

SCALAR AND MEAN CURVATURE COMPARISON ON COMPACT CYLINDER

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ABSTRACT. Let X be a closed, oriented Riemannian manifold. Denote by $(M = X \times I, \partial M = X \times \{0\} \cup X \times \{1\}, g)$ a compact cylinder with smooth boundary, $\dim M \geq 3$. In this article, we address the following question: If g is a Riemannian metric having (i) positive scalar curvature (PSC metric) on M and nonnegative mean curvature on ∂M ; and (ii) the g -angle between normal vector field ν_g along ∂M and $\partial_\xi \in \Gamma(TI)$ being less than $\frac{\pi}{4}$, then there exists a metric \tilde{g} on M such that $\tilde{g}|_{X \times \{0\}}$ is a PSC metric on $X \cong X \times \{0\}$. Equivalently, we show that if X admits no PSC metric, but M admits a PSC metric g satisfying the angle condition, then the mean curvature on ∂M must be negative somewhere. This generalizes a result of Gromov and Lawson [6] for $X = \mathbb{T}^n$.

1. INTRODUCTION

Let X be a closed, oriented manifold with $n - 1 := \dim X \geq 2$. Let $I = [0, 1]_\xi$ be the standard unit interval with ξ -variable. The relation between the geometry of X and the geometry of the compact cylinder $X \times I$ is an active research area.

In Gromov's seminal work [4], [5], the study of Riemannian bands on compact cylinders with lower scalar curvature bounds has been an active field for recent years. On compact cylinders $X \times I$, the Riemannian band is defined to be the distance between $X \times \{0\}$ and $X \times \{1\}$ with respect to g . Gromov conjectured an upper bound of Riemannian bands for a class of manifolds $(X \times I, g)$, where X admits no Riemannian metric with positive scalar curvature (PSC), but $X \times I$ admits a metric g whose associated scalar curvature is uniformly positive, i.e. there exists a positive constant $\kappa_0 > 0$ such that the scalar curvature is bounded below by κ_0 . The model case is $X = \mathbb{T}^n, n \geq 1$, see [5]. Recently, this conjecture has been proved for a wide class of compact cylinders with some extra hypotheses, especially when X is a spin manifold, see e.g. [2], [9], [12].

Gromov's conjecture on Riemannian bands gives a metric geometry information on compact cylinders. There is also a Riemannian geometry information in terms of mean curvature on compact cylinders. Gromov and Lawson [6] showed that if $X = \mathbb{T}^n, n \geq 1$, and $X \times I = \mathbb{T}^n \times I$ admits a PSC metric g , then the mean curvature h_g must be negative somewhere on $X \times \{0\} \cup X \times \{1\}$. Very recently, Råde [9] proved this for general compact cylinders up to $\dim X \leq 7$, except some cases when $\dim X = 4$. We point out that in this dimensions, a finite upper bound of Riemannian band implies that the mean curvature is negative somewhere, but the opposite direction is not true in general.

In this article, we generalize the result of Gromov and Lawson [6] on \mathbb{T}^n to all compact, oriented manifolds X with $\dim X \geq 2$ by a codimension two approach developed in [10], provided that the g -angle condition defined in (1) below is satisfied. The results in [2], [5], [6], [9], [12], use methods from spin geometry, Dirac operators, minimal surfaces, and μ -bubbles. Our method is different: we introduce an auxiliary one-dimensional space to construct a conformal factor for the original metric g on $X \times I$, which comes from the solution of a partial differential equation that is very closely related to the Gauss-Codazzi equation.

For notations throughout this article, let $(M = X \times I, \partial M = X \times \{0\} \cup X \times \{1\}, g)$ be a compact cylinder with $n := \dim M \geq 3$. Let R_g be the scalar curvature with respect to g on M , and let R_{ι^*g} be the scalar curvature for the induced metric R_{ι^*g} on the closed manifold $X \cong X \times \{0\}$ with respect to the natural inclusion $\iota : X \times \{0\} \rightarrow M$. Let ν_g be the positively oriented unit normal

vector field along ∂M , and inwarding along $X \times \{0\}$. Let h_g be the mean curvature with respect to g on ∂M . We denote the conformal class $[g]$ of g on M by

$$[g] = \{e^{2\phi}g : \phi \in \mathcal{C}^\infty(M)\}.$$

We sometimes will label I by I_ξ for clarity.

On $M = X \times I_\xi$, the g -angle between ν_g and the canonical vector field $\partial_\xi \in \Gamma(TI)$ on ∂M , which is defined to be

$$\angle_g(\nu_g, \partial_\xi) = \cos^{-1} \left(\frac{g(\nu_g, \partial_\xi)}{(g(\nu_g, \nu_g))^{\frac{1}{2}} (g(\partial_\xi, \partial_\xi))^{\frac{1}{2}}} \right) = \cos^{-1} \left(\frac{g(\nu_g, \partial_\xi)}{g(\partial_\xi, \partial_\xi)^{\frac{1}{2}}} \right) \in [0, \frac{\pi}{2}].$$

It is easy to see that the angle quantity is a conformal invariance. With $\tilde{g} = e^{2\phi}g \in [g]$, the unit normal vector field with \tilde{g} is $\nu_{\tilde{g}} = e^{-\phi}\nu_g$, hence

$$\cos(\angle_{\tilde{g}}(\nu_{\tilde{g}}, \partial_\xi)) = \frac{\tilde{g}(\nu_{\tilde{g}}, \partial_\xi)}{(\tilde{g}(\partial_\xi, \partial_\xi))^{\frac{1}{2}}} = \frac{e^{2\phi}g(e^{-\phi}\nu_g, \partial_\xi)}{(e^{2\phi}g(\partial_\xi, \partial_\xi))^{\frac{1}{2}}} = \cos(\angle_g(\nu_g, \partial_\xi)).$$

Therefore, such an angle condition is natural in conformal geometry setting. Our first main result gives the existence of PSC metrics on $X \cong X \times \{0\}_\xi$ along with the g -angle condition.

Theorem. *Assume that the compact cylinder $(M, \partial M)$, $\dim M \geq 3$ admits a metric g with positive scalar curvature $R_g > 0$ and nonnegative mean curvature $h_g \geq 0$. If g satisfies the g -angle condition*

$$(1) \quad \angle_g(\nu_g, \partial_\xi) < \frac{\pi}{4} \Leftrightarrow \cos(\angle_g(\nu_g, \partial_\xi)) > \frac{\sqrt{2}}{2} \Leftrightarrow \frac{g(\partial_\xi, \partial_\xi)}{g(\nu_g, \partial_\xi)^2} < 2$$

everywhere on $X \times \{0\}_\xi$, then there exists a metric \tilde{g} on $(M, \partial M)$ such that $\iota^\tilde{g}$ has positive scalar curvature on $X \times \{0\}_\xi \cong X$.*

The trivial case is $(M, \partial M, k)$ where $k = k_0 \oplus d\xi^2$ is a product metric for a metric k_0 on X . Clearly $h_k = 0$. If $R_k > 0$, it follows that $R_{\iota^*k} > 0$. In this situation, $\nu_k = \partial_\xi$. Therefore the k -angle $\angle_k(\nu_k, \partial_\xi) = 0 < \frac{\pi}{4}$. Denote by hypersurfaces $X_\xi := X \times \{\xi\}$, $\xi \in [0, 1]$. We can define the second fundamental form, mean curvature, and unit normal vector field $\nu_{g,\xi}$ along X_ξ . Analogously, we can define the g -angle on X_ξ by $\angle_g(\nu_{g,\xi}, \partial_\xi)$. If we choose the sign of $\nu_{g,\xi}$ such that $\angle_g(\nu_{g,\xi}, \partial_\xi) \in [0, \frac{\pi}{2}]$, the same argument in Theorem 3.1 implies that the same conclusion in the Theorem holds on $X_\xi \cong X$ provided that

$$(2) \quad \angle_g(\nu_{g,\xi}, \partial_\xi) < \frac{\pi}{4} \text{ on } X_\xi,$$

as we mentioned in Remark 3.1.

The contrapositive statement of the theorem generalizes the result of Gromov and Lawson [6] on compact cylinders $X \times I$ from the torus case $X = \mathbb{T}^n$ to all closed, oriented manifolds X with $\dim X \geq 2$ that having no PSC metrics, when the g -angle condition (1) is satisfied.

Corollary. *Let X be a closed, oriented manifold with $\dim X \geq 2$. If X admits no PSC metric, meanwhile $X \times I$ admits a PSC metric g , then either the g -angle condition fails on all hypersurfaces X_ξ , $\xi \in [0, 1]$, or the mean curvature h_g must be negative somewhere on $X \times \{0\} \cup X \times \{1\}$.*

The corollary also generalizes the result of Råde [9] by removing the dimension hypothesis.

We now outline why we need an auxiliary dimension in the context of conformal geometry and partial differential equations. The standard way to consider the sign of R_{ι^*g} on $X \times \{0\}$ is the Gauss-Codazzi equation:

$$(3) \quad R_g = R_{\iota^*g} + 2Ric_g(\nu_g, \nu_g) - h_g^2 + |A_g|^2.$$

Here A_g is the second fundamental form with respect to g . If we consider a conformal metric $\tilde{g} = e^{2\phi}g$ on $(M, \partial M)$, applying Gauss-Codazzi equation (3) and laws of conformal transformations, we have

$$(4) \quad \begin{aligned} R_{i^*\tilde{g}} &= R_{\tilde{g}} - 2Ric_{\tilde{g}}(\nu_{\tilde{g}}, \nu_{\tilde{g}}) + h_{\tilde{g}}^2 - |A_{\tilde{g}}|^2 \\ &= e^{-2\phi} (R_g - 2Ric_g(\nu_g, \nu_g) + 2(n-2)\nabla_{\nu_g}\nabla_{\nu_g}\phi - 2(n-2)\Delta_g\phi + h_g^2 - |A_g|^2) \\ &\quad - e^{-2\phi} (2(n-2)\nabla_{\nu_g}\phi\nabla_{\nu_g}\phi + (n-3)(n-2)|\nabla_g\phi|^2). \end{aligned}$$

In order to get $R_{i^*\tilde{g}} > 0$, we need to get a solution ϕ of some partial differential equation with second order differential operator $2(n-2)\nabla_{\nu_g}\nabla_{\nu_g} - 2(n-2)\Delta_g$, where the inhomogeneous term of the PDE is positive and large enough on $X \times \{0\}$; In addition, we need other ϕ -related terms in the conformal transformation of the Gauss-Codazzi equation (4) to be small, i.e. we require the solution—which will be used as the conformal factor—to have $\mathcal{C}^{1,\alpha}$ -smallness. The largeness of the inhomogeneous term on $X \times \{0\}$ thus dominates all other terms in (4) and therefore we obtain a PSC metric \tilde{g} , i.e. $R_{i^*\tilde{g}} > 0$. From this, a second conformal transformation will give a Riemannian metric g' with $R_{g'} > 0$ and $R_{i^*g'} > 0$ by an earlier work of the author [11].

Unfortunately, the operator $2(n-2)\nabla_{\nu_g}\nabla_{\nu_g} - 2(n-2)\Delta_g$ is never elliptic, which means that we may not be able to get a desired solution of such a PDE with desired inhomogeneous term. To resolve this issue, we introduce the auxiliary one-dimensional space, summarized by the following diagram. This codimension two approach is inspired by our recent work of \mathbb{S}^1 -stability conjecture [10], where the auxiliary space is $[-1, 1]$. The essential difference is that the auxiliary one-dimensional space we introduce here is \mathbb{S}^1 with t -variable and standard Riemannian metric dt^2 , since we require $M \times \mathbb{S}^1$ to be a manifold with boundary, but not a manifold with corner. Again, we will interchangeably use $\mathbb{S}^1 = \mathbb{S}_t^1$ for clarity.

$$(5) \quad \begin{array}{ccc} (M, g) & \xrightarrow{\tau_1} & (M \times \mathbb{S}_t^1, g \oplus dt^2) \\ \uparrow \iota & & \uparrow \sigma \\ (X \times \{0\}_\xi, \iota^*g) & \xrightarrow{\tau_2} & (W = X \times \mathbb{S}_t^1, \sigma^*(g \oplus dt^2)) \end{array}$$

We always identify $X \cong X \times \{0\}_\xi \subset M$ and $W \cong W \times \{0\}_\xi \subset M \times \mathbb{S}_t^1$. Here $\sigma : W \rightarrow M \times \mathbb{S}_t^1$ is given by $\sigma(w) = (w, 0)$ with fixed point $0 \in I_\xi$. The two inclusions τ_1, τ_2 are sending $M \ni x \mapsto (x, P) = \tau_1(x)$, $X \times \{0\}_\xi \ni y \mapsto (y, P) = \tau_2(y)$ for some fixed point $P \in \mathbb{S}^1$. With natural projections $\Pi_1 : M \times \mathbb{S}_t^1 \rightarrow M$ and $\Pi_2 : W \times \{0\}_\xi \rightarrow X \times \{0\}_\xi \times \{P\}$, we get $\Pi_i \circ \tau_i = Id, i = 1, 2$. Under local parametrization, we may identify P with point 0 in any local chart containing P . We use the labels $\{0\}_\xi \in I_\xi$ and $\{0\}_t \cong \{P\} \in \mathbb{S}_t^1$ to distinguish these two points.

With the help of the auxiliary space, we construct a partial differential equation with a modified, elliptic differential operator on the closed manifold W . The crucial point is that the PDE is elliptic when the g -angle condition holds. Such a PDE with desired inhomogeneous term admits a $\mathcal{C}^{1,\alpha}$ -small solution. We then lift a slight modification of this solution to the ambient space $M \times \mathbb{S}_t^1$, serving as a conformal factor of the metric $\bar{g} = g \oplus dt^2$ on $M \times \mathbb{S}_t^1$. When we restrict this conformal transformation to M , the conformal factor plays a role transfers some geometric data from W to M . With some work, the conformal factor and the original geometric information together yield the desired geometric information on $X \times \{0\}_\xi \cong X$.

As an outline of this article, we construct the conformal factor $u' : M \times \mathbb{S}_t^1 \rightarrow \mathbb{R}$ in §2 via a series of technical results. In §2, we also verify some properties the conformal factor u' must satisfy, especially its \mathcal{C}^0 -estimates and partial \mathcal{C}^2 -estimates. The g -angle condition is used to obtain the ellipticity of the differential operator, which comes from the conformal transformation of the Gauss-Codazzi equation.

In §3 we prove the main theorem Thm. 3.1, essentially by using Gauss-Codazzi on $\tilde{g} = (u')^{\frac{4}{n-2}}\bar{g}$ to first move from $M \times \mathbb{S}_t^1$ back to M and then back to $X \cong X \times \{0\}_\xi$. Corollary 3.1 generalizes

the results of Gromov and Lawson [6] and Råde [9]. With a direct application of a Yamabe-type problem with Dirichlet boundary condition [11], we show in Corollary 3.2 that there exists a metric g' in the conformal class $[g]$ such that $R_{g'} > 0$ and $R_{t^*g'} > 0$ under the same hypotheses of Theorem 3.1.

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2. TECHNICAL RESULTS

In this section, we give basic setup and all technical results that are essential for our main result. These technical results are analogous to the technical results of [10, Section 2]. Recall that X is a closed, oriented manifold, and $M = X \times I_\xi$ is a n -dimensional compact cylinder with Riemannian metric g such that $R_g > 0$ and $h_g \geq 0$. Without loss of generality, we assume, throughout this article, that $R_g > 0$ and $h_g \equiv 0$. This can be done by taking a conformal transformation of the original metric, see [3]. Throughout this article, we denote $\bar{g} = g \oplus dt^2$ be the Riemannian metric on $M \times \mathbb{S}_t^1$. It follows that $R_{\bar{g}} > 0$ on $M \times \mathbb{S}_t^1$ and $h_{\bar{g}} \equiv 0$ on $\partial(M \times \mathbb{S}_t^1) = X \times \{0\}_\xi \times \mathbb{S}_t^1 \cup X \times \{1\}_\xi \times \mathbb{S}_t^1$. We also assume the familiarity of the standard knowledge of Sobolev space $W^{k,p}$, and \mathcal{L}^p -elliptic regularity, see e.g. [1]. For any metric g_0 , we set $-\Delta_{g_0}$ to be the positive definite Laplace-Beltrami operator.

Choosing local coordinates $\{x^i, x^n = \xi\}$ of M and associated local frame $\{\partial_i, \partial_n = \partial_\xi\}$ such that $\{\partial_i\}_{i=1}^{n-1}$ are tangential to ∂M , we can write

$$\nu_g = a\partial_\xi + \sum_{i=1}^{n-1} b^i \partial_i.$$

This defines a g -related vector field $V = \nu_g - a\partial_\xi$, which is locally expressed by $V = \sum_{i=1}^{n-1} b^i \partial_i$. Note that

$$1 = g(\nu_g, \nu_g) = g(a\partial_\xi, \nu_g) + \sum_i g(b^i \partial_i, \nu_g) \Rightarrow a = g(\partial_\xi, \nu_g)^{-1}.$$

a is nowhere vanishing since both ν_g and ∂_ξ are nowhere tangent to ∂M .

Clearly $g(V, \nu_g) = g(\nu_g - a\partial_\xi, \nu_g) = 0$, thus the vector field V is tangential to ∂M , and can be extended to $(W \cong X \times \{0\}_\xi \times \mathbb{S}_t^1, \sigma^*g)$ via $\tau_{2,*}$. Still denoted by V as a global vector field on W , we apply the same argument in [10, Lemma 2.1] to show that:

Lemma 2.1. *If*

$$(6) \quad \frac{g(\partial_\xi, \partial_\xi)}{g(\nu_g, \partial_\xi)^2} < 2,$$

then the operator

$$L' := \nabla_V \nabla_V - \Delta_{\sigma^* \bar{g}}$$

is elliptic on W .

Proof. The same argument in [10, Lemma 2.1] follows. □

The next three lemmas and propositions are crucial to construct the conformal factor we will use in the main theorem. For the standard dt^2 -metric on \mathbb{S}_t^1 , we fix some chart $U \ni P$ with chart map $\Phi : \mathbb{S}^1 \supset U \rightarrow (-1, 1)$ such that $\Phi(P) = 0$ and still with locally t -variable. We start with the construction of the inhomogeneous term of our partial differential equation.

Lemma 2.2. *For any $p \in \mathbb{N}$, $C \gg 1$, and any positive $\delta \ll 1$. Then there exists a positive smooth function $F : W \rightarrow \mathbb{R}$ and small enough constant $\epsilon \ll 1$ such that $F|_{X \times (-\frac{\epsilon}{2}, \frac{\epsilon}{2})_t} = C + 1$, $F \equiv 0$ outside $X \times [-\epsilon, \epsilon]_t$ and $\|F\|_{\mathcal{L}^p(W, \sigma^* \bar{g})} < \delta$.*

Proof. Fix any p, C and δ . We take a constant function $f = C + 1 : W \rightarrow \mathbb{R}$. There exists a nonnegative smooth function $\phi : \mathbb{S}_t^1 \rightarrow \mathbb{R}$ such that

$$\phi(t) \equiv 1, t \in \left(-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right), \phi(t) = 0, t \notin [-\epsilon, \epsilon].$$

Set $F = f \cdot \phi : W \rightarrow \mathbb{R}$. It follows that

$$\|F\|_{\mathcal{L}^p(W, \sigma^* \bar{g})} < \delta$$

provided that $\epsilon \ll 1$ is small enough. \square

Remark 2.1. The function F in any local frame $\{\partial_1, \dots, \partial_{n-1}, \partial_{n+1} = \partial_t\}$ of W satisfies

$$\partial_i F \equiv 0, \forall i \in \{1, \dots, n-1\}$$

based on the construction of F in Lemma 2.2.

We now construct the first candidate of our conformal factor with small enough $\mathcal{C}^{1,\alpha}$ -norm. We define the $\mathcal{C}^{1,\alpha}$ -norm on W by fixing a chart cover $W = \bigcup_i (U_i, \varphi_i)$ such that

$$\|u\|_{\mathcal{C}^{1,\alpha}(W)} = \|u\|_{\mathcal{C}^0(W)} + \sup_{i,k} \|\partial_{x_i^k} u\|_{\mathcal{C}^0(U_i)} + \sup_{i,k} \sup_{x \neq x', x, x' \in U_i} \frac{|\partial_{x_i^k} u(x) - \partial_{x_i^k} u(x')|}{|x - x'|^\alpha}$$

where $\partial_{x_i^k} u$ are local representations in U_i with local coordinates $\{x_i^1, \dots, x_i^{n-1}, x_i^{n+1} = t\}$ and associated local frames $\{\partial_{x_i^k}\}$.

Proposition 2.1. *Let $(W, \sigma^* \bar{g})$ be as above. Assume that (6) holds. For any positive constant $\eta \ll 1$, any positive constant C , and any $p > n = \dim W$, there exists an associated F and δ in the sense of Lemma 2.2, such that the following partial differential equation*

$$(7) \quad Lu := 4\nabla_V \nabla_V u - 4\Delta_{\sigma^* \bar{g}} u + R_{\bar{g}}|_W u = F \text{ in } W$$

admits a unique smooth solution u with

$$(8) \quad \|u\|_{\mathcal{C}^{1,\alpha}(W)} < \eta$$

for some $\alpha \in (0, 1)$ such that $\alpha \geq 1 - \frac{n}{p}$.

Proof. Note that W is a closed, oriented Riemannian manifold. Fix $\eta \ll 1, C$, and $p > n$. We then fix some $\alpha > 0$ such that $1 + \alpha \geq 2 - \frac{n}{p}$. Denote $C = C(W, g, n, p, L)$ by the constant of \mathcal{L}^p elliptic regularity estimates with respect to L , and $C' = C'(W, g, n, p, \alpha)$ by the constant of the Sobolev embedding $W^{2,p} \hookrightarrow \mathcal{C}^{1,\alpha}$. Fix δ such that $\delta CC' < \eta$. Finally we choose an associated F in the sense of Lemma 2.2.

By assumption, $R_g > 0$ on M , therefore $R_{\bar{g}} > 0$ on $M \times \mathbb{S}_t^1$, hence $R_{\bar{g}}|_W > 0$ uniformly. By Lemma 2.1, the operator in (7) is elliptic. By maximum principle on closed manifolds, the elliptic operator is an injective operator. It follows that there exists a unique solution $u \in H^1(W, \sigma^* \bar{g})$ of (7) by Fredholm alternative. By standard H^s -type elliptic theory and the smoothness of F , $u \in \mathcal{C}^\infty(W)$.

By standard \mathcal{L}^p -regularity theory [1],

$$\|u\|_{W^{2,p}(W, \sigma^* \bar{g})} \leq C (\|F\|_{\mathcal{L}^p(W, \sigma^* \bar{g})} + \|u\|_{\mathcal{L}^p(W, \sigma^* \bar{g})}).$$

Due to the injectivity of the operator, a very similar argument of [10, Proposition 2.1] shows that the \mathcal{L}^p estimates can be improved by

$$\|u\|_{W^{2,p}(W, \sigma^* \bar{g})} \leq C \|F\|_{\mathcal{L}^p(W, \sigma^* \bar{g})}.$$

By Sobolev embedding, it follows that

$$\|u\|_{\mathcal{C}^{1,\alpha}(W)} \leq C' \|u\|_{W^{2,p}(W, \sigma^* \bar{g})} \leq CC' \|F\|_{\mathcal{L}^p(W, \sigma^* \bar{g})} < CC' \delta < \eta.$$

\square

Remark 2.2. By maximum principle and the nonnegativity of F , the solution of (7) satisfies $u \geq 0, u \not\equiv 0$ in W .

We now give our candidate of the conformal factor on $M \times \mathbb{S}_t^1$. To begin with, we define a new function

$$u_0 := u + 1, u_0 : W \rightarrow \mathbb{R}$$

where u is the solution of (7). With the natural projection $\pi : M \times \mathbb{S}_t^1 \rightarrow W$, we can pullback u_0 to $M \times \mathbb{S}_t^1$ by $u' := (\pi)^* u_0$. We also pullback u to $M \times \mathbb{S}_t^1$ by $\bar{u} := (\pi)^* u$ for later use. We have

$$(9) \quad u'|_W = u_0 = u + 1, \partial_\xi^l u' = 0 \text{ on } M \times \mathbb{S}_t^1, \forall l \in \mathbb{Z}_{>0}.$$

Denote $\tilde{u}' := \tau_1^* u' : M \rightarrow \mathbb{R}$. Following our diagram (3), we have

$$i^* \tilde{u}' = i^* \tau_1^* u' = \tau_2^* \sigma^* u' = \tau_2^* u_0 = u_0|_{X \times \{0\}_\xi \times \{P\}_t}.$$

Clearly all those functions defined above are positive functions on corresponding spaces. The function $(u')^{\frac{4}{n-2}}$ will be our choice of conformal factor on $M \times \mathbb{S}_t^1$. The relations above hold analogously for \bar{u} also.

Usually, the best estimate of the solution of (7) is given by (8). Due to the speciality of the product metric \bar{g} and the construction of F , we are able to give a partial \mathcal{C}^2 -estimate in terms of $\left\| \frac{\partial^2 u}{\partial t^2} \right\|_{\mathcal{C}^0(X \times \{0\}_\xi \times \{P\}_t)}$, as the next lemma shows.

Lemma 2.3. *Choosing ϵ defined in Lemma 2.2 to be small enough that will be determined below. Let $(W, \sigma^* \bar{g})$, p, C, δ and F be the same as in Proposition 2.1. Let u be the associated solution of (7). There exists a constant $\eta' \ll 1$ such that*

$$(10) \quad \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{\mathcal{C}^0(X \times \{0\}_\xi \times (-\frac{\epsilon}{4}, \frac{\epsilon}{4})_t)} < \eta'.$$

Proof. Set $U_1 = X \times \{0\}_\xi \times (-\frac{\epsilon}{4}, \frac{\epsilon}{4})$, $U_2 = X \times \{0\}_\xi \times (-\frac{\epsilon}{2}, \frac{\epsilon}{2})$, $O_1 = X \times \{0\}_\xi \times (-\frac{1}{4}, \frac{1}{4})$, $O_2 = X \times \{0\}_\xi \times (-\frac{1}{2}, \frac{1}{2})$. Since $\bar{g} = g \oplus dt^2$, the vector field V and $\Delta_{\sigma^* \bar{g}}$ are constant with respect to t ; furthermore the scalar curvature $R_{\bar{g}}$ is constant along t -direction. In addition, $F \equiv C + 1$ on U_2 by Lemma 2.2, we apply $\frac{\partial^2}{\partial t^2}$ on both sides of (7) in U , which follows that

$$L_{\sigma^* \bar{g}} \left(\frac{\partial^2 u}{\partial t^2} \right) := 4 \nabla_V \nabla_V \left(\frac{\partial^2 u}{\partial t^2} \right) - 4 \Delta_{\sigma^* \bar{g}} \left(\frac{\partial^2 u}{\partial t^2} \right) + R_{\bar{g}}|_W \left(\frac{\partial^2 u}{\partial t^2} \right) = 0 \text{ in } U_1.$$

Now consider $t = \epsilon t'$, the new t' -variable is associated with the metric $(dt')^2 = \epsilon^{-2} dt^2$ on \mathbb{S}^1 . We have

$$(\epsilon^{-1})^2 \left(4 \nabla_V \nabla_V \left(\frac{\partial^2 u}{\partial (t')^2} \right) - 4 \Delta_{\sigma^* \bar{g}} \left(\frac{\partial^2 u}{\partial (t')^2} \right) + R_{\bar{g}}|_W \left(\frac{\partial^2 u}{\partial (t')^2} \right) \right) = 0 \text{ in } O_1.$$

Equivalently, the following PDE holds in O_1 with the new metric $\bar{g} \mapsto \epsilon^{-2} \bar{g}$,

$$(11) \quad L_{\sigma^*(\epsilon^{-2} \bar{g})} \left(\frac{\partial^2 u}{\partial (t')^2} \right) := 4 \nabla_{V_{\epsilon^{-2} \bar{g}}} \nabla_{V_{\epsilon^{-2} \bar{g}}} \left(\frac{\partial^2 u}{\partial (t')^2} \right) - 4 \Delta_{\sigma^*(\epsilon^{-2} \bar{g})} \left(\frac{\partial^2 u}{\partial (t')^2} \right) + R_{\epsilon^{-2} \bar{g}}|_W \left(\frac{\partial^2 u}{\partial (t')^2} \right) = 0 \text{ in } O_1.$$

Here $V_{\epsilon^{-2} \bar{g}}$ is the vector field with respect to the new metric $\epsilon^{-2} \bar{g}$ satisfying $V_{\epsilon^{-2} \bar{g}} = \epsilon^{-1} V$.

We now estimate $\left\| \frac{\partial^2 u}{\partial (t')^2} \right\|_{\mathcal{C}^0(O_1)}$. We set $s > 0$ such that $s - \frac{n}{2} = 1 + \alpha \geq 2 - \frac{n}{p}$, where p and α are given in Proposition 2.1. With usual metric $\sigma^* \bar{g}$, we have H^s -type local elliptic regularity estimates for second order elliptic operator $L_{\sigma^* \bar{g}}$,

$$\|v\|_{H^s(U, \sigma^* \bar{g})} \leq D_s \left(\|L_{\sigma^* \bar{g}} u\|_{H^{s-2}(V, \sigma^* \bar{g})} + \|v\|_{\mathcal{L}^2(V, \sigma^* \bar{g})} \right), \forall v \in \mathcal{C}^\infty(V).$$

where the constant D_s only depends on the operator $L_{\sigma^*\bar{g}}$, the metric, the degree s and the domain U, V . For the scaled metric $\sigma^*(\epsilon^{-2}\bar{g})$, we apply the local estimates and the second order elliptic operator $L_{\sigma^*(\epsilon^{-2}\bar{g})}$ in (11),

$$(12) \quad \begin{aligned} \left\| \frac{\partial^2 u}{\partial(t')^2} \right\|_{H^s(O_1, \sigma^*(\epsilon^{-2}\bar{g}))} &\leq \epsilon^{-\frac{n}{2}} \left\| \frac{\partial^2 u}{\partial(t')^2} \right\|_{H^s(O_1, \sigma^*\bar{g})} \\ &\leq \epsilon^{-\frac{n}{2}} D_s \left\| \frac{\partial^2 u}{\partial(t')^2} \right\|_{L^2(O_2, \sigma^*\bar{g})} \leq D_s \left\| \frac{\partial^2 u}{\partial(t')^2} \right\|_{L^2(O_2, \sigma^*(\epsilon^{-2}\bar{g}))}. \end{aligned}$$

Applying the local H^2 -type local elliptic estimate for (7) and Hölder's inequality,

$$(13) \quad \begin{aligned} \left\| \frac{\partial^2 u}{\partial(t')^2} \right\|_{L^2(O_2, \sigma^*(\epsilon^{-2}\bar{g}))} &= \epsilon^{2-\frac{n}{2}} \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{L^2(O_2, \sigma^*\bar{g})} \leq \epsilon^{2-\frac{n}{2}} \|u\|_{H^2(O_2, \sigma^*\bar{g})} \\ &\leq \epsilon^{2-\frac{n}{2}} D_2 (\|F\|_{L^2(W, \sigma^*\bar{g})} + \|u\|_{L^2(W, \sigma^*\bar{g})}) \\ &\leq \epsilon^{2-\frac{n}{2}} D_2 \|F\|_{L^2(W, \sigma^*\bar{g})} + \epsilon^{2-\frac{n}{2}} D_1 D_2 \|u\|_{L^{\frac{2(n+1)}{n-1}}(W, \sigma^*\bar{g})} \\ &\leq \epsilon^2 D_2 \|F\|_{L^2(W, \sigma^*(\epsilon^{-2}\bar{g}))} + \epsilon^{2-\frac{n}{n+1}} D_1 D_2 \|u\|_{L^{\frac{2(n+1)}{n-1}}(W, \sigma^*(\epsilon^{-2}\bar{g}))}. \end{aligned}$$

Similarly, the Sobolev embedding inequality for the scaled metric $\sigma^*(\epsilon^{-2}\bar{g})$ says

$$(14) \quad \left\| \frac{\partial^2 u}{\partial(t')^2} \right\|_{C^0(O_1)} \leq D_0 \epsilon^{\frac{n}{2}} \left\| \frac{\partial^2 u}{\partial(t')^2} \right\|_{H^s(O_1, \sigma^*(\epsilon^{-2}\bar{g}))}.$$

Here D_0, D_1, D_2 are independent of ϵ and u .

On $M \times \mathbb{S}_t^1$ with $\dim(M \times \mathbb{S}_t^1) = n+1$, the metric \bar{g} has positive relative Yamabe constant $\lambda := \lambda(M \times \mathbb{S}_t^1, \partial(M \times \mathbb{S}_t^1), [\bar{g}]) > 0$. It follows from the definition of the relative Yamabe constant that for all nontrivial $v \in C^\infty(M \times \mathbb{S}_t^1)$, the Yamabe quotient gives

$$\begin{aligned} \|v\|_{L^{\frac{2(n+1)}{n-1}}(M \times \mathbb{S}_t^1, \bar{g})}^2 &\leq \lambda(M \times \mathbb{S}_t^1, \partial(M \times \mathbb{S}_t^1), [\bar{g}])^{-1} \left(\frac{4n}{n-1} \|\nabla_{\bar{g}} v\|_{L^2(M \times \mathbb{S}_t^1, \bar{g})}^2 \right) \\ &\quad + \lambda(M \times \mathbb{S}_t^1, \partial(M \times \mathbb{S}_t^1), [\bar{g}])^{-1} \left(\int_{M \times \mathbb{S}_t^1} R_{\bar{g}} v^2 d\text{Vol}_{\bar{g}} + 2n \int_{\partial(M \times \mathbb{S}_t^1)} h_{\bar{g}} v^2 \sigma_{\bar{g}} \right) \\ &= \lambda^{-1} \left(\frac{4n}{n-1} \|\nabla_{\bar{g}} v\|_{L^2(M \times \mathbb{S}_t^1, \bar{g})}^2 + \int_{M \times \mathbb{S}_t^1} R_{\bar{g}} v^2 d\text{Vol}_{\bar{g}} \right). \end{aligned}$$

Here $\sigma_{\bar{g}}$ is the volume form on the hypersurface $\partial(M \times \mathbb{S}_t^1)$. The boundary term is dropped off since we have assumed that $h_{\bar{g}} \equiv 0$. The relative Yamabe constant λ is invariant under any conformal transformation of the metric \bar{g} .

Since $M \times \mathbb{S}_t^1$ is compact, there exist two positive constants D_3 and D_4 such that two metrics \bar{g} and $\sigma^*\bar{g} \oplus d\xi^2$ satisfy

$$D_4^{-2} \sqrt{\det \bar{g}} \leq \sqrt{\det(\sigma^*\bar{g} \oplus d\xi^2)} \leq D_3^{\frac{2(n+1)}{n-1}} \sqrt{\det \bar{g}}.$$

Set $\xi' = \epsilon^{-1}\xi$, we have

$$D_4^{-2} \sqrt{\det(\epsilon^{-2}\bar{g})} \leq \sqrt{\det(\sigma^*(\epsilon^{-2}\bar{g}) \oplus d(\xi')^2)} \leq D_3^{\frac{2(n+1)}{n-1}} \sqrt{\det(\epsilon^{-2}\bar{g})}$$

for the same D_3, D_4 .

Denote the inclusion by $\sigma_\xi : X_\xi \times \mathbb{S}^1 \rightarrow M \times \mathbb{S}_t^1$, and the space $X_\xi \times \mathbb{S}^1$ by W_ξ . We now apply the three inequalities above for the function u on W and $\bar{u} = \pi^*u$ on $M \times \mathbb{S}_t^1$ with respect to the

metrics $\sigma^*(\epsilon^{-2}\bar{g})$ and $\epsilon^{-2}\bar{g}$ respectively. Recall that the function \bar{u} is constant along each ξ -fiber, it follows that

$$\begin{aligned}
& \|u\|_{\mathcal{L}^{\frac{2(n+1)}{n-1}}(W, \sigma^*(\epsilon^{-2}\bar{g}))} \\
&= \epsilon^{\frac{n-1}{2(n+1)}} \left(\int_0^{\epsilon^{-1}} \int_W |\bar{u}|^{\frac{2(n+1)}{n-1}} d\text{Vol}_{\sigma^*(\epsilon^{-2}\bar{g})} d\xi' \right)^{\frac{n-1}{2(n+1)}} \\
&= \epsilon^{\frac{n-1}{2(n+1)}} \left(\int_{M \times \mathbb{S}_{t'}^1} |\bar{u}|^{\frac{2(n+1)}{n-1}} d\text{Vol}_{\sigma^*\bar{g} \oplus d(\xi')^2} \right)^{\frac{n-1}{2(n+1)}} \leq \epsilon^{\frac{n-1}{2(n+1)}} D_3 \left(\int_{M \times \mathbb{S}_{t'}^1} |\bar{u}|^{\frac{2(n+1)}{n-1}} d\text{Vol}_{\epsilon^{-2}\bar{g}} \right)^{\frac{n-1}{2(n+1)}} \\
&\leq \epsilon^{\frac{n-1}{2(n+1)}} D_3 \lambda^{-1} \left(\frac{4n}{n-1} \|\nabla_{\epsilon^{-2}\bar{g}} \bar{u}\|_{\mathcal{L}^2(M \times \mathbb{S}_{t'}^1, \epsilon^{-2}\bar{g})} + \int_{M \times \mathbb{S}_{t'}^1} R_{\epsilon^{-2}\bar{g}} \bar{u}^2 d\text{Vol}_{\epsilon^{-2}\bar{g}} \right)^{\frac{1}{2}} \\
&\leq \epsilon^{\frac{n-1}{2(n+1)}} D_3 D_4 \lambda^{-1} \left(\frac{4n}{n-1} \int_0^{\epsilon^{-1}} \int_{W_\xi} |\nabla_{\sigma_\xi^*(\epsilon^{-2}\bar{g})} \bar{u}|^2 d\text{Vol}_{\sigma_\xi^*(\epsilon^{-2}\bar{g})} d\xi' \right)^{\frac{1}{2}} \\
&\quad + \epsilon^{\frac{n-1}{2(n+1)}} D_3 D_4 \lambda^{-1} \left(\int_0^{\epsilon^{-1}} \int_W R_{\epsilon^{-2}\bar{g}} \bar{u}^2 d\text{Vol}_{\sigma^*(\epsilon^{-2}\bar{g})} d\xi' \right)^{\frac{1}{2}} \\
&\leq 2D_3 D_4 \lambda^{-1} \epsilon^{-\frac{1}{n+1}} \left(\frac{4n}{n-1} \max_{\xi \in [0,1]} \left(\|\nabla_{\sigma_\xi^*(\epsilon^{-2}\bar{g})} \bar{u}\|_{\mathcal{L}^2(W_\xi, \sigma_\xi^*(\epsilon^{-2}\bar{g}))}^2 \right) + \epsilon^2 \max_W |R_{\bar{g}}| \|u\|_{\mathcal{L}^2(W, \sigma^*(\epsilon^{-2}\bar{g}))}^2 \right)^{\frac{1}{2}}
\end{aligned}$$

Note that $\|F\|_{\mathcal{L}^p(W, \sigma^*\bar{g})}$ is small in the sense that $(C+1)^p \epsilon \ll 1$, hence $(C+1)^2 \epsilon \ll 1$ since $p > n \geq 3$. Note also that there are trivial diffeomorphisms between W and W_ξ for every $\xi \in [0, 1]$. Since \bar{u} is constant for each ξ -fiber, and $M \times \mathbb{S}^1$ is compact, we have $\|\nabla_{\sigma_\xi^*\bar{g}} \bar{u}\|_{\mathcal{L}^2(W_\xi, \sigma_\xi^*\bar{g})} \leq D_5 \eta$ for some D_5 independent of \bar{u} and ϵ . Without loss of generality, we may assume $\max_W |R_{\bar{g}}| \leq D_5$ by increasing D_5 if necessary. Combining (8), (12), (13), (14) with the above inequality, we have

$$\begin{aligned}
& \left\| \frac{\partial^2 u}{\partial (t')^2} \right\|_{\mathcal{C}^0(O_1)} \leq D_0 \epsilon^{\frac{n}{2}} \left\| \frac{\partial^2 u}{\partial (t')^2} \right\|_{H^s(O_1, \sigma^*(\epsilon^{-2}\bar{g}))} \leq D_0 D_s \epsilon^{\frac{n}{2}} \left\| \frac{\partial^2 u}{\partial (t')^2} \right\|_{\mathcal{L}^2(O_2, \sigma^*(\epsilon^{-2}\bar{g}))} \\
& \leq \epsilon^{2+\frac{n}{2}} D_0 D_2 D_s \|F\|_{\mathcal{L}^2(W, \sigma^*(\epsilon^{-2}\bar{g}))} + \epsilon^{2-\frac{n}{n+1}+\frac{n}{2}} D_0 D_1 D_2 D_s \|u\|_{\mathcal{L}^{\frac{2(n+1)}{n-1}}(W, \sigma^*(\epsilon^{-2}\bar{g}))} \\
& \leq 2D_0 D_1 D_2 D_3 D_4 D_s \lambda^{-1} \epsilon^{1+\frac{n}{2}} \times \\
& \quad \times \left(\frac{4n}{n-1} \max_{\xi \in [0,1]} \left(\|\nabla_{\sigma_\xi^*(\epsilon^{-2}\bar{g})} \bar{u}\|_{\mathcal{L}^2(W_\xi, \sigma_\xi^*(\epsilon^{-2}\bar{g}))}^2 \right) + \epsilon^2 \max_W |R_{\bar{g}}| \|u\|_{\mathcal{L}^2(W, \sigma^*(\epsilon^{-2}\bar{g}))}^2 \right)^{\frac{1}{2}} \\
& \quad + \epsilon^{2+\frac{n}{2}} D_0 D_2 D_s \|F\|_{\mathcal{L}^2(W, \sigma^*(\epsilon^{-2}\bar{g}))} \\
& := \bar{D}_0 \epsilon^2 \left(\frac{4n}{n-1} \max_{\xi \in [0,1]} \left(\|\nabla_{\sigma_\xi^*\bar{g}} \bar{u}\|_{\mathcal{L}^2(W_\xi, \sigma_\xi^*\bar{g})}^2 \right) + \max_W |R_{\bar{g}}| \|u\|_{\mathcal{L}^2(W, \sigma^*\bar{g})}^2 \right)^{\frac{1}{2}} \\
& \quad + \epsilon^{2+\frac{n}{2}} D_0 D_2 D_s \|F\|_{\mathcal{L}^2(W, \sigma^*(\epsilon^{-2}\bar{g}))} \\
& < \frac{8n}{n-1} \bar{D}_0 D_5 \epsilon^2 \eta + D_0 D_2 D_s \epsilon^2 \text{Vol}_g(X) (C+1) \epsilon^{\frac{1}{2}} \\
& := \bar{D} \epsilon^2 \eta + \bar{D}' \epsilon^{2+\frac{1}{2}} (C+1).
\end{aligned}$$

It follows by the smallness of η in Proposition 2.1 that

$$\left\| \frac{\partial^2 u}{\partial t^2} \right\|_{\mathcal{C}^0(U_1)} = \epsilon^{-2} \left\| \frac{\partial^2 u}{\partial (t')^2} \right\|_{\mathcal{C}^0(O_1)} < \bar{D} \eta + \bar{D}' (C+1) \epsilon^{\frac{1}{2}} := \eta' \ll 1.$$

As desired. \square

Remark 2.3. Since $u_0 = u + 1$, Lemma 2.3 implies that

$$\left\| \frac{\partial^2 u_0}{\partial t^2} \right\|_{C^0(X \times \{0\}_\xi \times \{P\}_t)} < \eta' \Rightarrow \left| \frac{\partial^2 u'}{\partial t^2} \right|_{X \times \{0\}_\xi \times \{P\}_t} < \eta'.$$

We close this section by comparing $\Delta_{\bar{g}} u'$, $\Delta_{\sigma^* \bar{g}} u_0$ and $\Delta_{\tau_1^* \bar{g}} \tilde{u}'$ on $X \times \{0\}_\xi \times \{P\}_t \subset W \times \{0\}_\xi \subset M \times \mathbb{S}_t^1$.

Lemma 2.4. *Under the hypotheses of Lemma 2.3,*

$$(15) \quad \begin{aligned} \Delta_{\bar{g}} u' &= \Delta_{\tau_1^* \bar{g}} \tilde{u}' + \frac{\partial^2 u'}{\partial t^2} \text{ on } X \times \{0\}_\xi \times \{P\}_t, \\ \Delta_{\bar{g}} u' &= \Delta_{\sigma^* \bar{g}} u_0 + A_1(\bar{g}, M, u) \end{aligned}$$

where $A_1(\bar{g}, M, u)$ involves zeroth and first order derivatives of u , which does not contain ∂_t - terms.

Proof. We can check both equations of (15) for local coordinates and local frames on charts containing points of $Q \in X \times \{0\}_\xi \times \{P\}_t$. Note that $u' = u_0 \circ \pi$, $u_0 = u' \circ \sigma$, and $\tilde{u}' = u' \circ \tau_1$, the rest of the proof follows exactly the same as in [10, Lemma 2.4]. \square

3. POSITIVE SCALAR CURVATURE ON X

In this section, we show that for any metric g on the compact cylinder M with $R_g > 0$ and $h_g \geq 0$ and satisfying the g -angle condition (1), there exists a conformal metric $\tilde{g}' \in [g]$ such that $R_{\tilde{g}'} > 0$ and $R_{i^* \tilde{g}'} > 0$ simultaneously.

We define

$$e^{2\phi'} := (u')^{\frac{4}{n-2}} \Rightarrow \phi_0 := \sigma^* \phi', \tilde{\phi}' := \tau_1^* \phi' \Rightarrow e^{2\phi_0} := u_0^{\frac{4}{n-2}}, e^{\tilde{\phi}'} = (\tilde{u}')^{\frac{4}{n-2}}.$$

For any two functions φ, v with the relation $e^{2\varphi} = v^{\frac{4}{n-2}}$, $n \geq 3$ and any Riemannian metric g_0 , we have

$$\begin{aligned} v^{-\frac{n+2}{n-2}} \left(-\frac{4}{n-2} \Delta_{g_0} v \right) &= e^{-2\varphi} (-2\Delta_{g_0} \varphi - (n-2)|\nabla_{g_0} \varphi|^2), \\ e^{-2\varphi} \nabla_{g_0} \varphi &= \frac{2}{n-2} v^{-\frac{n+2}{n-2}} \nabla_{g_0} v, \quad e^{-2\varphi} |\nabla_{g_0} \varphi|^2 = \left(\frac{2}{n-2} \right)^2 v^{-\frac{2n}{n-2}} |\nabla_{g_0} v|^2, \\ e^{-2\tilde{\phi}'} \left(2(n-2) \nabla_V \nabla_V \tilde{\phi}' + (n-2)^2 \nabla_V \tilde{\phi}' \nabla_V \tilde{\phi}' \right) &= 4(\tilde{u}')^{-\frac{n+2}{n-2}} \nabla_V \nabla_V \tilde{u}'. \end{aligned}$$

Our main theorem gives the existence of the PSC metric on $X \cong X \times \{0\}_\xi$. We point out that the proof of Theorem 3.1 follows closely the argument of [10, Theorem 3.1].

Theorem 3.1. *Let X be a closed, oriented manifold with $\dim X \geq 2$, and $I_\xi = [0, 1]_\xi$ be the unit interval with ξ -variable. Let $M = X \times I_\xi$ be the compact cylinder. Assume that $(M, \partial M)$, $\dim M \geq 3$ admits a metric g with $R_g > 0$ and $h_g \geq 0$. If g satisfies the angle condition*

$$(16) \quad \cos(\angle_g(\nu_g, \partial_\xi)) > \frac{\sqrt{2}}{2} \Leftrightarrow \frac{g(\partial_\xi, \partial_\xi)}{g(\nu_g, \partial_\xi)^2} < 2$$

on $X \times \{0\}_\xi$, then there exists a metric \tilde{g} on $(M, \partial M)$ such that $i^ \tilde{g}$ has positive scalar curvature on $X \times \{0\}_\xi \cong X$.*

Proof. As we discussed before, we may assume that $R_g > 0$ and $h_g \equiv 0$ up to a possible conformal transformation. Recall that the g -angle condition is invariant under conformal transformation, hence is unchanged.

We set $p > n$ as in Proposition 2.1. Pick $C > 0$ such that

$$(17) \quad C > 2 \max_{\partial M} (|R_g| + 2|\text{Ric}_g(\nu_g, \nu_g)| + h_g^2 + |A_g|^2) + 3.$$

We choose $\eta, \eta' \ll 1$, and associated small enough δ, ϵ and associated F in Lemma 2.2, such that (i) the solution of (7) satisfies (8) and (10); (ii) $|A_1(\bar{g}, M, u)| < 1$ on $X \times \{0\}_\xi \times \{P\}_t$. Note that $\tau_1^* \bar{g} = g$, $\nabla_{\tau_1^* \bar{g}}$ and ∇_V do not contain ξ -derivative. It follows that

$$(18) \quad \frac{1}{2} < \|u_0\|_{C^0(X \times [-\frac{\epsilon}{2}, \frac{\epsilon}{2}]_t)} < \frac{3}{2} \Rightarrow \left| \frac{4}{n-2} \frac{|\nabla_{\tau_1^* \bar{g}} u_0|^2}{u_0} + \frac{4n}{n-2} \frac{\nabla_V u_0 \nabla_V u_0}{u_0} \right| < 1 \text{ on } X \cong X \times \{0\}_\xi.$$

For the metric $\tilde{g} = e^{2\phi'} \bar{g} = (u')^{\frac{4}{n-2}} \bar{g}$ on $M \times \mathbb{S}_t^1$, the normal vector field on $X \times \{0\}_\xi \times \{P\}_t$ becomes $e^{-\tilde{\phi}'} \nu_g$. Applying Gauss-Codazzi equation on $X \cong X \times \{0\}_\xi \times \{P\}_t$ with respect to the metric $\tau_1^* \tilde{g}$ on M ,

$$R_{\iota^* \tau_1^* \tilde{g}} = R_{\tau_1^* \tilde{g}} - 2\text{Ric}_{\tau_1^* \tilde{g}}(e^{-\tilde{\phi}'} \nu_g, e^{-\tilde{\phi}'} \nu_g) + h_{\tau_1^* \tilde{g}} - |A_{\tau_1^* \tilde{g}}|^2.$$

The conformal transformation of Ricci and scalar curvatures are given by

$$\begin{aligned} \text{Ric}_{\tau_1^* \tilde{g}}(e^{-\tilde{\phi}'} \nu_g, e^{-\tilde{\phi}'} \nu_g) &= e^{-2\tilde{\phi}'} \left(\text{Ric}_{\tau_1^* \bar{g}}(\nu_g, \nu_g) - (n-2)(\nabla_{\nu_g} \nabla_{\nu_g} \tilde{\phi}' - \nabla_{\nu_g} \tilde{\phi}' \nabla_{\nu_g} \tilde{\phi}') \right) \\ &\quad - e^{-2\tilde{\phi}'} \left(\Delta_{\tau_1^* \bar{g}} \tilde{\phi}' + (n-2) \left| \nabla_{\tau_1^* \bar{g}} \tilde{\phi}' \right|^2 \right) \bar{g}_{\xi\xi}, \end{aligned}$$

$$R_{\tau_1^* \tilde{g}} = e^{-2\tilde{\phi}'} \left(R_{\tau_1^* \bar{g}} - 2(n-1) \Delta_{\tau_1^* \bar{g}} \tilde{\phi}' - (n-2)(n-1) \left| \nabla_{\tau_1^* \bar{g}} \tilde{\phi}' \right|^2 \right).$$

The conformal transformation of the second fundamental form and mean curvature are given by

$$\begin{aligned} |A_{\tau_1^* \tilde{g}}|^2 &= e^{-2\tilde{\phi}'} \left(|A_{\tau_1^* \bar{g}}|^2 + 2nh_{\tau_1^* \bar{g}} \frac{\partial \tilde{\phi}'}{\partial \nu_g} + n^2 \left(\frac{\partial \tilde{\phi}'}{\partial \nu_g} \right)^2 \right), \\ h_{\tau_1^* \tilde{g}}^2 &= e^{-2\tilde{\phi}'} \left(h_{\tau_1^* \bar{g}}^2 + 2nh_{\tau_1^* \bar{g}} \frac{\partial \tilde{\phi}'}{\partial \nu_g} + n^2 \left(\frac{\partial \tilde{\phi}'}{\partial \nu_g} \right)^2 \right). \end{aligned}$$

By (9), $\nabla_{\nu_g} \nabla_{\nu_g} \tilde{\phi}' = \nabla_V \nabla_V \tilde{\phi}'$. According to the laws of conformal transformations listed above, and the differential relations before Theorem 3.1, the Gauss-Codazzi equation converts to

$$\begin{aligned} R_{\iota^* \tau_1^* \tilde{g}} &= e^{-2\tilde{\phi}'} \left(R_{\tau_1^* \bar{g}} - 2\text{Ric}_{\tau_1^* \bar{g}}(\nu_g, \nu_g) + h_{\tau_1^* \bar{g}}^2 - |A_{\tau_1^* \bar{g}}|^2 \right) \\ &\quad + e^{-2\tilde{\phi}'} \left(-2(n-2) \Delta_{\tau_1^* \bar{g}} \tilde{\phi}' - (n-3)(n-2) \left| \nabla_{\tau_1^* \bar{g}} \tilde{\phi}' \right|^2 \right) \\ &\quad + 2(n-2) e^{-2\tilde{\phi}'} \left(\nabla_V \nabla_V \tilde{\phi}' - \nabla_V \tilde{\phi}' \nabla_V \tilde{\phi}' \right) \\ (19) \quad &= (\tilde{u}')^{-\frac{n+2}{n-2}} \left(R_{\tau_1^* \bar{g}} \tilde{u}' - 2\text{Ric}_{\tau_1^* \bar{g}}(\nu_g, \nu_g) \tilde{u}' + h_{\tau_1^* \bar{g}}^2 \tilde{u}' - |A_{\tau_1^* \bar{g}}|^2 \tilde{u}' \right) \\ &\quad + (\tilde{u}')^{-\frac{n+2}{n-2}} \left(4\nabla_V \nabla_V \tilde{u}' - 4\Delta_{\tau_1^* \bar{g}} \tilde{u}' \right) \\ &\quad + (\tilde{u}')^{-\frac{n+2}{n-2}} \left(\frac{4}{n-2} \frac{|\nabla_{\tau_1^* \bar{g}} \tilde{u}'|^2}{\tilde{u}'} - \frac{4n}{n-2} \frac{\nabla_V \tilde{u}' \nabla_V \tilde{u}'}{\tilde{u}'} \right). \end{aligned}$$

Note that on $X \cong X \times \{0\}_\xi \times \{P\}_t \subset W \times \{0\}_\xi$, $\tilde{u}' = u_0$. Recall that $u_0 = u + 1$, which satisfies the following PDE

$$4\nabla_V \nabla_V u_0 - 4\Delta_{\sigma^* \bar{g}} u_0 + R_{\bar{g}}|_W u_0 = F + R_{\bar{g}}|_W u_0 \text{ on } X.$$

Since $\bar{g} = g \oplus dt^2$, it follows that $R_{\bar{g}} = R_{\tau_1^* \bar{g}} = R_g > 0$ on $X \times \{0\}_\xi$ by hypothesis. Applying (10) of Lemma 2.3, (15) of Lemma 2.4, Remark 2.3, and (18), the formula (19) on $X \cong X \times \{0\}_\xi \times \{P\}_t$ becomes

$$\begin{aligned}
 R_{\iota^* \tau_1^* \bar{g}} &= u_0^{-\frac{n+2}{n-2}} \left(-2\text{Ric}_{\tau_1^* \bar{g}}(\nu_g, \nu_g) u_0 + h_{\tau_1^* \bar{g}}^2 u_0 - |A_{\tau_1^* \bar{g}}|^2 u_0 \right) \\
 &\quad + u_0^{-\frac{n+2}{n-2}} \left(4\nabla_V \nabla_V u_0 - 4\Delta_{\tau_1^* \bar{g}} \tilde{u}' - 4\frac{\partial^2 \tilde{u}'}{\partial t^2} + 4\frac{\partial^2 \tilde{u}'}{\partial t^2} + R_{\tau_1^* \bar{g}} u_0 \right) \\
 &\quad + u_0^{-\frac{n+2}{n-2}} \left(\frac{4}{n-2} \frac{|\nabla_{\tau_1^* \bar{g}} u_0|^2}{u_0} - \frac{4n}{n-2} \frac{\nabla_V u_0 \nabla_V u_0}{u_0} \right) \\
 &\geq u_0^{-\frac{n+2}{n-2}} \left(-2\text{Ric}_{\tau_1^* \bar{g}}(\nu_g, \nu_g) u_0 + h_{\tau_1^* \bar{g}}^2 u_0 - |A_{\tau_1^* \bar{g}}|^2 u_0 \right) \\
 &\quad + u_0^{-\frac{n+2}{n-2}} (4\nabla_V \nabla_V u_0 - 4\Delta_{\bar{g}} u' - 4\eta' + R_{\bar{g}}|_W u_0) + u_0^{-\frac{n+2}{n-2}} \cdot (-1) \\
 &= u_0^{-\frac{n+2}{n-2}} \left(-2\text{Ric}_{\tau_1^* \bar{g}}(\nu_g, \nu_g) u_0 + h_{\tau_1^* \bar{g}}^2 u_0 - |A_{\tau_1^* \bar{g}}|^2 u_0 \right) \\
 &\quad + u_0^{-\frac{n+2}{n-2}} (4\nabla_V \nabla_V u_0 - 4\Delta_{\sigma^* \bar{g}} u_0 + R_{\bar{g}}|_W u_0) + u_0^{-\frac{n+2}{n-2}} \cdot (-1 - 4\eta' + A_1(\bar{g}, M, u)) \\
 &= u_0^{-\frac{n+2}{n-2}} \left(-2\text{Ric}_{\tau_1^* \bar{g}}(\nu_g, \nu_g) u_0 + h_{\tau_1^* \bar{g}}^2 u_0 - |A_{\tau_1^* \bar{g}}|^2 u_0 \right) \\
 &\quad + u_0^{-\frac{n+2}{n-2}} (F + R_{\bar{g}}|_W - 1 - 4\eta' + A_1(\bar{g}, M, u)).
 \end{aligned} \tag{20}$$

By Lemma 2.2, $F = C + 1$ on ∂M . By (17), (20) implies that

$$R_{\iota^* \tau_1^* \bar{g}} \geq u_0^{-\frac{n+2}{n-2}} \left(C + 1 + R_{\bar{g}} - 2 \max_{\partial M} (2|\text{Ric}_g(\nu_g, \nu_g)| + h_g^2 + |A_g|^2) - 3 \right) > 0.$$

Therefore, the metric

$$\tau_1^* \tilde{g} = \tau_1^*(u')^{\frac{4}{n-2}} \bar{g} = (\tilde{u}')^{\frac{4}{n-2}} g := g', g' \in [g]$$

induces a PSC metric $\iota^* g' \in [\iota^* g]$ on $X \cong X \times \{0\}_\xi$. \square

Remark 3.1. Set $X_\xi = X \times \{\xi\}$, $\xi \in [0, 1]$, we can define the second fundamental form, unit normal vector field, etc. on the hypersurface X_ξ also. We can then apply our g -angle condition on X_ξ . It follows that if M admits a metric g with $R_g > 0$ and $h_g \geq 0$, and the g -angle condition on X_ξ holds, then there exists a metric \tilde{g} such that $\iota_\xi^* \tilde{g}$ has positive scalar curvature on X_ξ . Here $\iota_\xi : X_\xi \rightarrow M$ is the natural inclusion.

With Remark 3.1, we generalize the scalar and mean curvature comparison result of Gromov and Lawson [6] for $X = \mathbb{T}^n$, $n \geq 1$, and the result of Råde for $\dim X \leq 7$, $\dim X \neq 4$.

Corollary 3.1. *Let X be a closed, oriented manifold with $\dim X \geq 2$. If X admits no PSC metric, meanwhile $X \times I$ admits a PSC metric g . If the g -angle condition holds on some hypersurface X_ξ , $\xi \in [0, 1]$, then the mean curvature h_g must be negative somewhere.*

We now give another geometric consequence of Theorem 3.1. Theorem 3.1 says that for PSC metric g on M , the Yamabe constant for the conformal class $[\iota^* g]$ on $X \cong X \times \{0\}_\xi$ is positive. The next result shows that there exists a metric in the conformal class $[g]$ which admits PSC metric on M and induced PSC metric on X .

Corollary 3.2. *Let $(M, \partial M, g)$ be the same as in Theorem 3.1, $\dim M = n$. If*

$$\cos(\angle_g(\partial_\xi, \nu_g)) > \frac{\sqrt{2}}{2}$$

on $X \times \{0\}_\xi$, then there exists a conformal metric $\tilde{g} \in [g]$ such that $R_{\tilde{g}} > 0$ and $R_{i^\tilde{g}} > 0$.*

Proof. Let $\eta_{1,g}$ be the first eigenvalue of the conformal Laplacian $-\frac{4(n-1)}{n-2}\Delta_g + R_g$ on M with Neumann boundary condition. It is well-known that $R_g > 0$ and $h_g \geq 0$ implies that $\eta_{1,g} > 0$.

Let ζ_{1,i^*g} be the first eigenvalue of the conformal Laplacian $-\frac{4(n-2)}{n-3}\Delta_{i^*g} + R_{i^*g}$. By Theorem 3.1, $R_{i^*g_1} > 0$ for some $g_1 \in [g]$. It follows that $\zeta_{1,i^*g_1} > 0$, see e.g. [8]. By [7, Theorem 3.2], $\zeta_{1,i^*g} > 0$ as the sign is preserved under conformal change.

Since $\eta_{1,g} > 0$, [11, Theorem 1.1] implies that there exists a metric $g' \in [g]$ such that $R_{\tilde{g}}$ is a positive constant, and $R_{i^*g'}$ is also a constant. Therefore $R_{i^*g'} > 0$ since otherwise it contradicts the positivity of $\zeta_{1,i^*g'}$. \square

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