Capacitary Muckenhoupt Weights and Weighted Norm Inequalities for Hardy-Littlewood Maximal Operators

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Abstract Let $\mathcal{H}_\infty^\delta$ denote the Hausdorff content of dimension $\delta \in (0, n]$ defined on subsets of $\mathbb{R}^n$. The principal problem, considered in this paper, is to characterize the non-negative function w for which there exists a positive constant K such that

$$(0.1) \qquad \int_{\mathbb{R}^n} \left[\mathcal{M}_{\mathcal{H}_\infty^\delta}(f)(x) \right]^p w(x) \, d\mathcal{H}_\infty^\delta \leq K \int_{\mathbb{R}^n} |f(x)|^p w(x) \, d\mathcal{H}_\infty^\delta,$$

and characterize the non-negative function w such that

$$(0.2) \qquad w_{\mathcal{H}_\infty^\delta}\left\{ x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}_\infty^\delta}(f)(x) > t \right\} \leq \frac{K}{t} \int_{\mathbb{R}^n} |f(x)| w(x) \, d\mathcal{H}_\infty^\delta, \quad \forall\, t \in (0, \infty),$$

where $p \in (1, \infty)$ and $\mathcal{M}_{\mathcal{H}_\infty^\delta}$ is the Hardy-Littlewood maximal operator associated with Hausdorff contents

$$\mathcal{M}_{\mathcal{H}_\infty^\delta} f(x) := \sup_{Q \ni x} \frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q |f(x)| \, d\mathcal{H}_\infty^\delta$$

with the supremum is taken over all cubes Q containing x. To achieve this, we introduce a class of capacitary Muckenhoupt weights depending on the dimension δ, denoted as $\mathcal{A}_{p,\delta}$, which enjoys the strict monotonicity on the dimension index δ. Then we show that, for any $p \in (1, \infty)$ and $\delta \in (0, n]$, the weighted norm inequality (0.1) holds true if and only if $w \in \mathcal{A}_{p,\delta}$, and the weighted norm inequality (0.2) holds true if and only if $w \in \mathcal{A}_{1,\delta}$ by a new approach developed in this paper. As the second objective, applying this new approach, the seminal properties of classical Muckenhoupt A_p weights, such as the reverse Hölder inequality (R. R. Coifman and C. Fefferman, Studia Math. 51 (1974), 241-250), the self-improving property (B. Muckenhoupt, Trans. Amer. Math. Soc. 165 (1972), 207-226), and the Jones factorization theorem (P. W. Jones, Ann. of Math. (2) 111 (1980), 511-530), are all established within the framework of capacitary Muckenhoupt weight class $\mathcal{A}_{p,\delta}$. Finally, we also show that the maximal operator $\mathcal{M}_{\mathcal{H}_\infty^\delta}$ is bounded on the weak weighted Choquet-Lebesgue space $L_w^{p,\infty}(\mathbb{R}^n, \mathcal{H}_\infty^\delta)$ if and only if $w \in \mathcal{A}_{p,\delta}$ with $p \in (1, \infty)$ and $\delta \in (0, n]$.

The main novelty of this paper lies in the fact that we propose a new approach in the Hausdorff content setting to build up a full analogue of the Muckenhoupt theorem, reverse Hölder inequality, the Jones factorization theorem, etc., where the classical methods fail. Consequently, the results established in this paper naturally extend the classical theory beyond measure-theoretic frameworks, while also provide innovative proofs—even when reduced to the classical Muckenhoupt A_p weight case (i.e., $\delta = n$)—that avoid the use of the Fubini theorem, the countable additivity of measures and linearity of the integral.

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1   Introduction and Main Results

Given a set $E \subset \mathbb{R}^n$ and $\delta \in (0, n]$, the *Hausdorff content* $\mathcal{H}_\infty^\delta(E)$ is defined by setting

$$\mathcal{H}_\infty^\delta(E) := \inf\left\{\sum_i [l(Q_i)]^\delta : E \subset \bigcup_i Q_i\right\},$$

where the infimum is taken over all finite or countable cubes coverings $\{Q_i\}_i$ of E and $l(Q)$ denotes the edge length of the cube Q. The quantity $\mathcal{H}_\infty^\delta(E)$ is also referred to as the *δ-Hausdorff content* or the *δ-Hausdorff capacity* or the *Hausdorff content of E of dimension δ*. Given $p \in (0, \infty)$ and any $\mathcal{H}_\infty^\delta$-almost everywhere defined function g on $\mathbb{R}^n$, its *Choquet integral* with respect to Hausdorff content $\mathcal{H}_\infty^\delta$ is defined to be

$$\int_{\mathbb{R}^n} |g(x)|^p \, d\mathcal{H}_\infty^\delta := p \int_0^\infty t^{p-1} \mathcal{H}_\infty^\delta(\{x \in \mathbb{R}^n : |g(x)| > t\}) \, dt.$$

The monotonicity of the set function $\mathcal{H}_\infty^\delta$ implies that the Lebesgue integral on the right hand side is always well defined, even when g is a non-measurable function in the sense of Lebesgue measures. Thus, the Choquet integral is well defined for all non-measurable functions with respect to the Lebesgue measure. It is worth noting that Hausdorff contents, originating from fractal geometry (see [11, 13]), play a crucial role in harmonic analysis and nonlinear potential theory (see [3]). The Choquet integral also has many essential and profound applications in quasilinear elliptic equations (see, for example, T. Kilpeläinen and J. Malý [22] as well as D. Labutin [23]), in continuous time dynamic and coherent risk measures in finance (see, for example, L. Denis, M. Hu and S. Peng [10]) and in Bayesian decision theory, subjective probability and robust optimization (see, for example, D. Bertsimas, D. Brown and C. Caramanis [4]). For the further theory of Choquet integrals with respect to Hausdorff contents, we refer the reader to the original work of

G. Choquet [6], the excellent survey by D. R. Adams [1] and the paper by L. Tang [34] for an overview.

Let $L^1_{\text{loc}}(\mathbb{R}^n)$ denote the set of all locally integrable functions on $\mathbb{R}^n$. The classical *Hardy-Littlewood maximal operator* $\mathcal{M}$ is defined by setting

$$\mathcal{M}(f)(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy, \quad \forall \, x \in \mathbb{R}^n \text{ and } f \in L^1_{\text{loc}}(\mathbb{R}^n).$$

Equivalently, the supremum may be taken over all cubes Q centered at x, or over all balls B containing x. Let $w \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $w \in (0, \infty)$ almost everywhere. Given $p \in [1, \infty)$ and a measurable set E, let $w(E) := \int_E w(x) \, dx$ and let f belong to weighted Lebesgue spaces $L^p_w(\mathbb{R}^n)$, i.e.,

$$\int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx < \infty.$$

As one of fundamental results in harmonic analysis, it is well known that the following weighted norm inequality for maximal operator $\mathcal{M}$,

(1.1)
$$\int_{\mathbb{R}^n} [\mathcal{M}(f)(x)]^p \, w(x) \, dx \le K \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx,$$

with $p \in (1, \infty)$ and K being a positive constant independent of f, and the weak-type weighted norm inequality in the endpoint case $p = 1$,

(1.2)
$$w(\{x \in \mathbb{R}^n : \mathcal{M}(f)(x) > t\}) \le \frac{K}{t} \int_{\mathbb{R}^n} |f(x)| w(x) \, dx, \quad \forall \, t \in (0, \infty)$$

are characterized, respectively, by the Muckenhoupt class A_p,

$$A_p := \left\{ w : \; [w]_{A_p} := \sup_Q \left[\frac{1}{|Q|} \int_Q w(x) \, dx \right] \left[\frac{1}{|Q|} \int_Q w(x)^{-\frac{1}{p-1}} \, dx \right]^{p-1} < \infty \right\}$$

with the supremum taken over all cubes Q, and the Muckenhoupt class A_1,

$$A_1 := \left\{ w : \; [w]_{A_1} := \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \frac{\mathcal{M}w(x)}{w(x)} < \infty \right\}.$$

Namely, for $p \in (1, \infty)$, the weighted norm inequality (1.1) holds true if and only if $w \in A_p$, and the weak-type weighted norm inequality (1.2) holds true if and only if $w \in A_1$. The original proofs were given by B. Muckenhoupt [30] based on an interpolation argument and a remarkable self-improving property of A_p weights, which says that $w \in A_p$ implies $w \in A_{p-\varepsilon}$ for some $\varepsilon > 0$ (i.e., openness property). Later, R. R. Coifman and C. Fefferman [8] gave a greatly simplified proof by a crucial reverse Hölder inequality for the A_p weight. After that, an elementary proof for (1.1) avoiding the reverse Hölder inequality was provided by M. Christ and R. Fefferman [7] based essentially on the Calderón-Zygmund decomposition. For further results concerning weighted norm inequalities on maximal operators and singular integral operators, we refer the reader to the series of contributions by T. Hytönen [19, 20] and A. K. Lerner [25, 28, 29].

In addition, there is an intimate connection between John-Nirenberg BMO functions, B. Muckenhoupt's A_p weights, reverse Hölder inequalities and C. Fefferman's $H^1 - BMO$ duality theorem. This connection can be roughly described by saying that BMO consists of the logarithms of A_p

weights or BMO consists of the logarithms of weights that satisfies the reverse Hölder inequality, which further gives a new proof of C. Fefferman's duality theorem; see [16, 33]. Indeed, the A_p weight theory has already reached a great level of perfection and has found applications in several branches of analysis, from complex function theory to partial differential equations; see, for example, [9, 15, 16, 18, 24, 27, 35, 36].

Due to the significance of Choquet integrals with respect to Hausdorff contents, A_p weights and the usefulness of the aforementioned weighted norm inequality for Hardy-Littlewood maximal operators in analysis, in this paper, we are interested in identifying the conditions on the function w under which the above weighted norm inequalities (1.1) and (1.2) remain true when Lebesgue measure dx is replaced by Hausdorff contents $d\mathcal{H}_\infty^\delta$, i.e., in the Choquet integral setting. Also, we are further devoted to build up the most important properties of the class of these functions w, including their analogues of self-improving property as in [30], reverse Hölder inequality as in [8] and P. W. Jones' factorization theorem as in [21].

More precisely, the first purpose of this paper is to characterize the function w on $\mathbb{R}^n$ such that there exists a positive constant K satisfying

$$(1.3) \qquad \int_{\mathbb{R}^n} \left[\mathcal{M}_{\mathcal{H}_\infty^\delta}(f)(x) \right]^p w(x) \, d\mathcal{H}_\infty^\delta \le K \int_{\mathbb{R}^n} |f(x)|^p w(x) \, d\mathcal{H}_\infty^\delta,$$

when $p \in (1, \infty)$, and when $p = 1$,

$$(1.4) \qquad w_{\mathcal{H}_\infty^\delta} \left\{ x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}_\infty^\delta}(f)(x) > t \right\} \le \frac{K}{t} \int_{\mathbb{R}^n} |f(x)| w(x) \, d\mathcal{H}_\infty^\delta, \quad \forall t \in (0, \infty).$$

Here and thereafter,

$$(1.5) \qquad w_{\mathcal{H}_\infty^\delta}(F) := \int_F w(x) \, d\mathcal{H}_\infty^\delta, \quad \forall F \subset \mathbb{R}^n,$$

and $\mathcal{M}_{\mathcal{H}_\infty^\delta}$ is the *capacitary Hardy-Littlewood maximal operator* with respect to $\mathcal{H}_\infty^\delta$ defined as

$$\mathcal{M}_{\mathcal{H}_\infty^\delta} f(x) := \sup_{Q \ni x} \frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q |f(x)| \, d\mathcal{H}_\infty^\delta$$

with the supremum taken over all cubes Q containing x. In what follows, we say a function w is a *capacitary weight* on $\mathbb{R}^n$, it always means that w is locally integrable and $w \in (0, \infty)$ almost everywhere with respect to the Hausdorff content $\mathcal{H}_\infty^\delta$. For simplicity, we will henceforth refer to a capacitary weight w simply as a weight w.

To settle this problem, it might seem intuitive and immediate to adapt the aforementioned methods for the classical A_p weight theory to the setting of Choquet integrals with respect to Hausdorff contents and capacitary maximal operators. However, upon closer examination, one can find that these approaches are not feasible and the thing is far away from immediate. A fundamental difficulty lies in the absence of linearity or even sublinear property for the Choquet integrals and the lack of countable additivity for Hausdorff contents, that is, for any non-negative functions $\{f_j\}_{j \in \mathbb{N}}$ defined on E,

$$\int_E \sum_{j \in \mathbb{N}} f_j(x) \, d\mathcal{H}_\infty^\delta \ne \sum_{j \in \mathbb{N}} \int_E f_j(x) \, d\mathcal{H}_\infty^\delta.$$

These two quantities are even not equivalent (see [32, p. 148]). Indeed, for any $M > 0$, there exist non-negative functions $\{f_j\}_{j \in \mathbb{N}}$ and E, such that

$$M \int_E \sum_{j \in \mathbb{N}} f_j(x) \, d\mathcal{H}_\infty^\delta < \sum_{j \in \mathbb{N}} \int_E f_j(x) \, d\mathcal{H}_\infty^\delta.$$

This makes that the aforementioned crucial techniques such as, self-improving property $A_p \Rightarrow A_{p-\varepsilon}$, reverse Hölder inequality and the Calderón-Zygmund decomposition, can not be extended to the current Hausdorff content and the Choquet integral setting. Consequently, new techniques and approaches are required to investigate the aforementioned weighted norm inequalities.

   In this paper, roughly speaking, we are inspired by the idea used in [26] to address the characterization of weight w satisfying the strong-type (p, p) inequality (1.3) and the weak-type $(1, 1)$ inequality (1.4). Note that A. K. Lerner [26] gave an extremely simple proof for Muckenhoupt's theorem that completely avoids additional ingredients and can be applied to maximal operators with respect to a general basis. But, differing with [26] and also differing with [8, 30], we do not use the linear properties for integrals, countable additivity for Hausdorff contents or the Fubini theorem. Instead, starting from covering arguments, we develop a new approach which shows that, under some suitable conditions, one can interchange the order of infinite sum with the Choquet integral. Precisely, inspired by a wonderful idea from Calderón-Zygmund decomposition technique, we formulate and prove a "sparse covering lemma" in the context of Hausdorff contents; see Proposition 2.10 below. Combining this crucial lemma and several other tricks, we discover that the weighted norm inequalities (1.3) and (1.4) are exactly characterized by the capacitary Muckenhoupt weights class $\mathcal{A}_{p,\delta}$ introduced in the present paper.

   The first finding of this paper is the following theorem.

Theorem 1.1. *Let $\delta \in (0, n]$, $p \in (1, \infty)$ and w be a weight. Then the following statements are equivalent*

   (i) *the strong-type (p, p) inequality (1.3) holds, i.e.,*

$$\mathcal{M}_{\mathcal{H}_\infty^\delta} : L_w^p(\mathbb{R}^n, \mathcal{H}_\infty^\delta) \to L_w^p(\mathbb{R}^n, \mathcal{H}_\infty^\delta)$$

   is bounded;

   (ii) *there exists a positive constant K such that*

$$w_{\mathcal{H}_\infty^\delta}\left\{ x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}_\infty^\delta}(f)(x) > t \right\} \le \frac{K}{t^p} \int_{\mathbb{R}^n} |f(x)|^p w(x) \, d\mathcal{H}_\infty^\delta, \quad \forall\, t \in (0, \infty),$$

   i.e., $\mathcal{M}_{\mathcal{H}_\infty^\delta} : L_w^p(\mathbb{R}^n, \mathcal{H}_\infty^\delta) \to L_w^{p,\infty}(\mathbb{R}^n, \mathcal{H}_\infty^\delta)$ is bounded;

   (iii) *$w \in \mathcal{A}_{p,\delta}$, i.e., there exists a positive constant A such that*

$$(1.6) \quad [w]_{\mathcal{A}_{p,\delta}} := \sup_{\text{cube } Q \subset \mathbb{R}^n} \left\{ \frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q w(x) \, d\mathcal{H}_\infty^\delta \right\} \left\{ \frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q w(x)^{-\frac{1}{p-1}} \, d\mathcal{H}_\infty^\delta \right\}^{p-1} \le A.$$

   Here and thereafter, a weight w satisfying (1.6) with $p \in (1, \infty)$ is called the *capacitary Muckenhoupt $\mathcal{A}_{p,\delta}$-weight* with respect to the Hausdorff content $\mathcal{H}_\infty^\delta$, denoted as $w \in \mathcal{A}_{p,\delta}$, and for the precise definitions of function spaces $L_w^p(\mathbb{R}^n, \mathcal{H}_\infty^\delta)$ and $L_w^{p,\infty}(\mathbb{R}^n, \mathcal{H}_\infty^\delta)$, we refer the reader to Section 2 below.

   For the endpoint $p = 1$, we also obtain the following characterization for weight w satisfying weak-type inequality (1.4).

Theorem 1.2. *Let $\delta \in (0, n]$ and w be a weight. Then the following statements are equivalent*

  (i) *the weak-type $(1, 1)$ inequality (1.4) holds, i.e.,*

$$\mathcal{M}_{\mathcal{H}_\infty^\delta} : L_w^1(\mathbb{R}^n, \mathcal{H}_\infty^\delta) \to L_w^{1,\infty}(\mathbb{R}^n, \mathcal{H}_\infty^\delta)$$

  is bounded;

  (ii) *$w \in \mathcal{A}_{1,\delta}$, i.e., there exists a positive constant A such that*

$$(1.7) \quad [w]_{\mathcal{A}_{1,\delta}} := \inf \left\{ K \in (0, \infty) : \ \mathcal{M}_{\mathcal{H}_\infty^\delta} w(x) \le K w(x) \text{ for } \mathcal{H}_\infty^\delta \text{--almost everywhere} \right\} \le A.$$

Here and thereafter, a weight w satisfying (1.7) is called the *capacitary Muckenhoupt $\mathcal{A}_{1,\delta}$-weight* with respect to the Hausdorff content, denoted as $w \in \mathcal{A}_{1,\delta}$.

Below are three comments on Theorems 1.1 and 1.2.

Remark 1.3. We point out that Theorems 1.1 and 1.2 revisit the classical Muckenhoupt theroem for maximal operator $\mathcal{M}$. Recall that when $\delta = n$, the Hausdorff content $\mathcal{H}_\infty^n$ is equivalent to the Lebesgue measure, namely, there exists positive constants $K_1(n)$ and $K_2(n)$ such that, for all measurable subset $E \subset \mathbb{R}^n$,

$$K_1(n) \mathcal{H}_\infty^n(E) \le |E| \le K_2(n) \mathcal{H}_\infty^n(E),$$

which is essential obtained by the equivalence between the n-Hausdorff measure and the Lebesgue measure; see L. C. Evans and R. F. Gariepy [11, Chapter 2.2] for the detail. In this case, the capacitary Hardy-Littlewood maximal operator $\mathcal{M}_{\mathcal{H}_\infty^n}$ goes back to the classical Hardy-Littlewood maximal operator $\mathcal{M}$ and the capacitary Muckenhoupt weights class $\mathcal{A}_{p,n}$ is just the classical Muckenhoupt class A_p for Lebesgue measurable weight functions. Thus, when $\delta = n$, Theorems 1.1 and 1.2 return exactly to the classical Muckenhoupt theorem, that is, the maximal operator $\mathcal{M}$ is of strong-type (p, p) for $p \in (1, \infty)$ if and only if $w \in A_p$, and is of weak-type $(1, 1)$ if and only if $w \in A_1$.

Remark 1.4. To show Theorems 1.1 and 1.2, we develop a new approach in the Choquet integral setting where the classical methods no longer apply. Moreover, even when returning to the classical A_p weight setting, the approach developed in this paper for proving Theorems 1.1 and 1.2 provides new and broadly applicable proofs, which avoid linearity of integrals, the countable additivity of measures, or the Fubini theorem.

Remark 1.5. Similar to the classical A_p weight, the new introduced capacitary Muckenhoupt weight class $\mathcal{A}_{p,\delta}$ is monotonically increasing on the first index p, i.e., $\mathcal{A}_{p_1,\delta} \subset \mathcal{A}_{p_2,\delta}$ when $1 \le p_1 \le p_2 < \infty$. An unexpected phenomenon is that the class $\mathcal{A}_{p,\delta}$ also enjoys the strict monotonicity on the dimension δ of Hausdorff contents; see Proposition 2.15 below.

One of the main novelties of Theorems 1.1 and 1.2 lie in the fact that the dimension δ of the Hausdorff content $\mathcal{H}_\infty^\delta$ is allowed to be strictly less than n. Indeed, the case of $\delta \in (0, n)$ is more delicate. In this setting, the Hausdorff content $\mathcal{H}_\infty^\delta$ may be identified with the standard definition of an outer measure. However, the set function $\mathcal{H}_\infty^\delta$, when $\delta \in (0, n)$, can not be restricted to a nontrivial sigma algebra so as to be an additive measure there. In fact, it fails to be, what in measure theory is called, a "metric outer measure"; see D. R. Adams [1]. Notice that the theory

of classical A_p weights has had quite a success, which is, in part, a consequence of the theory of classical A_p weights carries through to the situation in which Lebesgue measure dx is replaced by a general doubling measure $w(x)\,dx$. That is, under the Lebesgue integral setting, via the Fubini theorem, for any non-negative measurable function w and $f \in L^1_w(\mathbb{R}^n)$, it is easy to see

$$\int_{\mathbb{R}^n} |f(x)|w(x)\,dx = \int_{\mathbb{R}^n} |f(x)|\,dw.$$

Then one can show that, when $w \in A_p$, $w(F) := \int_F dw$, with $F \subset \mathbb{R}^n$, is a new measure satisfying doubling condition: $w(2B) \le Cw(B)$ for any ball $B \in \mathbb{R}^n$. But, in the case of Choquet integrals with $0 < \delta < n$, the following kind of the Fubini theorem

$$\int_{\mathbb{R}^n} \int_0^\infty f(x,t)w(x)\,dt\,d\mathcal{H}^\delta_\infty \sim \int_0^\infty \int_{\mathbb{R}^n} f(x,t)w(x)\,d\mathcal{H}^\delta_\infty\,dt.$$

fails in general [see Remark 2.9(ii) below], and hence the equivalence

(1.8) $$\int_{\mathbb{R}^n} |f(x)|w(x)\,d\mathcal{H}^\delta_\infty \sim \int_{\mathbb{R}^n} |f(x)|\,dw_{\mathcal{H}^\delta_\infty}$$

does not hold for general weight w [see Remark 2.9(i) below], where $w_{\mathcal{H}^\delta_\infty}$ is defined as in (1.5).

Surprisingly, given $p \in [1, \infty)$, we discover, in Proposition 2.7 below, that the condition $w \in \mathcal{A}_{p,\delta}$ ensures (1.8). This is unexpected compared to the classical situation and is also one of the key points in this work. The main idea behind the proof of (1.8) can be summarized as follows. Let $w \in \mathcal{A}_{p,\delta}$ and

$$E_k := \left\{ x \in \mathbb{R}^n : 2^{k-2} < |f(x)|w(x) \le 2^k \right\}, \quad \forall\, k \in \mathbb{Z}.$$

Then we find that the sets $G_j := \{x \in \mathbb{R}^n : 2^{j-1} < w(x) \le 2^j\}$, $\forall\, j \in \mathbb{Z}$, satisfy $E_k = \bigcup_{j \in \mathbb{Z}}(E_k \cap G_j)$ and, more importantly, we obtain the following inequality

$$\sum_{j \in \mathbb{N}} \mathcal{H}^\delta_\infty(E_k \cap G_j) \lesssim \mathcal{H}^\delta_\infty(E_k),$$

which may be invalid when $\delta < n$ for an arbitrary sequence $\{G_j\}_{j \in \mathbb{Z}}$.

As another obstacle for the case $\delta < n$, it is typically invalid that there exists a constant K such that if $Q_1, \ldots, Q_m$ are non-overlapping dyadic cubes and $f \ge 0$, then

$$\sum_{j=1}^m \int_{Q_j} |f(x)|w(x)\,d\mathcal{H}^\delta_\infty \le K \int_{\cup_{j=1}^m Q_j} |f(x)|w(x)\,d\mathcal{H}^\delta_\infty.$$

This can be shown by subdividing the interval $[0, 1]$ in 2^m equal intervals, with m large enough and taking $f \equiv w \equiv 1$ (see [32, p. 148]). Nevertheless, this inequality, even in the Lebesgue integral setting, is also essential both in establishing the characterization for weighted strong-type and weak-type boundedness. To overcome this difficulty, we propose a weighted packing condition (see Lemma 2.6) and prove a "sparse covering lemma" in the context of Hausdorff contents (see Proposition 2.10). Applying these key techniques, in Proposition 2.12, we show that there exists a positive constant K such that

(1.9) $$\sum_{j \in \mathbb{N}} \int_{Q_j} |f(x)|w(x)\,d\mathcal{H}^\delta_\infty \le K \int_{\cup_{j \in \mathbb{N}} Q_j} |f(x)|w(x)\,d\mathcal{H}^\delta_\infty,$$

where $\{Q_j\}_{j\in\mathbb{N}}$ is a family of non-overlapping dyadic cubes of $\mathbb{R}^n$ satisfying the *weighted packing condition*: there exists a constant $\beta \in (0, \infty)$ such that, for each dyadic cube Q,

$$\sum_{Q_j \subset Q} w_{\mathcal{H}^\delta_\infty}(Q_j) \le \beta\, w_{\mathcal{H}^\delta_\infty}(Q).$$

The inequality (1.9) plays a central role in partial substituting the linear property on the weighted Choquet integrals and hence we can prove Theorems 1.1 and 1.2. Here, we point out that the "sparse covering lemma" established in Proposition 2.10 seems to be new even when reduced to the classical setting, i.e., $\delta = n$, with independent significance and potential applicability to other contexts.

As an application, we obtain the following weighted norm inequalities for classical Hardy-Littlewood maximal operators $\mathcal{M}$ on Choquet integrals by an interpolation argument.

Corollary 1.6. *Let $\delta \in (0, n]$.*

(i) *If $w \in \mathcal{A}_{p,\delta}$ with $p \in (1, \infty)$ and $q \in [\frac{p\delta}{n}, \infty)$, then there exists a positive constant K such that*

$$\int_{\mathbb{R}^n} |\mathcal{M}f(x)|^q w(x)\, d\mathcal{H}^\delta_\infty \le K \int_{\mathbb{R}^n} |f(x)|^q w(x)\, d\mathcal{H}^\delta_\infty.$$

(ii) *If $w \in \mathcal{A}_{1,\delta}$ and $q \in (\frac{\delta}{n}, \infty)$, then there exists a positive constant K such that*

$$\int_{\mathbb{R}^n} |\mathcal{M}f(x)|^q w(x)\, d\mathcal{H}^\delta_\infty \le K \int_{\mathbb{R}^n} |f(x)|^q w(x)\, d\mathcal{H}^\delta_\infty.$$

(iii) *If $w \in \mathcal{A}_{p,\delta}$ with $p \in [1, \infty)$ and $q \in [\frac{p\delta}{n}, \infty)$, then there exists a positive constant K such that*

$$w_{\mathcal{H}^\delta_\infty}(\{x \in \mathbb{R}^n : \mathcal{M}f(x) > t\}) \le \frac{K}{t^q} \int_{\mathbb{R}^n} |f(x)|^q w(x)\, d\mathcal{H}^\delta_\infty.$$

The second objective of this paper is applying the new approach developed in this paper to further study the properties of capacitary Muckenhoupt weight class $\mathcal{A}_{p,\delta}$ with full range $p \in [1, \infty)$ and $\delta \in (0, n]$. To this end, we first establish the reverse Hölder inequality for $\mathcal{A}_{p,\delta}$—the deepest and most significant part of the whole theory—which naturally extends the seminal inequality of R. R. Coifman and C. Fefferman [8, Theorem IV]. It should be mentioned that reverse Hölder inequalities also serve an important role in such diverse areas as quasiconformal mappings (see, for example, F. W. Gehring [17]) and certain refined estimates for elliptic partial differential equations (see, for example, E. Fabes, D. Jerison and C. Kenig [12] and also R. Fefferman [14]). Indeed, the reverse Hölder inequality appeared in the renowned work of F. W. Gehring [17] in the following context: If F is a quasiconformal homeomorphism fron $\mathbb{R}^n$ into itself, then $|\det(\nabla F)|$ satisfies a reverse Hölder inequality.

The validity of the reverse Hölder inequality as follows for $w \in \mathcal{A}_{p,\delta}$ requires the weight w to be $\mathcal{H}^\delta_\infty$-quasicontinuous, which ensures that the class of bounded continuous functions is dense in $L^1(E, \mathcal{H}^\delta_\infty)$ with $\delta \in (0, n]$ and $E \subset \mathbb{R}^n$. This condition is natural (once this condition is removed, the density breaks down; see [1, p. 15, Proposition 1]) and we also show its necessity in some sense through Counterexample 4 below. For the precise definitions of $\mathcal{H}^\delta_\infty$-quasicontinuous, we refer the reader to Definition 4.1 below.

Theorem 1.7. *(The reverse Hölder inequality) Let $\delta \in (0, n]$, $p \in [1, \infty)$ and $w \in \mathcal{A}_{p,\delta}$. If w is $\mathcal{H}_\infty^\delta$-quasicontinuous, then there exists positive constants $K = K(n, \delta, p, [w]_{\mathcal{A}_{p,\delta}})$ and $\gamma = \gamma(n, \delta, p, [w]_{\mathcal{A}_{p,\delta}})$ such that, for every cube Q,*

$$(1.10) \qquad \left[\frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q w(x)^{1+\gamma} \, d\mathcal{H}_\infty^\delta \right]^{\frac{1}{1+\gamma}} \le \frac{K}{\mathcal{H}_\infty^\delta(Q)} \int_Q w(x) \, d\mathcal{H}_\infty^\delta.$$

We should point out employing this reverse Hölder inequality, one can also show the above Theorems 1.1 and 1.2 as done in [8]. Even along this route, one must still overcome the difficulties arising from the lack of the Fubini theorem and the nonlinearity of Choquet integrals, as discussed above.

Having established the crucial reverse Hölder inequality for the capacitary Muckenhoupt weight class $\mathcal{A}_{p,\delta}$, we now proceed to some important applications. Among them, the first result yields that if $w \in \mathcal{A}_{p,\delta}$ and w is $\mathcal{H}_\infty^\delta$-quasicontinuous, then automatically $w^{1+\gamma} \in \mathcal{A}_{p,\delta}$ for some $\gamma > 0$.

Corollary 1.8. *Let $\delta \in (0, n]$, $p \in [1, \infty)$ and $w \in \mathcal{A}_{p,\delta}$. If w is $\mathcal{H}_\infty^\delta$-quasicontinuous, then there exists a positive constant γ such that $w^{1+\gamma} \in \mathcal{A}_{p,\delta}$.*

Building upon the reverse Hölder inequality, we infer the following important self-improving property of $\mathcal{A}_{p,\delta}$, which generalizes the openness property of the classical Muckenhoupt A_p class originating from B. Muckenhoupt [30].

Theorem 1.9. *(Self-improving property, openness property) Let $\delta \in (0, n]$, $p \in (1, \infty)$ and $w \in \mathcal{A}_{p,\delta}$. If w is $\mathcal{H}_\infty^\delta$-quasicontinuous, then there is a $q = q(n, \delta, p, [w]_{\mathcal{A}_{p,\delta}})$ with $1 < q < p$ such that $w \in \mathcal{A}_{q,\delta}$.*

An application of Theorem 1.1 enables us to build up a complete analogue of the Jones factorization theorem within the $\mathcal{A}_{p,\delta}$ framework, which is a non-trivial extension of the celebrated result developed by P. W. Jones in [21].

Theorem 1.10. *(The Jones factorization theorem) Let $\delta \in (0, n]$ and $p \in [1, \infty)$. Then $w \in \mathcal{A}_{p,\delta}$ if and only if there exist two weights w_0, $w_1 \in \mathcal{A}_{1,\delta}$ such that $w = w_0 w_1^{1-p}$.*

Remark 1.11. We emphasize that the classical methods fail to establish the last three theorems due to bad properties of the Hausdorff content. Instead, we employ the new approach developed in the proofs of Theorems 1.1 and 1.2. Therefore, these theorems naturally extend the classical theory of [8, 21, 30], and simultaneously their proofs provide novel proofs—even when reduced to the classical Muckenhoupt A_p weight case (i.e., $\delta = n$)—that avoid the use of the Fubini theorem, the countable additivity of measures and linearity of the integral.

Finally, combining Theorems 1.1 and 1.7, we characterize the boundedness of the operator $\mathcal{M}_{\mathcal{H}_\infty^\delta}$ on the weak weighted Choquet-Lebesgue space $L_w^{p,\infty}(\mathbb{R}^n, \mathcal{H}_\infty^\delta)$ via capacitary Muckenhoupt weight class $\mathcal{A}_{p,\delta}$ as follows. To the best of our knowledge, this result appears to be new even when restricted to the classical Muckenhoupt A_p weight setting.

Theorem 1.12. *Let $\delta \in (0, n]$, $p \in (1, \infty)$ and w be a $\mathcal{H}_\infty^\delta$-quasicontinuous weight on $\mathbb{R}^n$. Then the maximal operator*

$$\mathcal{M}_{\mathcal{H}_\infty^\delta} : L_w^{p,\infty}(\mathbb{R}^n, \mathcal{H}_\infty^\delta) \to L_w^{p,\infty}(\mathbb{R}^n, \mathcal{H}_\infty^\delta)$$

is bounded, i.e., there exists a positive constant K such that, for any $f \in L_w^{p,\infty}(\mathbb{R}^n, \mathcal{H}_\infty^\delta)$,

$$(1.11) \qquad \left\|\mathcal{M}_{\mathcal{H}_\infty^\delta} f\right\|_{L_w^{p,\infty}(\mathbb{R}^n, \mathcal{H}_\infty^\delta)} \leq K \|f\|_{L_w^{p,\infty}(\mathbb{R}^n, \mathcal{H}_\infty^\delta)}$$

if and only if $w \in \mathcal{A}_{p,\delta}$.

The paper is organized as follows. In Section 2, we mainly give some fundamental properties of Choquet integrals, capacitary Muckenhoupt weights and several required lemmas in the following proofs. In particular, we give an example of capacitary Muckenhoupt $\mathcal{A}_{p,\delta}$-weight function and show that the new weight class $\mathcal{A}_{p,\delta}$ enjoys the strict monotonicity on the dimension index δ of Hausdorff contents in the end of Section 2. In Section 3, we prove Theorem 1.1, Theorem 1.2 and Corollary 1.6. Section 4 is devoted to showing the reverse Hölder inequality and self-improving property for the $\mathcal{A}_{p,\delta}$-weight. To achieve this, we establish a estimate for the capacity of a upper lever set for weight $w \in \mathcal{A}_{p,\delta}$ (see Lemma 4.9) by using the Calderón-Zygmund decomposition corresponding to Hausdorff contents. As applications, in Section 5, we build up the Jones factorization theorem within the $\mathcal{A}_{p,\delta}$ framework and show Theorem 1.12.

Throughout the paper, the notation $f \lesssim g$ (resp. $f \gtrsim g$) means $f \leq Kg$ (resp. $f \geq Kg$) for a positive constant K independent of the main parameters, and $f \sim g$ amounts to $f \lesssim g \lesssim f$. We also use the symbol $K(\alpha, \beta, \dots)$ to denote a positive constant which depends on the parameters $\alpha, \beta, \dots$ but may vary line to line. Also, we denote by $\mathbf{1}_E$ the characteristic function of set $E \subset \mathbb{R}^n$.

2   Preliminaries and Fundamental Tools

In this section, we first recall the definition of Choquet integral for a general capacity and then give some useful and basic properties with respect to Choquet integrals in Subsection 2.1. Then, via weighted packing lemma and sparse covering property, a substitute for the Fubini theorem and a substitute for the linearity of Choquet integrals are established in Subsection 2.2 and Subsection 2.3, respectively. Finally, we give an example of capacitary Muckenhoupt $\mathcal{A}_{p,\delta}$-weight function and show that the new weight class $\mathcal{A}_{p,\delta}$ enjoys the strict monotonicity both on the first index p and the dimension index δ of Hausdorff contents in the end of this section.

2.1   Capacity and Choquet Integrals

A real-valued set function C, defined on all subsets of $\mathbb{R}^n$, is called a capacity, if it satisfies the following conditions:

 (i) $C(\emptyset) = 0$ and, for any set $E \subset \mathbb{R}^n$, $C(E) \geq 0$;

 (ii) If $E \subset F$, then $C(E) \leq C(F)$;

 (iii) If $E = \bigcup_{i=1}^\infty E_i$, then $C(E) \leq \sum_{i=1}^\infty C(E_i)$,

i.e., C is a non-negative, monotone and countably subadditive set function.

Let C be a capacity. Then, for any subset $E \subset \mathbb{R}^n$, the *Choquet integral* of a non-negative function f on E is defined by setting

$$\int_E f(x)\, dC := \int_0^\infty C(\{x \in E : f(x) > \lambda\})\, d\lambda.$$

Since C is monotone, then, for any function g defined on E, its distribution function in the sense of the capacity

$$\lambda \mapsto C(\{x \in E : |g(x)| > \lambda\})$$

is decreasing on $\lambda \in [0, \infty)$. Thus, we easily find that the above distribution function is measurable with respect to the Lebesgue measure. Therefore,

$$\int_0^\infty C(\{x \in E : |g(x)| > \lambda\}) \, d\lambda$$

is a well-defined Lebesgue integral and hence the Choquet integral is also well defined. Based on this, for any $p \in (0, \infty)$, the *p-Choquet integral with respect to the capacity* of a function f on E is defined by setting

$$\|f\|_{L^p(E,C)} := \left[\int_E |f(x)|^p \, dC \right]^{\frac{1}{p}}.$$

In what follows, for any $p \in (0, \infty)$ and $E \subset \mathbb{R}^n$, we use $L^p(E, C)$ to denote the space of all functions f on E such that the quasi-norm $\|f\|_{L^p(E,C)} < \infty$, and use $L^\infty(E, C)$ to denote the space of all functions f on E such that the quasi-norm

$$\|f\|_{L^\infty(E,C)} := \inf_{t>0} \{t : C(\{x \in E : |f(x)| > t\}) = 0\} < \infty.$$

Denote by $L^1_{\mathrm{loc}}(\mathbb{R}^n, C)$ the set of all functions satisfying that, for any compact set $E \subset \mathbb{R}^n$, $f \in L^1(E, C)$. For a capacity C and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n, C)$, the *capacitary Hardy-Littlewood maximal function* $\mathcal{M}_C f$ of f is defined by setting, for any $x \in \mathbb{R}^n$,

$$\mathcal{M}_C f(x) := \sup_{Q \ni x} \frac{1}{C(Q)} \int_Q |f(y)| \, dC,$$

where the supremum is taken over all cubes Q of $\mathbb{R}^n$ containing x (see [5]), which seems to be more suitable for studying problems in capacitary setting.

The following basic properties of the Choquet integral can be found in [1].

Remark 2.1.   (i) (The Hölder inequality) Let C be a capacity and $p \in (1, \infty)$. Then, for any functions f and g on $\mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} |f(x)g(x)| \, dC \le 2 \left(\int_{\mathbb{R}^n} |f(x)|^p \, dC \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |g(x)|^q \, dC \right)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

(ii) Let C be a capacity, $N \in \mathbb{N}$, $E \subset \mathbb{R}^n$, and $\{f_j\}_{j=1}^N$ be a sequence of functions defined on E. Then

$$\int_E \left| \sum_{j=1}^N f_j(x) \right| dC \le N \sum_{j=1}^N \int_E |f_j(x)| \, dC.$$

Remark 2.2. Let C be a capacity and $E \subset \mathbb{R}^n$. For any $f \in L^\infty(E, C)$, it is not difficult to obtain

$$\|f\|_{L^\infty(E,C)} = \inf_{C(E_0)=0} \left\{ \sup_{x \in E \setminus E_0} |f(x)| \right\},$$

and, moreover, there exists a subset $E_f \subset E$ with $C(E_f) = 0$ such that

$$\|f\|_{L^\infty(E,C)} = \sup_{x \in E \setminus E_f} |f(x)|.$$

We point out that, compared with Riemann or Lebesgue integrals, the Choquet integral with respect to the Hausdorff content $\mathcal{H}^\delta_\infty$ has the following significant differences:

(i) The Choquet integral is a nonlinear integral, that is, for any non-negative functions f and g on E,

$$
\begin{aligned}
(2.1) \qquad \int_E f(x)\, d\mathcal{H}^\delta_\infty + \int_E g(x)\, d\mathcal{H}^\delta_\infty &\neq \int_E [f(x) + g(x)]\, d\mathcal{H}^\delta_\infty \\
&\leq 2 \left[\int_E f(x)\, d\mathcal{H}^\delta_\infty + \int_E g(x)\, d\mathcal{H}^\delta_\infty \right].
\end{aligned}
$$

(ii) Choquet integrals are well defined for all non-measurable functions with respect to the Lebesgue measure.

It is well known that some common examples of capacities include the Hausdorff content, the Riesz capacity and the Bessel capacity (see [1]). In this paper, we focus our attention on the Hausdorff content $\mathcal{H}^\delta_\infty$. Moreover, for any weight w on $\mathbb{R}^n$ and $\delta \in (0, n]$, it is not hard to see that $w_{\mathcal{H}^\delta_\infty}$ as in (1.5) is also a capacity.

Remark 2.3. Let $\delta \in (0, n]$.

(i) For any cube Q of $\mathbb{R}^n$, we have $\mathcal{H}^\delta_\infty(Q) = [l(Q)]^\delta$ and, for any ball $B(x, r)$ with $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, $\mathcal{H}^\delta_\infty(B(x, r)) \sim r^\delta$. Generally, for any $k \in \{1, 2, \cdots, n-1\}$, if W is a k-dimensional cube with side length l in $\mathbb{R}^n$, then

$$\mathcal{H}^\delta_\infty(W) = \begin{cases} l^\delta, & \text{when } \delta \in (0, k], \\ 0, & \text{when } \delta \in (k, n]; \end{cases}$$

moreover, if Q is a n-dimensional cube with side length l in $\mathbb{R}^n$ satisfying $W \subset Q$, then, for $\delta \in (0, k]$ and any subset E of $\mathbb{R}^n$ with $W \subset E \subset Q$, we have

$$\mathcal{H}^\delta_\infty(E) = l^\delta.$$

Indeed, when $\delta \in (0, k]$, obviously $\mathcal{H}^\delta_\infty(W) \leq l^\delta$. On the other hand, for any sequence $\{Q_j\}_j$ of cubes in $\mathbb{R}^n$ satisfying $W \subset \bigcup_j Q_j$, we have $l^k \leq \sum_j [l(Q_j)]^k$, which implies $l^\delta \leq \sum_j [l(Q_j)]^\delta$. Therefore, $\mathcal{H}^\delta_\infty(W) \geq l^\delta$ and consequently $\mathcal{H}^\delta_\infty(W) = l^\delta$. When $\delta \in (k, n]$, observing that, for any $m \in \mathbb{N}$, there exists a sequence $\{P_j\}_{j=1}^{m^k}$ of cubes in $\mathbb{R}^n$ with side length $\frac{l}{m}$ such that $W \subset \bigcup_{j=1}^{m^k} P_j$, we find that $\mathcal{H}^\delta_\infty(W) \leq m^{k-\delta} l^\delta$ and hence $\mathcal{H}^\delta_\infty(W) = 0$ by taking $m \to \infty$.

(ii) Let $\widetilde{\mathcal{H}}_{\infty}^{\delta}$ be the *dyadic Hausdorff content*, which is defined by the same way as $\mathcal{H}_{\infty}^{\delta}$ but with cubes coverings $\{Q_i\}_i$ replaced by dyadic cubes coverings. Then there exists a positive constant $K(n, \delta)$ such that, for any subset $E \subset \mathbb{R}^n$,

$$\mathcal{H}_{\infty}^{\delta}(E) \leq \widetilde{\mathcal{H}}_{\infty}^{\delta}(E) \leq K(n, \delta)\mathcal{H}_{\infty}^{\delta}(E).$$

This equivalent Hausdorff content was proved to be a capacity in the sense of Choquet; see [2] for $\delta \in (n-1, n]$ and [37] for $\delta \in (0, n-1]$.

(iii) By the definition of $\mathcal{H}_{\infty}^{\delta}$ and the Choquet integral, we have, for any sequence $\{E_j\}_{j \in \mathbb{N}}$ of subset in $\mathbb{R}^n$,

$$\int_{\bigcup_{j \in \mathbb{N}} E_j} |f(x)| \, d\mathcal{H}_{\infty}^{\delta} \leq \sum_{j \in \mathbb{N}} \int_{E_j} |f(x)| \, \mathcal{H}_{\infty}^{\delta}.$$

(iv) By (2.1), we know that the Choquet integral with respect to $\mathcal{H}_{\infty}^{\delta}$ is not sub-linear. However, using the equivalence of $\mathcal{H}_{\infty}^{\delta}$ and $\widetilde{\mathcal{H}}_{\infty}^{\delta}$, and the sub-linearity of the Choquet integral with respect to $\widetilde{\mathcal{H}}_{\infty}^{\delta}$ (see [1, p. 13, Theorem 1]), we conclude that there exists a positive constant $K(n, \delta)$ such that, for any non-negative functions $\{f_j\}_{j \in \mathbb{N}}$ defined on E,

$$\int_{E} \sum_{j \in \mathbb{N}} f_j(x) \, d\mathcal{H}_{\infty}^{\delta} \leq K(n, \delta) \sum_{j \in \mathbb{N}} \int_{E} f_j(x) \, d\mathcal{H}_{\infty}^{\delta}.$$

(v) For any weight w and subset $E \subset \mathbb{R}^n$, we define the *weighted Choquet-Lebesgue space* $L_w^p(E, \mathcal{H}_{\infty}^{\delta})$, with $p \in [1, \infty)$, as the set of all functions f satisfying

$$\|f\|_{L_w^p(E, \mathcal{H}_{\infty}^{\delta})} := \left\{ \int_{E} |f(x)|^p w(x) \, d\mathcal{H}_{\infty}^{\delta} \right\}^{\frac{1}{p}} < \infty,$$

and the *weak Choquet-Lebesgue space* $L_w^{p,\infty}(E, \mathcal{H}_{\infty}^{\delta})$ as the set of all functions f satisfying

$$\|f\|_{L_w^{p,\infty}(E, \mathcal{H}_{\infty}^{\delta})} := \sup_{t \in (0,\infty)} t \left(w_{\mathcal{H}_{\infty}^{\delta}}(\{x \in E : |f(x)| > t\}) \right)^{\frac{1}{p}} < \infty,$$

where $w_{\mathcal{H}_{\infty}^{\delta}}$ is as in (1.5). Also, we define the *space* $L_w^{\infty}(E, \mathcal{H}_{\infty}^{\delta})$ as the collection of all functions f satisfying

$$\|f\|_{L_w^{\infty}(E, \mathcal{H}_{\infty}^{\delta})} := \inf(\{t > 0 : w_{\mathcal{H}_{\infty}^{\delta}}(\{x \in E : |f(x)| > t\}) = 0\}) < \infty.$$

2.2 Weighted Packing Condition and Substitute for the Fubini Theorem

In this subsection, by proposing a weighted packing condition, we mainly prove, in Proposition 2.7, that the weighted Choquet integral can be equivalently viewed as a Choquet integral with respect to the new capacity $w_{\mathcal{H}_{\infty}^{\delta}}$. We begin with the following doubling properties of the capacitary Muckenhoupt weight.

Lemma 2.4. *(Doubling property) Let $\delta \in (0, n]$ and $p \in [1, \infty)$. If $w \in \mathcal{A}_{p,\delta}$, then for any cube Q of $\mathbb{R}^n$ and any subset $E \subset Q$,*

$$(2.2) \qquad \left[\frac{\mathcal{H}_{\infty}^{\delta}(E)}{\mathcal{H}_{\infty}^{\delta}(Q)} \right]^p \leq 2^p [w]_{\mathcal{A}_{p,\delta}} \frac{w_{\mathcal{H}_{\infty}^{\delta}}(E)}{w_{\mathcal{H}_{\infty}^{\delta}}(Q)};$$

moreover, for any $t \in [1, \infty)$, we have

$$w_{\mathcal{H}_\infty^\delta}(tQ) \leq 2^p [w]_{\mathcal{A}_{p,\delta}} t^{p\delta} w_{\mathcal{H}_\infty^\delta}(Q).$$

Proof. Let $w \in \mathcal{A}_{1,\delta}$. Then by definition, for any cube Q of $\mathbb{R}^n$ and $\mathcal{H}_\infty^\delta$-almost everywhere $x \in Q$, we have

$$\frac{w_{\mathcal{H}_\infty^\delta}(Q)}{\mathcal{H}_\infty^\delta(Q)} \leq [w]_{\mathcal{A}_{1,\delta}} w(x).$$

Then, for any subset $E \subset Q$, by taking Choquet integral over E with respect to $\mathcal{H}_\infty^\delta$, we obtain (2.2) in this case.

When $w \in \mathcal{A}_{p,\delta}$ with $p \in (1, \infty)$, applying the Hölder inequality, i.e., Remark 2.1(i), we find that, for any function f on $\mathbb{R}^n$,

$$\left(\frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q |f(x)| \, d\mathcal{H}_\infty^\delta \right)^p$$

$$\leq 2^p \frac{1}{[\mathcal{H}_\infty^\delta(Q)]^p} \left[\int_Q |f(x)|^p w(x) \, d\mathcal{H}_\infty^\delta \right] \left[\int_Q w(x)^{-\frac{1}{p-1}} \, d\mathcal{H}_\infty^\delta \right]^{p-1}$$

$$= 2^p \left\{ \frac{1}{w_{\mathcal{H}_\infty^\delta}(Q)} \int_Q |f(x)|^p w(x) \, d\mathcal{H}_\infty^\delta \right\} \left[\frac{w_{\mathcal{H}_\infty^\delta}(Q)}{\mathcal{H}_\infty^\delta(Q)} \right] \left\{ \frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q w(x)^{-\frac{1}{p-1}} \, d\mathcal{H}_\infty^\delta \right\}^{p-1}$$

$$\leq 2^p [w]_{\mathcal{A}_{p,\delta}} \frac{1}{w_{\mathcal{H}_\infty^\delta}(Q)} \int_Q |f(x)|^p w(x) \, d\mathcal{H}_\infty^\delta.$$

Applying this, we then obtain (2.2) by letting $f := \mathbf{1}_E$ for any subset $E \subset Q$.

On the other hand, observe that, for any cube Q of $\mathbb{R}^n$ and $t \in (0, \infty)$, $\mathcal{H}_\infty^\delta(tQ) = t^\delta \mathcal{H}_\infty^\delta(Q)$ due to Remark 2.3(i). Then it follows from (2.2) that, when $w \in \mathcal{A}_{p,\delta}$ with $p \in [1, \infty)$ and $t \in [1, \infty)$,

$$w_{\mathcal{H}_\infty^\delta}(tQ) \leq 2^p [w]_{\mathcal{A}_{p,\delta}} t^{p\delta} w_{\mathcal{H}_\infty^\delta}(Q).$$

This finishes the proof of Lemma 2.4. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Lemma 2.5. *Let $\delta \in (0, n]$, $\alpha \in (0, \infty)$, $w \in \mathcal{A}_{1,\delta}$, and Q be a cube of $\mathbb{R}^n$. If $\{E_k\}_{k \in \mathbb{N}}$ is a sequence of non-overlapping subsets of Q satisfying*

$$(2.3) \qquad\qquad\qquad \sum_{k \in \mathbb{N}} w_{\mathcal{H}_\infty^\delta}(E_k) \leq \alpha w_{\mathcal{H}_\infty^\delta}(Q),$$

then it holds that

$$\sum_{k \in \mathbb{N}} \mathcal{H}_\infty^\delta(E_k) \leq \alpha [w]_{\mathcal{A}_{1,\delta}} \mathcal{H}_\infty^\delta(Q).$$

Proof. By the definition of $\mathcal{A}_{1,\delta}$, it is not difficult to know that, for the given cube Q,

$$(2.4) \qquad\qquad\qquad w_{\mathcal{H}_\infty^\delta}(Q) \|w^{-1}\|_{L^\infty(Q, \mathcal{H}_\infty^\delta)} \leq [w]_{\mathcal{A}_{1,\delta}} \mathcal{H}_\infty^\delta(Q).$$

Then, by (2.3), we obtain

$$\sum_{k \in \mathbb{N}} \mathcal{H}_\infty^\delta(E_k) = \sum_{k \in \mathbb{N}} \int_{E_k} 1 \, d\mathcal{H}_\infty^\delta \leq \sum_{k \in \mathbb{N}} \int_{E_k} w(x) \, d\mathcal{H}_\infty^\delta \|w^{-1}\|_{L^\infty(E_k, \mathcal{H}_\infty^\delta)}$$

$$\leq \sum_{k \in \mathbb{N}} \int_{E_k} w(x) \, d\mathcal{H}_\infty^\delta \|w^{-1}\|_{L^\infty(Q, \mathcal{H}_\infty^\delta)} \leq \alpha w_{\mathcal{H}_\infty^\delta}(Q) \|w^{-1}\|_{L^\infty(Q, \mathcal{H}_\infty^\delta)}$$

$$\leq \alpha [w]_{\mathcal{A}_{1,\delta}} \mathcal{H}_\infty^\delta(Q),$$

which completes the proof of Lemma 2.5. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

We remark that it is interesting to clarify whether or not the corresponding conclusion in Lemma 2.5 holds true for $w \in \mathcal{A}_{p,\delta}$ with $p \in (1, \infty)$. The weighted packing condition is stated as follows. It is a generalization of J. Orobitg and J. Verdera [32, Lemma 2] and is of vital importance in this paper.

Lemma 2.6. *Let $\delta \in (0, n]$ and w be a weight. Assume that $\{Q_j\}_{j \in \mathbb{N}}$ is a family of non-overlapping dyadic cubes of $\mathbb{R}^n$, then there exists a subfamily $\{Q_{j_v}\}_{v=1}^N$, with positive integer $N < \infty$ or $N = \infty$, satisfying*

(i) *for any dyadic cubes Q,*

$$\sum_{Q_{j_v} \subset Q} w_{\mathcal{H}_\infty^\delta}(Q_{j_v}) \le 2 w_{\mathcal{H}_\infty^\delta}(Q);$$

(ii) *the inequality*

$$w_{\mathcal{H}_\infty^\delta}\left(\bigcup_{j \in \mathbb{N}} Q_j\right) \le 2 \sum_{v=1}^N w_{\mathcal{H}_\infty^\delta}(Q_{j_v}).$$

Moreover, if $w \in \mathcal{A}_{p,\delta}$ with $p \in [1, \infty)$, then

(iii) *for a family $\{P_j\}_{j \in \mathbb{N}}$ of non-overlapping dyadic cubes of $\mathbb{R}^n$ satisfying $P_j \subset t Q_j$ with any $j \in \mathbb{N}$ and some $t \in [1, \infty)$, it holds*

$$w_{\mathcal{H}_\infty^\delta}\left(\bigcup_{j \in \mathbb{N}} P_j\right) \le 2^{p+1} [w]_{\mathcal{A}_{p,\delta}} t^{p\delta} \sum_{v=1}^N w_{\mathcal{H}_\infty^\delta}(Q_{j_v}).$$

Proof. Let $j_1 = 1$. Then obviously we have, for any dyadic cube Q of $\mathbb{R}^n$ with $Q_{j_1} \subset Q$,

$$\sum_{Q_{j_v} \subset Q, v \le 1} w_{\mathcal{H}_\infty^\delta}(Q_{j_v}) = w_{\mathcal{H}_\infty^\delta}(Q_{j_1}) \le 2 w_{\mathcal{H}_\infty^\delta}(Q).$$

Now we assume that $\{j_1, j_2, \ldots, j_k\}$ have been chosen such that, for any dyadic cube Q,

$$\sum_{Q_{j_v} \subset Q, v \le k} w_{\mathcal{H}_\infty^\delta}(Q_{j_v}) \le 2 w_{\mathcal{H}_\infty^\delta}(Q).$$

Then j_{k+1} is defined to be the first index from $\{j_k + 1, j_k + 2, \ldots\}$ satisfying that, for any dyadic cube Q,

$$\sum_{Q_{j_v} \subset Q, v \le k+1} w_{\mathcal{H}_\infty^\delta}(Q_{j_v}) \le 2 w_{\mathcal{H}_\infty^\delta}(Q).$$

By induction, we obtain a subfamily $\{Q_{j_v}\}_{v=1}^N$ satisfying (i).

Next, we prove (ii). For any $Q_k \in \{Q_j\}_{j=1}^\infty \setminus \{Q_{j_v}\}_{v=1}^N$, we may assume that $j_m < k < j_{m+1}$ for some $m \in \mathbb{N}$. Then, by the proof of (i), we find that there exists a dyadic cube Q_k^* such that $Q_k \subset Q_k^*$ and

$$\sum_{Q_{j_v} \subset Q_k^*, v \le m} w_{\mathcal{H}_\infty^\delta}(Q_{j_v}) + w_{\mathcal{H}_\infty^\delta}(Q_k) > 2 w_{\mathcal{H}_\infty^\delta}(Q_k^*).$$

Thus,

$$w_{\mathcal{H}_\infty^\delta}(Q_k^*) \le \sum_{Q_{j_v} \subset Q_k^*, v \le m} w_{\mathcal{H}_\infty^\delta}(Q_{j_v}) \le \sum_{Q_{j_v} \subset Q_k^*} w_{\mathcal{H}_\infty^\delta}(Q_{j_v}).$$

Consider the maximal dyadic cubes of the family $\{Q_k^*\}_{k\in\mathbb{N}\setminus(\{j_v\}_{v=1}^N)}$, denoted by $\{\widetilde{Q}_k^*\}_k$. Then

$$(2.5) \qquad \bigcup_{j=1}^{\infty} Q_j \subset \left(\bigcup_{v=1}^{N} Q_{jv}\right) \bigcup \left(\bigcup_{k} \widetilde{Q}_k^*\right);$$

moreover, by the maximality of $\{\widetilde{Q}_k^*\}_k$ and the properties of dyadic cubes, we know that

$$(2.6) \qquad \sum_{k} w_{\mathcal{H}_\infty^\delta}(\widetilde{Q}_k^*) \le \sum_{k} \sum_{Q_{jv}\subset\widetilde{Q}_k^*} w_{\mathcal{H}_\infty^\delta}(Q_{jv}) \le \sum_{v=1}^{N} w_{\mathcal{H}_\infty^\delta}(Q_{jv}).$$

From this and (2.5), we infer that

$$w_{\mathcal{H}_\infty^\delta}\left(\bigcup_{j=1}^{\infty} Q_j\right) \le \sum_{v=1}^{N} w_{\mathcal{H}_\infty^\delta}(Q_{jv}) + \sum_{k} w_{\mathcal{H}_\infty^\delta}(\widetilde{Q}_k^*) \le 2\sum_{v=1}^{N} w_{\mathcal{H}_\infty^\delta}(Q_{jv}).$$

Therefore, (ii) is proved.

For (iii), applying Lemma 2.4, (2.5) and (2.6), we have

$$
\begin{aligned}
w_{\mathcal{H}_\infty^\delta}\left(\bigcup_{j\in\mathbb{N}} P_j\right) &\le w_{\mathcal{H}_\infty^\delta}\left(\bigcup_{j\in\mathbb{N}} tQ_j\right) \le \sum_{v=1}^{N} w_{\mathcal{H}_\infty^\delta}(tQ_{jv}) + \sum_{k} w_{\mathcal{H}_\infty^\delta}(t\widetilde{Q}_k^*) \\
&\le 2^p[w]_{\mathcal{A}_{p,\delta}} t^{p\delta} \left(\sum_{v=1}^{N} w_{\mathcal{H}_\infty^\delta}(Q_{jv}) + \sum_{k} w_{\mathcal{H}_\infty^\delta}(\widetilde{Q}_k^*)\right) \\
&\le 2^{p+1}[w]_{\mathcal{A}_{p,\delta}} t^{p\delta} \sum_{v=1}^{N} w_{\mathcal{H}_\infty^\delta}(Q_{jv}).
\end{aligned}
$$

This finishes the proof of Lemma 2.6. $\square$

The following proposition plays a pivotal role throughout the paper, which fills the gap left by the absence of the Fubini theorem in the weighted Choquet integral setting.

Proposition 2.7. *Let $\delta \in (0,n]$, $p \in [1,\infty)$ and $w \in \mathcal{A}_{p,\delta}$. Then there exists a positive constant $K(p)$ such that, for any $f \in L_w^1(\mathbb{R}^n, \mathcal{H}_\infty^\delta)$,*

$$\frac{1}{4}\int_{\mathbb{R}^n} |f(x)|w(x)\,d\mathcal{H}_\infty^\delta \le \int_{\mathbb{R}^n} |f(x)|\,dw_{\mathcal{H}_\infty^\delta} \le K(p)[w]_{\mathcal{A}_{p,\delta}}^{\frac{1}{p}} \int_{\mathbb{R}^n} |f(x)|w(x)\,d\mathcal{H}_\infty^\delta.$$

Proof. We prove this proposition by two steps. To this end, for any $j \in \mathbb{Z}$, let

$$F_j := \{x \in \mathbb{R}^n : \ 2^{j-1} < |f(x)| \le 2^j\}.$$

Step 1) We first show that

$$(2.7) \qquad \int_{\mathbb{R}^n} |f(x)|w(x)\,d\mathcal{H}_\infty^\delta \le \sum_{j\in\mathbb{Z}} \int_{F_j} |f(x)|w(x)\,d\mathcal{H}_\infty^\delta \le K(p)[w]_{\mathcal{A}_{p,\delta}}^{\frac{1}{p}} \int_{\mathbb{R}^n} |f(x)|w(x)\,d\mathcal{H}_\infty^\delta.$$

This is a consequence of

$$(2.8) \qquad \sum_{j \in \mathbb{Z}} \int_{F_j} |f(x)| w(x) \, d\mathcal{H}^\delta_\infty \leq K(p)[w]_{\mathcal{A}_{p,\delta}}^{\frac{1}{p}} \int_{\mathbb{R}^n} |f(x)| w(x) \, d\mathcal{H}^\delta_\infty,$$

since the first inequality of (2.7) is obvious by the facts that

$$\mathbb{R}^n = \bigcup_{j \in \mathbb{Z}} F_j \bigcup \{x \in \mathbb{R}^n : |f(x)| = \infty\} \bigcup \{x \in \mathbb{R}^n : |f(x)| = 0\},$$

and $\mathcal{H}^\delta_\infty(\{x \in \mathbb{R}^n : |f(x)| = \infty\}) = 0$.

Define

$$G_i := \{x \in \mathbb{R}^n : 2^{i-1} < w(x) \leq 2^i\}, \quad \forall i \in \mathbb{Z}$$

and

$$E_k := \{x \in \mathbb{R}^n : 2^{k-2} < |f(x)| w(x) \leq 2^k\}, \quad \forall k \in \mathbb{Z}.$$

Then we have

$$(2.9) \qquad \sum_{k \in \mathbb{Z}} 2^k \mathcal{H}^\delta_\infty(E_k) = 8 \sum_{k \in \mathbb{Z}} \int_{2^{k-3}}^{2^{k-2}} \mathcal{H}^\delta_\infty(\{x \in E_k : |f(x) w(x)| > t\}) \, dt$$

$$\leq 8 \int_{\mathbb{R}^n} |f(x)| w(x) \, d\mathcal{H}^\delta_\infty.$$

Moreover, for any $j \in \mathbb{Z}$,

$$F_j \subset \bigcup_{k \in \mathbb{Z}} (E_k \cap G_{k-j}) \bigcup \{x \in \mathbb{R}^n : w(x) = 0\} \bigcup \{x \in \mathbb{R}^n : w(x) = \infty\}$$

and

$$\mathcal{H}^\delta_\infty(\{x \in \mathbb{R}^n : w(x) = 0\}) = 0 = \mathcal{H}^\delta_\infty(\{x \in \mathbb{R}^n : w(x) = \infty\}).$$

Thus, we have

$$(2.10) \qquad \sum_{j \in \mathbb{Z}} \int_{F_j} |f(x)| w(x) \, d\mathcal{H}^\delta_\infty \leq \sum_{j \in \mathbb{Z}} 2^j w_{\mathcal{H}^\delta_\infty}(F_j) \leq \sum_{j \in \mathbb{Z}} 2^j \sum_{k \in \mathbb{Z}} w_{\mathcal{H}^\delta_\infty}(E_k \cap G_{k-j})$$

$$= \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \int_{E_k \cap G_{k-j}} 2^j w(x) \, d\mathcal{H}^\delta_\infty$$

$$\leq \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \int_{E_k \cap G_{k-j}} 2^j \cdot 2^{k-j} \, d\mathcal{H}^\delta_\infty$$

$$= \sum_{k \in \mathbb{Z}} 2^k \sum_{j \in \mathbb{Z}} \mathcal{H}^\delta_\infty(E_k \cap G_j).$$

Therefore, to prove (2.8), combining (2.9) and (2.10), we only need to verify that, for any $k \in \mathbb{Z}$,

$$(2.11) \qquad \sum_{j \in \mathbb{Z}} \mathcal{H}^\delta_\infty(E_k \cap G_j) \leq K(p)[w]_{\mathcal{A}_{p,\delta}}^{\frac{1}{p}} \mathcal{H}^\delta_\infty(E_k).$$

For any $k \in \mathbb{Z}$, by (2.9), we have $\mathcal{H}_\infty^\delta(E_k) < \infty$. Without loss of generality, we may assume that $\mathcal{H}_\infty^\delta(E_k) > 0$. Then there exists a family $\{Q_{k,m}\}_{m \in \mathbb{N}}$ of cubes of $\mathbb{R}^n$ such that

$$(2.12) \qquad E_k \subset \bigcup_{m \in \mathbb{N}} Q_{k,m}$$

and

$$(2.13) \qquad \sum_{m \in \mathbb{N}} \mathcal{H}_\infty^\delta(Q_{k,m}) \le 2 \mathcal{H}_\infty^\delta(E_k).$$

Furthermore, there exists a positive constant $K(p)$ such that, for any $k \in \mathbb{Z}$ and $m \in \mathbb{N}$,

$$(2.14) \qquad \sum_{j \in \mathbb{Z}} \mathcal{H}_\infty^\delta(Q_{k,m} \cap G_j) \le K(p)[w]_{\mathcal{A}_{p,\delta}}^{\frac{1}{p}} \mathcal{H}_\infty^\delta(Q_{k,m}).$$

Indeed, when $w \in \mathcal{A}_{1,\delta}$, since

$$\sum_{j \in \mathbb{Z}} \int_{Q_{k,m} \cap G_j} w(x) \, d\mathcal{H}_\infty^\delta \le 4 \int_{Q_{k,m}} w(x) \, d\mathcal{H}_\infty^\delta,$$

it follows from Lemma 2.5 that (2.14) is true. When $w \in \mathcal{A}_{p,\delta}$ with $p \in (1, \infty)$, we first have, for any $k \in \mathbb{Z}$ and $m \in \mathbb{N}$,

$$(2.15) \qquad \sum_{j \in \mathbb{Z}} 2^j \mathcal{H}_\infty^\delta(Q_{k,m} \cap G_j) \le 2 \sum_{j \in \mathbb{Z}} \int_{Q_{k,m} \cap G_j} w(x) \, d\mathcal{H}_\infty^\delta \le 8 \int_{Q_{k,m}} w(x) \, d\mathcal{H}_\infty^\delta$$

and

$$\sum_{j \in \mathbb{Z}} 2^{-\frac{j}{p-1}} \mathcal{H}_\infty^\delta(Q_{k,m} \cap G_j) \le \sum_{j \in \mathbb{Z}} \int_{Q_{k,m} \cap G_j} w(x)^{-\frac{1}{p-1}} \, d\mathcal{H}_\infty^\delta \le \frac{2^{\frac{2}{p-1}}}{2^{\frac{1}{p-1}} - 1} \int_{Q_{k,m}} w(x)^{-\frac{1}{p-1}} \, d\mathcal{H}_\infty^\delta.$$

Then, from this, (2.15) and the Hölder inequality with $\frac{1}{p} + \frac{1}{p'} = 1$, we deduce that, for any $k \in \mathbb{Z}$ and $m \in \mathbb{N}$,

$$\sum_{j \in \mathbb{Z}} \mathcal{H}_\infty^\delta(Q_{k,m} \cap G_j)$$

$$\le \left[\sum_{j \in \mathbb{Z}} 2^j \mathcal{H}_\infty^\delta(Q_{k,m} \cap G_j) \right]^{\frac{1}{p}} \left[\sum_{j \in \mathbb{Z}} 2^{-\frac{j}{p-1}} \mathcal{H}_\infty^\delta(Q_{k,m} \cap G_j) \right]^{\frac{1}{p'}}$$

$$\le K(p) \left[\int_{Q_{k,m}} w(x) \, d\mathcal{H}_\infty^\delta \right]^{\frac{1}{p}} \left[\int_{Q_{k,m}} w(x)^{-\frac{1}{p-1}} \, d\mathcal{H}_\infty^\delta \right]^{\frac{1}{p'}}$$

$$\le K(p)[w]_{\mathcal{A}_{p,\delta}}^{\frac{1}{p}} \mathcal{H}_\infty^\delta(Q_{k,m}),$$

which means that (2.14) also holds true for $p \in (1, \infty)$.

Combining (2.12), (2.13) and (2.14), we further conclude that, for any $k \in \mathbb{Z}$,

$$\sum_{j \in \mathbb{Z}} \mathcal{H}_\infty^\delta(E_k \cap G_j) \le \sum_{m \in \mathbb{N}} \sum_{j \in \mathbb{Z}} \mathcal{H}_\infty^\delta(Q_{k,m} \cap G_j)$$

$$\le K(p)[w]_{\mathcal{A}_{p,\delta}}^{\frac{1}{p}} \sum_{m \in \mathbb{N}} \mathcal{H}_\infty^\delta(Q_{k,m}) \le K(p)[w]_{\mathcal{A}_{p,\delta}}^{\frac{1}{p}} \mathcal{H}_\infty^\delta(E_k).$$

This is just (2.11). Therefore, (2.8) holds and hence (2.7) is true.

Step 2) We show that

$$(2.16) \qquad \sum_{j\in\mathbb{Z}}\int_{F_j}|f(x)|w(x)\,d\mathcal{H}_\infty^\delta \le 4\int_{\mathbb{R}^n}|f(x)|\,dw_{\mathcal{H}_\infty^\delta} \le 8\sum_{j\in\mathbb{Z}}\int_{F_j}|f(x)|w(x)\,d\mathcal{H}_\infty^\delta.$$

Obviously, we have

$$\sum_{j\in\mathbb{Z}}\int_{F_j}|f(x)|w(x)\,d\mathcal{H}_\infty^\delta \le \sum_{j\in\mathbb{Z}}2^j w_{\mathcal{H}_\infty^\delta}(F_j) \le 2\sum_{j\in\mathbb{Z}}\int_{F_j}|f(x)|w(x)\,d\mathcal{H}_\infty^\delta.$$

Thus, to verify (2.16), we only need to prove that

$$(2.17) \qquad \int_{\mathbb{R}^n}|f(x)|\,dw_{\mathcal{H}_\infty^\delta} \le \sum_{j\in\mathbb{Z}}2^j w_{\mathcal{H}_\infty^\delta}(F_j) \le 4\int_{\mathbb{R}^n}|f(x)|\,dw_{\mathcal{H}_\infty^\delta}.$$

Since

$$w_{\mathcal{H}_\infty^\delta}(\{x\in\mathbb{R}^n:\ |f(x)|=\infty\})=0,$$

it follows that

$$\int_{\mathbb{R}^n}|f(x)|\,dw_{\mathcal{H}_\infty^\delta} \le \sum_{j\in\mathbb{Z}}\int_{F_j}|f(x)|\,dw_{\mathcal{H}_\infty^\delta} \le \sum_{j\in\mathbb{Z}}2^j w_{\mathcal{H}_\infty^\delta}(F_j).$$

On the other hand, it is easy to know that

$$\sum_{j\in\mathbb{Z}}2^j w_{\mathcal{H}_\infty^\delta}(F_j) \le 4\sum_{j\in\mathbb{Z}}\int_{2^{j-2}}^{2^{j-1}} w_{\mathcal{H}_\infty^\delta}(\{x\in F_j:|f(x)|>t\})\,dt$$

$$\le 4\sum_{j\in\mathbb{Z}}\int_{2^{j-2}}^{2^{j-1}} w_{\mathcal{H}_\infty^\delta}(\{x\in\mathbb{R}^n:|f(x)|>t\})\,dt$$

$$= 4\int_{\mathbb{R}^n}|f(x)|\,dw_{\mathcal{H}_\infty^\delta},$$

which implies (2.17). This completes the proof of Proposition 2.7. □

Remark 2.8. Let $\delta\in(0,n]$ and $E\subset\mathbb{R}^n$. Then, as a special case of Proposition 2.7, we find that, for any $f\in L^1(E,\mathcal{H}_\infty^\delta)$,

$$\int_E|f(x)|\,d\mathcal{H}_\infty^\delta \le \sum_{k\in\mathbb{Z}}\int_{E_k}|f(x)|\,d\mathcal{H}_\infty^\delta \le 4\int_E|f(x)|\,d\mathcal{H}_\infty^\delta,$$

where, for any $k\in\mathbb{Z}$,

$$E_k := \left\{x\in E:\ 2^{k-1}<|f(x)|\le 2^k\right\}.$$

Moreover, the corresponding conclusion also holds true with $\mathcal{H}_\infty^\delta$ replaced by $w_{\mathcal{H}_\infty^\delta}$ as in (1.5).

Remark 2.9. (i) We point out that, in the case of $\delta=n$, it is easy to find that

$$(2.18) \qquad \int_{\mathbb{R}^n}|f(x)|\,dw_{\mathcal{H}_\infty^\delta} \le K\int_{\mathbb{R}^n}|f(x)|w(x)\,d\mathcal{H}_\infty^\delta$$

for any non-negative measurable function w and any measurable function f, where K is a positive constant independent of f and w. However, when $\delta \in (0, n)$, it is impossible to find a positive constant K such that (2.18) holds uniformly even for any capacitary Muckenhoupt weight w. Here we construct a counterexample.

Counterexample 1. Let $\{E_j\}_{j=1}^{\infty}$ be a sequence of non-overlapping left-open and right-closed cubes of $\mathbb{R}^n$ with side length 1 satisfying $\bigcup_{j=1}^{\infty} E_j = \mathbb{R}^n$. For any $m \in \mathbb{N}$, divide each E_j into m^n congruent left-open and right-closed subcubes $E_{m,k}^j$ $(k = 1, 2, \ldots, m^n)$ with side length $\frac{1}{m}$. For any $x \in \mathbb{R}^n$, let

$$f_m(x) := \sum_{k=1}^{m^n} 2^{k-1} \mathbf{1}_{E_{m,k}^1}(x) \quad \text{and} \quad w_m(x) := \sum_{j=1}^{\infty} \sum_{k=1}^{m^n} 2^{-(k-1)} \mathbf{1}_{E_{m,k}^j}(x).$$

Then

$$\int_{\mathbb{R}^n} f_m(x) w_m(x) \, d\mathcal{H}_\infty^\delta = \int_{\mathbb{R}^n} \mathbf{1}_{E_1}(x) \, d\mathcal{H}_\infty^\delta = \mathcal{H}_\infty^\delta(E_1) = 1$$

and, for any $m \in \mathbb{N}$, we have $w_m \in \mathcal{A}_{p,\delta}$ for all $p \in [1, \infty)$. Now, for any $k \in \mathbb{Z}$, let

$$F_{m,k} := \left\{ x \in \mathbb{R}^n : 2^{k-1} < |f_m(x)| \leq 2^k \right\}.$$

Then by Remark 2.8, we find that

$$\int_{\mathbb{R}^n} |f_m(x)| \, d(w_m)_{\mathcal{H}_\infty^\delta} \geq \frac{1}{4} \sum_{k \in \mathbb{Z}} \int_{F_{m,k}} |f_m(x)| \, d(w_m)_{\mathcal{H}_\infty^\delta}$$

$$\geq \frac{1}{8} \sum_{k=0}^{m^n-1} 2^k (w_m)_{\mathcal{H}_\infty^\delta}(F_{m,k}) = \frac{1}{8} \sum_{k=0}^{m^n-1} \mathcal{H}_\infty^\delta(E_{m,k+1}^1)$$

$$= \frac{1}{8} \sum_{k=0}^{m^n-1} \left(\frac{1}{m}\right)^\delta = \frac{1}{8} m^{n-\delta},$$

which tends to infinity as $m \to \infty$ due to $0 < \delta < n$. Hence, there is no constant K satisfying (2.18) for $\delta \in (0, n)$.

(ii) The reason why (2.18) does not hold uniformly for weights w in the case $\delta \in (0, n)$ seems to be the lack of the Fubini theorem in the framework of Choquet integrals with respect to Hausdorff contents. To better understand this, we construct the following counterexample.

Counterexample 2. For $n = 1$, let $\delta \in (0, 1)$ and $\beta \in (0, 1]$. For any $m \in \mathbb{N}$, let $E_{m,k} = (\frac{k-1}{m}, \frac{k}{m}]$, $\forall k \in \{1, 2, \ldots, m\}$, and, for any $x \in \mathbb{R}$, let

$$f_m(x) := \sum_{k=1}^{m} 2^{k-1} \mathbf{1}_{E_{m,k}}(x) \quad \text{and} \quad w_m(x) := \sum_{k=1}^{m} 2^{-(k-1)\beta} \mathbf{1}_{E_{m,k}}(x).$$

Additionally, we define

$$F_m(x, y) := \begin{cases} w_m(x) \mathbf{1}_{\{x \in \mathbb{R}:\, |f_m(x)| > y\}}(x, y), & \text{when } x \in \mathbb{R}, \ y \in (0, \infty), \\ 0, & \text{when } x \in \mathbb{R}, \ y \in (-\infty, 0]. \end{cases}$$

Then

$$\int_{\mathbb{R}}\left[\int_{\mathbb{R}} F_m(x,y)\,d\mathcal{H}_\infty^\beta(y)\right]d\mathcal{H}_\infty^\delta(x) = 1,$$

and

$$\int_{\mathbb{R}}\left[\int_{\mathbb{R}} F_m(x,y)\,d\mathcal{H}_\infty^\delta(x)\right]d\mathcal{H}_\infty^\beta(y) \geq \frac{2^\beta - 1}{2^{2\beta}}m^{1-\delta},$$

which tends to infinity as $m \to \infty$. Therefore, the following two Choquet integrals

$$\int_{\mathbb{R}}\left[\int_{\mathbb{R}} F_m(x,y)\,d\mathcal{H}_\infty^\beta(y)\right]d\mathcal{H}_\infty^\delta(x) \quad \text{and} \quad \int_{\mathbb{R}}\left[\int_{\mathbb{R}} F_m(x,y)\,d\mathcal{H}_\infty^\delta(x)\right]d\mathcal{H}_\infty^\beta(y)$$

are not equal even when a constant multiple is permitted.

However, we point out that, in the following special case, the order of integrals corresponding to the Hausdorff content $\mathcal{H}_\infty^\delta$ can be interchanged. Let $w \in \mathcal{A}_{p,\delta}$. Then, by Proposition 2.7, we have, for any function f on $\mathbb{R}^n$,

$$\begin{aligned}
\int_{\mathbb{R}^n}\int_0^\infty \mathbf{1}_{\{x\in\mathbb{R}^n:|f(x)|>t\}}(x)w(x)\,dt\,d\mathcal{H}_\infty^\delta &= \int_{\mathbb{R}^n}|f(x)|w(x)\,d\mathcal{H}_\infty^\delta \\
&\sim \int_0^\infty w_{\mathcal{H}_\infty^\delta}(\{x\in\mathbb{R}^n:|f(x)|>t\})\,dt \\
&\sim \int_0^\infty \int_{\mathbb{R}^n}\mathbf{1}_{\{x\in\mathbb{R}^n:|f(x)|>t\}}(x)w(x)\,d\mathcal{H}_\infty^\delta\,dt.
\end{aligned}$$

However, this estimation is obviously true in the Lebesgue integral setting for measurable function f and non-negative measurable function w and hence

$$\int_{\mathbb{R}^n}|f(x)|w(x)\,dx = \int_{\mathbb{R}^n}|f(x)|\,dw.$$

(iii) We remark that (2.18) may still not hold even for a fixed weight w. To show this, we give a counterexample as follows.

Counterexample 3. Let $\delta \in (0, n-1]$ with $n \geq 2$. For any $k \in \mathbb{N}$, define

$$E_k := \underbrace{(0,1]\times(0,1]\times\cdots\times(0,1]}_{n-1}\times\left(\frac{2^{k-1}-1}{2^{k-1}}, \frac{2^k-1}{2^k}\right)$$

and

$$E := \underbrace{(0,1]\times(0,1]\times\cdots\times(0,1]}_{n-1}\times(0,1).$$

Then, by Remark 2.3(i), we have $\mathcal{H}_\infty^\delta(E) = 1$ and, for any $k \in \mathbb{N}$, $\mathcal{H}_\infty^\delta(E_k) = 1$. Now, let

$$w(x) := \begin{cases} \sum_{k\in\mathbb{N}} 2^{-k}\mathbf{1}_{E_k}(x), & \text{when } x \in E, \\ 1, & \text{when } x \in \mathbb{R}^n\setminus E \end{cases}$$

and

$$f(x) := \sum_{k\in\mathbb{N}} 2^k\mathbf{1}_{E_k}(x), \quad \forall x \in \mathbb{R}^n.$$

Then it is not difficult to find that

$$\int_{\mathbb{R}^n} f(x)w(x)\,d\mathcal{H}_\infty^\delta = \int_{\mathbb{R}^n} \mathbf{1}_E(x)\,d\mathcal{H}_\infty^\delta = 1.$$

On the other hand, for any $k \in \mathbb{Z}$, let

$$F_k := \left\{ x \in \mathbb{R}^n : \ 2^{k-1} < |f(x)| \le 2^k \right\}.$$

Then $F_k = E_k$ for $k \in \mathbb{N}$ and, $F_k = \emptyset$ otherwise. Thus, by Remark 2.8, we have

$$\int_{\mathbb{R}^n} |f(x)|\,dw_{\mathcal{H}_\infty^\delta} \ge \frac{1}{4}\sum_{k\in\mathbb{Z}}\int_{F_k} |f(x)|\,dw_{\mathcal{H}_\infty^\delta} = \frac{1}{4}\sum_{k\in\mathbb{N}} 2^k w_{\mathcal{H}_\infty^\delta}(E_k) = \frac{1}{4}\sum_{k\in\mathbb{N}} 1 = \infty.$$

(iv) Let $\delta \in (0, n)$ and w be a given weight. Then (2.18) holds for any $f \in L_w^1(\mathbb{R}^n, \mathcal{H}_\infty^\delta)$ *if and only if* there exists a positive constant K such that, for any cube $Q \subset \mathbb{R}^n$,

$$(2.19) \qquad \int_Q \frac{1}{w(x)}\,dw_{\mathcal{H}_\infty^\delta} \le K\mathcal{H}_\infty^\delta(Q)$$

if and only if there exists a positive constant K such that for any $E \subset \mathbb{R}^n$,

$$(2.20) \qquad \sum_{k\in\mathbb{Z}} \mathcal{H}_\infty^\delta(E \cap E_k) \le K\mathcal{H}_\infty^\delta(E)$$

where, for any $k \in \mathbb{Z}$, $E_k := \{x \in \mathbb{R}^n : 2^{k-1} < w(x) \le 2^k\}$.

Proof. We complete the proof by showing (2.18)$\Longrightarrow$(2.19)$\Longrightarrow$(2.20)$\Longrightarrow$(2.18). Indeed, if (2.18) holds true, then for any cube $Q \subset \mathbb{R}^n$, by letting $f(x) := \frac{1}{w(x)}\mathbf{1}_Q(x)$, we obtain (2.19). If (2.19) holds true, then by an argument similar to that used in Remark 2.8, we know that, for any cube $Q \subset \mathbb{R}^n$,

$$(2.21) \qquad \sum_{k\in\mathbb{Z}} \mathcal{H}_\infty^\delta(Q \cap E_k) = \sum_{k\in\mathbb{Z}} \mathcal{H}_\infty^\delta(Q \cap E_{1-k}) \le 2\sum_{k\in\mathbb{Z}} 2^{k-1} w_{\mathcal{H}_\infty^\delta}(Q \cap E_{1-k})$$

$$\le 2\sum_{k\in\mathbb{Z}} \int_{Q\cap E_{1-k}} \frac{1}{w(x)}\,dw_{\mathcal{H}_\infty^\delta} \le 8\int_Q \frac{1}{w(x)}\,dw_{\mathcal{H}_\infty^\delta} \lesssim \mathcal{H}_\infty^\delta(Q).$$

For any $E \subset \mathbb{R}^n$, without loss of generality, we may assume that $\mathcal{H}_\infty^\delta(E) \in (0, \infty)$. Then there exists a family $\{Q_j\}_{j\in\mathbb{N}}$ of cubes in $\mathbb{R}^n$ such that $E \subset \bigcup_{j\in\mathbb{N}} Q_j$ and

$$\sum_{j\in\mathbb{N}} \mathcal{H}_\infty^\delta(Q_j) \le 2\mathcal{H}_\infty^\delta(E).$$

Combining this with (2.21), we conclude that

$$\sum_{k\in\mathbb{Z}} \mathcal{H}_\infty^\delta(E \cap E_k) \le \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{N}} \mathcal{H}_\infty^\delta(Q_j \cap E_k) \lesssim \sum_{j\in\mathbb{N}} \mathcal{H}_\infty^\delta(Q_j) \lesssim \mathcal{H}_\infty^\delta(E).$$

Therefore, (2.20) holds true. Finally, if (2.20) holds, then by (2.11) and an argument similar to that used in the proof of Proposition 2.7, we obtain (2.18). □

2.3 Sparse Covering Property and Substitute for the Linearity of Integrals

Inspired by the wonderful idea from Calderón-Zygmund decomposition technique, in this subsection, we formulate and prove a "sparse covering lemma" in the context of Hausdorff contents as follows, which is of central importance to this paper, and has its own independent significance and potential applicability to other problems. Using the sparse covering lemma, with a weighted packing condition, we build up Proposition 2.12. It serves as a partial substitute for the linearity property, which generally fails to hold for Choquet integrals.

Proposition 2.10. *Let $\delta \in (0, n]$ and E be a subset of $\mathbb{R}^n$ satisfying $\widetilde{\mathcal{H}}_\infty^\delta(E) < \infty$. Then there exists a subset $F \subset \mathbb{R}^n$ and a family $\{Q_j\}_{j\in\mathbb{N}}$ of non-overlapping dyadic cubes in $\mathbb{R}^n$ such that*

(i) *$E \subset (\bigcup_{j\in\mathbb{N}} Q_j) \cup F$ and $\widetilde{\mathcal{H}}_\infty^\delta(F) = 0$;*

(ii) *$\sum_{j\in\mathbb{N}} \widetilde{\mathcal{H}}_\infty^\delta(Q_j) \leq 2\widetilde{\mathcal{H}}_\infty^\delta(E)$;*

(iii) *for any $j \in \mathbb{N}$, we have $\widetilde{\mathcal{H}}_\infty^\delta(Q_j) \leq 3\widetilde{\mathcal{H}}_\infty^\delta(Q_j \cap E)$.*

Proof. We may assume that $\widetilde{\mathcal{H}}_\infty^\delta(E) > 0$, otherwise there is nothing to prove. According to the definition of the Hausdorff content $\widetilde{\mathcal{H}}_\infty^\delta$, we know that there exists a family $\{P_j\}_{j\in\mathbb{N}}$ of dyadic cubes such that

$$E \subset \bigcup_{j\in\mathbb{N}} P_j \quad \text{and} \quad \sum_{j\in\mathbb{N}} \widetilde{\mathcal{H}}_\infty^\delta(P_j) \leq 2\widetilde{\mathcal{H}}_\infty^\delta(E). \tag{2.22}$$

Let

$$A_1 := \left\{ j \in \mathbb{N} : \; \widetilde{\mathcal{H}}_\infty^\delta(P_j) \leq 3\widetilde{\mathcal{H}}_\infty^\delta(P_j \cap E) \right\},$$

$$A_2 := \left\{ j \in \mathbb{N} : \; \widetilde{\mathcal{H}}_\infty^\delta(P_j \cap E) = 0 \right\}$$

and

$$A_3 := \left\{ j \in \mathbb{N} : \; 0 < \widetilde{\mathcal{H}}_\infty^\delta(P_j \cap E) < \frac{1}{3}\widetilde{\mathcal{H}}_\infty^\delta(P_j) \right\}.$$

Then we have

$$E = \bigcup_{j\in\mathbb{N}}(P_j \cap E) \subset \left(\bigcup_{j\in A_1} P_j \right) \bigcup G_1 \bigcup B_1, \tag{2.23}$$

where

$$G_1 := \bigcup_{j\in A_3}(P_j \cap E) \quad \text{and} \quad B_1 := \bigcup_{j\in A_2}(P_j \cap E).$$

It is easy to see that $\widetilde{\mathcal{H}}_\infty^\delta(B_1) = 0$ and, by (2.22), we have

$$\widetilde{\mathcal{H}}_\infty^\delta(G_1) \leq \sum_{j\in A_3} \widetilde{\mathcal{H}}_\infty^\delta(P_j \cap E)) < \frac{1}{3}\sum_{j\in\mathbb{N}} \widetilde{\mathcal{H}}_\infty^\delta(P_j) \leq \frac{2}{3}\widetilde{\mathcal{H}}_\infty^\delta(E). \tag{2.24}$$

For any given $j \in A_3$, by the definition of the Hausdorff content $\widetilde{\mathcal{H}}_\infty^\delta(P_j \cap E)$, there exists a family $\{P_{j,k}\}_{k\in\mathbb{N}}$ of dyadic cubes such that

$$P_j \cap E \subset \bigcup_{k\in\mathbb{N}} P_{j,k} \quad \text{and} \quad \sum_{k\in\mathbb{N}} \widetilde{\mathcal{H}}_\infty^\delta(P_{j,k}) \leq 2\widetilde{\mathcal{H}}_\infty^\delta(P_j \cap E). \tag{2.25}$$

Moreover, we may assume that, for any $k \in \mathbb{N}$, $P_{j,k} \cap (P_j \cap E) \neq \emptyset$, and hence $P_{j,k} \subset P_j$. Similar to the above argument, we let

$$A_{j,1} := \left\{ k \in \mathbb{N} : \widetilde{\mathcal{H}}_{\infty}^{\delta}(P_{j,k}) \leq 3\widetilde{\mathcal{H}}_{\infty}^{\delta}(P_{j,k} \cap E) \right\},$$

$$A_{j,2} := \left\{ k \in \mathbb{N} : \widetilde{\mathcal{H}}_{\infty}^{\delta}(P_{j,k} \cap E) = 0 \right\}$$

and

$$A_{j,3} := \left\{ k \in \mathbb{N} : 0 < \widetilde{\mathcal{H}}_{\infty}^{\delta}(P_{j,k} \cap E) < \frac{1}{3}\widetilde{\mathcal{H}}_{\infty}^{\delta}(P_{j,k}) \right\}.$$

Then obviously

$$G_1 \subset \left(\bigcup_{j \in A_3, k \in A_{j,1}} P_{j,k} \right) \bigcup \left(\bigcup_{j \in A_3, k \in A_{j,3}} (P_{j,k} \cap E) \right) \bigcup \left(\bigcup_{j \in A_3, k \in A_{j,2}} (P_{j,k} \cap E) \right).$$

By this and (2.23), we further find that

$$E \subset \left(\bigcup_{j \in A_1} P_j \right) \bigcup \left(\bigcup_{j \in A_3, k \in A_{j,1}} P_{j,k} \right) \bigcup G_2 \bigcup B_2,$$

where

$$G_2 := \bigcup_{j \in A_3, k \in A_{j,3}} (P_{j,k} \cap E) \quad \text{and} \quad B_2 := B_1 \cup \left(\bigcup_{j \in A_3, k \in A_{j,2}} (P_{j,k} \cap E) \right).$$

Again, $\widetilde{\mathcal{H}}_{\infty}^{\delta}(B_2) = 0$. From $P_{j,k} \subset P_j$ for any $k \in \mathbb{N}$, it follows that

$$G_2 \subset \bigcup_{j \in A_3}(P_j \cap E) = G_1.$$

Moreover, combining (2.24) and (2.25), we obtain

$$\widetilde{\mathcal{H}}_{\infty}^{\delta}(G_2) \leq \sum_{j \in A_3} \sum_{k \in A_{j,3}} \widetilde{\mathcal{H}}_{\infty}^{\delta}(P_{j,k} \cap E) < \frac{1}{3} \sum_{j \in A_3} \sum_{k \in \mathbb{N}} \widetilde{\mathcal{H}}_{\infty}^{\delta}(P_{j,k}) \leq \frac{2}{3} \sum_{j \in A_3} \widetilde{\mathcal{H}}_{\infty}^{\delta}(E \cap P_j) \leq \left(\frac{2}{3}\right)^2 \widetilde{\mathcal{H}}_{\infty}^{\delta}(E),$$

and, by (2.22) and (2.25), we have

$$\sum_{j \in A_1} \widetilde{\mathcal{H}}_{\infty}^{\delta}(P_j) + \sum_{j \in A_3} \sum_{k \in A_{j,1}} \widetilde{\mathcal{H}}_{\infty}^{\delta}(P_{j,k})$$

$$\leq \sum_{j \in A_1} \widetilde{\mathcal{H}}_{\infty}^{\delta}(P_j) + 2 \sum_{j \in A_3} \widetilde{\mathcal{H}}_{\infty}^{\delta}(P_j \cap E) \leq \sum_{j \in A_1} \widetilde{\mathcal{H}}_{\infty}^{\delta}(P_j) + \frac{2}{3} \sum_{j \in A_3} \widetilde{\mathcal{H}}_{\infty}^{\delta}(P_j) \leq 2\widetilde{\mathcal{H}}_{\infty}^{\delta}(E).$$

Proceeding in the same manner, for any $m \in \mathbb{N}$, there exist subsets G_m, B_m and a sequence $\{P_j^m\}_{j \in \mathbb{N}}$ of dyadic cubes in $\mathbb{R}^n$ such that, for any $j \in \mathbb{N}$,

$$\widetilde{\mathcal{H}}_{\infty}^{\delta}(P_j^m) \leq 3\widetilde{\mathcal{H}}_{\infty}^{\delta}(P_j^m \cap E),$$

$$E \subset \left(\bigcup_{j \in \mathbb{N}} P_j^m \right) \bigcup G_m \bigcup B_m,$$

$\widetilde{\mathcal{H}}_\infty^\delta(B_m) = 0$ and

$$\sum_{j\in\mathbb{N}} \widetilde{\mathcal{H}}_\infty^\delta(P_j^m) \le 2\widetilde{\mathcal{H}}_\infty^\delta(E), \ \widetilde{\mathcal{H}}_\infty^\delta(G_m) \le \left(\frac{2}{3}\right)^m \widetilde{\mathcal{H}}_\infty^\delta(E).$$

Moreover,

$$\{P_j^m\}_{j\in\mathbb{N}} \subset \{P_j^{m+1}\}_{j\in\mathbb{N}}, \quad B_m \subset B_{m+1} \quad \text{and} \quad G_{m+1} \subset G_m.$$

Now, we rearrange $\{P_j^m\}_{j,m\in\mathbb{N}}$ as $\{\widetilde{Q}_j\}_{j\in\mathbb{N}}$. Let $\{Q_j\}_{j\in\mathbb{N}}$ be the maximal dyadic cubes of $\{\widetilde{Q}_j\}_{j\in\mathbb{N}}$,

$$B := \bigcup_{m\in\mathbb{N}} B_m \quad \text{and} \quad G := \bigcap_{m\in\mathbb{N}} G_m.$$

Then

$$\sum_{j\in\mathbb{N}} \widetilde{\mathcal{H}}_\infty^\delta(Q_j) \le 2\widetilde{\mathcal{H}}_\infty^\delta(E), \quad \widetilde{\mathcal{H}}_\infty^\delta(B) \le \sum_{m\in\mathbb{N}} \widetilde{\mathcal{H}}_\infty^\delta(B_m) = 0$$

and $\widetilde{\mathcal{H}}_\infty^\delta(G) \le (\frac{2}{3})^m \widetilde{\mathcal{H}}_\infty^\delta(E)$ for any $m \in \mathbb{N}$, which implies $\widetilde{\mathcal{H}}_\infty^\delta(G) = 0$. Moveover,

$$E \subset \left(\bigcup_{j\in\mathbb{N}} Q_j\right) \bigcup F,$$

where $F := B \cup G$ satisfying $\widetilde{\mathcal{H}}_\infty^\delta(F) = 0$. This finally completes the proof of Proposition 2.10. □

We point out that Proposition 2.10 seems to be new even when reduced to the classical Lebesgue measure setting, i.e., in the case of $\delta = n$. Furthermore, applying the "sparse covering lemma" in Proposition 2.10, we obtain the following conclusion, which realizes the interchange of summation and integration with respect to $\mathcal{H}_\infty^\delta$ in a certain sense.

Lemma 2.11. *Let $\delta \in (0, n]$, $p \in [1, \infty)$, $E \subset \mathbb{R}^n$ and $w \in \mathcal{A}_{p,\delta}$ satisfying $\int_E w(x)\, d\mathcal{H}_\infty^\delta < \infty$. Then there exist a family $\{Q_j\}_{j\in\mathbb{N}}$ of non-overlapping dyadic cubes and a subset $F \subset \mathbb{R}^n$ with $\mathcal{H}_\infty^\delta(F) = 0$ such that*

$$E \subset \left(\bigcup_{j\in\mathbb{N}} Q_j\right) \bigcup F \quad \text{and} \quad \sum_{j\in\mathbb{N}} \int_{Q_j} w(x)\, d\mathcal{H}_\infty^\delta \le K(n,\delta,p)[w]_{\mathcal{A}_{p,\delta}} \int_E w(x)\, d\mathcal{H}_\infty^\delta,$$

where $K(n, \delta, p)$ is a positive constant independent of E and w.

Proof. For any $k \in \mathbb{Z}$, let

$$E_k := \{x \in E : 2^{k-1} < w(x) \le 2^k\}.$$

Then, applying Remark 2.8, we have

$$(2.26) \qquad \sum_{k\in\mathbb{Z}} 2^k \mathcal{H}_\infty^\delta(E_k) \le 2 \sum_{k\in\mathbb{Z}} \int_{E_k} w(x)\, d\mathcal{H}_\infty^\delta \le 8 \int_E w(x)\, d\mathcal{H}_\infty^\delta$$

and hence, for any $k \in \mathbb{Z}$, $\widetilde{\mathcal{H}}_\infty^\delta(E_k) \le K(n,\delta)\mathcal{H}_\infty^\delta(E_k) < \infty$ due to Remark 2.3(ii). By this and Proposition 2.10, we know that, for any $k \in \mathbb{Z}$, there exist a subset $F_k \subset \mathbb{R}^n$ and a family $\{Q_{k,j}\}_{j\in\mathbb{N}}$ of non-overlapping dyadic cubes of $\mathbb{R}^n$ such that

(i) $E_k \subset (\bigcup_{j\in\mathbb{N}} Q_{k,j}) \cup F_k$, and $\mathcal{H}_\infty^\delta(F_k) = 0$;

(ii) $\sum_{j\in\mathbb{N}} \mathcal{H}^\delta_\infty(Q_{k,j}) \leq K(n,\delta)\mathcal{H}^\delta_\infty(E_k)$;

(iii) for any $j \in \mathbb{N}$, $\mathcal{H}^\delta_\infty(Q_{k,j}) \leq K(n,\delta)\mathcal{H}^\delta_\infty(Q_{k,j} \cap E_k)$.

Therefore, if we let $F := \bigcup_{k\in\mathbb{Z}} F_k$, then $\mathcal{H}^\delta_\infty(F) = 0$ and

$$E = \bigcup_{k\in\mathbb{Z}} E_k \subset \left(\bigcup_{k\in\mathbb{Z}}\bigcup_{j\in\mathbb{N}} Q_{k,j}\right)\bigcup F.$$

Moreover, by $w \in \mathcal{A}_{p,\delta}$, Lemma 2.4 and (2.26), we deduce that

$$\begin{aligned}
\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{N}} \int_{Q_{k,j}} w(x)\,d\mathcal{H}^\delta_\infty &\leq 2[w]_{\mathcal{A}_{p,\delta}} \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{N}} \left[\frac{\mathcal{H}^\delta_\infty(Q_{k,j})}{\mathcal{H}^\delta_\infty(E_k \cap Q_{k,j})}\right]^p \int_{E_k\cap Q_{k,j}} w(x)\,d\mathcal{H}^\delta_\infty \\
&\leq K(n,\delta,p)[w]_{\mathcal{A}_{p,\delta}} \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{N}} 2^k \mathcal{H}^\delta_\infty\left(E_k \cap Q_{k,j}\right) \\
&\leq K(n,\delta,p)[w]_{\mathcal{A}_{p,\delta}} \sum_{k\in\mathbb{Z}} 2^k \sum_{j\in\mathbb{N}} \mathcal{H}^\delta_\infty\left(Q_{k,j}\right) \\
&\leq K(n,\delta,p)[w]_{\mathcal{A}_{p,\delta}} \sum_{k\in\mathbb{Z}} 2^k \mathcal{H}^\delta_\infty\left(E_k\right) \\
&\leq K(n,\delta,p)[w]_{\mathcal{A}_{p,\delta}} \int_E w(x)\,d\mathcal{H}^\delta_\infty.
\end{aligned}$$

Finally, we finish the proof of Lemma 2.11 by rearranging $\{Q_{k,j}\}_{k\in\mathbb{Z},j\in\mathbb{N}}$ as $\{Q_j\}_{j\in\mathbb{N}}$. $\qquad\square$

Moreover, the following conclusion, with a weighted packing condition for a family $\{Q_j\}_j$ of non-overlapping cubes, becomes essential. It serves as a partial substitute for the linearity property, which generally fails to hold for Choquet integrals.

Proposition 2.12. *Let $\delta \in (0,n]$, $p \in [1,\infty)$ and $w \in \mathcal{A}_{p,\delta}$. Let $\{Q_j\}_{j\in\mathbb{N}}$ be a family of non-overlapping dyadic cubes of $\mathbb{R}^n$. If there exists a constant $\beta \in (0,\infty)$ such that, for each dyadic cube Q,*

$$\tag{2.27} \sum_{Q_j\subset Q} w_{\mathcal{H}^\delta_\infty}(Q_j) \leq \beta\, w_{\mathcal{H}^\delta_\infty}(Q),$$

then there exists a positive constant $K(n,\delta,p)$ such that, for any $f \in L^1_w(\cup_{j\in\mathbb{N}}Q_j, \mathcal{H}^\delta_\infty)$,

$$\tag{2.28} \sum_{j\in\mathbb{N}} \int_{Q_j} |f(x)|w(x)\,d\mathcal{H}^\delta_\infty \leq K(n,\delta,p)\max\{1,\beta\}[w]_{\mathcal{A}_{p,\delta}}^{1+\frac{1}{p}} \int_{\cup_{j\in\mathbb{N}}Q_j} |f(x)|w(x)\,d\mathcal{H}^\delta_\infty.$$

Proof. To prove (2.28), according to Proposition 2.7, we only need to show

$$\tag{2.29} \sum_{j\in\mathbb{N}} \int_{Q_j} |f(x)|\,dw_{\mathcal{H}^\delta_\infty} \leq K(n,\delta,p)\max\{1,\beta\}[w]_{\mathcal{A}_{p,\delta}} \int_{\cup_{j\in\mathbb{N}}Q_j} |f(x)|\,dw_{\mathcal{H}^\delta_\infty}.$$

For any given $t \in (0,\infty)$, applying Lemma 2.11, we know that there exist a family $\{P_i\}_i$ of non-overlapping dyadic cubes and a subset $F \subset \mathbb{R}^n$ with $w_{\mathcal{H}^\delta_\infty}(F) = 0$ such that

$$\left\{x \in \bigcup_{j\in\mathbb{N}} Q_j :\ |f(x)| > t\right\} \subset \left(\bigcup_i P_i\right)\bigcup F$$

and

$$\text{(2.30)} \qquad \sum_i w_{\mathcal{H}_\infty^\delta}(P_i) \le K(n, \delta, p)[w]_{\mathscr{A}_{p,\delta}} w_{\mathcal{H}_\infty^\delta}\left(\left\{x \in \bigcup_{j \in \mathbb{N}} Q_j : \ |f(x)| > t\right\}\right).$$

Moreover, we may assume, for any i, $(\{x \in \bigcup_{j \in \mathbb{N}} Q_j : \ |f(x)| > t\} \setminus F) \cap P_i \ne \emptyset$.

Let A_1 be the set of all $i \in \mathbb{N}$ such that there exists only one index i^* satisfying

$$\{x \in Q_{i^*} \setminus F : |f(x)| > t\} \subset P_i,$$

and A_2 the set of all $i \in \mathbb{N}$ such that there exist at least two indices i_v satisfying

$$\bigcup_v \{x \in Q_{i_v} \setminus F : |f(x)| > t\} \subset P_i.$$

For each $i \in A_2$, let $A_{2,i} := \{i_v\}_v$. Moreover, we let A be the set of all $j \in \mathbb{N}$ such that the subset $\{x \in Q_j \setminus F : \ |f(x)| > t\}$ only be covered by at least two cubes from $\{P_i\}_i$. Then

$$\text{(2.31)} \qquad \sum_{j \in \mathbb{N}} w_{\mathcal{H}_\infty^\delta}\left(\left\{x \in Q_j \setminus F : |f(x)| > t\right\}\right)$$

$$= \sum_{i \in A_1} w_{\mathcal{H}_\infty^\delta}\left(\{x \in Q_{i^*} \setminus F : \ |f(x)| > t\}\right) + \sum_{i \in A_2} \sum_{i_v \in A_{2,i}} w_{\mathcal{H}_\infty^\delta}\left(\{x \in Q_{i_v} \setminus F : \ |f(x)| > t\}\right)$$

$$+ \sum_{j \in A} w_{\mathcal{H}_\infty^\delta}\left(\left\{x \in Q_j \setminus F : |f(x)| > t\right\}\right).$$

If $i \in A_1$, it is clear that

$$\text{(2.32)} \qquad \sum_{i \in A_1} w_{\mathcal{H}_\infty^\delta}\left(\{x \in Q_{i^*} \setminus F : \ |f(x)| > t\}\right) \le \sum_{i \in A_1} w_{\mathcal{H}_\infty^\delta}(P_i).$$

We claim that $Q_{i_v} \subset P_i$ for any $i \in A_2$ and $i_v \in A_{2,i}$. Indeed, since

$$\{x \in Q_{i_v} \setminus F : \ |f(x)| > t\} \cap P_i \ne \emptyset,$$

it follows that $Q_{i_v} \cap P_i \ne \emptyset$. Thus, by the properties of dyadic cubes, we find that if $Q_{i_v} \not\subset P_i$, then $P_i \subset Q_{i_v}$. However, in this case, there exists another $i_k \in A_{2,i}$ such that $Q_{i_k} \cap P_i \ne \emptyset$, and hence $Q_{i_v} \cap Q_{i_k} \ne \emptyset$, which is contradictory to the fact that $\{Q_j\}_{j \in \mathbb{N}}$ are non-overlapping dyadic cubes. Therefore, for any $i \in A_2$,

$$\bigcup_{i_v \in A_{2,i}} Q_{i_v} \subset P_i.$$

By this claim and the condition (2.27), we have, for any $i \in A_2$,

$$\sum_{i_v \in A_{2,i}} w_{\mathcal{H}_\infty^\delta}(Q_{i_v}) \le \beta w_{\mathcal{H}_\infty^\delta}(P_i),$$

which implies that

$$\text{(2.33)} \qquad \sum_{i \in A_2} \sum_{i_v \in A_{2,i}} w_{\mathcal{H}_\infty^\delta}\left(\{x \in Q_{i_v} \setminus F : \ |f(x)| > t\}\right) \le \beta \sum_{i \in A_2} w_{\mathcal{H}_\infty^\delta}(P_i).$$

For any $j \in A$, we let $\{P_j^k\}_k$ be a sub-family of $\{P_i\}_i$ such that

$$\left\{x \in Q_j \backslash F : |f(x)| > t\right\} \subset \bigcup_k P_j^k.$$

Then

(2.34) $$\sum_{j \in A} w_{\mathcal{H}_\infty^\delta}\left(\left\{x \in Q_j \backslash F : |f(x)| > t\right\}\right) \leq \sum_{j \in A} \sum_k w_{\mathcal{H}_\infty^\delta}(P_j^k).$$

Therefore, from (2.31), (2.32), (2.33) and (2.34), we infer that

$$\sum_{j \in \mathbb{N}} w_{\mathcal{H}_\infty^\delta}\left(\left\{x \in Q_j \backslash F : |f(x)| > t\right\}\right) \leq \sum_{i \in A_1} w_{\mathcal{H}_\infty^\delta}(P_i) + \beta \sum_{i \in A_2} w_{\mathcal{H}_\infty^\delta}(P_i) + \sum_{j \in A} \sum_k w_{\mathcal{H}_\infty^\delta}(P_j^k)$$

$$\leq \max\{1, \beta\} \sum_i w_{\mathcal{H}_\infty^\delta}(P_i).$$

Combining this, (2.30) and the fact that $w_{\mathcal{H}_\infty^\delta}(F) = 0$, we further conclude that

$$\sum_{j \in \mathbb{N}} \int_{Q_j} |f(x)| \, dw_{\mathcal{H}_\infty^\delta} = \sum_{j \in \mathbb{N}} \int_0^\infty w_{\mathcal{H}_\infty^\delta}\left(\left\{x \in Q_j : |f(x)| > t\right\}\right) dt$$

$$= \int_0^\infty \sum_{j \in \mathbb{N}} w_{\mathcal{H}_\infty^\delta}\left(\left\{x \in Q_j \backslash F : |f(x)| > t\right\}\right) dt$$

$$\leq K(n, \delta, p) \max\{1, \beta\}[w]_{\mathcal{A}_{p,\delta}} \int_0^\infty w_{\mathcal{H}_\infty^\delta}\left(\left\{x \in \bigcup_{j \in \mathbb{N}} Q_j : |f(x)| > t\right\}\right) dt$$

$$= K(n, \delta, p) \max\{1, \beta\}[w]_{\mathcal{A}_{p,\delta}} \int_{\bigcup_{j \in \mathbb{N}} Q_j} |f(x)| \, dw_{\mathcal{H}_\infty^\delta},$$

which is (2.29). This completes the proof of Proposition 2.12. $\qquad \square$

Remark 2.13. When $w \in \mathcal{A}_{1,\delta}$, the proof of Proposition 2.12 can be simplified. In fact, according to Lemma 2.5 and (2.27), we know that

$$\sum_{Q_j \subset Q} \mathcal{H}_\infty^\delta(Q_j) \lesssim \mathcal{H}_\infty^\delta(Q).$$

We apply [32, (6) and (7)] to obtain

$$\sum_{j \in \mathbb{N}} \int_{Q_j} |g(x)| \, d\mathcal{H}_\infty^\delta \lesssim \int_{\bigcup_{j \in \mathbb{N}} Q_j} |g(x)| \, d\mathcal{H}_\infty^\delta$$

for any function g.

2.4   An Example and Monotonicity of Capacitary Muckenhoupt Weight $\mathcal{A}_{p,\delta}$

In this subsection, we first show the following specific example of capacitary Muckenhoupt $\mathcal{A}_{p,\delta}$-weight function.

Proposition 2.14. *Let $\delta \in (0, n]$ and $p \in [1, \infty)$. For any given $\alpha \in \mathbb{R}$, define $w(x) := |x|^\alpha$, $\forall x \in \mathbb{R}^n$. Then $w \in \mathcal{A}_{p,\delta}$ if and only if $\alpha \in (-\delta, \delta(p-1))$.*

Proof. We first prove the sufficiency for $p = 1$. To this end, we only need to show that, for any given cube $Q(y, r)$ with $y \in \mathbb{R}^n$ and $r \in (0, \infty)$,

$$(2.35) \qquad \frac{1}{\mathcal{H}_\infty^\delta (Q(y, r))} \int_{Q(y,r)} w(z) \, d\mathcal{H}_\infty^\delta \lesssim w(x), \quad \forall x \in Q(y, r).$$

If $|y| \leq \frac{3\sqrt{n}}{2} r$, then $Q(y, r) \subset B(0, 2\sqrt{n}r)$ and, for any $z \in Q(y, r)$, $w(z) = |z|^\alpha \gtrsim r^\alpha$. Thus, by Remark 2.3, we find that

$$\int_{Q(y,r)} w(z) \, d\mathcal{H}_\infty^\delta \leq \int_{B(0, 2\sqrt{n}r)} w(z) \, d\mathcal{H}_\infty^\delta$$
$$= \int_0^\infty \mathcal{H}_\infty^\delta \left(B\left(0, 2\sqrt{n}r\right) \cap B\left(0, t^{\frac{1}{\alpha}}\right) \right) dt$$
$$\sim \int_0^{(2\sqrt{n}r)^\alpha} (2\sqrt{n}r)^\delta \, dt + \int_{(2\sqrt{n}r)^\alpha}^{+\infty} t^{\frac{\delta}{\alpha}} \, dt$$
$$\sim r^{\delta+\alpha} \lesssim \mathcal{H}_\infty^\delta (Q(y, r)) w(x).$$

Consequently, (2.35) is true in this case.

If $|y| > \frac{3\sqrt{n}}{2} r$, then, for any $z \in Q(y, r)$, we have

$$|z| \geq |y| - |y - z| \geq \sqrt{n}r$$

and

$$|z| \leq |y| + |y - z| \leq 2\sqrt{n}r.$$

Thus, we obtain

$$\frac{1}{\mathcal{H}_\infty^\delta (Q(y, r))} \int_{Q(y,r)} w(z) \, d\mathcal{H}_\infty^\delta \lesssim \frac{1}{\mathcal{H}_\infty^\delta (Q(y, r))} \int_{Q(y,r)} r^\alpha \, d\mathcal{H}_\infty^\delta \lesssim w(x),$$

namely, (2.35) also holds true in this case. Therefore, $w \in \mathcal{A}_{1,\delta}$.

Now we prove sufficiency part for $p \in (1, \infty)$. When $\alpha \in (-\delta, 0)$, then by an argument similar to that used in the proof of the case $p = 1$, we find that, for any cube $Q(y, r)$ with $y \in \mathbb{R}^n$ and $r \in (0, \infty)$,

$$(2.36) \qquad \int_{Q(y,r)} |x|^\alpha \, d\mathcal{H}_\infty^\delta \lesssim r^{\delta+\alpha} \quad \text{and} \quad \left\{ \int_{Q(y,r)} [|x|^\alpha]^{-\frac{1}{p-1}} \, d\mathcal{H}_\infty^\delta \right\}^{p-1} \lesssim r^{\delta(p-1)-\alpha}.$$

When $\alpha \in (0, \delta(p-1))$, then $-\frac{\alpha}{p-1} \in (-\delta, 0)$. Thus, similarly, we also obtain (2.36). Consequently,

$$\int_{Q(y,r)} |x|^\alpha \, d\mathcal{H}_\infty^\delta \left\{ \int_{Q(y,r)} [|x|^\alpha]^{-\frac{1}{p-1}} \, d\mathcal{H}_\infty^\delta \right\}^{p-1} \lesssim \left[\mathcal{H}_\infty^\delta (Q(y, r)) \right]^p,$$

which means $|x|^\alpha \in \mathcal{A}_{p,\delta}$.

We next prove the necessity part for $p = 1$. It is enough to show that, when $\alpha \notin (-\delta, 0]$, $|x|^\alpha \notin \mathscr{A}_{1,\delta}$. If $\alpha \in (-\infty, -\delta]$, then

$$
\begin{aligned}
(2.37) \qquad \frac{1}{\mathcal{H}_\infty^\delta(Q(0,2))} \int_{Q(0,2)} |x|^\alpha \, d\mathcal{H}_\infty^\delta &\gtrsim \int_{B(0,1)} |x|^\alpha \, d\mathcal{H}_\infty^\delta \\
&= \int_0^\infty \mathcal{H}_\infty^\delta \left(B(0,1) \cap B\left(0, t^{\frac{1}{\alpha}}\right) \right) dt \\
&\sim \int_0^1 1 \, dt + \int_1^{+\infty} t^{\frac{\delta}{\alpha}} \, dt = \infty.
\end{aligned}
$$

Therefore, in this case, $|x|^\alpha \notin \mathscr{A}_{1,\delta}$.

If $\alpha \in (0, \infty)$, then

$$
\frac{1}{\mathcal{H}_\infty^\delta(Q(0,1))} \int_{Q(0,1)} |x|^\alpha \, d\mathcal{H}_\infty^\delta =: \lambda \in (0,1).
$$

Hence, for any positive constant N, when $x \in B(0, \frac{1}{2}) \cap B(0, (\frac{\lambda}{N})^{\frac{1}{\alpha}})$,

$$
\frac{1}{\mathcal{H}_\infty^\delta(Q(0,1))} \int_{Q(0,1)} |x|^\alpha \, d\mathcal{H}_\infty^\delta > N|x|^\alpha,
$$

which implies that, in this case, $|x|^\alpha \notin \mathscr{A}_{1,\delta}$.

Finally, we consider the necessity part for $p \in (1, \infty)$. To this end, we only need to prove that, when $\alpha \notin (-\delta, \delta(p-1))$, $|x|^\alpha \notin \mathscr{A}_{p,\delta}$. Indeed, if $\alpha \in (-\infty, -\delta]$, then by (2.37), we have

$$
\int_{Q(0,2)} |x|^\alpha \, d\mathcal{H}_\infty^\delta = \infty,
$$

but, obviously,

$$
\int_{Q(0,2)} |x|^{-\frac{\alpha}{p-1}} \, d\mathcal{H}_\infty^\delta > 0.
$$

Therefore, in this case,

$$
(2.38) \qquad \int_{Q(0,2)} |x|^\alpha \, d\mathcal{H}_\infty^\delta \left(\int_{Q(0,2)} |x|^{-\frac{\alpha}{p-1}} \, d\mathcal{H}_\infty^\delta \right)^{p-1} = \infty.
$$

If $\alpha \in [\delta(p-1), \infty)$, then $-\frac{\alpha}{p-1} \in (-\infty, -\delta]$ and hence (2.38) also holds true in this case. This completes the proof of the necessity part and hence of Proposition 2.14. $\qquad \square$

From the definition of $\mathscr{A}_{p,\delta}$, it is easy to verify that the new weight class $\mathscr{A}_{p,\delta}$ is monotonically increasing with respect to the first parameter p, i.e., $\mathscr{A}_{p_1,\delta} \subset \mathscr{A}_{p_2,\delta}$ when $1 \leq p_1 \leq p_2 < \infty$. Then, a natural question is whether the new weight class $\mathscr{A}_{p,\delta}$ also enjoys monotonicity with respect to the dimensional parameter δ. We end this subsection by showing the strict monotonicity of capacitary Muckenhoupt weight class $\mathscr{A}_{p,\delta}$ on the dimension δ of Hausdorff contents.

Proposition 2.15. *Let $0 < \delta < \beta \leq n$ and $p \in [1, \infty)$.*

(i) If $p \in (1, \infty)$, then

$$
\mathscr{A}_{p,\delta} \subsetneqq \mathscr{A}_{\frac{p\delta+\beta-\delta}{\beta}, \beta} \subsetneqq \mathscr{A}_{p,\beta}.
$$

More precisely, if $w \in \mathcal{A}_{p,\delta}$, then there exists a positive constant K such that

$$w \in \mathcal{A}_{\frac{p\delta+\beta-\delta}{\beta},\beta} \quad and \quad [w]_{\mathcal{A}_{\frac{p\delta+\beta-\delta}{\beta},\beta}} \le K[w]_{\mathcal{A}_{p,\delta}}.$$

If $w \in \mathcal{A}_{\frac{p\delta+\beta-\delta}{\beta},\beta}$, then there a positive constant K such that

$$w \in \mathcal{A}_{p,\beta} \quad and \quad [w]_{\mathcal{A}_{p,\beta}} \le K[w]_{\mathcal{A}_{\frac{p\delta+\beta-\delta}{\beta},\beta}}.$$

Additionally, there exists a weight $w_1 \in \mathcal{A}_{\frac{p\delta+\beta-\delta}{\beta},\beta}$, but $w_1 \notin \mathcal{A}_{p,\delta}$, and there exists a weight $w_2 \in \mathcal{A}_{p,\beta}$, but $w_2 \notin \mathcal{A}_{\frac{p\delta+\beta-\delta}{\beta},\beta}$.

(ii) For $p = 1$, then

$$\mathcal{A}_{1,\delta} \subsetneqq \mathcal{A}_{1,\beta}.$$

More precisely, if $w \in \mathcal{A}_{1,\delta}$, then there a positive constant K such that

$$w \in \mathcal{A}_{1,\beta} \quad and \quad [w]_{\mathcal{A}_{1,\beta}} \le K[w]_{\mathcal{A}_{1,\delta}}.$$

Additionally, there exists a weight $w \in \mathcal{A}_{1,\beta}$, but $w \notin \mathcal{A}_{1,\delta}$.

Proof. (i) We begin with proving that

$$(2.39) \qquad\qquad \mathcal{A}_{p,\delta} \subsetneqq \mathcal{A}_{\frac{p\delta+\beta-\delta}{\beta},\beta}.$$

For any $w \in \mathcal{A}_{p,\delta}$, from [5, Lemma 2.2], we can easily infer that for any cube Q of $\mathbb{R}^n$,

$$(2.40) \qquad\qquad \frac{1}{\mathcal{H}_\infty^\beta(Q)} \int_Q w(x)^{\frac{\beta}{\delta}} \, d\mathcal{H}_\infty^\beta \lesssim \left(\frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q w(x) \, d\mathcal{H}_\infty^\delta \right)^{\frac{\beta}{\delta}}$$

and hence

$$\frac{1}{\mathcal{H}_\infty^\beta(Q)} \int_Q w(x)^{-\frac{\beta}{(p-1)\delta}} \, d\mathcal{H}_\infty^\beta \lesssim \left(\frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q w(x)^{-\frac{1}{p-1}} \, d\mathcal{H}_\infty^\delta \right)^{\frac{\beta}{\delta}}.$$

Therefore, by this, (2.40) and the Hölder inequality, we have

$$\left(\frac{1}{\mathcal{H}_\infty^\beta(Q)} \int_Q w(x) \, d\mathcal{H}_\infty^\beta \right) \left(\frac{1}{\mathcal{H}_\infty^\beta(Q)} \int_Q w(x)^{-\frac{\beta}{(p-1)\delta}} \, d\mathcal{H}_\infty^\beta \right)^{\frac{(p-1)\delta}{\beta}}$$

$$\lesssim \left(\frac{1}{\mathcal{H}_\infty^\beta(Q)} \int_Q w(x)^{\frac{\beta}{\delta}} \, d\mathcal{H}_\infty^\beta \right)^{\frac{\delta}{\beta}} \left(\frac{1}{\mathcal{H}_\infty^\beta(Q)} \int_Q w(x)^{-\frac{\beta}{(p-1)\delta}} \, d\mathcal{H}_\infty^\beta \right)^{\frac{(p-1)\delta}{\beta}}$$

$$\lesssim \left(\frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q w(x) \, d\mathcal{H}_\infty^\delta \right) \left(\frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q w(x)^{-\frac{1}{p-1}} \, d\mathcal{H}_\infty^\delta \right)^{p-1} \lesssim [w]_{\mathcal{A}_{p,\delta}},$$

which implies that $w \in \mathcal{A}_{\frac{p\delta+\beta-\delta}{\beta},\beta}$. In addition, by Proposition 2.14, we have $|x|^\alpha \in \mathcal{A}_{p,\delta}$ if and only if $\alpha \in (-\delta, \delta(p-1))$, and $|x|^\alpha \in \mathcal{A}_{\frac{p\delta+\beta-\delta}{\beta},\beta}$ if and only if $\alpha \in (-\beta, \delta(p-1))$. Therefore, we choose $\lambda \in (-\beta, -\delta)$, so that $|x|^\lambda \in \mathcal{A}_{\frac{p\delta+\beta-\delta}{\beta},\beta}$, but $|x|^\lambda \notin \mathcal{A}_{p,\delta}$. For $\mathcal{A}_{\frac{p\delta+\beta-\delta}{\beta},\beta} \subsetneqq \mathcal{A}_{p,\beta}$, its validity can be established by applying the Hölder inequality and using Proposition 2.14 in a similar manner.

(ii) For any $w \in \mathcal{A}_{1,\delta}$ and any cube Q of $\mathbb{R}^n$, using the Hölder inequality, (2.40) and the definition of $\mathcal{A}_{1,\delta}$, for $\mathcal{H}_\infty^\delta$-almost every $x \in Q$, we have

$$\frac{1}{\mathcal{H}_\infty^\beta(Q)} \int_Q w(x) \, d\mathcal{H}_\infty^\beta \lesssim \left(\frac{1}{\mathcal{H}_\infty^\beta(Q)} \int_Q w(x)^{\frac{\beta}{\delta}} \, d\mathcal{H}_\infty^\beta \right)^{\frac{\delta}{\beta}} \lesssim \frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q w(x) \, d\mathcal{H}_\infty^\delta \lesssim [w]_{\mathcal{A}_{1,\delta}} w(x).$$

Additionally, for any subset E of $\mathbb{R}^n$, we know that $\mathcal{H}_\infty^\beta(E) \lesssim \mathcal{H}_\infty^\delta(E)^{\frac{\beta}{\delta}}$. Therefore, for $\mathcal{H}_\infty^\beta$-almost every $x \in Q$, we have

$$\frac{1}{\mathcal{H}_\infty^\beta(Q)} \int_Q w(x) \, d\mathcal{H}_\infty^\beta \lesssim [w]_{\mathcal{A}_{1,\delta}} w(x),$$

which implies that $w \in \mathcal{A}_{1,\beta}$. Similar to (i), take $\lambda \in (-\beta, -\delta)$ such that $|x|^\lambda \in \mathcal{A}_{1,\beta}$, but $|x|^\lambda \notin \mathcal{A}_{1,\delta}$. This finishes the proof of Proposition 2.15. $\qquad\square$

By Proposition 2.15, Theorem 1.2 and Theorem 1.1, it is easy to see the following boundedness.

Corollary 2.16. *Let $0 < \delta < \beta \leq n$, $p \in [1, \infty)$ and $w \in \mathcal{A}_{p,\delta}$.*

(i) If $p \in (1, \infty)$, then there exists a positive constant K such that, for any $f \in L_w^{\frac{p\delta+\beta-\delta}{\beta}}(\mathbb{R}^n, \mathcal{H}_\infty^\beta)$,

$$\int_{\mathbb{R}^n} |\mathcal{M}_{\mathcal{H}_\infty^\beta} f(x)|^{\frac{p\delta+\beta-\delta}{\beta}} w(x) \, d\mathcal{H}_\infty^\beta \leq K \int_{\mathbb{R}^n} |f(x)|^{\frac{p\delta+\beta-\delta}{\beta}} w(x) \, d\mathcal{H}_\infty^\beta.$$

(ii) For $p = 1$, then there exists a positive constant K such that, for any $f \in L_w^1(\mathbb{R}^n, \mathcal{H}_\infty^\beta)$ and $t \in (0, \infty)$,

$$w_{\mathcal{H}_\infty^\beta} \left(\left\{ x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}_\infty^\beta} f(x) > t \right\} \right) \leq \frac{K}{t} \int_{\mathbb{R}^n} |f(x)| w(x) \, d\mathcal{H}_\infty^\beta.$$

Remark 2.17. Observe that, in Corollary 2.16(i), $\frac{p\delta+\beta-\delta}{\beta} < p$. Thus, Corollary 2.16(i) provides a slight improvement for the boundedness established in Theorem 1.1 in the integrability index p, at the cost of a larger dimension index δ. This phenomenon can be viewed as a self-improving property on the integrability index of strong or weak type boundedness of capacitary Hardy-Littlewood maximal operators.

3   Proofs of Theorem 1.1, Theorem 1.2 and Corollary 1.6

The following interpolation result is an extension of [5, Lemma 2.5] from $p = 1$ to $p \in [1, \infty)$, and we omit the details here. In what follows, a property is said to hold C-quasieverywhere if the exceptional set has zero capacity.

Lemma 3.1. *Let $p \in [1, \infty)$ and C be a capacity. Suppose that T is a quasi-linear operator defined on $L^p(\mathbb{R}^n, C)$, that is,*

$$|T(f_1 + f_2)| \leq K [|T(f_1)| + |T(f_2)|]$$

for some constant K. If there exist $A_1, A_2 > 0$ such that, for any $t \in (0, \infty)$,

$$C(\{x \in \mathbb{R}^n : |Tf(x)| > t\}) \leq \left(\frac{A_1}{t} \|f\|_{L^p(\mathbb{R}^n, C)} \right)^p$$

for all C-quasieverywhere defined functions $f \in L^p(\mathbb{R}^n, C)$*, and*

$$\|Tf\|_{L^\infty(\mathbb{R}^n,C)} \le A_2 \|f\|_{L^\infty(\mathbb{R}^n,C)}$$

for all C-quasieverywhere defined functions $f \in L^\infty(\mathbb{R}^n, C)$*, then, for any* $q \in (p, \infty)$*, we have the estimate*

$$\|Tf\|_{L^q(\mathbb{R}^n,C)} \le A_3 \|f\|_{L^q(\mathbb{R}^n,C)}$$

for all C-quasieverywhere defined functions $f \in L^q(\mathbb{R}^n, C)$*, where* A_3 *depending only on* K, A_1, A_2, p *and* q. *Moreover,*

$$A_3 := \left(\frac{q}{q-p} \right)^{\frac{1}{q}} \left(2KA_1^{\frac{p}{q}} A_2^{1-\frac{p}{q}} \right)$$

Lemma 3.2. *Let* $\delta \in (0, n]$, $q \in [1, \infty)$ *and* $w \in \mathcal{A}_{q,\delta}$.

(i) *Then exists a positive constant* $K = K(n, \delta, q)$ *such that, for any* $t \in (0, \infty)$,

$$w_{\mathcal{H}_\infty^\delta} \left(\left\{ x \in \mathbb{R}^n : \mathcal{M}_{w_{\mathcal{H}_\infty^\delta}} f(x) > t \right\} \right) \le \frac{K[w]_{\mathcal{A}_{q,\delta}}^3}{t} \int_{\mathbb{R}^n} |f(x)| \, dw_{\mathcal{H}_\infty^\delta}.$$

(ii) *If* $p \in (1, \infty)$, *then there exists a positive constant* $K = K(n, \delta, p, q)$ *such that*

$$\int_{\mathbb{R}^n} \left[\mathcal{M}_{w_{\mathcal{H}_\infty^\delta}} f(x) \right]^p \, dw_{\mathcal{H}_\infty^\delta} \le K[w]_{\mathcal{A}_{q,\delta}}^3 \int_{\mathbb{R}^n} |f(x)|^p \, dw_{\mathcal{H}_\infty^\delta}.$$

To prove Lemma 3.2, we define the *dyadic capacitary Hardy-Littlewood maximal operator* associated with a capacity C by setting, for any $x \in \mathbb{R}^n$,

$$\mathcal{M}_C^{\mathrm{d}} f(x) = \sup_{\text{dyadic } Q \ni x} \frac{1}{C(Q)} \int_Q |f(x)| \, dC,$$

where the supremum is taken over all dyadic cubes Q of $\mathbb{R}^n$ containing x.

Proof of Lemma 3.2. We first prove (i). For any given $t \in (0, \infty)$, we first find that there exists a sequence $\{Q_j\}_{j \in \mathbb{N}}$ of dyadic cubes in $\mathbb{R}^n$ such that

$$(3.1) \qquad \left\{ x \in \mathbb{R}^n : \mathcal{M}_{w_{\mathcal{H}_\infty^\delta}}^{\mathrm{d}} f(x) > \frac{t}{2^{n+q} 5^{q\delta} [w]_{\mathcal{A}_{q,\delta}}} \right\} = \bigcup_{j \in \mathbb{N}} Q_j,$$

and, for any $j \in \mathbb{N}$,

$$(3.2) \qquad \frac{1}{w_{\mathcal{H}_\infty^\delta}(Q_j)} \int_{Q_j} |f(x)| \, dw_{\mathcal{H}_\infty^\delta} > \frac{t}{2^{n+q} 5^{q\delta} [w]_{\mathcal{A}_{q,\delta}}}.$$

Let $\{Q_j^*\}_{j \in \mathbb{N}}$ be the maximal dyadic cubes of $\{Q_j\}_{j \in \mathbb{N}}$. Then

$$(3.3) \qquad \left\{ x \in \mathbb{R}^n : \mathcal{M}_{w_{\mathcal{H}_\infty^\delta}}^{\mathrm{d}} f(x) > \frac{t}{2^{n+q} 5^{q\delta} [w]_{\mathcal{A}_{q,\delta}}} \right\} = \bigcup_{j \in \mathbb{N}} Q_j^*.$$

Fixed $x \in E_t := \{x \in \mathbb{R}^n : \mathcal{M}_{w_{\mathcal{H}^\delta_\infty}} f(x) > t\}$, let Q^x be a cube of $\mathbb{R}^n$ such that $x \in Q^x$ and

$$\frac{1}{w_{\mathcal{H}^\delta_\infty}(Q^x)} \int_{Q^x} |f(y)| \, dw_{\mathcal{H}^\delta_\infty} > t.$$

Let $m \in \mathbb{Z}$ be such that $2^m \le l(Q^x) < 2^{m+1}$, where $l(Q^x)$ denotes the side length of the cube Q^x. Then there are at most 2^n dyadic cubes $\{Q^x_j\}_{j=1}^{N_x}$, with side length 2^{m+1} and $1 \le N_x \le 2^n$, such that

$$Q^x \subset \bigcup_{j=1}^{N_x} Q^x_j,$$

and, for any $j \in \{1, \dots, N_x\}$, $Q^x \cap Q^x_j \ne \emptyset$. Therefore, $x \in 3Q^x_j$ and $Q^x_j \subset 5Q^x$ for any $j \in \{1, \dots, N_x\}$. Additionally, applying Lemma 2.4, we find that

$$\sum_{j=1}^{N_x} \frac{1}{w_{\mathcal{H}^\delta_\infty}(Q^x_j)} \int_{Q^x_j} |f(y)| \, dw_{\mathcal{H}^\delta_\infty} \ge \frac{1}{w_{\mathcal{H}^\delta_\infty}(5Q^x)} \int_{Q^x} |f(y)| \, dw_{\mathcal{H}^\delta_\infty}$$

$$\ge \frac{1}{2^q 5^{q\delta} [w]_{q,\delta} w_{\mathcal{H}^\delta_\infty}(Q^x)} \int_{Q^x} |f(y)| \, dw_{\mathcal{H}^\delta_\infty} > \frac{t}{2^q 5^{q\delta} [w]_{q,\delta}}.$$

Subsequently, there exists a $j_0 \in \{1, \dots, N_x\}$ such that

$$\frac{1}{w_{\mathcal{H}^\delta_\infty}(Q^x_{j_0})} \int_{Q^x_{j_0}} |f(y)| \, dw_{\mathcal{H}^\delta_\infty} > \frac{t}{2^q 5^{q\delta} N_x [w]_{q,\delta}} \ge \frac{t}{2^{n+q} 5^{q\delta} [w]_{q,\delta}}.$$

Therefore, $Q^x_{j_0}$ is one of the above dyadic cubes $\{Q_j\}_{j \in \mathbb{N}}$ as in (3.1), which, combined with (3.3), implies that, for any $t \in (0, \infty)$,

$$E_t = \left\{ x \in \mathbb{R}^n : \mathcal{M}_{w_{\mathcal{H}^\delta_\infty}} f(x) > t \right\} \subset \bigcup_{x \in E_t} 3Q^x_{j_0} \subset \bigcup_{j \in \mathbb{N}} 3Q^*_j.$$

For any $j \in \mathbb{N}$, there exists a sequence of dyadic cubes $\{Q^{*,i}_j\}_{i=1}^{3^n}$ such that $l(Q^{*,i}_j) = l(Q^*_j)$ and

$$3Q^*_j = \bigcup_{i=1}^{3^n} Q^{*,i}_j.$$

Hence, for given t, we have

$$(3.4) \qquad w_{\mathcal{H}^\delta_\infty}\left(\left\{ x \in \mathbb{R}^n : \mathcal{M}_{w_{\mathcal{H}^\delta_\infty}} f(x) > t \right\}\right) \le w_{\mathcal{H}^\delta_\infty}\left(\bigcup_{j \in \mathbb{N}} 3Q^*_j\right) \le \sum_{i=1}^{3^n} w_{\mathcal{H}^\delta_\infty}\left(\bigcup_{j \in \mathbb{N}} Q^{*,i}_j\right).$$

Now applying Lemma 2.6 to the dyadic cubes $\{Q^*_j\}_{j \in \mathbb{N}}$, we deduce that there exists a subfamily $\{Q^*_{j_v}\}_v$ such that, for any dyadic cube Q,

$$\sum_{Q^*_{j_v} \subset Q} w_{\mathcal{H}^\delta_\infty}(Q^*_{j_v}) \le 2 w_{\mathcal{H}^\delta_\infty}(Q),$$

and, for any $i \in \{1, \ldots, 3^n\}$, due to $Q_j^{*,i} \subset 3Q_j^*$, we obtain

$$w_{\mathcal{H}_\infty^\delta}\left(\bigcup_{j\in\mathbb{N}} Q_j^{*,i}\right) \leq K(\delta, q)[w]_{\mathcal{A}_{q,\delta}} \sum_v w_{\mathcal{H}_\infty^\delta}(Q_{j_v}^*).$$

From this, (3.4), (3.2) and (2.29), we deduce that

$$\begin{aligned} w_{\mathcal{H}_\infty^\delta}\left(\left\{x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}_\infty^\delta} f(x) > t\right\}\right) &\leq K(n, \delta, q)[w]_{\mathcal{A}_{q,\delta}} \sum_v w_{\mathcal{H}_\infty^\delta}(Q_{j_v}^*) \\ &\leq K(n, \delta, q)\frac{[w]_{\mathcal{A}_{q,\delta}}}{t} \sum_v \int_{Q_{j_v}^*} |f(x)| \, dw_{\mathcal{H}_\infty^\delta} \\ &\leq K(n, \delta, q)\frac{[w]_{\mathcal{A}_{q,\delta}}^2}{t} \int_{\cup_v Q_{j_v}^*} |f(x)| \, dw_{\mathcal{H}_\infty^\delta} \\ &\leq K(n, \delta, q)\frac{[w]_{\mathcal{A}_{q,\delta}}^2}{t} \int_{\mathbb{R}^n} |f(x)| \, dw_{\mathcal{H}_\infty^\delta}, \end{aligned}$$

which proves Lemma 3.2(i).

The conclusion (ii) is a consequence of Lemma 3.1, Lemma 3.2(i) and the fact that

$$\left\|\mathcal{M}_{w_{\mathcal{H}_\infty^\delta}} f\right\|_{L^\infty(\mathbb{R}^n, w_{\mathcal{H}_\infty^\delta})} \leq \|f\|_{L^\infty(\mathbb{R}^n, w_{\mathcal{H}_\infty^\delta})}.$$

This finishes the proof of Lemma 3.2.   □

Now we show Theorem 1.1.

Proof of Theorem 1.1. We complete the proof by showing (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii)$\Longrightarrow$(i). The proof of (i)$\Longrightarrow$(ii) is obvious.

Now we prove (ii)$\Longrightarrow$(iii). For any given cube Q of $\mathbb{R}^n$, let

$$f(x) := w(x)^{-\frac{1}{p-1}} \mathbf{1}_Q(x), \quad \forall x \in \mathbb{R}^n.$$

Then, for any

$$0 < t < \frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q w(x)^{-\frac{1}{p-1}} \, d\mathcal{H}_\infty^\delta,$$

we have $Q \subset \{x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}_\infty^\delta} f(x) > t\}$. Thus, applying the weak-type (p, p) inequality, we know

$$\begin{aligned} w_{\mathcal{H}_\infty^\delta}(Q) &\leq w_{\mathcal{H}_\infty^\delta}\left(\{x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}_\infty^\delta} f(x) > t\}\right) \\ &\lesssim \frac{1}{t^p} \int_{\mathbb{R}^n} |f(x)|^p w(x) \, d\mathcal{H}_\infty^\delta \sim \frac{1}{t^p} \int_Q w(x)^{-\frac{1}{p-1}} \, d\mathcal{H}_\infty^\delta. \end{aligned}$$

Letting $t \to \frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q w(x)^{-\frac{1}{p-1}} \, d\mathcal{H}_\infty^\delta$, we obtain

$$\left[\frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q w(x) \, d\mathcal{H}_\infty^\delta\right]\left(\frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q w(x)^{-\frac{1}{p-1}} \, d\mathcal{H}_\infty^\delta\right)^{p-1} \lesssim 1,$$

namely, $w \in \mathcal{A}_{p,\delta}$.

Next, we show (iii)$\Longrightarrow$(i). Let $w \in \mathcal{A}_{p,\delta}$ with $p \in (1, \infty)$. Define $u := w^{-\frac{1}{p-1}}$. Then, obviously,

$$u \in \mathcal{A}_{\frac{p}{p-1}} \quad \text{with} \quad [u]_{\mathcal{A}_{\frac{p}{p-1}}} = [w]_{\mathcal{A}_{p,\delta}}^{\frac{1}{p-1}},$$

and, for any cube Q of $\mathbb{R}^n$,

$$(3.5) \qquad w_{\mathcal{H}_\infty^\delta}(Q)^{\frac{1}{p-1}} u_{\mathcal{H}_\infty^\delta}(Q) \le [w]_{\mathcal{A}_{p,\delta}}^{\frac{1}{p-1}} \mathcal{H}_\infty^\delta(Q)^{\frac{p}{p-1}}.$$

For any fixed $x \in \mathbb{R}^n$ and for any cube Q containing x, by (3.5) and Proposition 2.7, we know that

$$\frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q |f(y)| \, d\mathcal{H}_\infty^\delta$$

$$= \frac{w_{\mathcal{H}_\infty^\delta}(Q)^{\frac{1}{p-1}} u_{\mathcal{H}_\infty^\delta}(Q)}{\mathcal{H}_\infty^\delta(Q)^{\frac{p}{p-1}}} \left\{ \frac{\mathcal{H}_\infty^\delta(Q)}{w_{\mathcal{H}_\infty^\delta}(Q)} \left(\frac{1}{u_{\mathcal{H}_\infty^\delta}(Q)} \int_Q |f(y)| \, d\mathcal{H}_\infty^\delta \right)^{p-1} \right\}^{\frac{1}{p-1}}$$

$$\le 4[w]_{\mathcal{A}_{p,\delta}}^{\frac{1}{p-1}} \left\{ \frac{1}{w_{\mathcal{H}_\infty^\delta}(Q)} \int_Q \left[\left(\frac{1}{u_{\mathcal{H}_\infty^\delta}(Q)} \int_Q |f(y)| w(y)^{\frac{1}{p-1}} \, du_{\mathcal{H}_\infty^\delta} \right)^{p-1} \right] d\mathcal{H}_\infty^\delta \right\}^{\frac{1}{p-1}}$$

$$\le 4[w]_{\mathcal{A}_{p,\delta}}^{\frac{1}{p-1}} \left\{ \frac{1}{w_{\mathcal{H}_\infty^\delta}(Q)} \left(\int_Q \left[\mathcal{M}_{u_{\mathcal{H}_\infty^\delta}} \left(|f| w^{\frac{1}{p-1}} \right)(z) \right]^{p-1} d\mathcal{H}_\infty^\delta \right) \right\}^{\frac{1}{p-1}}$$

$$\le 4^{\frac{p}{p-1}} [w]_{\mathcal{A}_{p,\delta}}^{\frac{1}{p-1}} \left\{ \frac{1}{w_{\mathcal{H}_\infty^\delta}(Q)} \left(\int_Q \left[\mathcal{M}_{u_{\mathcal{H}_\infty^\delta}} \left(|f| w^{\frac{1}{p-1}} \right)(z) \right]^{p-1} w(z)^{-1} \, dw_{\mathcal{H}_\infty^\delta} \right) \right\}^{\frac{1}{p-1}}$$

$$\le 4^{\frac{p}{p-1}} [w]_{\mathcal{A}_{p,\delta}}^{\frac{1}{p-1}} \left[\mathcal{M}_{w_{\mathcal{H}_\infty^\delta}} \left\{ \left[\mathcal{M}_{u_{\mathcal{H}_\infty^\delta}} \left(|f| w^{\frac{1}{p-1}} \right) \right]^{p-1} w^{-1} \right\}(x) \right]^{\frac{1}{p-1}},$$

which further implies that

$$\mathcal{M}_{\mathcal{H}_\infty^\delta} f(x) \le 4^{\frac{p}{p-1}} [w]_{\mathcal{A}_{p,\delta}}^{\frac{1}{p-1}} \left[\mathcal{M}_{w_{\mathcal{H}_\infty^\delta}} \left\{ \left[\mathcal{M}_{u_{\mathcal{H}_\infty^\delta}} \left(|f| w^{\frac{1}{p-1}} \right) \right]^{p-1} w^{-1} \right\}(x) \right]^{\frac{1}{p-1}}.$$

On the other hand, by Lemma 3.2, we know that, for any $q \in (1, \infty)$,

$$(3.6) \qquad \int_{\mathbb{R}^n} \left[\mathcal{M}_{u_{\mathcal{H}_\infty^\delta}} f(x) \right]^q du_{\mathcal{H}_\infty^\delta} \le K(n, \delta, p, q)[w]_{\mathcal{A}_{p,\delta}}^{\frac{3}{p-1}} \int_{\mathbb{R}^n} |f(x)|^q \, du_{\mathcal{H}_\infty^\delta}$$

and

$$\int_{\mathbb{R}^n} \left[\mathcal{M}_{w_{\mathcal{H}_\infty^\delta}} f(x) \right]^q dw_{\mathcal{H}_\infty^\delta} \le K(n, \delta, p, q)[w]_{\mathcal{A}_{p,\delta}}^3 \int_{\mathbb{R}^n} |f(x)|^q \, dw_{\mathcal{H}_\infty^\delta}.$$

From this, (3.6) and Proposition 2.7, we deduce that

$$\int_{\mathbb{R}^n} \left[\mathcal{M}_{\mathcal{H}_\infty^\delta} f(x) \right]^p w(x) \, d\mathcal{H}_\infty^\delta \le K(n, \delta, p,)[w]_{\mathcal{A}_{p,\delta}}^{3 + \frac{2}{p} + \frac{p+3}{p-1}} \int_{\mathbb{R}^n} |f(x)|^p w(x) \, d\mathcal{H}_\infty^\delta,$$

which implies that $\mathcal{M}_{\mathcal{H}_\infty^\delta}$ is bounded on $L_w^p(\mathbb{R}^n, \mathcal{H}_\infty^\delta)$. This finishes the proof of Theorem 1.1. $\qquad \square$

We next show Theorem 1.2.

Proof of Theorem 1.2. To prove (i)$\Longrightarrow$(ii), we need to show that, for any cube Q,

$$\frac{w_{\mathcal{H}^\delta_\infty}(Q)}{\mathcal{H}^\delta_\infty(Q)} \leq Kw(x) \text{ for } \mathcal{H}^\delta_\infty-\text{almost every } x \in Q,$$

where K is a positive constant independent of Q and x.

Let Q be a given cube of $\mathbb{R}^n$. For any $E \subset Q$, let $f := \mathbf{1}_E$. Then, for any $x \in Q$,

$$\mathcal{M}_{\mathcal{H}^\delta_\infty} f(x) \geq \frac{1}{\mathcal{H}^\delta_\infty(Q)} \int_Q |f(y)| \, d\mathcal{H}^\delta_\infty = \frac{\mathcal{H}^\delta_\infty(E)}{\mathcal{H}^\delta_\infty(Q)},$$

and hence, for any $t \in (0, \frac{\mathcal{H}^\delta_\infty(E)}{\mathcal{H}^\delta_\infty(Q)})$,

$$Q \subset \left\{ x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}^\delta_\infty} f(x) > t \right\}.$$

This, combined with the weak-type $(1, 1)$ inequality, implies that

$$w_{\mathcal{H}^\delta_\infty}(Q) \leq w_{\mathcal{H}^\delta_\infty}\left(\left\{ x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}^\delta_\infty} f(x) > t \right\} \right) \leq \frac{K}{t} w_{\mathcal{H}^\delta_\infty}(E).$$

Letting $t \to \frac{\mathcal{H}^\delta_\infty(E)}{\mathcal{H}^\delta_\infty(Q)}$, we obtain

$$(3.7) \qquad\qquad\qquad \frac{\mathcal{H}^\delta_\infty(E)}{\mathcal{H}^\delta_\infty(Q)} \leq K \frac{w_{\mathcal{H}^\delta_\infty}(E)}{w_{\mathcal{H}^\delta_\infty}(Q)}.$$

Now for any $a \in (0, \infty)$, let $E_a := \{x \in Q : w(x) < a\}$ and

$$\operatorname*{ess\,inf}_{x \in Q} w := \inf \left\{ a \in (0, \infty) : \mathcal{H}^\delta_\infty(E_a) > 0 \right\}.$$

Then, when $a > \operatorname{ess\,inf}_{x \in Q} w$, we have $\mathcal{H}^\delta_\infty(E_a) > 0$, which, together with (3.7), implies that

$$\frac{w_{\mathcal{H}^\delta_\infty}(Q)}{\mathcal{H}^\delta_\infty(Q)} \leq K \frac{w_{\mathcal{H}^\delta_\infty}(E_a)}{\mathcal{H}^\delta_\infty(E_a)}.$$

By this and the fact that

$$w_{\mathcal{H}^\delta_\infty}(E_a) = \int_{E_a} w(x) \, d\mathcal{H}^\delta_\infty \leq a \mathcal{H}^\delta_\infty(E_a),$$

we know that $\frac{w_{\mathcal{H}^\delta_\infty}(Q)}{\mathcal{H}^\delta_\infty(Q)} \leq aK$. Letting $a \to \operatorname{ess\,inf}_{x \in Q} w$, we further find that

$$\frac{w_{\mathcal{H}^\delta_\infty}(Q)}{\mathcal{H}^\delta_\infty(Q)} \leq K \operatorname*{ess\,inf}_{x \in Q} w,$$

namely, we have

$$\frac{w_{\mathcal{H}^\delta_\infty}(Q)}{\mathcal{H}^\delta_\infty(Q)} \leq Kw(x) \text{ for } \mathcal{H}^\delta_\infty-\text{almost every } x \in Q.$$

This means $w \in \mathcal{A}_{1,\delta}$.

Next we prove (ii)$\Longrightarrow$(i). To this end, we prove the following weak-type (p, p) inequality: if $w \in \mathcal{A}_{p,\delta}$ with $p \in [1, \infty)$, then, for any $f \in L_w^p(\mathbb{R}^n, \mathcal{H}_\infty^\delta)$ and any $t \in (0, \infty)$,

$$(3.8) \qquad w_{\mathcal{H}_\infty^\delta}\left(\left\{x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}_\infty^\delta} f(x) > t\right\}\right) \leq K(n, \delta, p)[w]_{\mathcal{A}_{p,\delta}}^{3+\frac{1}{p}} \frac{1}{t^p} \int_{\mathbb{R}^n} |f(x)|^p w(x) \, d\mathcal{H}_\infty^\delta.$$

For any given $t \in (0, \infty)$, let $\{Q_j\}_{j \in \mathbb{N}}$ be a sequence of dyadic cubes of $\mathbb{R}^n$ such that

$$\left\{x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}_\infty^\delta}^{\mathrm{d}} f(x) > \frac{t}{2^{n+\delta}}\right\} = \bigcup_{j \in \mathbb{N}} Q_j,$$

and, for any $j \in \mathbb{N}$,

$$(3.9) \qquad\qquad\qquad \frac{1}{\mathcal{H}_\infty^\delta(Q_j)} \int_{Q_j} |f(x)| \, d\mathcal{H}_\infty^\delta > \frac{t}{2^{n+\delta}}.$$

Let $\{Q_j^*\}_{j \in \mathbb{N}}$ be the maximal dyadic cubes of $\{Q_j\}_{j \in \mathbb{N}}$. Then

$$\left\{x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}_\infty^\delta}^{\mathrm{d}} f(x) > \frac{t}{2^{n+\delta}}\right\} = \bigcup_{j \in \mathbb{N}} Q_j^*.$$

Since $w \in \mathcal{A}_{p,\delta}$, then by (2.4), we conclude that, for any dyadic cube Q of $\mathbb{R}^n$,

$$(3.10) \qquad \int_Q |f(x)| \, d\mathcal{H}_\infty^\delta \leq 2[w]_{\mathcal{A}_{p,\delta}}^{\frac{1}{p}} \left(\int_Q |f(x)|^p w(x) \, d\mathcal{H}_\infty^\delta\right)^{\frac{1}{p}} \mathcal{H}_\infty^\delta(Q) \left(w_{\mathcal{H}_\infty^\delta}(Q)\right)^{-\frac{1}{p}}.$$

Combining (3.9) and (3.10), we further conclude that, for any $j \in \mathbb{N}$,

$$w_{\mathcal{H}_\infty^\delta}(Q_j^*) \leq K(n, \delta, p) \frac{[w]_{\mathcal{A}_{p,\delta}}}{t^p} \int_{Q_j^*} |f(x)|^p w(x) \, d\mathcal{H}_\infty^\delta.$$

Now by an argument similar to that used in the proof of Lemma 3.2(i), we find that

$$w_{\mathcal{H}_\infty^\delta}\left(\left\{x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}_\infty^\delta} f(x) > t\right\}\right) \leq w_{\mathcal{H}_\infty^\delta}\left(\bigcup_{j \in \mathbb{N}} 3Q_j^*\right).$$

Moreover, (3.8) holds true. This finishes the proof of Theorem 1.2. $\qquad\square$

We end this section by giving the proof of Corollary 1.6. To do this, we need the following lemma.

Lemma 3.3. *Let $\delta \in (0, n]$ and $f : \mathbb{R}^n \to \mathbb{R}$. Then, for any $x \in \mathbb{R}^n$,*

$$\left[\mathcal{M}(|f|^{\frac{n}{\delta}})(x)\right]^{\frac{\delta}{n}} \leq \left(\frac{n}{\delta}\right)^{\frac{\delta}{n}} \mathcal{M}_{\mathcal{H}_\infty^\delta} f(x).$$

Proof. This is an easy consequence of the inequality

$$\int_{\mathbb{R}^n} |f(x)| dx \leq \frac{n}{\delta} \left(\int_{\mathbb{R}^n} |f(x)|^{\frac{\delta}{n}} \, d\mathcal{H}_\infty^\delta\right)^{\frac{n}{\delta}},$$

which comes from [32, Lemma 3]. $\qquad\square$

We finally end this section by proving Corollary 1.6.

Proof of Corollary 1.6. We first prove (i). Let $w \in \mathcal{A}_{p,\delta}$ with $p \in (1, \infty)$. Then, by Theorem 1.1, we know that $\mathcal{M}_{\mathcal{H}_\infty^\delta}$ is bounded on $L_w^p(\mathbb{R}^n, \mathcal{H}_\infty^\delta)$. From this and Lemma 3.3, we deduce that, for any $f \in L_w^{\frac{p\delta}{n}}(\mathbb{R}^n, \mathcal{H}_\infty^\delta)$,

$$\int_{\mathbb{R}^n} [\mathcal{M}f(x)]^{\frac{p\delta}{n}} w(x) \, d\mathcal{H}_\infty^\delta \lesssim \int_{\mathbb{R}^n} |f(x)|^{\frac{p\delta}{n}} w(x) \, d\mathcal{H}_\infty^\delta,$$

which, combined with Proposition 2.7, implies that

$$(3.11) \qquad \int_{\mathbb{R}^n} [\mathcal{M}f(x)]^{\frac{p\delta}{n}} \, dw_{\mathcal{H}_\infty^\delta} \lesssim \int_{\mathbb{R}^n} |f(x)|^{\frac{p\delta}{n}} \, dw_{\mathcal{H}_\infty^\delta}.$$

On the other hand, from Lemma 3.3, it is not difficult to obtain

$$\|\mathcal{M}f\|_{L^\infty(\mathbb{R}^n, w_{\mathcal{H}_\infty^\delta})} \leq \frac{n}{\delta} \|f\|_{L^\infty(\mathbb{R}^n, w_{\mathcal{H}_\infty^\delta})}$$

By this, (3.11) and Lemma 3.1, we conclude that, for any $q \in (\frac{p\delta}{n}, \infty)$,

$$\|\mathcal{M}f\|_{L^q(\mathbb{R}^n, w_{\mathcal{H}_\infty^\delta})} \lesssim \|f\|_{L^q(\mathbb{R}^n, w_{\mathcal{H}_\infty^\delta})}.$$

The proof of (ii) is similar to that of (i) and we omit the details.

Finally, we prove (iii). According Theorems 1.1 and 1.2, we know that, for any $q \in [p, \infty)$, $t \in (0, \infty)$ and $f \in L_w^q(\mathbb{R}^n, \mathcal{H}_\infty^\delta)$,

$$w_{\mathcal{H}_\infty^\delta}\left(\left\{x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}_\infty^\delta} f(x) > t\right\}\right) \lesssim \frac{1}{t^q} \int_{\mathbb{R}^n} |f(x)|^q w(x) \, d\mathcal{H}_\infty^\delta.$$

Combining this with Lemma 3.3, we further conclude that, for any $q \in [\frac{p\delta}{n}, \infty)$ and any $f \in L_w^q(\mathbb{R}^n, \mathcal{H}_\infty^\delta)$,

$$w_{\mathcal{H}_\infty^\delta}\left(\{x \in \mathbb{R}^n : \mathcal{M}f(x) > t\}\right) \lesssim \frac{1}{t^q} \int_{\mathbb{R}^n} |f(x)|^q w(x) \, d\mathcal{H}_\infty^\delta, \quad \forall t \in (0, \infty).$$

The proof of Corollary 1.6 is completed. □

4 Proofs of Theorem 1.7 and Theorem 1.9

In this section, we first give the proof of the reverse Hölder inequality, Theorem 1.7, for capacitary Muckenhoupt weight class $\mathcal{A}_{p,\delta}$ with $p \in [1, \infty)$ and $\delta \in (0, n]$, which is inspired by the celebrated work of R. R. Coifman and C. Fefferman [8]. As an application, we then obtain the self-improving property of $\mathcal{A}_{p,\delta}$; see Theorem 1.9.

We first recall the definition of $\mathcal{H}_\infty^\delta$-quasicontinuous function; see [1, p. 15].

Definition 4.1. Let $\delta \in (0, n]$. We say that a function f is $\mathcal{H}_\infty^\delta$-*quasicontinuous* on $\mathbb{R}^n$, if for every $\varepsilon \in (0, \infty)$, there exists an open set O_ε such that $\mathcal{H}_\infty^\delta(O_\varepsilon) < \varepsilon$ and $f|_{\mathbb{R}^n \setminus O_\varepsilon}$ is continuous.

Remark 4.2. Observe that the classical Egorov theorem implies that any Lebesgue integrable function on $\mathbb{R}^n$ is $\mathcal{H}^n_\infty$-quasicontinuous. Moreover, from Proposition 2.14, we know that the capacitary Muckenhoupt $\mathcal{A}_{p,\delta}$-weight function $w(x) := |x|^\alpha$ defined on $\mathbb{R}^n$ with $\alpha \in (-\delta, \delta(p-1))$ is $\mathcal{H}^\delta_\infty$-quasicontinuous, where $p \in [1, \infty)$ and $\delta \in (0, n]$.

Remark 4.3. Let $\delta \in (0, n]$ and f be $\mathcal{H}^\delta_\infty$-quasicontinuous on $\mathbb{R}^n$. For any given $E \subset \mathbb{R}^n$, if $f \in L^1(E, \mathcal{H}^\delta_\infty)$, then there exists a sequence $\{f_k\}_{k\in\mathbb{N}}$ of bounded continuous functions on $\mathbb{R}^n$ such that

$$\lim_{k\to\infty} \int_E |f(x) - f_k(x)| \, d\mathcal{H}^\delta_\infty = 0.$$

For the proof, we refer the reader to [31, Proposition 2.2].

Remark 4.4. We remark that the $\mathcal{H}^\delta_\infty$-quasicontinuity serves as a Lusin-type theorem in the Hausdorff content setting, which states that an $\mathcal{H}^\delta_\infty$-quasicontinuous function is a continuous function on nearly all its domain. This condition is natural and ensures the denseness of continuous functions with compact support in Choquet spaces. Once this condition is removed, the density breaks down; see [1, p. 15, Proposition 1].

To prove Theorem 1.7, we need the following differential theorem on Choquet integrals with respect to Hausdorff contents.

Lemma 4.5. *Let $\delta \in (0, n]$ and f be $\mathcal{H}^\delta_\infty$-quasicontinuous on $\mathbb{R}^n$. For any given subset $E \subset \mathbb{R}^n$, if $f \in L^1(E, \mathcal{H}^\delta_\infty)$ then, for any $\mathcal{H}^\delta_\infty$-almost every $x \in E$,*

$$\lim_{\substack{Q\ni x \\ l(Q)\to 0}} \frac{1}{\mathcal{H}^\delta_\infty(Q)} \int_{E\cap Q} |f(y) - f(x)| \, d\mathcal{H}^\delta_\infty(y) = 0.$$

Proof. Let $f \in L^1(E, \mathcal{H}^\delta_\infty)$ with $E \subset \mathbb{R}^n$. Then there exists a sequence $\{f_k\}_{k\in\mathbb{N}}$ of bounded continuous functions on $\mathbb{R}^n$ such that

$$(4.1) \qquad \lim_{k\to\infty} \int_E |f_k(x) - f(x)| \, d\mathcal{H}^\delta_\infty = 0.$$

For any $x \in E$, any cube $Q \ni x$ and $k \in \mathbb{N}$, we have

$$(4.2) \qquad \int_{E\cap Q} |f(y) - f(x)| \, d\mathcal{H}^\delta_\infty(y)$$

$$\leq 3 \int_Q |f(y)\mathbf{1}_E(y) - f_k(y)\mathbf{1}_E(y)| \, d\mathcal{H}^\delta_\infty(y) + 3 \int_{E\cap Q} |f_k(y) - f_k(x)| \, d\mathcal{H}^\delta_\infty(y)$$

$$+ 3 \int_{E\cap Q} |f_k(x) - f(x)| \, d\mathcal{H}^\delta_\infty(y).$$

As f_k is a bounded continuous function, it is easy to see that

$$(4.3) \qquad \lim_{Q\ni x, l(Q)\to 0} \frac{1}{\mathcal{H}^\delta_\infty(Q)} \int_{E\cap Q} |f_k(y) - f_k(x)| \, d\mathcal{H}^\delta_\infty(y) = 0.$$

Therefore, by (4.2) and (4.3), we find that, for any $x \in E$,

$$\limsup_{Q\ni x, l(Q)\to 0} \frac{1}{\mathcal{H}^\delta_\infty(Q)} \int_{E\cap Q} |f(y) - f(x)| \, d\mathcal{H}^\delta_\infty(y)$$

$$\leq 3 \mathcal{M}_{\mathcal{H}_\infty^\delta}(f \mathbf{1}_E - f_k \mathbf{1}_E)(x) + 3|f_k(x) - f(x)| \mathbf{1}_E(x).$$

From this and Theorem 1.1 with $w \equiv 1$ therein, we deduce that, for any $\lambda > 0$,

$$\mathcal{H}_\infty^\delta \left(\left\{ x \in E : \limsup_{Q \ni x, l(Q) \to 0} \frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_{E \cap Q} |f(y) - f(x)| \, d\mathcal{H}_\infty^\delta(y) > \lambda \right\} \right)$$

$$\leq \mathcal{H}_\infty^\delta \left(\left\{ x \in E : 3 \mathcal{M}_{\mathcal{H}_\infty^\delta}(f \mathbf{1}_E - f_k \mathbf{1}_E)(x) + 3|f_k(x) - f(x)| \mathbf{1}_E(x) > \lambda \right\} \right)$$

$$\lesssim \frac{1}{\lambda} \int_E |f_k(x) - f(x)| \, d\mathcal{H}_\infty^\delta.$$

Thus, by (4.1), we conclude that, for any $\lambda > 0$,

$$\mathcal{H}_\infty^\delta \left(\left\{ x \in E : \limsup_{Q \ni x, l(Q) \to 0} \frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_{E \cap Q} |f(y) - f(x)| \, d\mathcal{H}_\infty^\delta(y) > \lambda \right\} \right) = 0,$$

which implies that

$$\mathcal{H}_\infty^\delta \left(\left\{ x \in E : \limsup_{Q \ni x, l(Q) \to 0} \frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_{E \cap Q} |f(y) - f(x)| \, d\mathcal{H}_\infty^\delta(y) > 0 \right\} \right) = 0.$$

This finishes the proof of Lemma 4.5. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark 4.6. It should be pointed out that the requirement of $\mathcal{H}_\infty^\delta$-quasicontiniuous for f in Lemma 4.5 is necessary in some sense. To show this, we construct the following counterexample.

Counterexample 4. Let $n \geq 2$ and $\delta \in (0, n - 1]$. If $P \subset \mathbb{R}^n$ is a closed cube with side length 1, then

   (i) $\mathbf{1}_P$ is not a $\mathcal{H}_\infty^\delta$-quasicontinous function;

  (ii) $\mathcal{H}_\infty^\delta(\partial P) = 1$ and, for any $x \in \partial P$,

$$\limsup_{Q \ni x, l(Q) \to 0} \frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q |\mathbf{1}_P(y) - \mathbf{1}_P(x)| \, d\mathcal{H}_\infty^\delta = 1,$$

     where ∂P denotes the boundary of P.

Proof. To prove (i), it is enough to show that, for any closed subset $F \subset \mathbb{R}^n$, if $\mathbf{1}_P$ is continuous on F, then $\mathcal{H}_\infty^\delta(\mathbb{R}^n \backslash F) \geq 1$. To this end, we claim that $F \cap (\mathbb{R}^n \backslash P)$ is closed. Otherwise, there exists $\{x_n\}_{n \in \mathbb{N}} \subset F \cap (\mathbb{R}^n \backslash P)$ and $x_0 \in \mathbb{R}^n \backslash [F \cap (\mathbb{R}^n \backslash P)]$ such that $|x_n - x_0| \to 0$ as $n \to \infty$. It follows, from F begin closed, that $x_0 \in F$ and hence $x_0 \notin \mathbb{R}^n \backslash P$. Therefore, $x_0 \in \partial P$. This further implies $\mathbf{1}_P(x_0) = 1$, but $\mathbf{1}_P(x_n) = 0$, which is contradiction to the continuity of $\mathbf{1}_P$ on F. This proves the above claim. By this claim, we know that

$$d(F \cap (\mathbb{R}^n \backslash P), P) := \inf \{|x - y| : \ x \in F \cap (\mathbb{R}^n \backslash P), y \in P\} > 0.$$

Let

$$E := \left\{ x \in \mathbb{R}^n \backslash P : \ d(x, P) = \frac{1}{2} d(F \cap (\mathbb{R}^n \backslash P), P) \right\},$$

where $d(x, P) := \inf\{|x - y| : y \in P\}$. Then $E \subset \mathbb{R}^n \backslash F$ and contains a $(n-1)$-dimensional cube W with side length 1 in $\mathbb{R}^n$. Thus, by Remark 2.3(i), we have $\mathcal{H}_\infty^\delta(E) \geq \mathcal{H}_\infty^\delta(W) = 1$ and hence $\mathcal{H}_\infty^\delta(\mathbb{R}^n \backslash F) \geq 1$.

For (ii), we first have $\mathcal{H}_\infty^\delta(\partial P) = 1$ by Remark 2.3(i) again. For any $x \in \partial P$, let $\{Q_n\}_{n \in \mathbb{N}}$ be a sequence of cubes of $\mathbb{R}^n$ satisfying that $\lim_{n \to \infty} l(Q_n) = 0$ and, for each $n \in \mathbb{N}$, $Q_n \ni x$ and one $(n-1)$-dimensional side of Q_n denoted by W_n, $W_n \subset \mathbb{R}^n \backslash P$. Then by Remark 2.3(i), we have

$$
\limsup_{Q \ni x, l(Q) \to 0} \frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q |\mathbf{1}_P(y) - \mathbf{1}_P(x)| \, d\mathcal{H}_\infty^\delta
$$
$$
\geq \frac{1}{\mathcal{H}_\infty^\delta(Q_n)} \int_{Q_n} |\mathbf{1}_P(y) - \mathbf{1}_P(x)| \, d\mathcal{H}_\infty^\delta
$$
$$
\geq \frac{1}{\mathcal{H}_\infty^\delta(Q_n)} \int_{W_n} |\mathbf{1}_P(y) - \mathbf{1}_P(x)| \, d\mathcal{H}_\infty^\delta
$$
$$
= \frac{\mathcal{H}_\infty^\delta(W_n)}{\mathcal{H}_\infty^\delta Q_n} = 1.
$$

Thus, the statement (ii) is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Using Lemma 4.5, we obtain the following Calderón-Zygmund decomposition with respect to the Hausdorff content $\mathcal{H}_\infty^\delta$.

Lemma 4.7. *Let $\delta \in (0, n]$ and w be a $\mathcal{H}_\infty^\delta$-quasicontinuous weigh function on $\mathbb{R}^n$. For any given cube Q of $\mathbb{R}^n$, if $\int_Q w(x) \, d\mathcal{H}_\infty^\delta < \infty$, then, for any*

$$
\lambda > \frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q w(x) \, d\mathcal{H}_\infty^\delta,
$$

there exists a collection $\{Q_j\}_{j \in \mathbb{N}}$ of non-overlapping dyadic subcubes of Q such that

  (i) *for any $j \in \mathbb{N}$, $\lambda < \frac{1}{\mathcal{H}_\infty^\delta(Q_j)} \int_{Q_j} w(x) \, d\mathcal{H}_\infty^\delta \leq 2^\delta \lambda$;*

  (ii) *for $\mathcal{H}_\infty^\delta$-almost every $x \in Q \backslash \bigcup_{j \in \mathbb{N}} Q_j$, $w(x) \leq 2\lambda$.*

Proof. We first divide the cube Q into a mesh of 2^n subcubes $\{R_j\}_{j=1}^{2^n}$ with equal side length. For any R_j, if it satisfies

$$
\frac{1}{\mathcal{H}_\infty^\delta(R_j)} \int_{R_j} w(x) \, d\mathcal{H}_\infty^\delta > \lambda,
$$

then it is selected as desired. Otherwise, we subdivide each unselected subcube R_j into 2^n cubes with equal side length and continue in this way indefinitely. We denote by $\{Q_j\}_{j \in \mathbb{N}}$ the collection of all selected subcubes of Q. Therefore, for any $j \in \mathbb{N}$, we have

$$
\frac{1}{\mathcal{H}_\infty^\delta(Q_j)} \int_{Q_j} w(x) \, d\mathcal{H}_\infty^\delta > \lambda.
$$

Moreover, if we denote by Q_j^* the parent cube of Q_j, then

$$
\frac{1}{\mathcal{H}_\infty^\delta(Q_j)} \int_{Q_j} w(x) \, d\mathcal{H}_\infty^\delta \leq \frac{2^\delta}{\mathcal{H}_\infty^\delta(Q_j^*)} \int_{Q_j^*} w(x) \, d\mathcal{H}_\infty^\delta \leq 2^\delta \lambda.
$$

For any $x \in Q \setminus \bigcup_{j \in \mathbb{N}} Q_j$ and any dyadic subcube P of Q with $x \in P$, we have

$$(4.4) \qquad \frac{1}{\mathcal{H}_\infty^\delta(P)} \int_P w(x) \, d\mathcal{H}_\infty^\delta \le \lambda.$$

Therefore, by Lemma 4.5 and (4.4), we have

$$w(x) \le 2 \lim_{\substack{P \ni x \\ l(P) \to 0}} \frac{1}{\mathcal{H}_\infty^\delta(P)} \int_{Q \cap P} |w(y) - w(x)| \, d\mathcal{H}_\infty^\delta(y) + 2 \lim_{\substack{P \ni x \\ l(P) \to 0}} \frac{1}{\mathcal{H}_\infty^\delta(P)} \int_{P \cap Q} w(y) \, d\mathcal{H}_\infty^\delta(y) \le 2\lambda,$$

which completes the proof of Lemma 4.7. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Lemma 4.8. *Let $\delta \in (0, n]$ and w be a $\mathcal{H}_\infty^\delta$-quasicontinuous function on $\mathbb{R}^n$. If $w \in \mathcal{A}_{p,\delta}$ with $p \in [1, \infty)$, then, for any $\beta \in (0, [w]_{\mathcal{A}_{p,\delta}}^{-1})$, there exists a positive constant $K(p, \beta, [w]_{\mathcal{A}_{p,\delta}})$ such that, for any cube Q of $\mathbb{R}^n$,*

$$\mathcal{H}_\infty^\delta(\{x \in Q : \; w(x) > \beta W_Q\}) \ge K(p, \beta, [w]_{\mathcal{A}_{p,\delta}}) \mathcal{H}_\infty^\delta(Q).$$

Here and thereafter, $W_Q := \frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q w(x) \, d\mathcal{H}_\infty^\delta$.

Proof. When $p = 1$, since $\beta \in (0, [w]_{\mathcal{A}_{p,\delta}}^{-1})$, by the definition of $\mathcal{A}_{1,\delta}$, it is clear that

$$\mathcal{H}_\infty^\delta(\{x \in Q : \; w(x) > \beta W_Q\}) = \mathcal{H}_\infty^\delta(Q).$$

When $p \in (1, \infty)$, let $E := \{x \in Q : \; w(x) \le \beta W_Q\}$. Then

$$\begin{aligned}
\frac{1}{\beta} \left(\frac{\mathcal{H}_\infty^\delta(E)}{\mathcal{H}_\infty^\delta(Q)} \right)^{p-1} &= W_Q \left[\frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_E (\beta W_Q)^{-\frac{1}{p-1}} \, d\mathcal{H}_\infty^\delta \right]^{p-1} \\
&\le W_Q \left[\frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q w(x)^{-\frac{1}{p-1}} \, d\mathcal{H}_\infty^\delta \right]^{p-1} \\
&\le [w]_{\mathcal{A}_{p,\delta}}.
\end{aligned}$$

Therefore, we have

$$\mathcal{H}_\infty^\delta(\{x \in Q : \; w(x) > \beta W_Q\}) \ge \mathcal{H}_\infty^\delta(Q) - \mathcal{H}_\infty^\delta(E) \ge \left(1 - (\beta [w]_{\mathcal{A}_{p,\delta}})^{\frac{1}{p-1}} \right) \mathcal{H}_\infty^\delta(Q).$$

This finishes the proof of Lemma 4.8. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Lemma 4.9. *Let $\delta \in (0, n]$ and w be a $\mathcal{H}_\infty^\delta$-quasicontinuous function on $\mathbb{R}^n$. If $w \in \mathcal{A}_{p,\delta}$ with $p \in [1, \infty)$, then, for any $\beta \in (0, [w]_{\mathcal{A}_{p,\delta}}^{-1})$, there exists a positive constant $K(n, \delta, p, \beta, [w]_{\mathcal{A}_{p,\delta}})$ such that, for any cube Q of $\mathbb{R}^n$ and any $\lambda > W_Q$,*

$$w_{\mathcal{H}_\infty^\delta}(\{x \in Q : \; w(x) > 2\lambda\}) \le K(n, \delta, p, \beta, [w]_{\mathcal{A}_{p,\delta}}) \lambda \mathcal{H}_\infty^\delta(\{x \in Q : w(x) > \beta\lambda\}).$$

Proof. By Lemma 4.7, we know that there exists a collection $\{Q_j\}_{j \in \mathbb{N}}$ of non-overlapping dyadic subcubes of Q such that

$$w_{\mathcal{H}_\infty^\delta}(\{x \in Q : \; w(x) > 2\lambda\}) \le w_{\mathcal{H}_\infty^\delta} \left(\bigcup_{j \in \mathbb{N}} Q_j \right),$$

and, for any $j \in \mathbb{N}$,

$$\lambda < \frac{1}{\mathcal{H}_\infty^\delta(Q_j)} \int_{Q_j} w(x) \, d\mathcal{H}_\infty^\delta \le 2^\delta \lambda.$$

Furthermore, although the Q_j here is a dyadic subcube of Q, the corresponding conclusions of Lemmas 2.6 and 2.12 still hold true. Therefore, by Lemma 4.8, we find that there exists a subfamily $\{Q_{j_\nu}\}_{\nu=1}^N$ such that

$$w_{\mathcal{H}_\infty^\delta}(\{x \in Q : w(x) > 2\lambda\}) \le 2 \sum_{\nu=1}^N w_{\mathcal{H}_\infty^\delta}(Q_{j_\nu}) \le K(\delta)\lambda \sum_{\nu=1}^N \mathcal{H}_\infty^\delta(Q_{j_\nu})$$

$$\le K(\delta, p, \beta, [w]_{\mathscr{A}_{p,\delta}})\lambda \sum_{\nu=1}^N \mathcal{H}_\infty^\delta(\{x \in Q_{j_\nu} : w(x) > \beta W_{Q_{j_\nu}}\})$$

$$\le K(\delta, p, \beta, [w]_{\mathscr{A}_{p,\delta}})\lambda \sum_{\nu=1}^N \int_{Q_{j_\nu}} \mathbf{1}_{\{x \in Q : w(x) > \beta\lambda\}}(x) \, d\mathcal{H}_\infty^\delta$$

$$\le K(n, \delta, p, \beta, [w]_{\mathscr{A}_{p,\delta}})\lambda \mathcal{H}_\infty^\delta(\{x \in Q : w(x) > \beta\lambda\}).$$

This finishes the proof of Lemma 4.9. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Now we are ready to prove Theorems 1.7 and 1.9.

Proof of Theorem 1.7. For any given cube Q of $\mathbb{R}^n$, let

$$W_Q := \frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q w(x) \, d\mathcal{H}_\infty^\delta \quad \text{and} \quad E := \{x \in Q : w(x) > 2W_Q\}.$$

We first show that there exists a positive constant $K = K(n, \delta, p, [w]_{\mathscr{A}_{p,\delta}})$ such that, for any $\gamma \in (0, 1)$,

$$(4.5) \qquad \frac{1}{\gamma} \int_E \left[\left(\frac{w(x)}{2} \right)^\gamma - (W_Q)^\gamma \right] w(x) \, d\mathcal{H}_\infty^\delta \le K \int_Q w(x)^{1+\gamma} \, d\mathcal{H}_\infty^\delta.$$

By Proposition 2.7 and Lemma 4.9, we have

$$\frac{1}{\gamma} \int_E \left[\left(\frac{w(x)}{2} \right)^\gamma - (W_Q)^\gamma \right] w(x) \, d\mathcal{H}_\infty^\delta$$

$$\le \frac{4}{\gamma} \int_E \left(\frac{w(x)}{2} \right)^\gamma - (W_Q)^\gamma \, dw_{\mathcal{H}_\infty^\delta}$$

$$= \frac{4}{\gamma} \int_0^\infty w_{\mathcal{H}_\infty^\delta} \left\{ x \in E : \left(\frac{w(x)}{2} \right)^\gamma - (W_Q)^\gamma > t \right\} dt$$

$$= 4 \int_{W_Q}^\infty t^{\gamma-1} w_{\mathcal{H}_\infty^\delta} \left\{ x \in E : \left(\frac{w(x)}{2} \right)^\gamma - (W_Q)^\gamma > t^\gamma - (W_Q)^\gamma \right\} dt$$

$$= 4 \int_{W_Q}^\infty t^{\gamma-1} w_{\mathcal{H}_\infty^\delta} \{ x \in Q : w(x) > 2t \} \, dt$$

$$\le K(n, \delta, p, [w]_{\mathscr{A}_{p,\delta}}) \int_{W_Q}^\infty t^\gamma \mathcal{H}_\infty^\delta \left\{ x \in Q : w(x) > \frac{1}{2[w]_{\mathscr{A}_{p,\delta}}} t \right\} dt$$

$$\leq K(n, \delta, p, [w]_{\mathcal{A}_{p,\delta}}) \int_Q w(x)^{1+\gamma} d\mathcal{H}_\infty^\delta,$$

which means that (4.5) holds true.

Next we prove that

$$(4.6) \qquad \int_Q w(x)^{1+\gamma} d\mathcal{H}_\infty^\delta \leq 4 \int_E \left[\left(\frac{w(x)}{2} \right)^\gamma - (W_Q)^\gamma \right] w(x) d\mathcal{H}_\infty^\delta + 8(W_Q)^{1+\gamma} \mathcal{H}_\infty^\delta(Q).$$

Indeed, we have

$$
\int_Q w(x)^{1+\gamma} d\mathcal{H}_\infty^\delta
$$

$$
\leq \int_{Q \setminus E} w(x)^{1+\gamma} d\mathcal{H}_\infty^\delta + \int_E w(x)^{1+\gamma} d\mathcal{H}_\infty^\delta
$$

$$
\leq \int_{Q \setminus E} w(x)^{1+\gamma} d\mathcal{H}_\infty^\delta + 2^{1+\gamma} \int_E \left[\left(\frac{w(x)}{2} \right)^\gamma - (W_Q)^\gamma \right] w(x) d\mathcal{H}_\infty^\delta + 2^{1+\gamma} \int_E (W_Q)^\gamma w(x) d\mathcal{H}_\infty^\delta
$$

$$
\leq 4 \int_E \left[\left(\frac{w(x)}{2} \right)^\gamma - (W_Q)^\gamma \right] w(x) d\mathcal{H}_\infty^\delta + 8(W_Q)^{1+\gamma} \mathcal{H}_\infty^\delta(Q),
$$

which implies (4.6).

Finally, by (4.5) and (4.6), we conclude that

$$(1 - 4K\gamma) \int_Q w(x)^{1+\gamma} d\mathcal{H}_\infty^\delta \leq 8(W_Q)^{1+\gamma} \mathcal{H}_\infty^\delta.$$

Choosing $\gamma \in (0, 1)$ such that $4K\gamma < 1$, then we obtain the reverse Hölder inequality (1.10). This finishes the proof of Theorem 1.7. $\qquad \square$

Corollary 1.8 is an immediate consequence of the reverse Hölder inequality, i.e., Theorem 1.7, and the definition of $\mathcal{A}_{p,\delta}$. We omit its proof here.

Proof of Theorem 1.9. It follows from $w \in \mathcal{A}_{p,\delta}$ that $w^{-\frac{1}{p-1}} \in \mathcal{A}_{\frac{p}{p-1},\delta}$. By this and Theorem 1.7, we know that there exist positive constants $\gamma \in (0, 1)$ and K such that, for any cube Q of $\mathbb{R}^n$,

$$(4.7) \qquad \left(\frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q w(x)^{-\frac{1+\gamma}{p-1}} d\mathcal{H}_\infty^\delta \right)^{\frac{1}{1+\gamma}} \leq \frac{K}{\mathcal{H}_\infty^\delta(Q)} \int_Q w(x)^{-\frac{1}{p-1}} d\mathcal{H}_\infty^\delta.$$

Choosing $q \in (1, p)$ such that $\frac{1+\gamma}{p-1} = \frac{1}{q-1}$, we then infer that $w \in \mathcal{A}_{q,\delta}$ from (4.7), which completes the proof of Theorem 1.9. $\qquad \square$

5 Applications

In this section, we give two applications of the main results obtained in the present paper, including the celebrated Jones factorization theorem for $\mathcal{A}_{p,\delta}$ and the capacitary Muckenhoupt weight characterization for the boundedness of maixmal operator $\mathcal{M}_{\mathcal{H}_\infty^\delta}$ on weak weighted Choquet-Lebesgue spaces.

5.1 The Jones Factorization Theorem within the $\mathcal{A}_{p,\delta}$ Framework

Using the boundedness of the operator $\mathcal{M}_{\mathcal{H}^{\delta}_{\infty}}$ established in Theorem 1.1, we prove the Jones factorization theorem within the $\mathcal{A}_{p,\delta}$ framework stated in Theorem 1.10, which presents the property of decomposition and synthesis of capacitary Muckenhoupt weight class $\mathcal{A}_{p,\delta}$ for all $p \in [1, \infty)$ and $\delta \in (0, n]$.

Let $\delta \in (0, n]$, $p \in (1, \infty)$ and $w \in \mathcal{A}_{p,\delta}$. Define three operators by setting, for any $f \in L^{pq}(\mathbb{R}^n, \mathcal{H}^{\delta}_{\infty})$ with $q := \frac{p}{p-1}$, and $x \in \mathbb{R}^n$,

$$T_1(f)(x) := \left[\mathcal{M}_{\mathcal{H}^{\delta}_{\infty}}\left(|f|^q w^{-\frac{1}{p}}\right)(x)\right]^{\frac{1}{q}} w(x)^{\frac{1}{pq}},$$

$$T_2(f)(x) := \left[\mathcal{M}_{\mathcal{H}^{\delta}_{\infty}}\left(|f|^p w^{\frac{1}{p}}\right)(x)\right]^{\frac{1}{p}} w(x)^{-\frac{1}{p^2}}$$

and

$$T_3 := T_1 + T_2.$$

Then we have the following conclusions.

Lemma 5.1. *Let $\delta \in (0, n]$, $p \in (1, \infty)$ and $w \in \mathcal{A}_{p,\delta}$. Then there exists a positive constant K such that*

(i) *for any sequence $\{f_j\}_{j \in \mathbb{N}}$ of functions in $L^{pq}(\mathbb{R}^n, \mathcal{H}^{\delta}_{\infty})$ and $i \in \{1, 2, 3\}$,*

$$T_i\left(\sum_{j \in \mathbb{N}} f_j\right)(x) \leq K \sum_{j \in \mathbb{N}} T_i(f_j)(x), \quad \forall\, x \in \mathbb{R}^n;$$

(ii) *for any $f \in L^{pq}(\mathbb{R}^n, \mathcal{H}^{\delta}_{\infty})$ and $i \in \{1, 2, 3\}$,*

$$\int_{\mathbb{R}^n} |T_i(f)(x)|^{pq}\, d\mathcal{H}^{\delta}_{\infty} \leq K \int_{\mathbb{R}^n} |f(x)|^{pq}\, d\mathcal{H}^{\delta}_{\infty}.$$

Proof. To prove (i), we first note that the following Minkowski inequality

$$\left\{\int_{\mathbb{R}^n}\left(\sum_{j \in \mathbb{N}} |f_j(x)|\right)^p d\mathcal{H}^{\delta}_{\infty}\right\}^{\frac{1}{p}} \lesssim \sum_{j \in \mathbb{N}}\left\{\int_{\mathbb{R}^n} |f_j(x)|^p\, d\mathcal{H}^{\delta}_{\infty}\right\}^{\frac{1}{p}}$$

holds true. By this, we find that, for any $x \in \mathbb{R}^n$ and any cube Q containing x,

$$\left(\frac{1}{\mathcal{H}^{\delta}_{\infty}(Q)} \int_Q \left(\sum_{j \in \mathbb{N}} |f_j(y)| w(y)^{-\frac{1}{pq}}\right)^q d\mathcal{H}^{\delta}_{\infty}\right)^{\frac{1}{q}} \lesssim \sum_{j \in \mathbb{N}}\left(\frac{1}{\mathcal{H}^{\delta}_{\infty}(Q)} \int_Q |f_j(y)|^q w(y)^{-\frac{1}{p}}\, d\mathcal{H}^{\delta}_{\infty}\right)^{\frac{1}{q}}$$

$$\lesssim \sum_{j \in \mathbb{N}}\left[\mathcal{M}_{\mathcal{H}^{\delta}_{\infty}}(|f_j|^q w^{-\frac{1}{p}})(x)\right]^{\frac{1}{q}}.$$

Therefore, (i) is true for T_1 and, similarly for T_2 and T_3.

(ii) Since $w \in \mathcal{A}_{p,\delta}$ and hence $w^{-q/p} \in \mathcal{A}_{q,\delta}$, it follows from Theorem 1.1 that, for $i = 1, 2$,

$$\int_{\mathbb{R}^n} |T_i(f)(x)|^{pq}\, d\mathcal{H}^{\delta}_{\infty} \lesssim \int_{\mathbb{R}^n} |f(x)|^{pq}\, d\mathcal{H}^{\delta}_{\infty},$$

which further implies that (ii) is also true for T_3. This finishes the proof of Lemma 5.1. $\qquad\square$

We now give the proof of Theorem 1.10.

Proof of Theorem 1.10. To prove the necessity, we only need to consider the case of $p \in (1, \infty)$, since it is easy to see that, for $p = 1$, $w_0 := w$ and $w_1 :\equiv 1$ satisfy the requirement.

Let $w \in \mathcal{A}_{p,\delta}$ and fixed a function $g \in L^{pq}(\mathbb{R}^n, \mathcal{H}_\infty^\delta)$, where $\frac{1}{p} + \frac{1}{q} = 1$. We define

$$\varphi(x) := \sum_{k=0}^{\infty} \frac{1}{A^k} T_3^k g(x), \quad \forall x \in \mathbb{R}^n,$$

where T_3 is defined above and $A > K^{\frac{1}{pq}}$ with K being the constant in Lemma 5.1(i). Then

$$\left[\int_{\mathbb{R}^n} |\varphi(x)|^{pq} d\mathcal{H}_\infty^\delta \right]^{\frac{1}{pq}} \lesssim \sum_{k=0}^{\infty} \frac{1}{A^k} \left[\int_{\mathbb{R}^n} |T_3^k(g)(x)|^{pq} d\mathcal{H}_\infty^\delta \right]^{\frac{1}{pq}}$$

$$\lesssim \sum_{k=0}^{\infty} \frac{K^{\frac{k}{pq}}}{A^k} \left[\int_{\mathbb{R}^n} |g(x)|^{pq} d\mathcal{H}_\infty^\delta \right]^{\frac{1}{pq}} \sim \|g\|_{L^{pq}(\mathbb{R}^n, \mathcal{H}_\infty^\delta)},$$

namely, $\varphi \in L^{pq}(\mathbb{R}^n, \mathcal{H}_\infty^\delta)$. From this and Lemma 5.1, we infer that, for any $x \in \mathbb{R}^n$,

$$T_1 \varphi(x) \le T_3 \left(\sum_{k=0}^{\infty} \frac{1}{A^k} T_3^k g(x) \right) \lesssim \sum_{k=0}^{\infty} \frac{1}{A^k} T_3^{k+1}(g)(x)$$

$$\sim A \sum_{k=1}^{\infty} \frac{1}{A^k} T_3^k(g)(x) \sim \varphi(x),$$

and, similarly $T_2 \varphi(x) \lesssim \varphi(x)$. Therefore, by the definitions of T_1 and T_2, we further find that

$$\mathcal{M}_{\mathcal{H}_\infty^\delta}(\varphi^q w^{-\frac{1}{p}}) \lesssim \varphi^q w^{-\frac{1}{p}} \quad \text{and} \quad \mathcal{M}_{\mathcal{H}_\infty^\delta}(\varphi^p w^{\frac{1}{p}}) \lesssim \varphi^p w^{\frac{1}{p}}.$$

Let $w_0 := \varphi^p w^{\frac{1}{p}}$ and $w_1 := \varphi^q w^{-\frac{1}{p}}$. Then $w_0, w_1 \in \mathcal{A}_{1,\delta}$ and

$$w_0 w_1^{1-p} = \varphi^p w^{\frac{1}{p}} \varphi^{q(1-p)} w^{-\frac{1-p}{p}} = w.$$

Conversely, let $w_0, w_1 \in \mathcal{A}_{1,\delta}$ and $w := w_0 w_1^{1-p}$. Then, for any cube Q of $\mathbb{R}^n$, we have

$$\frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q w_i(x) d\mathcal{H}_\infty^\delta \|w_i^{-1}\|_{L^\infty(Q, \mathcal{H}_\infty^\delta)} \le [w_i]_{\mathcal{A}_{1,\delta}}, \quad i \in \{0, 1\}.$$

By this, we know that

$$\frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q w(x) d\mathcal{H}_\infty^\delta \left(\frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q w(x)^{-\frac{1}{p-1}} d\mathcal{H}_\infty^\delta \right)^{p-1}$$

$$\le \frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q w_0(x) d\mathcal{H}_\infty^\delta \|w_1^{-(p-1)}\|_{L^\infty(Q, \mathcal{H}_\infty^\delta)} \left(\frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q w_1(x) d\mathcal{H}_\infty^\delta \|w_0^{-\frac{1}{p-1}}\|_{L^\infty(Q, \mathcal{H}_\infty^\delta)} \right)^{p-1}$$

$$= \frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q w_0(x) d\mathcal{H}_\infty^\delta \|w_0^{-1}\|_{L^\infty(Q, \mathcal{H}_\infty^\delta)} \left(\frac{1}{\mathcal{H}_\infty^\delta(Q)} \int_Q w_1(x) d\mathcal{H}_\infty^\delta \|w_1^{-1}\|_{L^\infty(Q, \mathcal{H}_\infty^\delta)} \right)^{p-1}$$

$$\le [w_0]_{\mathcal{A}_{1,\delta}} [w_1]_{\mathcal{A}_{1,\delta}}^{p-1}.$$

Therefore, $w \in \mathcal{A}_{p,\delta}$. This finishes the proof of Theorem 1.10. □

5.2 Characterization for the Boundedness of $\mathcal{M}_{\mathcal{H}^\delta_\infty}$ on Weak-Type Spaces

In this subsection, by using Theorems 1.1 and 1.9, we prove Theorem 1.12, which characterizes the boundedness of the operator $\mathcal{M}_{\mathcal{H}^\delta_\infty}$ on the weak weighted Choquet-Lebesgue space $L^{p,\infty}_w(\mathbb{R}^n, \mathcal{H}^\delta_\infty)$ via capacitary Muckenhoupt weight class $\mathcal{A}_{p,\delta}$.

Proof of Theorem 1.12. If (1.11) holds true, then by the Chebyshev inequality, we know that the maximal operator $\mathcal{M}_{\mathcal{H}^\delta_\infty}$ is of weak-type (p, p). Therefore, by Theorem 1.1, we obtain $w \in \mathcal{A}_{p,\delta}$.

Conversely, let $w \in \mathcal{A}_{p,\delta}$. We only need to show that, for any $\lambda \in (0, \infty)$,

$$(5.1) \qquad w_{\mathcal{H}^\delta_\infty}\left(\left\{x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}^\delta_\infty} f(x) > 4\lambda\right\}\right) \lesssim \frac{1}{\lambda^p} \|f\|^p_{L^{p,\infty}_w(\mathbb{R}^n, \mathcal{H}^\delta_\infty)}.$$

For any $x \in \mathbb{R}^n$, define

$$f^\lambda(x) := \begin{cases} f(x), & \text{when } |f(x)| > \lambda, \\ 0, & \text{when } |f(x)| \le \lambda, \end{cases}$$

and $f_\lambda(x) := f(x) - f^\lambda(x)$. Then $\mathcal{M}_{\mathcal{H}^\delta_\infty} f_\lambda \le \lambda$ and

$$\mathcal{M}_{\mathcal{H}^\delta_\infty} f \le 2\left[\mathcal{M}_{\mathcal{H}^\delta_\infty} f^\lambda + \mathcal{M}_{\mathcal{H}^\delta_\infty} f_\lambda\right].$$

Thus,

$$w_{\mathcal{H}^\delta_\infty}\left(\left\{x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}^\delta_\infty} f(x) > 4\lambda\right\}\right) \le w_{\mathcal{H}^\delta_\infty}\left(\left\{x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}^\delta_\infty} f^\lambda(x) > \lambda\right\}\right).$$

Since $w \in \mathcal{A}_{p,\delta}$, it follows, from Theorem 1.9, that we can choose a $p_1 \in (1, p)$ such that $w \in \mathcal{A}_{p_1,\delta}$, which, combined with Theorem 1.1, implies

$$w_{\mathcal{H}^\delta_\infty}\left(\left\{x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}^\delta_\infty} f(x) > 4\lambda\right\}\right) \lesssim \frac{1}{\lambda^{p_1}} \int_{\mathbb{R}^n} |f^\lambda(x)|^{p_1} w(x) \, d\mathcal{H}^\delta_\infty.$$

In addition, by Proposition 2.7, we know that

$$\begin{aligned}
\int_{\mathbb{R}^n} |f^\lambda(x)|^{p_1} w(x) \, d\mathcal{H}^\delta_\infty &\le 4 \int_{\mathbb{R}^n} |f^\lambda(x)|^{p_1} \, dw_{\mathcal{H}^\delta_\infty} \\
&= 4p_1 \int_0^\infty t^{p_1-1} w_{\mathcal{H}^\delta_\infty}(\{x \in \mathbb{R}^n : |f(x)| > \max(t, \lambda)\}) \, dt \\
&= 4p_1 \int_0^\lambda t^{p_1-1} w_{\mathcal{H}^\delta_\infty}(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}) \, dt \\
&\quad + 4p_1 \int_\lambda^\infty t^{p_1-1} w_{\mathcal{H}^\delta_\infty}(\{x \in \mathbb{R}^n : |f(x)| > t\}) \, dt \\
&\le 4p_1 \|f\|^p_{L^{p,\infty}_w(\mathbb{R}^n, \mathcal{H}^\delta_\infty)} \left(\int_0^\lambda \frac{t^{p_1-1}}{\lambda^p} \, dt + \int_\lambda^\infty t^{p_1-p-1} \, dt\right) \\
&= \frac{4p}{p - p_1} \lambda^{p_1-p} \|f\|^p_{L^{p,\infty}_w(\mathbb{R}^n, \mathcal{H}^\delta_\infty)}.
\end{aligned}$$

Therefore,

$$w_{\mathcal{H}^\delta_\infty}\left(\left\{x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}^\delta_\infty} f(x) > 4\lambda\right\}\right) \lesssim \frac{4p}{p - p_1} \frac{1}{\lambda^p} \|f\|^p_{L^{p,\infty}_w(\mathbb{R}^n, \mathcal{H}^\delta_\infty)}.$$

Thus, (5.1) is proved, and hence completes the proof of Theorem 1.12. $\qquad \square$

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