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ON THE ASYMPTOTIC PROFILE OF SOLUTIONS TO SEMILINEAR DAMPED WAVE EQUATIONS WITH CRITICAL NONLINEARITIES

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ABSTRACT. In this paper, we consider the Cauchy problem for a semilinear damped wave equation with the nonlinear term $|u|^{1+\frac{2}{n}}\mu(|u|)$, where μ is a modulus of continuity. In recent papers by Ebert–Girardi–Reissig [8] and Girardi [11], the authors obtained a sharp critical condition on μ in low space dimensions $n = 1, 2, 3$, which determines the threshold between global (in time) existence of small data solutions and blow-up of solutions in finite time. Our new results are to prove that this condition remains valid in dimension $n = 4$, together with the asymptotic profiles of global solutions. From this, we see that the behavior of the solution at $t \to \infty$ is identified by the Gauss kernel. Finally, a sharp lifespan estimate for local solutions is also derived in the case when blow-up occurs.

1. INTRODUCTION

In this work, we consider the following Cauchy problem for the semilinear damped wave equation:

$$\begin{cases} u_{tt} - \Delta u + u_t = \mathcal{N}(u), & x \in \mathbb{R}^n,\ t > 0, \\ (u, u_t)(0, x) = \varepsilon(u_0, u_1)(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $\mathcal{N}(s) := |s|^{1+\frac{2}{n}}\mu(|s|)$ and the positive constant ε describes the size of the initial data. The function $\mu : [0, \infty) \to [0, \infty)$ is a modulus of continuity, that is, μ is a continuous, concave, and increasing function satisfying $\mu(0) = 0$.

To begin with, we present the motivation for studying this model and review several well-known results related to it. The damped wave equation is a fundamental mathematical model describing wave motion when energy loss is present. Unlike the pure wave equation, it includes a damping term that accounts for friction, viscosity, or absorption in the medium. In the last half century, the following Cauchy problem has been extensively investigated by many mathematicians (see, for instance, [30, 14, 32, 33] and the references therein):

$$\begin{cases} u_{tt} - \Delta u + u_t = |u|^p, & x \in \mathbb{R}^n,\ t > 0, \\ (u, u_t)(0, x) = \varepsilon(u_0, u_1)(x), & x \in \mathbb{R}^n, \end{cases} \tag{2}$$

where $p > 1$. In the classical paper [30], Matsumura was the first to consider the Cauchy problem of nonlinear wave equations with dissipation terms. Their main tools are some basic decay estimates established by the Fourier splitting method for the following linear damped wave equation:

$$\begin{cases} u_{tt} - \Delta u + u_t = 0, & x \in \mathbb{R}^n,\ t > 0, \\ (u, u_t)(0, x) = \varepsilon(u_0, u_1)(x), & x \in \mathbb{R}^n. \end{cases} \tag{3}$$

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In addition, they concluded that the damped wave equation has a diffusive structure as $t \to \infty$. This phenomenon has been further investigated in many subsequent papers, such as [6, 23, 31, 14, 2] and the references therein. From this, they conclude that the diffusion phenomenon bridges decay properties of solutions to the Cauchy problem for the classical damped wave equation (3) and solutions to the following Cauchy problem for the heat equation:

$$\begin{cases} v_t - \Delta v = 0, & x \in \mathbb{R}^n,\ t > 0, \\ v(0,x) = \varepsilon(u_0 + u_1)(x), & x \in \mathbb{R}^n. \end{cases} \tag{4}$$

Specifically, they obtain the approximation of the solution u by the Gauss kernel when t is large

$$\mathcal{G}(t,x) := (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} = \mathfrak{F}^{-1}\big(e^{-|\xi|^2 t}\big)(t,x).$$

More extended, Michihisa [23] discovered the higher-order asymptotic behavior of solution to problem (3) in the L^2 framework for any spatial dimension n by using some tools including Taylor series expansion and Faà di Bruno's formula. Takeda [31] found the higher-order asymptotic behavior of solution to problem (3) in L^q framework for some $q \in (1,\infty)$ and concluded that the effect of u_{tt} in (3) is not negligible for the third-order expansion.

For the semilinear problem (2) under the additional regularity L^1 for the initial data, in [15, 32, 16, 18], the authors showed that the Fujita exponent $p_F(n) := 1 + 2/n$ is the critical exponent of (2) and the paper [33] showed that the case $p = p_F(n)$ belongs to the blow-up range. Here, the critical exponent is understood as the threshold between the global (in time) existence of small data mild solutions and the blow-up of solutions even for small data. The Fujita exponent p_F is also the critical exponent for the semilinear problem (4), whose right-hand side is the nonlinear term $|v|^p$. Specifically, Ikehata et al. [15] succeeded in proving the global existence and optimal decay estimates of the total energy of the weak solutions to problem (2) with the power $p > p_F(n)$ for $n = 1, 2$, and $p > 2$ for $n \geqslant 3$. It should be mentioned that the results in [32] fully depend on the compactness assumption on the support of initial data, while in [15] these were removed. Next, Gallay-Raugel [10] proved that global solutions of nonlinear damped wave equation behaves like those of nonlinear heat equations with suitable data, including more general nonlinearity for $n = 1$. Karch [21] proved the approximation of the solution to (2) by the Gauss kernel for $p \geqslant 1 + 4/n$. Nishihara [27] proved it for $p > 1+2/n$ when $n = 3$, followed by [26] for $n = 1$, [20] for $n = 2$, [28] for $n = 4, 5$ and [19] for all $n \geqslant 1$. More gererally, Kawakami-Takeda [22] established the higher order asymptotic expansion of the solution to (2) under suitable assumptions for the nonlinearity and the initial data. For the blow-up range $p \leqslant 1 + 2/n$, according to the works [24, 13, 25], the sharp lifespan estimates for blow-up solutions to (2) in all spatial dimensions have been investigated. Here, we denote that T_ε is the lifespan of the local solution in the following sense:

$$\begin{aligned} T_\varepsilon := \sup\{T \in (0,\infty) :\ &\text{there exists a unique local (in time) solution } u \text{ on } [0,T) \\ &\text{with a fixed parameter } \varepsilon > 0\}. \end{aligned}$$

Consequently, these papers provided the following sharp lifespan estimate for blow-up solutions:

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-\frac{2(p-1)}{2-n(p-1)}} & \text{if } 1 < p < 1 + \dfrac{2}{n}, \\ \exp\big(C\varepsilon^{-\frac{2}{n}}\big) & \text{if } p = 1 + \dfrac{2}{n}, \end{cases}$$

where C is a positive constant independent of ε.

Under additional regularity L^m for the initial data, with $m \in [1,2]$, papers [16, 7] and references therein identified the critical exponent of problem (2) as $p_c(m) = p_F(n/m) = 1 + 2m/n$. However, the authors did not provide conclusions regarding the solution's properties when $p = p_c(m)$. Paper [14] has proved the existence of global mild solutions when $p = p_c(m)$ for all $m \in (1,2]$ in small dimentional spaces along with some other conditions. In addition, they proved that the global solution is approximated by that of the linear heat equation (4) for all dimension spaces $n \geqslant 1$.

From the above discussion, we can see that the Fujita exponent $p_F = 1 + 2/n$ serves as the threshold separating the blow-up range from the global existence range of global (in time) solutions to problem (2) with small data. Moreover, the critical case $p = p_F$ belongs to the blow-up range. Recently, Ebert et al. [8] investigated the Cauchy problem (1) with nonlinear term $|u|^{1+\frac{2}{n}}\mu(|u|)$ and established the sharp condition on $\mu(s)$ that separates the blow-up case from the global existence case. Specifically, the problem (1) has a unique global (in time) small data solution if $n = 1, 2$ and the *Dini condition* satisfied. Moreover, problem (1) admits a blow-up solution in finite time for all spatial dimensions $n \geqslant 1$ if μ satisfies the *non-Dini condition.*

Definition 1.1. *Let $\mu : [0, +\infty) \to [0, +\infty)$ be a modulus of continuity. Then, μ satisfies the Dini condition if*

$$\int_0^1 \frac{\mu(s)}{s} ds < +\infty. \tag{5}$$

On the other hand, μ satisfies the non-Dini condition if (5) does not hold, i.e., if

$$\int_0^1 \frac{\mu(s)}{s} ds = +\infty. \tag{6}$$

Extending the work [8], Girardi [11] showed that the results obtained in [8] can be generalized to more general semilinear evolution models with the Fujita-type critical exponent; in particular, the critical behavior of nonlinearity is still described by the Dini condition (5). From this, they extended the global existence result of (1) to the case $n = 3$ by using the estimates of Nishihara in [27]. Continuing with this topic, Dao-Reissig [4] studied for semilinear damped wave systems, while the papers of Chen-Girardi [3] and Dao-Son [5] investigated it in the framework of evolution equations. They also proved that, under the condition (5), these problems admit a unique global (in time) solution for small data. However, their assumptions on the spatial dimension are rather restrictive, since they are constrained by tools from harmonic analysis. Therefore, our first main goal in this paper not only extends the global existence property of (1) to dimensions $1 \leqslant n \leqslant 4$ under condition (5) by applying the linear estimates obtained in [14] , but also provides a new approach to relax the dimensional restrictions in [4, 5, 3]. Next, our second main objective is to determine the asymptotic behavior of global (in time) solutions to (1) in spatial dimensions $1 \leqslant n \leqslant 4$. More specifically, we show that the behavior of the solution at $t \to \infty$ is identified by the Gauss kernel under the Dini condition (5) for the global existence

$$\lim_{t\to\infty} t^{\frac{n}{2}(1-\frac{1}{q})} \left\| u(t,\cdot) - M\mathfrak{F}^{-1}\big(e^{-|\xi|^2 t}\big)(t,\cdot) \right\|_{L^q} = 0,$$

for all $q \in [\min\{2, 1 + 2/n\}, +\infty]$ and M defined by (8). Finally, we will provide sharp lifespan estimates for local solutions of problem (1) under the non-Dini condition (6) in spatial dimensions $1 \leqslant n \leqslant 4$.

Notations.

- We write $f \lesssim g$ when there exists a constant $C > 0$ such that $f \leqslant Cg$, and $f \sim g$ when $g \lesssim f \lesssim g$.
- For any $\eta \in \mathbb{R}$, we denote by $[\eta]^+ := \max\{0, \eta\}$, its positive part.
- In addition, we denote $\widehat{w}(t, \xi) := \mathfrak{F}_{x\to\xi}\big(w(t, x)\big)$ as the Fourier transform with respect to the spatial variable of a function $w(t, x)$ and $\mathfrak{F}^{-1}$ represents the inverse Fourier transform.
- As usual, H^a_m and $\dot{H}^a_m$, with $m \in (1, \infty), a \geqslant 0$, denote potential spaces based on L^m spaces. Here $\big\langle\nabla\big\rangle^a$ and $|\nabla|^a$ stand for the pseudo-differential operators with symbols $\big\langle\xi\big\rangle^a$ and $|\xi|^a$, respectively, where the symbol $\langle x\rangle := \sqrt{|x|^2 + 1}$ denotes the Japanese bracket.
- For later convenience, we denote by C and C_ℓ with $\ell \in \mathbb{N}$ positive constants independent of t, which may be changed from line to line.

Main results. Let us state the global (in time) existence of small data solutions, along with asymptotic behaviors for global solution and sharp lifespan estimates for blow-up solutions , which will be proved in this paper.

Theorem 1.1 (Global existence). *Let $1 \leqslant n \leqslant 4$. The modulus of continuity $\mu(s)$ satisfies the Dini condition (5) and*

$$s|\mu'(s)| \lesssim |\mu(s)| \quad \text{for } s \in (0,1]. \tag{7}$$

In addition, we fix

$$\alpha := \min\left\{2, 1+\frac{2}{n}\right\},\ \beta_\alpha := (n-1)\left(\frac{1}{\alpha}-\frac{1}{2}\right)$$

and assume that

$$r \in \left(2, \frac{2n}{[n-2]^+}\right).$$

Furthermore, the initial data (u_0, u_1) satisfies

$$(u_0,u_1) \in \mathcal{D} := \left(H^2 \cap H^2_r \cap H^{\beta_\alpha}_\alpha \cap L^1\right) \times \left(H^1 \cap H^1_r \cap L^\alpha \cap L^1\right).$$

Then, there exists a constant $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon}]$, problem (1) admits a unique global (in time) Sobolev solution

$$u \in \mathcal{C}([0,\infty), H^2 \cap L^\alpha \cap L^\infty)$$

satisfying the following estimates:

$$\begin{aligned}
\|u(t,\cdot)\|_{\dot{H}^2} &\lesssim \varepsilon(1+t)^{-\frac{n}{4}-1}\|(u_0,u_1)\|_{\mathcal{D}},\\
\|u(t,\cdot)\|_{L^\alpha} &\lesssim \varepsilon(1+t)^{-\frac{n}{2}(1-\frac{1}{\alpha})}\|(u_0,u_1)\|_{\mathcal{D}},\\
\|u(t,\cdot)\|_{L^\infty} &\lesssim \varepsilon(1+t)^{-\frac{n}{2}}\|(u_0,u_1)\|_{\mathcal{D}}.
\end{aligned}$$

Remark 1.1. From the statements of Theorem 1.1, we assert that the Dini condition (5) remains indispensable for the global (in time) existence of small data solutions to problem (1) in 4-dimension space. This result extends Theorem 3 in [8] and Proposition 4.1 in [11]. For some examples of modulus of continuity μ satisfying the conditions of Theorem 1.1, the reader may refer to Example 1 in [8].

Remark 1.2. The approach used to prove Theorem 1.1 can still be effectively applied to the weakly coupled system of equations (1). Namely, we can extend the global existence result in [4] to higher dimensions $n = 3, 4$.

Theorem 1.2 (Asymptotic profiles). *Consider that the assumptions of Theorem 1.1 hold, the global (in time) small data solution to (1) satisfies the following estimates for $t \gg 1$:*

$$\begin{aligned}
\|u(t,\cdot) - M\mathcal{G}(t,\cdot)\|_{L^\alpha} &= o(t^{-\frac{n}{2}(1-\frac{1}{\alpha})}),\\
\|u(t,\cdot) - M\mathcal{G}(t,\cdot)\|_{\dot{H}^2} &= o(t^{-\frac{n}{4}-1}),\\
\|u(t,\cdot) - M\mathcal{G}(t,\cdot)\|_{L^\infty} &= o(t^{-\frac{n}{2}}),
\end{aligned}$$

where

$$M := \varepsilon\int_{\mathbb{R}^n}(u_0(x)+u_1(x))dx + \int_0^\infty\int_{\mathbb{R}^n}\mathcal{N}(u(\tau,x))dxd\tau. \tag{8}$$

Remark 1.3. From the statements of Theorem 1.2, we obtain the asymptotic behavior of the global solution at $t \to \infty$ is identified by the Gauss kernel under the Dini condition (5) as follow:

$$u(t,\cdot) \sim \left(\varepsilon \int_{\mathbb{R}^n} (u_0(x) + u_1(x))dx + \int_0^\infty \int_{\mathbb{R}^n} \mathcal{N}(u(\tau,x))dxd\tau\right) \mathfrak{F}^{-1}\big(e^{-|\xi|^2 t}\big)(t,\cdot),$$

in the L^q framework, for all $q \in [\min\{2, 1+2/n\}, +\infty]$. From this, we can conclude the diffusive structure of problem (1) when it admits a unique global solution because it presents the following optimal decay estimates:

$$\|u(t,\cdot)\|_{L^q} \sim t^{-\frac{n}{2}(1-\frac{1}{q})} \quad \text{and} \quad \|u(t,\cdot)\|_{\dot{H}^2} \sim t^{-\frac{n}{4}-1},$$

where $t \gg 1$. These estimates can be easily derived from the conclusion of Theorem 1.2 together with the triangle inequality.

If the non-Dini condition (6) occurs, then Theorem 5 in [8] shows that the solution to (1) blows up in finite time. Now, we define the following function:

$$\Psi(R) := \int_1^R \frac{\mu\left(Cr^{-\frac{n}{2}}\right)}{r} dr \tag{9}$$

for a sufficiently large positive constant C. The sharp lifespan estimates for blow-up solutions are determined as follows.

Theorem 1.3 (Sharp lifespan estimates for blow-up solutions)**.** *Let us consider the problem (1) with the dimension spaces $1 \leqslant n \leqslant 4$. The modulus of continuity $\mu(s)$ satisfies the non-Dini condition (6) and the function*

$$\mathcal{N} : s \in \mathbb{R} \to \mathcal{N}(s) := |s|^{1+\frac{2}{n}} \mu(|s|) \in [0,\infty) \text{ is convex }. \tag{10}$$

In addition, we assume that the initial data $(u_0, u_1) \in \mathcal{D}$ satisfies

$$\int_{\mathbb{R}^n} (u_0(x) + u_1(x))dx > 0. \tag{11}$$

Then, there exists a positive constant ε_0 such that for any $\varepsilon \in (0, \varepsilon_0]$, the solution u blows up in finite time and the upper bound estimate for the lifespans

$$T_\varepsilon \lesssim \Psi^{-1}\left(C\varepsilon^{-\frac{2}{n}}\right)$$

holds. Furthermore, if we replace the condition (10) by (7), the lifespan of solution

$$u \in \mathcal{C}([0, T_\varepsilon), H^2 \cap L^\alpha \cap L^\infty)$$

satisfies the lower bound estimate

$$T_\varepsilon \gtrsim \Psi^{-1}\left(c\varepsilon^{-\frac{2}{n}}\right),$$

for any $\varepsilon \in (0, \varepsilon_0]$. Here C, c are positive constants depending only on n, u_0, u_1, μ and $\Psi^{-1}(\tau)$ is the inverse function of the function $\Psi(\tau)$ defined in (9).

Remark 1.4. Linking the achieved estimates in Theorem 1.3, one recognizes that the sharp estimates for the lifespan T_ε of blow-up solutions to the Cauchy problem (1) in spatial dimensions $1 \leqslant n \leqslant 4$ under the non-Dini condition are determined by the following relations:

$$T_\varepsilon \sim \Psi^{-1}\big(c\varepsilon^{-\frac{2}{n}}\big).$$

This is also a new result for spatial dimensional spaces $1 \leqslant n \leqslant 4$.

This paper is organized as follows: In Section 2, we provide the proof of global (in time) existence results for solutions to the problem (1). Subsequently, in Section 3, we describe the asymptotic behavior of the global solution to (1). Finally, we establish sharp lifespan estimates for blow-up solutions in Section 4.

2. Global Existence

2.1. **Philosophy of our approach.** In this section, our main aim is to prove Theorem 1.1. To begin with, we can write the solution to (3) by the formula

$$u^{\text{lin}}(t,x)=\varepsilon(\mathcal{K}(t,x)+\partial_t\mathcal{K}(t,x))*_x u_0(x)+\varepsilon\mathcal{K}(t,x)*_x u_1(x),$$

so that the solution to (1) becomes

$$u(t,x)=u^{\text{lin}}(t,x)+u^{\text{non}}(t,x),$$

thanks to Duhamel's principle, where

$$u^{\text{non}}(t,x):=\int_0^t\mathcal{K}(t-\tau,x)*_x\mathcal{N}(u(\tau,x))d\tau$$

and

$$\mathcal{K}(t,x):=\begin{cases}\mathfrak{F}^{-1}\left(\dfrac{e^{-\frac{t}{2}}\sinh\left(t\sqrt{\frac14-|\xi|^2}\right)}{\sqrt{\frac14-|\xi|^2}}\right)(t,x) & \text{if } |\xi|\leqslant\dfrac12,\\[2ex] \mathfrak{F}^{-1}\left(\dfrac{e^{-\frac{t}{2}}\sin\left(t\sqrt{|\xi|^2-\frac14}\right)}{\sqrt{|\xi|^2-\frac14}}\right)(t,x) & \text{if } |\xi|>\dfrac12.\end{cases}$$

Let $\chi_k=\chi_k(r)$ with $k\in\{\mathrm{L},\mathrm{H}\}$ be smooth cut-off functions having the following properties:

$$\chi_{\mathrm{L}}(r)=\begin{cases}1 & \text{if } r\leqslant\varepsilon^*/2,\\ 0 & \text{if } r\geqslant\varepsilon^*,\end{cases}\quad\text{and}\qquad \chi_{\mathrm{H}}(r)=1-\chi_{\mathrm{L}}(r),$$

where ε^* is a sufficiently small constant. It is obvious to see that $\chi_{\mathrm{H}}(r)=1$ if $r\geqslant\varepsilon^*$ and $\chi_{\mathrm{H}}(r)=0$ if $r\leqslant\varepsilon^*/2$.
Now, we restate the important result as follows.

Lemma 2.1 (Linear Estimates)**.** *Let $n\geqslant1$, $j\in\{0,1\}$, $1\leqslant\rho\leqslant q<\infty$, $q\neq1$, $\beta_q:=(n-1)\left|\frac12-\frac1q\right|$ and $s_1\geqslant s_2\geqslant0$. Then, the following estimate holds for all $t>0$:*

$$\begin{aligned}&\|\partial_t^j|\nabla|^{s_1}\mathcal{K}(t,x)*_x\varphi(x)\|_{L^q}\\ &\quad\lesssim(1+t)^{-\frac{n}{2}(\frac1\rho-\frac1q)-\frac{s_1-s_2}{2}-j}\||\nabla|^{s_2}\chi_{\mathrm{L}}(|\nabla|)\varphi\|_{L^\rho}+e^{-ct}\||\nabla|^{s_1}\chi_{\mathrm{H}}(|\nabla|)\varphi\|_{H_q^{\beta_q+j-1}},\end{aligned}$$

where c is a suitable positive constant. Furthermore, we obtain the following estimate for $1\leqslant\rho\leqslant\infty, 1<r<\infty$ and $d>\dfrac{n}{r}$:

$$\begin{aligned}&\|\partial_t^j|\nabla|^{s_1}\mathcal{K}(t,x)*_x\varphi(x)\|_{L^\infty}\\ &\quad\lesssim(1+t)^{-\frac{n}{2\rho}-\frac{s_1-s_2}{2}-j}\||\nabla|^{s_2}\chi_{\mathrm{L}}(|\nabla|)\varphi\|_{L^\rho}+e^{-ct}\||\nabla|^{s_1+d}\chi_{\mathrm{H}}(|\nabla|)\varphi\|_{H_r^{\beta_r+j-1}}.\end{aligned}$$

Proof. Due to the similarity in the proof and the fact that the result is completely analogous to Theorem 1.1 in [14], we present only the proof of the case $j = 0$. Specifically, using Proposition 2.4 in [14], we gain

$$\||\nabla|^{s_1}\chi_{\rm L}(|\nabla|)\mathcal{K}(t,x) *_x \varphi(x)\|_{L^q} \lesssim (1+t)^{-\frac{n}{2}(\frac{1}{\rho}-\frac{1}{q})-\frac{s_1-s_2}{2}}\||\nabla|^{s_2}\chi_{\rm L}(|\nabla|)\varphi\|_{L^\rho}, \text{ for all } 1 \leqslant \rho \leqslant q \leqslant \infty.$$

In addition, from Proposition 2.5 in [14], we obtain

$$\||\nabla|^{s_1}\chi_{\rm H}(|\nabla|)\mathcal{K}(t,x) *_x \varphi(x)\|_{L^q} \lesssim e^{-ct}\||\nabla|^{s_1}\chi_{\rm H}(|\nabla|)\varphi\|_{H_q^{\beta_q-1}} \text{ for } 1 < q < \infty.$$

Moreover, it is a fact that

$$\begin{aligned}&\||\nabla|^{s_1}\chi_{\rm H}(|\nabla|)\mathcal{K}(t,x) *_x \varphi(x)\|_{L^\infty}\\ &\quad \lesssim \||\nabla|^{s_1}\langle\nabla\rangle^d\chi_{\rm H}(|\nabla|)\mathcal{K}(t,x) *_x \varphi(x)\|_{L^r} \lesssim e^{-ct}\||\nabla|^{s_1+d}\chi_{\rm H}(|\nabla|)\varphi\|_{H_r^{\beta_r-1}},\end{aligned}$$

for all $1 < r < \infty$ and $d > n/r$. This completes the proof. $\square$

Under the assumptions of Theorem 1.1, we define the following function spaces for $T > 0$

$$X(T) := L^\infty([0,T], H^2 \cap L^\alpha \cap L^\infty),$$

with the norm

$$\|\varphi\|_{X(T)} := \sup_{t\in[0,T]}\left\{(1+t)^{\frac{n}{2}(1-\frac{1}{\alpha})}\|\varphi(t,\cdot)\|_{L^\alpha} + (1+t)^{\frac{n}{4}+1}\|\varphi(t,\cdot)\|_{\dot{H}^2} + (1+t)^{\frac{n}{2}}\|\varphi(t,\cdot)\|_{L^\infty}\right\}$$

and

$$Y(T) := L^\infty([0,T], L^2),$$

with the norm

$$\|\varphi\|_{Y(T)} := \sup_{t\in[0,T]}\big\{\|\varphi(t,\cdot)\|_{L^2}\big\}.$$

Moreover, we denote that

$$X(T,M) := \{\varphi \in X(T) : \|\varphi\|_{X(T)} \leqslant M\},$$

for all $M > 0$. We now proceed to the following important property.

Lemma 2.2. *$X(T,M)$ is a closed subset of $Y(T)$ with respect to the metric $Y(T)$.*

Proof. Firstly, we can easily see that $X(T,M) \subset Y(T)$ by interpolation. Therefore, it suffices to show that if a sequence in $X(T,M)$ converging in $Y(T)$, its limit will belong to $X(T,M)$. More specifically, we assume that $\{\varphi_j\}_{j=1}^\infty \subset X(T,M)$ and $\varphi_j \to \varphi \in Y(T)$ as $j \to \infty$. It is a fact that

$$L^\infty\left([0,T], H^2 \cap L^\infty \cap L^\alpha\right) = \left(L^1([0,T], H^{-2} + L^1 + L^{\alpha'})\right)^*,$$

where $\alpha' = \alpha/(\alpha-1)$. Due to the separability of $L^1([0,T], H^{-2} + L^1 + L^{\alpha'})$, we apply Banach-Alaoglu theorem (see Theorem 3.16 or Corollary 3.30 in [1]) to take a subsequence $\{\varphi_{j(k)}\}_{k=1}^\infty$ and $\psi \in L^\infty\left([0,T], H^2 \cap L^\infty \cap L^\alpha\right)$ such that

$$\varphi_{j(k)} \overset{*}{\rightharpoonup} \psi \quad \text{as} \quad k \to \infty.$$

In addition, we have

$$\|\psi\|_{X(T)} \leqslant \liminf_{k\to\infty}\|\varphi_{j(k)}\|_{X(T)} \leqslant M,$$

that is, $\psi \in X(T, M)$. On the other hand, both $\{\varphi_j\}_{j=1}^{\infty}$ and $\{\varphi_{j(k)}\}_{k=1}^{\infty}$ converge in the space of the distribution $\mathcal{D}'([0,T] \times \mathbb{R}^n)$, that is,

$$\begin{aligned} \varphi_{j(k)} &\to \psi \in \mathcal{D}'([0,T] \times \mathbb{R}^n) \quad \text{as} \quad k \to \infty, \\ \varphi_j &\to \varphi \in \mathcal{D}'([0,T] \times \mathbb{R}^n) \quad \text{as} \quad j \to \infty. \end{aligned}$$

As a result, the uniqueness of the limit of distribution implies $\psi \equiv \varphi$, which shows $\varphi \in X(T, M)$. □

Next, we define the operator Φ on the space $X(T)$

$$\Phi[u] := u^{\text{lin}} + u^{\text{non}} \tag{12}$$

and denote the quantity

$$\mathcal{I}(\varepsilon_0) := \int_0^{\varepsilon_0} \frac{\mu(s)}{s} ds,$$

for $\varepsilon_0 \in (0,1]$. Due to the condition (5), we have $\mathcal{I}(\varepsilon_0) \to 0$ as $\varepsilon_0 \to 0^+$. Finally, we introduce an important tool from Harmonic Analysis.

Proposition 2.1 (see Corollary 2.4 in [12])**.** *Let $1 < p,\ p_0,\ p_1 < \infty$, $a > 0$ and $\theta \in [0, a)$. Then, it holds the following fractional Gagliardo-Nirenberg inequality:*

$$\|u\|_{\dot{H}^{\theta}_p} \lesssim \|u\|_{L^{p_0}}^{1-\omega(\theta,a)} \|u\|_{\dot{H}^{a}_{p_1}}^{\omega(\theta,a)},$$

where $\omega(\theta, a) = \dfrac{\frac{1}{p_0} - \frac{1}{p} + \frac{\theta}{n}}{\frac{1}{p_0} - \frac{1}{p_1} + \frac{a}{n}}$ *and* $\dfrac{\theta}{a} \leqslant \omega(\theta, a) \leqslant 1$.

2.2. **Proof of Theorem 1.1.** Firstly, we will prove the following auxiliary results.

Lemma 2.3. *Under the assumptions of Theorem 1.1, the following estimates hold for all $u \in X(T, \varepsilon_0)$ and $T > 0,\ \varepsilon_0 \in (0,1]$:*

$$\begin{aligned} \|\mathcal{N}(u(\tau,\cdot))\|_{L^\gamma} &\lesssim \mu\left(\varepsilon_0 (1+\tau)^{-\frac{n}{2}}\right)(1+\tau)^{-\frac{n}{2}(1+\frac{2}{n}-\frac{1}{\gamma})} \|u\|_{X(T)}^{1+\frac{2}{n}} \quad \text{for } \gamma \geqslant 1, \\ \|\nabla \mathcal{N}(u(\tau,\cdot))\|_{L^\theta} &\lesssim \mu\left(\varepsilon_0 (1+\tau)^{-\frac{n}{2}}\right)(1+\tau)^{-\frac{n}{2}(1+\frac{2}{n}-\frac{1}{\theta})-\frac{1}{2}} \|u\|_{X(T)}^{1+\frac{2}{n}} \quad \text{for } \theta \in \left[\alpha, \frac{2n}{[n-2]^+}\right). \end{aligned}$$

Proof. Firstly, applying interpolation, we can easily see that

$$\begin{aligned} \|\mathcal{N}(u(\tau,\cdot))\|_{L^\gamma} &\leqslant \mu(\|u(\tau,\cdot)\|_{L^\infty}) \||u(\tau,\cdot)|^{1+\frac{2}{n}}\|_{L^\gamma} = \mu(\|u(\tau,\cdot)\|_{L^\infty}) \|u(\tau,\cdot)\|_{L^{\gamma(1+\frac{2}{n})}}^{1+\frac{2}{n}} \\ &\lesssim \mu\left((1+\tau)^{-\frac{n}{2}} \|u\|_{X(T)}\right) \|u(\tau,\cdot)\|_{L^\alpha}^{\frac{\alpha}{\gamma}} \|u(\tau,\cdot)\|_{L^\infty}^{1+\frac{2}{n}-\frac{\alpha}{\gamma}} \\ &\lesssim \mu\left(\varepsilon_0 (1+\tau)^{-\frac{n}{2}}\right)(1+\tau)^{-\frac{n}{2}(1+\frac{2}{n}-\frac{1}{\gamma})} \|u\|_{X(T)}^{1+\frac{2}{n}}. \end{aligned}$$

Moreover, using the condition (7) yields

$$|\nabla \mathcal{N}(u(\tau,\cdot))| \lesssim \mu(|u(\tau,\cdot)|) |u(\tau,\cdot)|^{\frac{2}{n}} |\nabla u(\tau,\cdot)|.$$

Therefore, we can obtain an immediate result

$$\|\nabla \mathcal{N}(u(\tau,\cdot))\|_{L^\theta} \lesssim \|\mu(|u(\tau,\cdot)|)\|_{L^\infty} \|u(\tau,\cdot)\|_{L^\infty}^{\frac{2}{n}} \|\nabla u(\tau,\cdot)\|_{L^\theta}.$$

Thanks to Proposition 2.1, we arrive at

$$\begin{aligned} \|\nabla u(\tau,\cdot)\|_{L^\theta} &\lesssim \|u(\tau,\cdot)\|_{L^\alpha}^{1-\omega_1} \|u(\tau,\cdot)\|_{\dot{H}^2}^{\omega_1} \\ &\lesssim (1+\tau)^{-\frac{n}{2}(1-\frac{1}{\theta})-\frac{1}{2}} \|u\|_{X(T)}, \end{aligned}$$

where

$$\omega_1 := \frac{\frac{1}{\alpha} - \frac{1}{\theta} + \frac{1}{n}}{\frac{1}{\alpha} - \frac{1}{2} + \frac{2}{n}} \in [0,1] \quad \text{for all} \quad \theta \in \left[\alpha, \frac{2n}{[n-2]^+}\right).$$

Therefore, we get

$$\|\nabla \mathcal{N}(u(\tau,\cdot))\|_{L^\theta} \lesssim \mu\left(\varepsilon_0(1+\tau)^{-\frac{n}{2}}\right)(1+\tau)^{-\frac{n}{2}(1+\frac{2}{n}-\frac{1}{\theta})-\frac{1}{2}}\|u\|_{X(T)}^{1+\frac{2}{n}}.$$

Thus, we completed the proof of Lemma 2.3. □

Proposition 2.2. *Under the assumptions of Theorem 1.1, the following estimate holds for all $u \in X(T,\varepsilon_0)$, $\varepsilon_0 \in (0,1]$:*

$$\|u^{\mathrm{non}}\|_{X(T)} \lesssim \mathcal{I}(\varepsilon_0)\|u\|_{X(T)}^{1+\frac{2}{n}}.$$

Proof. Firstly, using Lemma 2.3 and Lemma 2.1 with $j=0$, $\rho=1, q=2$, $(s_1,s_2)=(2,0)$ for $\tau \in [0,t/2]$, and $\rho=q=2, (s_1,s_2)=(2,1)$ for $\tau\in(t/2,t]$ to derive

$$\begin{aligned}
\|u^{\mathrm{non}}(t,\cdot)\|_{\dot{H}^2} &\lesssim \int_0^{t/2}(1+t-\tau)^{-\frac{n}{4}-1}\|\mathcal{N}(u(\tau,\cdot))\|_{L^1\cap \dot{H}^1}d\tau \\
&\quad + \int_{t/2}^t (1+t-\tau)^{-\frac{1}{2}}\|\mathcal{N}(u(\tau,\cdot))\|_{\dot{H}^1}d\tau \\
&\lesssim (1+t)^{-\frac{n}{4}-1}\|u\|_{X(T)}^{1+\frac{2}{n}}\int_0^{t/2}(1+\tau)^{-1}\mu\left(\varepsilon_0(1+\tau)^{-\frac{n}{2}}\right)d\tau \\
&\quad + (1+t)^{-\frac{n}{4}-1}\|u\|_{X(T)}^{1+\frac{2}{n}}\int_{t/2}^t(1+t-\tau)^{-1}\mu\left(\varepsilon_0(1+t-\tau)^{-\frac{n}{2}}\right)d\tau,
\end{aligned}$$

where we note that $1+\tau \geqslant 1+t-\tau$ for all $\tau\in[t/2,t]$. By using the change of variables $\theta_1=\varepsilon_0(1+\tau)^{-\frac{n}{2}}$ and $\theta_2=\varepsilon_0(1+t-\tau)^{-\frac{n}{2}}$, we can easily see that

$$\int_0^{t/2}(1+\tau)^{-1}\mu\left(\varepsilon_0(1+\tau)^{-\frac{n}{2}}\right)d\tau + \int_{t/2}^t(1+t-\tau)^{-1}\mu\left(\varepsilon_0(1+t-\tau)^{-\frac{n}{2}}\right)d\tau \lesssim \mathcal{I}(\varepsilon_0).$$

Therefore, we can conclude that

$$\|u^{\mathrm{non}}(t,\cdot)\|_{\dot{H}^2} \lesssim \mathcal{I}(\varepsilon_0)(1+t)^{-\frac{n}{4}-1}\|u\|_{X(T)}^{1+\frac{2}{n}}. \tag{13}$$

Next, thanks to again Lemma 2.1 for $j=0$, $\rho=1,q=\infty$, $(s_1,s_2)=(0,0)$ for $\tau\in[0,t/2]$, and $\rho=r,q=\infty$, $(s_1,s_2)=(0,0)$ for $\tau\in(t/2,t]$ to obtain

$$\begin{aligned}
\|u^{\mathrm{non}}(t,\cdot)\|_{L^\infty} &\lesssim \int_0^{t/2}(1+t-\tau)^{-\frac{n}{2}}\|\mathcal{N}(u(\tau,\cdot))\|_{L^1\cap H_r^{d+\beta_r-1}}d\tau \\
&\quad + \int_{t/2}^t(1+t-\tau)^{-\frac{n}{2r}}\|\mathcal{N}(u(\tau,\cdot))\|_{L^r\cap H_r^{d+\beta_r-1}}d\tau,
\end{aligned}$$

where we choose $d := n/r+\delta$ with δ is a sufficiently small constant. Noting that the conditions $r\in\left(2,\frac{2n}{[n-2]^+}\right)$ and $1\leqslant n\leqslant 4$ imply

$$0 < d+\beta_r = \frac{n}{r}+\delta+(n-1)\left(\frac{1}{2}-\frac{1}{r}\right) < 2,$$

that is,

$$\begin{aligned}\|\mathcal{N}(u(\tau,\cdot))\|_{H^{d+\beta_r-1}_r} &\lesssim \|\mathcal{N}(u(\tau,\cdot))\|_{H^1_r} \approx \|\mathcal{N}(u(\tau,\cdot))\|_{L^r} + \|\nabla\mathcal{N}(u(\tau,\cdot))\|_{L^r}\\ &\lesssim (1+\tau)^{-\frac{n}{2}(1+\frac{2}{n}-\frac{1}{r})}\mu\left(\varepsilon_0(1+\tau)^{-\frac{n}{2}}\right)\|u\|^{1+\frac{2}{n}}_{X(T)},\end{aligned}$$

due to using again Lemma 2.3. As a result, we get

$$\begin{aligned}\|u^{\mathrm{non}}(t,\cdot)\|_{L^\infty} &\lesssim \|u\|^{1+\frac{2}{n}}_{X(T)}\int_0^{t/2}(1+t-\tau)^{-\frac{n}{2}}(1+\tau)^{-1}\mu\left(\varepsilon_0(1+\tau)^{-\frac{n}{2}}\right)d\tau\\ &\quad + \|u\|^{1+\frac{2}{n}}_{X(T)}\int_{t/2}^{t}(1+t-\tau)^{-\frac{n}{2r}}(1+\tau)^{-\frac{n}{2}(1+\frac{2}{n}-\frac{1}{r})}\mu\left(\varepsilon_0(1+\tau)^{-\frac{n}{2}}\right)d\tau\\ &\lesssim (1+t)^{-\frac{n}{2}}\|u\|^{1+\frac{2}{n}}_{X(T)}\int_0^{t/2}(1+\tau)^{-1}\mu\left(\varepsilon_0(1+\tau)^{-\frac{n}{2}}\right)d\tau\\ &\quad + (1+t)^{-\frac{n}{2}}\|u\|^{1+\frac{2}{n}}_{X(T)}\int_{t/2}^{t}(1+t-\tau)^{-\frac{n}{2r}}(1+\tau)^{-1+\frac{n}{2r}}\mu\left(\varepsilon_0(1+\tau)^{-\frac{n}{2}}\right)d\tau\\ &\lesssim (1+t)^{-\frac{n}{2}}\|u\|^{1+\frac{2}{n}}_{X(T)}\int_0^{t/2}(1+\tau)^{-1}\mu\left(\varepsilon_0(1+\tau)^{-\frac{n}{2}}\right)d\tau\\ &\quad + (1+t)^{-\frac{n}{2}}\|u\|^{1+\frac{2}{n}}_{X(T)}\int_{t/2}^{t}(1+t-\tau)^{-1}\mu\left(\varepsilon_0(1+t-\tau)^{-\frac{n}{2}}\right)d\tau\\ &\lesssim \mathcal{I}(\varepsilon_0)(1+t)^{-\frac{n}{2}}\|u\|^{1+\frac{2}{n}}_{X(T)}.\end{aligned}$$

From this, we obtain the following estimate:

$$\|u(t,\cdot)\|_{L^\infty} \lesssim \mathcal{I}(\varepsilon_0)(1+t)^{-\frac{n}{2}}\|u\|^{1+\frac{2}{n}}_{X(T)}. \tag{14}$$

Finally, we need to estimate $\|u^{\mathrm{non}}(\tau,\cdot)\|_{L^\alpha}$. We note that $\beta_\alpha < 1$ for all $1 \leqslant n \leqslant 4$. Therefore, employing again Lemma 2.3 combined with Lemma 2.1 with $j = 0$, $\rho = 1, q = \alpha, (s_1, s_2) = (0,0)$ for $\tau \in [0, t/2)$ and $\rho = q = \alpha, (s_1, s_2) = (0,0)$ for $\tau \in [t/2, t]$ we have

$$\begin{aligned}\|u^{\mathrm{non}}(t,\cdot)\|_{L^\alpha} &\lesssim \int_0^{t/2}(1+t-\tau)^{-\frac{n}{2}(1-\frac{1}{\alpha})}\|\mathcal{N}(u(\tau,\cdot))\|_{L^1\cap L^\alpha}d\tau + \int_{t/2}^{t}\|\mathcal{N}(u(\tau,\cdot))\|_{L^\alpha}d\tau\\ &\lesssim \|u\|^{1+\frac{2}{n}}_{X(T)}\int_0^{t/2}(1+t-\tau)^{-\frac{n}{2}(1-\frac{1}{\alpha})}(1+\tau)^{-1}\mu\left(\varepsilon_0(1+\tau)^{-\frac{n}{2}}\right)d\tau\\ &\quad + \|u\|^{1+\frac{2}{n}}_{X(T)}\int_{t/2}^{t}(1+\tau)^{-\frac{n}{2}(1+\frac{2}{n}-\frac{1}{\alpha})}\mu\left(\varepsilon_0(1+\tau)^{-\frac{n}{2}}\right)d\tau\\ &\lesssim (1+t)^{-\frac{n}{2}(1-\frac{1}{\alpha})}\|u\|^{1+\frac{2}{n}}_{X(T)}\int_0^{t/2}(1+\tau)^{-1}\mu\left(\varepsilon_0(1+\tau)^{-\frac{n}{2}}\right)d\tau\\ &\quad + (1+t)^{-\frac{n}{2}(1-\frac{1}{\alpha})}\|u\|^{1+\frac{2}{n}}_{X(T)}\int_{t/2}^{t}(1+t-\tau)^{-1}\mu\left(\varepsilon_0(1+t-\tau)^{-\frac{n}{2}}\right)d\tau\\ &\lesssim \mathcal{I}(\varepsilon_0)(1+t)^{-\frac{n}{2}(1-\frac{1}{\alpha})}\|u\|^{1+\frac{2}{n}}_{X(T)}.\end{aligned}$$

In summary, we have the following estimate:

$$\|u^{\mathrm{non}}(t,\cdot)\|_{L^\alpha} \lesssim \mathcal{I}(\varepsilon_0)(1+t)^{-\frac{n}{2}(1-\frac{1}{\alpha})}\|u\|^{1+\frac{2}{n}}_{X(T)}. \tag{15}$$

The estimates (13)-(15) complete the proof of Proposition 2.2. $\square$

Proposition 2.3. *Under the assumptions of Theorem 1.1, the following estimate holds for all* $u, v \in X(T, \varepsilon_0)$, $\varepsilon_0 \in (0,1]$*:*

$$\|\Phi[u]-\Phi[v]\|_{Y(T)} \lesssim \mathcal{I}(\varepsilon_0)\|u-v\|_{Y(T)}\left(\|u\|_{X(T)}^{p-1}+\|v\|_{X(T)}^{p-1}\right).$$

Proof. From the definition of the spaces $Y(T)$ combined with Lemma 2.1 with $j=0$, $\rho = m \in (1,2), q=2$ and $(s_1,s_2)=(0,0)$, we obtain

$$\begin{aligned}\|\Phi[u]-\Phi[v]\|_{Y(T)} &\lesssim \sup_{t\in[0,T]}\left\{\int_0^t (1+t-\tau)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})}\|\mathcal{N}(u(\tau,\cdot))-\mathcal{N}(v(\tau,\cdot))\|_{L^m\cap H^{-1}}d\tau\right\}\\ &\lesssim \sup_{t\in[0,T]}\left\{\int_0^t (1+t-\tau)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})}\|\mathcal{N}(u(\tau,\cdot))-\mathcal{N}(v(\tau,\cdot))\|_{L^m}d\tau\right\},\end{aligned}$$

where we have the embedding

$$\|\varphi\|_{L^2} \lesssim \|\langle\nabla\rangle\varphi\|_{L^m} \quad \text{with} \quad \frac{1}{m} \leqslant \frac{1}{2}+\frac{1}{n} \text{ and } m \in (1,2).$$

Additionally, using the assumption (7) we gain

$$\begin{aligned}|\mathcal{N}(u(\tau,\cdot))-\mathcal{N}(v(\tau,\cdot))| &= \left|(u-v)(\tau,\cdot)\times\left(\int_0^1\left(\frac{\partial\mathcal{N}}{\partial s}\right)(v+\kappa(u-v))d\kappa\right)(\tau,\cdot)\right|\\ &\lesssim |(u-v)(\tau,\cdot)|\times\left|\left(\int_0^1 |v+\kappa(u-v)|^{\frac{2}{n}}\mu\left(|v+\kappa(u-v)|\right)d\kappa\right)(\tau,\cdot)\right|\\ &\lesssim |(u-v)(\tau,\cdot)|\left(|u(\tau,\cdot)|^{\frac{2}{n}}+|v(\tau,\cdot)|^{\frac{2}{n}}\right)\int_0^1 \|\mu(v+\kappa(u-v))(\tau,\cdot)\|_{L^\infty}d\kappa.\end{aligned}$$

Therefore, thanks to Hölder's inequality and interpolation one has

$$\begin{aligned}&\|\mathcal{N}(u(\tau,\cdot))-\mathcal{N}(v(\tau,\cdot))\|_{L^m}\\ &\quad\lesssim \|u(\tau,\cdot)-v(\tau,\cdot)\|_{L^2}\left(\|u(\tau,\cdot)\|_{L^{\frac{2}{n}\eta}}^{\frac{2}{n}}+\|v(\tau,\cdot)\|_{L^{\frac{2}{n}\eta}}^{\frac{2}{n}}\right)\int_0^1\|\mu(v+\kappa(u-v))(\tau,\cdot)\|_{L^\infty}d\kappa\\ &\quad\lesssim (1+\tau)^{-1+\frac{n}{2\eta}}\mu\left(\varepsilon_0(1+\tau)^{-\frac{n}{2}}\right)\|u-v\|_{Y(T)}\left(\|u\|_{X(T)}^{\frac{2}{n}}+\|v\|_{X(T)}^{\frac{2}{n}}\right)\\ &\quad\lesssim (1+\tau)^{-1+\frac{n}{2}(\frac{1}{m}-\frac{1}{2})}\mu\left(\varepsilon_0(1+\tau)^{-\frac{n}{2}}\right)\|u-v\|_{Y(T)}\left(\|u\|_{X(T)}^{\frac{2}{n}}+\|v\|_{X(T)}^{\frac{2}{n}}\right),\end{aligned}$$

provided that the following conditions are satisfied:

$$m\in(1,2),\quad \frac{1}{m}=\frac{1}{2}+\frac{1}{\eta}\leqslant\frac{1}{2}+\frac{1}{n}\quad \text{and } \alpha\leqslant\frac{2\eta}{n}<\infty.$$

Thus, one can choose $1/\eta = \delta \ll 1$ to ensure the existence of the parameter m. Summarily, we obtain the following estimate:

$$\|\Phi[u]-\Phi[v]\|_{Y(T)} \lesssim \sup_{t\in[0,T]}\{\mathcal{J}_1(t)+\mathcal{J}_2(t)\}\|u-v\|_{Y(T)}\left(\|u\|_{X(T)}^{\frac{2}{n}}+\|v\|_{X(T)}^{\frac{2}{n}}\right),$$

where

$$\begin{aligned}\mathcal{J}_1(t) &:= \int_0^{t/2}(1+t-\tau)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})}(1+\tau)^{-1+\frac{n}{2}(\frac{1}{m}-\frac{1}{2})}\mu\left(\varepsilon_0(1+\tau)^{-\frac{n}{2}}\right)d\tau,\\ \mathcal{J}_2(t) &:= \int_{t/2}^t(1+t-\tau)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})}(1+\tau)^{-1+\frac{n}{2}(\frac{1}{m}-\frac{1}{2})}\mu\left(\varepsilon_0(1+\tau)^{-\frac{n}{2}}\right)d\tau.\end{aligned}$$

We can easily see that

$$\begin{aligned}\mathcal{J}_1(t) &\lesssim (1+t)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})}\int_0^{t/2}(1+\tau)^{-1+\frac{n}{2}(\frac{1}{m}-\frac{1}{2})}\mu\left(\varepsilon_0(1+\tau)^{-\frac{n}{2}}\right)d\tau\\ &\lesssim \int_0^{t/2}(1+\tau)^{-1}\mu\left(\varepsilon_0(1+\tau)^{-\frac{n}{2}}\right)d\tau \lesssim \mathcal{I}(\varepsilon_0),\end{aligned}$$

and

$$\mathcal{J}_2(t) \lesssim \int_{t/2}^{t}(1+t-\tau)^{-1}\mu\left(\varepsilon_0(1+t-\tau)^{-\frac{n}{2}}\right)d\tau \lesssim \mathcal{I}(\varepsilon_0),$$

for all $t>0$ due to $n\,(1/m-1/2)<2$ and $1+\tau \geqslant 1+t-\tau$ for $\tau \in [t/2,t]$. Thus, we completed the proof of Proposition 2.3. $\square$

Proof of Theorem 1.1. Using again Lemma 2.1, it is the fact that

$$\|u^{\mathrm{lin}}\|_{X(T)} \lesssim \varepsilon\|(u_0,u_1)\|_{\mathcal{D}},$$

for all $T>0$. For this reason, combining with Propositions 2.2 and 2.3, we obtain the following two important estimates for all $u,v \in X(T,\varepsilon_0)$, $\varepsilon_0 \in (0,1]$:

$$\|\Phi[u]\|_{X(T)} \leqslant C_1\varepsilon\|(u_0,u_1)\|_{\mathcal{D}} + C_1\mathcal{I}(\varepsilon_0)\|u\|_{X(T)}^{1+\frac{2}{n}}, \tag{16}$$

$$\|\Phi[u]-\Phi[v]\|_{Y(T)} \leqslant C_2\mathcal{I}(\varepsilon_0)\|u-v\|_{Y(T)}\left(\|u\|_{X(T)}^{\frac{2}{n}}+\|v\|_{X(T)}^{\frac{2}{n}}\right). \tag{17}$$

Let us fix

$$\bar{\varepsilon} := \frac{\varepsilon_0}{2C_1\|(u_0,u_1)\|_{\mathcal{D}}} \tag{18}$$

and ε_0 satisfies

$$\max\{C_1,2C_2\}\mathcal{I}(\varepsilon_0)\varepsilon_0^{\frac{2}{n}} \leqslant \frac{1}{2}.$$

As a result, the inequalities (16) and (17) become

$$\|\Phi[u]\|_{X(T)} \leqslant \varepsilon_0, \tag{19}$$

$$\|\Phi[u]-\Phi[v]\|_{Y(T)} \leqslant \frac{1}{2}\|u-v\|_{Y(T)}, \tag{20}$$

for all $u,v \in X(T,\varepsilon_0)$, $\varepsilon \in (0,\bar{\varepsilon}]$. Next, taking the recurrence sequence $\{u_j\}_{j=0}^{\infty}$ with $u_0=0$; $u_j=\Phi[u_{j-1}]$ for $j=1,2,...$, we employ (19) to conclude that

$$\{u_j\}_{j=0}^{\infty} \subset X(T,\varepsilon_0),$$

for all $\varepsilon \in (0,\bar{\varepsilon}]$. Moreover, using (20) implies that $\{u_j\}_{j=0}^{\infty}$ is a Cauchy sequence in the Banach space $Y(T)$. Furthermore, $X(T,\varepsilon_0)$ is a closed subset of $Y(T)$ with respect to the metric $Y(T)$ (see Lemma 2.2). Therefore, there exists a solution $u=\Phi[u]$ in $X(T,\varepsilon_0)$ for all $T>0$. Since T is arbitrary, the solution is global, that is, $u \in X(\infty,\varepsilon_0)$.

Next, we need to show that $u \in \mathcal{C}([0,\infty),H^2 \cap L^{\alpha} \cap L^{\infty})$. To prove this property, we recall that

$$u(t,x)=\Phi[u](t,x)=u^{\mathrm{lin}}(t,x)+\int_0^t \mathcal{K}(t-\tau,x)*_x\mathcal{N}(u(\tau,x))d\tau.$$

Since the linear part $u^{\mathrm{lin}}(t,x)$ obviously satisfies continuity, it suffices to show that

$$\int_0^t \mathcal{K}(t-\tau,x)*_x\mathcal{N}(u(\tau,x))d\tau \in \mathcal{C}([0,\infty),H^s \cap L^{\alpha} \cap L^{\infty}). \tag{21}$$

Noting that $u \in X(\infty, \varepsilon_0)$ and using again Lemmas 2.1 and 2.3, we have the following estimates:

$$\begin{aligned}
&\|\mathcal{K}(t-\tau,x) *_x \mathcal{N}(u(\tau,x))\|_{L^\alpha} \lesssim \mu\left(\varepsilon_0(1+\tau)^{-\frac{n}{2}}\right)(1+\tau)^{-\frac{n}{2}(1+\frac{2}{n}-\frac{1}{\alpha})}\|u\|^{1+\frac{2}{n}}_{X(\infty)},\\
&\|\mathcal{K}(t-\tau,x) *_x \mathcal{N}(u(\tau,x))\|_{\dot{H}^2} \lesssim \mu\left(\varepsilon_0(1+\tau)^{-\frac{n}{2}}\right)(1+\tau)^{-\frac{n}{4}-\frac{3}{2}}\|u\|^{1+\frac{2}{n}}_{X(\infty)},\\
&\|\mathcal{K}(t-\tau,x) *_x \mathcal{N}(u(\tau,x))\|_{L^\infty} \lesssim \mu\left(\varepsilon_0(1+\tau)^{-\frac{n}{2}}\right)(1+\tau)^{-\frac{n}{2}-1}\|u\|^{1+\frac{2}{n}}_{X(\infty)}.
\end{aligned}$$

Therefore, the Lebesgue convergence theorem in the Bochner integral immediately implies (21).

Finally, we establish the uniqueness of the global solution u , which lies in the space $\mathcal{C}([0,\infty), H^2 \cap L^\alpha \cap L^\infty)$. Let u, v be solutions to (1) belonging to this space. Using again the arbitrary positive number T and performing the same proof steps of Proposition 2.3, we obtain

$$\|u-v\|_{Y(t)} \lesssim \int_0^t \|u-v\|_{Y(\tau)}\big(\|u\|^{\frac{2}{n}}_{X(\tau)} + \|v\|^{\frac{2}{n}}_{X(\tau)}\big)d\tau,$$

for all $t \in [0,T]$. One can easily see that there exists a constant $M(T)$ such that $\|u\|^{\frac{2}{n}}_{X(T)} + \|v\|^{\frac{2}{n}}_{X(T)} \leqslant M(T)$. Therefore, we immediately have

$$\|u-v\|_{Y(t)} \lesssim M(T)\int_0^t \|u-v\|_{Y(\tau)}d\tau,$$

for all $t \in [0,T]$. From this, the Gronwall inequality implies $u \equiv v$ on $[0,T]$. Because T is an arbitrary positive number, $u \equiv v$ on $[0,\infty)$. Hence, the proof of Theorem 1.1 is completed.

3. Asymptotic behaviors of global solution

In this section, our main aim is to prove Theorem 1.2. To do this, we recall the following important results.

Lemma 3.1. *Let $1 \leqslant \rho \leqslant q < \infty$, $q \neq 1$, $\beta_q := (n-1)\big|\frac{1}{2}-\frac{1}{q}\big|$ and $s_1 \geqslant s_2 \geqslant 0$. Then, the following estimate holds for $t \geqslant 1$ (see Theorem 1.2 in [14]):*

$$\begin{aligned}
&\||\nabla|^{s_1}(\mathcal{K}(t,x)-\mathcal{G}(t,x)) *_x \varphi(x)\|_{L^q}\\
&\qquad\lesssim (1+t)^{-\frac{n}{2}(\frac{1}{\rho}-\frac{1}{q})-\frac{s_1-s_2}{2}-1}\||\nabla|^{s_2}\chi_{\mathrm{L}}(|\nabla|)\varphi\|_{L^\rho} + e^{-ct}\||\nabla|^{s_1}\chi_{\mathrm{H}}(|\nabla|)\varphi\|_{H^{\beta_q-1}_q},
\end{aligned}$$

where c is a suitable positive constant. Furthermore, we obtain the following estimate for $1 \leqslant \rho \leqslant \infty, 1 < r < \infty$ and $d > \dfrac{n}{r}$:

$$\begin{aligned}
&\||\nabla|^{s_1}(\mathcal{K}(t,x)-\mathcal{G}(t,x)) *_x \varphi(x)\|_{L^\infty}\\
&\qquad\lesssim (1+t)^{-\frac{n}{2\rho}-\frac{s_1-s_2}{2}-1}\||\nabla|^{s_2}\chi_{\mathrm{L}}(|\nabla|)\varphi\|_{L^\rho} + e^{-ct}\||\nabla|^{s_1+d}\chi_{\mathrm{H}}(|\nabla|)\varphi\|_{H^{\beta_r-1}_r}.
\end{aligned}$$

Proof. We need to prove the estimate of the L^∞ norm. Proposition 2.10 in [14] implies

$$\||\nabla|^{s_1}\big(\chi_{\mathrm{L}}(|\nabla|)\mathcal{K}(t,x)-\mathcal{G}(t,x)\big) *_x \varphi(x)\|_{L^\infty} \lesssim (1+t)^{-\frac{n}{2\rho}-\frac{s_1-s_2}{2}-1}\||\nabla|^{s_2}\chi_{\mathrm{L}}(|\nabla|)\varphi\|_{L^\rho},$$

for all $1 \leqslant \rho \leqslant \infty$ and $s_1 \geqslant s_2 \geqslant 0$. In addition, from Proposition 2.5 in [14], we obtain

$$\||\nabla|^{s_1}\chi_{\mathrm{H}}(|\nabla|)\mathcal{K}(t,x) *_x \varphi(x)\|_{L^q} \lesssim e^{-ct}\||\nabla|^{s_1}\chi_{\mathrm{H}}(|\nabla|)\varphi\|_{H^{\beta_q-1}_q} \text{ for } 1<q<\infty.$$

Moreover, it is the fact that

$$\begin{aligned}
&\||\nabla|^{s_1}\chi_{\mathrm{H}}(|\nabla|)\mathcal{K}(t,x) *_x \varphi(x)\|_{L^\infty}\\
&\quad\lesssim \||\nabla|^{s_1}\langle\nabla\rangle^d\chi_{\mathrm{H}}(|\nabla|)\mathcal{K}(t,x) *_x \varphi(x)\|_{L^r} \lesssim e^{-ct}\||\nabla|^{s_1+d}\chi_{\mathrm{H}}(|\nabla|)\varphi\|_{H^{\beta_r-1}_r},
\end{aligned}$$

for all $1 < r < \infty$ and $d > n/r$. This completes the proof. $\square$

Lemma 3.2. *Let $1 \leqslant q \leqslant \infty$, $\gamma \geqslant 0$, $j \in \mathbb{N}$. Then, we have the following estimates for all $\varphi \in L^1$ and $t \gg 1$:*

$$\|\partial_t^j |\nabla|^\gamma \mathcal{G}(t,\cdot)\|_{L^q} \lesssim t^{-\frac{n}{2}(1-\frac{1}{q})-\frac{\gamma}{2}-j} \tag{22}$$

and if $\gamma \in \mathbb{N}$, we have the following relation:

$$\left\| |\nabla|^\gamma \mathcal{G}(t,x) *_x \varphi(x) - \left(\int_{\mathbb{R}^n} \varphi(x)dx\right) |\nabla|^\gamma \mathcal{G}(t,x)\right\|_{L^q} = o(t^{-\frac{n}{2}(1-\frac{1}{q})-\frac{\gamma}{2}}). \tag{23}$$

Proof. To prove the estimate (22), we apply Theorem 24.2.4 in [9] as follows:

$$\|\partial_t^j |\nabla|^\gamma \mathcal{G}(t,\cdot)\|_{L^q} = C\left\|\mathfrak{F}^{-1}\left(|\xi|^{2j+\gamma} e^{-|\xi|^2 t}\right)(t,\cdot)\right\|_{L^q} \lesssim t^{-\frac{n}{2}(1-\frac{1}{q})-\frac{\gamma}{2}-j}.$$

The relation (23) follows directly from Lemma A.1 in [17]. □

Proof of Theorem 1.2. Firstly, we recall that the global solution to (1) satisfies the following integral equation:

$$\begin{aligned} u(t,x) &= u^{\text{lin}}(t,x) + u^{\text{non}}(t,x) \\ &= \varepsilon(\mathcal{K}(t,x) + \partial_t \mathcal{K}(t,x)) *_x u_0(x) + \varepsilon \mathcal{K}(t,x) *_x u_1(x) + \int_0^t \mathcal{K}(t-\tau,x) *_x \mathcal{N}(u(\tau,x))d\tau. \end{aligned}$$

We shall use the notation and results established in Section 2 and fix (θ, s) as one of the two pairs $(\alpha, 0)$ and $(2,2)$. From Lemmas 3.1 and 3.2, we can easily arrive at

$$\begin{aligned} &\left\| |\nabla|^s \left(u^{\text{lin}}(t,x) - \varepsilon\left(\int_{\mathbb{R}^n} (u_0(x)+u_1(x))dx\right) \mathcal{G}(t,x)\right)\right\|_{L^\theta} \\ &\quad \lesssim \varepsilon \left\| |\nabla|^s \big(\mathcal{K}(t,x) - \mathcal{G}(t,x)\big) *_x (u_0+u_1)(x)\right\|_{L^\theta} \\ &\qquad + \varepsilon \left\| |\nabla|^s \left(\mathcal{G}(t,x) *_x (u_0+u_1)(x) - \left(\int_{\mathbb{R}^n} (u_0(x)+u_1(x))dx\right)\mathcal{G}(t,x)\right)\right\|_{L^\theta} \\ &\qquad + \varepsilon \|\partial_t |\nabla|^s \mathcal{K}(t,x) *_x u_0(x)\|_{L^\theta} \\ &\quad = o(t^{-\frac{n}{2}(1-\frac{1}{\theta})-\frac{s}{2}}), \quad t \gg 1, \end{aligned}$$

where we used Lemma 3.1 for the first term, the relation (23) for the second term, and Lemma 2.1 for the third term. Therefore, we need to prove the following relation:

$$\left\| |\nabla|^s \left(u^{\text{non}}(t,x) - \left(\int_0^\infty \int_{\mathbb{R}^n} \mathcal{N}(u(t,x))dxdt\right) \mathcal{G}(t,x)\right)\right\|_{L^\theta} = o(t^{-\frac{n}{2}(1-\frac{1}{\theta})-\frac{s}{2}}), \quad t \gg 1, \tag{24}$$

for $(\theta,s)\in\{(2,2),(\alpha,0)\}$. Let us now divide the left-hand side term of (24) in the L^θ norm for $s=0$ into five parts as follows:

$$\begin{aligned}
&u^{\rm non}(t,x)-\left(\int_0^\infty\int_{\mathbb{R}^n}\mathcal{N}(u(\tau,x))dxd\tau\right)\mathcal{G}(t,x)\\
&\quad=\int_0^t\mathcal{K}(t-\tau,x)\ast_x\mathcal{N}(u(\tau,x))d\tau-\left(\int_0^\infty\int_{\mathbb{R}^n}\mathcal{N}(u(\tau,x))dxd\tau\right)\mathcal{G}(t,x)\\
&\quad=\int_0^{t/2}(\mathcal{K}(t-\tau,x)-\mathcal{G}(t-\tau,x))\ast_x\mathcal{N}(u(\tau,x))d\tau+\int_{t/2}^t\mathcal{K}(t-\tau,x)\ast_x\mathcal{N}(u(\tau,x))d\tau\\
&\qquad+\int_0^{t/2}(\mathcal{G}(t-\tau,x)-\mathcal{G}(t,x))\ast_x\mathcal{N}(u(\tau,x))d\tau\\
&\qquad+\int_0^{t/2}\mathcal{G}(t,x)\ast_x\mathcal{N}(u(\tau,x))-\left(\int_{\mathbb{R}^n}\mathcal{N}(u(\tau,x))dx\right)\mathcal{G}(t,x)d\tau\\
&\qquad-\left(\int_{t/2}^\infty\int_{\mathbb{R}^n}\mathcal{N}(u(\tau,x))dxd\tau\right)\mathcal{G}(t,x)\\
&\quad=:K_1(t,x)+K_2(t,x)+K_3(t,x)+K_4(t,x)-K_5(t,x). \qquad (25)
\end{aligned}$$

We proved that Theorem 1.1 ensures the existence of a unique global solution u satisfying

$$\|u\|_{X(\infty)}\leqslant\varepsilon_0.$$

Therefore, thanks to again Lemma 2.3 and Lemma 3.1 for $\rho=1$, $q=\theta$ and $(s_1,s_2)=(s,0)$, we obtain

$$\begin{aligned}
\||\nabla|^sK_1(t,\cdot)\|_{L^\theta}&\lesssim\int_0^{t/2}(1+t-\tau)^{-\frac{n}{2}(1-\frac{1}{\theta})-\frac{s}{2}-1}\|\mathcal{N}(u(\tau,\cdot))\|_{L^1\cap H^{s+\beta_\theta-1}_\theta}d\tau\\
&\lesssim\varepsilon_0^{1+\frac{2}{n}}(1+t)^{-\frac{n}{2}(1-\frac{1}{\theta})-\frac{s}{2}-1}\int_0^{t/2}(1+\tau)^{-1}\mu\big(\varepsilon_0(1+\tau)^{-1}\big)d\tau\\
&\lesssim\mathcal{I}(\varepsilon_0)\varepsilon_0^{1+\frac{2}{n}}(1+t)^{-\frac{n}{2}(1-\frac{1}{\theta})-\frac{s}{2}-1}=o\big(t^{-\frac{n}{2}(1-\frac{1}{\theta})-\frac{s}{2}}\big),\quad t\gg1, \qquad (26)
\end{aligned}$$

for $(\theta,s)=(\alpha,0)$ or $(2,2)$. Now, we consider the term $K_2(t,x)$. Specicially, employing again Lemma 2.3 and Lemma 2.1 for $\rho=q=\theta$ and $(s_1,s_2)=(s,s/2)$ we get

$$\begin{aligned}
\||\nabla|^sK_2(t,\cdot)\|_{L^\theta}&\lesssim\int_{t/2}^t(1+t-\tau)^{-\frac{s}{4}}\|\chi_{\rm L}(|\nabla|)\mathcal{N}(u(\tau,\cdot))\|_{\dot{H}^{s/2}_\theta}\\
&\qquad\qquad+e^{-c(t-\tau)}\|\chi_{\rm H}(|\nabla|)\mathcal{N}(u(\tau,\cdot))\|_{\dot{H}^{s+\beta_\theta-1}_\theta}d\tau\\
&\lesssim\varepsilon_0^{1+\frac{2}{n}}\int_{t/2}^t(1+t-\tau)^{-\frac{s}{4}}(1+\tau)^{-\frac{n}{2}(1+\frac{2}{n}-\frac{1}{\theta})-\frac{s}{4}}\mu\left(\varepsilon_0(1+\tau)^{-\frac{n}{2}}\right)d\tau\\
&\lesssim\varepsilon_0^{1+\frac{2}{n}}(1+t)^{1-\frac{n}{2}(1+\frac{2}{n}-\frac{1}{\theta})-\frac{s}{2}}\mu(\varepsilon_0(1+t/2)^{-\frac{n}{2}})\\
&=\varepsilon_0^{1+\frac{2}{n}}(1+t)^{-\frac{n}{2}(1-\frac{1}{\theta})-\frac{s}{2}}\mu(\varepsilon_0(1+t/2)^{-\frac{n}{2}})=o(t^{-\frac{n}{2}(1-\frac{1}{\theta})-\frac{s}{2}}),\quad t\gg1, \qquad (27)
\end{aligned}$$

for $(\theta,s)=(\alpha,0)$ or $(2,2)$, where we note that $\mu(0)=0$, μ is continuity and $\beta_\alpha<1$, $\beta_2=0$. Taking account of $K_3(t,x)$ we use the mean value theorem on t to get the following representation:

$$\mathcal{G}(t-\tau,x)-\mathcal{G}(t,x)=-\tau\partial_t\mathcal{G}(t-\lambda_1\tau,x),$$

with $\lambda_1 = \lambda_1(t,\tau) \in [0,1]$. Thus, using the relation $t - \lambda_1\tau \sim t$ for $\tau \in [0, t/2]$ and the estimate (22), we arrive at

$$\begin{aligned}\||\nabla|^s K_3(t,\cdot)\|_{L^\theta} &\lesssim \int_0^{t/2} \tau(t-\lambda_1\tau)^{-\frac{n}{2}(1-\frac{1}{\theta})-\frac{s}{2}-1}\|\mathcal{N}(u(\tau,\cdot))\|_{L^1}d\tau\\
&\lesssim \varepsilon_0^{1+\frac{2}{n}} t^{-\frac{n}{2}(1-\frac{1}{\theta})-\frac{s}{2}-1}\int_0^{t/2}\tau(1+\tau)^{-1}\mu\big(\varepsilon_0(1+\tau)^{-\frac{n}{2}}\big)d\tau\\
&\lesssim \varepsilon_0^{1+\frac{2}{n}} t^{-\frac{n}{2}(1-\frac{1}{\theta})-\frac{s}{2}-1}\int_0^{t/2}\mu\big(\varepsilon_0(1+\tau)^{-\frac{n}{2}}\big)d\tau = o(t^{-\frac{n}{2}(1-\frac{1}{\theta})-\frac{s}{2}}), \quad t \gg 1, \qquad (28)\end{aligned}$$

due to applying the L'Hospital rule, we get

$$\lim_{t\to\infty} t^{-1}\int_0^{t/2}\mu\big(\varepsilon_0(1+\tau)^{-\frac{n}{2}}\big)d\tau = 0.$$

Next, we estimate the quantity $K_4(t,x)$ by dividing two parts as follows:

$$\begin{aligned}K_4(t,x) &= \int_0^{t/2}\int_{\mathbb{R}^n}\big(\mathcal{G}(t,x-y)-\mathcal{G}(t,x)\big)\mathcal{N}(u(\tau,y))dyd\tau\\
&= \int_0^{t/2}\int_{|y|\leqslant t^{1/4}}\big(\mathcal{G}(t,x-y)-\mathcal{G}(t,x)\big)\mathcal{N}(u(\tau,y))dyd\tau\\
&\quad + \int_0^{t/2}\int_{|y|\geqslant t^{1/4}}\big(\mathcal{G}(t,x-y)-\mathcal{G}(t,x)\big)\mathcal{N}(u(\tau,y))dyd\tau\\
&=: K_{41}(t,x) + K_{42}(t,x).\end{aligned}$$

For the term $K_{41}(t,x)$, we can present the following formula:

$$\mathcal{G}(t,x-y)-\mathcal{G}(t,x) = \int_0^1 y\nabla\mathcal{G}(t,x-\eta y)d\eta.$$

From this, employing Lemma 3.1 to have

$$\begin{aligned}\||\nabla|^s K_{41}(t,\cdot)\|_{L^\theta} &\lesssim \int_0^1\int_0^{t/2}\int_{|y|\leqslant t^{1/4}}|y|\||\nabla|^{s+1}\mathcal{G}(t,x-\eta y)\|_{L^\theta}|\mathcal{N}(u(\tau,y))|dyd\eta\\
&\lesssim t^{-\frac{n}{2}(1-\frac{1}{\theta})-\frac{s+1}{2}}\int_0^{t/2}t^{\frac{1}{4}}\|\mathcal{N}(u(\tau,\cdot))\|_{L^1}d\tau\\
&\lesssim \varepsilon_0^{1+\frac{2}{n}} t^{-\frac{n}{2}(1-\frac{1}{\theta})-\frac{s}{2}-\frac{1}{4}}\int_0^{t/2}(1+\tau)^{-1}\mu\big(\varepsilon_0(1+\tau)^{-\frac{n}{2}}\big)d\tau\\
&= o(t^{-\frac{n}{2}(1-\frac{1}{\theta})-\frac{s}{2}}), \quad t \gg 1.\end{aligned}$$

Next, using a change of variable $\lambda := \varepsilon_0(1+\tau)^{-\frac{n}{2}}$, it is a fact that

$$\begin{aligned}\int_0^\infty\int_{\mathbb{R}^n}\mathcal{N}(u(\tau,y))dyd\tau &= \int_0^\infty \|\mathcal{N}(u(\tau,\cdot))\|_{L^1}d\tau\\
&\lesssim \int_0^\infty (1+\tau)^{-1}\mu\big(\varepsilon_0(1+\tau)^{-\frac{n}{2}}\big)d\tau = C\int_0^{\varepsilon_0}\frac{\mu(\lambda)}{\lambda}d\lambda < \infty.\end{aligned}$$

This immediately implies

$$\lim_{t\to\infty}\int_0^{t/2}\int_{|y|\geqslant t^{1/4}}\mathcal{N}(u(\tau,y))dyd\tau = 0.$$

Therefore, the term $K_{42}(t,x)$ will be estimated as follows:

$$\begin{aligned}\||\nabla|^s K_{42}(t,\cdot)\|_{L^\theta} &\lesssim \int_0^{t/2}\int_{|y|\geqslant t^{1/4}} \||\nabla|^s(\mathcal{G}(t,x-y)-\mathcal{G}(t,x))\|_{L^\theta}\mathcal{N}(u(\tau,y))dyd\tau\\ &\lesssim t^{-\frac{n}{2}(1-\frac{1}{\theta})-\frac{s}{2}}\int_0^{t/2}\int_{|y|\geqslant t^{1/4}}\mathcal{N}(u(\tau,y))dyd\tau\\ &= o(t^{-\frac{n}{2}(1-\frac{1}{\theta})-\frac{s}{2}}),\quad t\gg 1.\end{aligned}$$

In summary, we obtain

$$\||\nabla|^s K_4(t,\cdot)\|_{L^\theta} = o(t^{-\frac{n}{2}(1-\frac{1}{\theta})-\frac{s}{2}}),\quad t\gg 1. \tag{29}$$

Finally, we can control $K_5(t,x)$ as follows:

$$\begin{aligned}\||\nabla|^s K_5(t,\cdot)\|_{L^\theta} &\lesssim \||\nabla|^s\mathcal{G}(t,\cdot)\|_{L^\theta}\int_{t/2}^\infty \|\mathcal{N}(u(\tau,\cdot))\|_{L^1}d\tau\\ &\lesssim \varepsilon_0^{1+\frac{2}{n}}t^{-\frac{n}{2}(1-\frac{1}{\theta})-\frac{s}{2}}\int_{t/2}^\infty(1+\tau)^{-1}\mu\big(\varepsilon_0(1+\tau)^{-\frac{n}{2}}\big)d\tau\\ &= o(t^{-\frac{n}{2}(1-\frac{1}{\theta})-\frac{s}{2}}),\quad t\gg 1,\end{aligned} \tag{30}$$

due to

$$\lim_{t\to\infty}\int_{t/2}^\infty(1+\tau)^{-1}\mu\big(\varepsilon_0(1+\tau)^{-\frac{n}{2}}\big)d\tau = C\lim_{t\to\infty}\int_0^{\varepsilon_0(1+t/2)^{-n/2}}\frac{\mu(\lambda)}{\lambda}d\lambda = 0.$$

The relation (25)-(30) immediately imply (24). Therefore, we obtain the asymptotic behavior of the solution in the norms L^α and $\dot{H}^2$. As for the asymptotic behavior in the L^∞ norm, some minor adjustments are needed due to the estimate in Lemma 3.1. Nevertheless, the proof proceeds in exactly the same way as in the L^α case and Theorem 1.1. Hence, we conclude the proof of Theorem 1.2 here.

4. Sharp lifespan estimate for blow-up solutions

In this section, our main aim is to prove Theorem 1.3. Following the argument in the proof of Theorem 1.1, one obtains the existence of local (in time) solutions to the Cauchy problem (1). Moreover, these solutions blow-up in finite time when the non-Dini condition (6) occurs. The remaining open problem is to determine a sharp lifespan estimate for such local solutions. Specifically, we divide the proof of Theorem 1.3 into two parts as follows.

4.1. **Upper bound estimate of lifespan.** In this subsection, we establish the upper bound estimate of the lifespan

$$T_\varepsilon \lesssim \Psi^{-1}\left(C\varepsilon^{-\frac{2}{n}}\right),$$

under the conditions stated in Theorem 1.3. Our proof is based on the estimates established in the proof of [8, Theorem 5]. Let us recall the test functions constructed as in [8, Theorem 5]:

$$\eta(t) = \begin{cases}1 & \text{if } 0\leqslant t\leqslant 1/2,\\ \text{decreasing} & \text{if } 1/2\leqslant t\leqslant 1,\\ 0 & \text{if } t\geqslant 1\end{cases}\quad\text{and}\quad \eta^*(t) = \begin{cases}0 & \text{if } 0\leqslant t<1/2,\\ \eta(t) & \text{if } t\geqslant 1/2.\end{cases}$$

For all R satisfies $1\leqslant R_0\leqslant R\leqslant T_\varepsilon$, we define the following functions:

$$\psi_R = \psi_R(t,x) = \eta\left(\frac{|x|^2+t}{R}\right)^{n+2}\ \text{and}\ \psi_R^* = \psi_R^*(t,x) = \eta^*\left(\frac{|x|^2+t}{R}\right)^{n+2}.$$

We can easily see that the support of ψ_R is contained in

$$Q_R := [0,R] \times B_{\sqrt{R}} \quad \text{with} \quad B_{\sqrt{R}} := \left\{ x \in \mathbb{R}^n,\ |x| \leqslant \sqrt{R} \right\}.$$

Next, we define the quantities

$$I_R := \int_{Q_R} \mathcal{N}(u(t,x))\psi_R(t,x)d(t,x),$$

$$y = y(r) := \int_{Q_r} \mathcal{N}(u(t,x))\psi_r^*(t,x)d(t,x)$$

and

$$Y(R) := \int_0^R y(r)r^{-1}dr.$$

Multiplying both sides of the main equation in (1) by the test function $\psi_R(t,x)$ and performing integration by parts over $[0,\infty) \times \mathbb{R}^n$, we get

$$I_R + \varepsilon \int_{\mathbb{R}^n} (u_0(x) + u_1(x))\psi_R(0,x)dx = \int_{Q_R} u(t,x) \left(\partial_t^2 - \Delta - \partial_t \right) \psi_R(t,x)d(t,x).$$

Using the dominated convergence theorem combined with the condition (11), there exists a constant $R_0 \geqslant 1$ satisfies

$$\int_{\mathbb{R}^n} (u_0(x) + u_1(x))\psi_R(0,x)dx > \frac{1}{2} \int_{\mathbb{R}^n} (u_0(x) + u_1(x))dx > 0, \tag{31}$$

for all $R \geqslant R_0$. Without loss of generality, we may assume that $T_\varepsilon \geqslant R_0$. In [8, Theorem 5], the authors proved that

$$\begin{aligned} \frac{Y(R)}{\log(2)} + \varepsilon \int_{\mathbb{R}^n} (u_0(x) + u_1(x))\psi_R(0,x)dx &\leqslant I_R + \varepsilon \int_{\mathbb{R}^n} (u_0(x) + u_1(x))\psi_R(0,x)dx \\ &\leqslant C_1 R^{\frac{n}{2}} \mathcal{N}^{-1} \left(\frac{Y'(R)}{C_0 R^{\frac{n}{2}}} \right), \end{aligned} \tag{32}$$

for all $R \geqslant R_0$. As a direct consequence, the relations (31) and (32) yield

$$\mathcal{N} \left(C_2(\varepsilon + Y(R))R^{-\frac{n}{2}} \right) \leqslant C_0 R^{-\frac{n}{2}} Y'(R) \text{ for any } R_0 \leqslant R \leqslant T_\varepsilon$$

and

$$\mathcal{N} \left(C_3 R^{-\frac{n}{2}} Y(R) \right) \leqslant C_0 R^{-\frac{n}{2}} Y'(R) \text{ for any } R_0 \leqslant R \leqslant T_\varepsilon.$$

The first estimate deduces

$$Y'(R) \geqslant C_4(\varepsilon + Y(R_0))^{1+\frac{2}{n}} R^{-1} \mu \left(C_2(\varepsilon + Y(R_0))R^{-\frac{n}{2}} \right) \gtrsim \varepsilon^{1+\frac{2}{n}} R^{-1} \mu \left(C_5 R^{-\frac{n}{2}} \right) \tag{33}$$

and the second deduces

$$R^{-1} \mu \left(C_5 R^{-\frac{n}{2}} \right) \lesssim \frac{Y'(R)}{Y(R)^{1+\frac{2}{n}}}. \tag{34}$$

We introduce the following functions

$$\Psi_{R_0}(s) = \int_{R_0}^s \frac{\mu \left(C r^{-\frac{n}{2}} \right)}{r} dr \text{ and } \phi(s) := \Psi_{R_0}^{-1} \left(\frac{\Psi_{R_0}(s)}{2} \right) \text{ for } s \geqslant R_0.$$

The second definition implies

$$\Psi_{R_0}(R) = 2\Psi_{R_0}(\phi(R)), \quad \text{that is,} \quad \frac{\Psi_{R_0}(R)}{2} = \int_{R_0}^{\phi(R)} \frac{\mu \left(C r^{-\frac{n}{2}} \right)}{r} dr = \int_{\phi(R)}^{R} \frac{\mu \left(C r^{-\frac{n}{2}} \right)}{r} dr.$$

After denoting $r := R$ and integrating two sides of (33) over $[R_0, \phi(R)]$, one finds

$$\varepsilon^{1+\frac{2}{n}} \int\limits_{R_0}^{\phi(R)} \frac{\mu\left(Cr^{-\frac{n}{2}}\right)}{r} dr \lesssim Y(\phi(R)) - Y\left(R_0\right) \lesssim Y(\phi(R)). \tag{35}$$

In the same way, denoting $r := R$ and integrating two sides of (34) over $[\phi(R), R]$, one has

$$\int\limits_{\phi(R)}^{R} \frac{\mu\left(Cr^{-\frac{n}{2}}\right)}{r} dr \lesssim \frac{n}{2} Y(\phi(R))^{-\frac{2}{n}}. \tag{36}$$

Combining (35) and (36), we obtain

$$\Psi_{R_0}(R) \lesssim \varepsilon^{-\frac{2}{n}\left(1+\frac{2}{n}\right)} (\Psi_{R_0}(R))^{-\frac{2}{n}}, \text{ that is, } \quad \Psi_{R_0}(R) \lesssim \varepsilon^{-\frac{2}{n}}.$$

Therefore, we can easily see that

$$\Psi(R) = \Psi_{R_0}(R) + \int_1^{R_0} \frac{\mu\left(Cr^{-\frac{n}{2}}\right)}{r} \, dr \lesssim \varepsilon^{-\frac{2}{n}}.$$

Finally, letting $R \to T_\varepsilon$ in the last inequality one arrives at

$$T_\varepsilon \lesssim \Psi^{-1}\left(C\varepsilon^{-\frac{2}{n}}\right),$$

which we wanted to prove.

4.2. **Lower bound estimate of lifespan.** In this subsection, we establish the lower bound estimate of the lifespan

$$T_\varepsilon \gtrsim \Psi\left(c\varepsilon^{-\frac{2}{n}}\right),$$

under the conditions stated in Theorem 1.3. The proof relies on the argument introduced in [13], together with the solution spaces and the quantities established in the proof of Theorem 1.1. Specifically, we recall some estimates related to the global existence of small data solutions.

$$\|u^{\text{lin}}\|_{X(T)} \leqslant c_0^* \varepsilon \|(u_0, u_1)\|_{\mathcal{D}}.$$

On the other hand, we see that

$$\begin{aligned}
\|u^{\text{non}}(t,\cdot)\|_{\dot{H}^2} &\lesssim (1+t)^{-\frac{n}{4}-1} \|u\|_{X(T)}^{1+\frac{2}{n}} \int_0^{t/2} (1+\tau)^{-1} \mu\left(C(1+\tau)^{-\frac{n}{2}}\right) d\tau \\
&\quad + (1+t)^{-\frac{n}{4}-1} \|u\|_{X(T)}^{1+\frac{2}{n}} \int_{t/2}^{t} (1+t-\tau)^{-1} \mu\left(C(1+t-\tau)^{-\frac{n}{2}}\right) d\tau \\
&\lesssim (1+t)^{-\frac{n}{4}-1} \Psi(1+T) \|u\|_{X(T)}^{1+\frac{2}{n}}.
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
\|u^{\text{non}}(t,\cdot)\|_{L^\infty} &\lesssim (1+t)^{-\frac{n}{2}} \Psi(1+T) \|u\|_{X(T)}^{1+\frac{2}{n}}, \\
\|u^{\text{non}}(t,\cdot)\|_{L^\alpha} &\lesssim (1+t)^{-\frac{n}{2}\left(1-\frac{1}{\alpha}\right)} \Psi(1+T) \|u\|_{X(T)}^{1+\frac{2}{n}}.
\end{aligned}$$

As a result, we gain

$$\|\Phi[u]\|_{X(T)} \leqslant \frac{c_0 \varepsilon}{2} + c_1 \Psi(1+T) \|u\|_{X(T)}^{1+\frac{2}{n}},$$

where $c_0 = 2c_0^*\|(u_0,u_1)\|_{\mathcal{D}}$. Similarly, we can immediately conclude that

$$\|\Phi[u]-\Phi[v]\|_{Y(T)} \leqslant c_2\Psi(1+T)\|u-v\|_{Y(T)}\left(\|u\|_{X(T)}^{\frac{2}{n}}+\|v\|_{X(T)}^{\frac{2}{n}}\right).$$

Therefore, for all $u,v \in X(T,c_0\varepsilon)$, the last two estimates imply that

$$\|\Phi[u]\|_{X(T)} \leqslant \frac{c_0\varepsilon}{2} + c_1\Psi(1+T)c_0^{1+\frac{2}{n}}\varepsilon^{1+\frac{2}{n}},$$
$$\|\Phi[u]-\Phi[v]\|_{Y(T)} \leqslant 2c_2c_0^{\frac{2}{n}}\varepsilon^{\frac{2}{n}}\Psi(1+T)\|u-v\|_{Y(T)},$$

where constants c_1 and c_2 independent of M,ε and T. Therefore, if we assume that

$$\max\{c_1,2c_1\}c_0^{\frac{2}{n}}\varepsilon^{\frac{2}{n}}\Psi(1+T) < \frac{1}{4},$$

then we may construct a unique local solution $u \in X(T,c_0\varepsilon)$ by an argument entirely analogous to the proof of Theorem 1.1. Moreover, the following estimate holds:

$$\|u\|_{X(T)} = \|\Phi[u]\|_{X(T)} < \frac{3c_0\varepsilon}{4}. \tag{37}$$

Afterwards, motivated by the approach in [13], we determine

$$T^* := \sup\left\{T\in[0,T_\varepsilon) \text{ such that } F(T):=\|u\|_{X(T)} \leqslant c_0\varepsilon\right\}.$$

If T^* satisfies the inequality (37), then from the previous estimate it follows that $F(T^*) < 3c_0\varepsilon/4$. Since $F(T)$ is a continuous function, there exists $\overline{T} \in (T^*,T_\varepsilon)$ satisfies $F(\overline{T}) \leqslant c_0\varepsilon$, which contradicts the definition of T^*. Hence, we deduce that

$$\max\{c_1,2c_2\}\Psi\left(T^*+1\right)c_0^{\frac{2}{n}}\varepsilon^{\frac{2}{n}} \geqslant \frac{1}{4},$$

which leads to

$$T_\varepsilon \geqslant T^* \gtrsim \Psi^{-1}\left(c\varepsilon^{-\frac{2}{n}}\right).$$

This completes the proof of the lower bound for the lifespan T_ε.

Proof of Theorem 1.3. Combining the results from subsections 4.1 and 4.2, we conclude the proof of Theorem 1.3.

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