z' with $\text{codim}(Z') \geq \text{codim}(Z)$. Then for each nonempty open subset Z^* of Z , there are $\nu \in \mathbb{R}$, and a nonempty open subset L^* of $\prod_i L_i$ such that under (condition $r./r._+ > \nu$), for each $(g_1, \dots, g_n) \in L^*$, the normalization \tilde{U}'_{n+1} of \tilde{U}_{n+1} is nonsingular at each point of $V(t, u_1 - 1, \dots, u_n - 1)$, and*

$$(F_1, \dots, F_n)_Z^{\text{ord}} = \sum_{u \in \tilde{U}'_{n+1} \cap \pi^{-1}(Z^* \setminus \check{Z})} (t, u_1 - 1, \dots, u_n - 1)_u$$

where $\pi : \tilde{U}'_{n+1} \rightarrow \bar{X}$ is the natural projection.

Proof. This follows from arguments as in the proof of Claim 6.3.1 and Proposition 6.3.4 and the observation that a normal variety is nonsingular in codimension one. \square

10.2. An expression in positive characteristic. Cowsik and Nori [CN78] showed that affine curves are set theoretic complete intersections in positive characteristic. It follows that in positive characteristic each “summand” of $(F_1, \dots, F_n)_{\mathbb{A}^n}^{\text{ord}}$ can be represented as an intersection multiplicity up to a multiple; more precisely, for each $\vec{i} = (i_1, \dots, i_\rho)$, consider the curves $C_{\vec{i}, \vec{g}} = V(I_{i_1, \dots, i_\rho}^{(\rho-1)}, g_{i'_1}, \dots, g_{i'_{n-\rho}})$ considered in Section 7.5, where i'_j are as usual the elements of $\{1, \dots, n\} \setminus \{i_1, \dots, i_\rho\}$, and $g_{i'_j}$ are generic elements from $\mathcal{L}_{i'_j}$. By Cowsik and Nori’s theorem, for each irreducible component $C_{\vec{i}, \vec{g}; k}$ of $C_{\vec{i}, \vec{g}}$, one can find $h_{i_1, \dots, i_\rho; k, l}$, $l = 1, \dots, n - 1$, such that set theoretically

$$C_{\vec{i}, \vec{g}; k} = \bigcap_{l=1, \dots, n-1} V(h_{i_1, \dots, i_\rho; k, l})$$

Proposition 10.2.1. *With the above set up,*

$$\begin{aligned} \text{len}(\mathcal{O}_{Z, X} / I_{i_1, \dots, i_\rho}^{(\rho)} \mathcal{O}_{Z, X}) \deg_{i'_1, \dots, i'_{n-\rho}}(Z) &= \sum_k \frac{1}{n_{i_1, \dots, i_\rho; k}} \sum_{z \in Z} (h_{i_1, \dots, i_\rho; k, 1}, \dots, h_{i_1, \dots, i_\rho; k, n-1}, f_{i_\rho})_z \\ (F_1, \dots, F_n)_{\mathbb{A}^n}^{\text{ord}} &= \sum_{\substack{i_1, \dots, i_\rho, k \\ z \in Z}} \frac{1}{n_{i_1, \dots, i_\rho; k}} (h_{i_1, \dots, i_\rho; k, 1}, \dots, h_{i_1, \dots, i_\rho; k, n-1}, f_{i_\rho})_z \end{aligned}$$

Proof. This follows immediately from Proposition 7.6. \square

10.3. Expression for irreducible crossing divisors. Given a Cartier divisor D on X , we say that D is an *irreducible crossing divisor* if for each $z \in \text{Supp}(D)$, there is an open neighborhood U of z in X , and regular functions h_1, \dots, h_k on U such that

- (1) D is represented on U by a monomial in h_1, \dots, h_k ,
- (2) $V(h_{k_1}, \dots, h_{k_m}) \cap U$ is irreducible for each choice of k_1, \dots, k_m .

We say that D is a *simplicial crossing divisor* if the following stronger version of property (2) is satisfied:

- (2') if k_1, \dots, k_m are pairwise distinct, then $V(h_{k_1}, \dots, h_{k_m}) \cap U$ is irreducible of *codimension* m if $m \leq n$, and empty if $m > n$.

We now show that the ordered intersection multiplicity of F_1, \dots, F_n along a variety Z can be expressed in terms of the usual intersection multiplicities if $\bigcup \text{Supp}(F_i)$ is the support of an irreducible crossing divisor near generic points of Z . Since a strict normal crossing divisor is a simplicial crossing divisor, and in particular, an irreducible crossing divisor, this gives another inductive route to the affine Bézout problem when resolution of singularities is available (e.g. in characteristic zero).

Proposition 10.3.1. *Let Z be an irreducible component of $Z_{i_1, \dots, i_\rho}^{(\rho)}$ (as defined in Theorem 2.7) such that $Z \not\subseteq \text{Sing}(X)$. Assume there is an open subset \tilde{U} of X such that $\tilde{U} \cap Z \neq \emptyset$, and $\tilde{U} \cap \sum_i F_i$ is an irreducible crossing divisor. Then there is an open subset U of \tilde{U} and $f_1, \dots, f_n \in \mathbb{k}[U]$ such that*

- (1) $U \cap Z \neq \emptyset$,
- (2) each F_i is represented by f_i on U , and
- (3) there are divisors $f_{i_1, \dots, i_\rho; j, s_j}$ of f_{i_j} in $\mathcal{O}_{Z, X}$, $j = 1, \dots, \rho$, such that

$$\text{len}(\mathcal{O}_{Z, X} / I_{i_1, \dots, i_\rho}^{(\rho)} \mathcal{O}_{Z, X}) = \sum_{s_1, \dots, s_\rho} (f_{i_1, \dots, i_\rho; 1, s_1}, \dots, f_{i_1, \dots, i_\rho; \rho, s_\rho})_Z$$

If $\sum_i F_i$ is a simplicial crossing divisor, then there is only one possible choice of s_j for each j , and more precisely, there are irreducible elements $f'_{i_1, \dots, i_\rho; j} \in \mathcal{O}_{Z, X}$ and positive integers $\alpha_{i_1, \dots, i_\rho; j}$, $j = 1, \dots, \rho$, such that for each j ,

- (4) up to multiplication by a unit in $\mathcal{O}_{Z, X}$, f_{i_j} equals $(f'_{i_1, \dots, i_\rho; j})^{\alpha_{i_1, \dots, i_\rho; j}}$ times a monomial in $f'_{i_1, \dots, i_\rho; j+1}, \dots, f'_{i_1, \dots, i_\rho; \rho}$; in particular,
- (5) f_{i_ρ} equals $(f'_{i_1, \dots, i_\rho; \rho})^{\alpha_{i_1, \dots, i_\rho; \rho}}$ times a unit in $\mathcal{O}_{Z, X}$, and

$$\text{len}(\mathcal{O}_{Z, X} / I_{i_1, \dots, i_\rho}^{(\rho)} \mathcal{O}_{Z, X}) = ((f'_{i_1, \dots, i_\rho; 1})^{\alpha_{i_1, \dots, i_\rho; 1}}, \dots, (f'_{i_1, \dots, i_\rho; \rho})^{\alpha_{i_1, \dots, i_\rho; \rho}})_Z$$

Proof. Fix an open subset U of \tilde{U} such that $U \cap Z \neq \emptyset$, and each F_i is represented by some $f_i \in \mathbb{k}[U]$, and there are regular functions h_1, \dots, h_k on U such that

- (i) $\sum_i F_i$ is represented on U by a monomial in h_1, \dots, h_k , and
- (ii) $V(h_{k_1}, \dots, h_{k_m}) \cap U$ is irreducible for each choice of k_1, \dots, k_m .

Choosing U sufficiently small if necessary, we may also assume that

- (iii) each h_i is an irreducible element in $\mathcal{O}_{Z, X}$, and
- (iv) $Z \cap U \subseteq V(h_1, \dots, h_k)$.

Now consider the construction of the ideals $I_{i_1, \dots, i_\rho}^{(j)}$ from Theorem 2.11. Since $Z \not\subseteq \text{Sing}(X)$, the Auslander–Buchsbaum theorem implies that $\mathcal{O}_{Z, X}$ is a UFD, so that

- (v) each f_i is a monomial in h_1, \dots, h_k .

Since all sets of the form $V(h_{s_1}, \dots, h_{s_m}) \cap U$ are irreducible, this implies that for each $j = 1, \dots, \rho$,

- (vi) each primary component of $I_{i_1, \dots, i_\rho}^{(j)} \cap U$ is of the form $V(h_{s_1}^{m_1}, \dots, h_{s_j}^{m_j})$ where s_1, \dots, s_j are pairwise distinct, and $h_{s_l}^{m_l}$ are divisors of f_{i_l} , $l = 1, \dots, j$.

The preceding observation, applied to $j = \rho$, implies that

$$\text{len}(\mathcal{O}_{Z, X} / I_{i_1, \dots, i_\rho}^{(\rho)} \mathcal{O}_{Z, X}) = \sum_{s_1, \dots, s_\rho} (h_{s_1}^{m_1}, \dots, h_{s_\rho}^{m_\rho})_Z$$

where the sum is over all s_1, \dots, s_ρ such that $h_{s_1}^{m_1}, \dots, h_{s_\rho}^{m_\rho}$ generate a primary component of $I_{i_1, \dots, i_\rho}^{(\rho)}$. This proves the first assertion of Proposition 10.3.1. Now since Z is an irreducible component of $V(I_{i_1, \dots, i_\rho}^{(\rho)})$, observations (iv) and (vi) imply that

- (vii) $Z = V(h_1, \dots, h_k)$.

Replacing U by a smaller open subset if necessary, we may assume that $Z \cap U$ is the only irreducible component of $Z_{i_1, \dots, i_\rho}^{(\rho)} \cap U = Z_{i_1, \dots, i_\rho}^{(\rho-1)} \cap V(f_{i_\rho}) \cap U$. Now assume $\sum_i F_i$ is a simplicial crossing divisor. Then

- (viii) $\rho = \text{codim}(Z) = \text{codim}(h_1, \dots, h_k) = k$.

On the other hand, it follows from the definition of the $I_{i_1, \dots, i_\rho}^{(j)}$ that

- (ix) if $h_{s_1}^{m_1}, \dots, h_{s_j}^{m_j}$ generate a primary component of $I_{i_1, \dots, i_\rho}^{(j)}$, then none of the h_{s_l} divides $\prod_{j' > j} f_{i_{j'}}$.

Since $\rho = k$, applying the preceding observation to $j = \rho - 1$, then to $j = \rho - 2$, and so on up to $j = 1$, it follows after a renumbering of the h_s if necessary that for each $j = 1, \dots, \rho$,

- (x) up to multiplication by a unit in $\mathcal{O}_{Z, X}$, f_{i_j} equals $h_j^{m_j}$ times a monomial in h_{j+1}, \dots, h_ρ ;
- (xi) $I_{i_1, \dots, i_\rho}^{(j)}$ is generated by $h_1^{m_1}, \dots, h_j^{m_j}$.

This completes the proof of Proposition 10.3.1. \square

Part 3. Appendix

APPENDIX A. MISCELLANEOUS ALGEBRAIC RESULTS USED IN THE ARTICLE

Lemma A.1. *Let C be a (possibly non-reduced) curve, $z \in C$, and t be a non zero-divisor and non-unit in $\mathcal{O}_{z,C}$, so that $m := \text{len}(\hat{\mathcal{O}}_{z,C}/t\hat{\mathcal{O}}_{z,C})$ is positive and finite. If $h \in \mathcal{O}_{z,C}$ is nilpotent, then $h^m = 0$.*

Proof. Since t is a non zero-divisor in $\mathcal{O}_{z,C}$, it follows from standard commutative algebra that there is a $\mathbb{k}[[t]]$ -module isomorphism

$$\hat{\mathcal{O}}_{z,C} \cong \mathbb{k}[[t]]^m$$

(see e.g. [Mon21, Proof of Proposition C.6]). Since t is a non zero-divisor, it follows that

$$(\hat{\mathcal{O}}_{z,C})_t \cong \mathbb{k}((t))^m$$

as a vector space over the field $\mathbb{k}((t))$ of Laurent series in t . Now pick the smallest $s \geq 1$ such that $h^s = 0$. We claim that $1, h, \dots, h^{s-1}$ are linearly independent over $\mathbb{k}((t))$. Indeed, otherwise there is a relation of the form

$$h^{s'}(\phi_0(t) + \sum_{j=1}^{s''} h^j \phi_j(t)) = 0$$

with $0 \leq s' < s$ and each $\phi_j \in \mathbb{k}[[t]]$. However, since t is a non zero-divisor and h is nilpotent, it follows that $\phi_0(t) + \sum_{j=1}^{s''} h^j \phi_j(t)$ is also a non zero-divisor. Then $h^{s'} = 0$, contradicting the choice of s . This proves the claim. It follows that $s \leq m$, as required. \square

Proposition A.2. *Let I be an ideal of pure codimension k in a Noetherian ring R . Let $f \in R$ and $I' := IR_f \cap R$.*

- (1) *If $f \in \sqrt{I}$, then $I' = R$.*
- (2) *If $f \notin \sqrt{I}$, then I' has pure codimension k and $I' + fR$ has pure codimension $k + 1$ in R . Moreover, for each codimension $k + 1$ prime \mathfrak{p}' of R ,*

$$\begin{aligned} \text{len}(R_{\mathfrak{p}'} / (I' R_{\mathfrak{p}'} + f R_{\mathfrak{p}'})) &= \sum_{\substack{\mathfrak{p} \subseteq \mathfrak{p}' \\ \text{codim}(\mathfrak{p})=k}} \text{len}(R_{\mathfrak{p}} / I' R_{\mathfrak{p}}) \text{len}(R_{\mathfrak{p}'} / (\mathfrak{p} R_{\mathfrak{p}'} + f R_{\mathfrak{p}'})) \\ (40) \quad &= \sum_{\substack{f \notin \mathfrak{p} \subseteq \mathfrak{p}' \\ \text{codim}(\mathfrak{p})=k}} \text{len}(R_{\mathfrak{p}} / I R_{\mathfrak{p}}) \text{len}(R_{\mathfrak{p}'} / (\mathfrak{p} R_{\mathfrak{p}'} + f R_{\mathfrak{p}'})) \end{aligned}$$

Proof. The first assertion is immediate. So assume $f \notin \sqrt{I}$. Then $I' + fR$ has pure codimension $k + 1$ in R due to Krull's principal ideal theorem. So for each prime ideal \mathfrak{p} associated to I , we have

$$\text{codim}(I' + fR) = k + 1 > k = \text{codim}(\mathfrak{p})$$

so that

$$I' + fR \not\subseteq \mathfrak{p}$$

In particular, if $f \in \mathfrak{p}$, then $I' \not\subseteq \mathfrak{p}$, so that $I' R_{\mathfrak{p}} = R_{\mathfrak{p}}$. The second equality of (40) then follows from the following observation that

$$I' R_{\mathfrak{p}} = \begin{cases} R_{\mathfrak{p}} & \text{if } f \in \mathfrak{p}, \\ I R_{\mathfrak{p}} & \text{if } f \notin \mathfrak{p}. \end{cases}$$

Fix a codimension $k + 1$ prime \mathfrak{p}' of R containing $I' + fR$. An application of [Ful98, Lemma A.2.7] with $M = A = R_{\mathfrak{p}'} / I' R_{\mathfrak{p}'}$ shows that

$$\sum_{\substack{I' \subseteq \mathfrak{p} \subseteq \mathfrak{p}' \\ \text{codim}(\mathfrak{p})=k}} \text{len}(R_{\mathfrak{p}} / I' R_{\mathfrak{p}}) \text{len}(R_{\mathfrak{p}'} / (\mathfrak{p} R_{\mathfrak{p}'} + f R_{\mathfrak{p}'})) = e_{R_{\mathfrak{p}'} / I' R_{\mathfrak{p}'}}(f)$$

where $e(\cdot)$ is the “multiplicity” defined by

$$e_A(a) := \text{len}(A/aA) - \text{len}(0 : a)$$

However, it follows from the definition of I' that f is a non zero-divisor in $R_{\mathfrak{p}'} / I' R_{\mathfrak{p}'}$, and consequently,

$$e_{R_{\mathfrak{p}'} / I' R_{\mathfrak{p}'}}(f) = \text{len}(R_{\mathfrak{p}'} / (I' R_{\mathfrak{p}'} + f R_{\mathfrak{p}'}))$$

as required. \square

APPENDIX B. A RESULT ABOUT COMPLETIONS OF LOCAL RINGS AT A SUBVARIETY

The goal of this section is to prove Lemma **B.2** which is used in Appendix **C** to prove Claim **6.4.2**.

Theorem B.1. *Let X be an irreducible affine variety, Z be an irreducible subvariety of X . Let $\phi_1, \dots, \phi_q, f_1, \dots, f_r \in \mathbb{k}[X]$ such that*

- (1) $\mathbb{k}[Z]$ is algebraic over $\mathbb{k}[\phi_1|_Z, \dots, \phi_q|_Z]$,
- (2) Z is set-theoretically the closed subset $V(f_1, \dots, f_r)$ of X .

Then

- (i) $\mathbb{k}[\phi_1, \dots, \phi_q][[f_1, \dots, f_r]]$ is a subring of $\hat{\mathcal{O}}_{Z,X}$, and there is a nonzero element $\phi \in \mathbb{k}[\phi_1, \dots, \phi_q]$ such that $\mathbb{k}[X]$ is integral over $\mathbb{k}[\phi_1, \dots, \phi_q]_{\phi}[[f_1, \dots, f_r]]$ in $\hat{\mathcal{O}}_{Z,X}$.
- (ii) If $\mathbb{k}[Z]$ is integral over $\mathbb{k}[\phi_1|_Z, \dots, \phi_q|_Z]$, then
 - (a) we can take $\phi = 1$ in the preceding sentence, i.e. $\mathbb{k}[X]$ is integral over $\mathbb{k}[\phi_1, \dots, \phi_q][[f_1, \dots, f_r]]$ in $\hat{\mathcal{O}}_{Z,X}$,
 - (b) in addition, the completion $\widehat{\mathbb{k}[X]}$ of $\mathbb{k}[X]$ with respect to the prime ideal of Z is integral over the subring $\mathbb{k}[\phi_1, \dots, \phi_q][[f_1, \dots, f_r]]$.

Proof. Assume X is a subvariety of \mathbb{k}^N with coordinates (x_1, \dots, x_N) . We abuse the notation to use ϕ_i (respectively, f_j) also for a representative of ϕ_i (respectively, f_j) in $\mathbb{k}[x_1, \dots, x_N]$. Let J be the ideal of $\mathbb{k}[x_1, \dots, x_N]$ generated by f_1, \dots, f_r , so that

$$I(Z) = \sqrt{I(X) + J}$$

For each i , $x_i|_Z$ is algebraic over $\mathbb{k}[\phi_1|_Z, \dots, \phi_q|_Z]$. Taking a power of the minimal equation of $x_i|_Z$ over $\mathbb{k}[\phi_1|_Z, \dots, \phi_q|_Z]$ if necessary, we have that

$$a_{i,0}(\phi_1, \dots, \phi_q) x_i^{d_i} + \sum_{j=1}^{d_i} a_{i,j}(\phi_1, \dots, \phi_q) x_i^{d_i-j} \in (I(X) + J)$$

for some $a_{i,0}, \dots, a_{i,d_i} \in \mathbb{k}[\phi_1, \dots, \phi_q]$ with $a_{i,d_i} \neq 0$. It follows that

$$x_i^{d_i} + \sum_{j=1}^{d_i} \frac{a_{i,j}}{a_{i,0}} x_i^{d_i-j} = \frac{1}{a_{i,0}} \left(\sum_{j=1}^r g_{i,j} f_j + g'_i \right)$$

where $g_{i,1}, \dots, g_{i,r}, g'_i$ are polynomials in (x_1, \dots, x_N) and $g'_i \in I(X)$. Let

$$\phi := \prod_{i=1}^N a_{i,0} \in \mathbb{k}[\phi_1, \dots, \phi_q]$$

and \mathcal{A} be the (finite) set of all $\alpha = (\alpha_1, \dots, \alpha_N) \in (\mathbb{Z}_{\geq 0})^N$ such that $\alpha_i < d_i$ for each i .

Claim B.1.1. *Each $h \in \mathbb{k}[X]$ can be represented in $\hat{\mathcal{O}}_{Z,X}$ as a $\mathbb{k}[\phi_1, \dots, \phi_q]_\phi[[f_1, \dots, f_r]]$ -linear combination of x^α , $\alpha \in \mathcal{A}$.*

Proof. Given $h \in \mathbb{k}[x_1, \dots, x_N]$, replacing each $(x_i)^{d_i}$ using the above equation and repeating as many times as necessary, one has that

$$h = \sum_{\alpha \in \mathcal{A}} b_\alpha x^\alpha + \frac{1}{\phi^{m_1}} \left(\sum_{j=1}^r f_j h_j + h'_1 \right)$$

where the $b_\alpha \in \mathbb{k}[\phi_1, \dots, \phi_q]_\phi$, h_1, \dots, h_r, h'_1 are polynomials in (x_1, \dots, x_N) and $h'_1 \in I(X)$. Then continue the same process with each h_j to yield

$$h = \sum_{\alpha \in \mathcal{A}} b_\alpha x^\alpha + \sum_{j=1}^r \frac{1}{\phi^{m_1}} f_j \left(\sum_{\alpha \in \mathcal{A}} b_{\alpha,j} x^\alpha + \frac{1}{\phi^{m_2}} \left(\sum_{k=1}^r f_k h_{j,k} + h'_{1,j} \right) \right) + \frac{h'_1}{\phi^{m_1}}$$

which can be rewritten as

$$h = \sum_{\alpha \in \mathcal{A}} \left(b_\alpha + \sum_{j=1}^r \frac{b_{\alpha,j}}{\phi^{m_1}} f_j \right) x^\alpha + \frac{1}{\phi^{m_1+m_2}} \left(\sum_{j,k} f_j f_k h_{j,k} + h'_2 \right)$$

This can be continued with

$$h = \sum_{\alpha \in \mathcal{A}} \left(b_\alpha + \sum_j \frac{b_{\alpha,j}}{\phi^{m_1}} f_j + \sum_{j,k} \frac{b_{\alpha,j,k}}{\phi^{m_1+m_2}} f_j f_k \right) x^\alpha + \frac{1}{\phi^{m_1+m_2+m_3}} \left(\sum_{j,k,l} f_j f_k f_l h_{j,k,l} + h'_3 \right)$$

and so on to obtain a sum $\sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha$ with $c_\alpha \in \mathbb{k}[\phi_1, \dots, \phi_q]_\phi[[f_1, \dots, f_r]]$ such that for each M , $h|_X \equiv \sum_{\alpha \in \mathcal{A}} (c_\alpha x^\alpha)|_X$ modulo the ideal in $\hat{\mathcal{O}}_{Z,X}$ generated by f^α with $|\alpha| > M$. Since each f_j is in the maximal ideal of $\hat{\mathcal{O}}_{Z,X}$, it follows (e.g. since the intersection of all powers of a proper ideal in a Noetherian ring is zero) that

$$h = \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha \in \hat{\mathcal{O}}_{Z,X}$$

This completes the proof. \square

Assertions (i) and (ii.a) of Theorem B.1 immediately follow from Claim B.1.1 via the “determinantal trick”. For the remaining assertion (ii.b) it suffices to show that Claim B.1.1 continues to hold if we replace $\mathbb{k}[X]$ by $\widehat{\mathbb{k}[X]}$, since then we can use the determinantal trick to prove integrality. So assume $\phi = 1$ as in assertion (ii), and pick $h \in \widehat{\mathbb{k}[X]}$. Assume h is the inverse limit of $(h_n)_{n \geq 0}$, with

$$h_n \equiv h_m \pmod{(I(Z))^n}$$

for all $m \geq n$. We are going to construct $c_{\alpha,\beta} \in \mathbb{k}[\phi_1, \dots, \phi_q]$ such that $h = \sum_{\alpha,\beta} c_{\alpha,\beta} x^\alpha f^\beta$, where the α vary over \mathcal{A} and the β over $(\mathbb{Z}_{\geq 0})^r$, and x^α, f^β are short-hands for respectively $\prod_{i=1}^N x_i^{\alpha_i}$ and $\prod_{i=1}^r f_i^{\beta_i}$. Fix a monomial ordering

$$0 = \beta^0 \prec \beta^1 \prec \dots$$

of $(\mathbb{Z}_{\geq 0})^r$ which has “finite depth”, i.e. for each $\beta \in (\mathbb{Z}_{\geq 0})^r$, there are only finitely many $\beta' \in (\mathbb{Z}_{\geq 0})^r$ such that $\beta' \preceq \beta$. For each $k \geq 0$, let J_k be the ideal generated by $\{f^{\beta^j} : j \geq k\}$. Fix a set of generators g_1, \dots, g_s of $I(Z)$. Since \prec has finite depth, we can pick $0 < n_0 < n_1 < \dots$ such that for each $\gamma \in (\mathbb{Z}_{\geq 0})^s$ with $|\gamma| := \sum_i \gamma_i = n_k$, the “ g monomial” $g^\gamma := \prod_i (g_i)^{\gamma_i}$ is in J_k . Due to Claim **B.1.1**, h_{n_1} has an expression of the form

$$h_{n_1} = \sum_{\alpha,\beta} c_{n_1,\alpha,\beta}(\phi_1, \dots, \phi_q) x^\alpha f^\beta$$

For each $\alpha \in \mathcal{A}$, define

$$c_{\alpha,\beta^0} := c_{n_1,\alpha,\beta^0}$$

For each $n \geq n_1$, since $h_n - h_{n_1} \in (I(Z))^{n_1}$, by definition of n_1 and using Claim **B.1.1** we can write

$$h_{n_2} - h_{n_1} = \sum_{\alpha \in \mathcal{A}} \sum_{\beta \succeq \beta^1} c_{n_2,\alpha,\beta}(\phi_1, \dots, \phi_q) x^\alpha f^\beta$$

For each $\alpha \in \mathcal{A}$, define

$$c_{\alpha,\beta^1} := c_{n_1,\alpha,\beta^1} + c_{n_2,\alpha,\beta^1}$$

In the same way, for each k , we write

$$h_{n_{k+1}} - h_{n_k} = \sum_{\alpha \in \mathcal{A}} \sum_{\beta \succeq \beta^k} c_{n_{k+1},\alpha,\beta}(\phi_1, \dots, \phi_q) x^\alpha f^\beta,$$

$$c_{\alpha,\beta^k} := c_{n_1,\alpha,\beta^k} + \dots + c_{n_{k+1},\alpha,\beta^k}$$

Finally, define

$$h' := \sum_{\alpha \in \mathcal{A}} \sum_{k \geq 0} c_{\alpha,\beta^k} x^\alpha f^{\beta^k},$$

It is straightforward to check that

$$h' \equiv h_{n_k} \pmod{J_k}$$

which implies that $h' = h$, as required to complete the proof. \square

Lemma B.2. *In the set up of Theorem **B.1**, assume $\mathbb{k}[Z]$ is integral over $\mathbb{k}[\phi_1|_Z, \dots, \phi_q|_Z]$. Let $s \leq r$ and $f \in \mathbb{k}[X]$ be such that f does not vanish identically on any irreducible component of $V_s := V(f_1, \dots, f_s)$ or equivalently, f is not in any minimal prime ideal in $\mathbb{k}[X]$ of the radical of the ideal \mathfrak{q}_s generated by f_1, \dots, f_s . Then f satisfies an identity of the form*

$$f^d + \sum_{j=1}^d \xi_j f^{d-j} + \sum_{k=1}^s f_k \chi_k = 0$$

in the completion $\widehat{\mathbb{k}[X]}$ of $\mathbb{k}[X]$ with respect to $I(Z)$, where

- (1) each $\xi_j \in \mathbb{k}[\phi_1, \dots, \phi_q][[f_1, \dots, f_r]]$,
- (2) ξ_d is nonzero and has a monomial in (f_1, \dots, f_r) in which f_k does not appear for $k \leq s$,

(3) each $\chi_k \in \widehat{\mathbb{k}[X]}$.

Proof. By Theorem **B.1**, f satisfies an integral equation in $\widehat{\mathbb{k}[X]}$ of the form

$$f^d + \sum_{j=1}^d \xi_j f^{d-j} = 0$$

with each $\xi_j \in \mathbb{k}[\phi_1, \dots, \phi_q][[f_1, \dots, f_r]]$. If ξ_d has a monomial in (f_1, \dots, f_r) in which f_k does not appear for $k \leq s$, then Lemma **B.2** clearly holds. So assume either $\xi_d = 0$ or each monomial in ξ_d contains some f_i with $i \leq s$. Now, by assumption, f is a non zero-divisor in $\mathbb{k}[X]/\sqrt{\mathfrak{q}_s}$ (indeed, if $fg \in \sqrt{\mathfrak{q}_s}$, then fg vanishes identically on every irreducible component of $V(f_1, \dots, f_s)$, and consequently, g vanishes identically on every irreducible component of $V(f_1, \dots, f_s)$). It then follows from the flatness of completion that f is also a non zero-divisor in $\widehat{\mathbb{k}[X]}/\widehat{\sqrt{\mathfrak{q}_s}} \cong \widehat{\mathbb{k}[X]}/\sqrt{\mathfrak{q}_s}\widehat{\mathbb{k}[X]}$. However, by the assumption on ξ_d , the above integral equation of f can be written as

$$f^e(f^{d-e} + \sum_{j=1}^{d-e} \xi_j f^{d-e-j}) \in \mathfrak{q}_s \widehat{\mathbb{k}[X]}$$

where $d-e$ is the largest index such that $\xi_{d-e} \notin \mathfrak{q}_s \widehat{\mathbb{k}[X]}$. Since f is non zero-divisor in $\widehat{\mathbb{k}[X]}/\sqrt{\mathfrak{q}_s}\widehat{\mathbb{k}[X]}$, it follows that

$$f^{d-e} + \sum_{j=1}^{d-e} \xi_j f^{d-e-j} \in \sqrt{\mathfrak{q}_s} \widehat{\mathbb{k}[X]}$$

and consequently,

$$(f^{d-e} + \sum_{j=1}^{d-e} \xi_j f^{d-e-j})^m \in \mathfrak{q}_s \widehat{\mathbb{k}[X]}$$

for some $m \geq 1$. This completes the proof. \square

APPENDIX C. PROOF OF CLAIM **6.4.2**

In this section we prove Claim **6.4.2**. Given an n -dimensional affine variety X and fixed $f_1, \dots, f_n, g_1, \dots, g_n \in \mathbb{k}[X]$, we will consider $C(g_1, \dots, g_n; r_1, \dots, r_n)$ (defined as in Section **3.1**) for various r_1, \dots, r_n . As in the proof of Theorem **6.4**, we assume Z is an irreducible component of $Z_{i_1, \dots, i_\rho}^{(\rho)}$, and let $I_{i_1, \dots, i_\rho}^{(j)}, I_{i_1, \dots, i_\rho}^{\prime\prime(j)}$ be as in Paragraph **6.4.1**.

Lemma C.1. *There are $\nu_0^* \in \mathbb{R}$, a finite set $Q \subseteq \mathbb{Q}$, and an open subset U^* of X , and $h_j'' \in I_{i_1, \dots, i_\rho}^{\prime\prime(j)} \mathbb{k}[U^*]$, $j = 1, \dots, \rho - 1$, such that*

- (1) $U^* \cap Z \neq \emptyset$,
- (2) *if $r_1, \dots, r_n \in \mathbb{R}$ and a branch B of $C = C(g_1, \dots, g_n; r_1, \dots, r_n)$ centered at $(U^* \cap Z) \times \{0\}$ satisfy condition $(\emptyset(\nu))$ for $\nu \geq \nu_0^*$, then for each $j = 1, \dots, \rho - 1$,*

$$\text{ord}_B(h_j'') = \sum_{k=1}^{\rho-j} q_k r_{i_{j+k}} \text{ord}_B(t) = \sum_{k=1}^{\rho-j} q_k \text{ord}_B(f_{i_{j+k}})$$

for $q_1, \dots, q_{\rho-j} \in Q$.

In addition, U^ does not change (but ν_0^* possibly changes) if the f_i are replaced by $(f_i)^{m_i}$ for $m_i \geq 1$.*

Proof. The first ingredient of our proof of Lemma C.1 is Lemma C.1.1 below. Here we consider a stronger version of the condition $(\emptyset(\mu^*, \nu^*))$ from Section 4.3: pick $1 \leq i_1 < i_2 < \dots < i_\rho \leq n$, a nonempty open subset Z_0^* of an irreducible subvariety Z of X , and a collection \mathcal{H}^* of pairs (h, i) such that $1 \leq i \leq n$, and h is a regular function on an open subset of X containing Z_0^* . We say that \mathcal{H}^*, Z_0^* satisfy (\emptyset_+) if the following holds:

- (\emptyset_+) for each $\mu^*, \nu^* \in \mathbb{R}$ and each nonempty subset Z^* of Z_0^* , condition $(\emptyset(\mu^*, \nu^*))$ holds for \mathcal{H}^*, Z^* , i.e. there are $\nu \geq \nu^*$, $r_1, \dots, r_n \in \mathbb{R}$ and a branch B of $C = C(g_1, \dots, g_n; r_1, \dots, r_n)$ centered at $Z^* \times \{0\}$ such that $(\emptyset(\nu))$ holds, and in addition, $\text{ord}_B(h) > \mu^* \text{ord}_B(f_i) \geq \mu^* r_i \text{ord}_B(t)$ for each $(h, i) \in \mathcal{H}^*$.

We also consider a stronger version of (\emptyset_+) : we say that \mathcal{H}^*, Z_0^* satisfy (\emptyset_{++}) if the following holds:

- for each $\mu^*, \nu^* \in \mathbb{R}$ and each nonempty subset Z^* of Z_0^* , there is $\nu_+^* \geq \nu^*$ such that for each $\nu \geq \nu_+^*$,
- (\emptyset_{++}) (a) there are $r_1, \dots, r_n \in \mathbb{R}$ and a branch B of $C = C(g_1, \dots, g_n; r_1, \dots, r_n)$ centered at $Z^* \times \{0\}$ such that $(\emptyset(\nu))$ holds, and in addition,
- (b) for each such B , $\text{ord}_B(h) > \mu^* \text{ord}_B(f_i) \geq \mu^* r_i \text{ord}_B(t)$ for each $(h, i) \in \mathcal{H}^*$.

Lemma C.1.1. Fix $i^* = i_1 < i_2 < \dots < i_\rho \leq n$ (where i^* , as in Remark 3.6, is the smallest index such that f_{i^*} is not identically zero), and an irreducible subvariety Z of X . Let U^* be an open subset of X such that $U^* \cap Z \neq \emptyset$, and \mathcal{H} be a collection of pairs (h, i) , where $h \in \mathbb{k}[U^*]$ and $1 \leq i \leq n$, such that

- (a) condition (\emptyset_+) (respectively, (\emptyset_{++})) is satisfied with $\mathcal{H}^* = \mathcal{H}$ and $Z_0^* = U^* \cap Z$.

Assume in addition that

- (b) there is j , $1 \leq j \leq \rho - 1$, such that
- (1) \mathcal{H} consists of j pairs (h_k, \tilde{i}_k) , $k = 1, \dots, j$;
 - (2) $\tilde{i}_k \leq i_{j+1}$ for each $k = 1, \dots, j$;
 - (3) $V(h_1, \dots, h_j) \cap \bigcup_{k=j+1}^\rho V(f_{i_k})$ is (set-theoretically) a codimension $j + 1$ subvariety of U^* containing $U^* \cap Z$.

Then

- (1) There are $h_{j+1}, \dots, h_\rho \in \mathbb{k}[U^*]$ such that
- (a) $V(h_1, \dots, h_k)$ has codimension k for each $k = j + 1, \dots, \rho$,
 - (b) $V(h_1, \dots, h_k) \cap \bigcup_{k'=k+1}^\rho V(f_{i_{k'}})$ has codimension $k + 1$ for each $k = j, \dots, \rho - 1$,
 - (c) (\emptyset_+) (respectively, (\emptyset_{++})) continues to hold with $Z_0^* = U^* \cap Z$ and $\mathcal{H}^* = \mathcal{H} \cup \{(h_{j+1}, i_{j+2}), \dots, (h_{\rho-1}, i_\rho)\}$.

In addition, one can take

- (d) $h_\rho := f_{i_\rho}$.

Pick $h \in \mathbb{k}[U^*]$ which does not vanish identically on any irreducible component of $V(h_1, \dots, h_j)$ containing $U^* \cap Z$.

- (2) If the (\emptyset_+) -version of assumption (a) holds, then there is a finite set $Q \subseteq \mathbb{Q}$, and an open subset $U'^* \subseteq U^*$ intersecting Z such that for each $\nu^* \in \mathbb{R}$ and each nonempty open subset Z^* of $U'^* \cap Z$, there are $\nu \geq \nu^*$, $r_1, \dots, r_n \in \mathbb{Z}$ and a branch B of $C = C(g_1, \dots, g_n; r_1, \dots, r_n)$ centered at $Z^* \times \{0\}$ such that $(\emptyset(\nu))$ holds, and

$$\text{ord}_B(h) = \sum_{k=1}^{\rho-j} q_k r_{i_{j+k}} \text{ord}_B(t) = \sum_{k=1}^{\rho-j} q_k \text{ord}_B(f_{i_{j+k}})$$

for some $q_1, \dots, q_{\rho-j} \in Q$.

- (3) If the (\emptyset_{++}) -version of assumption (a) holds, then there are $\nu^* \in \mathbb{R}$, a finite set $Q \subseteq \mathbb{Q}$, and an open subset $U'^* \subseteq U^*$ intersecting Z such that for each $\nu \geq \nu^*$ and each nonempty open subset Z^* of $U'^* \cap Z$,

- (a) there are $r_1, \dots, r_n \in \mathbb{R}$ and a branch B of $C = C(g_1, \dots, g_n; r_1, \dots, r_n)$ centered at $Z^* \times \{0\}$ such that $(\emptyset(\nu))$ holds, and in addition,
 (b) for each such B ,

$$\text{ord}_B(h) = \sum_{k=1}^{\rho-j} q_k r_{i_{j+k}} \text{ord}_B(t) = \sum_{k=1}^{\rho-j} q_k \text{ord}_B(f_{i_{j+k}})$$

for some $q_1, \dots, q_{\rho-j} \in Q$.

- (4) In addition, U'^* from assertions (2) and (3) does not change if the f_i and h_j are replaced respectively by $(f_i)^{m_i}$ and $(h_j)^{n_j}$ for $m_i, n_j \geq 1$.

Proof. Note that $Z \cap U^* \subseteq V(f_1, \dots, f_n) \cap V(h_1, \dots, h_j)$ (due to assumptions (a) and (b.3), and the definition of $C(g_1, \dots, g_n; r_1, \dots, r_n)$). Lemma 4.4 implies that $V(h_1, \dots, h_j) \not\subseteq V(f_{i_{j+1}})$. Pick h_{j+1} such that

- (i) h_{j+1} vanishes identically on each irreducible component of $V(h_1, \dots, h_j) \cap V(f_{i_{j+1}})$ not contained in $\bigcup_{k=j+1}^{\rho} V(f_{i_k})$,
 (ii) $V(h_1, \dots, h_{j+1}) \cap \bigcup_{k=j+2}^{\rho} V(f_{i_k})$ has codimension $j+2$ in U^* .

It follows from the arguments of the proof of Lemma 4.4 that the property (a) from the statement of Lemma C.1.1 continues to hold if \mathcal{H} is replaced by $\mathcal{H} \cup \{(h_{j+1}, i_{j+2})\}$. Repeating this process we obtain $h_{j+1}, \dots, h_{\rho} \in \mathbb{k}[U^*]$ such that

- (iii) $V(h_1, \dots, h_k)$ has codimension k for each $k = j+1, \dots, \rho$,
 (iv) $V(h_1, \dots, h_k) \cap \bigcup_{k'=k+1}^{\rho} V(f_{i_{k'}})$ has codimension $k+1$ for each $k = j, \dots, \rho-1$,
 (v) property (a) continues to hold with $Z_0^* = U^* \cap Z$ and $\mathcal{H}^* = \mathcal{H} \cup \{(h_{j+1}, i_{j+2}), \dots, (h_{\rho-1}, i_{\rho})\}$.

It is straightforward to check that we can in fact take

- (vi) $h_{\rho} := f_{i_{\rho}}$.

In particular, this implies assertion (1) of Lemma C.1.1. Now we prove assertions (2) and (3). Due to property (iii), we can pick an open subset $U_0'^*$ of U^* such that $U_0'^* \cap Z \neq \emptyset$ and $U_0'^* \cap V(h_1, \dots, h_{\rho})$ has only one irreducible component, namely $Z \cap U_0'^*$.

Subclaim C.1.1.1. *There are $\mu'^*, \nu'^* \in \mathbb{R}$, a finite set $Q' \subseteq \mathbb{Q}$ and an open subset $U_0''^*$ of $U_0'^*$ such that if $(\emptyset(\mu^*, \nu^*))$ is satisfied for some $\mu^* \geq \mu'^*$ and $\nu^* \geq \nu'^*$ with $\mathcal{H}^* = \mathcal{H} \cup \{(h_{j+1}, i_{j+2}), \dots, (h_{\rho-1}, i_{\rho})\}$ and $Z^* = U_0''^* \cap Z$, then for each $k = j+1, \dots, \rho$,*

$$\text{ord}_B(h_k) = \sum_{l=k}^{\rho} q'_l r_{i_l} \text{ord}_B(t) = \sum_{l=k}^{\rho} q'_l \text{ord}_B(f_{i_l})$$

for some $q'_k, \dots, q'_{\rho} \in Q'$.

Proof. Since $h_{\rho} = f_{i_{\rho}}$, the claim is true for $k = \rho$. We proceed by backward induction on k and assume that

- (vii) Subclaim C.1.1.1 holds for $k = \rho, \rho-1, \dots, k'+1$ with a finite subset $Q'_{k'+1}$ in place of Q' , and a nonempty subset $U_{k'+1}''^*$ of $U_0''^*$ in place of $U_0''^*$.

By Noether normalization there are $u_1, \dots, u_{n-\rho} \in \mathbb{k}[U'^*]$ such that

- (viii) $u_1|_Z, \dots, u_{n-\rho}|_Z$ are algebraically independent over \mathbb{k} , and

(ix) $\mathbb{k}[Z \cap U'^*]$ is integral over $\mathbb{k}[u_1|_Z, \dots, u_{n-\rho}|_Z]$.

Due to properties (iv) and (ix), Lemma B.2 applies (with h_i in place of the f_i , the u_i in place of the ϕ_i , and $f = f_{i_{k'}}$ and $s = k' - 1$) and implies that $f_{i_{k'}}$ satisfies an identity in the completion $\widehat{\mathbb{k}[U'^*]}$ of $\mathbb{k}[U'^*]$ with respect to the ideal $I(Z)$ of Z of the form

$$f_{i_{k'}}^d + \sum_{e=1}^d \xi_e f_{i_{k'}}^{d-e} + \sum_{k=1}^{k'-1} h_k \chi_k = 0 \in \widehat{\mathbb{k}[U'^*]}$$

where

- (x) each $\xi_e \in \mathbb{k}[u_1, \dots, u_{n-\rho}][[h_1, \dots, h_\rho]]$,
- (xi) ξ_d is nonzero and has a monomial in (h_1, \dots, h_ρ) in which h_k does not appear for $k \leq k' - 1$,
- (xii) each $\chi_k \in \widehat{\mathbb{k}[X]}$.

For each e , let $\alpha_e := (\alpha_{e,1}, \dots, \alpha_{e,\rho})$ denote the *lexicographically minimal* element of $\text{Supp}(\xi_d)$ (which is the set of all $(\beta_1, \dots, \beta_\rho)$ such that the coefficient of $\prod_{k=1}^\rho h_k^{\beta_k}$ in ξ_e is nonzero). Then the above integral equation of $f_{i_{k'}}$ can be expressed as

$$f_{i_{k'}}^d + \sum_l (c_{e_l} \prod_{k=k'}^\rho h_k^{\alpha_{e_l,k}}) f_{i_{k'}}^{d-e_l} + \sum_l \xi'_{e_l} f_{i_{k'}}^{d-e_l} + \sum_{k=1}^{k'-1} h_k \chi'_k = 0$$

where the second (and the third) sum is over all l such that $\alpha_{e_l,k} = 0$ for $k = 1, \dots, k' - 1$, and for each such l ,

- (xiii) $c_{e_l} \in \mathbb{k}[u_1, \dots, u_{n-\rho}]$ is the *coefficient* $\prod_{k=1}^\rho h_k^{\alpha_{e_l,k}}$ in ξ_{e_l} is nonzero, and
- (xiv) $\xi'_{e_l} := \xi_{e_l} - c_{e_l} \prod_{k=1}^\rho h_k^{\alpha_{e_l,k}}$.

Since $u_1|_Z, \dots, u_n|_Z$ are algebraically independent over \mathbb{k} , none of the c_{e_l} vanishes identically on Z . Consequently, $U''^*_{k'} := U''^*_{k'+1} \setminus V(\prod_l c_{e_l})$ has a nonempty intersection with Z (where $U''^*_{k'+1}$ is from the inductive hypothesis). It then follows from the inductive hypothesis and property (v) that if $r_1, \dots, r_n \in \mathbb{Z}$ and $C(g_1, \dots, g_n; r_1, \dots, r_n)$ centered at $(U''^*_{k'} \cap Z) \times \{0\}$ satisfy condition $(\emptyset(\nu))$ for $\nu \gg 1$, then there must be two distinct $l \neq m$, such that

$$\text{ord}_B(f_{i_{k'}}^{d-e_l} \prod_{k=k'}^\rho h_k^{\alpha_{e_l,k}}) = \text{ord}_B(f_{i_{k'}}^{d-e_m} \prod_{k=k'}^\rho h_k^{\alpha_{e_m,k}})$$

and in addition, $\alpha_{e_l,k'} \neq \alpha_{e_m,k'}$. It is then straightforward to check that Subclaim C.1.1.1 holds for $k = k'$ with $U''^*_{k'}$ in place of U''^*_0 , and \mathcal{Q}' equal to the union of $\mathcal{Q}'_{k'+1}$ (from the inductive hypothesis) and at most finitely many rational numbers. \square

Assertions (2) and (3) of Lemma C.1.1 now follows from applying Lemma B.2 (with h_i in place of the f_i , the u_i in place of the ϕ_i , and $f = h$ and $s = j$), and following the arguments from the proof of Subclaim C.1.1.1. \square

C.1.2. We go back to the proof of Lemma C.1. Due to Theorem 4.8 we may assume without loss of generality that $Z_{i_1, \dots, i_\rho}^{(\rho)} \neq \emptyset$.

C.1.3. For the case of $j = 1$, pick an irreducible component W of $V(I_{i_1, \dots, i_\rho}^{(1)})$. We will find $h_W \in \mathbb{k}[X]$ as follows:

- (1) If $W \subseteq V(f_{i_2})$, then set $h_W := f_{i_2}$.

- (2) Otherwise there is $i_W, i_1 < i_W < i_2$, such that $f_{i_W}|_W \not\equiv 0$. Pick h_W such that h_W vanishes on W but does not vanish identically on any irreducible component of $V(f_{i_W})$.

We claim that assertion (2) of Lemma C.1 holds for $j = 1$ if we take $h_1'' = h_W$. Indeed, this is clear in the first case when $h_W = f_{i_2}$, and in the second case this follows from applying the (\emptyset_{++}) version of Lemma C.1.1 with $\mathcal{H} = \{(f_{i_W}, i_2)\}$ and $h = h_W$. Take $h_1'' := \prod_W h_W$. Then it is easy to see that

- (3) Lemma C.1 holds for $j = 1$ with h_1'' .

We now choose $h_1', \tilde{h}_1' \in I_{i_1, \dots, i_\rho}^{(1)}$ as follows (these elements will help inductive construction of h_j'' for $j > 1$):

- (4) (a) $h_1' \in I_{i_1, \dots, i_\rho}^{(1)}$,
 (b) h_1' does not vanish identically on any irreducible component of $V(\prod_{j \geq 2} f_{i_j})$,
 (c) h_1' does not vanish identically on any irreducible component of $V(I_{i_1, \dots, i_\rho}^{(1)})$.
 (5) (a) $\tilde{h}_1' \in I_{i_1, \dots, i_\rho}^{(1)}$,
 (b) \tilde{h}_1' does not vanish identically on any irreducible component of $V(\prod_{j \geq 2} f_{i_j})$,
 (c) \tilde{h}_1' does not vanish identically on any irreducible component of $V(h_1')$ which is not an irreducible component of $V(I_{i_1, \dots, i_\rho}^{(1)})$,
 (d) \tilde{h}_1' does not vanish identically on any irreducible component of $V(h_1', f_{i_2})$ which is not contained in $V(I_{i_1, \dots, i_\rho}^{(1)})$.

Since $h_1' h_1''$ and $\tilde{h}_1' h_1''$ are elements of $\sqrt{I_{i_1, \dots, i_\rho}^{(1)}} = \sqrt{\langle f_{i_1} \rangle}$, the arguments from the proof of Lemma 4.4 imply that

- (6) Condition (\emptyset_{++}) holds with $\mathcal{H}^* = \{(h_1', i_2)\}$ and also with $\mathcal{H}^* = \{(\tilde{h}_1', i_2)\}$.

C.1.4. By induction, assume there are $h_1'', \dots, h_k'', 1 \leq k \leq \rho - 2$, such that for each $j = 1, \dots, k$,

- (1) $h_j'' \in I_{i_1, \dots, i_\rho}^{(j)}$,
 (2) Lemma C.1 holds for $j = 1, \dots, k$ (with h_1'', \dots, h_j'')

Assume in addition that there are $h_1', \dots, h_k', 1 \leq k \leq \rho - 2$, such that

- (3) $h_j' \in I_{i_1, \dots, i_\rho}^{(j)}$ for each j ,
 (4) $V(h_1', \dots, h_k')$ does not contain any irreducible component of $V(I_{i_1, \dots, i_\rho}^{(k)})$.
 (5) $V(h_1', \dots, h_k')$ has pure codimension k ,
 (6) $V(h_1', \dots, h_k', \prod_{j=k+1}^\rho f_{i_j})$ has pure codimension $k + 1$,
 (7) $\mathcal{H}^* = \{(h_1', i_2), \dots, (h_k', i_{k+1})\}$ satisfies (\emptyset_{++}) with an appropriate Z_0^* .

For technical reasons we also assume that there is $\tilde{h}_k' \in I_{i_1, \dots, i_\rho}^{(k)}$ such that

- (8) each of the properties (3) to (7) continues to be satisfied if h_k' is replaced by \tilde{h}_k' ,
 (9) \tilde{h}_k' does not vanish identically on any irreducible component of $V(h_1', \dots, h_k')$ which is not an irreducible component of $V(I_{i_1, \dots, i_\rho}^{(k)})$.
 (10) \tilde{h}_k' does not vanish identically on any irreducible component of $V(h_1', \dots, h_k', f_{i_{k+1}})$ which is not contained in $V(I_{i_1, \dots, i_\rho}^{(k)})$.

We will construct $h_{k+1}', \tilde{h}_{k+1}', h_{k+1}''$ extending the above properties.

C.1.5. Let $h'_{k+1} \in I'_{i_1, \dots, i_\rho}(k+1)$ be such that

- (1) h'_{k+1} does not vanish identically on any irreducible component of $V(h'_1, \dots, h'_k)$,
- (2) h'_{k+1} does not vanish identically on any irreducible component of $V(h'_1, \dots, h'_k, \prod_{j=k+2}^\rho f_{i_j})$,
- (3) h'_{k+1} does not vanish identically on any irreducible component of $V(I''_{i_1, \dots, i_\rho}(k+1))$.

Pick $\tilde{h}'_{k+1} \in I'_{i_1, \dots, i_\rho}(k+1)$ be such that

- (4) \tilde{h}'_{k+1} satisfies each of the above properties of h'_{k+1} , and in addition,
- (5) \tilde{h}'_{k+1} does not vanish identically on any irreducible component of $V(h'_1, \dots, h'_{k+1})$ which is not an irreducible component of $V(I'_{i_1, \dots, i_\rho}(k+1))$.
- (6) \tilde{h}'_{k+1} does not vanish identically on any irreducible component of $V(h'_1, \dots, h'_{k+1}, f_{i_{k+2}})$ which is not contained in $V(I'_{i_1, \dots, i_\rho}(k+1))$.

Claim C.1.6. Condition (\emptyset_{++}) holds with $\mathcal{H}^* = \{(h'_{k+1}, i_{k+2})\}$ and also with $\mathcal{H}^* = \{(\tilde{h}'_{k+1}, i_{k+2})\}$.

Proof. We only show that $\mathcal{H}^* = \{(h'_{k+1}, i_{k+2})\}$ satisfies (\emptyset_{++}) , since the proof of the case of $\{(\tilde{h}'_{k+1}, i_{k+2})\}$ is completely analogous, and follows from running the arguments below after swapping h'_{k+1} with \tilde{h}'_{k+1} (and vice versa).

C.1.6.1. Pick an irreducible component W of $V(h'_1, \dots, h'_k, f_{i_{k+1}})$ such that $h'_{k+1}|_W \not\equiv 0$ (which means in particular that $W \not\subseteq V(I'_{i_1, \dots, i_\rho}(k+1))$). We will construct regular functions $h_{W,j}$, $j = 1, \dots, n_W$, which vanish identically on W , and satisfy the following condition:

there are real numbers μ_W^*, ν_W^* and a nonempty open subset Z_W^* of Z such that if $(\emptyset(\nu))$ holds with $\nu \geq \nu_W^*$ and $\text{center}(B) \in Z_W^*$, then $\text{ord}_B(h_{W,j}|_C) \leq \mu_W^* r_{i_{k+2}} \text{ord}_B(t)$ for some $j = 1, \dots, n_W$.

There are three possible options for W :

C.1.6.2. First consider the case that $W \subseteq V(\prod_{l=k+2}^\rho f_{i_l})$. In this case take $n_W := 1$ and $h_{W,1} := \prod_{l=k+2}^\rho f_{i_l}$.

C.1.6.3. The second possibility is that $W \not\subseteq V(\prod_{l=k+2}^\rho f_{i_l})$, and there is $i_W, i_{k+1} < i_W < i_{k+2}$, such that $f_{i_W}|_W \not\equiv 0$. In this case take $n_W := k+1$, and inductively choose $h_{W,1}, \dots, h_{W,k+1}$ such that

- (1) $h_{W,1}$ vanishes identically on W but does not vanish identically on any irreducible component of $V(f_{i_W} \prod_{l=k+2}^\rho f_{i_l})$,
- (2) for each $j = 1, \dots, k$,
 - (a) $h_{W,j+1}$ vanishes identically on W ,
 - (b) $h_{W,j+1}$ does not vanish identically on any irreducible component of $V(h_{W,1}, \dots, h_{W,j})$,
 - (c) $h_{W,j+1}$ does not vanish identically on any irreducible component of $V(h_{W,1}, \dots, h_{W,j}, f_{i_W} \prod_{l=k+2}^\rho f_{i_l})$.

We claim that $h_{W,j}$, $j = 1, \dots, k+1$, satisfy the condition from Paragraph **C.1.6.1** above. Indeed, otherwise due to the assumption in Appendix **C.1.2**, condition (\emptyset_+) is satisfied with $\mathcal{H}^* = \{(h_{W,1}, i_{k+2}), \dots, (h_{W,k+1}, i_{k+2})\}$. Consequently, the (\emptyset_+) version of Lemma **C.1.1** applies with $h = f_{i_W}$ and yields a contradiction.

C.1.6.4. In the remaining scenario, $W \not\subseteq V(\prod_{l=k+2}^{\rho} f_{i_l} I''_{i_1, \dots, i_{\rho}}^{(k+1)})$. Then in particular, $W \not\subseteq V(I''_{i_1, \dots, i_{\rho}}^{(k+1)}) \cup V(I''_{i_1, \dots, i_{\rho}}^{(k+1)}) = V(I_{i_1, \dots, i_{\rho}}^{(k+1)})$. Since $W \subseteq V(f_{i_{k+1}})$, it follows that $W \not\subseteq V(I_{i_1, \dots, i_{\rho}}^{(k)})$ (see the definition of these ideals in Paragraph 6.4.1). Due to inductive property (10) of Appendix C.1.4 we may inductively pick $h_{W,1}, \dots, h_{W,k+1}$ which vanish identically on W , and satisfy does both properties (1) and (2) from Paragraph C.1.6.3 with f_{i_W} replaced by \tilde{h}_k . An application of the (\emptyset_+) version of Lemma C.1.1 with $\mathcal{H}^* = \{(h_{W,1}, i_{k+2}), \dots, (h_{W,k+1}, i_{k+2})\}$ then shows that $h_{W,1}, \dots, h_{W,k+1}$ satisfy the condition from Paragraph C.1.6.1 above with $n_W = k + 1$.

C.1.6.5. For each tuple $\sigma = (\sigma_W)_W$ with $\sigma_W \in \{1, \dots, n_W\}$, define

$$h_{\sigma} := \prod_W h_{W, \sigma_W}$$

If $(\emptyset(\nu))$ holds with $\nu \geq \max_W \nu_W^*$ and $\text{center}(B) \in \bigcap_W Z_W^*$, then there is σ such that $\text{ord}_B(h_{\sigma}|_C) \leq \sum_W \mu_W^* r_{i_{k+2}} \text{ord}_B(t)$. Since $h'_{k+1} h_{\sigma} \in \sqrt{\langle h'_1, \dots, h'_k, f_{i_{k+1}} \rangle}$ for each σ , it follows from the inductive property (7) of Appendix C.1.4 and the arguments from the proof of Lemma 4.4 that (\emptyset_{++}) holds with $\mathcal{H}^* = \{(h'_{k+1}, i_{k+2})\}$, as required. \square

C.1.7. Now pick $h''_{k+1} \in I''_{i_1, \dots, i_{\rho}}^{(k+1)}$ which does not vanish identically on any irreducible component of $V(h'_1, \dots, h'_{k+1})$ (this is possible due to property (3) from Appendix C.1.5). Then due to the inductive hypothesis and Claim C.1.6, the (\emptyset_{++}) version of Lemma C.1.1 can be applied with $\mathcal{H}^* = \{(h'_1, i_2), \dots, (h'_{k+1}, i_{k+2})\}$ and $h = h''_{k+1}$, and implies that Lemma C.1 holds for $j = 1, \dots, k + 1$ with h'_1, \dots, h''_{k+1} . This completes the inductive step and concludes the proof of Lemma C.1. \square

Claim C.2 (Claim 6.4.2). *There is a nonempty open subset Z_0^* of Z and $\nu_0^* \in \mathbb{R}$ such that if $z \in Z_0^*$ and $g_1, \dots, g_n, r_1, \dots, r_n$ satisfy $(\emptyset(\nu))$ with $\nu \geq \nu_0^*$, then*

(1) *for each j , $1 \leq j \leq \rho$,*

$$I_{i_1, \dots, i_{\rho}}^{(j)} \hat{\mathcal{O}}_{(z,0),C} \subseteq t^{r_{i_j}} \hat{\mathcal{O}}_{(z,0),C}$$

In addition, Z_0^ does not change (but ν_0^* possibly changes) if the f_i are replaced by $(f_i)^{m_i}$ for $m_i \geq 1$.*

Pick $i' \in \{1, \dots, n\} \setminus \{i_1, \dots, i_{\rho}\}$. Recall that $i_1 = i^$, where i^* is the smallest index such that F_{i^*} is not identically zero.*

(2) *If $i' < i^*$, then $g_{i'}|_C \equiv 0 \in t \hat{\mathcal{O}}_{(z,0),C}$.*

(3) *Assume $i' > i^*$. Pick the largest j such that $i_j < i'$. Then $f_{i'} \in \sqrt{I_{i_1, \dots, i_{\rho}}^{(j)}}$. If $f_{i'} \in I_{i_1, \dots, i_{\rho}}^{(j)}$, then $g_{i'} \in t \hat{\mathcal{O}}_{(z,0),C}$.*

Proof. We start with the proof of assertion (1). Fix $\phi_j \in I_{i_1, \dots, i_{\rho}}^{(j)} \mathcal{O}_{(z,0),C}$. Then $\phi_j = \phi'_{j-1} + f_{i_j} \psi_j$ for some $\phi'_{j-1} \in I_{i_1, \dots, i_{\rho}}^{(j-1)} \mathcal{O}_{(z,0),C}$ and $\psi_j \in \mathcal{O}_{(z,0),C}$. However, then

$$h''_{j-1} \phi_j = \phi_{j-1} + h''_{j-1} f_{i_j} \psi_j$$

where h''_{j-1} is from Lemma C.1 and $\phi_{j-1} := h''_{j-1} \phi'_{j-1} \in I_{i_1, \dots, i_{\rho}}^{(j-1)} I_{i_1, \dots, i_{\rho}}^{(j-1)} \subseteq I_{i_1, \dots, i_{\rho}}^{(j-1)}$. Lemma C.1 implies that for each branch B of C at $(z, 0)$, there are integers $m_{j-1,B}, n_{j-1,B}$ such that

$$\text{ord}_B(h''_{j-1}) = \frac{m_{j-1,B}}{n_{j-1,B}} r_{i_j} \text{ord}_B(t)$$

Claim C.2.1. *There are invertible elements $\xi_{j-1,B,k} \in \hat{\mathcal{O}}_{(z,0),C}$ and an integer m'_{j-1} such that*

$$(t^{r_{i_j} m_{j-1,B} N_{j-1,B}} + \sum_{k=1}^{N_{j-1,B}} \xi_{j-1,B,k} t^{r_{i_j} m_{j-1,B} (N_{j-1,B}-k)} (h''_{j-1})^{k n_{j-1,B}})^{m'_{j-1}} = 0 \in \hat{\mathcal{O}}_{(z,0),C}$$

Proof. Let C' be the (unique) reduced curve with support equal to $\text{Supp}(C)$. Let $\pi : \tilde{C} \rightarrow C'$ be a desingularization of C' , and \tilde{z}_B be the center of B on \tilde{C} . Since $\hat{\mathcal{O}}_{\tilde{z}_B, \tilde{C}}$ is integral over $\pi^*(\hat{\mathcal{O}}_{(z,0),C'})$, we may take an integral equation of $t^{m_{j-1,B} r_{i_j}} / (h''_{j-1})^{n_{j-1,B}}$ over $\pi^*(\hat{\mathcal{O}}_{(z,0),C'})$ and clear out the denominators to obtain an identity of the following form in $\hat{\mathcal{O}}_{\tilde{z}_B, \tilde{C}}$:

$$t^{r_{i_j} m_{j-1,B} N_{j-1,B}} + \sum_{k=1}^{N_{j-1,B}} \pi^*(\xi_{j-1,B,k}) t^{r_{i_j} m_{j-1,B} (N_{j-1,B}-k)} (h''_{j-1})^{k n_{j-1,B}} = 0$$

Due to exactness of completions (i.e. the injectivity of the map $\hat{\mathcal{O}}_{(z,0),C'} \rightarrow \prod_B \hat{\mathcal{O}}_{\tilde{z}_B, \tilde{C}}$), it then follows that the product over all B of the left hand side of the above equation is zero in $\hat{\mathcal{O}}_{(z,0),C'}$. Since the kernel of the natural map from $\hat{\mathcal{O}}_{(z,0),C} \rightarrow \hat{\mathcal{O}}_{(z,0),C'}$ is nilpotent, it becomes zero in $\hat{\mathcal{O}}_{(z,0),C}$ after raising to a power, as required to prove the claim. \square

Claim C.2.2. *The total number of branches B , and all $m_{j-1,B}$, $n_{j-1,B}$, and m'_{j-1} can be bounded above by an integer which is independent of ν_0^* and B .*

Proof. The total number of branches B is clearly $\leq (F_1, \dots, F_n)$. Lemma C.1 implies that each $m_{j-1,B}$, $n_{j-1,B}$ can be chosen to be bounded above by a quantity independent of ν_0^* and B . On the other hand, Lemma A.1 implies that m'_{j-1} is bounded by above $\text{ord}_B(t)$. Since $\text{ord}_B(t) \leq (F_1, \dots, F_n)$, this completes the proof. \square

C.2.3. It follows that for each $j = 2, \dots, \rho$, there is an identity of the form

$$t^{N r_{i_j}} - \sum_{k=1}^N \xi_{j-1,k} t^{(N-k) r_{i_j}} (h''_{j-1})^{\alpha_{j-1,k}} \equiv 0 \in \hat{\mathcal{O}}_{(z,0),C}$$

where N is independent of ν_0^* and B , and each $\alpha_{j-1,k}$ is *positive*. Then in $\hat{\mathcal{O}}_{(z,0),C}$ one has

$$\begin{aligned} t^{N r_{i_j}} \phi_j &= t^{N r_{i_j}} (\phi'_{j-1} + f_{i_j} \psi_j) \\ &= \sum_{k=1}^N \xi_{j-1,k} t^{(N-k) r_{i_j}} (h''_{j-1})^{\alpha_{j-1,k}} \phi'_{j-1} + t^{N r_{i_j}} f_{i_j} \psi_j \\ &= \chi_{j-1} \phi_{j-1} + t^{N r_{i_j} + r_{i_j}} g_{i_j} \psi_j \end{aligned}$$

where

$$\begin{aligned} \chi_{j-1} &:= \sum_{k=1}^N \xi_{j-1,k} t^{(N-k) r_{i_j}} (h''_{j-1})^{\alpha_{j-1,k}-1} \in \hat{\mathcal{O}}_{(z,0),C}, \text{ and} \\ \phi_{j-1} &:= h''_{j-1} \phi'_{j-1} \in I_{i_1, \dots, i_\rho}^{(j-1)} \end{aligned}$$

C.2.4. Repeating the arguments of the preceding paragraph for $j-1, j-2, \dots, 2$ yields, with some $N' \geq N$, that

$$\begin{aligned} t^{N' \sum_{k=2}^j r_{i_k}} \phi_j &= t^{N'(\sum_{k=2}^j r_{i_k})+r_{i_j}} g_{i_j} \psi_j + \chi_{j-1} t^{N'(\sum_{k=2}^{j-1} r_{i_k})+r_{i_{j-1}}} g_{i_{j-1}} \psi_{j-1} \\ &\quad + \dots + \left(\prod_{k=2}^{j-1} \chi_k \right) t^{N' r_{i_2} + r_{i_2}} g_{i_2} \psi_2 + \left(\prod_{k=1}^{j-1} \chi_k \right) \phi_1 \\ &= t^{N'(\sum_{k=2}^j r_{i_k})+r_{i_j}} g_{i_j} \psi_j + \chi_{j-1} t^{N'(\sum_{k=2}^{j-1} r_{i_k})+r_{i_{j-1}}} g_{i_{j-1}} \psi_{j-1} \\ &\quad + \dots + \left(\prod_{k=2}^{j-1} \chi_k \right) t^{N' r_{i_2} + r_{i_2}} g_{i_2} \psi_2 + \left(\prod_{k=1}^{j-1} \chi_k \right) t^{r_{i_1}} g_{i_1} \psi_1 \end{aligned}$$

where ψ_1, \dots, ψ_ρ are elements of $\mathcal{O}_{(z,0),C}$ which depend on ϕ_ρ . Now if r_1, \dots, r_n satisfy (condition $r./r._+ > \nu$) for $\nu > N' + 1$, then

$$r_{i_1} > N' r_{i_2} + r_{i_2} > N'(r_{i_2} + r_{i_3}) + r_{i_3} > \dots > N' \left(\sum_{j=2}^{\rho} r_{i_j} \right) + r_{i_\rho}$$

so that

$$t^{N' \sum_{k=2}^j r_{i_k}} \phi_j = t^{N'(\sum_{k=2}^j r_{i_k})+r_{i_j}} \phi'_j$$

for some $\phi'_j \in \hat{\mathcal{O}}_{(z,0),C}$. Then $t^{N' \sum_{k=2}^j r_{i_k}} (\phi_j - t^{r_{i_j}} \phi'_j) = 0$. Since t is a non zero-divisor in $\hat{\mathcal{O}}_{(z,0),C}$, it follows that $\phi_j = t^{r_{i_j}} \phi'_j$. Consequently, $I_{i_1, \dots, i_\rho}^{(j)} \hat{\mathcal{O}}_{(z,0),C} \subseteq t^{r_{i_j}} \hat{\mathcal{O}}_{(z,0),C}$, which completes the proof of assertion (1) of Claim **C.2**.

C.2.5. Assertion (2) of Claim **C.2** follows directly from the definition of i^* . For assertion (3), assume $i'_k > i^*$. Pick the largest j such that $i_j < i'_k$. Then $g_{i'_k} t^{r_{i'_k}} = f_{i'_k} \in \sqrt{I_{i_1, \dots, i_\rho}^{(j)}}$. Assume now that $f_{i'_k} \in I_{i_1, \dots, i_\rho}^{(j)}$. Then we claim that $g_{i'_k} \in t \hat{\mathcal{O}}_{\tilde{z}_B, \tilde{C}}$. Indeed, it follows from the arguments in Appendices **C.2.3** and **C.2.4** that

$$t^{r_{i'_k} + N' r_{i_{j+1}}} g_{i'_k} = t^{N' r_{i_{j+1}}} f_{i'_k} = \chi_j \phi_j$$

with $\phi_j \in I$, and in turn,

$$t^{N' \sum_{j'=2}^j r_{i_{j'}} + r_{i'_k} + N' r_{i_{j+1}}} g_{i'_k} = t^{N' \sum_{j'=2}^j r_{i_{j'}} + r_{i_j}} \phi'_j$$

for some $\phi'_j \in \hat{\mathcal{O}}_{(z,0),C}$. Then under (condition $r./r._+ >> 1$), $r_{i_j} > r_{i'_k} + N' r_{i_{j+1}}$ and consequently, $g_{i'_k} \in \hat{\mathcal{O}}_{\tilde{z}_B, \tilde{C}}$, as required. □

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