A NOTE ON A DIFFEOMORPHISM CRITERION VIA LONG-TIME RICCI FLOW

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ABSTRACT. In this note, we give a diffeomorphism (to $\mathbb{R}^n$) criterion via long-time Ricci flow and show some applications. In particular, we provide an affirmative answer that the conclusion in [3] and [17] about manifolds with small curvature concentration can be improved to diffeomorphism in dimension 4.

1. INTRODUCTION

Ricci flow, which was introduced by R. Hamilton in 1982[12], is a powerful tool to study the topology and geometry of the underlying manifold, see [20],[1],[2] for example. In this note, we will focus on studying the topology of complete non-compact manifolds via Ricci flow. For the compact case, the model is the standard sphere $\mathbb{S}^n$ or the complex projective space $\mathbb{CP}^n$; while for the non-compact case, the model is the standard Euclidean space $\mathbb{R}^n$ or $\mathbb{C}^n$. In [21], W.-X. Shi initiates a programme to study uniformization conjecture on Kähler geometry via Ricci flow under the supervision by S.-T. Yau. The uniformization conjecture states that any n-dimensional complete non-compact Kähler manifold with positive holomorphic bisectional curvature must be bi-holomorphic to the standard complex Euclidean space $\mathbb{C}^n$. Shi aims to construct a complete long-time solution to the Ricci flow starting from such Kähler manifold in order to study the topology of this underlying manifold. This programme is now achieved by many works under the maximal volume growth assumption, see [8],[5],[14],[15] for example.

On the other hand, it is also natural to study the topology of a complete non-compact Riemannian manifold via Ricci flow. In fact, Chen, Tang and Zhu first showed that the topology of certain Kähler manifolds with positive holomorphic bisectional curvature must be $\mathbb{R}^{2n}$ via Ricci flow in [10],[8], see also [19]. In [24], B. Wang investigates the local entropy and pseudo-locality theorems for the Ricci flow. Then he applies these to provide a Ricci flow proof for an important result, which was first proved by J. Cheeger and T. Colding [6] via their theory on studying Gromov-Hausdorff convergence for Riemannian manifolds with Ricci curvature bounded from below, stating that an

n-dimensional complete non-compact Riemannian manifold with non-negative Ricci curvature and almost Euclidean volume ratio must be diffeomorphic to $\mathbb{R}^n$. In [13], F. He and M.-C. Lee show that an n-dimensional complete non-compact Riemannian manifold with weakly PIC_1 and maximal volume growth must be diffeomorphic to $\mathbb{R}^n$ via Ricci flow. In [3], Chan, Lee and the first named author show that an n-dimensional complete non-compact Riemannian manifold with small curvature concentration under certain assumptions must be homeomorphic to $\mathbb{R}^n$ if $n \geq 4$ and diffeomorphic to $\mathbb{R}^n$ if $n \geq 5$. Later, A. Martens [17] removes the scalar curvature assumption in [3].

The initial motivation of this note is to investigate whether in dimension 4, the conclusion in [3] and [17] can be improved to diffeomorphism. We will provide an affirmative answer by showing the following diffeomorphism (to $\mathbb{R}^n$) criterion via long-time Ricci flow.

Theorem 1.1. *Suppose $(M^n, g(t))$ is a smooth complete long-time solution to the Ricci flow on $M^n \times [0, +\infty)$ satisfying*

(i) $\operatorname{Ric}(g(t)) \geq -\dfrac{\psi}{t}$ *for some constant* $0 < \psi < \dfrac{1}{2}$*;*
(ii) $\operatorname{inj}(g(t)) \geq \beta\sqrt{t}$ *for some constant* $\beta > 0$*.*

Then M^n is diffeomorphic to $\mathbb{R}^n$.

Remark 1.1. We can see from the proof of Method 1 of Theorem 1.1 that if the injectivity radius condition (ii) is changed to $\operatorname{inj}(g(t)) \geq \beta t^\kappa$ for some $\kappa > 0$, then we only need to require $0 < \psi < \kappa$ in the curvature condition (i).

As applications, we first obtain the following, which confirms that the conclusion in [3] and [17] can be improved to diffeomorphism in dimension 4.

Corollary 1.1. *For $n \geq 4$ and $A, v_0 > 0$, there is $\sigma > 0$ depending only on n, A, v_0 such that if (M^n, g_0) is an n-dimensional complete non-compact manifold satisfying*

(i) $\operatorname{Vol}_{g_0}(B_{g_0}(x, r)) \leq v_0 r^n$ *for all* $r > 0$*;*
(ii) $\bar{\nu}(M, g_0) \geq -A$*;*
(iii) $\left(\displaystyle\int_M |\operatorname{Rm}(g_0)|^{n/2} d\operatorname{vol}_{g_0} \right)^{2/n} \leq \sigma$*;*
(iv) $\operatorname{Ric}(g_0) \geq -k$ *for some* $k \in \mathbb{R}$*,*

for all $x \in M$. Then M^n is diffeomorphic to $\mathbb{R}^n$.

Another interesting application is the following, which removes the almost Euclidean assumption in the main result of [6] and [24] mentioned above in dimension 3.

Corollary 1.2. *Suppose (M^3, g_0) is a 3-dimensional complete non-compact manifold satisfying*

(i) $\operatorname{Ric}(g_0) \geq 0$*;*
(ii) g_0 *has maximal volume growth.*

Then M^3 is diffeomorphic to $\mathbb{R}^3$.

Remark 1.2. This result should be known by experts. Indeed, G.Liu has classified the topology of all 3-dimensional complete non-compact manifold with non-negative Ricci curvature in [16] via minimal surface theory; if one further assume the manifold has maximal volume growth, then a simple argument can rule out other cases and the topology of the manifold is the trivial one. We provide a Ricci flow proof here by combining Theorem 1.1 and a short-time existence result of Ricci flow for this setting in [22]. Note that this result is not true in higher dimension, since Eguchi-Hanson metric is a counter-example, see [23] for some related discussion. In higher dimension, one should assume stronger curvature condition or almost Euclidean condition.

The next application studies the topology of complete non-compact manifold with small curvature concentration related to $W^{1,2}$-Sobolev inequality, which improves the main result in [4] where some L^p-bound for Riemann curvature tensor is needed.

Corollary 1.3. *For $n \geq 3$, there is a dimensional constants $\sigma_n > 0$ such that if (M^n, g_0) is an n-dimensional complete non-compact manifold with **bounded curvature** satisfying*

(i) $\left(\int_M u^{\frac{2n}{n-2}} d\operatorname{vol}_{g_0} \right)^{\frac{n-2}{n}} \leq C_S \int_M |\nabla u|^2 d\operatorname{vol}_{g_0}$ *for all $u \in W^{1,2}(M, g_0)$;*

(ii) $\left(\int_M |\operatorname{Rm}(g_0)|^{n/2} d\operatorname{vol}_{g_0} \right)^{2/n} \leq \sigma_n \dfrac{1}{C_S}$,

where C_S is a positive constant. Then M^n is diffeomorphic to $\mathbb{R}^n$.

Note that the condition (ii) in Corollary 1.3 is explicit about the dependence of the Sobolev constant C_S, while Corollary 1.1 merely implies the case of an implicit dependence of the Sobolev constant C_S in condition (ii). We wonder that whether one can remove the bounded curvature assumption in Corollary 1.3, see [11] for the compact case.

The paper is organized as follows. In Section 2, we will prove the diffeomorphism (to $\mathbb{R}^n$) criterion via long-time Ricci flow, namely Theorem 1.1. In Section 3, we will show the applications, namely Corollary 1.1, 1.2, 1.3.

Acknowledgement: The authors would like to thank Liang Cheng and Xian-Tao Huang for some useful discussion.

2. Diffeomorphism criterion via long-time Ricci flow

In this section, we will provide two methods to prove the diffeomorphism (to $\mathbb{R}^n$) criterion via long-time Ricci flow, namely Theorem 1.1.

Proof of Theorem 1.1 (Method 1). This method adapts the idea in the proof of [13, Theorem 1.1], which is more or less standard in differential topology.

Fix $x_0 \in M$. Let $exp_t : \mathbb{R}^n \to M$ be the exponential map w.r.t $g(t)$ at x_0 and denote $\mathcal{E}_t = exp_t^{-1}$. For simplicity, we denote

$$\tilde{B}(r) := \{x \in \mathbb{R}^n \,\big|\, |x| < r\},$$
$$B_t(r) := B_{g(t)}(x_0, r),$$

then $exp_t(\tilde{B}(r)) = B_t(r)$ as long as $r < inj(g(t))$.

Let $t_i := i$ and $R_i := \left(\dfrac{1}{2} - \dfrac{1}{2^{i+1}}\right)\beta\sqrt{t_i}$ for $i = 1, 2, 3, \cdots$. Denote $\varepsilon_i := \dfrac{1}{2^{i+2}}\beta\sqrt{t_{i+1}}$. We claim that for each $t \in [t_i, t_{i+1}]$,

$$(2.1) \qquad B_{t_1}\left(\frac{\beta}{4}t_i^{\frac{1}{2}-\psi}\right) \subset B_{t_i}(R_i) \subset B_t\left(R_i\left(\frac{t}{t_i}\right)^{\frac{1}{2}}\right) \subset B_{t_{i+1}}(R_{i+1} - \varepsilon_i).$$

To see (2.1), we first note that by the geodesic length evolution equation along the Ricci flow and (i), we have

$$\frac{d}{dt}\log d_{g(t)}(x, x_0) = -\frac{1}{|\gamma|}\int_\gamma \mathrm{Rc}(\dot{\gamma}, \dot{\gamma})d\theta \leq \frac{\psi}{t}.$$

It follows that

$$(2.2) \qquad \frac{d_{g(t')}(x, x_0)}{d_{g(t)}(x, x_0)} \leq \left(\frac{t'}{t}\right)^\psi$$

for any $t' \geq t$.

For all $x \in B_{t_1}\left(\frac{\beta}{4}t_i^{\frac{1}{2}-\psi}\right)$, by (2.2),

$$d_{t_i}(x, x_0) \leq d_{t_1}(x, x_0)t_i^\psi < \frac{\beta}{4}\sqrt{t_i} \leq R_i.$$

For all $x \in B_{t_i}(R_i)$, by (2.2),

$$d_t(x, x_0) \leq d_{t_i}(x, x_0)\left(\frac{t}{t_i}\right)^\psi < R_i\left(\frac{t}{t_i}\right)^{\frac{1}{2}}.$$

For all $x \in B_t\left(R_i\left(\frac{t}{t_i}\right)^{\frac{1}{2}}\right)$, by (2.2),

$$d_{t_{i+1}}(x, x_0) \leq d_t(x, x_0)\left(\frac{t_{i+1}}{t}\right)^\psi < R_i\left(\frac{t_{i+1}}{t_i}\right)^{\frac{1}{2}} = R_{i+1} - \varepsilon_i.$$

By the first inclusion in (2.1), $\{B_{t_i}(R_i)\}$ is an exhaustion of M, and obviously $\{\tilde{B}(R_i)\}$ is an exhaustion of $\mathbb{R}^n$.

Claim. For each i, there is a diffeomorphism $\Phi_i : \mathbb{R}^n \to \mathbb{R}^n$, such that $\Phi_i(\mathcal{E}_{t_{i+1}}(B_{t_i}(R_i))) = \tilde{B}(R_i) = \mathcal{E}_{t_i}(B_{t_i}(R_i))$ and Φ_i is identity on $\mathbb{R}^n \setminus \tilde{B}(R_{i+1} - \frac{\varepsilon_i}{2})$.

Proof of claim. For each $t \in [t_i, t_{i+1}]$, by (2.1), we know that $\mathcal{E}_t$ embeds $B_{t_i}(R_i)$ into $\mathbb{R}^n$, and $\mathcal{E}_t(B_{t_i}(R_i)) \subset \tilde{B}(R_{i+1} - \varepsilon_i)$.

For an interior point x in the domain of $\mathcal{E}_t$, $t \in [t_i, t_{i+1}]$, let $V(x, t)$ be the tangent vector of the t-parameterized curve $\mathcal{E}_t(x) \subset \mathbb{R}^n$,

$$\text{(2.3)} \qquad \frac{d}{dt}\mathcal{E}_t(x) = V(\mathcal{E}_t(x), t).$$

Now choose a smooth time-independent cutoff function φ such that $\varphi = 1$ on $\tilde{B}(R_{i+1} - \varepsilon_i)$ and $\varphi = 0$ on $\mathbb{R}^n \setminus \tilde{B}(R_{i+1} - \frac{\varepsilon_i}{2})$. Let Ψ_s, $s \in [0, 1]$ be the family of diffeomorphisms generated by $-(t_{i+1} - t_i)\varphi V(x, t_{i+1} - (t_{i+1} - t_i)s)$ with $\Psi_0 = id$, namely

$$\begin{cases} \dfrac{d}{ds}\Psi_s(y) = -(t_{i+1} - t_i)\varphi V(\Psi_s(y), t_{i+1} - (t_{i+1} - t_i)s), \\ \Psi_0(y) = y. \end{cases}$$

For all $x \in B_{t_i}(R_i)$, let $\gamma(s) := \Psi_s(\mathcal{E}_{t_{i+1}}(x))$. Then $\varphi(\gamma(s)) = 1$ if $s \geq 0$ sufficiently small. Denote $\eta(\tau) := \gamma\left(\dfrac{t_{i+1} - \tau}{t_{i+1} - t_i}\right)$ and $s = \frac{t_{i+1} - \tau}{t_{i+1} - t_i}$. We have

$$\begin{cases} \dfrac{d}{d\tau}\eta(\tau) = \dfrac{d}{ds}\gamma(s) \cdot (-\dfrac{1}{t_{i+1} - t_i}) = V(\eta(\tau), \tau), \\ \eta(t_{i+1}) = \gamma(0) = \mathcal{E}_{t_{i+1}}(x). \end{cases}$$

This is exactly (2.3). By the uniqueness of ODE, $\eta(\tau) = \mathcal{E}_\tau(x)$ for small $s \geq 0$. Since $\varphi(\mathcal{E}_t(x)) = 1$ for all $t \in [t_i, t_{i+1}]$, we have $\eta(\tau) = \mathcal{E}_\tau(x)$ for all $s \in [0, 1]$. Hence

$$\Psi_1(\mathcal{E}_{t_{i+1}}(x)) = \gamma(1) = \eta(t_i) = \mathcal{E}_{t_i}(x).$$

So we have $\Psi_1 \circ \mathcal{E}_{t_{i+1}} = \mathcal{E}_{t_i}$ on $B_{t_i}(R_i)$. By the construction of the cutoff function φ, Ψ_1 is obviously identity on $\mathbb{R}^n \setminus \tilde{B}(R_{i+1} - \frac{\varepsilon_i}{2})$. Then $\Phi_i = \Psi_1$ is the desired diffeomorphism. $\qquad\square$

Define $\Theta_i : B_{t_i}(R_i) \to \mathbb{R}^n$ by

$$\Theta_i := \Phi_1 \cdots \Phi_{i-1}\mathcal{E}_{t_i}.$$

By the definition of Φ_i, Θ_i agrees with Θ_j on $B_{t_j}(R_j)$ for each $j < i$. Then we can let $i \to +\infty$ to obtain a diffeomorphism from M to $\mathbb{R}^n$. $\qquad\square$

Proof of Theorem 1.1 (Method 2). The proof is just a slight modification from the construction in [24, Theorem 5.7]. We provide details here for completeness.

Fix $x_0 \in M$. Define

$$M' := \{(x, t) \big| d_{g(t)}(x, x_0) = \frac{\beta}{2}\sqrt{t}\}.$$

By the injectivity condition (ii), $M' = \bigcup_{t \geq 0} \partial B_{g(t)}(x_0, \frac{\beta}{2}\sqrt{t})$ is a smooth manifold. Define a space-time exponential map as follows

$$Exp: \quad \mathbb{R}^n \times [0, +\infty) \to M \times [0, +\infty),$$
$$(v, t) \mapsto (exp_{x_0, g(t)} v, t).$$

Let $\Omega := \{(x, t) \big| d_{g(t)}(x, x_0) \leq \frac{3\beta}{4}\sqrt{t}\}$. Then by the injectivity condition (ii) again, $Exp|_\Omega$ is a diffeomorphism. Since $M' \subset \Omega$, we know that M' is diffeomorphic to $Exp^{-1}(M') = \{(v, t) \big| |v| = \frac{\beta}{2}\sqrt{t}\}$ via the diffeomorphism map Exp^{-1}. Note that $\{(v, t) \big| |v| = \frac{\beta}{2}\sqrt{t}\}$ is diffeomorphic to $\mathbb{R}^n$ through the projection P' defined as

$$P'(v, t) := v \in \mathbb{R}^n.$$

So we obtained the diffeomorphism from M' to $\mathbb{R}^n$: $P' \circ Exp^{-1}$.

To show M is diffeomorphic to M', we define the projection map P from M' to M by

$$P(x, t) := x, \quad \forall (x, t) \in M'.$$

Since the projection map P is smooth and non-degenerate everywhere, it suffices to show that P is both injective and surjective.

We prove the injectivity first. Suppose $P(x, t_1) = P(y, t_2)$ for some $x, y \in M$ and $t_1 \leq t_2$. It is clear that $x = y$. If $t_1 = 0$, then $x = y = x_0$ and hence $t_2 = 0$. For $0 < t_1 \leq t_2$, we have

$$(2.4) \qquad \frac{d_{g(t_1)}(x, x_0)}{\sqrt{t_1}} = \frac{d_{g(t_2)}(x, x_0)}{\sqrt{t_2}} = \frac{\beta}{2}.$$

Combining with (2.2), we arrive at

$$\left(\frac{t_2}{t_1}\right)^{\frac{1}{2}} = \frac{d_{g(t_2)}(x, x_0)}{d_{g(t_1)}(x, x_0)} \leq \left(\frac{t_2}{t_1}\right)^{\psi},$$

which implies $t_2 \leq t_1$ since $\psi < \frac{1}{2}$. Then we have $t_1 = t_2$ and the injectivity is obtained.

We now show that P is surjective. It is trivial that $P(x_0, 0) = x_0$. We need to prove that for each $x \in M \setminus \{x_0\}$, there exists $t > 0$ such that

$$d_{g(t)}(x, x_0) = \frac{\beta}{2}\sqrt{t}.$$

This is true by noting that the function $\frac{d_{g(t)}(x, x_0)}{\sqrt{t}}$ is continuous in t and the fact that

$$\lim_{t \to 0} \frac{d_{g(t)}(x, x_0)}{\sqrt{t}} = +\infty$$

and

$$\lim_{t \to +\infty} \frac{d_{g(t)}(x, x_0)}{\sqrt{t}} \leq \lim_{t \to +\infty} \frac{d_{g(1)}(x, x_0) \cdot t^{\psi}}{\sqrt{t}} = 0,$$

where we used (2.2).

Therefore, we prove that $P : M' \to M$ is a diffeomorphism. Combining with the one we have already obtained from M' to $\mathbb{R}^n$, we obtain the desired diffeomorphism map from M to $\mathbb{R}^n$: $P' \circ Exp^{-1} \circ P^{-1}$. This completes the proof. $\qquad\square$

3. Applications

Proof of Corollary 1.1. By the proof of [3, Theorem 1.3] and [17, Theorem 1.4], there exists a long-time solution $g(t)$ to the Ricci flow with $g(0) = g_0$ and

$$\begin{cases} |\operatorname{Rm}(g(t))| \leq C_0 \sigma t^{-1}; \\ \operatorname{inj}(g(t)) \geq C_0^{-1}\sqrt{t}, \end{cases}$$

where $C_0 = C_0(n, A, v_0) > 0$. Since σ can be small, the result follows from Theorem 1.1. $\qquad\square$

Proof of Corollary 1.2. By assumptions, there exists $v_0 > 0$ such that for all $x \in M$ and $r > 0$, we have

$$(3.1) \qquad \operatorname{Vol}_{g_0}(B_{g_0}(x,r)) \geq v_0 r^n.$$

By [22, Theorem 1.7], there exists $T = T(v_0) > 0$ and a smooth complete Ricci flow $\tilde{g}(t)$ defined on $M \times [0, T]$ with $\tilde{g}(0) = g_0$. It follows from [9] that non-negativity of Ricci curvature is preserved. By [22, Lemma 4.1], there exist $\beta = \beta(v_0) > 0$, $C_0 = C_0(v_0) > 0$ and $\hat{T} = \hat{T}(v_0) \in (0, T]$ such that

$$(3.2) \qquad \begin{cases} |\operatorname{Rm}(\tilde{g}(t))| \leq \dfrac{C_0}{t}; \\ \operatorname{inj}(\tilde{g}(t)) \geq \beta\sqrt{t} \end{cases}$$

for all $t \in [0, \hat{T}]$.

Since the assumptions and (3.2) are scaling invariant, by applying the argument above to $R^{-2}g_0$ for $R \to +\infty$ and re-scaling it back, we obtain a sequence of Ricci flow $g_R(t)$ on $[0, \hat{T}R^2]$ with $g_R(0) = g_0$ for any $R > 0$. Using the argument in the proof of [3, Theorem 1.1], we can extract convergent sub-sequence in locally smooth sense to obtain a smooth complete long-time solution $g(t)$ to the Ricci flow with $g(0) = g_0$ and

$$\begin{cases} \operatorname{Ric}(g(t)) \geq 0; \\ \operatorname{inj}(g(t)) \geq \beta\sqrt{t}. \end{cases}$$

Then the result follows from Theorem 1.1 or the proof of [13, Theorem 1.1]. $\quad\square$

Proof of Corollary 1.3. By [4, Theorem 1.2], there exists a smooth complete long-time solution $g(t)$ to the Ricci flow with $g(0) = g_0$ and

$$\begin{cases} |\operatorname{Rm}(g(t))| \leq (3e)^{\frac{n}{2}} t^{-1}; \\ \displaystyle\lim_{t \to +\infty} t \sup_{M} |\operatorname{Rm}(g(t))| = 0. \end{cases}$$

By [4, Corollary 4.3] and [18, Lemma 3.1], there exists $v_0 = v_0(n, C_{g_0}) > 0$ such that
$$\mathrm{Vol}_{g(t)} B_{g(t)}(x, \sqrt{t}) \geq v_0 t^{\frac{n}{2}} \text{ for all } t > 0.$$
Let $\tilde{g} := \frac{1}{t} g(t)$. Then we have
$$\begin{cases} |\mathrm{Rm}(\tilde{g})| \leq (3e)^{\frac{n}{2}}; \\ \mathrm{Vol}_{\tilde{g}} B_{\tilde{g}}(x, 1) \geq v_0. \end{cases}$$

By [7, Theorem 4.7], there exists $\beta = \beta(n, C_{g_0}) > 0$ such that $\mathrm{inj}(\tilde{g}) \geq \beta$, which implies $\mathrm{inj}(g(t)) \geq \beta\sqrt{t}$.

Since $\lim_{t \to +\infty} t \sup_M |\mathrm{Rm}(g(t))| = 0$, then the curvature condition in Theorem 1.1 is satisfied for any $\psi > 0$. Indeed, since there exists $T_0 > 0$ such that $|\mathrm{Rm}(g(t)| \leq \frac{\psi}{(n-1)t}$ for all $t \geq T_0$, it suffices to consider $\tilde{g}(t) = g(T_0 + t)$. By Theorem 1.1, the result follows. $\square$

References

1. Christoph Böhm and Burkhard Wilking, *Manifolds with positive curvature operators are space forms*, Ann. of Math. (2) **167** (2008), no. 3, 1079–1097. MR 2415394
2. Simon Brendle and Richard Schoen, *Manifolds with 1/4-pinched curvature are space forms*, J. Amer. Math. Soc. **22** (2009), no. 1, 287–307. MR 2449060
3. Pak-Yeung Chan, Shaochuang Huang, and Man-Chun Lee, *Manifolds with small curvature concentration*, Ann. PDE **10** (2024), no. 2, Paper No. 23, 31. MR 4804227
4. Albert Chau and Adam Martens, *Long-time Ricci flow existence and topological rigidity from manifolds with pinched scale-invariant integral curvature*, 2024, arXiv:2403.02564.
5. Albert Chau and Luen-Fai Tam, *On the complex structure of Kähler manifolds with nonnegative curvature*, J. Differential Geom. **73** (2006), no. 3, 491–530. MR 2228320
6. Jeff Cheeger and Tobias H. Colding, *On the structure of spaces with Ricci curvature bounded below. I*, J. Differential Geom. **46** (1997), no. 3, 406–480. MR 1484888
7. Jeff Cheeger, Mikhail Gromov, and Michael Taylor, *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds*, J. Differential Geometry **17** (1982), no. 1, 15–53. MR 658471
8. Bing-Long Chen, Siu-Hung Tang, and Xi-Ping Zhu, *A uniformization theorem for complete non-compact Kähler surfaces with positive bisectional curvature*, J. Differential Geom. **67** (2004), no. 3, 519–570. MR 2153028
9. Bing-Long Chen, Guoyi Xu, and Zhuhong Zhang, *Local pinching estimates in 3-dim Ricci flow*, Math. Res. Lett. **20** (2013), no. 5, 845–855. MR 3207356
10. Bing-Long Chen and Xi-Ping Zhu, *On complete noncompact Kähler manifolds with positive bisectional curvature*, Math. Ann. **327** (2003), no. 1, 1–23. MR 2005119
11. Eric Chen, Guofang Wei, and Rugang Ye, *Ricci flow and Gromov almost flat manifolds*, 2022, arXiv:2203.05107.
12. Richard S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geometry **17** (1982), no. 2, 255–306. MR 664497
13. Fei He and Man-Chun Lee, *Weakly PIC1 manifolds with maximal volume growth*, J. Geom. Anal. **31** (2021), no. 11, 10868–10885. MR 4310158
14. Shaochuang Huang and Luen-Fai Tam, *Kähler-Ricci flow with unbounded curvature*, Amer. J. Math. **140** (2018), no. 1, 189–220. MR 3749193
15. Man-Chun Lee and Luen-Fai Tam, *Chern-Ricci flows on noncompact complex manifolds*, J. Differential Geom. **115** (2020), no. 3, 529–564. MR 4120818

16. Gang Liu, *3-manifolds with nonnegative Ricci curvature*, Invent. Math. **193** (2013), no. 2, 367–375. MR 3090181
17. Adam Martens, *Removing scalar curvature assumption for Ricci flow smoothing*, Bull. Lond. Math. Soc. **57** (2025), no. 7, 1968–1989. MR 4936721
18. _____, *Sharpening a gap theorem: nonnegative Ricci and small curvature concentration*, Calc. Var. Partial Differential Equations **64** (2025), no. 3, Paper No. 79, 30. MR 4861062
19. Lei Ni and Luen-Fai Tam, *Kähler-Ricci flow and the Poincaré-Lelong equation*, Commun. Anal. Geom. **12** (2004), no. 1-2, 111–141. MR 2074873
20. Grisha Perelman, *The entropy formula for the Ricci flow and its geometric applications*, 2002, arXiv:math/0211159.
21. Wan-Xiong Shi, *Ricci deformation of the metric on complete noncompact Kähler manifolds*, ProQuest LLC, Ann Arbor, MI, 1990, Thesis (Ph.D.)–Harvard University. MR 2638857
22. Miles Simon and Peter M. Topping, *Local mollification of Riemannian metrics using Ricci flow, and Ricci limit spaces*, Geom. Topol. **25** (2021), no. 2, 913–948. MR 4251438
23. _____, *Local control on the geometry in 3D Ricci flow*, J. Differential Geom. 122 (2022), no. 3, 467–518 **122** (2022), no. 3, 467–518. MR 4544560
24. Bing Wang, *The local entropy along Ricci flow—Part B: the pseudo-locality theorems*, 2020, arXiv:2010.09981.

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