MASSES OF BLOCKS OF THE Λ -COALESCENT WITH DUST VIA STOCHASTIC FLOWS

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ABSTRACT. We study the masses of blocks of the Λ -coalescent with dust and some aspects of their large and small time behaviors. To do so, we start by associating the Λ -coalescent to a nested interval-partition constructed from the flow of inverses, introduced by Bertoin and Le Gall in [15], of the Λ -Fleming-Viot flow, and prove Poisson representations for the masses of blocks in terms of the flow of inverses. The representations enable us to use the power of stochastic calculus to study the masses of blocks. We apply this method to study the long and small time behaviors. In particular, for all k > 1, we determine the decay rate of the expectation of the k-th largest block as time goes to infinity and find that a cut-off phenomenon, related to the presence of dust, occurs: the decay rate is increasing for small indices k but remains constant after a fixed index depending on the measure Λ .

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1. Introduction

For a finite and non-zero measure Λ on [0,1], the Λ -coalescent is a Markovian process $(\Pi_t)_{t\geq 0}$ on partitions of $\mathbb{N}=\{1,2,\ldots\}$ introduced by [62] and [64]. For any $n\geq 1$ let Π_t^n denote the restriction of the partition Π_t to $\{1,\ldots,n\}$. Then $(\Pi_t^n)_{t\geq 0}$ is a Markov process on a finite state space which has the following dynamics: if there are currently p blocks in the partition, for $k\in\{2,\ldots,p\}$, any k of the blocks merge into one with rate

$$\lambda_{p,k}(\Lambda) := \int_{[0,1]} r^{k-2} (1-r)^{p-k} \Lambda(dr). \tag{1.1}$$

This and the starting condition $\Pi_0 = \{\{1\}, \{2\}, \ldots\}$ characterize the law of the partition process $(\Pi_t)_{t\geq 0}$. The Λ -coalescent is an exchangeable coalescent that generalizes the classical Kingman coalescent (which corresponds to the case $\Lambda = \delta_0$) by allowing multiple mergers instead of only binary mergers. It can also be seen as a particular case of coalescents with simultaneous multiple collisions [66]. Background on the Λ -coalescent can be found in [11, 12, 37].

The Λ -coalescent is related to the genealogy of several population models [61, 68, 28, 27, 46, 19], to the genealogy of Continuous State Branching Processes (CSBPs) [13, 17, 16, 10, 8], to stable Continuous Random Trees (CRTs) [9], or also to pruning of trees [39, 1, 2]. One of its most fundamental connection is with the Λ -Fleming-Viot flow, of which it provides the genealogy [14]. The later process is valued in probability measures on [0,1] and models an infinite population with constant size, with genotypes indexed by [0,1], and that is subject to random neutral reproductions determined by the measure Λ . These flows have been introduced in [14] and implicitly in the lookdown construction of [25] which, in [55], is unified with the construction of [14]. They have been studied by many authors [15, 17, 16, 21, 44, 41, 56] and can be seen as a multi-type version of Λ -Wright-Fisher diffusions, as the frequency of any hereditary subpopulation of a Λ -Fleming-Viot flow follows a Λ -Wright-Fisher diffusions [15].

For a block $\mathcal{B} \in \Pi_t$, its mass (also called asymptotic frequency) is defined as $|\mathcal{B}| := \lim_{n \to \infty} \sharp(\mathcal{B} \cap \{1,\ldots,n\})/n$. It is shown in [55, Prop. 2.13] that, almost surely, this limit exists for all blocks in Π_t at all t. A block $\mathcal{B} \in \Pi_t$ is called massive if $|\mathcal{B}| > 0$. From a Λ -coalescent process $(\Pi_t)_{t \geq 0}$, one can define the process $(\{|\mathcal{B}|, \mathcal{B} \in \Pi_t \text{ s.t. } |\mathcal{B}| > 0\})_{t \geq 0}$ of the collection of masses of its massive blocks, which is a process on mass partitions. Looking at one or the other process is equivalent thanks to Kingman's correspondence [11, 12, 63]. Therefore, understanding the behavior of the collection of masses of blocks is essential to understand the Λ -coalescent. Unfortunately, the distribution of masses of blocks at a given time t (the entrance law from dust) has no known explicit expression, except in the remarkable case of the Bolthausen-Sznitman coalescent [62, 63, 11], and little is known about the precise behavior of masses of blocks as t goes to infinity. This problem is a focus of the present paper.

We say that a Λ -coalescent process $(\Pi_t)_{t\geq 0}$ has dust if the cumulative mass of the collection of singletons in Π_t is positive. There are four parameter regimes for the Λ -coalescent [62, 67] (see also [11, 37]) and, equivalently, for the Λ -Fleming-Viot flow [55, Prop. 1.3], namely (i) the case with finitely many massive blocks and dust, (ii) the case with infinitely many massive blocks and dust, (iii) the case with infinitely many massive blocks and no dust, (iv) the case with finitely many massive blocks and no dust. The processes $(\Pi_t)_{t\geq 0}$ and $(\Pi^n)_{n\geq 1}$ have been intensely studied in cases (iii) and (iv) [39, 16, 48, 26, 22, 9, 10, 6, 69, 49, 38, 8, 45, 57, 24, 23]. We are here interested in the case referred in the literature as the Λ -coalescent with dust, corresponding to the union of cases (i) and (ii) of the above classification. As discussed in Section 1.1 below, this is equivalent to the measure Λ satisfying the assumption (1.2). In the case with dust, interesting properties of the sequence of partition processes $(\Pi^n)_{n\geq 1}$ have been established as n goes to infinity by many authors [59, 36, 43, 45, 1, 2, 35, 51, 52, 60] (see also [37, 53]) but, unlike cases (iii) and (iv), little attention has been given to the process $(\Pi_t)_{t>0}$ of infinite partitions. The case with dust allows one to define models in which massive blocks emerge (in a discontinuous way) from dispersed matter. Moreover, it has been observed recently that this case may be used in models displaying mathematically and biologically interesting behaviors. For example, in [20], the author studies a family of Λ-Wright-Fisher diffusions with frequency dependent selection and environmental effects, and where the measure Λ satisfies the assumption (1.2). This leads to four possible regimes that include in particular a regime of coexistence. The later had been observed empirically by biologists but could not have been captured by simple mathematical models before. Motivated by this, we believe that a thorough study of the case with dust is in order, even in the classical models of the Λ -coalescent and Λ -Fleming-Viot flow.

In the present paper we study the entrance law from dust at time t of the masses of blocks of a Λ -coalescent with dust and some aspects of its asymptotic behavior as t is large or small. We denote by $W_k(t)$ the mass of the k^{th} largest block at time t (note that $W_k(t_1)$ and $W_k(t_2)$ may correspond to completely unrelated blocks since, at any time, blocks are ordered by non-increasing masses). It is intuitively clear that, as t goes to infinity, one large block occupies a proportion of the mass increasing to 1 as other massive blocks progressively merge with it, thus $W_1(t) \to 1$. Meanwhile, as t increases, the k^{th} largest block (for $k \ge 2$) is found among the smaller and smaller remaining blocks of "rare genotypes" whose total mass is less than $1 - W_1(t)$, thus $W_k(t) \to 0$ for $k \ge 2$. One of our goals is to determine how fast these convergences occur.

A useful point of view on the Λ -coalescent is as follows. Consider a population subject to a Λ -Fleming-Viot flow dynamic on $(-\infty,0]$ and divide the population at time 0 into *t-families*, where two individuals belong to the same *t*-family if and only if their ancestor from time -t is common. It can be seen from Bertoin and Le Gall's correspondence [14] that the sizes of *t*-families are the masses of blocks in an associated Λ -coalescent. The process of the *t*-families is a *nested interval-partition* in the framework of [33], where such an object is constructed and shown to be associated to the Λ -coalescent. In other words, it is perfectly equivalent to study the

masses of blocks in a Λ -coalescent or to study the sizes of t-families in a population subject to a Λ -Fleming-Viot flow dynamic on $(-\infty,0]$ (then, $W_k(t)$ is the size of the k^{th} largest t-family). We take the later point of view and start by associating the Λ -coalescent with dust to a nested interval-partition arising naturally from the flow of inverses introduced in [15]. Our construction method is different from the one in [33] but the underlying objects are the same. This allows us to prove Poisson representations for the masses of blocks in terms of the flow of inverses. The representations enable us to use the power of stochastic calculus to study aspects of the entrance law from dust of the masses of blocks. We apply this method to study the long and small time behaviors. This allows to study the Λ -coalescent with dust without passing by the classical approximation of $(\Pi_t)_{t>0}$ by $(\Pi_t^n)_{t>0}$.

Our results display an interesting cut-off phenomenon that is related to the presence of dust. More precisely, there is an integer $N(\Lambda) > 2$ such that the decay rate of $\mathbb{E}[W_k(t)]$ (as $t \to \infty$) is increasing in k for $k \in \{2, \ldots, N(\Lambda)\}$ and constant in k for $k \geq N(\Lambda)$. This has the following interpretation: new blocks whose masses are proportional to the total mass of the dust regularly emerge from the dust; therefore, the mass of the k^{th} largest block cannot decay faster than the mass of the dust. The decay rate can thus increase with k until it reaches the decay rate of the dust mass, and it then remains constant.

We now briefly discuss some related models.

Flows of subordinators. While Λ -Fleming-Viot flows model constant size populations, flows of subordinators (the flow version of CSBPs) model populations with similar dynamics but non-constant size. Their genealogy has also been studied [13] and, in that context, a similar problem to ours is studied in [31, 32] via the inverse flow, using technics available in that context. They study the non-exchangeable coalescent process, called consecutive coalescent, describing the genealogy in their case and show that, as t goes to infinity, the sizes of t-families converge to the jumps sizes of an explicit subordinator. That interesting behavior, that differs strongly from our case, seems to be related to the independence of subpopulations enjoyed by the case of non-constant size populations, leading to such Poisson structure for families. In our case, a difficulty is that we are considering a sequence of ordered random variables arising from a complex structure that results in strong dependency in the system. This feature appears in several probabilistic models and developing tools that can efficiently address it is a motivation in itself.

Evolving coalescent. A realization of the Λ -coalescent can be seen as the genealogical structure of a population sampled at a given time. By letting the sampling time evolve forward in time, one obtains a Markov process of genealogical structures called the evolving coalescent, see for example [69, 50, 53]. Finding the appropriate representation and spate space for this Markov process is non-trivial and several approaches have been proposed [40, 42, 33]. In the present paper, the Λ -coalescent (in the case with dust) is seen as a function of the flow of inverses starting at 0. Provided that one consistently constructs the flow of inverses starting at all times (along with its Poisson background), this would yield another construction of the evolving coalescent in the case with dust, and the Poisson representation from Theorem 1.8 would naturally extend to the masses of blocks of the evolving coalescent.

Infinite allele model. Consider the Λ -coalescent restricted to $\{1, \ldots, n\}$, let every block freeze at some rate $\theta > 0$, and only allow mergings between unfrozen blocks. This is interpreted biologically as looking at the allelic types of n individuals, with merging events corresponding to coalescents of ancestral lineages and freezing events corresponding to appearances of mutations leading to new allelic types. After letting time go to infinity, the resulting partition is called the allelic partition and the sizes of its blocks are called the allele frequency spectrum. These objects have attracted a lot of interest [58, 9, 5, 34, 59, 7] (see also [11, 37]). We believe that the

construction and representation of the present paper could possibly be adapted to include the effect of mutations and study properties of the allele frequency spectrum in the case with dust.

1.1. Assumptions, particular cases, and notations. Let Λ be a non-zero finite measure on [0,1]. In all this paper we assume that

$$\Lambda(\{1\}) = 0, \ H(\Lambda) := \int_{[0,1)} r^{-1} \Lambda(dr) < \infty.$$
 (1.2)

If $\Lambda(\{1\}) > 0$, all blocks merge into a single one at rate $\Lambda(\{1\})$. The condition $\Lambda(\{1\}) = 0$ is meant to eliminate this degenerate case. Being assumed that $\Lambda(\{1\}) = 0$, $H(\Lambda) < \infty$ is equivalent to the almost sure presence of dust, i.e. cases (i) and (ii) mentioned above, as was shown by [62] (see also [11, 37], and see [55, Prop. 1.3] for the Λ -Fleming-Viot flow). We also note that (1.2) implies $\Lambda(\{0\}) = 0$ so there is no Kingman component in the coalescent process. The case (i) "finitely many massive blocks and dust" is a sub-case of the case with dust. It correspond to the condition

$$\Lambda(\{1\}) = 0, \ \int_{[0,1)} r^{-2} \Lambda(dr) < \infty.$$
 (1.3)

Example 1 (Beta-coalescent). For any a, b > 0 we set $\Lambda_{a,b}(dr) := r^{a-1}(1-r)^{b-1}dr$. Then the transitions rates $\lambda_{p,k}(\Lambda_{a,b})$ from (1.1) can be expressed as $\lambda_{p,k}(\Lambda_{a,b}) = B(a+k-2,b+p-k)$ where $B(\cdot,\cdot)$ is the beta function. The condition (1.2) holds true if and only if a > 1 and the condition (1.3) holds true if and only if a > 2. An important case is when $a = 2 - \alpha, b = \alpha$ for some $\alpha \in (0,2)$. That case is called the Beta $(2-\alpha,\alpha)$ -coalescent.

We write $U \sim \mathcal{U}([0,1])$ (resp. $(U_i)_{i\geq 1} \sim \mathcal{U}([0,1])^{\times \mathbb{N}}$) if U is a uniform random variable (resp. a sequence of iid uniform random variables) on (0,1). Similarly we write $Z \sim \mathcal{B}(r)$ (resp. $(Z_i)_{i\geq 1} \sim \mathcal{B}(r)^{\times \mathbb{N}}$) if Z is a Bernoulli random variable (resp. a sequence of iid Bernoulli random variables) with parameter r. We use the notation $Q := [0,1] \cap \mathbb{Q}$. For any set $A \subset [0,1]$ we write \overline{A} and A^c for respectively the closure and interior of A, and $A^c := [0,1] \setminus A$. We denote by $\mathcal{B}([0,1])$ the family of Borel sets in [0,1].

1.2. Λ -Fleming-Viot flow and flow of inverses. Let N(ds, dr, du) be a Poisson random measure on $(0, \infty) \times (0, 1)^2$ with intensity measure $ds \times r^{-2}\Lambda(dr) \times du$. N can be seen has a random collection of mass 1 atoms $(s, r, u) \in (0, \infty) \times (0, 1)^2$. We refer to an atom $(s, r, u) \in N$ as a jump, to s as its time component, and to r and u as respectively its r-component and its u-component. We define the set of jumping times by $J_N := \{s > 0, \exists (r, u) \in (0, 1)^2 \text{ s.t. } (s, r, u) \in N\}$. For any $t \geq 0$, \mathcal{F}_t denotes the sigma-field generated by the random measure $N(\cdot \cap (0, t] \times (0, 1)^2)$.

The Λ -Fleming-Viot flow is defined as the solution of the following SDE:

$$X_w(x) = x + \int_{(0,w]\times(0,1)^2} r\left(\mathbb{1}_{\{u \le X_{s-}(x)\}} - X_{s-}(x)\right) N(ds, dr, du), \tag{1.4}$$

almost surely for all $x \in [0,1]$ and $w \ge 0$. By [21, Thm. 4.4] this SDE defines a unique flow $(X_w(x), x \in [0,1], w \ge 0)$ that is called the Λ -process in [15]. A jump $(s,r,u) \in N$ has the following interpretation: at time s the individual "located" at $u \in [0,1]$ produces an offspring of size r that replaces an identical amount of individuals chosen uniformly in the population. The quantity $X_w(x)$ represents the amount of individuals, in the population at time w, whose ancestor from time 0 lies in [0,x]. For any $x \in [0,1]$, the process $(X_w(x))_{w\ge 0}$ is the so-called Λ -Wright-Fisher diffusion with initial value x.

In this paper, we are interested in the genealogy of a population that underwent the dynamic (1.4) from a very long time until present. The designated tool to study this is the so-called

flow of inverses, see [15]. Its heuristic definition and interpretation is as follows. We fix t > 0 and consider a flow $(X_{-t,w}(x), x \in [0,1], w \in [-t,0])$ representing a population undergoing the dynamics (1.4) on the time interval [-t,0], 0 representing present day and -t representing the starting time in the past. In practice it is not trivial to define a consistent collection $X_{-t,\cdot}(\cdot)$ that solves (1.4) for all $t \in \mathbb{R}_+$ but, in the terminology of [14], one can consider the dual flow associated with the Λ -coalescent; the latter is well-defined and represents a population undergoing the same dynamic. We then denote by $X_{-t,0}^{-1}(\cdot)$ the generalized inverse of the non-decreasing function $X_{-t,0}(\cdot)$. The link with the genealogy is now clear: a subinterval of [0,1] on which $X_{-t,0}^{-1}(\cdot)$ is constant corresponds to a set of individuals, in the population at present time, whose ancestor from time -t in the past is common. We refer to such subinterval as a t-family. By increasing t we look further in the past and obverse mergers of t-families when they get connected by new potential ancestors.

In [15] (see also [14, Sec. 3.2]), the coalescing flow related to the genealogy of the Λ -process, called *flow of inverses*, is defined as described above, by taking the generalized inverse in flows of bridges. In our case, we define it from an SDE and then justify that it is equal in law to the one from [15]. More precisely, we consider the stochastic flow $(Y_{0,t}(y), y \in [0, 1], t \geq 0)$ solving

$$Y_{0,t}(y) = y + \int_{(0,t]\times(0,1)^2} \left(m_{r,u}(Y_{0,s-}(y)) - Y_{0,s-}(y) \right) N(ds, dr, du), \tag{1.5}$$

almost surely for all $y \in [0,1]$ and $t \geq 0$, where $m_{r,u}(z) := \text{Median}\{\frac{z-r}{1-r}, \frac{z}{1-r}, u\}$ for $z \in [0,1]$. For a population that underwent the dynamic (1.4) on [-t,0], $Y_{0,t}(y)$ represents the position of the ancestor from time -t of an individual "located" at position y at time 0. In particular, a jump $(s,r,u) \in N$ in (1.5) has the following interpretation: at time -s a new potential ancestor appears at the location u, all individuals from time -(s-) that are located in the interval $I_{r,u} := [u(1-r), u(1-r) + r]$ adopt this ancestor, so the corresponding ancestral lines coalesce at u. It particular, each jump of N results in a merger of families (with a fraction of the dust). It is justified in the following proposition that the stochastic flow solving (1.5) is well-defined.

Proposition 1.1. Assume that (1.2) holds true. There exists a unique stochastic flow $(Y_{0,t}(y), y \in [0,1], t \ge 0)$ with the following properties:

- (i) almost surely, (1.5) holds for all $y \in [0, 1]$ and $t \ge 0$;
- (ii) almost surely, for every $y \in [0,1]$, the trajectory $t \mapsto Y_{0,t}(y)$ is càd-làg;
- (iii) almost surely, for every $t \ge 0$, the map $y \mapsto Y_{0,t}(y)$ is non-decreasing and continuous, and $Y_{0,t}(0) = 0$, $Y_{0,t}(1) = 1$.

Proposition 1.1 is proved in Appendix A. Also, Proposition A.9 from Appendix A shows that the p-point motion of this flow solves the martingale problem satisfied by the p-point motion of the flow of inverses of the Λ -process (see [15, Thm. 5]) and that, in our case, that martingale problem is well posed. This shows that, in our case, the process $(Y_{0,t}(\cdot))_{t\geq 0}$, defined as the solution of (1.5), is indeed equal in law to the flow of inverses of the Λ -process, defined in [15].

Remark 1.2. That $Y_{0,\cdot}(\cdot)$ is equal in law to the flow of inverses of the Λ -process implies in particular that, for any $x, y \in [0,1]$ and $t \geq 0$, we have $\mathbb{P}(X_t(x) \geq y) = \mathbb{P}(x \geq Y_{0,t}(y))$. In other words, the one-point motions of the flows (1.4) and (1.5) are Siegmund duals. This last point was already observed in [20, Thm. 2.5].

Remark 1.3 (A population model for Y). Even if Y is a tool to understand the genealogy of X, it also has a population model interpretation of its own. Consider an infinite population that is continuously distributed in [0,1]. If $(s,r,u) \in N$, at time s a catastrophe occurs and kills all individuals in the sub-interval $I_{r,u}$. Since u is uniformly distributed, the affected interval is, given

its size r, located uniformly at random inside [0,1]. After the catastrophe, the remaining individuals in the population reproduce uniformly (preserving their order) such as to instantaneously refill all the interval [0,1]. Then, for any $z \in [0,1]$, the descents at time s of the individuals that occupied [0,z] at time s— can be seen to be $[0, m_{r,u}(z)]$. For any $t \geq 0$, the descents at time t of the individuals that occupied the interval [0,y] at time 0 is then $[0,Y_{0,t}(y)]$.

It is also of interest to start the flows Y at specific times and to compose them. We show in Proposition B.1 from Appendix B that a countable family of stochastic flows $\{(Y_{s,t}(y), y \in [0,1], t \geq s), s \in J_N \cup \{0\}\}$ can be defined on the same probability space, such that we have the following composition property: almost surely, for any $s_1, s_2 \in J_N \cup \{0\}$ with $s_1 < s_2$,

$$\forall t \ge s_2, \forall y \in [0, 1], \ Y_{s_1, t}(y) = Y_{s_2, t}(Y_{s_1, s_2}(y)). \tag{1.6}$$

Moreover, each flow in this family satisfies a shifted version of (1.5) (see Proposition B.1). A consequence of (1.5) and the above is that the following property holds true almost surely: for any t > 0 and $x \in [0, 1]$, if for some $(s, r, u) \in N$ with $s \in (0, t]$ we have $x \in Y_{0,s-}^{-1}(I_{r,u})$ then

$$Y_{0,s}(x) = m_{r,u}(Y_{0,s-}(x)) = u \text{ and } Y_{0,t}(x) = Y_{s,t}(Y_{0,s}(x)) = Y_{s,t}(u).$$
 (1.7)

In other words, each jumping time $s \in J_N$ yields a merger event for some trajectories of the flow Y. The following proposition shows the important property that mergers of trajectories of the flow Y cannot occur continuously but only at jumping time $s \in J_N$.

Proposition 1.4. Assume that (1.2) holds true. We have

$$\mathbb{P}\left(\forall t > 0, \forall y_1 \neq y_2 \in [0, 1], \ Y_{0,t}(y_1) \neq Y_{0,t}(y_2) \ or \ \exists (s, r, u) \in N \ s.t. \ s \leq t, \ y_1, y_2 \in Y_{0,s-}^{-1}(I_{r,u})\right) = 1$$

The same property holds with $Y_{0,t}(\cdot)$ replaced by $Y_{0,t-}(\cdot)$ and $s \leq t$ replaced by s < t.

Proposition 1.4 is proved in Section 2.1.

1.3. From flow of inverses to Λ -coalescent. A fruitful approach to study a process valued on decreasing sequences is to build a random structure containing a process on interval partitions that allows one to recover the target process, see for example [30]. We now explain how the flow Y from Proposition 1.1 naturally provides such a construction of the process of masses of blocks of a Λ -coalescent with dust. To a realization of the flow Y we first associate a partition process via the sampling procedure of [14]: let $(U_i)_{i\geq 1} \sim \mathcal{U}([0,1])^{\times \mathbb{N}}$ that is independent of N (and therefore of Y), we define a process $(\pi_t^Y)_{t\geq 0}$ of random partitions of \mathbb{N} by the equivalence relation $i \sim_{\pi_t^Y} j \Leftrightarrow Y_{0,t}(U_i) = Y_{0,t}(U_j)$. The following lemma is in the line with [14, Thm. 1] and relates Y to the Λ -coalescent.

Lemma 1.5. Assume that (1.2) holds true. The partition process $(\pi_t^Y)_{t\geq 0}$ is a Λ -coalescent.

For all $t \geq 0$ let m_t (resp. m_{t-}) be the Stiljes measure on [0,1] associated to the non-decreasing function $Y_{0,t}(\cdot)$ (resp. $Y_{0,t-}(\cdot)$), i.e. $m_t(A) := \int_{[0,1]} \mathbbm{1}_A(x) dY_{0,t}(x)$ (resp. $m_{t-}(A) := \int_{[0,1]} \mathbbm{1}_A(x) dY_{0,t-}(x)$) for $A \in \mathcal{B}([0,1])$. Proposition 1.1 ensures that, almost surely, for all $t \geq 0$ the measures m_t and m_{t-} are well-defined. It will turn out later that these measures have a simple expression (see Section 2.6). Let $C_t := Supp(m_t)^c$ (resp. $C_{t-} := Supp(m_{t-})^c$). It will be a consequence of Proposition 2.7 that, almost surely for all t > 0, $C_{t-} = \bigcup_{s \in (0,t)} C_s$. Let $(\mathcal{O}_k(t))_{k \geq 1}$ be an enumeration of the open connected components of C_t such that $|\mathcal{O}_1(t)| \geq |\mathcal{O}_2(t)| \geq \ldots$ and such that, if several components have the same length, their order of appearance in the sequence is determined by the U_i' of smallest index that they contain (where $(U_i')_{i \geq 1} \sim \mathcal{U}([0,1])^{\times \mathbb{N}}$ is independent from everything else). We set $W_k(t) := |\mathcal{O}_k(t)|$. Since $Y_{0,t}(\cdot)$ is constant on each

 $\mathcal{O}_k(t)$, let us denote by $V_k(t)$ the value taken by $Y_{0,t}(\cdot)$ on $\mathcal{O}_k(t)$; we then have $Y_{0,t}^{-1}(\{V_k(t)\}) = \overline{\mathcal{O}_k(t)}$. We note that $k_1 \neq k_2 \Rightarrow V_{k_1}(t) \neq V_{k_2}(t)$. We enlarge the probability space by adding a new sequence $(\tilde{U}_k)_{k\geq 1}$ that is independent from everything else and, if for some t, C_t has only finitely many open components $\mathcal{O}_1(t), \ldots, \mathcal{O}_K(t)$ (which occurs if the measure Λ satisfies (1.3)) then for all k > K, we set $\mathcal{O}_k(t) := \emptyset$ (so $W_k(t) = 0$) and $V_k(t) := \tilde{U}_k$. We similarly define $(\mathcal{O}_k(t-), W_k(t-), V_k(t-))_{k\geq 1}$ from C_{t-} instead of C_t . What we informally called t-families earlier are the open connected components of C_t or, equivalently, the jump intervals of $X_{-t,0}(\cdot)$ (note however that $X_{-t,0}(\cdot)$ is not formally defined in our framework).

Proposition 1.6. Assume that (1.2) holds true. Almost surely, the following holds

- $(C_t)_{t\geq 0}$ is a nested interval-partition in the sense of [33, Def. 1.3];
- $(\pi_t^Y)_{t\geq 0}$ is the partition process obtained from the paintbox based on $(C_t)_{t\geq 0}$ in the sense of [33, eq. (2)];
- For any $k \ge 1$, $t \mapsto W_k(t)$ is càd-làg and $\lim_{s \to t, s < t} W_k(s) = W_k(t-)$ for all t > 0;
- For any $k \geq 1$, $\{t \geq 0 \text{ s.t. } W_k(t) \neq W_k(t-)\} \subset J_N$; in particular, for any fixed $T \geq 0$ and $k \geq 1$, $(W_k(t))_{t\geq 0}$ is almost surely continuous at T;
- $t \mapsto (W_k(t))_{k\geq 1}$ is a càd-làg (for the topology considered in [12, Prop. 1.1]) version of the process of ordered non-zero masses of blocks of a Λ -coalescent.

Lemma 1.5 and Proposition 1.6 are rather intuitive but we provide justifications for them in Sections 2.2 and 2.5 respectively. Based on Proposition 1.6, $(W_k(t))_{k\geq 1,t\geq 0}$ is distributed as the process of ordered masses of blocks of a Λ -coalescent; we thus often refer to it as such. That relation can be seen as a re-statement of the well-known fact that the jumps sizes of $X_{-t,0}(\cdot)$ are the masses of blocks of a Λ -coalescent at time t, [14, Sec. 3.3], [55, (1.4)]. The combination of Proposition 1.6 with Lemma 1.5 also shows that $(C_t)_{t\geq 0}$ is a nested interval-partition associated with a Λ -coalescent. We thus recover [33, Prop. 1.11]. Our construction is slightly different from the one in [33], as it is based on the flow of inverses Y (itself constructed in Proposition 1.1), while their construction is based on nested compositions of \mathbb{N} , see [33, Sec. 3]. However, the Λ -Fleming-Viot flow underlies both constructions [33, Cor. 3.6]. Working with the flow of inverses will allow us to derive a useful Poisson representation in Theorem 1.8.

1.4. Some more definitions. Let $(S_t)_{t\geq 0}$ be the subordinator defined by

$$S_t := -\int_{(0,t]\times(0,1)^2} \log(1-r)N(ds,dr,du). \tag{1.8}$$

By [65, Thm. 19.3] and (1.2) we see that $(S_t)_{t\geq 0}$ is well-defined. For any $\lambda, t\geq 0$ we have $\mathbb{E}[e^{-\lambda S_t}] = e^{-t\phi_S(\lambda)}$ where $\phi_S(\cdot)$ is the Laplace exponent of $(S_t)_{t\geq 0}$. According to the Lévy-Kintchine formula we have

$$\phi_S(\lambda) = \int_{(0,1)} \left(1 - (1-r)^{\lambda} \right) r^{-2} \Lambda(dr). \tag{1.9}$$

Note that $H(\Lambda) = \phi_S(1)$. It is well-known that, for Λ satisfying (1.2), the total mass of the dust of the Λ at time t is given by e^{-S_t} [62, Prop. 26], see also [12, 36, 37]. This classical fact, which can be stated as $\sum_{k\geq 1} W_k(t) = 1 - e^{-S_t}$, is, not surprisingly, also recovered from our construction, see Corollary 2.16 and Remark 2.17 from Section 2.6. More importantly, $(S_t)_{t\geq 0}$, will play a role in the Poisson representations from Theorems 1.8 and 1.9, and the formula from Theorem 1.10.

For any $t \geq 0$ and $r, u \in (0,1)$ let $Z_k(t,r,u) := \mathbbm{1}_{V_k(t) \in I_{r,u}}$ (resp. $Z_k(t-,r,u) := \mathbbm{1}_{V_k(t-) \in I_{r,u}}$) and $\beta_k(t,r,u) := \sum_{j=1}^k Z_j(t,r,u)$ (resp. $\beta_k(t-,r,u) := \sum_{j=1}^k Z_j(t-,r,u)$). The following lemma is a consequence of [14, Lem. 2]. A detailed justification is given in Section 2.5.

Lemma 1.7. For any fixed $t \geq 0$, $(V_k(t))_{k\geq 1}$ and $(W_k(t))_{k\geq 1}$ are independent, and $(V_k(t))_{k\geq 1} \sim \mathcal{U}([0,1])^{\times \mathbb{N}}$. In particular, for any $(t,r,u) \in [0,\infty) \times (0,1)^2$, $(Z_k(t,r,u))_{k\geq 1}$ and $(W_k(t))_{k\geq 1}$ are independent and $(Z_k(t,r,u))_{k\geq 1} \sim \mathcal{B}(r)^{\times \mathbb{N}}$.

For any $t \geq 0$, $r, u \in (0, 1)$, and $k \geq 1$ we set

$$H_t^k(r,u) := \mathbb{1}_{\beta_k(t,r,u)=0} \left(e^{-S_t} r + \sum_{j>k} Z_j(t,r,u) W_j(t) - W_k(t) \right)_+$$

$$+ \mathbb{1}_{\beta_k(t,r,u)\neq 0} \left(e^{-S_t} r + \sum_{j>k} Z_j(t,r,u) W_j(t) \right)_+$$

$$+ \sum_{j>k} (1 - Z_j(t,r,u)) W_j(t) \mathbb{1}_{\sum_{i=1}^j (1 - Z_i(t,r,u)) \leq k - 1}.$$

$$(1.10)$$

For any $t \geq 0$, $r, u \in (0, 1)$, and $k \geq 2$ we set

$$\begin{split} K_t^k(r,u) := & \mathbb{1}_{\beta_k(t,r,u)=0} \\ & \times \left(\text{Median} \left\{ W_{k-1}(t), W_k(t), e^{-S_t} r + \sum_{j > k} Z_j(t,r,u) W_j(t) \right\} - W_k(t) \right) \\ & + \mathbb{1}_{\beta_k(t,r,u)=1, Z_k(t,r,u)=1} \\ & \times \left(\text{Min} \left\{ W_k(t) + e^{-S_t} r + \sum_{j > k} Z_j(t,r,u) W_j(t), W_{k-1}(t) \right\} - W_k(t) \right) \\ & + \mathbb{1}_{\beta_k(t,r,u) \geq 2} \left(\sum_{j > k} (1 - Z_j(t,r,u)) W_j(t) \mathbb{1}_{\sum_{i=1}^j (1 - Z_i(t,r,u)) = k - 1} - W_k(t) \right). \end{split} \tag{1.11}$$

For k=1 we set $K^1_t(r,u):=H^1_t(r,u)$. We similarly define $H^k_{t-}(r,u)$ and $K^k_{t-}(r,u)$. It will be shown in Section 3 that the quantities $H^k_{t-}(r,u)$ and $K^k_{t-}(r,u)$ represent the increment of, respectively, $\sum_{1\leq j\leq k}W_j(\cdot)$ and $W_k(\cdot)$ at t if (t,r,u) is a jump; thus they will play a role in the representation of Theorem 1.9. We also defined these quantities at times t that are not jumping time to make separation of randomness possible, as in the formula of Theorem 1.10.

1.5. Main results. In order to get insights on the sequence $(W_k(t))_{k\geq 1}$, we study a random measure μ_t defined for any $t\geq 0$ by

$$\mu_t := \sum_{j \ge 1} W_j(t) \delta_{V_j(t)}. \tag{1.12}$$

For any t > 0 we similarly define $\mu_{t-} := \sum_{j \geq 1} W_j(t-)\delta_{V_j(t-)}$. The random measure μ_t encodes the information on the lengths of all intervals that are coalesced by Y at time t (equivalently, of the masses of blocks of the Λ -coalescent). A measure similar to μ_t appeared in [55, eq. (1.4)] in the lookdown representation of the Λ -Fleming-Viot flow, but little is known about the measure itself. Our first main result provides an almost sure representation of μ_t in terms of the flow Y.

Theorem 1.8. Assume that (1.2) holds true. For any fixed $t \geq 0$ we have almost surely

$$\mu_t = \sum_{(s,r,u)\in N, s\in(0,t]} re^{-S_{s-}} \delta_{Y_{s,t}(u)}.$$
(1.13)

Moreover, we have almost surely that, for all $t \in J_N$,

$$\mu_t = \sum_{(s,r,u)\in N, s\in(0,t]} re^{-S_{s-}} \delta_{Y_{s,t}(u)} \text{ and } \mu_{t-} = \sum_{(s,r,u)\in N, s\in(0,t)} re^{-S_{s-}} \delta_{Y_{s,t-}(u)}.$$
 (1.14)

It is transparent from the Poisson representation of Theorem 1.8 that each jump $(s,r,u) \in N$ represents a merger of existing blocks together with a fraction r of the dust. The representation from Theorem 1.8 yields $W_k(t) = \sum_{(s,r,u)\in N, s\in(0,t]} re^{-S_{s-}} \mathbb{1}_{Y_{s,t}(u)=V_k(t)}$. Unfortunately, this expression is not a stochastic integral because the integrand is not progressively measurable. However, Theorem 1.8 leads to the following expression of $W_k(t)$ as a stochastic integral.

Theorem 1.9 (Stochastic integral representation for $W_k(t)$). Assume that (1.2) holds true. We have almost surely that for all $t \geq 0$,

$$W_k(t) = \int_{(0,t]\times(0,1)^2} K_{s-}^k(r,u) N(\mathrm{d}s,\mathrm{d}r,\mathrm{d}u). \tag{1.15}$$

Thanks to the above result, we are now able to use stochastic calculus to study $W_k(t)$.

Theorem 1.10 (Pseudo-generator formula for $W_k(t)$). Assume that (1.2) holds true. For any Lipschitz function $f:[0,1] \to \mathbb{R}$ and $k \ge 1$, $(t \mapsto \mathbb{E}[f(W_k(t))])$ is of class \mathcal{C}^1 and for any $t \ge 0$,

$$\frac{d}{dt}\mathbb{E}[f(W_k(t))] = \mathbb{E}\left[\int_{(0,1)^2} \left(f(W_k(t) + K_t^k(r,u)) - f(W_k(t))\right)r^{-2}\Lambda(dr)du\right]. \tag{1.16}$$

Theorems 1.8–1.10 provide insights on the entrance law from dust of the masses of blocks and allow us to derive some large and small time asymptotics for it. We now present such results, which can be useful for parameter inference of Λ -coalescent and Λ -Fleming-Viot flows. We start with the large time asymptotics of the expectations $\mathbb{E}[W_k(t)]$. For $k \geq 1$ let

$$\lambda_k(\Lambda) := \int_{(0,1)} (1 - (1-r)^k - kr(1-r)^{k-1})r^{-2}\Lambda(dr).$$

Note from (1.1) that $\lambda_k(\Lambda) = \sum_{\ell=2}^k \binom{k}{\ell} \lambda_{k,\ell}(\Lambda)$, that is, for any $n \geq k$, $\lambda_k(\Lambda)$ is the total transition rate of $(\Pi_t^n)_{t\geq 0}$ when the partition currently contains k blocks. Also, the sequence $(\lambda_k(\Lambda))_{k\geq 1}$ is increasing, since $(1-(1-r)^k-kr(1-r)^{k-1})$ is the probability for a binomial random variable with parameter (k,r) to be larger or equal to 2. Finally we note that $\lambda_k(\Lambda) \to \int_{(0,1)} r^{-2} \Lambda(dr) \in (0,\infty]$ as $k\to\infty$. Let also

$$N(\Lambda) := \inf \left\{ k \ge 1 \text{ s.t. } \lambda_k(\Lambda) \ge H(\Lambda) \right\}. \tag{1.17}$$

Since $\int_{(0,1)} r^{-2} \Lambda(dr) > H(\Lambda)$, the above shows that $N(\Lambda)$ is finite and, since $\lambda_2(\Lambda) = \Lambda((0,1)) < H(\Lambda)$, we necessarily have $N(\Lambda) \geq 3$. The following result provides the asymptotic behaviors of the expectations of the ordered masses of blocks. It shows in particular that a cutoff phenomenon occurs, as the decay rate of $\mathbb{E}[W_k(t)]$ first increases with k but then remains constant for all $k \geq N(\Lambda)$. If a and b go to infinity with t, $a \sim b$ means that b/a converges to 1 as t goes to infinity, $a \approx b$ means that there are constants C, c > 0 such that ca < b < Ca for all large t, and $a \approx b$ means that $\log(a) \sim \log(b)$. Note that, if $a, b \to 0$, $a \sim b \Rightarrow a \approx b \Rightarrow a \approx b$.

Theorem 1.11 (Long time behavior). Assume that (1.2) holds true. We have

$$1 - \mathbb{E}[W_1(t)] \simeq e^{-t\lambda_2(\Lambda)}. (1.18)$$

$$\forall k \in \{2, ..., N(\Lambda) - 1\}, \ \mathbb{E}[W_k(t)] \simeq e^{-t\lambda_k(\Lambda)}. \tag{1.19}$$

$$\forall k \ge N(\Lambda), \ \mathbb{E}[W_k(t)] \approx e^{-tH(\Lambda)}.$$
 (1.20)

Note that (1.20) does not contradict the fact that $\sum_{k\geq 1} \mathbb{E}[W_k(t)] \leq 1$ since (1.20) allows, for example, $\mathbb{E}[W_k(t)]$ to be of order $c_k e^{-tH(\Lambda)}$ for all $k\geq N(\Lambda)$ with $\sum_{k\geq N(\Lambda)} c_k < \infty$.

Remark 1.12. In [36, 52] it is shown that the absorption time τ_n of the process $(\Pi_t^n)_{t\geq 0}$ has order $\log n$ when n is large. This heuristically agrees with (1.18). Indeed, the restriction of $(\pi_t^Y)_{t\geq 0}$ to $\{1,\ldots,n\}$ is, by Lemma 1.5, a realization of $(\Pi_t^n)_{t\geq 0}$. Then we see from Proposition 1.6 (second point) that $\{t\geq \tau_n\} = \bigcup_{k\geq 1} \{U_1,\ldots,U_n\in \mathcal{O}_k(t)\}$. Therefore, $\mathbb{P}(t\geq \tau_n) = \sum_{k\geq 1} \mathbb{E}[W_k(t)^n] \approx \mathbb{E}[W_1(t)^n]$. By the results of [36, 52], $\mathbb{P}(t\geq \tau_n)$ goes to 1 or 0 depending on whether $t>>\log n$ or $t<<\log n$. This suggests that $W_1(t)$ is of order $1-e^{-ct}$ for some constant c.

Remark 1.13. The coefficients $\lambda_k(\Lambda)$ appeared in a study of very different aspects of Λ -Fleming-Viot flows in [44], where they are called pushing rates and used to produce martingales that allow changes of measure. They also appear in [41] as the eigenvalues of the generator of the Λ -Wright-Fisher diffusion. The appearance of these coefficients in Theorem 1.11 raise the question of whether the decays observed there can be related to objects considered in [44] and [41], and in particular on the relation between the eigenfunctions of the generator of the Λ -Wright-Fisher diffusion (or other processes involved) and ordered masses of blocks of the Λ -coalescent. Theorem 1.10 may provide support to study this question, which would give insights on these eigenfunctions on which very little is known. It would also be interesting if the cut-off phenomenon from Theorem 1.11, that does not appear in [41, 44], can be given a spectral interpretation.

Remark 1.14 $(\lambda_k(\Lambda))$ and $N(\Lambda)$. Assume that (1.2) holds true. We have $\lambda_k(\Lambda) < (k-1)^2 \lambda_2(\Lambda)$ for $k \geq 3$ and $N(\Lambda) > 2 \vee \sqrt{H(\Lambda)/\lambda_2(\Lambda)}$. This shows in particular that, the more Λ has mass around 0, the larger $N(\Lambda)$ is. The proof of these bound is given in Appendix C.1.

Remark 1.15 ($Beta(2-\alpha,\alpha)$ -coalescent). In the case of the $Beta(2-\alpha,\alpha)$ -coalescent (see Example 1) with $\alpha \in (0,1)$, Theorem 1.11 applies and we have

$$\lambda_k(\Lambda_{2-\alpha,\alpha}) = \frac{\Gamma(1-\alpha)\Gamma(k+\alpha)(k-1)(1-\alpha)}{\Gamma(k)\alpha(\alpha+k-1)}, \qquad H(\Lambda_{2-\alpha,\alpha}) = \Gamma(1-\alpha)\Gamma(\alpha), \tag{1.21}$$

$$\left(\frac{\Gamma(\alpha+1)}{1-\alpha}\right)^{1/\alpha} - \alpha \le N(\Lambda_{2-\alpha,\alpha}) \le \left[\left(\frac{(2+\alpha)\Gamma(\alpha+1)}{2(1-\alpha)}\right)^{1/\alpha} - \alpha + 1\right].$$
(1.22)

These identities are justified in Appendix C.2.

Remark 1.16 (Bolthausen-Sznitman coalescent). The Bolthausen-Sznitman coalescent is the $Beta(2-\alpha,\alpha)$ -coalescent with $\alpha=1$. It falls under case (iii) of the classification (see the Introduction). In this case, if we still denote by $W_k(t)$ the mass of the k^{th} largest block in Π_t , then $(W_k(t))_{k\geq 1}$ follows the Poisson-Dirchlet distribution with parameters $(e^{-t},0)$, see [11, Thm. 6.2]. This yields that, in this case, $1-\mathbb{E}[W_1(t)]=e^{-t}$ and

$$\forall k \ge 2, \ \mathbb{E}[W_k(t)] = \frac{1 - e^{-t}}{1 + (k-1)e^{-t}} \times \frac{(k-1)!e^{-t(k-1)}}{\prod_{j=1}^{k-1}(1 + (j-1)e^{-t})} \underset{t \to \infty}{\sim} (k-1)!e^{-t(k-1)}.$$

Moreover, it is not difficult to see that $\lambda_{k,\ell}(\Lambda_{1,1}) = \frac{(\ell-2)!(k-\ell)!}{(k-1)!}$ so $\lambda_k(\Lambda_{1,1}) = k-1$, and $H(\Lambda_{1,1}) = \infty$, so $N(\Lambda_{1,1}) = \infty$. Therefore, the result of Theorem 1.11 also holds in this case. It would be interesting to see if Theorem 1.11 extends to the full generality of case (iii).

We now turn to the small time behavior of the sizes of blocks and start with the following consequence of Theorem 1.10.

Corollary 1.17 (Small time behavior). Assume that (1.2) holds true. For any Lipschitz function $f:[0,1] \to \mathbb{R}$ we have,

$$\mathbb{E}[f(W_1(t))] = f(0) + t \int_{(0,1)} (f(r) - f(0))r^{-2}\Lambda(dr) + \underset{t \to 0}{o}(t), \tag{1.23}$$

$$\forall k \ge 2, \ \mathbb{E}[f(W_k(t))] = f(0) + \underset{t \to 0}{o}(t).$$
 (1.24)

It is suggested by (1.23) that, in general, $W_1(t)$ does not converge to a non-trivial limit as t goes to zero after any renormalization. Therefore, we rather study the expectations of $W_1(t)$ and $W_2(t)$ as t is small. For $W_1(t)$, Corollary 1.17 yields

$$\mathbb{E}[W_1(t)] \underset{t \to 0}{\sim} tH(\Lambda). \tag{1.25}$$

For a Λ -coalescent $(\Pi_t)_{t\geq 0}$ with Λ satisfying (1.2) we see, proceeding as in Remark 1.12, that the size K(t) of the block containing 1 at time t (also called *size-biased picked block*) has the same expectation as the quantity $\sum_{k\geq 1} W_k(t)^2$ (called the *Simpson index* in biology). Therefore,

$$\mathbb{E}[K(t)] = \mathbb{E}\left[\sum_{k>1} W_k(t)^2\right] \underset{t\to 0}{\sim} t\lambda_2(\Lambda), \tag{1.26}$$

where the last estimates comes from the combination of Lemma 5.1 with Corollary 1.17. Thus, (1.25) and (1.26) show that, in our case, both the largest block and a typical block are, in expectation, of order $const. \times t$ when t is small, but with different constants.

Remark 1.18. The Beta(2- α , α)-coalescent with $\alpha \in (1,2)$ falls in case (iv) of the classification (see the Introduction). That coalescent is known for being embedded in a α -stable CSBP [13, 17, 10] and in a α -stable CRT [9]. In that case, the sizes $W_1(t)$ and K(t) of respectively the largest block and a typical block are studied in [10] where it is shown that $t^{-1/\alpha}W_1(t)$ and $t^{-1/(\alpha-1)}K(t)$ converge in distribution as t goes to zero. This is in contrast with the order t from (1.25)-(1.26).

We now provide an equivalent of $\mathbb{E}[W_2(t)]$ for small times. For this we set

$$h_{\Lambda}(x) := \int_{(0,1)} (a \wedge x) a^{-2} \Lambda(da), \qquad k_{\Lambda}(x) := \int_{(0,1)} ((1-x)a \wedge x) (1-a) a^{-2} \Lambda(da). \tag{1.27}$$

Note that $h_{\Lambda}(\cdot)$ and $k_{\Lambda}(\cdot)$ are well-defined because of (1.2). We have $0 \le k_{\Lambda}(x) \le h_{\Lambda}(x)$ for any $x \in [0,1]$. Moreover, $h_{\Lambda}(0) = k_{\Lambda}(0) = k_{\Lambda}(1) = 0$. $h_{\Lambda}(\cdot)$ is non-decreasing on [0,1]. By (1.2) and dominated convergence we see that $h_{\Lambda}(\cdot)$ and $k_{\Lambda}(\cdot)$ are continuous on [0,1]. Our next result requires the condition

$$\int_{(0,1)} h_{\Lambda}(r)r^{-2}\Lambda(dr) < \infty. \tag{1.28}$$

Note that, under (1.2), (1.28) is equivalent to $\int_{(0,1)} k_{\Lambda}(r) r^{-2} \Lambda(dr) < \infty$. Also, the condition (1.3) implies (1.28). As shown in Remark 1.20 below, the assumption "(1.2) and (1.28)" is strictly stronger than (1.2) alone and strictly weaker than (1.3).

Theorem 1.19 (Small time behavior of second largest block). Assume that (1.2) and (1.28) hold true. Then,

$$\mathbb{E}[W_2(t)] \underset{t \to 0}{\sim} \frac{t^2}{2} \int_{(0.1)} k_{\Lambda}(r) r^{-2} \Lambda(dr), \tag{1.29}$$

where $k_{\Lambda}(\cdot)$ is defined in (1.27).

Note that, if (1.28) does not hold, then $\int_{(0,1)} k_{\Lambda}(r)r^{-2}\Lambda(dr) = \infty$. This splits case (ii) of the classification into two sub-cases. Let us also mention that the methodology we used for Theorem 1.19 seems to apply to study $\mathbb{E}[W_k(t)]$ as $t \to 0$ for every $k \ge 2$ but we are so far unaware of a way to simultaneously cover all integers $k \ge 2$.

Remark 1.20. For the beta-coalescent from Example 1 (with a > 1), as x goes to 0 we have $h_{\Lambda_{a,b}}(x) \sim (\frac{1}{a-1} + \frac{1}{2-a})x^{a-1}$ if $a \in (1,2)$, $h_{\Lambda_{a,b}}(x) \sim x \log(1/x)$ if a = 2, and $h_{\Lambda_{a,b}}(x) \sim x B(a-2,b)$ if a > 2, so (1.28) holds true if and only if a > 3/2. Theorem 1.19 thus suggests that the beta-coalescent has another phase transition at a = 3/2.

Remark 1.21. For the Bolthausen-Sznitman coalescent, the expressions from Remark 1.16 show that, for any $k \geq 1$, $\mathbb{E}[W_k(t)] \sim t/k$. Therefore, at short times scales, the Bolthausen-Sznitman coalescent behaves differently from cases covered by our results ((1.25) and Theorem 1.19), while at large time scales their behavior is similar by Remark 1.16.

The rest of the paper is organized as follows. In Section 2 we study the processes $(\pi_t^Y)_{t\geq 0}$ and $(C_t)_{t\geq 0}$ (via the flow Y) and conclude the section by proving the results stated in Sections 1.3 and 1.4, and Theorem 1.8. In Section 3 we prove Theorems 1.9 and 1.10 and use the latter in a few applications, including Corollary 1.17. In Section 4 we study the long time asymptotics of $\mathbb{E}[W_k(t)]$ and prove Theorem 1.11. In Section 5 we study the small time asymptotics of $\mathbb{E}[W_2(t)]$ and prove Theorem 1.19. Relevant analytical properties of the SDE (1.5) are studied in Appendices A and B. Some remarks from Section 1.5 are proved in Appendix C.

2. Construction and representation

Recall that we always assume that (1.2) holds true.

2.1. No continuous mergers: proof of Proposition 1.4. We start with a preliminary lemma. Recall that $Q := [0,1] \cap \mathbb{Q}$.

Lemma 2.1. For $y \in [0,1]$, let $N_{t_1,t_2}(y) := \{(s,r,u) \in N \text{ s.t. } s \in (t_1,t_2], Y_{0,s-}(y) \in I_{r,u}\}.$

- Almost surely, $N_{t_1,t_2}(y)$ is finite for all $y \in Q$ and t_1,t_2 with $0 \le t_1 \le t_2 < \infty$,
- Almost surely, for almost every $z \in [0,1]$, $N_{t_1,t_2}(z)$ is finite for all t_1, t_2 with $0 \le t_1 \le t_2 < \infty$.

Proof. First note that for any $y \in [0,1]$, and $r \leq 1/2$,

$$\int_{(0,1)} \mathbb{1}_{y \in I_{r,u}} du = \left[\frac{y-r}{1-r}, \frac{y}{1-r} \right] \cap [0,1] = \left(\frac{y}{1-r} \wedge 1 \right) - \left(\frac{y-r}{1-r} \vee 0 \right) \le \frac{r}{1-r} \le 2r. \quad (2.30)$$

Fix $y \in [0,1]$ and let $\tilde{N}_{t_1,t_2}(y) := \{(s,r,u) \in N_{t_1,t_2}(y) \text{ s.t. } r \leq 1/2\}$. By the compensation formula and (2.30) we get

$$\mathbb{E}\left[\sharp \tilde{N}_{t_{1},t_{2}}(y)\right] = \mathbb{E}\left[\sum_{(s,r,u)\in N, s\in(t_{1},t_{2}],r\leq1/2} \mathbb{1}_{Y_{0,s-}(y)\in I_{r,u}}\right] \\
= \int_{t_{1}}^{t_{2}} \mathbb{E}\left[\int_{(0,1/2]} \left(\int_{(0,1)} \mathbb{1}_{Y_{0,s}(y)\in I_{r,u}} du\right) r^{-2} \Lambda(dr)\right] ds \leq 2(t_{2}-t_{1}) \int_{(0,1/2]} r^{-1} \Lambda(dr) < \infty, \tag{2.31}$$

where the finiteness comes from (1.2). The combination of (2.31) with the monotonicity of $\tilde{N}_{t_1,t_2}(y)$ with respect to the time interval $(t_1,t_2]$ show that $\tilde{N}_{t_1,t_2}(y)$ is almost surely finite for

all $t_1, t_2 \geq 0$. Clearly $N_{t_1,t_2}(y) \setminus \tilde{N}_{t_1,t_2}(y) \subset \{(s,r,u) \in N \text{ s.t. } s \in (t_1,t_2], r > 1/2\}$, which is also almost surely finite for all $t_1, t_2 \geq 0$. The first statement is thus proved for a fixed $y \in [0,1]$ and, since Q is countable, the statement follows. Then, using Fubini's theorem and (2.31),

$$\mathbb{E}\left[\int_{[0,1]} \sharp \tilde{N}_{t_1,t_2}(z) dz\right] = \int_{[0,1]} \mathbb{E}\left[\sharp \tilde{N}_{t_1,t_2}(z)\right] dz \le 2(t_2 - t_1) \int_{(0,1/2)} r^{-1} \Lambda(dr) < \infty.$$

We thus get that $\int_{[0,1]} \sharp \tilde{N}_{t_1,t_2}(z) dz$ is almost surely finite so, for almost every $z \in [0,1]$, $\tilde{N}_{t_1,t_2}(z)$ is finite. By the monotonicity of $\tilde{N}_{t_1,t_2}(y)$ with respect to the time interval $(t_1,t_2]$ we get that the second statement of the lemma holds for $\tilde{N}_{t_1,t_2}(z)$. Then, $N_{t_1,t_2}(z) \setminus \tilde{N}_{t_1,t_2}(z) \subset \{(s,r,u) \in N \text{ s.t. } s \in (t_1,t_2], r > 1/2\}$, which does not depend on z and is also almost surely finite for all $t_1,t_2 \geq 0$. This concludes the proof of the second statement of the lemma for $N_{t_1,t_2}(z)$.

By Lemma 2.1 we know that, on a probability one event, $N_{0,\infty}(y)$ is discrete for all $y \in Q$. On this event, for any $y_1, y_2 \in Q$ let $(T_k(y_1, y_2))_{k \geq 1}$ be the increasing enumeration of the time components of $N_{0,\infty}(y_1) \cup N_{0,\infty}(y_2) \cup \{(s,r,u) \in N \text{ s.t. } r \geq 1/2\}$. For convenience we also set $T_0(y_1, y_2) := 0$. The following lemma says that trajectories starting from distinct points in Q cannot merge outside $\{T_k(y_1, y_2), k \geq 1\}$.

Lemma 2.2. For any $y_1, y_2 \in Q$ with $y_1 \neq y_2$ and $k \geq 0$ we have

$$\mathbb{P}(\exists t \in (T_k(y_1, y_2), T_{k+1}(y_1, y_2)) \ s.t. \ Y_{0,t}(y_1) = Y_{0,t}(y_2) | Y_{0,T_k(y_1, y_2)}(y_1) \neq Y_{0,T_k(y_1, y_2)}(y_2)) = 0,$$

$$\mathbb{P}(\exists t \in (T_k(y_1, y_2), T_{k+1}(y_1, y_2)] \ s.t. \ Y_{0,t-}(y_1) = Y_{0,t-}(y_2) | Y_{0,T_k(y_1, y_2)}(y_1) \neq Y_{0,T_k(y_1, y_2)}(y_2)) = 0.$$

Proof. Fix $y_1, y_2 \in Q$ such that $y_1 \geq y_2$ and $k \geq 1$. Let us set $\Delta_t := Y_{0,t}(y_1) - Y_{0,t}(y_2)$. By (1.5), we have that, almost surely, for any $t \in [T_k(y_1, y_2), T_{k+1}(y_1, y_2)), \Delta_t - \Delta_{T_k(y_1, y_2)}$ equals

$$\begin{split} \int_{(T_k(y_1,y_2),t]\times(0,1/2)\times(0,1)} \left(\mathbf{m}_{r,u}(Y_{0,s-}(y_1)) - \mathbf{m}_{r,u}(Y_{0,s-}(y_2)) - Y_{0,s-}(y_1) + Y_{0,s-}(y_2)\right) N(ds,dr,du) \\ = & \int_{(T_k(y_1,y_2),t]\times(0,1/2)\times(0,1)} \left(\frac{r\Delta_{s-}}{1-r}\mathbb{1}_{u\in(0,\frac{Y_{0,s-}(y_2)-r}{1-r}]\cup(\frac{Y_{0,s-}(y_1)}{1-r},1)} - \frac{r(1-\Delta_{s-})}{1-r}\mathbb{1}_{u\in(\frac{Y_{0,s-}(y_2)}{1-r},\frac{Y_{0,s-}(y_1)-r}{1-r}]}\right) N(ds,dr,du). \end{split}$$

Fix $M, T \in (0, \infty) \cap \mathbb{Q}$ and set $f_M(x) := -\log(x \vee e^{-M})$. It is not difficult to see that, for the Itô's formula from [47, Thm. II.5.1], if all terms composing the process are null except the uncompensated Poisson stochastic integral, then the formula holds true for Lipschitz functions instead of functions of class \mathcal{C}^2 . Since $f_M(\cdot)$ is a Lipschitz function and $(\Delta_t)_{t\geq 0}$ is of the right form, we can apply the formula to $f_M(\cdot)$ and $(\Delta_t)_{t\geq 0}$ and get that, almost surely, for any $t \in [T_k(y_1, y_2), T_{k+1}(y_1, y_2)), f_M(\Delta_t) - f_M(\Delta_{T_k(y_1, y_2)})$ equals

$$\int_{(T_{k}(y_{1},y_{2}),t]\times(0,1/2)\times(0,1)} \left(f_{M}\left(\frac{\Delta_{s-}}{1-r}\right) - f_{M}(\Delta_{s-}) \right) \mathbb{1}_{u\in(0,\frac{Y_{0,s-}(y_{2})-r}{1-r}]\cup(\frac{Y_{0,s-}(y_{1})}{1-r},1)} N(ds,dr,du)
+ \int_{(T_{k}(y_{1},y_{2}),t]\times(0,1/2)\times(0,1)} \left(f_{M}\left(\frac{\Delta_{s-}-r}{1-r}\right) - f_{M}(\Delta_{s-}) \right) \mathbb{1}_{u\in(\frac{Y_{0,s-}(y_{2})}{1-r},\frac{Y_{0,s-}(y_{1})-r}{1-r}]} N(ds,dr,du).$$

For any $a \in [0,1]$ and $r \in (0,1/2)$ we have $\frac{a-r}{1-r} \le a \le \frac{a}{1-r}$ so, since f_M is non-increasing, we see that the integrand of the first integral is non-positive while the integrand of the second integral is non-negative. We thus get that almost surely,

$$\sup_{t \in [T_k(y_1, y_2), T_{k+1}(y_1, y_2) \land T)} f_M(\Delta_t) \le f_M(\Delta_{T_k(y_1, y_2)}) + B_M(T), \tag{2.32}$$

where

$$B_M(T) := \int_{(0,T]\times(0,1/2)\times(0,1)} \left(f_M\left(\frac{\Delta_{s-}-r}{1-r}\right) - f_M(\Delta_{s-}) \right) \mathbbm{1}_{u \in (\frac{Y_{0,s-}(y_2)}{1-r},\frac{Y_{0,s-}(y_1)-r}{1-r}]} N(ds,dr,du).$$

By the compensation formula and Fubini's theorem we get that $\mathbb{E}[B_M(T)]$ equals

$$\int_{(0,T]} \int_{(0,1/2)} \mathbb{E} \left[\int_{(0,1)} \left(f_M \left(\frac{\Delta_{s-} - r}{1 - r} \right) - f_M(\Delta_{s-}) \right) \mathbb{1}_{u \in (\frac{Y_{0,s-}(y_2)}{1 - r}, \frac{Y_{0,s-}(y_1) - r}{1 - r}]} du \right] r^{-2} \Lambda(dr) ds
= \int_{(0,T]} \int_{(0,1/2)} \mathbb{E} \left[\frac{\Delta_{s-} - r}{1 - r} \left(f_M \left(\frac{\Delta_{s-} - r}{1 - r} \right) - f_M(\Delta_{s-}) \right) \mathbb{1}_{r < \Delta_{s-}} \right] r^{-2} \Lambda(dr) ds
\leq CT \int_{(0,1/2)} r^{-1} \Lambda(dr) < \infty,$$
(2.33)

where the last inequality comes from Lemma A.3 and the finiteness from (1.2). Let us denote by \mathcal{E}_k the event $\{Y_{0,T_k(y_1,y_2)}(y_1) \neq Y_{0,T_k(y_1,y_2)}(y_2)\}$ and

$$\mathcal{A}_k(T) := \{ \exists t \in (T_k(y_1, y_2), T_{k+1}(y_1, y_2) \land T) \text{ s.t. } Y_{0,t}(y_1) = Y_{0,t}(y_2) \}$$
$$\cup \{ \exists t \in (T_k(y_1, y_2), T_{k+1}(y_1, y_2) \land T] \text{ s.t. } Y_{0,t-}(y_1) = Y_{0,t-}(y_2) \}.$$

Using the definitions of Δ_t and $f_M(\cdot)$, (2.32), Markov inequality, and (2.33) we get that $\mathbb{P}(\mathcal{A}_k(T)|\mathcal{E}_k)$ is smaller than

$$\mathbb{P}\left(\sup_{t\in[T_{k}(y_{1},y_{2}),T_{k+1}(y_{1},y_{2})\wedge T)}f_{M}(\Delta_{t})\geq M|\mathcal{E}_{k}\right)\leq\mathbb{1}_{f_{M}(\Delta_{T_{k}(y_{1},y_{2})})\geq M/2}+\frac{\mathbb{P}(B_{M}(T)\geq M/2)}{\mathbb{P}(\mathcal{E}_{k})}$$

$$\leq\mathbb{1}_{\Delta_{T_{k}(y_{1},y_{2})}\leq e^{-M/2}}+\frac{2\mathbb{E}[B_{M}(T)]}{M\times\mathbb{P}(\mathcal{E}_{k})}\leq\mathbb{1}_{\Delta_{T_{k}(y_{1},y_{2})}\leq e^{-M/2}}+\frac{2CT\int_{(0,1/2)}r^{-1}\Lambda(dr)}{M\times\mathbb{P}(\mathcal{E}_{k})}.$$

Letting M go to infinity, we get that the right-hand side goes to zero almost surely on \mathcal{E}_k . Then letting T go to infinity we get the result.

The following lemma says that trajectories starting from distinct points in Q only merge at specific jump times.

Lemma 2.3. We have

$$\mathbb{P}\left(\forall t > 0, \forall y_1 \neq y_2 \in Q, \ Y_{0,t}(y_1) \neq Y_{0,t}(y_2) \ or \ \exists (s,r,u) \in N \ s.t. \ s \leq t, \ y_1, y_2 \in Y_{0,s-}^{-1}(I_{r,u})\right) = 1.$$
The same property holds with $Y_{0,t}(\cdot)$ replaced by $Y_{0,t-}(\cdot)$ and $s \leq t$ replaced by $s < t$.

Proof. Let $y_1, y_2 \in Q$ with $y_1 \neq y_2$ and $T(y_1, y_2) := \inf\{t \geq 0, Y_{0,t}(y_1) = Y_{0,t}(y_2) \text{ or } Y_{0,t-}(y_1) = Y_{0,t-}(y_2)\}$. By Lemma 2.1 the sequence $(T_k(y_1, y_2))_{k\geq 1}$ is well-defined almost surely and by Lemma 2.2 we have almost surely $T(y_1, y_2) \in \{T_k(y_1, y_2), k \geq 1\} \cup \{\infty\}$ and $Y_{0,T(y_1,y_2)-}(y_1) \neq Y_{0,T(y_1,y_2)-}(y_2)$ on $\{T(y_1, y_2) < \infty\}$. On $\{T(y_1, y_2) < \infty\}$, let us denote by $(s, r, u) \in N$ the jump at time $T(y_1, y_2)$ (then $s = T(y_1, y_2)$). Since $Y_{0,s-}(y_1) \neq Y_{0,s-}(y_2)$ and $Y_{0,s}(y_1) = Y_{0,s}(y_2)$, we see from (1.5) that we have necessarily $Y_{0,s-}(y_1), Y_{0,s-}(y_2) \in I_{r,u}$. Since Q is countable the result follows. □

We can now conclude the proof of Proposition 1.4.

Proof of Proposition 1.4. Assume we are on the probability one event where the claims from Lemmas 2.3 and 2.1 hold true. By contradiction, we assume that there are t > 0 and $y_1, y_2 \in [0, 1]$ with $y_1 < y_2$ such that $Y_{0,t}(y_1) = Y_{0,t}(y_2)$ (resp. $Y_{0,t-}(y_1) = Y_{0,t-}(y_2)$) and such that there is no jump $(s, r, u) \in N$ with $s \le t$ (resp. s < t) for which both y_1 and y_2 belong to $Y_{0,s-}^{-1}(I_{r,u})$.

Then, $Y_{0,t}([y_1,y_2])$ (resp. $Y_{0,t-}([y_1,y_2])$) is a singleton. Fix $n_1 > m_0 := 3/(y_2 - y_1)$ and choose $q_1, \tilde{q}_1 \in Q$ such that $q_1 \in (y_1, y_1 + 1/n_1)$ and $\tilde{q}_1 \in (y_2 - 1/n_1, y_2)$. By Lemma 2.3, there is $(s_1, r_1, u_1) \in N$ such that $s_1 \leq t$ (resp. $s_1 < t$) and $q_1, \tilde{q}_1 \in Y_{0,s_1-}^{-1}(I_{r_1,u_1})$. We can choose a rational $q \in (y_1 + 1/m_0, y_2 - 1/m_0) \subset [q_1, \tilde{q}_1] \subset Y_{0,s_1-}^{-1}(I_{r_1,u_1})$. By assumption, it is not possible that both y_1 and y_2 belong to the interval $Y_{0,s_1-}^{-1}(I_{r_1,u_1})$ so there is $n_2 > n_1$ such that $y_1 + 1/n_2 < \inf Y_{0,s_1-}^{-1}(I_{r_1,u_1})$ or $y_2 - 1/n_2 > \sup Y_{0,s_1-}^{-1}(I_{r_1,u_1})$. We can choose $q_2, \tilde{q}_2 \in Q$ such that $q_2 \in (y_1, y_1 + 1/n_2)$ and $\tilde{q}_2 \in (y_2 - 1/n_2, y_2)$ and proceed as before. Iterating this procedure (but with always the same q) we construct an infinite sequence of distinct jumps $(s_j, r_j, u_j)_{j \geq 1}$ such that $q \in Y_{0,s_j-}^{-1}(I_{r_j,u_j})$ for all $j \geq 1$. This contradicts the assumption that we are on the probability one event given by Lemma 2.1. The result follows.

Remark 2.4. Proposition 1.4 and (1.7) yield that, almost surely, if $Y_{0,t_2}(y_1) = Y_{0,t_2}(y_2)$ (resp. $Y_{0,t_2-}(y_1) = Y_{0,t_2-}(y_2)$) and $Y_{0,t_1}(y_1) \neq Y_{0,t_1}(y_2)$ for some $t_2 > t_1 \geq 0$ and $y_1, y_2 \in [0,1]$, then there is $(s,r,u) \in N$ such that $s \in (t_1,t_2]$ (resp. $s \in (t_1,t_2)$) and $y_1,y_2 \in Y_{0,s-}^{-1}(I_{r,u})$.

2.2. **Proof of Lemma 1.5.** Let $(U_i)_{i\geq 1}$ be as in Sections 1.3. We define the process $(\pi_t^N)_{t\geq 0}$ of random partitions of $\mathbb N$ by the equivalence relation $i\sim_{\pi_t^N} j\Leftrightarrow \exists (s,r,u)\in N$ such that $s\in (0,t]$ and $U_i,U_j\in Y_{0,s-}^{-1}(I_{r,u})$. For two partitions P_1 and P_2 of $\mathbb N$, we say that P_2 is a coagulation of P_1 and note $P_1\leq P_2$ if each block of P_2 is a union of blocks from P_1 .

Remark 2.5. We note from respectively (1.7) and Proposition 1.4 that we have almost surely that $\pi_t^N \leq \pi_t^Y$ and $\pi_t^Y \leq \pi_t^N$ for all $t \geq 0$. We thus have almost surely $\pi_t^Y = \pi_t^N$ for all $t \geq 0$.

By Corollary 2.16 from Section 2.6 we see that, almost surely, for all $t \geq 0$ there are infinitely many $i \geq 1$ such that $U_i \in C_t^c$ and $U_i \in C_{t-}^c$, which implies that π_t^Y and π_{t-}^Y (and therefore π_t^N and π_{t-}^N) have infinitely many singletons blocks. Since Lemma 1.5 is nowhere used to prove the results from Section 2.6, we take the infiniteness of blocks into account in the following proof.

Proof of Lemma 1.5. Thanks to Remark 2.5, we only need to prove the statement for $(\pi_t^N)_{t\geq 0}$. For any $t\geq 0$ we denote by $A_t^1,A_t^2,...$ the blocks of π_t^N ordered by their lowest elements. We see from Remark 2.5 that $j_1 \sim_{\pi_t^N} j_2 \Leftrightarrow Y_{0,t}(U_{j_1}) = Y_{0,t}(U_{j_2})$. For $i\geq 1$ we can thus set $U_i(t):=Y_{0,t}(U_j)$ where j is any element of A_t^i . For any $(s,r,u)\in N$ we set $Z_{s,i}:=\mathbbm{1}_{U_i(s-)\in I_{r,u}}$. We see from the definition of $(\pi_t^N)_{t\geq 0}$ that merging events in that coalescent process only occur at jumping times $s\in J_N$ and that, for any $s\in J_N$, the non-empty blocks A_{s-}^i involved in the merging are exactly those for which $Z_{s,i}=1$. We show below that the random set $\{(s,r,(Z_{s,i})_{i\geq 1})\}$ defines a Poisson random measure on $(0,\infty)\times(0,1)\times(\{0,1\}^{\mathbb{N}})$ with intensity measure $m(ds,dr,dz):=ds\times(\mathcal{B}(r)^{\times\mathbb{N}}(dz))r^{-2}\Lambda(dr)$. By [62, Cor. 3] it will follow that $(\pi_t^N)_{t\geq 0}$ is a Λ -coalescent.

Let us fix some $\eta \in (0,1/2)$ and let $(s_i^\eta, r_i^\eta, u_i^\eta)_{i \geq 1}$ be the enumeration of $\{(s,r,u) \in N, r > \eta\}$ such that $s_1^\eta < s_2^\eta < \ldots$ and for convenience we set $s_0^\eta := 0$. We note from the definition of a bridge (see for example [14, Sec. 2.1]) that any integral of parametrized laws of bridges with respect to a probability measure is the law of a bridge. Since the law of $Y_{0,s_1^\eta-}(\cdot)$ is the integral of the law of $Y_{0,t_1^\eta}(\cdot)$ with respect to the measure $\Lambda((\eta,1))e^{-\Lambda((\eta,1))t}\mathbbm{1}_{t>0}dt$, we get from the discussion after Proposition 1.1 that the generalized inverse of $Y_{0,s_1^\eta-}(\cdot)$ is a bridge B (even after conditioning with respect to $(s_1^\eta, r_1^\eta, u_1^\eta)$). We see that B, $\pi_{s_1^\eta}^N$ and $(U_j(s_1^\eta-))_{j\geq 1}$ are as the bridge, the partition and the sequence considered in [14, Lem. 2]. By that lemma we get that, conditionally on $(s_1^\eta, r_1^\eta, u_1^\eta)$, $(U_j(s_1^\eta-))_{j\geq 1} \sim \mathcal{U}([0,1])^{\times \mathbb{N}}$ so $(Z_{s_1^\eta,j})_{j\geq 1} \sim \mathcal{B}(r_1^\eta)^{\times \mathbb{N}}$. Then, conditionally on (s_1^η, r_1^η) , the generalized inverse of $m_{r_1^\eta, u_1^\eta}(\cdot)$ is a bridge \tilde{B} independent of $(U_j(s_1^\eta-))_{j\geq 1}$. Let us define

a partition π of $\mathbb N$ by the equivalence relation $i \sim_{\pi} j \Leftrightarrow \mathrm{m}_{r_1^{\eta},u_1^{\eta}}(U_i(s_1^{\eta}-)) = \mathrm{m}_{r_1^{\eta},u_1^{\eta}}(U_j(s_1^{\eta}-)).$ We note from (1.7) that $\{j \geq 1, Z_{s_1^{\eta},j} = 1\}$ is a block of the partition π and that, for all $j \geq 1$ such that $Z_{s_1^{\eta},j} = 0$, $\{j\}$ is a block of the partition π . In particular, $(Z_{s_1^{\eta},j})_{j\geq 1}$ is a function of the partition π . We see that \tilde{B} , π and $(U_j(s_1^{\eta}))_{j\geq 1}$ are as the bridge, the partition and the sequence considered in [14, Lem. 2]. By that lemma we get that, conditionally on (s_1^{η}, r_1^{η}) , the sequences $(U_j(s_1^{\eta}))_{j\geq 1}$ and $(Z_{s_1^{\eta},j})_{j\geq 1}$ are independent and that $(U_j(s_1^{\eta}))_{j\geq 1} \sim \mathcal{U}([0,1])^{\times \mathbb{N}}$. Then iterating the above arguments applied to $Y_{s_i^{\eta},s_{i+1}^{\eta}-}(\cdot)$ and $(U_j(s_i^{\eta}))_{j\geq 1}$ instead of $Y_{0,s_1^{\eta}-}(\cdot)$ and $(U_j)_{j\geq 1}$, and then to $\mathrm{m}_{r_{i+1}^{\eta},u_{i+1}^{\eta}}(\cdot)$ and $(U_j(s_{i+1}^{\eta}-))_{j\geq 1}$ instead of $\mathrm{m}_{r_1^{\eta},u_1^{\eta}}(\cdot)$ and $(U_j(s_1^{\eta}-))_{j\geq 1}$, we get that, conditionally on $(s_i^{\eta},r_i^{\eta})_{i\geq 1}$, the sequences $(Z_{s_i^{\eta},j})_{j\geq 1}$ are independent and satisfy $(Z_{s_i^{\eta},j})_{j\geq 1} \sim \mathcal{B}(r_i^{\eta})^{\times \mathbb{N}}$.

For $\eta \in (0, 1/2)$ we let $\mathcal{D}_{\eta} := (0, \infty) \times (\eta, 1) \times (\{0, 1\}^{\mathbb{N}})$. The above shows that $\{(s, r, (Z_{s,i})_{i \geq 1})\} \cap \mathcal{D}_{\eta}$ defines a Poisson random measure on \mathcal{D}_{η} with intensity measure $m(\mathcal{D}_{\eta} \cap \cdot)$. We thus get that $\{(s, r, (Z_{s,i})_{i \geq 1})\}$ defines a Poisson random measure on $(0, \infty) \times (0, 1) \times (\{0, 1\}^{\mathbb{N}})$ with intensity measure m(ds, dr, dz). This concludes the proof.

Remark 2.6. The above proof shows that the Λ -coalescent process $(\pi_t^N)_{t\geq 0}$ only has transitions at times $s\in J_N$ and, if p is the corresponding r-component of the jump, blocks take part in the merging event independently with probability p. This will be useful later on.

2.3. A first representation of families. In this subsection we prove Proposition 2.7 (see below) which provides a representation for the sets C_t from Section 1.3. For any t > 0 we set

$$D_t := \bigcup_{(s,r,u)\in N, s\in(0,t]} Y_{0,s-}^{-1}(I_{r,u}^{o}), \qquad D_{t-} := \bigcup_{s\in(0,t)} D_s = \bigcup_{(s,r,u)\in N, s\in(0,t)} Y_{0,s-}^{-1}(I_{r,u}^{o}). \tag{2.34}$$

Proposition 2.7. We have almost surely that $C_t = D_t$ and $C_{t-} = D_{t-}$ for all $t \ge 0$. In particular $C_{t-} = \bigcup_{s \in (0,t)} C_s$.

The proof of Proposition 2.7 requires some preliminary lemmas. Recall that $Q := [0,1] \cap \mathbb{Q}$.

Lemma 2.8. Almost surely, for all $(s_1, r_1, u_1), (s_2, r_2, u_2) \in N$ with $s_1 < s_2$ we have $Y_{s_1, s_2-}(u_1) \notin I_{r_2, u_2} \setminus I_{r_2, u_2}^o$, and for all $(s, r, u) \in N$ and $y \in Q$, $Y_{0, s_-}(y) \notin I_{r, u} \setminus I_{r, u}^o$.

Proof. Let \tilde{N} be a Poisson point process on $(0,\infty)\times(0,1)$ with intensity measure $dt\times r^{-2}\Lambda(dr)$ and let $(V_i)_{i\geq 1}\sim \mathcal{U}([0,1])^{\times\mathbb{N}}$. We set a deterministic total order on $(0,\infty)\times(0,1)$ such that, almost surely, the elements of \tilde{N} can be enumerated in a sequence that respects that order. We denote by n(s,r) the position of the element $(s,r)\in \tilde{N}$. Note that $\{(s,r,V_{n(s,r)}),(s,r)\in \tilde{N}\}$ is equal in law to N so, in this proof, we assume that N is built in this way. We work conditionally on \tilde{N} and pick $(s_1,r_1),(s_2,r_2)\in \tilde{N}$ with $s_1< s_2$. Then

$$\begin{split} \left\{ Y_{s_1,s_2-}(V_{n(s_1,r_1)}) \in I_{r_2,V_{n(s_2,r_2)}} \setminus I^{\text{o}}_{r_2,V_{n(s_2,r_2)}} \right\} \\ &= \left\{ Y_{s_1,s_2-}(V_{n(s_1,r_1)}) \in \{(1-r_2)V_{n(s_2,r_2)}, (1-r_2)V_{n(s_2,r_2)} + r_2 \} \right\}. \end{split}$$

Since $V_{n(s_2,r_2)}$ is independent of $(Y_{s_1,s_2-}(V_{n(s_1,r_1)}),r_2)$, the above event has null probability. Then intersecting over all choices of $(s_1,r_1),(s_2,r_2) \in \tilde{N}$ we get that the result holds, conditionally on \tilde{N} , and integrating with respect to \tilde{N} we get the first statement.

Proceeding as above we can show that, almost surely, for all $(s, r, u) \in N$, $u \notin \{Y_{0,s-}(y)/(1-r), y \in Q\} \cup \{(Y_{0,s-}(y)-r)/(1-r), y \in Q\}$. For $(s, r, u) \in N$ and $y \in Q$, we have $Y_{0,s-}(y) \in I_{r,u} \setminus I_{r,u}^o \Leftrightarrow Y_{0,s-}(y) \in \{(1-r)u, (1-r)u+r\}$. The second claim follows.

Lemma 2.9. Almost surely, for all $(s, r, u) \in N$ we have $Y_{0,s-}^{-1}(I_{r,u}^{o}) = (Y_{0,s-}^{-1}(I_{r,u}))^{o}$.

Proof. Let us assume that we are on the probability one event provided by Lemma 2.8, and assume that there is $(s,r,u) \in N$ such that $Y_{0,s-}^{-1}(I_{r,u}^{\circ}) \neq (Y_{0,s-}^{-1}(I_{r,u}))^{\circ}$. Since $Y_{0,s-}^{-1}(I_{r,u}^{\circ}) \subset (Y_{0,s-}^{-1}(I_{r,u}))^{\circ}$, there exists a nonempty open interval $J \subset (Y_{0,s-}^{-1}(I_{r,u}))^{\circ} \setminus Y_{0,s-}^{-1}(I_{r,u}^{\circ})$. We have $Y_{0,s-}(J) \subset I_{r,u} \setminus I_{r,u}^{\circ}$. Then, for $y \in J \cap \mathbb{Q}$ we have $Y_{0,s-}(y) \in I_{r,u} \setminus I_{r,u}^{\circ}$, which contradicts the assumption that we are one the probability one events provided by Lemma 2.8.

The combination of Proposition 1.4 with Lemma 2.9 yields the following lemma.

Lemma 2.10. Almost surely, if $Y_{0,t}(y_a) = Y_{0,t}(y_b)$ (resp. $Y_{0,t-}(y_a) = Y_{0,t-}(y_b)$) for some t > 0 and $y_a, y_b \in [0, 1]$ such that $y_a < y_b$, then $(y_a, y_b) \subset D_t$ (resp. $(y_a, y_b) \subset D_{t-}$).

Proof of Proposition 2.7. Note from (1.7) that, almost surely, for any $(s,r,u) \in N$ and $t \geq s$, $Y_{0,t}(\cdot)$ is constant on $Y_{0,s-}^{-1}(I_{r,u}^{o})$. Therefore, $m_t(Y_{0,s-}^{-1}(I_{r,u}^{o})) = 0$ and $Y_{0,s-}^{-1}(I_{r,u}^{o}) \subset C_t$. We thus get that almost surely, for all $t \geq 0$, $D_t \subset C_t$.

We now assume that we are on the probability one events provided by Proposition 1.4 and Lemma 2.10, fix $t \geq 0$ and prove that $C_t \subset D_t$. Note that $\{0,1\} \subset D_t^c$ and Proposition 1.4 implies that $\{0,1\} \subset C_t^c$. Now let any $x \in D_t^c \cap (0,1)$ and $\epsilon \in (0,x \wedge (1-x))$. Since $(x-\epsilon,x+\epsilon) \not\subseteq D_t$, by Lemma 2.10 we have $Y_{0,t}(x+\epsilon) > Y_{0,t}(x-\epsilon)$ so $m_t([x-\epsilon,x+\epsilon]) > 0$. Since this is satisfied for any $x \in D_t^c \cap (0,1)$ and small $\epsilon > 0$ we get that any $D_t^c \cap (0,1) \subset Supp(m_t) = C_t^c$ so $C_t \subset D_t$.

We get that, almost surely, $C_t = D_t$ for all t > 0. The proof for $C_{t-} = D_{t-}$ is the same.

2.4. **Some more lemmas.** In this subsection we prove some more lemmas that will come useful for the proof of Theorem 1.8.

Lemma 2.11. Almost surely, the set $\bigcup_{(s,r,u)\in N} (Y_{0,s-}^{-1}(I_{r,u}) \setminus Y_{0,s-}^{-1}(I_{r,u}^{\circ}))$ is countable and $\mathbb{Q} \cap \bigcup_{(s,r,u)\in N} (Y_{0,s-}^{-1}(I_{r,u}) \setminus Y_{0,s-}^{-1}(I_{r,u}^{\circ})) = \emptyset$.

Proof. Let us assume that we are on the probability one event provided by Lemmas 2.8 and 2.9. Since each set $Y_{0,s-}^{-1}(I_{r,u})$ is a closed interval, Lemma 2.9 shows that each set $Y_{0,s-}^{-1}(I_{r,u}) \setminus Y_{0,s-}^{-1}(I_{r,u}^{o})$ has exactly two elements, yielding the first claim. Let $y \in Q$, then $y \in \bigcup_{(s,r,u)\in N}(Y_{0,s-}^{-1}(I_{r,u})\setminus Y_{0,s-}^{-1}(I_{r,u}^{o})) \Rightarrow \exists (s,r,u)\in N \text{ s.t. } Y_{0,s-}(y)\in I_{r,u}\setminus I_{r,u}^{o}$. By Lemma 2.8, the later does not occur. This proves the second claim.

For any t > 0 and $y \in [0,1]$ let

$$J_t(y) := Y_{0,t}^{-1}(\{Y_{0,t}(y)\}) = \{z \in [0,1] \text{ s.t. } Y_{0,t}(z) = Y_{0,t}(y)\}.$$
(2.35)

By Proposition 1.1 we have that, almost surely, $Y_{0,t}(\cdot)$ is non-decreasing and continuous for all $t \geq 0$, so each set $J_t(y)$ is a closed interval (possibly equal to the singleton $\{y\}$). We note that for each $y \in C_t$, $J_t(y)^{\circ}$ is the open connected component of C_t containing y so, in particular, $J_t(y)^{\circ} \in \{\mathcal{O}_k(t), k \geq 1\}$ and $|J_s(y)| = |J_s(y)^{\circ}| \in \{W_k(s), k \geq 1\}$.

Lemma 2.12. Almost surely, for every $y \in Q$, the interval-valued process $(J_t(y))_{t\geq 0}$ is non-decreasing, piecewise constant and right-continuous, increase times s of $(J_t(y))_{t\geq 0}$ are exactly the time components of jumps $(s, r, u) \in N$ such that $y \in Y_{0,s-}^{-1}(I_{r,u}^{\circ})$. This claim is also true when "for every $y \in Q$ " is replaced by "for almost every $y \in [0,1]$ ".

Proof. We assume that we are on the probability one events from Proposition 1.4, (1.7), Remark 2.4, Lemma 2.1 and Lemma 2.11. Let $\mathcal{U} \subset [0,1]$ be the set of measure one produced by Lemma 2.1. Let $t_2 > t_1 \geq 0$ and $y \in [0,1]$. If $z \in J_{t_1}(y)$ then $Y_{0,t_1}(z) = Y_{0,t_1}(y)$ so, by Proposition 1.4, either z = y, in which case $Y_{0,t_2}(z) = Y_{0,t_2}(y)$, or $\exists (s,r,u) \in N$ such that $s \leq t_1$ and $y,z \in S$

 $Y_{0,s-}^{-1}(I_{r,u})$. In this case, we get from (1.7) that $Y_{0,t_2}(z) = Y_{0,t_2}(y)$. Therefore $z \in J_{t_2}(y)$. This proves that $(J_t(y))_{t\geq 0}$ is non-decreasing for all $y \in [0,1]$. If $J_{t_1}(y) \neq J_{t_2}(y)$ for some $t_2 > t_1 \geq 0$ and $y \in Q$ (resp. $y \in \mathcal{U} \cap (\bigcup_{(s,r,u)\in N}(Y_{0,s-}^{-1}(I_{r,u})\setminus Y_{0,s-}^{-1}(I_{r,u}^{o}))^c)$ then let $z \in J_{t_2}(y)\setminus J_{t_1}(y)$. By definition of $J_{\cdot}(y)$ we have $Y_{0,t_1}(z) \neq Y_{0,t_1}(y)$ and $Y_{0,t_2}(z) = Y_{0,t_2}(y)$. By Remark 2.4 we get that there is $(s,r,u)\in N$ with $s\in (t_1,t_2]$ and $y\in Y_{0,s-}^{-1}(I_{r,u})$ (so $(s,r,u)\in N_{t_1,t_2}(y)$) and, by Lemma 2.11, we even have $y\in Y_{0,s-}^{-1}(I_{r,u}^{o})$. In conclusion, we can have $J_{t_1}(y)\neq J_{t_2}(y)$ only if $N_{t_1,t_2}(y)$ is non-empty. By Lemma 2.1, the sets $N_{t_1,t_2}(y)$ are finite for all $0\leq t_1< t_2$ so let $T_1< T_2<\ldots$ be the ordered sequence of time components of elements $(s,r,u)\in N_{0,\infty}(y)$ (and set $T_0:=0$ for convenience). We thus get that $(J_t(y))_{t\geq 0}$ is constant on intervals $[T_i,T_{i+1})$. This concludes the proof.

2.5. Relation between flow of inverses and Λ -coalescent. In this section we prove Proposition 1.6 and Lemma 1.7 from Sections 1.3 and 1.4. We recall that $(U_i)_{i\geq 1}$ and π_t^Y are defined in Section 1.3.

Proof of Proposition 1.6. First point. Let us fix a realization in the probability one events from Propositions 1.1 and 2.7. Thanks to Proposition 2.7, we only need to prove the claim for $(D_t)_{t\geq 0}$. For this, we need to verify that almost surely $(D_t)_{t\geq 0}$ is non-decreasing and $(D_t^c)_{t\geq 0}$ is càd-làg for the Hausdorff distance $d_H(\cdot,\cdot)$. The non-decreasing property follows from the definition of $(D_t)_{t\geq 0}$ in (2.34). The existence of left limits for $(D_t^c)_{t\geq 0}$ in the $d_H(\cdot,\cdot)$ topology follows from the non-decreasing property of $(D_t)_{t\geq 0}$ and [29, Prop. 2.5.6]. Note that these left limits are equal to the sets D_t^c from (2.34). We now show the right continuity. If there are $t\geq 0$, $\epsilon>0$ and a sequence $(t_n)_{n\geq 1}$ decreasing to t such that $d_H(D_t^c, D_{t_n}^c) > \epsilon$ for all $n\geq 1$, then for each $n\geq 1$ there is $x_n\in D_t^c$ such that $d_{i}(x_n,D_{t_n}^c)>\epsilon$. By compactness of D_t^c , there is a subsequence $(x_{n(i)})_{i\geq 1}$ converging to some $x\in D_t^c=C_t^c$. If $x\in (0,1)$, let $\tilde{\epsilon}\in (0,\epsilon)$ be such that $(x-\tilde{\epsilon},x+\tilde{\epsilon})\subset (0,1)$. There is $i_0\geq 1$ such that $|x_{n(i)}-x|<\epsilon-\tilde{\epsilon}$ for all $i\geq i_0$. We get that $i\geq i_0\Rightarrow d(x,D_{t_{n(i)}}^c)>\tilde{\epsilon}$ $(x-\tilde{\epsilon},x+\tilde{\epsilon})\subset D_{t_{n(i)}}=C_{t_{n(i)}}$. In particular, $Y_{0,t_{n(i)}}(x+\tilde{\epsilon}/2)-Y_{0,t_{n(i)}}(x-\tilde{\epsilon}/2)=0$ for all $i\geq i_0$. By Proposition 1.1(ii) we get $Y_{0,t}(x+\tilde{\epsilon}/2)-Y_{0,t}(x-\tilde{\epsilon}/2)=0$, which contradicts $x\in C_t^c$. The case $x\in \{0,1\}$ is treated similarly. This ends the proof of the first point.

Second point. Thanks to Lemma 2.11, the event $\mathcal{E} := \{\{U_i, i \geq 1\} \cap \bigcup_{(s,r,u) \in N} (Y_{0,s-}^{-1}(I_{r,u}) \setminus Y_{0,s-}^{-1}(I_{r,u}^{\circ})) = \emptyset\}$ has probability one. Let us consider a realization in this event and in the probability one events from Propositions 1.1, 2.7 and 1.4. Recall that, by definition of $(\pi_t^Y)_{t\geq 0}$ in Section 1.3, $i \sim_{\pi_t^Y} j$ for some t if and only if $Y_{0,t}(U_i) = Y_{0,t}(U_j)$. By Proposition 1.4 and the definition of \mathcal{E} , this implies that $U_i, U_j \in Y_{0,s-}^{-1}(I_{r,u}^{\circ})$ for some $(s,r,u) \in N$ with $s \in (0,t]$. All sets $Y_{0,s-}^{-1}(I_{r,u}^{\circ})$ are open intervals by Proposition 1.1 so U_i and U_j lie in the same open connected component of D_t and therefore of C_t by Proposition 2.7. Reciprocally, if U_i and U_j lie in the same open connected component of C_t then $m_t([U_i, U_j]) = 0$ by definition of C_t in Section 1.3 so $Y_{0,t}(U_i) = Y_{0,t}(U_j)$ and $i \sim_{\pi_t^Y} j$. This concludes the proof of the second point.

Third point. The first point of the proposition and its proof (together with Proposition 2.7) imply that, almost surely, $(C_t)_{t\geq 0}$ is càd-làg for the topology on interval partitions considered in [12, Sec. 1.1.2] and that the left limit at any t>0 is given by C_{t-} . By [12, Prop. 1.2], the function that maps C_t (resp. C_{t-}) to the sequence $(W_k(t))_{k\geq 1}$ (resp. $(W_k(t-))_{k\geq 1}$) is continuous. This yields the third point.

Fourth point. The combination of the proof of the third point with Proposition 2.7 shows that, almost surely, we have $\{t \geq 0 \text{ s.t. } W_k(t) \neq W_k(t-)\} \subset \{t \geq 0 \text{ s.t. } C_t \neq C_{t-}\} = \{t \geq 0 \text{ s.t. } D_t \neq D_{t-}\} \subset J_N$, where the last inclusion is a consequence of (2.34). This yields the fourth point.

Fifth point. The combination of the third point with [12, Prop. 1.1] show that $t \mapsto (W_k(t))_{k\geq 1}$ is càd-làg. By the second point, [12, Prop 1.3], and the definition of $(W_k(t))_{k\geq 1}$ in Section 1.3, we get that, for any $t\geq 0$, the partition π_t^Y almost surely possesses asymptotic frequencies and the ordered non-zero masses of its blocks are given by $(W_k(t))_{k\geq 1}$. Since $(\pi_t^Y)_{t\geq 0}$ is a Λ -coalescent by Lemma 1.5 the fifth point follows.

Proof of Lemma 1.7. By the second point of Proposition 1.6, the non-singleton blocks of π_t^Y are given by $(A_k)_{k\geq 1}$ where $A_k:=\{i\geq 1, U_i\in \mathcal{O}_k(t)\}$. By the law of large numbers, $\lim_{n\to\infty}\sharp(A_k\cap \llbracket 1,n\rrbracket)/n=P(U_1\in \mathcal{O}_k(t))=W_k(t)$. In particular, $(W_k(t))_{k\geq 1}$ is a function of the partition π_t^Y . Moreover, for any k such that $A_k\neq\emptyset$ and any $i\in A_k$ we have $Y_{0,t}(U_i)=V_k(t)$. If $A_k=\emptyset$ then $V_k(t)=\tilde{U}_k$ (see Section 1.3). By the discussion after Proposition 1.1, $Y_{0,t}(\cdot)$ is equal in law to the inverse of a bridge B. From the definition of π_t^Y and the above, we see that B, π_t^Y and $(V_k(t))_{k\geq 1}$ are as the bridge, the partition and a subfamily to the sequence considered in [14, Lem. 2] (the subfamily associated to non-singleton blocks of π_t^Y). By that lemma, $(V_k(t))_{k\geq 1}$ and π_t^Y are independent and $(V_k(t))_{k\geq 1}\sim \mathcal{U}([0,1])^{\times\mathbb{N}}$. All the claims of the lemma follow. \square

Remark 2.13. For $\eta \in (0,1)$, let $(s_i^{\eta}, r_i^{\eta}, u_i^{\eta})_{i\geq 1}$ be the enumeration of $\{(s,r,u) \in N, r > \eta\}$ as in the proof of Lemma 1.5. For any $i \geq 1$, the argument in the proof of Lemma 1.7 can be applied at s_i^{η} (instead of a fixed t) and shows that, conditionally on s_i^{η} , $(V_k(s_i^{\eta}))_{k\geq 1}$ is independent of $(W_k(s_i^{\eta}))_{k\geq 1}$ and $(V_k(s_i^{\eta}))_{k\geq 1} \sim \mathcal{U}([0,1])^{\times \mathbb{N}}$.

2.6. On the Stiljes measures arising from the flow. In this subsection we provide an expression for the measure m_t from Section 1.3 and for the measure of the sets composing D_t . First, recall the subordinator $(S_t)_{t\geq 0}$ defined in (1.8) and define another subordinator $(L_t)_{t\geq 0}$ by

$$L_t := \int_{(0,t]\times(0,1)^2} rN(ds,dr,du). \tag{2.36}$$

By [65, Thm. 19.3] and (1.2) we see that $(L_t)_{t\geq 0}$ is well-defined. By Itô's formula (see e.g. [47, Thm. II.5.1]) we have almost surely that for all $t\geq 0$,

$$e^{-S_t} = 1 - \int_{(0,t]\times(0,1)^2} e^{-S_{s-}} rN(ds, dr, du) = 1 - \int_0^t e^{-S_{s-}} dL_s.$$
 (2.37)

Finally, recall that $\mathcal{B}([0,1])$ denotes the family of Borel sets in [0,1].

Proposition 2.14. We have,

$$\mathbb{P}\left(\forall t \geq 0, \forall A \in \mathcal{B}([0,1]) \text{ s.t. } A \subset D_t^c, \ m_t(A) = |A|e^{S_t}\right) = 1,$$
$$\mathbb{P}\left(\forall t \geq 0, \forall A \in \mathcal{B}([0,1]) \text{ s.t. } A \subset D_{t-}^c, \ m_{t-}(A) = |A|e^{S_{t-}}\right) = 1.$$

Proof. Recall that $Q := [0,1] \cap \mathbb{Q}$ and fix $y \in Q$. Using integration by part (see e.g. [3, Thm. 4.4.13]), (2.37) and (1.5), we get that, almost surely for all $t \geq 0$,

$$e^{-S_{t}}Y_{0,t}(y) - y = -\int_{(0,t]\times(0,1)^{2}} e^{-S_{s-}}rY_{0,s-}(y)N(ds,dr,du)$$

$$+ \int_{(0,t]\times(0,1)^{2}} e^{-S_{s-}} \left(m_{r,u}(Y_{0,s-}(y)) - Y_{0,s-}(y) \right) N(ds,dr,du)$$

$$- \int_{(0,t]\times(0,1)^{2}} e^{-S_{s-}}r \left(m_{r,u}(Y_{0,s-}(y)) - Y_{0,s-}(y) \right) N(ds,dr,du)$$

$$= \int_{(0,t]\times(0,1)^{2}} e^{-S_{s-}} \left((1-r)m_{r,u}(Y_{0,s-}(y)) - Y_{0,s-}(y) \right) N(ds,dr,du)$$

$$= -\int_{(0,t]\times(0,1)^{2}} e^{-S_{s-}} \left(\int_{0}^{Y_{0,s-}(y)} \mathbb{1}_{z\in I_{r,u}^{\circ}} dz \right) N(ds,dr,du), \qquad (2.38)$$

where we have used Lemma A.2 from Appendix A.1 for the last equality.

Let us now consider a realization in the probability one events given by Proposition 1.1 and Remark A.8, and in the probability one event where $(S_t)_{t\geq 0}$ and $(L_t)_{t\geq 0}$ are well-defined and càd-làg, where (2.37) holds true, and where (2.38) holds true for all $y \in Q$ and $t \geq 0$. Note that for any $y \in [0,1]$ and $(s,r,u) \in N$ we have

$$0 \le e^{-S_{s-}} \int_0^{Y_{0,s-}(y)} \mathbb{1}_{z \in I_{r,u}^{\circ}} dz \le r \tag{2.39}$$

For any $t \ge 0$, $y \in [0, 1]$ and $(y_n)_{n\ge 1}$ in Q converging to y, Remark A.8 and (2.39) allow to apply dominated convergence in the right-hand side of (2.38) (applied at y_n) while Proposition 1.1 yields convergence of the left-hand side. We thus get that, on the above probability one events, (2.38) holds true for all $y \in [0, 1]$ and $t \ge 0$.

We still fix a realization in the above mentioned events. Let $\varphi \in \mathcal{C}^{\infty}([0,1])$. Using the definition of m_t and integration by parts for Stieltjes integrals we get that for all $t \geq 0$,

$$e^{-S_t} \int_0^1 \varphi(x) m_t(dx) = e^{-S_t} \int_0^1 \varphi(x) dY_{0,t}(x) = e^{-S_t} \varphi(1) - \int_0^1 \varphi'(x) e^{-S_t} Y_{0,t}(x) dx.$$

Plugging (2.38) into the above we get

$$e^{-S_t} \int_0^1 \varphi(x) m_t(dx) = e^{-S_t} \varphi(1) - \int_0^1 \varphi'(x) x dx + \int_0^1 \int_{(0,t] \times (0,1)^2} \varphi'(x) e^{-S_{s-}} \left(\int_0^{Y_{0,s-}(x)} \mathbb{1}_{z \in I_{r,u}^{\circ}} dz \right) N(ds, dr, du) dx.$$
 (2.40)

Using (2.39) we get

$$\int_{0}^{1} \int_{(0,t]\times(0,1)^{2}} \left| \varphi'(x)e^{-S_{s-}} \left(\int_{0}^{Y_{0,s-}(x)} \mathbb{1}_{z\in I_{r,u}^{\circ}} dz \right) \right| N(ds,dr,du) dx \leq L_{t} \int_{0}^{1} \left| \varphi'(x) \right| dx < \infty,$$

where $(L_t)_{t\geq 0}$ is defined in (2.36). We can thus use Fubini's theorem for the last term in (2.40). Using that along with integration by part we get that the last term in (2.40) equals

$$\begin{split} &\int_{(0,t]\times(0,1)^2} e^{-S_{s-}} \left(\int_0^1 \varphi'(x) \left(\int_0^{Y_{0,s-}(x)} \mathbbm{1}_{z\in I_{r,u}^{\circ}} dz \right) dx \right) N(ds,dr,du) \\ &= \int_{(0,t]\times(0,1)^2} e^{-S_{s-}} \left(\varphi(1)r - \int_0^1 \varphi(x) \mathbbm{1}_{Y_{0,s-}(x)\in I_{r,u}^{\circ}} dY_{0,s-}(x) \right) N(ds,dr,du) \\ &= \varphi(1)(1-e^{-S_t}) - \int_{(0,t]\times(0,1)^2} e^{-S_{s-}} \left(\int_0^1 \varphi(x) \mathbbm{1}_{Y_{0,s-}(x)\in I_{r,u}^{\circ}} dY_{0,s-}(x) \right) N(ds,dr,du) \end{split}$$

where we have used (2.37). Plugging this in (2.40) and using that $\varphi(1) - \int_0^1 \varphi'(x) x dx = \int_0^1 \varphi(x) dx$ we get

$$e^{-S_t} \int_0^1 \varphi(x) m_t(dx) + \int_{(0,t] \times (0,1)^2} e^{-S_{s-}} \left(\int_0^1 \varphi(x) \mathbb{1}_{Y_{0,s-}(x) \in I_{r,u}^{\circ}} dY_{0,s-}(x) \right) N(ds, dr, du)$$

$$= \int_0^1 \varphi(x) dx. \quad (2.41)$$

Each of the two sides of (2.41) is the integral of φ with respect to a finite positive measure. Since the above holds for all $t \geq 0$ and $\varphi \in \mathcal{C}^{\infty}([0,1])$, we get that the underlying positive measures on [0,1] are equal for all $t \geq 0$. Finally, note that for all $(s,r,u) \in N$ with $s \in (0,t]$ and $x \in D_t^c$ we have $x \notin Y_{0,s-}^{-1}(I_{r,u}^o)$ so the second measure in the left-hand side of (2.41) does not charge D_t^c . Using this together with the equality of measures implied by (2.41) we get that, for any $t \geq 0$ and any Borel set $A \subset D_t^c$, $e^{-S_t}m_t(A) = |A|$. The same reasoning works with t replaced by t-so the result follows.

An important consequence of Proposition 2.14 is the following:

Proposition 2.15. We have,

$$\mathbb{P}\left(\forall t \ge 0, \forall A \in \mathcal{B}([0,1]), |Y_{0,t}^{-1}(A) \cap D_t^c| = e^{-S_t}|A|, |Y_{0,t-}^{-1}(A) \cap D_{t-}^c| = e^{-S_{t-}}|A|\right) = 1. \quad (2.42)$$

Proof. We fix a realization in the probability one events given by Propositions 2.7 and 2.14. Let $t \geq 0$ and $A \subset [0,1]$ be a Borel set. Note that $m_t(D_t) = 0$ for all $t \geq 0$ by Proposition 2.7 and the definition of C_t . Using Proposition 2.14, $m_t(D_t) = 0$, the definition of m_t and the change of variable $y = Y_{0,t}(x)$, we get

$$|Y_{0,t}^{-1}(A) \cap D_t^c| = e^{-S_t} m_t(Y_{0,t}^{-1}(A) \cap D_t^c) = e^{-S_t} m_t(Y_{0,t}^{-1}(A))$$

$$= e^{-S_t} \int_0^1 \mathbb{1}_{Y_{0,t}(x) \in A} dY_{0,t}(x) = e^{-S_t} \int_0^1 \mathbb{1}_{y \in A} dy = e^{-S_t} |A|.$$

The same reasoning works with t replaced by t- so the result follows.

Applying Proposition 2.15 with A = [0,1], together with Proposition 2.7, we obtain the following corollary.

Corollary 2.16. We have almost surely $|C_t| = |D_t| = 1 - e^{-S_t}$ and $|C_{t-}| = |D_{t-}| = 1 - e^{-S_{t-}}$ for all $t \ge 0$.

Remark 2.17. Corollary 2.16 allows in particular to recover the classical fact, mentioned in Section 1.4, that, almost surely, $\sum_{k\geq 1} W_k(t) = 1 - e^{-S_t}$ for all $t\geq 0$. Combining with Lemma 1.7 we deduce that for any $t\geq 0$ the sequence $(Z_k(t,r,u))_{k\geq 1}$ is independent of S_t .

2.7. Poisson representation: Proof of Theorem 1.8.

Proof of Theorem 1.8. We only prove (1.13) as the proof of (1.14) is similar (and uses that N has countably many jumps). We know from Proposition 2.7 that $C_t = D_t = \bigcup_{(s,r,u) \in N, s \in (0,t]} Y_{0,s-}^{-1}(I_{r,u}^o)$. However, the open intervals appearing in this countable union are not disjoint as two such intervals can be included in one-another. Let us assume that we are on the probability one events from (1.7), Remark 2.4, Lemma 2.8 and Propositions 1.1 and 2.7. In order to separate the open connected components of C_t we define an equivalence relation \simeq on $\{(s,r,u) \in N \text{ s.t. } s \in (0,t]\}$ by writing $(s_1,r_1,u_1) \simeq (s_2,r_2,u_2)$ if and only if $Y_{0,s_1-}^{-1}(I_{r_1,u_1}^o)$ and $Y_{0,s_2-}^{-1}(I_{r_2,u_2}^o)$ are in the same open connected component of $D_t = C_t$.

For $j \geq 1$ such that $\mathcal{O}_j(t) \neq \emptyset$ let us denote by $\mathcal{C}_j(t)$ the equivalence class for \simeq that is associated to the open connected component $\mathcal{O}_j(t)$ of $C_t = D_t$ (and let $\mathcal{C}_j(t) := \emptyset$ otherwise). We see from (1.7) that $V_j(t)$, the value taken by $Y_{0,t}(\cdot)$ on $\mathcal{O}_j(t)$, is given by $Y_{s,t}(u)$ for any choice of $(s,r,u) \in \mathcal{C}_j(t)$. Let us denote by ν_t the measure defined by the right-hand side of (1.13). We thus get

$$\nu_t = \sum_{j \ge 1} \left(\sum_{(s,r,u) \in \mathcal{C}_j(t)} re^{-S_{s-}} \right) \delta_{V_j(t)}. \tag{2.43}$$

We are thus left to prove that $\sum_{(s,r,u)\in\mathcal{C}_j(t)}re^{-S_{s-}}=W_j(t)$ for all $j\geq 1$. The case $\mathcal{O}_j(t)=\emptyset$ is trivial so we only consider $j\geq 1$ such that $\mathcal{O}_j(t)\neq\emptyset$. For this we further assume that we are on the probability one event from Lemma 2.12, denote by $\mathcal{U}\subset[0,1]$ the set of measure one produced by Lemma 2.12, and show that

$$\mathcal{U} \cap \left(\cup_{(s,r,u) \in \mathcal{C}_j(t)} Y_{0,s-}^{-1}(I_{r,u}^{o}) \right) \subset \cup_{(s,r,u) \in \mathcal{C}_j(t)} \left(Y_{0,s-}^{-1}(I_{r,u}^{o}) \cap D_{s-}^{c} \right) \subset \cup_{(s,r,u) \in \mathcal{C}_j(t)} Y_{0,s-}^{-1}(I_{r,u}^{o}). \tag{2.44}$$

The second inclusion of (2.44) is trivial so we only prove the first one. Let z belong to the set in the left-hand side of (2.44). By definition there exists $(\tilde{s}, \tilde{r}, \tilde{u}) \in \mathcal{C}_j(t)$ such that $z \in Y_{0,\tilde{s}^-}^{-1}(I_{\tilde{r},\tilde{u}}^o)$. By Lemma 2.12, there is $(\hat{s}, \hat{r}, \hat{u}) \in N$ such that $\hat{s} \in (0, \tilde{s}]$ is the smallest increase time of the process $(J_s(z))_{s\geq 0}$. By Lemma 2.12 we have $z \in Y_{0,\hat{s}^-}^{-1}(I_{\hat{r},\hat{u}}^o)$ and $z \notin Y_{0,s^-}^{-1}(I_{r,u}^o)$ for any $(s, r, u) \in N$ such that $s \in (0, \hat{s})$ so $z \in D_{\hat{s}^-}^c$. We thus get $z \in Y_{0,\hat{s}^-}^{-1}(I_{\hat{r},\hat{u}}^o) \cap D_{\hat{s}^-}^c$. The intersection $Y_{0,\hat{s}^-}^{-1}(I_{\tilde{r},\hat{u}}^o) \cap Y_{0,\hat{s}^-}^{-1}(I_{\hat{r},\hat{u}}^o)$ is non-empty as it contains z so $(\tilde{s}, \tilde{r}, \tilde{u}) \simeq (\hat{s}, \hat{r}, \hat{u})$. In particular $(\hat{s}, \hat{r}, \hat{u}) \in \mathcal{C}_j(t)$ so $z \in \bigcup_{(s,r,u)\in\mathcal{C}_j(t)}(Y_{0,s^-}^{-1}(I_{r,u}^o) \cap D_{s^-}^c)$. This proves (2.44). Then, for $(s_1, r_1, u_1), (s_2, r_2, u_2) \in \mathcal{C}_j(t)$ with $s_1 < s_2$, we have

$$Y_{0,s_1-}^{-1}(I_{r_1,u_1}^{\text{o}})\cap D_{s_1-}^c\subset Y_{0,s_1-}^{-1}(I_{r_1,u_1}^{\text{o}})\subset D_{s_1}\subset D_{s_2-}\subset \left(Y_{0,s_2-}^{-1}(I_{r_2,u_2}^{\text{o}})\cap D_{s_2-}^c\right)^c.$$

Therefore, for any $(s_1, r_1, u_1), (s_2, r_2, u_2) \in \mathcal{C}_i(t)$.

$$s_1 \neq s_2 \Rightarrow \left(Y_{0,s_1-}^{-1}(I_{r_1,u_1}^{\text{o}}) \cap D_{s_1-}^c\right) \cap \left(Y_{0,s_2-}^{-1}(I_{r_2,u_2}^{\text{o}}) \cap D_{s_2-}^c\right) = \emptyset. \tag{2.45}$$

Let us now further assume that are on the probability one event from Proposition 2.15. By definition of $C_j(t)$ we have $\mathcal{O}_j(t) = \bigcup_{(s,r,u)\in C_j(t)} Y_{0,s-}^{-1}(I_{r,u}^o)$ so, using (2.44), (2.45), and Proposition 2.15 we get that, for our fixed realization of the process,

$$W_{j}(t) = |\mathcal{O}_{j}(t)| = \left| \bigcup_{(s,r,u)\in\mathcal{C}_{j}(t)} Y_{0,s-}^{-1}(I_{r,u}^{o}) \right| = \left| \bigcup_{(s,r,u)\in\mathcal{C}_{j}(t)} \left(Y_{0,s-}^{-1}(I_{r,u}^{o}) \cap D_{s-}^{c} \right) \right|$$

$$= \sum_{(s,r,u)\in\mathcal{C}_{j}(t)} \left| Y_{0,s-}^{-1}(I_{r,u}^{o}) \cap D_{s-}^{c} \right| = \sum_{(s,r,u)\in\mathcal{C}_{j}(t)} e^{-S_{s-}} |I_{r,u}^{o}| = \sum_{(s,r,u)\in\mathcal{C}_{j}(t)} e^{-S_{s-}} r. \quad (2.46)$$

Then, the combination of (2.43) with (2.46) and (1.12) yields (1.13), which concludes the proof.

3. Stochastic integral representation for $W_k(t)$

In this Section we prove Theorems 1.9 and 1.10 and then use the latter in a few applications, such as Corollary 1.17. Recall that we always assume that (1.2) holds true.

3.1. First step: behavior of $W_k(t)$ at a jump. For $k \ge 1$, let $M_k(t) := W_1(t) + \cdots + W_k(t)$.

Lemma 3.1. For any $(s,r,u) \in N$ we have almost surely $W_k(s) = W_k(s-) + K_{s-}^k(r,u)$ and $M_k(s) = M_k(s-) + H_{s-}^k(r,u)$, where $K_{s-}^k(\cdot,\cdot)$ and $H_{s-}^k(\cdot,\cdot)$ are defined in (1.11) and (1.10).

Proof. Let $(s,r,u) \in N$. Combining (1.12) with (1.14) from Theorem 1.8 and using (B.117) and the definition of $Z_j(s-,r,u)$ in Section 1.4 we get $\mu_{s-} = \sum_{j\geq 1} W_j(s-)\delta_{V_j(s-)}$ and

$$\begin{split} \sum_{j\geq 1} W_{j}(s) \delta_{V_{j}(s)} &= \mu_{s} = re^{-S_{s-}} \delta_{u} + \sum_{(\tilde{s},\tilde{r},\tilde{u}) \in N, \tilde{s} \in (0,s)} \tilde{r}e^{-S_{\tilde{s}-}} \delta_{\mathbf{m}_{r,u}(Y_{\tilde{s},s-}(\tilde{u}))} \\ &= re^{-S_{s-}} \delta_{u} + \sum_{j\geq 1} W_{j}(s-) \delta_{\mathbf{m}_{r,u}(V_{j}(s-))} \\ &= \sum_{j\geq 1, Z_{j}(s-,r,u) = 0} W_{j}(s-) \delta_{\mathbf{m}_{r,u}(V_{j}(s-))} + \left(e^{-S_{s-}}r + \sum_{j\geq 1} Z_{j}(s-,r,u)W_{j}(s-)\right) \delta_{u}. \end{split}$$

$$(3.47)$$

Note that all the Dirac measures appearing in the above expression are distinct.

We first assume that we are on the event $\{\beta_k(s-,r,u)=0\}$. We thus have $Z_j(s-,r,u)=0$ for all $j\in\{1,\ldots,k\}$ so all terms $W_j(s-)\delta_{\mathrm{m}_{r,u}(V_j(s-))}$ for $j\in\{1,\ldots,k\}$ appear in the sum $\sum_{j\geq 1,Z_j(s-,r,u)=0}\cdots$ from (3.47). If the factor of δ_u in (3.47) is in $(0,W_k(s-))$ then the k larger factors appearing in (3.47) are $W_1(s-),\ldots,W_k(s-)$. We thus have $W_k(s)-W_k(s-)=0$ and $M_k(s)-M_k(s-)=0$ in that case, which agrees with the expressions of $K_s^k(r,u)$ and $H_s^k(r,u)$ (see (1.11) and (1.10)). If the factor of δ_u in (3.47) is in $[W_k(s-),W_{k-1}(s-))$ then this factor is the k^{th} larger factors appearing in (3.47) (while the first k-1 are $W_1(s-),\ldots,W_{k-1}(s-)$). In this case we thus have $W_k(s)-W_k(s-)=M_k(s)-M_k(s-)=$ (factor of $\delta_u)-W_k(s-)$, which agrees with the expressions of $K_s^k(r,u)$ and $H_s^k(r,u)$. Finally, if the factor of δ_u in (3.47) is in $[W_{k-1}(s-),1]$ then this factor is one of the $(k-1)^{th}$ largest factors appearing in (3.47) and the k^{th} is $W_{k-1}(s-)$. In this case we thus have $W_k(s)-W_k(s-)=W_{k-1}(s-)-W_k(s-)$ and that $M_k(s)-M_k(s-)$ is as in the previous case. This agrees with the expressions of $K_s^k(r,u)$ and $H_s^k(r,u)$.

We now assume that we are on the event $\{\beta_k(s-,r,u)\neq 0\}$ and study $M_k(s)$. In this case a number equal to $\beta_k(s-,r,u)$ of the terms $W_j(s-)$, for $j\in\{1,\ldots,k\}$, appears in the factor of δ_u in (3.47) so this factor is one of the k largest factors appearing in (3.47). $M_k(s)$ is thus equal to the factor of δ_u in (3.47) plus the sum of the $k-\beta_k(s-,r,u)$ terms $W_j(s-)$, for $j\in\{1,\ldots,k\}$ such that $Z_j(s-,r,u)=0$, plus the $\beta_k(s-,r,u)-1$ largest terms $W_j(s-)$, for j>k such that $Z_j(s-,r,u)=0$. This agrees with the expression of $M_k(s-)+H_{s-}^k(r,u)$ and completes the proof of $M_k(s)=M_k(s-)+H_{s-}^k(r,u)$.

We now assume that we are on the event $\{\beta_k(s-,r,u)=1\}$ and study $W_k(s)$. Let us denote by J the unique $j \in \{1,\ldots,k\}$ that is such that $Z_j(s-,r,u)=1$. If $J \neq k$ then the k-1 largest factors appearing in (3.47) are the factor of δ_u in (3.47) and the terms $W_j(s-)$ for

 $j \in \{1, \ldots, k-1\} \setminus \{J\}$ and $W_k(s-)$ is the k^{th} largest factors appearing in (3.47) so $W_k(s) = W_k(s-)$. If J=k then the factor of δ_u in (3.47) contains $W_k(s-)$ but no other $W_j(s-)$ for $j \in \{1, \ldots, k-1\}$. If the factor of δ_u in (3.47) is smaller than $W_{k-1}(s-)$ then it is the k^{th} largest factors appearing in (3.47) and is thus equal to $W_k(s)$, if it is larger than $W_{k-1}(s-)$ then $W_{k-1}(s-)$ is the k^{th} largest factors appearing in (3.47) so $W_k(s) = W_{k-1}(s-)$. In all cases the obtained expression of $W_k(s)$ agrees with the expression of $W_k(s-) + K_{s-}^k(r,u)$.

We now assume that we are on the event $\{\beta_k(s-,r,u) \geq 2\}$ and study $W_k(s)$. In this case a number equal to $\beta_k(s-,r,u) \geq 2$ of the terms $W_j(s-)$, for $j \in \{1,\ldots,k\}$, appears in the factor of δ_u in (3.47) so this factor is one of the k-1 largest factors appearing in (3.47). The $k-\beta_k(s-,r,u)$ terms $W_j(s-)$, for $j \in \{1,\ldots,k\}$, such that $Z_j(s-,r,u)=0$ are all part of the k-1 largest factors appearing in (3.47), and so are the $\beta_k(s-,r,u)=0$ largest terms $W_j(s-)$, for j>k, such that $Z_j(s-,r,u)=0$. $W_k(s)$ is thus equal to the $\beta_k(s-,r,u)-1$ largest terms $W_j(s-)$, for j>k, such that $Z_j(s-,r,u)=0$. This agrees with the expression of $W_k(s-)+K_{s-}^k(r,u)=0$ and completes the proof of $W_k(s)=W_k(s-)+K_{s-}^k(r,u)$.

Remark 3.2. Let us fix $t \geq 0$, $(r, u) \in (0, 1)^2$, and let $\tilde{\mu}_t$ be obtained from $(\mu_t, S_t, (Z_j(t, r, u))_{j \geq 1})$ just like μ_s is obtained from $(\mu_{s-}, S_{s-}, (Z_j(s-, r, u))_{j \geq 1})$ in (3.47). For $k \geq 1$ we denote by $\tilde{W}_k(t)$ the k^th largest mass of $\tilde{\mu}_t$ and $\tilde{M}_k(t) := \sum_{j=1}^k \tilde{W}_j(t)$. Since the reasoning of the above proof relies only on (3.47) and on the expressions (1.11) and (1.10) of $K_c^k(\cdot, \cdot)$ and $H_c^k(\cdot, \cdot)$, it also shows that, almost surely, $\tilde{W}_j(t) = W_j(t) + K_t^j(r, u)$ for all j and $\tilde{M}_k(t) = M_k(t) + H_t^k(r, u)$ for all k. We deduce that we have almost surely $H_t^k(r, u) = \sum_{j=1}^k K_t^j(r, u)$.

3.2. Stochastic integral representation for $W_k(t)$: Proof of Theorem 1.9.

Lemma 3.3. For any $k \ge 1$ there is a constant C_k such that for any $\eta \in (0,1]$, and $t \ge 0$,

$$\mathbb{E}\left[\int_{(0,t]\times(0,\eta]\times(0,1)} \left| K_{s-}^k(r,u) \right| N(\mathrm{d}s,\mathrm{d}r,\mathrm{d}u) \right] \le tC_k \int_{(0,\eta]} r^{-1} \Lambda(dr) < \infty.$$
 (3.48)

In particular we have almost surely for any $k \geq 1$, $\eta \in (0,1]$ and $t \geq 0$,

$$\int_{(0,t]\times(0,\eta]\times(0,1)} \left| K_{s-}^k(r,u) \right| N(\mathrm{d}s,\mathrm{d}r,\mathrm{d}u) < \infty.$$
 (3.49)

Proof. We fix $k \ge 1$ and denote by $A_{f,k}^{\eta}(t)$ the quantity in (3.49). (3.49) follows easily from (3.48) and from the fact that $A_{f,k}^{\eta}(t)$ is almost surely non-decreasing in t and η . By the compensation formula we get

$$\mathbb{E}\left[A_{f,k}^{\eta}(t)\right] = \int_{(0,t]} \left(\int_{(0,\eta]\times(0,1)} \mathbb{E}\left[\left|K_s^k(r,u)\right|\right] r^{-2} \Lambda(dr) du\right) ds. \tag{3.50}$$

Using the expression of $K_{\cdot}^{k}(\cdot,\cdot)$ in (1.11) we can see that

$$\left| K_s^k(r,u) \right| \le \left(e^{-S_s} r + \sum_{j>k} Z_j(s,r,u) W_j(s) \right) + \mathbb{1}_{\beta_k(s,r,u) \ge 2},$$

so, taking expectation and using Lemma 1.7 and Remark 2.17, we get

$$\mathbb{E}\left[\left|K_s^k(r,u)\right|\right] \le r\mathbb{E}\left[1 - \sum_{j=1}^k W_j(s)\right] + (1 - (1-r)^k - kr(1-r)^{k-1}) \le C_k r,$$

for some constant $C_k > 0$. Plugging the above into (3.50) we obtain (3.48), where the finiteness of this upper bound comes from (1.2). This concludes the proof.

Remark 3.4. For later use, note that the above proof also shows that, for a Lipschitz function $f:[0,1] \to \mathbb{R}$, an integer $k \ge 1$, $t \ge 0$ and $\eta \in (0,1]$ we have

$$\mathbb{E}\left[\int_{(0,\eta]\times(0,1)}\left|f\left(W_k(t)+K_t^k(r,u)\right)-f\left(W_k(t)\right)\right|r^{-2}\Lambda(dr)du\right]\leq C_{f,k}\int_{(0,\eta]}r^{-1}\Lambda(dr),$$

for some constant $C_{f,k}>0$.

Proof of Theorem 1.9. Let us fix $t \geq 0$ and $k \geq 1$. The set $\{(s, r, u) \in N \text{ s.t. } r > \eta\}$ is almost surely discrete for all $\eta \in (0, 1)$. Let us enumerate its elements by $(s_i^{\eta}, r_i^{\eta}, u_i^{\eta})$ where $s_1^{\eta} < s_2^{\eta} < \dots$ and for convenience set $s_0^{\eta} := 0$. Our first step is to control $\mathbb{E}[|\Sigma_{\eta}(t)|]$ where

$$\Sigma_{\eta}(t) := \sum_{i \ge 1} \left(W_k((s_{i+1}^{\eta} \wedge t) -) - W_k(s_i^{\eta}) \right) \mathbb{1}_{s_i^{\eta} < t}.$$
(3.51)

We define $D_{s_i^{\eta}+w}^{s_i^{\eta}}=\cup_{(s,r,u)\in N,s\in(s_i^{\eta},s_i^{\eta}+w]}Y_{s_i^{\eta},s-}^{-1}(I_{r,u}^{\text{o}})$ for $w\geq 0$. Note that $D_{s_i^{\eta}+.}^{s_i^{\eta}}$ is obtained from $Y_{s_i^{\eta},s_i^{\eta}+.}(\cdot)$ in the same way D. is obtained from $Y_{0,\cdot}(\cdot)$. Therefore, by Appendice B, we get

$$(D_{s_i^{\eta}+w}^{s_i^{\eta}}, S_{s_i^{\eta}+w} - S_{s_i^{\eta}})_{w \ge 0} \stackrel{(d)}{=} (D_w, S_w)_{w \ge 0}, \qquad (D_{s_i^{\eta}+w}^{s_i^{\eta}})_{w \ge 0} \perp \mathcal{F}_{s_i^{\eta}}. \tag{3.52}$$

By Corollary 2.16 we deduce that, on $\{s_i^{\eta} < t\}$, we have almost surely

$$\begin{split} |D_{(s_{i+1}^{\eta} \wedge t)^{-}}^{s_{i}^{\eta}}| &= 1 - e^{-(S_{(s_{i+1}^{\eta} \wedge t)^{-}} - S_{s_{i}^{\eta}})} \le S_{(s_{i+1}^{\eta} \wedge t)^{-}} - S_{s_{i}^{\eta}} \\ &= - \int_{(s_{i}^{\eta}, s_{i+1}^{\eta} \wedge t) \times (0, \eta] \times (0, 1)} \log(1 - r) N(ds, dr, du). \end{split}$$
 (3.53)

Let $\mathcal{E}_i(\eta) := \{s_i^{\eta} \geq t\} \cup \{s_i^{\eta} < t, V_1(s_i^{\eta}), \dots, V_k(s_i^{\eta}) \notin D_{(s_{i+1}^{\eta} \wedge t)^{-}}^{s_i^{\eta}}\}$. Using (3.52), Remark 2.13, and (3.53) we get

$$\sum_{i\geq 1} \mathbb{P}(\mathcal{E}_{i}(\eta)^{c}) \leq \sum_{i\geq 1} \sum_{j=1}^{k} \mathbb{P}\left(s_{i}^{\eta} < t, V_{j}(s_{i}^{\eta}) \in D_{(s_{i+1}^{\eta} \wedge t)-}^{s_{i}^{\eta}}\right) = \sum_{i\geq 1} k \mathbb{E}\left[|D_{(s_{i+1}^{\eta} \wedge t)-}^{s_{i}^{\eta}}|\mathbb{1}_{s_{i}^{\eta} < t}\right] \\
\leq -k \mathbb{E}\left[\int_{(0,t] \times (0,\eta] \times (0,1)} \log(1-r)N(ds, dr, du)\right] = -kt \int_{(0,\eta]} \log(1-r)r^{-2}\Lambda(dr). \tag{3.54}$$

Let $E_i(j) := \{(s, r, u) \in N \text{ s.t. } s \in (0, s_i^{\eta}], Y_{s, s_i^{\eta}}(u) = V_j(s_i^{\eta})\}$. By (1.14) we have $W_j(s_i^{\eta}) = \sum_{(s, r, u) \in E_i(j)} re^{-S_{s-}}$. Using Theorem 1.8 and (1.6) we get that, on $\{s_i^{\eta} < t\}, \mu_{(s_{i+1}^{\eta} \wedge t)-} \text{ equals }$

$$\begin{split} & \sum_{(s,r,u) \in N, s \in (0,s_{i+1}^{\eta} \wedge t)} re^{-S_s -} \delta_{Y_{s,(s_{i+1}^{\eta} \wedge t) -}(u)} \\ &= \sum_{(s,r,u) \in N, s \in (s_i^{\eta}, s_{i+1}^{\eta} \wedge t)} re^{-S_s -} \delta_{Y_{s,(s_{i+1}^{\eta} \wedge t) -}(u)} + \sum_{j \geq 1} \sum_{(s,r,u) \in E_i(j)} re^{-S_s -} \delta_{Y_{s_i^{\eta},(s_{i+1}^{\eta} \wedge t) -}(Y_{s,s_i^{\eta}}(u))} \\ &= \sum_{(s,r,u) \in N, s \in (s_i^{\eta}, s_{i+1}^{\eta} \wedge t)} re^{-S_s -} \delta_{Y_{s,(s_{i+1}^{\eta} \wedge t) -}(u)} + \sum_{j \geq 1} W_j(s_i^{\eta}) \delta_{Y_{s_i^{\eta},(s_{i+1}^{\eta} \wedge t) -}(V_j(s_i^{\eta}))}. \end{split}$$

Let $\mathcal{U} := (\bigcup_{(s,r,u)\in N} (Y_{0,s-}^{-1}(I_{r,u}) \setminus Y_{0,s-}^{-1}(I_{r,u}^{o})))^c$ and note that $|\mathcal{U}| = 1$ by Lemma 2.11. For $(s,r,u)\in N$ such that $s\in (s_i^{\eta},s_{i+1}^{\eta}\wedge t)$ we choose $x_s\in Y_{0,s-}^{-1}(I_{r,u}^{o})\cap D_{s-}^{c}\cap \mathcal{U}$. By Proposition 2.15 the latter set is non-empty so such a choice of x_s exists. Note from (1.7) that $Y_{s,(s_{i+1}^{\eta}\wedge t)-}(u)=$

 $Y_{0,(s_{i+1}^{\eta} \wedge t)-}(x_s)$. Then, for any $j \geq 1$ such that $W_j(s_i^{\eta}) > 0$ we choose $x_j \in \mathcal{O}_j(s_i^{\eta}) \cap \mathbb{Q}$ and note that $V_j(s_i^{\eta}) = Y_{0,s_i^{\eta}}(x_j)$. By this and (1.6) we get that, on $\{s_i^{\eta} < t\}$, we have almost surely

$$\mu_{(s_{i+1}^{\eta} \wedge t)-} = \sum_{(s,r,u) \in N, s \in (s_{i}^{\eta}, s_{i+1}^{\eta} \wedge t)} re^{-S_{s-}} \delta_{Y_{0,(s_{i+1}^{\eta} \wedge t)-}(x_{s})} + \sum_{j \ge 1} W_{j}(s_{i}^{\eta}) \delta_{Y_{0,(s_{i+1}^{\eta} \wedge t)-}(x_{j})}.$$
(3.55)

On $\{s_i^{\eta} < t\}$, let $\mathcal{L}_i := \{j \geq 1, W_j(s_i^{\eta}) > 0\}$ and $\mathcal{J}_i(s_{i+1}^{\eta} \wedge t) := \mathcal{L}_i \cap \{j \geq 1, V_j(s_i^{\eta}) \notin D_{(s_{i+1}^{\eta} \wedge t)-1}^{s_i^{\eta}}\}$. We note that on $\mathcal{E}_i(\eta) \cap \{s_i^{\eta} < t\}$ we have $\{1, \dots, k\} \subset \mathcal{J}_i(s_{i+1}^{\eta} \wedge t) \cup (\mathbb{N} \setminus \mathcal{L}_i)$. We now justify that, on $\{s_i^{\eta} < t\}$, for any $j \in \mathcal{J}_i(s_{i+1}^{\eta} \wedge t)$, the atom of $\mu_{(s_{i+1}^{\eta} \wedge t)-}$ at $Y_{0,(s_{i+1}^{\eta} \wedge t)-}(x_j)$ (see (3.55)) is of size exactly $W_j(s_i^{\eta})$. Since the values $V_j(s_i^{\eta})$'s are distinct, we get from Remark 2.4 and Lemma 2.11 that, for any $j \in \mathcal{J}_i(s_{i+1}^{\eta} \wedge t)$ and $\ell \in \mathcal{L}_i \setminus \{j\}$, if we have $Y_{0,(s_{i+1}^{\eta} \wedge t)-}(x_j) = Y_{0,(s_{i+1}^{\eta} \wedge t)-}(x_\ell)$, then there is $(s, r, u) \in N$ such that $s \in (s_i^{\eta}, s_{i+1}^{\eta} \wedge t)$ and $x_j, x_\ell \in Y_{0,s-}^{-1}(I_{r,u}^{\circ})$. This implies that $Y_{s_i^{\eta},s-}(V_j(s_i^{\eta})) = Y_{0,s-}(x_j) \in I_{r,u}^{\circ}$ so $V_j(s_i^{\eta}) \in Y_{s_i^{\eta},s-}^{-1}(I_{r,u}^{\circ})$. This contradicts $j \in \mathcal{J}_i(s_{i+1}^{\eta} \wedge t)$. Similarly, by Proposition 1.4 and Lemma and 2.11, if there is $(s, r, u) \in N$ with $s \in (s_i^{\eta}, s_{i+1}^{\eta} \wedge t)$ and $j \in \mathcal{J}_i(s_{i+1}^{\eta} \wedge t)$ such that $Y_{0,(s_{i+1}^{\eta} \wedge t)-}(x_s) = Y_{0,(s_{i+1}^{\eta} \wedge t)-}(x_j)$, then there is $(\tilde{s}, \tilde{r}, \tilde{u}) \in N$ such that $\tilde{s} \in (0, s_{i+1}^{\eta} \wedge t)$ and $x_s, x_j \in Y_{0,\tilde{s}-}^{-1}(I_{\tilde{r},\tilde{u}}^{\circ})$. Since $x_s \in D_{s-}^c$ we have necessarily $\tilde{s} \in (s, s_{i+1}^{\eta} \wedge t) \subset (s_i^{\eta}, s_{i+1}^{\eta} \wedge t)$ and $Y_{s_i^{\eta}, \tilde{s}-}(V_j(s_i^{\eta})) = Y_{0,\tilde{s}-}(x_j) \in I_{\tilde{r},\tilde{u}}^{\circ}$ so $V_j(s_i^{\eta}) \in Y_{s_i^{\eta}, \tilde{s}-}^{-1}(I_{\tilde{r},\tilde{u}}^{\circ})$. Again, this contradicts $j \in \mathcal{J}_i(s_{i+1}^{\eta} \wedge t)$.

The previous discussion and (3.55) yield that, on $\mathcal{E}_i(\eta) \cap \{s_i^{\eta} < t\} \cap \{W_k(s_i^{\eta}) > 0\}$, $\mu_{(s_{i+1}^{\eta} \wedge t) -}$ has, for each $j \in \{1, \dots, k\}$, an atom at $Y_{0,(s_{i+1}^{\eta} \wedge t) -}(x_j)$ with weight $W_j(s_i^{\eta})$. In particular $\mathcal{E}_i(\eta) \cap \{s_i^{\eta} < t\} \subset \{W_k((s_{i+1}^{\eta} \wedge t) -) \geq W_k(s_i^{\eta})\}$. On $\mathcal{E}_i(\eta) \cap \{s_i^{\eta} < t\} \cap \{W_k(s_i^{\eta}) > 0\}$, if additionally to these k atoms, $\mu_{(s_{i+1}^{\eta} \wedge t) -}$ has another atom at a point $a \in [0,1]$ with weight strictly larger than $W_k(s_i^{\eta})$, then the above discussion and (3.55) show that for all $j \in \mathcal{L}_i$ such that $Y_{0,(s_{i+1}^{\eta} \wedge t) -}(x_j) = a$ we have $j \notin \mathcal{J}_i(s_{i+1}^{\eta} \wedge t)$ so $V_j(s_i^{\eta}) \in D_{(s_{i+1}^{\eta} \wedge t) -}^{s_i^{\eta}}$. Combining with (1.12) we get that, on $\mathcal{E}_i(\eta) \cap \{s_i^{\eta} < t\} \cap \{W_k((s_{i+1}^{\eta} \wedge t) -) > W_k(s_i^{\eta}) > 0\}$, we have almost surely

$$W_k((s_{i+1}^{\eta} \wedge t) -) \le \int_{(s_i^{\eta}, s_{i+1}^{\eta} \wedge t) \times (0, \eta] \times (0, 1)} rN(ds, dr, du) + \sum_{j > k} W_j(s_i^{\eta}) \mathbb{1}_{V_j(s_i^{\eta}) \in D_{(s_{i+1}^{\eta} \wedge t) -}^{s_i^{\eta}}}.$$
(3.56)

On $\{s_i^{\eta} < t\} \cap \{W_k(s_i^{\eta}) = 0\}$, we see from (3.55) that (3.56) still holds, the second term in the right-hand side being null. In conclusion we get that, almost surely,

$$\begin{split} |\Sigma_{\eta}(t)| &\leq \sum_{i \geq 1} |W_{k}((s_{i+1}^{\eta} \wedge t) -) - W_{k}(s_{i}^{\eta})|\mathbb{1}_{s_{i}^{\eta} < t} \\ &\leq \sum_{i \geq 1} \mathbb{1}_{\mathcal{E}_{i}(\eta)^{c}} + \int_{(0,t) \times (0,\eta] \times (0,1)} rN(ds,dr,du) + \sum_{i \geq 1} \mathbb{1}_{s_{i}^{\eta} < t} \sum_{j > k} W_{j}(s_{i}^{\eta})\mathbb{1}_{V_{j}(s_{i}^{\eta}) \in D_{(s_{i+1}^{\eta} \wedge t) -}^{s_{i}^{\eta}}}. \end{split}$$

Taking the expectation and using the compensation formula, (3.52), Remark 2.13, (3.54), (3.53) and that $\sum_{j>k} W_j(s_i^{\eta}) \leq 1$ almost surely,

$$\mathbb{E}[|\Sigma_{\eta}(t)|] \leq \sum_{i \geq 1} \mathbb{P}(\mathcal{E}_{i}(\eta)^{c}) + t \int_{(0,\eta]} r^{-1} \Lambda(dr) + \sum_{i \geq 1} \sum_{j > k} \mathbb{E}[W_{j}(s_{i}^{\eta}) \mathbb{1}_{s_{i}^{\eta} < t} | D_{(s_{i+1}^{\eta} \wedge t)_{-}}^{s_{i}^{\eta}} |] \\
\leq -3kt \int_{(0,\eta]} \log(1-r) r^{-2} \Lambda(dr) \underset{\eta \to 0}{\longrightarrow} 0, \tag{3.57}$$

where the convergence toward 0 comes from (1.2).

We denote by $B_{f,k}(t)$ the right-hand side of (1.15). Lemma 3.3 shows that $B_{f,k}(t)$ is well-defined almost surely. We note that, almost surely, $W_k(t) = W_k(t-)$ and $B_{f,k}(t) = B_{f,k}(t-)$ by respectively Proposition 1.6 and the fact that $t \notin J_N$ almost surely. Using this and Lemma 3.1 we get that almost surely,

$$\begin{split} W_k(t) &= W_k(t-) = \sum_{i \geq 1} \left(W_k^{\eta}(s_i^{\eta}) - W_k^{\eta}(s_i^{\eta}-) \right) \mathbb{1}_{s_i^{\eta} < t} + \Sigma_{\eta}(t) \\ &= \sum_{i \geq 1} K_{s_i^{\eta}-}^k(r_i^{\eta}, u_i^{\eta}) \mathbb{1}_{s_i^{\eta} < t} + \Sigma_{\eta}(t) = B_{f,k}(t-) + \Sigma_{\eta}(t) - I_{\eta}(t-) = B_{f,k}(t) + \Sigma_{\eta}(t) - I_{\eta}(t), \end{split}$$

where we have set $I_{\eta}(t) := \int_{(0,t]\times(0,\eta]\times(0,1)} K_{s-}^k(r,u) N(\mathrm{d}s,\mathrm{d}r,\mathrm{d}u)$. By (3.48) from Lemma 3.3 we get $\mathbb{E}[|I_{\eta}(t)|] \to 0$ as $\eta \to 0$. Combining with (3.57) we get that $\Sigma_{\eta}(t) - I_{\eta}(t)$ converges to 0 in probability as η goes to 0 so $W_k(t-) = B_{f,k}(t)$ almost surely. This proves that, for every fixed $t \geq 0$, (1.15) holds almost surely. Finally, since both sides of (1.15) are càd-làg in t almost surely by Proposition 1.6 and by properties of Poisson integrals. We deduce that (1.15) holds almost surely for all $t \geq 0$ simultaneously.

3.3. Pseudo-generator formula for $W_k(t)$: Proof of Theorem 1.10.

Proof of Theorem 1.10. Let f be as in the statement of the theorem and $k \geq 1$. Applying Itô's formula from [47, Thm. II.5.1] to f and the stochastic integral from Theorem 1.9 (as in the proof of Lemma 2.2, this is a case where Itô's formula is valid for Lipschitz functions instead of functions of class C^2) we get that, almost surely, for all $t \geq 0$,

$$f(W_k(t)) = f(0) + \int_{(0,t]\times(0,1)^2} \left(f(W_k(s-) + K_{s-}^k(r,u)) - f(W_k(s-)) \right) N(\mathrm{d}s,\mathrm{d}r,\mathrm{d}u).$$
 (3.58)

We note from Lemma 3.3 and the fact that f is Lipschitz that

$$\mathbb{E}\left[\int_{(0,t]\times(0,1)^2} \left| f(W_k(s-) + K_{s-}^k(r,u)) - f(W_k(s-)) \right| N(\mathrm{d}s,\mathrm{d}r,\mathrm{d}u) \right] < \infty.$$
 (3.59)

We then take the expectation on both sides of (3.58). Thanks to (3.59) we can apply the compensation formula for the right-hand side. Taking the obtained expression at t_1 and t_2 , and taking the difference, we obtain that for any $k \ge 1$ and $t_2 > t_1 \ge 0$,

$$\mathbb{E}[f(W_k(t_2))] - \mathbb{E}[f(W_k(t_1))]$$

$$= \int_{(t_1, t_2]} \mathbb{E}\left[\int_{(0, 1)^2} \left(f(W_k(s) + K_s^k(r, u)) - f(W_k(s))\right) r^{-2} \Lambda(dr) du\right] ds = \int_{(t_1, t_2]} G_0^f(s) ds,$$
(3.60)

where we have set $G^f_{\epsilon}(s) := \mathbb{E}[\int_{(\epsilon,1)\times(0,1)} (f(W_k(s) + K^k_s(r,u)) - f(W_k(s)))r^{-2}\Lambda(dr)du]$ for any $\epsilon \in [0,1)$. We need to show that $G^f_0(\cdot)$ is continuous. For this we first show that $G^f_{\epsilon}(\cdot)$ is continuous when $\epsilon \in (0,1)$. Let us fix $s \geq 0$. Note from Lemma 1.7 and Remark 2.17 that $(Z_j(s,r,u))_{j\geq 1}$ is independent of $(W_j(s))_{j\geq 1}$ and S_s , and that $(Z_j(s,r,u))_{j\geq 1} \sim \mathcal{B}(r)^{\times \mathbb{N}}$. Using

the definition of $K_s^k(r,u)$ in (1.11) together with this we get that, if $k \geq 2$, $G_{\epsilon}^f(s)$ equals

$$\begin{split} & \int_{(\epsilon,1)} (1-r)^k \mathbb{E} \left[f \left(\text{Median} \left\{ W_{k-1}(s), W_k(s), e^{-S_s} r + \sum_{j>k} Z_j W_j(s) \right\} \right) - f(W_k(s)) \right] r^{-2} \Lambda(dr) \\ & + \int_{(\epsilon,1)} r (1-r)^{k-1} \mathbb{E} \left[f \left(\text{Min} \left\{ W_k(s) + e^{-S_s} r + \sum_{j>k} Z_j W_j(s), W_{k-1}(s) \right\} \right) - f(W_k(s)) \right] r^{-2} \Lambda(dr) \\ & + \sum_{\ell=2}^k \binom{k}{\ell} \int_{(\epsilon,1)} r^\ell (1-r)^{k-\ell} \mathbb{E} \left[f \left(\sum_{j>k} (1-Z_j) W_j(s) \mathbbm{1}_{\sum_{i=k+1}^j (1-Z_i) = \ell-1} \right) - f(W_k(s)) \right] r^{-2} \Lambda(dr), \end{split}$$

where $(Z_j)_{j\geq 1}$ is independent of $(W_j(s))_{j\geq 1}$ and S_s , and $(Z_j)_{j\geq 1}\sim \mathcal{B}(r)^{\times \mathbb{N}}$. Since S is a Lévy process, S is almost surely continuous at our fixed s. By Remark 2.17, we have almost surely $\sum_{j\geq 1}W_j(t)=1-e^{-S_t}$ for all t and, by Proposition 1.6, that all $W_j(\cdot)$ are non-negative and continuous at s. We deduce that, almost surely, the series appearing in the above three expectations are continuous at s. Combining with the continuity of f we get that, for any $s\geq 0$ and $r\in (0,\epsilon]$, the terms in the above three expectations are continuous at s. Note that those terms are all bounded by $2||f||_{\infty}$. Therefore, by dominated converges, $G_{\epsilon}^f(\cdot)$ is continuous at any $s\geq 0$, so it is continuous. If k=1, the continuity of $G_{\epsilon}^f(\cdot)$ follows along the same lines, using that $K_s^1(r,u)=H_s^1(r,u)$, and the definition of $H_s^1(r,u)$ in (1.10). We have

$$|G_0^f(s) - G_{\epsilon}^f(s)| \le \int_{(0,\epsilon)\times(0,1)} \mathbb{E}\left[\left|f(W_k(s) + K_s^k(r,u)) - f(W_k(s))\right|\right] r^{-2}\Lambda(dr) \times du$$

$$\le tC_{f,k} \int_{(0,\epsilon]} r^{-1}\Lambda(dr) < \infty \underset{\epsilon \to 0}{\longrightarrow} 0,$$

where we have used Remark 3.4 for the last inequality and (1.2) for the convergence toward 0. Therefore $G_{\epsilon}^f(\cdot)$ converges uniformly to $G_0^f(\cdot)$ as ϵ goes to 0. Since all functions $G_{\epsilon}^f(\cdot)$ are continuous, we get that $G_0^f(\cdot)$ is continuous. Combining this continuity with (3.60) we get that $(t \mapsto \mathbb{E}[f(W_k(t))])$ is of class \mathcal{C}^1 and that (1.16) holds true.

3.4. Direct applications of Theorem 1.10. Recall from Section 3.1 that $M_k(t) := W_1(t) + \cdots + W_k(t)$ for $k \geq 1$. Let also $Y_k(t) := \mathbb{E}[M_k(t)]$ and $w_k(t) := \mathbb{E}[W_k(t)]$ (note that $Y_1(t) = w_1(t)$). The following result tells that Theorem 1.10 allows us to explicitly differentiate these functions, which will be useful to prove Theorem 1.11.

Corollary 3.5. Assume that (1.2) holds true. For any $k \ge 1$, $Y_k(\cdot)$ and $w_k(\cdot)$ is of class C^1 and for any $t \ge 0$,

$$Y'_{k}(t) = \mathbb{E} \left| \int_{(0,1)^{2}} H_{t}^{k}(r,u) r^{-2} \Lambda(dr) du \right|.$$
 (3.61)

$$w'_k(t) = \mathbb{E}\left[\int_{(0,1)^2} K_t^k(r,u)r^{-2}\Lambda(dr)du\right].$$
 (3.62)

Proof. Applying (1.16) with f(y) := y we get (3.62). Summing (3.62) for indices $1, \ldots, k$ we get (3.61), but with $H_t^k(r,u)$ replaced by $\sum_{j=1}^k K_t^j(r,u)$. Remark 3.2 shows that for any fixed (t,r,u), we have almost surely $H_t^k(r,u) = \sum_{j=1}^k K_t^j(r,u)$. Combining with Fubini's theorem we see that $\sum_{j=1}^k K_t^j(r,u)$ can be replaced by $H_t^k(r,u)$ in the expression, yielding (3.61).

Another direct application of Theorem 1.10 is Corollary 1.17 which we prove now.

Proof of Corollary 1.17. Note that $\mathbb{E}[f(W_k(0))] = f(0)$ so, using Theorem 1.10, we get

$$\mathbb{E}[f(W_k(t))] = f(0) + t(\frac{d}{ds}\mathbb{E}[f(W_k(s))]_{|s=0}) + o(t)$$

and

$$\frac{d}{ds}\mathbb{E}[f(W_1(s))]_{|s=0} = \mathbb{E}[\int_{(0,1)^2} (f(H_0^1(r,u)) - f(0))r^{-2}\Lambda(dr)du],$$

$$\forall k \ge 2, \ \frac{d}{ds}\mathbb{E}[f(W_k(s))]_{|s=0} = \mathbb{E}[\int_{(0,1)^2} (f(K_0^k(r,u)) - f(0))r^{-2}\Lambda(dr)du].$$

We have almost surely $H_0^1(r,u) = r$ and $K_0^k(r,u) = 0$ for all $k \ge 2$ and $(r,u) \in (0,1)^2$ so $\frac{d}{ds}\mathbb{E}[f(W_1(s))]_{|s=0} = \int_{(0,1)} (f(r) - f(0))r^{-2}\Lambda(dr)$ and, for $k \ge 2$, $\frac{d}{ds}\mathbb{E}[f(W_k(s))]_{|s=0} = 0$. This yields (1.23) and (1.24).

3.5. Another application of Theorem 1.10. Let us study the expectations of some specific functions of $W_1(t)$ and $W_2(t)$. Recall the functions $h_{\Lambda}(\cdot)$ and $k_{\Lambda}(\cdot)$ defined by (1.27). The quantities $\mathbb{E}[h_{\Lambda}(W_2(t))]$ and $\mathbb{E}[k_{\Lambda}(W_1(t))]$ will appear in the study of the small time behavior of $\mathbb{E}[W_2(t)]$ in Section 5. In this subsection, additionally to assuming (1.2), we also assume (1.28).

Proposition 3.6. The functions $(t \mapsto \mathbb{E}[h_{\Lambda}(W_2(t))])$ and $(t \mapsto \mathbb{E}[k_{\Lambda}(W_1(t))])$ are of class \mathcal{C}^1 and for any $t \geq 0$,

$$\frac{d}{dt}\mathbb{E}[h_{\Lambda}(W_2(t))] = \mathbb{E}\left[\int_{(0,1)^2} \left(h_{\Lambda}\left(W_2(t) + K_t^2(r,u)\right) - h_{\Lambda}\left(W_2(t)\right)\right)r^{-2}\Lambda(dr)du\right],\tag{3.63}$$

$$\frac{d}{dt}\mathbb{E}[k_{\Lambda}(W_1(t))] = \mathbb{E}\left[\int_{(0,1)^2} \left(k_{\Lambda}\left(W_1(t) + K_t^1(r,u)\right) - k_{\Lambda}\left(W_1(t)\right)\right)r^{-2}\Lambda(dr)du\right],\tag{3.64}$$

where the multiple integrals in the right-hand sides of (3.63) and (3.64) are well-defined for each $t \geq 0$.

We note that $h_{\Lambda}(\cdot)$ and $k_{\Lambda}(\cdot)$ are in general not Lipschitz under the assumptions (1.2) and (1.28). Indeed, differentiating $h_{\Lambda}(\cdot)$ in the sense of distributions on (0,1) yields $h'_{\Lambda}(x) = \int_{[x,1)} r^{-2} \Lambda(dr)$, which is bounded only under (1.3), but that condition is strictly stronger than "(1.2) and (1.28)", see Remark 1.20. However we also note that $h_{\Lambda}(x) = \int_{(0,1)} h_a(x) a^{-2} \Lambda(da)$ and $k_{\Lambda}(x) = \int_{(0,1)} k_a(x) (1-a) a^{-2} \Lambda(da)$ where $h_a(x) := a \wedge x$ and $k_a(x) := ((1-x)a) \wedge x$. We thus use that, for each $a \in (0,1)$, the functions $h_a(\cdot)$ and $k_a(\cdot)$ satisfy the requirements of Theorem 1.10 and justify that the results of that theorem can be transferred to $h_{\Lambda}(\cdot)$ and $k_{\Lambda}(\cdot)$.

Proof of Proposition 3.6. By Fubini's theorem we have, for all $t \geq 0$,

$$\mathbb{E}[h_{\Lambda}(W_2(t))] = \int_{(0,1)} \mathbb{E}[h_a(W_2(t))] a^{-2} \Lambda(da), \ \mathbb{E}[k_{\Lambda}(W_1(t))] = \int_{(0,1)} \mathbb{E}[k_a(W_1(t))] (1-a) a^{-2} \Lambda(da).$$
(3.65)

Fix $a \in (0,1)$. Applying Theorem 1.10 to $W_2(t)$ and $W_1(t)$ with $f(\cdot) = h_a(\cdot)$ and $f(\cdot) = k_a(\cdot)$ we get that for any $t \ge 0$,

$$\frac{d}{dt}\mathbb{E}[h_a(W_2(t))] = \mathbb{E}\left[\int_{(0,1)^2} \left(h_a\left(W_2(t) + K_t^2(r,u)\right) - h_a\left(W_2(t)\right)\right)r^{-2}\Lambda(dr)du\right],\tag{3.66}$$

$$\frac{d}{dt}\mathbb{E}[k_a(W_1(t))] = \mathbb{E}\left[\int_{(0,1)^2} \left(k_a\left(W_1(t) + H_t^1(r,u)\right) - k_a\left(W_1(t)\right)\right)r^{-2}\Lambda(dr)du\right].$$
 (3.67)

One can easily check that $|h_a(x_1) - h_a(x_2)| \le h_a(|x_1 - x_2|)$ and $|k_a(x_1) - k_a(x_2)| \le h_a(|x_1 - x_2|)$. We thus get $|h_a(W_2(t) + K_t^2(r, u)) - h_a(W_2(t))| \le h_a(|K_t^2(r, u)|)$ and $|k_a(W_1(t) + H_t^1(r, u)) - k_a(W_1(t))| \le h_a(|K_t^1(r, u)|)$. Combining with the definitions of $K_t^2(r, u)$ and $H_t^1(r, u)$ from (1.11) and (1.10) we get

$$\left| h_a \left(W_2(t) + K_t^2(r, u) \right) - h_a \left(W_2(t) \right) \right| \le h_a \left(e^{-S_t} r + \sum_{j>2} Z_j(t, r, u) W_j(t) \right) + a \mathbb{1}_{\beta_2(t, r, u) \ge 1},$$

$$\left| k_a \left(W_1(t) + H_t^1(r, u) \right) - k_a \left(W_1(t) \right) \right| \le h_a \left(e^{-S_t} r + \sum_{j>1} Z_j(t, r, u) W_j(t) \right).$$

Taking expectation we get

$$\mathbb{E}\left[\left|h_a\left(W_2(t) + K_t^2(r, u)\right) - h_a\left(W_2(t)\right)\right|\right]$$

$$\leq \mathbb{E}\left[h_a\left(\mathbb{E}\left[e^{-S_t}r + \sum_{j>2} Z_j(t, r, u)W_j(t)|S_t, (W_j(t))_{j\geq 1}\right]\right)\right] + 2ar$$

$$= \mathbb{E}\left[h_a\left(e^{-S_t}r + r\sum_{j>2} W_j(t)\right)\right] + 2ar.$$

where we have used Jensen's inequality together with the concavity of h_a and then Lemma 1.7 and Remark 2.17. Since, by Remark 2.17, $e^{-S_t} + \sum_{j>2} W_j(t) = 1 - (W_1(t) + W_2(t)) \le 1$ and since $h_a(\cdot)$ is non-decreasing we get $\mathbb{E}[|h_a(W_2(t) + K_t^2(r, u)) - h_a(W_2(t))|] \le 3(a \wedge r)$. Therefore, for any $\eta \in (0, 1]$,

$$\int_{(0,1)^2} \mathbb{E}\left[\left|h_a\left(W_2(t) + K_t^2(r,u)\right) - h_a\left(W_2(t)\right)\right|\right] r^{-2} \Lambda(dr) du \le 3h_{\Lambda}(a), \tag{3.68}$$

where we have used the definition of $h_{\Lambda}(\cdot)$ in (1.27). Similarly we get

$$\int_{(0,1)^2} \mathbb{E}\left[\left|k_a\left(W_1(t) + H_t^1(r,u)\right) - k_a\left(W_1(t)\right)\right|\right] r^{-2} \Lambda(dr) du \le h_{\Lambda}(a). \tag{3.69}$$

The bounds (3.68) and (3.69) show that for any $t \geq 0$, the absolute values of the derivatives from (3.66) and (3.67) are bounded by $3h_{\Lambda}(a)$. Combining this, the assumption (1.28), and the continuity of the functions $(t \mapsto \frac{d}{dt}\mathbb{E}[h_a(W_2(t))])$ and $(t \mapsto \frac{d}{dt}\mathbb{E}[k_a(W_1(t))])$ (which follow from Theorem 1.10), we get from differentiation under the integrals in (3.65) that the functions

 $(t \mapsto \mathbb{E}[h_{\Lambda}(W_2(t))])$ and $(t \mapsto \mathbb{E}[k_{\Lambda}(W_1(t))])$ are of class \mathcal{C}^1 and that for any $t \geq 0$,

$$\frac{d}{dt}\mathbb{E}[h_{\Lambda}(W_{2}(t))] = \int_{(0,1)} \mathbb{E}\left[\int_{(0,1)^{2}} \left(h_{a}\left(W_{2}(t) + K_{t}^{2}(r,u)\right) - h_{a}\left(W_{2}(t)\right)\right)r^{-2}\Lambda(dr)du\right]a^{-2}\Lambda(da),\tag{3.70}$$

$$\frac{d}{dt}\mathbb{E}[k_{\Lambda}(W_{1}(t))] = \int_{(0,1)} \mathbb{E}\left[\int_{(0,1)^{2}} \left(k_{a}\left(W_{1}(t) + H_{t}^{1}(r,u)\right) - k_{a}\left(W_{1}(t)\right)\right)r^{-2}\Lambda(dr)du\right] (1-a)a^{-2}\Lambda(da).$$
(3.71)

Then, (3.68), (3.69) and (1.28) allow to use Fubini's theorem in (3.70)-(3.71), yielding the result.

4. Long time behavior

Recall that we always assume that (1.2) holds true. For $k \geq 1$ let

$$C_k := \int_{(0,1)} (1 - (1-r)^k) r^{-1} \Lambda(dr), \ E_k := \int_{(0,1)} (1-r) (1 - (1-r)^k - kr(1-r)^{k-1}) r^{-2} \Lambda(dr).$$

Note that $C_1 = \lambda_2(\Lambda) = \Lambda((0,1))$ and that the sequence $(C_k)_{k\geq 1}$ is increasing. Moreover, $E_1 = 0$ and a straightforward calculation shows that for any $k \geq 1$, $C_k + E_k = \lambda_{k+1}(\Lambda)$. Finally we have that $C_k \to H(\Lambda)$ as $k \to \infty$.

4.1. Some analytic bounds. In this subsection we prove some bounds that result from Theorem 1.10.

Lemma 4.1. For any $k \ge 1$ and $t \ge 0$ we have

$$Y_k(t) \ge 1 - e^{-tC_k}. (4.72)$$

Proof. From the definition of $H_{\cdot}^{k}(\cdot,\cdot)$ in (1.10) we have almost surely

$$H_t^k(r,u) \ge (1 - \mathbb{1}_{Z_1(t,r,u) = \dots = Z_k(t,r,u) = 0}) \left(e^{-S_t} r + \sum_{j>k} Z_j(t,r,u) W_j(t) \right) + (1 - Z_{k+1}(t,r,u)) W_{k+1}(t) \mathbb{1}_{\sum_{i=1}^{k+1} (1 - Z_i(t,r,u)) \le k - 1}.$$

Note that the second term equals $W_{k+1}(t)\mathbbm{1}_{Z_{k+1}(t,r,u)=0,\beta_k(t,r,u)\geq 2}$. Recall from Lemma 1.7 and Remark 2.17 that $(Z_j(t,r,u))_{j\geq 1}$ and $\beta_k(t,r,u)$ are independent of $(W_j(t))_{j\geq 1}$ and S_t and that $(Z_j(t,r,u))_{j\geq 1}\sim \mathcal{B}(r)^{\times \mathbb{N}}$. We thus get

$$\mathbb{E}\left[\int_{(0,1)^2} H_t^k(r,u) r^{-2} \Lambda(dr) du\right] \ge \mathbb{E}\left[e^{-S_t} + \sum_{j>k} W_j(t)\right] \int_{(0,1)} (1 - (1-r)^k) r^{-1} \Lambda(dr)$$

$$+ \mathbb{E}\left[W_{k+1}(t)\right] \int_{(0,1)} (1-r) (1 - (1-r)^k - kr(1-r)^{k-1}) r^{-2} \Lambda(dr)$$

$$= C_k \mathbb{E}\left[1 - \sum_{j=1}^k W_j(t)\right] + E_k \mathbb{E}[M_{k+1}(t) - M_k(t)]$$

$$= C_k \mathbb{E}[1 - M_k(t)] + E_k (M_{k+1}(t) - M_k(t)).$$

Plugging this into (3.61) from Corollary 3.5 and using that $Y_k(t) = \mathbb{E}[M_k(t)]$ we obtain the following two bounds:

$$Y'_k(t) \ge C_k(1 - Y_k(t)), \qquad Y'_k(t) \ge \lambda_{k+1}(\Lambda)(1 - Y_k(t)) - E_k(1 - Y_{k+1}(t)).$$
 (4.73)

For the first bound we have used $E_k(M_{k+1}(t) - M_k(t)) \ge 0$, and for the second bound we have used the identity $C_k + E_k = \lambda_{k+1}(\Lambda)$.

The first bound in (4.73) yields $\frac{d}{dt}\log(1-Y_k(t)) \leq -C_k$ so $1-Y_k(t) \leq (1-Y_k(0))e^{-C_kt}$. Since $Y_k(0) = 0$ we get (4.72).

Lemma 4.2. For any $k \in \{1,...,N(\Lambda)-2\}$ there is $q_k > 0$ such that for any $t \geq 0$ we have

$$Y_k(t) \ge 1 - q_k e^{-\lambda_{k+1}(\Lambda)t}. (4.74)$$

For any $k \geq N(\Lambda) - 1$ and $\epsilon > 0$, there is $q_{k,\epsilon} > 0$ such that for any $t \geq 0$ we have

$$Y_k(t) \ge 1 - q_{k,\epsilon} e^{-t(H(\Lambda) - \epsilon)}. \tag{4.75}$$

Proof. Let us fix $k \ge 1$. We set $f_k(s) := e^{\lambda_{k+1}(\Lambda)s}(1 - Y_k(s))$. We have $f_k(0) = 1$ and

$$f'_k(s) = \lambda_{k+1}(\Lambda)e^{\lambda_{k+1}(\Lambda)s}(1 - Y_k(s)) - Y'_k(s)e^{\lambda_{k+1}(\Lambda)s} \le E_k e^{\lambda_{k+1}(\Lambda)s}(1 - Y_{k+1}(s)),$$

where we have used the second bound in (4.73). Integrating the above inequality on [0,t] and multiplying both sides by $e^{-\lambda_{k+1}(\Lambda)t}$ we get that for any $t \geq 0$,

$$1 - Y_k(t) \le e^{-\lambda_{k+1}(\Lambda)t} \left(1 + E_k \int_0^t e^{\lambda_{k+1}(\Lambda)s} (1 - Y_{k+1}(s)) ds \right). \tag{4.76}$$

Iterating (4.76) we get that for any $n \ge 1$ and $t \ge 0$, $e^{\lambda_{k+1}(\Lambda)t}(1-Y_k(t))$ is smaller than

$$1 + \sum_{j=1}^{n-1} \left(\prod_{i=1}^{j} E_{k+i-1} \right) \int_{[0,t]^{j}} \mathbb{1}_{s_{j} \leq \dots \leq s_{1}} \left(\prod_{i=1}^{j} e^{(\lambda_{k+i}(\Lambda) - \lambda_{k+1+i}(\Lambda))s_{i}} \right) ds_{1} \dots ds_{j}$$

$$+ \left(\prod_{i=1}^{n} E_{k+i-1} \right) \int_{[0,t]^{n}} \mathbb{1}_{s_{n} \leq \dots \leq s_{1}} \left(\prod_{i=1}^{n-1} e^{(\lambda_{k+i}(\Lambda) - \lambda_{k+1+i}(\Lambda))s_{i}} \right) e^{\lambda_{k+n}(\Lambda)s_{n}} (1 - Y_{k+n}(s_{n})) ds_{1} \dots ds_{n},$$

$$(4.77)$$

with the conventions $\sum_{j=1}^{0} \cdots = 0$ and $\prod_{i=1}^{0} \cdots = 1$. Since $\int_{[0,t]^j} \mathbb{1}_{s_j \leq \cdots \leq s_1} (\ldots) ds_1 \ldots ds_j \leq \int_{[0,\infty)^j} (\ldots) ds_1 \ldots ds_j$ we have

$$\int_{[0,t]^j} \mathbb{1}_{s_j \le \dots \le s_1} \left(\prod_{i=1}^j e^{(\lambda_{k+i}(\Lambda) - \lambda_{k+1+i}(\Lambda))s_i} \right) ds_1 \dots ds_j \le \prod_{i=1}^j \frac{1}{\lambda_{k+1+i}(\Lambda) - \lambda_{k+i}(\Lambda)}.$$
(4.78)

Let us fix $k \geq N(\Lambda) - 1$ and $\epsilon > 0$. Then we have $\lambda_{k+1}(\Lambda) \geq H(\Lambda)$. We fix n large enough so that $H(\Lambda) - \epsilon < C_{k+n}$. We thus have $H(\Lambda) - \epsilon < C_{k+n} < H(\Lambda) \leq \lambda_{k+1}(\Lambda)$. Using Lemma 4.1 and integrating the variables one by one we get that the second integral in (4.77) is smaller than

$$\int_{[0,t]^{n-1}} \mathbb{1}_{s_{n-1} \le \dots \le s_{1}} \left(\prod_{i=1}^{n-1} e^{(\lambda_{k+i}(\Lambda) - \lambda_{k+1+i}(\Lambda))s_{i}} \right) \frac{e^{(\lambda_{k+n}(\Lambda) - C_{k+n})s_{n-1}}}{\lambda_{k+n}(\Lambda) - C_{k+n}} ds_{1} \dots ds_{n-1} \\
\le \left(\prod_{i=1}^{n} \frac{1}{\lambda_{k+i}(\Lambda) - C_{k+n}} \right) e^{(\lambda_{k+1}(\Lambda) - C_{k+n})t} \le e^{\lambda_{k+1}(\Lambda)t} \left(\prod_{i=1}^{n} \frac{1}{\lambda_{k+i}(\Lambda) - C_{k+n}} \right) e^{-(H(\Lambda) - \epsilon)t}. \tag{4.79}$$

Combining (4.78) and (4.79) with (4.77) we get that (4.75) holds for some choice of $q_{k,\epsilon}$.

Let us now fix $k \in \{1, ..., N(\Lambda) - 2\}$. Then we have $\lambda_{k+1}(\Lambda) < H(\Lambda)$. Since $C_{k+n} \to H(\Lambda)$ as $n \to \infty$, for n large enough we have $\lambda_{k+1}(\Lambda) < C_{k+n} < H(\Lambda)$ and $C_{k+n} < \lambda_{k+n}(\Lambda)$. Note also that, since $N(\Lambda)$ is finite and $(C_{\ell})_{\ell \geq 1}$ is increasing and bounded by $H(\Lambda)$, for n large enough C_{k+n} does not coincide with any coefficient $\lambda_j(\Lambda)$. We assume that n is chosen such that the above requirements are satisfied. Let $m := \min\{j \leq n, \lambda_{k+j}(\Lambda) > C_{k+n}\}$. Note that $m \geq 2$ and $\lambda_{k+m-1}(\Lambda) < C_{k+n} < \lambda_{k+m}(\Lambda)$. Proceeding as in (4.79) for the variables s_n, \ldots, s_m and as in (4.78) for the variables s_{m-1}, \ldots, s_1 we get that the second integral in (4.77) is smaller than

$$\left(\prod_{i=m}^{n} \frac{1}{\lambda_{k+i}(\Lambda) - C_{k+n}}\right) \times \frac{1}{C_{k+n} - \lambda_{k+m-1}(\Lambda)} \times \left(\prod_{i=1}^{m-2} \frac{1}{\lambda_{k+1+i}(\Lambda) - \lambda_{k+i}(\Lambda)}\right), \tag{4.80}$$

where we have used the convention $\int_{[0,t]^{m-2}}(\cdots)ds_1...ds_{m-2}=1$ if m=2. Combining (4.78) and (4.80) with (4.77) we get that (4.74) holds.

Lemma 4.3. For any $k \geq 2$ there is $Q_k > 0$ such that for any $t \geq 1$ we have

$$w_k(t) \ge Q_k e^{-\lambda_k(\Lambda)t}. (4.81)$$

Proof. From the definition of $K_{\cdot}^{k}(\cdot,\cdot)$ in (1.11) we have almost surely $K_{t}^{k}(r,u) \geq -\mathbb{1}_{\beta_{k}(t,r,u)\geq 2}W_{k}(t)$. Recall from Lemma 1.7 that $\beta_{k}(t,r,u)$ is independent of $(W_{j}(t))_{j\geq 1}$ and that $\beta_{k}(t,r,u)$ follows the binomial distribution with parameter (k,r). Plugging this into (3.62) from Corollary 3.5 and using that $w_{k}(t) = \mathbb{E}[W_{k}(t)]$ we obtain

$$w'_k(t) \ge -\mathbb{E}[W_k(t)] \int_{(0,1)} (1 - (1-r)^k - kr(1-r)^{k-1}) r^{-2} \Lambda(dr) = -\lambda_k(\Lambda) w_k(t).$$

Therefore $\frac{d}{dt}\log(w_k(t)) \geq -\lambda_k(\Lambda)$ for any t > 0 so $w_k(t) \geq w_k(1)e^{-\lambda_k(\Lambda)(t-1)}$. We have clearly $\mathbb{P}(W_k(1) > 0) > 0$ so $w_k(1) = \mathbb{E}[W_k(1)] > 0$. We thus get (4.81) with $Q_k := w_k(1)e^{\lambda_k(\Lambda)}$.

4.2. A probabilistic bound. In this subsection we use Theorem 1.8 to derive the following bound.

Proposition 4.4. For any $k \geq 3$ there is $c_k > 0$ such that for all $t \geq 1$, $\mathbb{E}[W_k(t)] \geq c_k e^{-tH(\Lambda)}$.

Let $t_1, t_2 \geq 0$ with $t_1 < t_2, k \geq 3, \eta \in (0, 1)$ and $\alpha \in (0, \infty]$. We set

$$E_{t_1,t_2}^{\eta} := \{ (s,r,u) \in N \text{ s.t. } s \in (t_1,t_2], \ r > \eta \}, \qquad M_{t_1,t_2}^{\eta} := \sharp E_{t_1,t_2}^{\eta}.$$

For $i \in \{1, \ldots, M_{t_1, t_2}^{\eta}\}$, let (s_i, r_i, u_i) be the i^{th} element of E_{t_1, t_2}^{η} , where the ordering is such that $s_1 < s_2 < \ldots < s_{M_{t_1, t_2}^{\eta}}$. Let

$$\mathcal{E}(t_1, t_2, k, \alpha, \eta) := \left\{ M_{t_1, t_2}^{\eta} = k, S_{t_2} - S_{t_1} \le \alpha, \forall i \ne j \in \{1, \dots, k\}, Y_{s_i, t_2}(u_i) \ne Y_{s_j, t_2}(u_j) \right\}. \tag{4.82}$$

Note that the event $\mathcal{E}(t_1, t_2, k, \alpha, \eta)$ is independent from the sigma-field \mathcal{F}_{t_1} from Section 1.2.

Lemma 4.5. Let $t_1, t_2 \geq 0$ with $t_1 < t_2$, $k \geq 3$, $\eta \in (0,1)$ and $\alpha > 0$. On the event $\mathcal{E}(t_1, t_2, k, \alpha, \eta)$ we have almost surely $W_k(t_2) \geq \eta e^{-\alpha} e^{-S_{t_1}}$.

Proof. Assume we are on the event $\mathcal{E}(t_1, t_2, k, \alpha, \eta)$ and on the probability one event where (1.13) holds true at $t = t_2$ (see Theorem 1.8). Note from (1.13) that for any $i \in \{1, \ldots, k\}$,

$$\mu_{t_2}(\{Y_{s_i,t_2}(u_i)\}) \ge r_i e^{-S_{s_i}} > \eta e^{-(S_{t_2} - S_{t_1})} e^{-S_{t_1}} \ge \eta e^{-\alpha} e^{-S_{t_1}}.$$

This shows that, for each $i \in \{1, ..., k\}$, $Y_{s_i, t_2}(u_i)$ is an atom of μ_{t_2} with weight larger than $\eta e^{-\alpha} e^{-S_{t_1}}$, and, since $Y_{s_i, t_2}(u_i) \neq Y_{s_j, t_2}(u_j)$ for $i, j \in \{1, ..., k\}$ with $i \neq j$, these atoms are

pairwise distinct. Therefore μ_{t_2} has at least k atoms with weight larger than $\eta e^{-\alpha} e^{-S_{t_1}}$. Since, by (1.12), $W_k(t_2)$ is the k^{th} larger weight of atoms of μ_{t_2} , we get $W_k(t_2) \ge \eta e^{-\alpha} e^{-S_{t_1}}$.

Lemma 4.6. Let h > 0, $k \ge 3$, $\eta \in (0, \max \operatorname{supp} \Lambda)$. For all $\alpha > 0$ large enough there is $c(h, k, \eta, \alpha) > 0$ such that for all $t \ge 0$ we have $\mathbb{P}(\mathcal{E}(t, t + h, k, \alpha, \eta)) \ge c(h, k, \eta, \alpha)$.

Proof. Let h > 0 and let $(U_i)_{i \ge 1}$, $(\pi_t^Y)_{t \ge 0}$ and $(\pi_t^N)_{t \ge 0}$ be as in Sections 1.3 and 2.2. We consider the event where 1) there are exactly k jumps $(s_i, r_i, u_i) \in N$ such that $r_i > \eta$ and $s_i \in (0, h]$ (we order them via $s_1 < \ldots < s_k$), 2) for any $i \in \{1, \ldots, k\}$, the block of $(\pi_t^N)_{t \ge 0}$ containing i is not involved in any merger event in $(0, h] \setminus \{s_i\}$ but it takes part in the merger event at time s_i . We see from Remark 2.6 that this event has a positive probability. The event implies that, at time h, the blocks of $(\pi_t^N)_{t \ge 0}$ containing 1,...,k are all distinct (so, by Remark 2.5, the $Y_{0,h}(U_i)$ are distinct for $i \in \{1, \ldots, k\}$) and, via (1.7), that $Y_{0,h}(U_i) = Y_{s_i,h}(u_i)$. The event is thus included into $\mathcal{E}(0, h, k, \infty, \eta)$ so, in particular, $\mathbb{P}(\mathcal{E}(0, h, k, \infty, \eta)) > 0$. Since $\mathbb{P}(S_h > \alpha)$ converges to 0 as α goes to infinity, we can choose $\alpha > 0$ such that $\mathbb{P}(S_h > \alpha) < \mathbb{P}(\mathcal{E}(0, h, k, \infty, \eta))$. For such α we have $\mathbb{P}(\mathcal{E}(0, h, k, \alpha, \eta)) \ge \mathbb{P}(\mathcal{E}(0, h, k, \infty, \eta)) - \mathbb{P}(S_h > \alpha) > 0$. We have clearly that for all $t \ge 0$, $\mathbb{P}(\mathcal{E}(t, t + h, k, \alpha, \eta)) = \mathbb{P}(\mathcal{E}(0, h, k, \alpha, \eta))$, so the result follows.

Proof of Proposition 4.4. Fix $k \geq 3$ and $\eta \in (0, \max \operatorname{supp}\Lambda)$. According to Lemma 4.6 there is $\alpha > 0$ and a constant c > 0 such that for any $t \geq 1$ we have $\mathbb{P}(\mathcal{E}(t-1,t,k,\alpha,\eta)) \geq c$. According to Lemma 4.5 we have $W_k(t) \geq \eta e^{-\alpha} e^{-S_{t-1}}$ almost surely on $\mathcal{E}(t-1,t,k,\alpha,\eta)$. We thus get that for any $t \geq 1$,

$$\mathbb{E}[W_k(t)] \ge \eta e^{-\alpha} \mathbb{E}[e^{-S_{t-1}} \mathbb{1}_{\mathcal{E}(t-1,t,k,\alpha,\eta)}] = \eta e^{-\alpha} \mathbb{E}[e^{-S_{t-1}}] \times \mathbb{P}(\mathcal{E}(t-1,t,k,\alpha,\eta)),$$

where we have used that S_{t-1} is measurable with respect to \mathcal{F}_{t-1} while the event $\mathcal{E}(t-1, t, k, \alpha, \eta)$ is independent of \mathcal{F}_{t-1} . Combining with the bound $\mathbb{P}(\mathcal{E}(t-1, t, k, \alpha, \eta)) \geq c$, the definition of $\phi_{S}(\cdot)$ in Section 1.4, and (1.9), we get

$$\mathbb{E}[W_k(t)] \ge c\eta e^{-\alpha} \mathbb{E}[e^{-S_{t-1}}] = c\eta e^{-\alpha} e^{-(t-1)\phi_S(1)} = c\eta e^{-\alpha} e^{-(t-1)H(\Lambda)}.$$

4.3. Conclusion: Proof of Theorem 1.11.

Proof of Theorem 1.11. Since $Y_1(t) = \mathbb{E}[W_1(t)]$ and $C_1 = \lambda_2(\Lambda)$, Lemma 4.1 applied at k = 1 yields $1 - \mathbb{E}[W_1(t)] \le e^{-t\lambda_2(\Lambda)}$. Then for $k \ge 2$ we have $Y_{k-1}(t) + \mathbb{E}[W_k(t)] = \mathbb{E}[M_k(t)] \le 1$ so

$$\mathbb{E}[W_k(t)] \le 1 - Y_{k-1}(t). \tag{4.83}$$

Since $w_2(t) = \mathbb{E}[W_2(t)]$, the combination of (4.83) with (4.81) (both applied at k = 2) yields $1 - \mathbb{E}[W_1(t)] \ge Q_2 e^{-t\lambda_2(\Lambda)}$ for $t \ge 1$, completing the proof of (1.18).

We now fix $k \in \{2, ..., N(\Lambda)-1\}$. Combining (4.83) with (4.74) yields $\mathbb{E}[W_k(t)] \leq q_{k-1}e^{-\lambda_k(\Lambda)t}$. Since $w_k(t) = \mathbb{E}[W_k(t)]$, (4.81) yields $\mathbb{E}[W_k(t)] \geq Q_k e^{-\lambda_k(\Lambda)t}$ for $t \geq 1$, completing the proof of (1.19).

We now fix $k \geq N(\Lambda)$ and $\epsilon > 0$. Combining (4.83) with (4.75) yields $\mathbb{E}[W_k(t)] \leq q_{k-1,\epsilon}e^{-t(H(\Lambda)-\epsilon)}$. Combining with Proposition 4.4 we get

$$H(\Lambda) - \epsilon \le \liminf_{t \to \infty} -\frac{1}{t} \log(\mathbb{E}[W_k(t)]) \le \limsup_{t \to \infty} -\frac{1}{t} \log(\mathbb{E}[W_k(t)]) \le H(\Lambda).$$

Since ϵ can be chosen arbitrarily small we get (1.20). This concludes the proof.

5. Small time behavior: Proof of Theorem 1.19

We start with a preliminary lemma.

Lemma 5.1. We have

$$\mathbb{E}[W_1(t)] = tH(\Lambda) + o(t), \tag{5.84}$$

$$\mathbb{E}\left[\sum_{j\geq 2} W_j(t)\right] = \underset{t\to 0}{o}(t). \tag{5.85}$$

Proof. Applying (1.23) with f(y) := y yields (5.84). Then, by Remark 2.17 and (1.9),

$$\mathbb{E}\left[\sum_{j\geq 1} W_j(t)\right] = \mathbb{E}[1 - e^{-S_t}] = 1 - e^{-t\Phi_S(1)} = 1 - e^{-tH(\Lambda)} = tH(\Lambda) + o(t).$$

Combining with (5.84) we get (5.85).

In the rest of this section, additionally to assuming (1.2), we assume that (1.28) holds true. The next step in proving Theorem 1.19 is to establish, in the following proposition, a Taylor expansions of order 2, near 0, of $Y_2(t) = \mathbb{E}[W_1(t) + W_2(t)]$.

Proposition 5.2. We have

$$Y_2(t) = tH(\Lambda) - \frac{t^2}{2}H(\Lambda)^2 + o_{t\to 0}(t^2).$$
 (5.86)

Proof. Using the expression of $Y_2'(t)$ given by Corollary 3.5 (together with the expression (1.10) of $H_t^k(r,u)$) and that, for $a,b \ge 0$, $(a-b)_+ = a - (a \land b)$, we get

$$Y_{2}'(t) = \mathbb{E}\left[\int_{(0,1)^{2}} \left(e^{-S_{t}}r + \sum_{j>2} Z_{j}(t,r,u)W_{j}(t)\right) r^{-2}\Lambda(dr)du\right]$$

$$- \mathbb{E}\left[\int_{(0,1)^{2}} \mathbb{1}_{\beta_{2}(t,r,u)=0} \left(\left(e^{-S_{t}}r + \sum_{j>2} Z_{j}(t,r,u)W_{j}(t)\right) \wedge W_{2}(t)\right) r^{-2}\Lambda(dr)du\right]$$

$$+ \mathbb{E}\left[\int_{(0,1)^{2}} \left(\sum_{j>2} (1 - Z_{j}(t,r,u))W_{j}(t)\mathbb{1}_{\sum_{i=1}^{j} (1 - Z_{i}(t,r,u)) \le 1}\right) r^{-2}\Lambda(dr)du\right]$$

$$=: E_{1}(t) - E_{2}(t) + E_{3}(t). \tag{5.87}$$

Recall from Lemma 1.7 and Remark 2.17 that $(Z_j(t,r,u))_{j\geq 1}$ is independent of $(W_j(t))_{j\geq 1}$ and S_t and that $(Z_j(t,r,u))_{j\geq 1}\sim \mathcal{B}(r)^{\times\mathbb{N}}$. We thus get

$$E_{1}(t) = \mathbb{E}\left[\int_{(0,1)} \left(1 - \sum_{j \ge 1} W_{j}(t) + \sum_{j > 2} W_{j}(t)\right) r^{-1} \Lambda(dr)\right]$$

$$= \mathbb{E}[1 - M_{2}(t)]H(\Lambda) = (1 - Y_{2}(t))H(\Lambda). \tag{5.88}$$

Since $Y_2(t) = \mathbb{E}[W_1(t)] + \mathbb{E}[W_2(t)]$ we get from (5.84) and (5.85) that $Y_2(t) = tH(\Lambda) + o(t)$. Combining with the above we get

$$E_1(t) = H(\Lambda) - tH(\Lambda)^2 + \underset{t \to 0}{o}(t).$$
 (5.89)

Note that the integrand of the term $E_3(t)$ can be non-zero only if $Z_1(t,r,u) = Z_2(t,r,u) = 1$. We thus get

$$0 \le E_3(t) \le \mathbb{E}\left[\left(\sum_{j \ge 2} W_j(t)\right) \int_{(0,1)^2} \mathbb{1}_{Z_1(t,r,u) = Z_2(t,r,u) = 1} r^{-2} \Lambda(dr) du\right].$$

Using Lemma 1.7 again and (5.85), we get

$$0 \le E_3(t) \le \mathbb{E}\left[\sum_{j \ge 2} W_j(t)\right] \Lambda((0,1)) = \underset{t \to 0}{o}(t). \tag{5.90}$$

For any $a,b,c \ge 0$ we have $(a \land c) \le (a+b) \land c \le (a \land c) + b$ so $|((a+b) \land c) - (a \land c)| \le b$. We deduce that

$$\left| E_{2}(t) - \mathbb{E} \left[\int_{(0,1)^{2}} \mathbb{1}_{\beta_{2}(t,r,u)=0} \left(e^{-S_{t}} r \wedge W_{2}(t) \right) r^{-2} \Lambda(dr) du \right] \right| \\
\leq \mathbb{E} \left[\int_{(0,1)^{2}} \mathbb{1}_{\beta_{2}(t,r,u)=0} \left(\sum_{j>2} Z_{j}(t,r,u) W_{j}(t) \right) r^{-2} \Lambda(dr) du \right] \\
= \mathbb{E} \left[\sum_{j>2} W_{j}(t) \right] \int_{(0,1)} (1-r)^{2} r^{-1} \Lambda(dr) = \underset{t \to 0}{o}(t), \tag{5.91}$$

where we have used Lemma 1.7 again for the penultimate equality and (5.85) for the last equality. Since $e^{-S_t} \leq 1$ we get

$$0 \leq \mathbb{E} \left[\int_{(0,1)^2} \mathbb{1}_{\beta_2(t,r,u)=0} \left(e^{-S_t} r \wedge W_2(t) \right) r^{-2} \Lambda(dr) du \right]$$

$$\leq \mathbb{E} \left[\int_{(0,1)^2} \left(r \wedge W_2(t) \right) r^{-2} \Lambda(dr) du \right] = \mathbb{E} [h_{\Lambda}(W_2(t))], \tag{5.92}$$

where $h_{\Lambda}(\cdot)$ is defined in (1.27). We have $\mathbb{E}[h_{\Lambda}(W_2(0))] = 0$ and, by Proposition 3.6, the function $(t \mapsto \mathbb{E}[h_{\Lambda}(W_2(t))])$ is of class \mathcal{C}^1 and evaluating (3.63) at t = 0 we get

$$\frac{d}{dt}\mathbb{E}[h_{\Lambda}(W_{2}(t))]_{|t=0} = \mathbb{E}\left[\int_{(0,1)^{2}} \left(h_{\Lambda}\left(W_{2}(0) + K_{0}^{2}(r,u)\right) - h_{\Lambda}\left(W_{2}(0)\right)\right)r^{-2}\Lambda(dr)du\right] = 0,$$

where we have used (1.11) evaluated at k=2 and t=0. We thus get that, as t is small, $\mathbb{E}[h_{\Lambda}(W_2(t))] = o(t)$. Combining with (5.91) and (5.92) we obtain

$$E_2(t) = \underset{t \to 0}{o}(t). \tag{5.93}$$

Combining (5.89), (5.90) and (5.93) with (5.87) we get

$$Y_2'(t) = H(\Lambda) - tH(\Lambda)^2 + o_0(t).$$

Since
$$Y_2(0) = \mathbb{E}[W_1(0) + W_2(0)] = 0$$
 we get (5.86).

In the following proposition, we establish a Taylor expansions of order 2 of $Y_1(t) = \mathbb{E}[W_1(t)]$.

Proposition 5.3. We have

$$Y_1(t) = tH(\Lambda) - \frac{t^2}{2} \left(H(\Lambda)^2 + \int_{(0,1)} k_{\Lambda}(r) r^{-2} \Lambda(dr) \right) + \underset{t \to 0}{o}(t^2), \tag{5.94}$$

where $k_{\Lambda}(\cdot)$ is defined in (1.27).

Proof. We use the expression of $Y_1'(t)$ given by Corollary 3.5, together with the expression (1.10) of $H_t^k(r, u)$. Note that the third term in the expression of $H_t^k(r, u)$ vanishes when k = 1. Using also that, for $a, b \ge 0$, $(a - b)_+ = a - (a \land b)$, we get

$$Y_{1}'(t) = \mathbb{E}\left[\int_{(0,1)^{2}} \left(e^{-S_{t}}r + \sum_{j>1} Z_{j}(t,r,u)W_{j}(t)\right)r^{-2}\Lambda(dr)du\right]$$

$$-\mathbb{E}\left[\int_{(0,1)^{2}} \mathbb{1}_{\beta_{1}(t,r,u)=0} \left(\left(e^{-S_{t}}r + \sum_{j>1} Z_{j}(t,r,u)W_{j}(t)\right)\wedge W_{1}(t)\right)r^{-2}\Lambda(dr)du\right]$$

$$=: \tilde{E}_{1}(t) - \tilde{E}_{2}(t). \tag{5.95}$$

Proceeding as in (5.88) we get $\tilde{E}_1(t) = (1 - Y_1(t))H(\Lambda)$. Combining with (5.84) we get

$$\tilde{E}_1(t) = H(\Lambda) - tH(\Lambda)^2 + o_0(t). \tag{5.96}$$

Proceeding as in (5.91) we get

$$\left| \tilde{E}_{2}(t) - \mathbb{E} \left[\int_{(0,1)^{2}} \mathbb{1}_{\beta_{1}(t,r,u)=0} \left(e^{-S_{t}} r \wedge W_{1}(t) \right) r^{-2} \Lambda(dr) du \right] \right| = \underset{t \to 0}{o}(t), \tag{5.97}$$

For any $a,b,c\geq 0$ we have $(a\wedge c)-b\leq (a-b)\wedge c\leq (a\wedge c)$ so $|((a-b)\wedge c)-(a\wedge c)|\leq b$. Since, by Remark 2.17, $e^{-S_t}=1-W_1(t)-\sum_{j\geq 2}W_j(t)$, we deduce that

$$\left| \mathbb{E} \left[\int_{(0,1)^2} \mathbb{1}_{\beta_1(t,r,u)=0} \left(e^{-S_t} r \wedge W_1(t) \right) r^{-2} \Lambda(dr) du \right] \right. \\
\left. - \mathbb{E} \left[\int_{(0,1)^2} \mathbb{1}_{\beta_1(t,r,u)=0} \left((1 - W_1(t)) r \wedge W_1(t) \right) r^{-2} \Lambda(dr) du \right] \right| \\
\leq \mathbb{E} \left[\int_{(0,1)^2} \mathbb{1}_{\beta_1(t,r,u)=0} \left(r \sum_{j \geq 2} W_j(t) \right) r^{-2} \Lambda(dr) du \right] \\
= \mathbb{E} \left[\sum_{j \geq 2} W_j(t) \right] \int_{(0,1)} (1 - r) r^{-1} \Lambda(dr) = \underset{t \to 0}{o}(t), \tag{5.98}$$

where we have used Lemma 1.7 for the penultimate equality and (5.85) for the last equality. Then, using Lemma 1.7,

$$\mathbb{E}\left[\int_{(0,1)^2} \mathbb{1}_{\beta_1(t,r,u)=0} \left((1 - W_1(t))r \wedge W_1(t) \right) r^{-2} \Lambda(dr) du \right] = \mathbb{E}[k_{\Lambda}(W_1(t))], \tag{5.99}$$

where $k_{\Lambda}(\cdot)$ is defined in (1.27). We have $\mathbb{E}[k_{\Lambda}(W_1(0))] = 0$ and, by Corollary 3.6, the function $(t \mapsto \mathbb{E}[k_{\Lambda}(W_1(t))])$ is of class \mathcal{C}^1 and evaluating (3.64) at t = 0 we get

$$\frac{d}{dt} \mathbb{E}[k_{\Lambda}(W_{1}(t))]_{|t=0} = \mathbb{E}\left[\int_{(0,1)^{2}} \left(k_{\Lambda}\left(W_{1}(0) + H_{0}^{1}(r,u)\right) - k_{\Lambda}\left(W_{1}(0)\right)\right) r^{-2} \Lambda(dr) du\right]
= \int_{(0,1)} k_{\Lambda}(r) r^{-2} \Lambda(dr),$$

where we have used (1.10) evaluated at k=1 and t=0. Recall from the discussion after (1.28) that, under the assumption (1.28), $\int_{(0,1)} k_{\Lambda}(r) r^{-2} \Lambda(dr)$ is indeed finite. This yields $\mathbb{E}[k_{\Lambda}(W_1(t))] = t \int_{(0,1)} k_{\Lambda}(r) r^{-2} \Lambda(dr) + o(t)$. Combining with (5.97), (5.98) and (5.99) we obtain

$$\tilde{E}_2(t) = t \int_{(0,1)} k_{\Lambda}(r) r^{-2} \Lambda(dr) + \mathop{o}_{t \to 0}(t). \tag{5.100}$$

Combining (5.96) and (5.100) with (5.95) we get

$$Y_1'(t) = H(\Lambda) - t \left(H(\Lambda)^2 + \int_{(0,1)} k_{\Lambda}(r) r^{-2} \Lambda(dr) \right) + \underset{t \to 0}{o}(t).$$

Since $Y_1(0) = \mathbb{E}[W_1(0)] = 0$ we get (5.94).

We can now prove Theorem 1.19.

Proof of Theorem 1.19. Since $\mathbb{E}[W_2(t)] = Y_2(t) - Y_1(t)$, combining Propositions 5.2 and 5.3 yields

$$\mathbb{E}[W_2(t)] = \frac{t^2}{2} \int_{(0,1)} k_{\Lambda}(r) r^{-2} \Lambda(dr) + \underset{t \to 0}{o} (t^2),$$

which yields (1.29).

APPENDIX A.

In this appendix we study in details the SDE (1.5) and establish Propositions 1.1 and A.9.

A.1. **Preliminary:** some estimates. We start by proving some estimates on the function $m_{r,u}(\cdot)$ appearing in the SDE (1.5).

Lemma A.1. For any $r, u \in (0,1)$ and $z \in [0,1]$ we have

$$|\mathbf{m}_{r,u}(z) - z| \le \frac{r}{1 - r}.$$
 (A.101)

For any $r \in (0,1)$ and $1 \ge z_1 \ge z_2 \ge 0$ we have

$$\int_0^1 |\mathbf{m}_{r,u}(z_1) - \mathbf{m}_{r,u}(z_2) - z_1 + z_2| du \le \frac{4r}{(1-r)^2} |z_1 - z_2|. \tag{A.102}$$

Proof. Let $r, u \in (0, 1)$ and $z \in [0, 1]$ and let us study $|\mathbf{m}_{r,u}(z) - z|$. If $u \leq \frac{z-r}{1-r}$ then $|\mathbf{m}_{r,u}(z) - z| = \frac{r}{1-r}(1-z) \leq \frac{r}{1-r}$. If $\frac{z-r}{1-r} \leq u \leq \frac{z}{1-r}$ then $|\mathbf{m}_{r,u}(z) - z| = |u-z| \leq (z/(1-r)-z) \vee (z-(z-r)/(1-r)) \leq \frac{r}{1-r}$. If $\frac{z}{1-r} \leq u$ then $|\mathbf{m}_{r,u}(z) - z| = \frac{r}{1-r}z \leq \frac{r}{1-r}$. We thus get (A.101) in all cases.

Let now $r, u \in (0,1)$ and $1 \ge z_1 \ge z_2 \ge 0$. We study $m_{r,u}(z_1) - m_{r,u}(z_2)$. If $u \le \frac{z_2 - r}{1 - r}$ then

$$m_{r,u}(z_1) - m_{r,u}(z_2) = \frac{z_1 - r}{1 - r} - \frac{z_2 - r}{1 - r} = \frac{z_1 - z_2}{1 - r}.$$

If $\frac{z_2-r}{1-r} \le u \le \frac{z_1-r}{1-r} \land \frac{z_2}{1-r}$ then $m_{r,u}(z_1) - m_{r,u}(z_2) = \frac{z_1-r}{1-r} - u$ so

$$\frac{z_1 - z_2}{1 - r} - \frac{r}{1 - r} \le m_{r,u}(z_1) - m_{r,u}(z_2) \le \frac{z_1 - r}{1 - r} - \frac{z_2 - r}{1 - r} = \frac{z_1 - z_2}{1 - r}.$$

If $\frac{z_1-r}{1-r} \le u \le \frac{z_2}{1-r}$ then

$$m_{r,u}(z_1) - m_{r,u}(z_2) = u - u = 0.$$

If $\frac{z_2}{1-r} \le u \le \frac{z_1-r}{1-r}$ then

$$m_{r,u}(z_1) - m_{r,u}(z_2) = \frac{z_1 - r}{1 - r} - \frac{z_2}{1 - r} = \frac{z_1 - z_2}{1 - r} - \frac{r}{1 - r}.$$

If $\frac{z_1-r}{1-r} \vee \frac{z_2}{1-r} \leq u \leq \frac{z_1}{1-r}$ then $m_{r,u}(z_1) - m_{r,u}(z_2) = u - \frac{z_2}{1-r}$ so

$$\frac{z_1 - z_2}{1 - r} - \frac{r}{1 - r} \le m_{r,u}(z_1) - m_{r,u}(z_2) \le \frac{z_1}{1 - r} - \frac{z_2}{1 - r} = \frac{z_1 - z_2}{1 - r}.$$

If $\frac{z_1}{1-r} \leq u$ then

$$m_{r,u}(z_1) - m_{r,u}(z_2) = \frac{z_1}{1-r} - \frac{z_2}{1-r} = \frac{z_1 - z_2}{1-r}.$$

In conclusion we get

$$|\mathbf{m}_{r,u}(z_1) - \mathbf{m}_{r,u}(z_2) - z_1 + z_2| \le \frac{r}{1-r} |z_1 - z_2| + \frac{r}{1-r} \mathbb{1}_{u \in [\frac{z_2-r}{1-r}, \frac{z_1-r}{1-r}] \cup [\frac{z_2}{1-r}, \frac{z_1}{1-r}]} + |z_1 - z_2| \mathbb{1}_{u \in [\frac{z_1-r}{1-r}, \frac{z_2}{1-r}]}.$$

Integrating with respect to u we get (A.102).

Lemma A.2. For $y \in [0,1]$ we have $(1-r) m_{r,u}(y) - y = -\int_0^y \mathbb{1}_{z \in I_{r,u}^o} dz$.

Proof. Using the definition of $m_{r,u}(\cdot)$ we get

$$(1-r)\mathbf{m}_{r,u}(y) - y = y \mathbb{1}_{y \le (1-r)u} + (1-r)u\mathbb{1}_{y \in I_{r,u}^{\circ}} + (y-r)\mathbb{1}_{y \ge (1-r)u+r} - y$$
$$= ((1-r)u - y) \mathbb{1}_{y \in I_{r,u}^{\circ}} - r\mathbb{1}_{y \ge (1-r)u+r} = -\int_{0}^{y} \mathbb{1}_{z \in I_{r,u}^{\circ}} dz.$$

Lemma A.3. For any M > 0 let $f_M : [0,1] \to \mathbb{R}$ be defined by $f_M(x) := -\log(x \vee e^{-M})$. Then there is a constant C independent of M such that for any $a \in (0,1)$ and $r \in (0,a \wedge (1/2))$,

$$\frac{a-r}{1-r}\left(f_M\left(\frac{a-r}{1-r}\right) - f_M(a)\right) \le Cr. \tag{A.103}$$

Proof. Let $a \in (0,1)$ and $r \in (0, a \land (1/2))$. Distinguishing the three cases $a < e^{-M}$, $\frac{a-r}{1-r} \le e^{-M} \le a$, and $e^{-M} < \frac{a-r}{1-r}$ we get that in any case

$$\frac{a-r}{1-r} \left(f_M \left(\frac{a-r}{1-r} \right) - f_M(a) \right) \le -\frac{a-r}{1-r} \log \left(\frac{a-r}{a(1-r)} \right) \le -2(a-r) \log \left(1 - \frac{r}{a} \right) =: u(a,r). \tag{A.104}$$

Let $C_1 := \sup_{x \in (0,1/2]} -\log(1-x)/x$ and $C_2 := \sup_{x \in (0,1]} -x\log(x)$. If $r \in (0,a/2]$ we have $u(a,r) \leq 2C_1r$. If $r \in (a/2,a)$ we have $u(a,r) \leq 2(a-r) \times C_2/(1-\frac{r}{a}) = 2C_2a \leq 4C_2r$. Setting $C := \max\{2C_1, 4C_2\}$ we get that, in both cases $r \in (a/2,a]$ and $r \in (a/2,a)$ we have $u(a,r) \leq Cr$. Combining with (A.104) we get (A.103).

A.2. **Proof of Propositions 1.1 and A.9.** Our approach requires to prove some regularity of single trajectories with respect to their initial condition but is rather different from the approach used in [21] as, in our case, we use approximations by the case with finitely many jumps. In this appendix we always assume that (1.2) holds true.

We denote by D([0,T]) (resp. $D([0,\infty))$) the space of càd-làg functions from [0,T] (resp. $[0,\infty)$) to \mathbb{R} . We sometimes use the metrics d_T and d_∞ on D([0,T]) and $D([0,\infty))$ defined by

$$d_T(f,g) := 1 \wedge \sup_{t \in [0,T]} |f(t) - g(t)|, \ d_{\infty}(f,g) := \int_0^\infty e^{-T} d_T(f_{|[0,T]}, g_{|[0,T]}) dT. \tag{A.105}$$

We note that $(D([0,T]), d_T)$ (resp. $(D([0,\infty)), d_\infty)$) is a complete metric space. Moreover, the topology induced by d_T (resp. d_∞) is stronger than the usual Skorokhod topology.

For $\delta \in (0,1)$, we consider the flow Y^{δ} defined by

$$Y_{0,t}^{\delta}(y) = y + \int_{(0,t]\times(\delta,1)\times(0,1)} \left(\mathbf{m}_{r,u}(Y_{0,s-}^{\delta}(y)) - Y_{0,s-}^{\delta}(y) \right) N(ds, dr, du), \tag{A.106}$$

for all $y \in [0,1]$ and $t \ge 0$. Comparing with (1.5) we see that the flow Y^{δ} is obtained similarly as Y, but by keeping only the jumps of N with a r-component larger than δ . Since such jumps occur at finite rate, the flow Y^{δ} is much simpler than the flow Y. It can be defined as follows. Let $\delta \in (0,1)$ and $(s_k, r_k, u_k)_{k\ge 1}$ be the enumeration of the discrete set $\{(s, r, u) \in N, r > \delta\}$ such that $s_1 < s_2 < \ldots$. We set $Y_{0,t}^{\delta}(\cdot) := m_{r_n,u_n} \circ \ldots \circ m_{r_1,u_1}$ if $t \in [s_n, s_{n+1})$ and $Y_{0,t}^{\delta}(\cdot)$ to be the identity function if $t < s_1$. It is easy to see by induction that, for any $\delta \in (0,1)$, the flow such defined is the unique solution of (A.106).

Lemma A.4. We have almost surely that, for all T > 0, $\delta \in (0,1)$, and $a,b \in [0,1]$,

$$d_T(Y_{0,\cdot}^{\delta}(a), Y_{0,\cdot}^{\delta}(b)) \le e^{S_T}|a-b|.$$
 (A.107)

Proof. Each function $m_{r,u}(\cdot)$ is Lipschitz continuous with Lipschitz constant 1/(1-r). We thus get that, for $t \in [s_n, s_{n+1})$ (resp. for $t < s_1$), then $Y_{0,t}^{\delta}(\cdot)$ is Lipschitz continuous with Lipschitz constant $\prod_{k=1}^n 1/(1-r_k) \le e^{S_t}$ (resp. $1 \le e^{S_t}$). This proves (A.107).

In order to prove the existence of a unique stochastic flow satisfying (1.5), we first turn our attention to a simpler SDE. If there exist a flow $(Y_{0,t}(y), y \in [0,1], t \ge 0)$ satisfying (1.5), then a single trajectory $Y_{0,\cdot}(y)$ (also called *one-point motion*) is solution of the SDE

$$Y_t = y + \int_{(0,t]\times(0,1)^2} \left(m_{r,u}(Y_{s-}) - Y_{s-} \right) N(ds, dr, du), \ t \ge 0.$$
 (A.108)

We say that a process $(Y_t)_{t\geq 0}$ satisfying (A.108) almost surely for all $t\geq 0$ is a solution of (A.108) with initial value y. We call it a *strong solution* if it is càd-làg and adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ (defined a little before (1.4)). We say that pathwise uniqueness holds if any two solutions with same initial values are almost surely equal for all $t\geq 0$. The following lemma lays out some facts about SDE (A.108) and follows from [20, Prop. B.5].

Lemma A.5. For any $y \in [0,1]$, there exists a pathwise unique strong solution $(Y_t)_{t\geq 0}$ of (A.108) with initial value y. Moreover it satisfies $Y_t \in [0,1]$ for all $t \geq 0$. If $0 \leq y_1 \leq y_2 \leq 1$ and Y^1 and Y^2 are the solutions of (A.108) with initial values y_1 and y_2 respectively, then $\mathbb{P}(Y_t^1 \leq Y_t^2)$ for all $t \geq 0$ = 1.

The following lemma allows to control approximations of Y by Y^{δ} .

Lemma A.6. Let $y \in [0,1]$ and $(Y_t)_{t\geq 0}$ be the unique strong solution of (A.108) with initial value y. Let also $Y_t^{\delta} := Y_{0,t}^{\delta}(y)$. For any T > 0, $\delta \in (0,1/2)$, and $\rho \in [1/2,1)$, we have

$$\mathbb{E}\left[d_T(Y, Y^{\delta})\mathbb{1}_{N((0,T]\times(\rho,1)\times(0,1))=0}\right] \le C_T(\rho) \times K(\delta), \tag{A.109}$$

where we have set $K(\delta) := \int_{(0,\delta]} (1-r)^{-1} r^{-1} \Lambda(\mathrm{d}r)$ and $C_T(\rho) := \frac{e^{4T \int_{(0,\rho]} (1-r)^{-2} r^{-1} \Lambda(\mathrm{d}r)}}{4 \int_{(0,\rho]} (1-r)^{-2} r^{-1} \Lambda(\mathrm{d}r)}$.

Proof. We fix y, T, δ, ρ as in the statement of the lemma. Using (A.108) and (A.106) we get

$$Y_{t} - Y_{t}^{\delta} = \int_{(0,t] \times (0,\delta] \times (0,1)} \left(\mathbf{m}_{r,u}(Y_{s-}) - Y_{s-} \right) N(\mathrm{d}s, \mathrm{d}r, \mathrm{d}u)$$

$$+ \int_{(0,t] \times (\delta,1) \times (0,1)} \left(\mathbf{m}_{r,u}(Y_{s-}) - \mathbf{m}_{r,u}(Y_{s-}^{\delta}) - Y_{s-} + Y_{s-}^{\delta} \right) N(\mathrm{d}s, \mathrm{d}r, \mathrm{d}u),$$

for $t \in [0, T]$. We get that almost surely,

$$\sup_{t \in [0,T]} |Y_t - Y_t^{\delta}| \le \int_{(0,T] \times (0,\delta] \times (0,1)} |\mathbf{m}_{r,u}(Y_{s-}) - Y_{s-}| N(\mathrm{d}s, \mathrm{d}r, \mathrm{d}u)
+ \int_{(0,T] \times (\delta,1) \times (0,1)} |\mathbf{m}_{r,u}(Y_{s-}) - \mathbf{m}_{r,u}(Y_{s-}^{\delta}) - Y_{s-} + Y_{s-}^{\delta} |N(\mathrm{d}s, \mathrm{d}r, \mathrm{d}u).$$
(A.110)

Let $(\tilde{Y}_t)_{t\geq 0}$ and $(\tilde{Y}_t^{\delta})_{t\geq 0}$ be defined as $(Y_t)_{t\geq 0}$ and $(Y_t^{\delta})_{t\geq 0}$ but where $(0,1)^2$ and $(\delta,1)\times(0,1)$ from (A.108) and (A.106) are replaced by respectively $(0,\rho)\times(0,1)$ and $(\delta,\rho)\times(0,1)$. We note that, for any measurable function $f:[0,1]^2\times(0,1)^2\to\mathbb{R}_+$ we have

$$\mathbb{E}\left[\mathbb{1}_{N((0,T]\times(\rho,1)\times(0,1))=0}\int_{(0,T]\times(\delta,1)\times(0,1)}f(Y_{s-},Y_{s-}^{\delta},r,u)N(\mathrm{d}s,\mathrm{d}r,\mathrm{d}u)\right]$$

$$=\mathbb{E}\left[\mathbb{1}_{N((0,T]\times(\rho,1)\times(0,1))=0}\int_{(0,T]\times(\delta,\rho]\times(0,1)}f(\tilde{Y}_{s-},\tilde{Y}_{s-}^{\delta},r,u)N(\mathrm{d}s,\mathrm{d}r,\mathrm{d}u)\right]$$

$$=e^{-T\int_{(\rho,1)}r^{-2}\Lambda(\mathrm{d}r)}\int_{0}^{T}\mathbb{E}\left[\int_{(\delta,\rho]\times(0,1)}f(\tilde{Y}_{s},\tilde{Y}_{s}^{\delta},r,u)r^{-2}\Lambda(\mathrm{d}r)du\right]ds$$

$$\leq \int_{0}^{T}\mathbb{E}\left[\mathbb{1}_{N((0,s]\times(\rho,1)\times(0,1))=0}\int_{(\delta,\rho]\times(0,1)}f(Y_{s},Y_{s}^{\delta},r,u)r^{-2}\Lambda(\mathrm{d}r)du\right]ds,$$
(A.111)

where we have used that $\int_{(0,T]\times(\delta,\rho]\times(0,1)} f(\tilde{Y}_{0,s-}(a),\tilde{Y}_{0,s-}^{\delta}(b),r,u)N(\mathrm{d} s,\mathrm{d} r,\mathrm{d} u)$ is a measurable function of $N((0,T]\times(\delta,\rho]\times(0,1)\cap\cdot)$, which is independent of $N((0,T]\times(\rho,1)\times(0,1)\cap\cdot)$, and the compensation formula. Using (A.101) and (A.102) from Lemma A.1 we get that,

$$\int_{(0,\delta]\times(0,1)} |\mathbf{m}_{r,u}(Y_s) - Y_s| \, r^{-2} \Lambda(\mathrm{d}r) du \le \int_{(0,\delta]} \frac{\Lambda(\mathrm{d}r)}{(1-r)r}, \tag{A.112}$$

$$\int_{(\delta,\rho]\times(0,1)} \left| \mathbf{m}_{r,u}(Y_s) - \mathbf{m}_{r,u}(Y_s^{\delta}) - Y_s + Y_s^{\delta} \right| r^{-2} \Lambda(\mathrm{d}r) du \le 4 \left(\int_{(\delta,\rho]} \frac{\Lambda(\mathrm{d}r)}{(1-r)^2 r} \right) |Y_s - Y_s^{\delta}|. \tag{A.113}$$

Multiplying each term in (A.110) by $\mathbb{1}_{N((0,T]\times(\rho,1)\times(0,1))=0}$, taking the expectation and using (A.111) (and the compensation formula for the term $\int_{(0,T]\times(0,\delta]\times(0,1)}\dots$) and (A.112)-(A.113), we get that the left-hand side of (A.109) is smaller than

$$T \int_{(0,\delta]} \frac{\Lambda(\mathrm{d}r)}{(1-r)r} + 4 \left(\int_{(0,\rho]} \frac{\Lambda(\mathrm{d}r)}{(1-r)^2 r} \right) \times \int_0^T \mathbb{E} \left[\left(\sup_{t \in [0,s]} |Y_t - Y_t^{\delta}| \right) \mathbb{1}_{N((0,s] \times (\rho,1) \times (0,1)) = 0} \right] ds.$$

We then get (A.109) using Gronwall's lemma.

Recall that $Q := [0,1] \cap \mathbb{Q}$. By Lemma A.5 a flow $(Y_{0,t}(y), y \in Q, t \geq 0)$ can be defined which satisfies (1.5) (with "for all $y \in [0,1]$ " replaced by "for all $y \in Q$ "), is càd-làg in t, is non-decreasing in y, and $Y_{0,t}(0) = 0$, $Y_{0,t}(1) = 1$. In order to extend this flow to $[0,1] \times [0,\infty)$,

we need the following lemma that builds on Lemma A.6 and shows that the flow of Y can be approximated by Y^{δ} .

Lemma A.7. There is a decreasing sequence $(\delta_n)_{n\geq 1}$ in (0,1/2) such that for any T>0 we have almost surely that for all $y\in Q$, $d_T(Y_{0,\cdot}(y),Y_{0,\cdot}^{\delta_n}(y))\to 0$ as $n\to\infty$.

Proof. By (1.2) and the definition of $K(\delta)$ in Lemma A.6 we have $K(\delta) \to 0$ as $\delta \to 0$. We can thus choose a decreasing sequence $(\delta_n)_{n\geq 1}$ such that for any $n\geq 1$ we have $K(\delta_n)\leq 2^{-n}$. Fix T>0 and $y\in Q$. Applying Lemma A.6 we get that for any $\rho\in[1/2,1)\cap\mathbb{Q}, n\geq 1$,

$$\mathbb{E}\left[d_T(Y_{0,\cdot}(y), Y_{0,\cdot}^{\delta_n}(y))\mathbb{1}_{N((0,T]\times(\rho,1)\times(0,1))=0}\right] \leq 2^{-n}C_T(\rho).$$

Combining with Markov inequality we get that for any $\rho \in [1/2, 1) \cap \mathbb{Q}$, $\epsilon \in (0, 1) \cap \mathbb{Q}$, and $n \geq 1$,

$$\mathbb{P}\left(d_T(Y_{0,\cdot}(y), Y_{0,\cdot}^{\delta_n}(y)) > \epsilon, N((0,T] \times (\rho,1) \times (0,1)) = 0\right) \le 2^{-n} C_T(\rho) / \epsilon.$$

By the Borel-Cantelli lemma we get that, on $\{N((0,T]\times(\rho,1)\times(0,1))=0\}$, we have almost surely $d_T(Y_{0,\cdot}(y),Y_{0,\cdot}^{\delta_n}(y))\leq\epsilon$ for all large n. Since this is true for all $\rho\in[1/2,1)\cap\mathbb{Q}$ and $\epsilon\in(0,1)\cap\mathbb{Q}$, we get that $d_T(Y_{0,\cdot}(y),Y_{0,\cdot}^{\delta_n}(y))$ converges almost surely to 0 as n goes to infinity. Since Q is countable the result follows.

Proof of Proposition 1.1. If two such flows exist, they almost surely coincide on all $(y,t) \in Q \times [0,\infty)$ by Lemma A.5 and then on all $(y,t) \in [0,1] \times [0,\infty)$ by continuity with respect to y. This proves uniqueness. We now prove existence. We consider the flow $(Y_{0,t}(y), y \in Q, t \ge 0)$ as defined before Lemma A.7. By that lemma, there is a decreasing sequence $(\delta_n)_{n\ge 1}$ in (0,1) such that for any T>0 we have almost surely that for all $y\in Q$, $d_T(Y_0,(y),Y_0^{\delta_n}(y))$ converges to 0 as n goes to infinity. Combining with Lemma A.4 we get that, almost surely,

$$\forall T \in (0, \infty) \cap \mathbb{Q}, \forall a, b \in Q, \ d_T(Y_{0, \cdot}(a), Y_{0, \cdot}(b)) \le e^{S_T} |a - b|. \tag{A.114}$$

Therefore, there is a probability one event on which the function $y \mapsto Y_{0,\cdot}(y)$ is uniformly continuous from Q to $(D([0,\infty)), d_{\infty})$ which is complete. For any fixed realization of this event, we can thus define $Y_{0,\cdot}(y)$ for all $y \in [0,1]$ by extension and obtain a flow satisfying (ii),(iii). To show that it satisfies (i) we consider $y \in [0,1]$ and $(y_n)_{n\geq 1}$ in Q converging to y. Then, for each $n \geq 1$, $Y_{0,\cdot}(y_n)$ satisfies (A.108), i.e.

$$Y_{0,t}(y_n) = y_n + \int_{(0,t]\times(0,1)^2} \left(m_{r,u}(Y_{0,s-}(y_n)) - Y_{0,s-}(y_n) \right) N(ds, dr, du), \ t \ge 0.$$
 (A.115)

Since $Y_{0,\cdot}(y_n)$ converges to $Y_{0,\cdot}(y)$ in $(D([0,\infty)), d_\infty)$, the left-hand side of (A.115) converges to $Y_{0,t}(y_n)$ while the integrand in the right-hand side converges to $m_{r,u}(Y_{0,s-}(y)) - Y_{0,s-}(y)$. By (A.101) from Lemma A.1, the absolute value of the integrand is bounded by r/(1-r) and, by (1.2), we have $\int_{(0,t]\times(0,1)^2} \frac{r}{1-r} N(ds,dr,du) < \infty$ (after intersection with another probability one event). Therefore, by dominated convergence, the right-hand side of (A.115) converges to the right-hand side of (1.5). We get that the flow we just defined satisfies (i).

Remark A.8. The above proof shows that, for the flow $(Y_{0,t}(y), y \in [0,1], t \ge 0)$ from Proposition 1.1, we have almost surely that $y \mapsto Y_{0,\cdot}(y)$ is continuous from [0,1] to $(D([0,\infty)), d_\infty)$.

The following proposition allows to identify (in law) the flow $(Y_{0,t}(y), y \in [0,1], t \geq 0)$ with the flow of inverses of the Λ -process (see (1.4)).

Proposition A.9. For any $p \ge 1$ and $y_1, \dots, y_p \in [0, 1]$ with $y_1 \le \dots \le y_p$, $(Y_{0,t}(y_1), \dots, Y_{0,t}(y_p))_{t \ge 0}$ is solution to the martingale problem from [15, Thm. 5] and, under the assumption (1.2), this martingale problem is well posed.

Proof. By (1.5), the \mathbb{R}^p -valued process $(Y_{0,t}(y_1), \cdots, Y_{0,t}(y_p))_{t>0}$ satisfies the SDE

$$Z_t^j = y_j + \int_{(0,t]\times(0,1)^2} \left(\mathbf{m}_{r,u}(Z_{s-}^j) - Z_{s-}^j \right) N(ds, dr, du), \ j \in \{1,\dots,p\}, t \ge 0.$$
 (A.116)

Applying Itô's formula (see e.g. [47, Thm. II.5.1]) we get that this process solves the martingale problem. Note that " $(Z_t^1, \ldots, Z_t^p)_{t\geq 0}$ solves the SDE (A.116) with initial value (y_1, \ldots, y_p) " is equivalent to "for each $i \in \{1, \ldots, p\}$, $(Z_t^i)_{t\geq 0}$ solves (A.108) with initial value y_i . We deduce from Lemma A.5 that there exists a pathwise unique strong solution of (A.116) with initial value (y_1, \ldots, y_p) . Moreover it satisfies $(Z_t^1, \ldots, Z_t^p) \in [0, 1]^p$ for all $t \geq 0$. By [54, Thm. 2.3], every solution to the martingale problem is a weak solution to the SDE (A.116). Since pathwise uniqueness implies weak uniqueness (see e.g. [4, Thm. 1]), the martingale problem is well posed. This completes the proof.

Appendix B.

In this appendix we define the flows $(Y_{s,t}(y), y \in [0,1], t \geq s)$ for jumping times s and show the composition property. We assume that (1.2) holds true. Our argument is similar to the one from [21, Thm. 4.5]. If τ is a stopping time with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ we consider stochastic flows $(Y_{\tau,t}(y), y \in [0,1], t \geq \tau)$ that satisfies

$$Y_{\tau,t}(y) = y + \int_{(\tau,t]\times(0,1)^2} \left(\mathbf{m}_{r,u}(Y_{\tau,s-}(y)) - Y_{\tau,s-}(y) \right) N(ds, dr, du), \tag{B.117}$$

almost surely for all $y \in [0, 1]$ and $t \ge \tau$. This flow is well-defined almost surely by Proposition 1.1 applied to the shifted measure $N(\tau + ds, dr, du)$. Moreover the flow $(Y_{\tau,\tau+t}(y), y \in [0, 1], t \ge 0)$ is equal in law to $(Y_{0,t}(y), y \in [0, 1], t \ge 0)$ from Proposition 1.1 and independent of \mathcal{F}_{τ} .

For $\eta \in (0,1/2)$, let $(s_i^\eta, r_i^\eta, u_i^\eta)_{i \geq 1}$ be the enumeration of $\{(s,r,u) \in N, r > \eta\}$ such that $s_1^\eta < s_2^\eta < \dots$ and for convenience we set $s_0^\eta := 0$. Note that for any $j \geq 1$, s_j^η is a stopping time. We can thus define the countable collection of flows $\{(Y_{s_j^\eta,t}(y),y \in [0,1],t \geq s_j^\eta),\eta \in (0,1/2) \cap \mathbb{Q}, j \geq 1\}$ on the same probability space.

Now let $\eta \in (0,1/2) \cap \mathbb{Q}$ and $0 \leq i < j$ and let us define $Z_{s_i^{\eta},t}^{\eta,i,j}(y) := Y_{s_i^{\eta},t}(\cdot)$ for $t \in [s_i^{\eta},s_j^{\eta})$ and $Z_{s_i^{\eta},t}^{\eta,i,j}(y) := Y_{s_i^{\eta},t}(Y_{s_i^{\eta},s_j^{\eta}}(\cdot))$ for $t \in [s_j^{\eta},\infty)$. It is not difficult to see that this flow satisfies (B.117) with $\tau = s_i^{\eta}$ and properties (ii) and (iii) from Proposition 1.1 (for $t \geq s_i^{\eta}$ instead of $t \geq 0$). By uniqueness from Proposition 1.1 we get that $Z_{s_i^{\eta},\cdot}^{\eta,i,j}(\cdot) = Y_{s_i^{\eta},\cdot}(\cdot)$ so we get the composition property $Y_{s_i^{\eta},t}(\cdot) = Y_{s_i^{\eta},t}(Y_{s_i^{\eta},s_i^{\eta}}(\cdot))$ for all $t \geq s_j^{\eta}$.

Since, for any pair of times $s_1, s_2 \in J_N \cup \{0\}$, we have $s_1 = s_i^{\eta}$ and $s_2 = s_j^{\eta}$ for some $\eta \in (0, 1/2) \cap \mathbb{Q}$ and $i, j \geq 1$, the above discussion results in the following proposition.

Proposition B.1. One can define a countable family of stochastic flows $\{(Y_{s,t}(y), y \in [0,1], t \geq s), s \in J_N \cup \{0\}\}$ such that, almost surely, each flow $(Y_{s,t}(y), y \in [0,1], t \geq s)$ from this family satisfies the following properties:

- (i) (B.117) holds with $\tau = s$, for all $y \in [0, 1]$ and $t \geq s$;
- (ii) for every $y \in [0,1]$, the trajectory $t \mapsto Y_{s,t}(y)$ is càd-làg;
- (iii) for every $t \ge s$, the map $y \mapsto Y_{s,t}(y)$ is non-decreasing and continuous, and $Y_{s,t}(0) = 0$, $Y_{s,t}(1) = 1$.

Moreover, almost surely, for any $s_1, s_2 \in J_N \cup \{0\}$ with $s_1 < s_2$, (1.6) holds true.

Appendix C.

C.1. Lower bound for $N(\Lambda)$: Proof of Remark 1.14. It has been justified after (1.17) that $N(\Lambda) > 2$. Let us denote $M(\Lambda) := \sqrt{H(\Lambda)/\lambda_2(\Lambda)}$. For $k \geq 3$, using the definition of $\lambda_k(\Lambda)$ and that $(1-r)^{k-1} > 1 - (k-1)r$ for all r > 0 we get

$$\lambda_k(\Lambda) = \int_{(0,1)} [1 - (1-r)^{k-1} (1 + (k-1)r)] r^{-2} \Lambda(dr)$$

$$< \int_{(0,1)} [1 - (1 - (k-1)r)(1 + (k-1)r)] r^{-2} \Lambda(dr) = (k-1)^2 \lambda_2(\Lambda).$$
(C.118)

Set $K := \lfloor 1 + M(\Lambda) \rfloor$. Then $(K - 1)^2 \lambda_2(\Lambda) \leq H(\Lambda)$ so, by (C.118), $\lambda_K(\Lambda) < H(\Lambda)$. By the definition of $N(\Lambda)$ in (1.17) the later implies $K < N(\Lambda)$. Since $K > M(\Lambda)$ we get $N(\Lambda) > M(\Lambda)$, which concludes the proof.

C.2. Case of $Beta(2-\alpha,\alpha)$ -coalescent: Proof of Remark 1.15. Using $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ we get $H(\Lambda_{2-\alpha,\alpha}) = B(1-\alpha,\alpha) = \Gamma(1-\alpha)\Gamma(\alpha)$. Re-writing $1-(1-r)^k$ as $r\sum_{j=0}^{k-1}(1-r)^j$ and using the definitions of $\lambda_k(\cdot)$ and $\Lambda_{2-\alpha,\alpha}$ we get

$$\lambda_k(\Lambda_{2-\alpha,\alpha}) = -kB(1-\alpha,k-1+\alpha) + \sum_{j=0}^{k-1} B(1-\alpha,j+\alpha)$$

$$= -k\frac{\Gamma(1-\alpha)\Gamma(k-1+\alpha)}{\Gamma(k)} + \sum_{j=0}^{k-1} \frac{\Gamma(1-\alpha)\Gamma(j+\alpha)}{\Gamma(j+1)}$$

$$= -k\frac{\Gamma(1-\alpha)\Gamma(k+\alpha)}{\Gamma(k)(\alpha+k-1)} + \frac{\Gamma(1-\alpha)\Gamma(k+\alpha)}{\alpha\Gamma(k)}$$

$$= \frac{\Gamma(1-\alpha)\Gamma(k+\alpha)(k-1)(1-\alpha)}{\Gamma(k)\alpha(\alpha+k-1)},$$

where we have used [18, Lem. A.1]. This completes the proof of (1.21). Combining this expression of $\lambda_k(\Lambda_{2-\alpha,\alpha})$ with Gautschi's inequality we get that for any $k \geq 3$,

$$\frac{2(1-\alpha)\Gamma(1-\alpha)}{\alpha(2+\alpha)}(k+\alpha-1)^{\alpha} \le \lambda_k(\Lambda_{2-\alpha,\alpha}) \le \frac{(1-\alpha)\Gamma(1-\alpha)}{\alpha}(k+\alpha)^{\alpha}.$$
 (C.119)

From (1.17) and the discussion after we have that $N(\Lambda_{2-\alpha,\alpha})$ is the smallest $k \geq 3$ such that $\lambda_k(\Lambda_{2-\alpha,\alpha}) \geq H(\Lambda_{2-\alpha,\alpha})$. Combining with $H(\Lambda_{2-\alpha,\alpha}) = \Gamma(1-\alpha)\Gamma(\alpha)$ and (C.119) we obtain (1.22).

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