Parameter estimation for the stochastic Burgers equation driven by white noise from local measurements

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Abstract

For one dimensional stochastic Burgers equation driven by space-time white noise we consider the problem of estimation of the diffusivity parameter in front of the second-order spatial derivative. Based on local observations in space, we study the estimator derived in [3] for linear stochastic heat equation that has also been used in [2] to cover large class of semilinear SPDEs and has been examined for the stochastic Burgers equation driven by trace class noise. We extend the achieved results by considering the space-time white noise case which has also relevant physical motivations. After we establish new regularity results for the solution, we are able to show that our proposed estimator is strongly consistent and asymptotically normal.

Keywords: Stochastic Burgers equation; space-time white noise; local measurements; diffusivity estimation; augmented MLE; central limit theorem.

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1 Introduction

We consider estimation of the diffusivity parameter $\vartheta>0$ in the stochastic Burgers equation driven by space-time white noise

$$\begin{cases}
dX(t) &= \vartheta \partial_{xx}^2 X(t) dt + \frac{1}{2} \partial_x \left(X^2(t) \right) dt + dW(t), \\
X(0) &= X_0, \\
X(t)|_{\partial \Lambda} &= 0, \quad 0 < t \le T.
\end{cases}$$
(1.1)

Here W is a cylindrical Wiener process on $L^2(\Lambda)$, where $\Lambda = (0,1) \subset \mathbb{R}$, dW(t)/dt is also referred to as space-time white noise. We consider Dirichlet boundary conditions and a smooth and deterministic initial condition X_0 .

Note that (1.1) is often written as

$$dX(t,x) = \vartheta \frac{\partial^2}{\partial x^2} X(t,x) dt + \frac{1}{2} \frac{\partial}{\partial x} X^2(t,x) dt + dW_t(x), \quad X(0,x) = X_0(x), \quad (1.2)$$

 $x \in (0,1), \ X(t,0) = X(t,1) = 0, \ t \in (0,T].$

The stochastic Burgers equation has an important role in fluid dynamics and has been studied by several authors. We mainly refer to the works [12] and [13], where the existence and uniqueness of the global solution as well as the existence of an invariant measure has been established, Chapter 14 from [14], which provides not only a self-contained overview, but also strong Feller and irreducibility properties of the corresponding transition semigroup, and [11], where moment estimates for the solution were established.

The estimation of the diffusivity (or the drift part) has become a standard inference problem for stochastic partial differential equations (SPDEs) in both parametric and nonparametric setting, but it is worth noting that parameter estimation for the stochastic Burgers equation has not been much investigated. In that regard, we may mention the work [29], where stochastic Burgers equation driven by a trace class noise fits into a broader setting of semilinear stochastic evolution equations.

In general, the classical methods mostly rely on the observations in the Fourier space over some time interval (with the number of modes $N \to \infty$ in so-called spectral approach, see, e.g. [20]) or on the ergodic properties of the solution (with $T \to \infty$, see, e.g. [26]) or on discrete sampling (see, e.g. [7] and [8]), however in this paper we continue to expand the methodology that is based on local measurements. This novel approach was first introduced in [3] and relies on a local observation in space of the solution against some small "observational window". That is represented by a function called kernel K that is scaled and localized around certain observation point $x_0 \in \Lambda$. The observational data then comes in the form of convolution $\langle X(t), K_{\delta, x_0} \rangle_{L^2(\Lambda)}$ and the asymptotics is studied for $\delta \to 0^+$.

The augmented maximum likelihood estimator (augmented MLE) of ϑ was introduced in [3] for linear SPDEs in nonparametric setting. Since then, it was used in the parametric setting to experimental data from cell biology (see [1] and note that such usage is the practical motivation of local measurements) and from the theoretical point of view, the estimator is remarkably flexible and robust to some misspecifications. For instance, [23] studied the augmented MLE and its variants in the case of stochastic heat equation driven by the multiplicative noise, while the authors in [2] focused on a case of semilinear SPDEs. Their work covers a wide range of nonlinearities (such as stochastic reaction-diffusion equations) assuming that the SPDE is driven by additive noise B dW(t), where B^* scales as the fractional operator $(-\partial_{xx}^2)^{-\gamma}$. However, the stochastic Burgers equation is still in some sense a "borderline case" of their presented theory. The augmented MLE is strongly consistent, but the analysis of the asymptotic normality must have been done separately and only in the case of a trace class noise: $(-\partial_{xx}^2)^{-\gamma} dW(t)$ for $\gamma > 1/4$ (that is the case when B is Hilbert-Schmidt). This is the starting point of our work.

We study the augmented MLE for stochastic Burgers equation that is driven by space-time white noise (i.e., the case when $\gamma = 0$). We believe that this case is challenging and from the theoretical point of view also very interesting (for physical motivations see [6], [9] and [25]; for numerical approximations see [17]).

Our results (and similar arguments as in [2]) can be further extended to cover the case $\gamma \in [0, 1/4]$ but, in fact, to cover all $\gamma \geq 0$.

At first, we consider the notion of mild solution to our equation (1.1) using so-called "splitting technique", where we split the solution to the linear part \bar{X} and the nonlinear part \tilde{X} . For the analysis of a (possible) asymptotic bias, we need to assert the regularities of both parts. The results on regularity of the stochastic convolution \bar{X} are basically known, but for the nonlinear part \tilde{X} , the results seem to be new and we formulate them in Propositions 2.1 and 4.6.

For the analysis of the error $\delta^{-1}\left(\hat{\vartheta}_{\delta}-\vartheta\right)$, we follow general ideas from [2] and its supplement, where many auxiliary lemmas (e.g., for scaling) and propositions do hold true even in our case $\gamma=0$. It must be noted though, that we use different proof techniques to adjust to our space-time white noise case and to clarify some points of the original paper; see also Remark 3.3. We believe that Lemma 4.11 is pivotal to our paper; in its proof we show a suitable representation for $\widetilde{X}(t,x_0)=\widetilde{X}(t)(x_0)$.

In conclusion, we find that the augmented MLE is unbiased, asymptotically normal and with the rate of convergence and the asymptotic variance that align to previous results from the theory of estimation using local measurements. The precise statement is formulated in Theorem 3.4.

Based on the asymptotic result, we can deduce (data-driven) confidence intervals for ϑ (that also match the developed theory). We mention that numerical simulations for stochastic Burgers equations (1.1) driven by space-time white noise were already presented in Section 4 of [2].

Our paper is organized as follows. The notation and the exact setting is introduced in Section 2, where we also discuss the notion of mild and weak solution. The results on the regularity of the nonlinear part \widetilde{X} can be also found here. In Section 3 we construct the estimator, revisit assumptions of the model and present the main asymptotic results. The proofs and auxiliary results are delegated to Section 4, where they are assorted by the topic.

2 The model

2.1 Notation

Consider $\Lambda = (0,1)$ and the space $L^2(\Lambda)$ equipped with the usual L^2 -norm $\|\cdot\| := \|\cdot\|_{L^2(\Lambda)}$ and the scalar product $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{L^2(\Lambda)}$. Even though the set Λ is only one-dimensional, we use the standard Laplace operator notation

$$\Delta u = \partial_{xx}^2 u = u''$$

for a function u which is regular enough. We denote $(\lambda_n, e_n)_{n=1}^{\infty}$ the eigensystem of the positive, self-adjoint operator $-\Delta$ on Λ with Dirichlet boundary

conditions. That is

$$\lambda_n = \pi^2 n^2$$
, $e_n(x) = \sqrt{2}\sin(nx\pi)$, $x \in [0, 1]$.

To describe higher regularities, we consider for $s \in \mathbb{R}$ the fractional Laplacians $(-\Delta)^{s/2}$ and denote their domains by $H^s = H^s(\Lambda)$ with the norm $\|\cdot\|_s := \|\cdot\|_{H^s} := \|(-\Delta)^{s/2} \cdot \|_{L^2(\Lambda)}$. The Hilbert spaces

$$H^{s} = \operatorname{Dom}\left((-\Delta)^{s/2}\right) = \left\{ u \in L^{2}(\Lambda); \sum_{n=1}^{\infty} \lambda_{n}^{s} u_{n}^{2} = \sum_{n=1}^{\infty} \lambda_{n}^{s} \left\langle u, e_{n} \right\rangle < \infty \right\}$$

are equipped with the scalar product $\langle u,v\rangle_{H^s}=\sum_{n=1}^\infty \lambda_n^s u_n v_n$ for $u,v\in H^s$. We also set $H:=L^2(\Lambda)$. Recall that $H^1=H^1_0$ with equivalence of norms, where H^1_0 denotes the space of all $f\in H$ with weak derivative $f'\in H$ and such that f(x)=0 for $x\in\partial\Lambda$ (cf. [17]).

For $s \in (0,1)$ and $p \geq 2$ we consider also $W^{s,p}(\Lambda) := \{u \in L^p(\Lambda) : ||u||_{s,p} < \infty\}$, where $||\cdot||_{W^{s,p}(\Lambda)} := ||(-\Delta)^{s/2} \cdot ||_{L^p(\Lambda)}$. These are fractional Sobolev spaces associated to Δ defined as Bessel potentials spaces.

The space $C(\bar{\Lambda})$ of all continuous functions on $\bar{\Lambda}$ is equipped with the supremum norm $\|\cdot\|_{\infty}$. The spaces $C^{k,\alpha}(\Lambda)$ for $k \geq 0$ integer and $0 < \alpha \leq 1$ are classical Hölder spaces equipped with usual norms.

We fix a constant $\vartheta > 0$, that is the parameter of interest, and denote by $(S_{\vartheta}(t), t > 0)$ the strongly continuous semigroup on $L^2(\Lambda)$ generated by $\vartheta \Delta$.

Throughout this work we fix a finite time horizon T>0 and let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis (satisfying the usual hypothesis) with a cylindrical Wiener process W on $L^2(\Lambda)$ (cf. [15]). The process W(t) is formally given by " $W(t) = \sum_{k\geq 1} \beta_k(t) e_k$ " where (β_k) are independent one dimensional Wiener processes adapted to the previous filtration; we are using the previous orthonormal basis (e_k) in $L^2(\Lambda)$.

Throughout, all equalities and inequalities, unless otherwise mentioned, will be understood in the \mathbb{P} -a.s. sense. We write $\mathbb{Q}_+ := \mathbb{Q} \cap (0, \infty)$. By $A_\delta \lesssim B_\delta$ we mean that there exists some constant C>0 such that $A_\delta \leq CB_\delta$ for all values δ under consideration. Here, we work with $\delta \in (0,1)$ or with the convergence $\delta \to 0$. Convergence in probability and convergence in distribution are denoted by $\stackrel{\mathbb{P}}{\to}$ and $\stackrel{d}{\to}$, respectively. The symbol $A_\delta = O_{\mathbb{P}}(B_\delta)$ for random variables A_δ, B_δ means that A_δ/B_δ is tight, that is, $\sup_\delta \mathbb{P}(|A_\delta| > C|B_\delta|) \to 0$ as $C \to \infty$. The notation $A_\delta = o_{\mathbb{P}}(B_\delta)$ stands for $A_\delta/B_\delta \stackrel{\mathbb{P}}{\to} 0$ as $\delta \to 0$.

2.2 The stochastic Burgers equation

We study the stochastic Burgers equation (1.1) driven by additive space-time white noise. The initial value $X_0 \in H^{3/2}$ is supposed to be deterministic.

By Theorem 14.2.4 in [14] (see also [12], [13] and Remark 2.3 below), there exists a unique mild solution to the equation (1.1).

Namely, for any T > 0, there exists an $L^2(\Lambda)$ -valued adapted process $X = (X(t), 0 \le t \le T)$ with continuous paths such that

$$X(t) = \underbrace{S_{\vartheta}(t)X_0 + \frac{1}{2} \int_0^t S_{\vartheta}(t-s)\partial_x \left(X^2(s)\right) ds}_{=:\widetilde{X}(t)} + \underbrace{\int_0^t S_{\vartheta}(t-s) dW(s)}_{=:\widetilde{X}(t)}, \quad (2.1)$$

for any $t \in [0,T]$, \mathbb{P} -a.s. In what follows, we use the 'splitting' of the solution into a linear part $(\bar{X}(t), 0 \le t \le T)$ and a nonlinear part $(\bar{X}(t), 0 \le t \le T)$ as indicated. The $L^2(\Lambda)$ -valued process $\bar{X} = (\bar{X}(t), 0 \le t \le T)$ is the so-called stochastic convolution and it is the unique mild solution to the following linear equation

$$d\bar{X}(t) = \vartheta \Delta \bar{X}(t) dt + dW(t), \quad 0 < t \le T, \quad \bar{X}(0) = 0. \tag{2.2}$$

(cf. Chapter 5 in [15]). On the other hand, the nonlinear part $\widetilde{X} = X - \overline{X}$ solves formally the PDE with random coefficients given by

$$\frac{d}{dt}\widetilde{X}(t) = \vartheta \Delta \widetilde{X}(t) + \frac{1}{2}\partial_x \left((\bar{X}(t) + \widetilde{X}(t))^2 \right), \quad 0 < t \le T, \quad \widetilde{X}(0) = X_0. \quad (2.3)$$

In the estimation procedure, we exploit the fact that \widetilde{X} has higher regularity than \overline{X} . While the results on the regularity of \overline{X} are known, the regularity of the nonlinear part \widetilde{X} seems to be new in the present space-time white noise case. We summarize them in the following proposition.

Proposition 2.1. For any $X_0 \in H^{3/2}$ there exists a pathwise unique mild solution X to the equation (1.1) that satisfies $X \in C([0,T];H^s)$, \mathbb{P} -a.s., for any $s \in [0,1/2)$. The nonlinear part \widetilde{X} satisfies $\widetilde{X} \in C([0,T];H^s)$, \mathbb{P} -a.s., for any $s \in [0,3/2)$.

We also need to consider the notion of weak solution. In that regard, we formulate the following lemma.

Lemma 2.2. The mild solution X(t) verifies, for any $z \in H_0^1 \cap H^2$, \mathbb{P} -a.s., for any $t \in [0, T]$,

$$\langle z, X(t) \rangle = \langle z, X_0 \rangle + \vartheta \int_0^t \langle \Delta z, X(s) \rangle \ ds - \frac{1}{2} \int_0^t \langle \partial_x z, X^2(s) \rangle \ ds + \langle z, W(t) \rangle.$$
(2.4)

Proofs of Proposition 2.1 and Lemma 2.2 are deferred to Section 4.

Remark 2.3. We point out that well-posedness for the Burgers equation (2.1) is treated differently in the papers [12] (see also [14]) and [13]. Let us briefly review the approach of [13] in our case of additive noise; this will be also useful in the proof of Lemma 4.11.

In [13] they consider, for any $n \geq 1$, the projection $\pi_n : H = L^2(\Lambda) \rightarrow B(0,n)$, with $B(0,n) = \{f \in H : ||f|| = ||f||_H \leq n\}$ (here $\pi_n(f) = f$ if

 $f \in B(0,n)$ and $\pi_n(f) = \frac{nf}{\|f\|}$ if $\|f\| > n$). They show that one can solve for any $n \ge 1$, $X_0 \in H$, T > 0, the equation

$$X_n(t) = S_{\vartheta}(t)X_0 + \frac{1}{2} \int_0^t S_{\vartheta}(t-s)\partial_x \left([\pi_n(X_n(s))]^2 \right) ds + \bar{X}(t), \quad t \in [0,T],$$
(2.5)

 \mathbb{P} -a.s.. To this purpose they first prove that (2.5) has a unique solution on a small time interval $[0,R],\ R\leq T$, by the contraction principle, using the space $Z_R;\ Z_R$ is the Banach space of all continuous adapted H-valued processes Y such that $\|Y\|_{Z_R}^2 = \mathbb{E}[\sup_{t\in[0,R]}\|Y(t)\|^2] < \infty$.

They also require that $R \leq (Cn)^{-4}$ where C is a deterministic constant independent of n. After a finite numbers of steps they prove that the solution X_n exists and it is unique on [0,T] (for any T>0). Clearly, this solution is an adapted stochastic process. Indeed we can repeatedly apply the contraction principle on a finite number of closed time intervals $I_{R,k} \subset [0,T]$, k=1,...,N=N(n), with length less or equal than R (clearly, $\bigcup_{k=1}^N I_{R,k} = [0,T]$). Thus on each $I_{R,k}$ we have that X_n can be obtained, \mathbb{P} -a.s., as limit on $C(I_{R,k};H)$ of an approximating sequence of adapted stochastic processes.

Solutions of (2.5) can be defined for all $t \ge 0$. In [13] they also define the usual stopping times:

$$\tau_n = \inf\{t \ge 0 : ||X_n(t)|| \ge n\}, \quad n \ge 1.$$
(2.6)

They prove that $\tau_n \to \infty$, \mathbb{P} -a.s.. Since $X_n(t) = X_m(t)$, $m \ge n$, $t \le \tau_n$ one can set $X(t) = X_n(t)$, $t \le \tau_n$, and obtain a mild solution to the initial Burgers equation.

2.3 The observation scheme

As motivated in [2], [3], [23], we observe the solution process $(X(t,x), t \in [0,T], x \in \Lambda)$ only locally in space around some point $x_0 \in \Lambda$. That point x_0 as well as the terminal time T > 0 remain fixed.

More precisely, the observations are given by a spatial convolution of the solution process with a kernel K_{δ,x_0} , localising at x_0 as the resolution δ tends to zero. This kernel might for instance model the *point spread function* in microscopy.

For $z \in L^2(\mathbb{R})$ and $\delta \in (0,1]$ introduce the scalings

$$\Lambda_{\delta,x_0} := \delta^{-1} (\Lambda - x_0) = \{ \delta^{-1} (x - x_0) : x \in \Lambda \},$$

$$z_{\delta,x_0}(x) := \delta^{-1/2} z \left(\delta^{-1} (x - x_0) \right), \quad x \in \mathbb{R},$$

and we also set $\Lambda_{0,x_0} := \mathbb{R}$. For $\delta \in (0,1]$, we denote Δ_{δ,x_0} the Laplace operator on $L^2(\Lambda_{\delta,x_0})$.

Throughout this paper, $K \in H^2(\mathbb{R})$ stands for a fixed function of compact support in Λ_{δ,x_0} , for some $0 < \delta \leq 1$, called kernel (here $H^2(\mathbb{R})$ is the usual L^2 -Sobolev space on \mathbb{R}). The compact support ensures that K_{δ,x_0} is localising

around x_0 as $\delta \to 0$ and that $K_{\delta,x_0} \in H^2 = H^2(\Lambda)$. The scaling with $\delta^{-1/2}$ simplifies calculations due to $||K_{\delta,x_0}|| = ||K||_{L^2(\mathbb{R})}$, while the proposed estimator is invariant with respect to kernel scaling.

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Local measurements of (1.1) at the point x_0 with resolution level $\delta \in (0,1)$ are described by the real-valued processes $(X_{\delta,x_0}(t), 0 \leq t \leq T)$ and $(X_{\delta,x_0}^{\Delta}(t), 0 \leq t \leq T)$ given by

$$X_{\delta,x_0}(t) = \langle X(t), K_{\delta,x_0} \rangle, \qquad (2.7)$$

$$X_{\delta,x_0}^{\Delta}(t) = \langle X(t), \Delta K_{\delta,x_0} \rangle. \tag{2.8}$$

These two processes are the data for our estimation procedure. In fact, since $X_{\delta,x_0}^{\Delta}(t) = \Delta X_{\delta,\cdot}(t)|_{x=x_0}$ by convolution, $X_{\delta,x_0}^{\Delta}(t)$ can be computed by observing $X_{\delta,x}(t)$ for x in a neighborhood of x_0 .

3 Estimation method and main results

3.1 The estimator

We use the augmented maximum likelihood estimator $\hat{\vartheta}_{\delta}$ of ϑ introduced in [3]. This (nonparametric) estimator was derived for a linear stochastic heat equation with additive space-time white noise, but [2] studied it in an abstract nonlinear (and parametric) setting that also covers the stochastic Burgers equation with trace class noise.

Definition 3.1. The augmented maximum likelihood estimator (augmented MLE) $\hat{\vartheta}_{\delta}$ of the parameter $\vartheta > 0$ is defined as

$$\hat{\vartheta}_{\delta} = \frac{\int_{0}^{T} X_{\delta, x_{0}}^{\Delta}(t) dX_{\delta, x_{0}}(t)}{\int_{0}^{T} (X_{\delta, x_{0}}(t))^{2} dt}.$$
(3.1)

As discussed in [3], this estimator is closely related to, but different from, the actual MLE, which cannot be computed in closed form, even for linear equations and constant ϑ .

From (2.4), the dynamics of X_{δ,x_0} is given by

$$dX_{\delta,x_0} = \vartheta X_{\delta,x_0}^{\Delta} dt + \frac{1}{2} \left\langle \partial_x \left(X^2(t) \right), K_{\delta,x_0} \right\rangle dt + \|K\|_{L^2(\mathbb{R})} d\bar{w}(t), \tag{3.2}$$

where $\bar{w}(t) := \langle W(t), K_{\delta,x_0} \rangle / \|K_{\delta,x_0}\|$ is a scalar Brownian motion. (Note that $\|K_{\delta,x_0}\| = \|K\|_{L^2(\mathbb{R})} > 0$.) Using (3.2) in the numerator on the right-hand side of (3.1), we obtain the fundamental error decomposition

$$\delta^{-1}(\hat{\vartheta}_{\delta} - \vartheta) = \delta^{-1}(\mathcal{I}_{\delta})^{-1}\mathcal{R}_{\delta} + \delta^{-1}(\mathcal{I}_{\delta})^{-1}\mathcal{M}_{\delta}, \tag{3.3}$$

where

$$\mathcal{I}_{\delta} := \|K\|_{L^{2}(\mathbb{R})}^{-2} \int_{0}^{T} \left(X_{\delta,x_{0}}^{\Delta}(t)\right)^{2} dt, \qquad \text{(observed Fisher information)}$$

$$\mathcal{R}_{\delta} := \|K\|_{L^{2}(\mathbb{R})}^{-2} \frac{1}{2} \int_{0}^{T} X_{\delta,x_{0}}^{\Delta}(t) \left\langle \partial_{x} \left(X^{2}(t)\right), K_{\delta,x_{0}} \right\rangle dt, \qquad \text{(nonlinear bias)}$$

$$\mathcal{M}_{\delta} := \|K\|_{L^{2}(\mathbb{R})}^{-1} \int_{0}^{T} X_{\delta,x_{0}}^{\Delta}(t) d\bar{w}(t). \qquad \text{(martingale part)}$$

The observed Fisher information \mathcal{I}_{δ} does not correspond to the Fisher information of the statistical model, although it plays a similar role. It also gives the quadratic variation of the martingale \mathcal{M}_{δ} in time.

To prove consistency, it is enough to show that $(\mathcal{I}_{\delta})^{-1}\mathcal{R}_{\delta}$ and $(\mathcal{I}_{\delta})^{-1}\mathcal{M}_{\delta}$ vanish, as $\delta \to 0$, and to prove asymptotic normality, we will show that $\delta^{-1}(\mathcal{I}_{\delta})^{-1}\mathcal{R}_{\delta} \to 0$, while $\delta^{-1}(\mathcal{I}_{\delta})^{-1}\mathcal{M}_{\delta}$ converges in distribution to a Gaussian random variable.

To analyze these terms, we use the 'splitting technique' of the solution (i.e., we write $X = \bar{X} + \widetilde{X}$ as above) and we study separately the linear parts

$$\begin{split} \bar{X}_{\delta,x_0}(t) &:= \left\langle \bar{X}(t), K_{\delta,x_0} \right\rangle, \\ \bar{X}_{\delta,x_0}^{\Delta}(t) &:= \left\langle \bar{X}(t), \Delta K_{\delta,x_0} \right\rangle \end{split}$$

and the corresponding nonlinear parts $\widetilde{X}_{\delta,x_0}(t)$, $\bar{X}_{\delta,x_0}^{\Delta}(t)$. (Note that since the stochastic convolution \bar{X} is centered Gaussian process, the \bar{X}_{δ,x_0} and $\bar{X}_{\delta,x_0}^{\Delta}$ are also centered Gaussian processes.) We also define the observed Fisher information $\bar{\mathcal{I}}_{\delta}$ that corresponds to the linear part

$$\bar{\mathcal{I}}_{\delta} := \|K\|_{L^{2}(\mathbb{R})}^{-2} \int_{0}^{T} \left(\bar{X}_{\delta, x_{0}}^{\Delta}(t)\right)^{2} dt.$$

For the analysis, we follow [2], where the stochastic Burgers equation with spatially 'smoothed' noise $(-\Delta)^{-\gamma} dW(t)$, $\gamma > 1/4$ is also discussed. (It is a borderline case of the presented more general nonlinear SPDE and as that, it is analyzed separately.) A lot of general and auxiliary results can be used also in our case $\gamma = 0$, so first, let us revisit the assumptions of the model from [2]:

Assumption B: Since we have $\gamma=0,\ B=I,\ B^*=I,\ B^*_{\delta,x_0}=I,$ this assumption trivially fulfilled.

Assumption K: Since we have $\gamma=0,\,\lceil\gamma\rceil=0,$ this assumption is fulfilled by taking $\widetilde{K}:=K.$

Assumption ND: Since we have $\gamma = 0$, $\lceil \gamma \rceil = 0$, $B_{\delta,x_0}^* = I$, it is required that $||K||_{L^2(\mathbb{R})} > 0$ and $\frac{1}{2}||K'||_{L^2(\mathbb{R})}^2 > 0$. We assume K with a compact support, so this non-degeneracy condition is fulfilled.

Assumption F: Since we analyze the nonlinear parts concretely (and not abstractly for rather general nonlinearity F), only the first part of this assumption

is needed, i.e., by Lemma (4.7)(i) below we show that the following holds

$$\int_0^T \left(\widetilde{X}_{\delta, x_0}(t) \right)^2 dt = o_{\mathbb{P}}(\delta^{-2}), \tag{3.4}$$

which is enough to obtain the point (iii) in Proposition 3 in [2].

With this, the only assumption that we pose on our model is about the kernel K and comes from Theorem 13 in [2]. It is rather technical and perhaps could be even weakened.

Assumption (L). There exists a function $L \in H^3(\mathbb{R})$ with compact support in Λ_{δ,x_0} , for some $\delta \leq 1$, such that $K = \partial_x L$.

Recall that $x_0 \in \Lambda = (0,1)$. We provide an easy example of kernel K which verifies Assumption (L).

Example 3.2. Consider a smooth compactly supported bump function

$$f(x) := \exp\left(-\frac{10}{1-x^2}\right) \mathbf{1}_{[-1,1]}(x), \quad x \in \mathbb{R}.$$

Since $[-1,1] \subset \Lambda_{\delta,x_0} = (-\delta^{-1}x_0,\delta^{-1}(1-x_0))$ for some $\delta \leq 1$ and $f \in H^3(\mathbb{R})$, we can take L := f and the kernel K for the statistical procedure as K := f'. Assumption (L) is satisfied. Note though, that different choices and setups are also possible.

In order to prove the asymptotic normality of our estimator, we follow [2] in the analysis of the martingale term $\delta^{-1}(\mathcal{I}_{\delta})^{-1}\mathcal{M}_{\delta}$ as $\delta \to 0$, but we deviate for the analysis of the bias term $\delta^{-1}(\mathcal{I}_{\delta})^{-1}\mathcal{R}_{\delta}$ mainly for two reasons. First, we need to adjust Lemma S.3 of the supplement of [2] to our case $\gamma = 0$, p = 2 for \widetilde{X} and the regularity of \widetilde{X} and \widetilde{X} (for the comparison, see Lemma 4.7 below.) Second, we want to clarify some points in the paper (see the comments below).

Remark 3.3. From the calculations in Lemma S.4 of the supplement of [2], it seems there should be a factor $\lambda_k^{-\gamma-1/2}$ (instead of just $\lambda_k^{-\gamma}$) in (S.4) and this is then too big for the sum in (S.3) to converge. (Such result comes from Lemma S.2(ii) and it is not quite clear if Lemma S.1 is able to tame it even in the case $\gamma > 1/4$.) Moreover, in the proof of Lemma S.8, it is not completely clear why the random variable $\widetilde{X}(t,0)$ (that is basically $\widetilde{X}(t,x_0)$) is \mathcal{G} -measurable. The isonormal Gaussian process $\widetilde{W}(z)$ is defined such that it does not correspond with the norm $\|\cdot\|_{\mathcal{H}}$ of the presented Hilbert space \mathcal{H} . (Basically, some sort of integration with respect to x is missing.) However, if the definition of $\widetilde{W}(z)$ is changed, the σ -field \mathcal{G} is changed and then one should verify that the random variable $\widetilde{X}(t,0)$ is \mathcal{G} -measurable.

We therefore develope new arguments that can overcome these difficulties. Our presented proofs of Lemmas 4.10 and 4.11 rely on a different technique and

cover not only our case $\gamma = 0$, but can be also adjusted to cover the case of any $\gamma > 0$.

Finally, when all convergences in (3.3) are assembled, we can formulate our main result in the following theorem.

Theorem 3.4. Grant Assumption (L). Then the augmented maximum likelihood estimator $\hat{\vartheta}_{\delta}$ is strongly consistent and asymptotically normal estimator of the parameter ϑ satisfying

$$\delta^{-1}\left(\hat{\vartheta}_{\delta} - \vartheta\right) \stackrel{d}{\to} N\left(0, \frac{2\vartheta \|K\|_{L^{2}(\mathbb{R})}^{2}}{T \|K'\|_{L^{2}(\mathbb{R})}^{2}}\right),\tag{3.5}$$

as $\delta \to 0$.

Proof. The proof is deferred to Section 4.

The asymptotic normality of the estimator allows us to prescribe asymptotic confidence intervals for the parameter ϑ .

Corollary 3.5. Let $\alpha \in (0,1)$. Based on the estimator $\hat{\vartheta}_{\delta}$, the confidence interval

$$I_{1-\alpha} = \left[\hat{\vartheta}_{\delta} - \mathcal{I}_{\delta}^{-1/2} q_{1-\alpha/2}, \hat{\vartheta}_{\delta} + \mathcal{I}_{\delta}^{-1/2} q_{1-\alpha/2} \right],$$

with the standard Gaussian $(1 - \alpha/2)$ -quantile $q_{1-\alpha/2}$ has asymptotic coverage $1 - \alpha$ as $\delta \to 0$ under the assumptions of Theorem 3.4.

Such results are satisfactory, because they match not only to the results in [2], but also to [3], [23] and other works that studied the asymptotic behaviour of the augmented MLE in the framework of local measurements.

4 Proofs

4.1 The algebra property of Sobolev spaces

This section will be also used in the proof of Proposition 2.1. It is well known that the space H^s is an algebra with respect to pointwise multiplication for s > 1/2 (in our case d = 1). For bounded functions it holds true for any $s \ge 0$. We believe the next result is known, but we have not found a precise reference. We provide a proof for the sake of completeness.

Proposition 4.1. For any $s \in [0,1]$, we have

$$||uv||_s \le ||u||_{\infty} ||v||_s + ||v||_{\infty} ||u||_s, \tag{4.1}$$

for any $u, v \in H^s \cap L^{\infty}(\Lambda)$.

Proof. We will use the theory of Dirichlet forms, see, e.g. [16].

Consider the negative definite, self-adjoint linear operator $A = \Delta$ with Dirichlet boundary conditions on Λ . It is well known that the semigroup

 $e^{tA} = T_t$ is symmetric and Markovian on $L^2(X, m) = L^2(\Lambda)$ (according to the definition used in Theorem 1.4.1 of [16] or the one in [19]). We will consider the Dirichlet form \mathcal{E} associated to T_t . By Theorem 1.3.1 in [16], there is a biunivocal correspondence: $\mathcal{E} \leftrightarrow e^{tA} \leftrightarrow A$. Moreover,

$$\mathrm{Dom}(\mathcal{E}) = \mathrm{Dom}(\sqrt{-A}), \quad \mathcal{E}(u,v) = \left\langle \sqrt{-A}u, \sqrt{-A}v \right\rangle.$$

Recall Theorem 1.4.2 in [16]: For any $u, v \in \text{Dom}(\mathcal{E}) \cap L^{\infty}(X, m)$ (in our case $L^{\infty}(X, m) = L^{\infty}(\Lambda)$), we have

$$\sqrt{\mathcal{E}(uv, uv)} \le ||u||_{\infty} \sqrt{\mathcal{E}(v, v)} + ||v||_{\infty} \sqrt{\mathcal{E}(u, u)}.$$

By Remark 1 on page 417 in [19], we know that for any $0 < s \le 1$, the operator $-(-A)^s$ is also the generator of a symmetric Markovian semigroup T_t^s on $L^2(\Lambda)$ obtained by subordination of order s of T_t . (For more on subordination like $T_t^f = \int_0^\infty T_s \, \mu_t^f(ds)$, see for instance [4].)

The Dirichlet form associated to T_t^s is \mathcal{E}_s with $\text{Dom}(\mathcal{E}_s) = \text{Dom}((-A)^{s/2})$. We will apply Theorem 1.4.2 of [16] to the Dirichlet form \mathcal{E}_s . We find

$$\sqrt{\left\langle (-A)^{s/2}(uv), (-A)^{s/2}(uv)\right\rangle} \le \|u\|_{\infty} \|(-A)^{s/2}v\| + \|v\|_{\infty} \|(-A)^{s/2}u\|,$$

for any $u, v \in \text{Dom}(\mathcal{E}_s) \cap L^{\infty}(X, m) = H^s \cap L^{\infty}(\Lambda)$ for $s \in (0, 1]$. Since the inequality (4.1) holds elementarily for s = 0, the proof is complete.

4.2 Regularity of the linear part

We consider path regularity of the stochastic convolution \bar{X} (that is the linear part of the solution defined in (2.1)).

Proposition 4.2. The process \bar{X} has a continuous version with values in H. Moreover, for all $2 \leq p < \infty$, $0 \leq s < 1/2$: $\bar{X} \in C([0,T];W^{s,p}(\Lambda))$, \mathbb{P} -a.s.. In particular, we have $\bar{X} \in C([0,T];C(\bar{\Lambda}))$, \mathbb{P} -a.s.

For the proof we can follow Section 5 in [15] (using the so-called factorization method) or Proposition S.9 in [2] with $\gamma = 0$, d = 1 and B = I.

The last assertion follows by the well-known embedding: $W^{s,p}(\Lambda) \subset C^{0,s-1/p}(\Lambda) \subset C(\bar{\Lambda})$ which holds if in addition p > 1/s.

As a consequence of Proposition 4.1 we obtain the following additional regularity result.

Proposition 4.3. The process \bar{X} considered in Proposition 4.2 has the following property: for any $s \in [0, 1/2)$, we have

$$\bar{X}^2 \in C([0,T]; H^s), \mathbb{P}$$
-a.s.

Proof. Fix $s \in [0, 1/2)$. By Proposition 4.1, we write for any $t, r \in [0, T]$, \mathbb{P} -a.s.,

$$\|\bar{X}^{2}(t) - \bar{X}^{2}(r)\|_{s} = \|(\bar{X}(t) + \bar{X}(r))(\bar{X}(t) - \bar{X}(r))\|_{s}$$

$$\leq \|\bar{X}(t) + \bar{X}(r)\|_{\infty} \|\bar{X}(t) - \bar{X}(r)\|_{s} + \|\bar{X}(t) - \bar{X}(r)\|_{\infty} \|\bar{X}(t) + \bar{X}(r)\|_{s}$$

$$\leq C_{T} \|\bar{X}(t) - \bar{X}(r)\|_{s} + C_{T} \|\bar{X}(t) - \bar{X}(r)\|_{\infty},$$

for some positive constant C_T possibly depending on $\omega \in \Omega$. Passing to the limit $t \to r$ gives the assertion by Proposition 4.2.

4.3 Regularity of the nonlinear part

In this section we prove Proposition 2.1. We follow the ideas from Section 7.1 in [30] and, in particular, the proof of Proposition 24 in [30].

For $g \in C([0,T];H)$, we define the operator S by

$$(Sg)(t) := \int_0^t S_{\vartheta}(t-r)g(r) \, dr, \ t \in [0,T]$$

and we recall the assertions (7.7) and (7.8) from [30] in our notation.

Lemma 4.4.

- (i) $T := (-A)^{-1/2} \partial_x$ is a bounded linear operator from H^s into H^s , $s \in [0,1]$.
- (ii) S is a bounded linear operator from C([0,T];H) into $C([0,T];H^s)$, $s \in [0,2)$.

For $u \in C([0,T];H^1)$, define the operator R by

$$(Ru)(t) := \int_0^t S_{\vartheta}(t-r)\partial_x u(r) dr, \quad t \in [0,T]. \tag{4.2}$$

We summarize the properties of the operator R.

Lemma 4.5.

- (i) The operator R can be extended to a linear, bounded operator from $C([0,T];L^1(\Lambda))$ into $C([0,T];H^s), s \in [0,1/2)$.
- (ii) The operator R can be extended to a linear, bounded operator from C([0,T];H) into $C([0,T];H^s)$, $s \in [0,1)$.
- (iii) The operator R can be extended to a linear, bounded operator from $C([0,T];H^s)$, $s \in [0,1/2)$ into $C([0,T];H^s)$, $s \in [0,3/2)$.
- (iv) The mapping $u\mapsto R(u^2)$ is continuous from C([0,T];H) into $C([0,T];H^s),\ s\in [0,1/2).$

Proof. (i). This assertion follows from Lemma 14.2.1 of [14].

(ii). Using Lemma 4.4(i) we can write, for $u \in C([0,T]; H)$,

$$(Ru)(t) = \int_0^t S_{\vartheta}(t-r)\partial_x u(r) dr = \int_0^t S_{\vartheta}(t-r)(-A)^{1/2} [(-A)^{-1/2}\partial_x] u(r) dr, \quad t \in [0,T].$$

Note that $[(-A)^{-1/2}\partial_x]u \in C([0,T];H)$. By Lemma 4.4(ii) we know that, for any $\varepsilon \in (0,1]$,

$$t \mapsto (-A)^{1-\varepsilon} \int_0^t S_{\vartheta}(t-r)[(-A)^{-1/2}\partial_x]u(r) dr$$

belongs to C([0,T];H). Hence

$$(-A)^s Ru \in C([0,T]; H), s \in [0,1/2), \text{ i.e., } Ru \in C([0,T]; H^s), s \in [0,1).$$

(iii). Fix $s \in [0, 1/2)$ and take $u \in C([0, T]; H^s)$. We can write

$$(Ru)(t) = \int_0^t S_{\vartheta}(t-r)\partial_x u(r) dr$$

$$= (-A)^{1/2} \int_0^t S_{\vartheta}(t-r)(-A)^{-s/2} \underbrace{(-A)^{s/2}[(-A)^{-1/2}\partial_x]u(r)}_{=:g(r)} dr$$

$$= (-A)^{1/2-s/2} \int_0^t S_{\vartheta}(t-r)g(r) dr. \tag{4.3}$$

By Lemma 4.4(i), the function $g \in C([0,T];H)$ and consequently the integral on the right-hand side of (4.3) belongs to $C([0,T];H^{2-\varepsilon})$ for any small $\varepsilon > 0$ by Lemma 4.4(ii). But that means that $Ru \in C([0,T];H^{1+s-\varepsilon})$. Since we can take s as close to 1/2 as we wish, we conclude with $Ru \in C([0,T];H^s)$ for $s \in [0,3/2)$.

(iv). The mapping: $h\mapsto h^2$ is continuous from C([0,T];H) into $C([0,T];L^1(\Lambda))$, so the assertion follows from (i).

Proof of Proposition 2.1. We deduce from Lemma 14.2.1 of [14] that the solution $X \in C([0,T];H^s)$, \mathbb{P} -a.s., for $s \in [0,1/2)$. For the stochastic convolution \bar{X} , we know that from Proposition 4.2, for the nonlinear part \tilde{X} , it follows from Lemma 4.5(iv). Recall that the initial condition $X_0 \in H^{3/2}$.

To get more spatial regularity for \widetilde{X} we proceed in two steps.

Step 1. We show that $\widetilde{X} \in C([0,T]; H^s)$, \mathbb{P} -a.s., for $s \in [0,1)$.

Fix s=1/4. By the continuous inclusion $H^{1/4}\subset L^4(\Lambda)$, we get $X^2\in C([0,T];H)$ and using Lemma 4.5(ii) with $u=X^2$, we obtain $R(X^2)\in C([0,T];H^s)$ for any $s\in [0,1)$.

Taking into account the initial condition X_0 , we obtain that $\widetilde{X} \in C([0,T];H^s)$, \mathbb{P} -a.s., for $s \in [0,1)$. By a well known Sobolev embedding this implies that $\widetilde{X} \in C([0,T];C(\overline{\Lambda}))$, \mathbb{P} -a.s.

Step 2. We show that $\widetilde{X} \in C([0,T]; H^s)$, \mathbb{P} -a.s., for $s \in [0,3/2)$.

We know that in particular $\bar{X}, \tilde{X} \in C([0,T]; H^s) \cap C([0,T]; C(\bar{\Lambda})), \mathbb{P}\text{-a.s.},$ for $s \in [0, 1/2)$. Since

$$X^{2}(t) = (\bar{X}(t) + \tilde{X}(t))^{2}, \ t \in [0, T],$$

we can argue as in the proof of Proposition 4.3 using Proposition 4.1. We deduce easily that it is also indeed true for the solution to (1.1) that $X^2 \in$ $C([0,T]; H^s) \cap C([0,T]; C(\bar{\Lambda})), \mathbb{P}$ -a.s., for $s \in [0,1/2)$.

Using Lemma 4.5(iii) with $u = X^2$, we obtain $R(X^2) \in C([0,T]; H^s)$ for any $s \in [0, 3/2).$

Taking into account the initial condition X_0 , we obtain that $\widetilde{X} \in$ $C([0,T];H^s)$, P-a.s., for $s \in [0,3/2)$. The proof is complete.

Integrability of the nonlinear part 4.4

Proposition 4.6. The nonlinear part of the solution \widetilde{X} defined in (2.1) satisfies $\widetilde{X} \in L^2(\Omega \times [0,T]; C(\overline{\Lambda})).$

Proof. We first find an upper bound for $\mathbb{E} \int_0^T \|\widetilde{X}(t)\|_1^2 dt$. To this purpose we modify the proof of Theorem 14.2.4 in [14]. Following the argument of [14] we first consider a regular approximation of $\bar{X}(t)$. With such approximation we can work with equation (2.3) which holds for a.e. $t \in (0,T]$. Note that the final obtained estimates do not require the presence of the regular approximation of $\bar{X}(t)$; they hold with $\bar{X}(t)$.

Multiplying both sides of equation (2.3) by X, integrating with respect to xin $\Lambda = (0,1)$ and using the integration by parts we get

$$\begin{split} \frac{1}{2}\frac{d}{dt}\int_0^1 \widetilde{X}^2(t,x)\,dx + \vartheta \int_0^1 (\partial_x \widetilde{X}(t,x))^2\,dx \\ &= -\frac{1}{2}\int_0^1 \left(\widetilde{X}(t,x) + \bar{X}(t,x)\right)^2 \partial_x \widetilde{X}(t,x)\,dx \\ &= -\frac{1}{2}\int_0^1 \widetilde{X}^2(t,x)\partial_x \widetilde{X}(t,x)\,dx - \int_0^1 \bar{X}(t,x)\widetilde{X}(t,x)\partial_x \widetilde{X}(t,x)\,dx \\ &- \frac{1}{2}\int_0^1 \bar{X}^2(t,x)\partial_x \widetilde{X}(t,x)\,dx. \end{split}$$

The first term on the right-hand side equals to zero

$$\int_0^1 \widetilde{X}^2(t, x) \partial_x \widetilde{X}(t, x) \, dx = \frac{1}{3} \left[\widetilde{X}^3(t, x) \right]_{x=0}^{x=1} = 0,$$

so we obtain

$$\begin{split} &\frac{1}{2}\frac{d}{dt}\|\widetilde{X}(t)\|^2 + \vartheta\|\widetilde{X}(t)\|_1^2 \\ &\leq \|\bar{X}(t)\|_{L^4(\Lambda)}\|\widetilde{X}(t)\|_{L^4(\Lambda)}\|\widetilde{X}(t)\|_1 + \frac{1}{2}\|\bar{X}(t)\|_{L^4(\Lambda)}^2\|\widetilde{X}(t)\|_1 \\ &\leq \frac{\vartheta}{4}\|\widetilde{X}(t)\|_1^2 + \frac{2}{\vartheta}\|\bar{X}(t)\|_{L^4(\Lambda)}^4 + \frac{2}{\vartheta}\|\widetilde{X}(t)\|_{L^4(\Lambda)}^4 + \frac{\vartheta}{4}\|\widetilde{X}(t)\|_1^2 + \frac{1}{\vartheta}\|\bar{X}(t)\|_{L^4(\Lambda)}^4. \end{split}$$

Therefore, we find

$$\frac{d}{dt} \|\widetilde{X}(t)\|^2 + \vartheta \|\widetilde{X}(t)\|_1^2 \leq \frac{6}{\vartheta} \|\bar{X}(t)\|_{L^4(\Lambda)}^4 + \frac{4}{\vartheta} \|\widetilde{X}(t)\|_{L^4(\Lambda)}^4,$$

which we integrate with respect to t in [0,T]

$$\|\widetilde{X}(T)\|^2 + \vartheta \int_0^T \|\widetilde{X}(t)\|_1^2 dt \leq \|X_0\|^2 + \frac{6}{\vartheta} \int_0^T \|\bar{X}(t)\|_{L^4(\Lambda)}^4 dt + \frac{4}{\vartheta} \int_0^T \|\widetilde{X}(t)\|_{L^4(\Lambda)}^4 dt$$

(note that the corresponding estimate in the proof of Theorem 14.2.4 in [14] is different). Now we apply expectation and obtain the following bound

$$\mathbb{E} \int_{0}^{T} \|\widetilde{X}(t)\|_{1}^{2} dt \leq \frac{\|X_{0}\|^{2}}{\vartheta} + \frac{6}{\vartheta^{2}} \mathbb{E} \int_{0}^{T} \|\bar{X}(t)\|_{L^{4}(\Lambda)}^{4} dt + \frac{4}{\vartheta^{2}} \mathbb{E} \int_{0}^{T} \|\widetilde{X}(t)\|_{L^{4}(\Lambda)}^{4} dt. \tag{4.4}$$

We show that the right hand side of (4.4) is finite. Recall that $\widetilde{X}(t) = X(t) - \overline{X}(t)$. We examine separately X and \overline{X} . The fact that

$$\mathbb{E} \int_0^T \|X(t)\|_{L^4(\Lambda)}^4 dt < \infty$$

follows by Proposition 2.2 in [11] with p = k = 4.

It is straightforward to check that

$$\mathbb{E} \int_0^T \|\bar{X}(t)\|_{L^4(\Lambda)}^4 dt = \mathbb{E} \int_0^T \int_{\Lambda} |\bar{X}(t,x)|^4 dt < \infty;$$

for instance, one can argue as in Remark 5.2.10 of [14] which allows to get $\sup_{t\in[0,T]}\mathbb{E}\|\bar{X}(t,\cdot)\|_{C(\bar{\Lambda})}^4<\infty$.

The overall result

$$\mathbb{E} \int_0^T \|\widetilde{X}(t)\|_1^2 dt < \infty$$

means that $\widetilde{X} \in L^2(\Omega \times [0,T]; H^1)$. The Sobolev embedding $H^1 \subset C^{0,1/2}(\bar{\Lambda}) \subset C(\bar{\Lambda})$ finishes the proof.

4.5 The notion of weak solution

Proof of Lemma 2.2. The proof is not difficult, but rather lengthy. We give a sketch of proof.

For the nonlinear part \widetilde{X} we have by (2.1) \mathbb{P} -a.s.

$$\widetilde{X}(t) = S_{\vartheta}(t)X_0 + \frac{1}{2} \int_0^t S_{\vartheta}(t-s)\partial_x \left((\bar{X}(s) + \widetilde{X}(s))^2 \right) ds. \tag{4.5}$$

Now we work with ω fixed (i.e., we fix $\omega \in \Omega'$ for some event Ω' which verifies $\mathbb{P}(\Omega') = 1$). We first prove that

$$\langle z, \widetilde{X}(t) \rangle = \langle z, X_0 \rangle + \vartheta \int_0^t \langle \Delta z, \widetilde{X}(s) \rangle ds - \frac{1}{2} \int_0^t \langle \partial_x z, (\overline{X}(s) + \widetilde{X}(s))^2 \rangle ds,$$

 $t \in [0, T]$ (this is so-called weak formulation of (4.5)). To this purpose we will apply a classical result by [5], c.f. Proposition A.6 in [15]. Setting

$$\bar{X}(t,\omega) = y(t), \ t \in [0,T],$$

we choose $(\varphi_n) \subset C([0,T]; H_0^1)$ such that $\varphi_n(t) \to y(t)$ in H, uniformly in $t \in [0,T]$. (See, for instance, Corollary 0.1.3 in [28].) Then define

$$\widetilde{X}_n(t) = S_{\vartheta}(t)X_0 + \frac{1}{2} \int_0^t S_{\vartheta}(t-s)\partial_x \left((\varphi_n(s) + \widetilde{X}_n(s))^2 \right) ds$$

and note that $\partial_x \left((\varphi_n(s) + \widetilde{X}_n(s))^2 \right) \in C([0,T];H)$ (see also Proposition 2.1). Hence we have

$$\left\langle z, \widetilde{X}_n(t) \right\rangle = \left\langle z, X_0 \right\rangle + \vartheta \int_0^t \left\langle \Delta z, \widetilde{X}_n(s) \right\rangle \, ds - \frac{1}{2} \int_0^t \left\langle \partial_x z, (\varphi_n(s) + \widetilde{X}_n(s))^2 \right\rangle \, ds. \tag{4.6}$$

Recall that by the estimates in the proof of Theorem 14.2.4 in [14] (see in particular estimate (14.2.12)), we know that

$$\sup_{n \ge 1} \sup_{t \in [0,T]} \|\widetilde{X}_n(t)\| = C_0 < \infty \tag{4.7}$$

where C_0 depends on ω, T , $\|X_0\|$ and $\sup_{n\geq 1} \|\varphi_n\|_{C([0,T];H)}$ which is finite. Note that

$$\widetilde{X}_n(t) - \widetilde{X}(t) = \frac{1}{2} \int_0^t S_{\vartheta}(t-s) \partial_x [(\varphi_n(s) + \widetilde{X}_n(s))^2 - (\widetilde{X}(s) + y(s))^2] ds.$$

Arguing as in the proof of Lemma 14.2.1 in [14] (see also page 393 in [12]) we obtain, for $0 < s \le t$, $t \in (0,T]$,

$$||S_{\vartheta}(t-s)\partial_{x}[(\varphi_{n}(s)+\widetilde{X}_{n}(s))^{2}-(\widetilde{X}(s)+y(s))^{2}]||$$

$$\leq \frac{C_{1}}{(t-s)^{3/4}}||(\widetilde{X}_{n}(s)+\varphi_{n}(s))^{2}-(\widetilde{X}(s)+y(s))^{2}||_{L^{1}(\Lambda)}$$

for some constant $C_1 > 0$. We are using that

$$||S_{\vartheta}(t)\partial_x \phi|| \le \frac{C_1}{t^{3/4}} ||\phi||_{L^1(\Lambda)}, \ \phi \in H^1, \ t \in (0, T].$$

Now it is not difficult to see that, using also the bound (4.7),

$$\|(\widetilde{X}_n(s) + \varphi_n(s))^2 - (\widetilde{X}(s) + y(s))^2\|_{L^1(\Lambda)} \le C_2 \|\widetilde{X}_n(s) - \widetilde{X}(s)\| + C_3 \|\varphi_n - y\|_{C([0,T];H)},$$

where constants $C_2, C_3 > 0$ are independent of n (they depends also on ω and T). Recall that $\bar{X}(t,\omega) = y(t)$. We have

$$\|\widetilde{X}_{n}(t) - \widetilde{X}(t)\| \leq \frac{1}{2} \int_{0}^{t} \|S_{\vartheta}(t-s)\partial_{x}[(\varphi_{n}(s) + \widetilde{X}_{n}(s))^{2} - (\widetilde{X}(s) + y(s))^{2}]\| ds$$

$$\leq C_{4} \int_{0}^{t} \frac{1}{(t-s)^{3/4}} \|(\widetilde{X}_{n}(s) - \widetilde{X}(s)\| ds + C_{5}\|\varphi_{n} - y\|_{C([0,T];H)},$$

for some constants $C_4, C_5 > 0$ independent of n (they depend also on ω and T). Applying the Henry-Gronwall lemma, we find, for any $t \in [0, T]$,

$$\|\widetilde{X}_n(t) - \widetilde{X}(t)\| \le C_5 M \|\varphi_n - y\|_{C([0,T];H)},$$

where $M = M(C_4, T) > 0$. We obtain that \mathbb{P} -a.s.

$$\lim_{n\to\infty} \|\widetilde{X}_n(t) - \widetilde{X}(t)\|_{C([0,T];H)} = 0.$$

Now it is easy to pass the limit in (4.6) as $n \to \infty$ and get \mathbb{P} -a.s.

$$\left\langle z, \widetilde{X}(t) \right\rangle = \left\langle z, X_0 \right\rangle + \vartheta \int_0^t \left\langle \Delta z, \widetilde{X}(s) \right\rangle ds - \frac{1}{2} \int_0^t \left\langle \partial_x z, (\overline{X}(s) + \widetilde{X}(s))^2 \right\rangle ds. \tag{4.8}$$

On the other hand, we have \mathbb{P} -a.s. for any $z \in H_0^1 \cap H^2$, $t \in [0,T]$

$$\langle z, \bar{X}(t) \rangle = \vartheta \int_0^t \langle \Delta z, \bar{X}(s) \rangle ds + \langle z, W(t) \rangle.$$
 (4.9)

Summing up equalities (4.8) and (4.9) gives the result, since $X = \bar{X} + \tilde{X}$.

4.6 Proof of the main theorems

In this section we prove our main result, that is the central limit theorem (see Theorem 3.4).

We consider the error decomposition (3.3). While the asymptotics of the term $\delta^{-1}(\mathcal{I}_{\delta})^{-1}\mathcal{M}_{\delta}$ for $\delta \to 0$ can be analyzed as in [2], we use different techniques to tackle the term $\delta^{-1}(\mathcal{I}_{\delta})^{-1}\mathcal{R}_{\delta}$. However, we start similarly.

From Proposition 3(i)-(iii) in [2], that can be also used for $\gamma = 0$, it follows that $\mathcal{I}_{\delta} = O_{\mathbb{P}}(\delta^{-2})$. Therefore, to obtain $\delta^{-1}(\mathcal{I}_{\delta})^{-1}\mathcal{R}_{\delta} \to 0$, it is sufficient to show that

$$\delta \|K\|_{L^2(\mathbb{R})}^2 \mathcal{R}_\delta \stackrel{\mathbb{P}}{\to} 0, \quad \delta \to 0.$$

To ease the notation, we omit the x_0 in the lower index and, by integration by parts, we write

$$||K||_{L^2(\mathbb{R})}^2 \mathcal{R}_{\delta} = \frac{1}{2} \int_0^T X_{\delta}^{\Delta}(t) \left\langle X^2(t), \partial_x K_{\delta} \right\rangle dt = \frac{1}{2} (U_{1,\delta} + U_{2,\delta} + U_{3,\delta}),$$

where

$$\begin{split} U_{1,\delta} &:= \int_0^T \bar{X}_\delta^\Delta(t) \left\langle \bar{X}^2(t), \partial_x K_\delta \right\rangle \, dt, \\ U_{2,\delta} &:= \int_0^T \widetilde{X}_\delta^\Delta(t) \left\langle X^2(t), \partial_x K_\delta \right\rangle \, dt + \int_0^T \bar{X}_\delta^\Delta(t) \left\langle \widetilde{X}^2(t), \partial_x K_\delta \right\rangle \, dt \\ &+ 2 \int_0^T \bar{X}_\delta^\Delta(t) \left\langle \bar{X}(t) (\widetilde{X}(t) - \widetilde{X}(t, x_0)), \partial_x K_\delta \right\rangle \, dt =: V_{1,\delta} + V_{2,\delta} + V_{3,\delta}, \\ U_{3,\delta} &:= 2 \int_0^T \widetilde{X}(t, x_0) \bar{X}_\delta^\Delta(t) \left\langle \bar{X}(t), \partial_x K_\delta \right\rangle \, dt. \end{split}$$

In the next, we will show that $\delta U_{1,\delta} \stackrel{\mathbb{P}}{\to} 0$ using the Wick's theorem and $\delta V_{j,\delta} \stackrel{\mathbb{P}}{\to} 0$, for j=1,2,3, using the excess spatial regularity of \widetilde{X} over \overline{X} . The convergence $\delta U_{3,\delta} \stackrel{\mathbb{P}}{\to} 0$ will be established by a specific representation of $\widetilde{X}(t,x_0)$. First, we establish upper bounds for relevant terms (c.f. Lemma S.3 in [2]).

Lemma 4.7. For any small $\varepsilon > 0$, uniformly in $0 \le t \le T$, $k \ge 1$, $r \le 1$:

(i)
$$\left| \widetilde{X}_{\delta}^{\Delta}(t) \right| \lesssim \delta^{-1/2 - \varepsilon}, \left| \overline{X}_{\delta}^{\Delta}(t) \right| \lesssim \delta^{-1 - \varepsilon},$$

 $\left| \left\langle \widetilde{X}^{2}(t), \partial_{x} K_{\delta} \right\rangle \right| \lesssim \delta^{1/2 - \varepsilon}, \left| \left\langle X^{2}(t), \partial_{x} K_{\delta} \right\rangle \right| \lesssim \delta^{-\varepsilon},$

(ii)
$$\left|\left\langle e_k^2, \partial_x K_\delta \right\rangle\right| \lesssim \lambda_k^r \delta^{r-1/2-\varepsilon}, \left|\left\langle \widetilde{X}(t), e_k \right\rangle\right| \lesssim \lambda_k^{-3/4+\varepsilon}.$$

Proof. The proof is based on Lemma 19 from [2] and the regularity results on \bar{X} and \tilde{X} established in Propositions 4.2 and 2.1.

For $\varepsilon \in (0, 1/2)$ and $p \geq 2$, we have, \mathbb{P} -a.s.,

$$\bar{X} \in C([0,T]; W^{1/2-\varepsilon,p}(\Lambda)), \quad \widetilde{X} \in C([0,T]; H^{3/2-\varepsilon}). \tag{4.10}$$

We know that \widetilde{X}^2 still belongs to $C([0,T];H^{3/2-\varepsilon})$.

Concerning \bar{X} we can use a classical result: $W^{s,p}(\Lambda)$ is an algebra for p > 1/s (see [31]). It follows that for p > 2 large enough (depending on ϵ) we also have

$$\bar{X}^2 \in C([0,T]; W^{1/2-\varepsilon,p}(\Lambda)).$$

This allows to perform the estimates in [2] and to obtain the required inequalities involving \bar{X} . Concerning \tilde{X} , for the first inequality, we use Lemma 15(i) in [2] for scaling and Lemma 19 with p = q = 2, $r = 3/2 - \varepsilon$, d = 1 applied to $z_{\delta} = \delta^{-2}(\Delta_{\delta}K)_{\delta}$. We obtain

$$\begin{split} \left| \widetilde{X}_{\delta}^{\Delta}(t) \right| &= \left| \left\langle \widetilde{X}(t), \Delta K_{\delta} \right\rangle \right| = \left| \left\langle \widetilde{X}(t), \delta^{-2}(\Delta_{\delta}K)_{\delta} \right\rangle \right| \\ &\lesssim \delta^{-2} \delta^{3/2 - \varepsilon} \| \widetilde{X}(t) \|_{3/2 - \varepsilon} \| (-\Delta_{\delta})^{1/4 + \varepsilon/2} K \|_{L^{2}(\Lambda_{\delta})}. \end{split}$$

The finiteness of $\sup_{t\in[0,T]}\|\widetilde{X}(t)\|_{3/2-\varepsilon}$ comes from (4.10) and the finiteness of the last factor is provided by Lemma 20 in [2]. This gives the result, but note that it is a different result that would come from Lemma S.3 in [2] just by allowing $\gamma = 0$.

The remaining inequalities are shown analogously.

Remark 4.8. The inequality $\left|\widetilde{X}_{\delta}^{\Delta}(t)\right| \lesssim \delta^{-1/2-\varepsilon}$, that we just showed, holds for any $\varepsilon > 0$, P-a.s., uniformly in $t \in [0,T]$ and gives (3.4) from Assumption F discussed above.

Now we assert the needed convergences.

Lemma 4.9. As $\delta \to 0$, we have that $\delta U_{1.\delta} \stackrel{\mathbb{P}}{\to} 0$.

Proof. The processes $\bar{X}(t)$ and $\bar{X}^{\Delta}_{\delta}$ are centered Gaussian processes, so we may use the Wick's formula (c.f. Theorem 1.28 in [24]) and repeat the proof as for Lemma S.7 in [2] with $\gamma = 0$. The preparatory Lemmas S.5 and S.6 do work with $\gamma = 0$ the same in our case.

Lemma 4.10. As $\delta \to 0$, we have that $\delta U_{2,\delta} \stackrel{\mathbb{P}}{\to} 0$.

Proof. According to Remark 3.3 we change the argument of the proof of Lemma S.4 (see the supplement of [2]).

Lemma 4.7(i) yields $V_{1,\delta} = O_{\mathbb{P}}(\delta^{-1/2-\varepsilon}), V_{2,\delta} = O_{\mathbb{P}}(\delta^{-1/2-\varepsilon})$ for any small $\varepsilon > 0$. As for the term $V_{3,\delta}$, we show the desired convergence in the \mathbb{P} -a.s. sense.

Fix $\omega \in \Omega$, \mathbb{P} -a.s. From Proposition 2.1, we know that $\widetilde{X} \in C([0,T]; H^{3/2-\varepsilon})$, \mathbb{P} -a.s., for any small $\varepsilon > 0$. However, the space $H^{3/2-\varepsilon}$ continuously embedds into $H^{3/2-\varepsilon} \subset C^{0,1-\varepsilon}(\Lambda) = C^{1-\varepsilon}(\Lambda)$. Hence, we have

$$\sup_{t \in [0,T]} |\widetilde{X}(t,x) - \widetilde{X}(t,y)| \le C(\omega)|x - y|^{1-\varepsilon}, \tag{4.11}$$

for any $x,y\in\Lambda$ and some finite positive constant $C(\omega)$. Since $\left|\bar{X}_{\delta}^{\Delta}(t)\right|\lesssim\delta^{-1-\varepsilon'},\ \mathbb{P}$ -a.s., uniformly in t, for any small $\varepsilon'>0$ by

Lemma 4.7(i), we start with the following upper bound:

$$\delta|V_{3,\delta}(\omega)| \leq 2\delta \int_0^T \left| \bar{X}_{\delta}^{\Delta}(t,\omega) \right| \left| \left\langle \bar{X}(t,\omega)(\tilde{X}(t,\omega) - \tilde{X}(t,x_0,\omega)), \partial_x K_{\delta} \right\rangle \right| dt
\lesssim \delta^{-\varepsilon'} \int_0^T \left| \int_{\Lambda} \bar{X}(t,x,\omega)(\tilde{X}(t,x,\omega) - \tilde{X}(t,x_0,\omega))\partial_x K_{\delta}(x) dx \right| dt
\lesssim \delta^{-\varepsilon'} \int_0^T \left| \int_{\Lambda_{\delta,x_0}} \bar{X}(t,\delta y + x_0,\omega)(\tilde{X}(t,\delta y + x_0,\omega) - \tilde{X}(t,x_0,\omega)) \frac{K'(y)\delta}{\delta^{1/2}\delta} dy \right| dt
\leq \delta^{-1/2-\varepsilon'} \int_0^T \int_{\Lambda_{\delta,x_0}} \left| \bar{X}(t,\delta y + x_0,\omega) \right| \left| \tilde{X}(t,\delta y + x_0,\omega) - \tilde{X}(t,x_0,\omega) \right| \left| K'(y) \right| dy dt
\lesssim \delta^{1/2-\varepsilon'-\varepsilon} \tilde{C}(\omega) \int_0^T \int_{\Lambda_{\delta,x_0}} |y|^{1-\varepsilon} |K'(y)| dy dt,$$
(4.12)

where we used (4.11) and the fact that $|\bar{X}(t, \delta y + x_0, \omega)| \to |\bar{X}(t, x_0, \omega)|$ as $\delta \to 0$ by continuity, uniformly in $t \in [0, T]$ (cf. Proposition 4.2); therefore this term is bounded by some positive random constant.

Since the kernel K has compact support, the right-hand side of (4.12) is finite and tends to zero as $\delta \to 0$ for almost all $\omega \in \Omega$, which finishes the proof.

Now we consider the most difficult term:

$$U_{3,\delta} = 2 \int_0^T \widetilde{X}(t, x_0) \bar{X}_{\delta}^{\Delta}(t) \left\langle \bar{X}(t), \partial_x K_{\delta} \right\rangle dt.$$

Lemma 4.11. As $\delta \to 0$, we have that $\delta U_{3,\delta} \stackrel{\mathbb{P}}{\to} 0$.

Proof. We cannot adapt the method used in the proof of Lemma S.8 (see supplement of [2]) which uses Malliavin Calculus. Indeed we do not have a Wiener process (W_t) with values in $H = L^2(\Lambda)$.

We argue differently. The main point of the proof will be to represent $\widetilde{X}(t,x_0), x_0 \in \Lambda, t \in [0,T]$ a.e., in the form

$$\widetilde{X}(t,x_0) = \sum_{i=1}^{\infty} b_i(t)\nu_i, \tag{4.13}$$

where $b_i \in L^2([0,T])$ are deterministic functions and $(\nu_i)_{i=1}^{\infty}$ form an orthonormal basis in a separable space $L^2(\Omega, \mathcal{F}') = L^2(\Omega, \mathcal{F}', \mathbb{P})$, with a σ -field $\mathcal{F}' \subset \mathcal{F}$ that will be specified later (the previous series converges in $L^2(\Omega, \mathcal{F}')$).

Then we will show the desired convergence of $\delta U_{3,\delta}$ in $L^1(\Omega)$. To that end, we proceed in several steps.

Step 1. First, recall the formal representation of the cylindrical Wiener process W(t): $W(t) = \sum_{n=1}^{\infty} e_n \beta_n(t)$, where β_n , $n \ge 1$, are independent real standard Brownian motions and $(e_n)_{n=1}^{\infty}$ is the fixed orthonormal basis of eigenfunctions in $L^2(\Lambda)$.

We start by considering the regular version of \bar{X} given by Proposition 4.2. Let us consider

$$Y_n(t) = \langle \bar{X}(t), e_n \rangle \tag{4.14}$$

It is well know that this is a one dimensional Ornstein-Uhlenbeck processes:

$$Y_n(t) = \int_0^t e^{-\vartheta \lambda_n(t-s)} d\beta_n(s), \quad t \ge 0, \quad n \ge 1.$$

We also have, \mathbb{P} -a.s, for any $t \geq 0$, $n \geq 1$, $Y_n(t) = -\vartheta \lambda_n \int_0^t Y_n(s) ds + \beta_n(t)$ and so, \mathbb{P} -a.s.,

$$Y_n(t) = -\vartheta \lambda_n \int_0^t e^{-\vartheta \lambda_n(t-s)} \beta_n(s) ds + \beta_n(t), \quad t \ge 0, \quad n \ge 1.$$
 (4.15)

Step 2. Example 5.a in [22] provides the following representation for a real Brownian motion $\beta(t)$:

$$\beta(t) = \sum_{k=1}^{\infty} \int_0^t \varphi_k(u) \, du \cdot \int_0^T \varphi_k(u) \, d\beta(u), \quad \mathbb{P} - a.s., \tag{4.16}$$

where $(\varphi_k)_{k=1}^{\infty}$ is any orthonormal basis in $L^2([0,T])$. Moreover, Theorem 5.1 in [22] yields that the infinite series in (4.16) converges uniformly in $t \in [0,T]$ to $\beta(t)$, \mathbb{P} -a.s. Hence, we have, for any $t \in [0,T]$, $n \geq 1$, \mathbb{P} -a.s.,

$$Y_n(t) = \sum_{k=1}^{\infty} (-\vartheta \lambda_n) \int_0^t e^{-\vartheta \lambda_n(t-s)} \left(\int_0^s \varphi_k(u) \, du \right) ds \int_0^T \varphi_k(u) \, d\beta_n(u)$$
$$+ \sum_{k=1}^{\infty} \int_0^t \varphi_k(u) \, du \cdot \int_0^T \varphi_k(u) \, d\beta_n(u).$$

Let us define $\xi_{k,n} = \int_0^T \varphi_k(u) d\beta_n(u)$, $k, n \ge 1$. Note that $\xi_{k,n}$ are i.i.d. random variables, $\xi_{k,n} \sim N(0,1)$ for every $k, n \ge 1$. Let us define deterministic functions $g_k(t) = \int_0^t \varphi_k(u) du$, $t \in [0,T]$. Let $n \ge 1$. We have that

$$Y_n(t)(\omega) = \sum_{k=1}^{\infty} \left(-\vartheta \lambda_n \int_0^t e^{-\vartheta \lambda_n(t-s)} g_k(s) ds - g_k(t) \right) \xi_{k,n}(\omega). \tag{4.17}$$

It is easy to see that there exists an event Ω' with $\mathbb{P}(\Omega') = 1$ (independent of $n \geq 1$ and $t \in [0, T]$) such that (4.17) holds for any $\omega \in \Omega'$, $t \in [0, T]$, and the random series converges uniformly on [0, T], T > 0.

Step 3. Let us fix T > 0. We introduce the σ -field

$$\mathcal{G}'$$
 generated by the random variables $\xi_{k,n}, k \ge 1, n \ge 1.$ (4.18)

Moreover \mathcal{F}' is obtained completing \mathcal{G}' by adding all the \mathbb{P} -null sets of $(\Omega, \mathcal{F}, \mathbb{P})$.

It turns out that by (4.17), for any $t \in [0, T]$, $n \ge 1$,

$$Y_n(t) = \langle \bar{X}(t), e_n \rangle$$
 is \mathcal{F}' -measurable.

Applying Proposition 1.1 in [15] we obtain that, for any $t \in [0, T]$,

$$\bar{X}(t)$$
 is \mathcal{F}' -measurable with values in H . (4.19)

Step 4. Using the approach of [13] as explained in Remark 2.3 one can show that, for any $t \in [0, T]$,

$$\widetilde{X}(t)$$
 is \mathcal{F}' -measurable. (4.20)

To clarify (4.20) consider first the equation (2.5). Since we can solve by the contraction principle this equation on a finite number of "small" deterministic time intervals $I_{R,k} \subset [0,T]$, k=1,...,N, we obtain, \mathbb{P} -a.s., the solution X_n as limit in $C(I_{R,k};H)$, k=1,...,N, of approximating processes which at any fixed time are \mathcal{F}' -measurable random variable.

For instance, let us fix $n \ge 1$ and consider the first time interval $[0, R] = I_{R,1}$. The solution $X_n(t)$ can be obtained as limit, \mathbb{P} -a.s., in $C(I_{R,1}; H)$ of processes $Z_k(t)$ obtained by the usual iteration scheme (we omit the dependence on n to simplify notation):

$$Z_0(t) = 0, \ t \in [0, R],$$

$$Z_{k+1}(t) = S_{\vartheta}(t)X_0 + \frac{1}{2} \int_0^t S_{\vartheta}(t-s)\partial_x \left([\pi_n(Z_k(s))]^2 \right) ds + \bar{X}(t), \ t \in [0, R],$$

 $k \geq 0$. Arguing by induction it is not difficult to prove that each $Z_k(t)$ is \mathcal{F}' -measurable. Consequently, $X_n(t)$ is \mathcal{F}' -measurable for any $t \in [0, R]$. We can argue similarly on the other intervals $I_{R,k}$.

It follows that, for any $n \geq 1$, $t \in [0,T]$, $X_n(t)$ is \mathcal{F}' -measurable. We also note that the solution X to the Burgers equation verifies, \mathbb{P} -a.s., for any $t \in [0,T]$, $n \geq 1$,

$$X(t \wedge \tau_n) = X_n(t \wedge \tau_n)$$

 $(\tau_n \text{ is defined in (2.6)})$. We deduce that $X(t \wedge \tau_n)$ is \mathcal{F}' -measurable, $t \in [0, T]$, $n \geq 1$.

Passing to the limit, \mathbb{P} -a.s. as $n \to \infty$ (since $\tau_n \to \infty$) we obtain that for each $t \in [0, T]$ the H-valued random variable X(t) is \mathcal{F}' -measurable. Assertion (4.20) follows recalling that $\widetilde{X}(t) = X(t) - \overline{X}(t)$.

Step 5. Recall that Proposition 4.6 provides $\widetilde{X} \in L^2(\Omega \times [0,T]; C(\overline{\Lambda}))$ and therefore, for any $x_0 \in \Lambda$, $\widetilde{X}(x_0) \in L^2(\Omega \times [0,T], \mathcal{F}' \otimes \mathcal{B}([0,T]))$. As the result, for almost all t (with respect to the Lebesgue measure on [0,T]) we have

$$\widetilde{X}(t,x_0) \in L^2(\Omega,\mathcal{F}').$$

Recall the σ -field \mathcal{G}' in (4.18). It is straightforward to check that \mathcal{G}' is countably generated. This means that there exists a countable set $\mathcal{E} = (E_n)_{n=1}^{\infty} \subset \mathcal{G}'$ such that $\sigma(\mathcal{E}) = \mathcal{G}'$.

By Proposition 3.4.5 in [10], $L^2(\Omega, \mathcal{G}')$ is a separable Hilbert space. We deduce easily that also

 $L^2(\Omega, \mathcal{F}')$ is a separable Hilbert space.

There exists an orthonormal basis $(\nu_i)_{i=1}^{\infty}$ (different from $(\xi_{k,n})_{k,n=1}^{\infty}$) such that for any $Y \in L^2(\Omega, \mathcal{F}')$ we have

$$Y = \sum_{i=1}^{\infty} \langle Y, \nu_i \rangle_{L^2(\Omega)} \nu_i. \tag{4.21}$$

Hence, for almost all $t \in [0, T]$, we have the desired representation

$$\widetilde{X}(t,x_0) = \sum_{i=1}^{\infty} b_i(t)\nu_i, \tag{4.22}$$

with $b_i \in L^2([0,T])$ being deterministic functions.

Step 6. We finish the proof of the convergence in $L^1(\Omega)$ of $\delta U_{3,\delta}$ proceeding analogously as in the second part of the proof of Lemma S.8 in supplement of [2]. Consider the representation (4.22) and, for $N \in \mathbb{N}$, set

$$U_{3,\delta,N} := 2 \sum_{i=1}^{N} \nu_i \underbrace{\int_0^T b_i(t) \bar{X}_{\delta}^{\Delta}(t) \left\langle \bar{X}(t), \partial_x K_{\delta} \right\rangle dt}_{=:s_{i,\delta}} = 2 \sum_{i=1}^{N} \nu_i s_{i,\delta}.$$

Fix $\eta > 0$ and choose $N = N(\eta) \in \mathbb{N}$ large enough such that

$$\int_0^T \mathbb{E}\Big(\widetilde{X}(t,x_0) - \sum_{i=1}^N b_i(t)\nu_i\Big)^2 dt = \int_0^T \mathbb{E}\Big(\sum_{i=N+1}^\infty b_i(t)\nu_i\Big)^2 dt$$
$$\lesssim \int_0^T \sum_{i=N+1}^\infty b_i^2(t) dt < \eta.$$

We decompose $\delta \mathbb{E}|U_{3,\delta}| \leq \delta \mathbb{E}|U_{3,\delta} - U_{3,\delta,N}| + \delta \mathbb{E}|U_{3,\delta,N}|$ and we handle both terms separately. By the Cauchy-Schwarz inequality and Gaussianity (recall that if Z is Gaussian, then $\mathbb{E}|Z|^p \lesssim (\mathbb{E}Z^2)^{p/2}$), we obtain

$$\delta^{2} \left(\mathbb{E} |U_{3,\delta} - U_{3,\delta,N}| \right)^{2} = \delta^{2} \left(\mathbb{E} \left| 2 \int_{0}^{T} \sum_{i=N+1}^{\infty} \nu_{i} b_{i}(t) \bar{X}_{\delta}^{\Delta}(t) \left\langle \bar{X}(t), \partial_{x} K_{\delta} \right\rangle dt \right| \right)^{2}$$

$$\leq 4\delta^{2} \int_{0}^{T} \mathbb{E} \left(\sum_{i=N+1}^{\infty} \nu_{i} b_{i}(t) \right)^{2} dt \cdot \int_{0}^{T} \mathbb{E} \left(\bar{X}_{\delta}^{\Delta}(t) \left\langle \bar{X}(t), \partial_{x} K_{\delta} \right\rangle \right)^{2} dt$$

$$\lesssim \delta^{2} \eta \int_{0}^{T} \mathbb{E} \left\langle \bar{X}(t), \Delta K_{\delta} \right\rangle^{2} \mathbb{E} \left\langle \bar{X}(t), \partial_{x} K_{\delta} \right\rangle^{2} dt$$

$$= \delta^{-4} \eta \int_{0}^{T} \mathbb{E} \left\langle \bar{X}(t), (\Delta_{\delta} K)_{\delta} \right\rangle^{2} \mathbb{E} \left\langle \bar{X}(t), (\partial_{x} K)_{\delta} \right\rangle^{2} dt$$

$$\lesssim \eta \| (-\Delta_{\delta})^{1/2} K \|_{L^{2}(\Lambda_{\delta})}^{2} \| (-\Delta_{\delta})^{-1/2} \partial_{x} K \|_{L^{2}(\Lambda_{\delta})}^{2} \lesssim \eta,$$

using also Lemma 22, Lemma 20(i) and Lemma S.2(i) from [2] for scaling and a uniform boundedness (in $\delta > 0$) of the two terms in the second line from the end. (Here we also use Assumption (L).) As the result we obtain

$$\sup_{\delta \in (0,1]} \delta \mathbb{E}|U_{3,\delta} - U_{3,\delta,N}| \lesssim \eta^{1/2}. \tag{4.23}$$

Now we focus on $\delta \mathbb{E}[U_{3,\delta,N}]$. By the Cauchy-Schwarz inequality, we have

$$\delta \mathbb{E}|U_{3,\delta,N}| = 2\delta \mathbb{E}\left|\sum_{i=1}^N \nu_i s_{i,\delta}\right| \le 2\delta \left(\sum_{i=1}^N \mathbb{E}\nu_i^2\right)^{1/2} \cdot \left(\sum_{i=1}^N \mathbb{E} s_{i,\delta}^2\right)^{1/2}.$$

The first sum on the right-hand side in the above expression is of order $N^{1/2}$. Since N is fixed (because small η is fixed), this factor only contributes to a constant. Next, we will prove that

$$\delta^2 \mathbb{E} s_{i,\delta}^2 \to 0, \quad i \in \mathbb{N}.$$
 (4.24)

Using Wick's formula and taking advantage of the symmetry in t, t', we obtain

$$\mathbb{E}s_{i,\delta}^{2} = 2 \int_{0}^{T} \int_{0}^{t} b_{i}(t)b_{i}(t')(\rho_{1,\delta}(t,t') + \rho_{2,\delta}(t,t') + \rho_{3,\delta}(t,t')) dt' dt,$$

$$\rho_{1,\delta}(t,t') = \mathbb{E}[\bar{X}_{\delta}^{\Delta}(t) \langle \bar{X}(t), \partial_{x}K_{\delta} \rangle] \mathbb{E}[\bar{X}_{\delta}^{\Delta}(t') \langle \bar{X}(t'), \partial_{x}K_{\delta} \rangle],$$

$$\rho_{2,\delta}(t,t') = \mathbb{E}[\bar{X}_{\delta}^{\Delta}(t') \langle \bar{X}(t), \partial_{x}K_{\delta} \rangle] \mathbb{E}[\bar{X}_{\delta}^{\Delta}(t) \langle \bar{X}(t'), \partial_{x}K_{\delta} \rangle],$$

$$\rho_{3,\delta}(t,t') = \mathbb{E}[\langle \bar{X}(t), \partial_{x}K_{\delta} \rangle \langle \bar{X}(t'), \partial_{x}K_{\delta} \rangle] \mathbb{E}[\bar{X}_{\delta}^{\Delta}(t) \bar{X}_{\delta}^{\Delta}(t')].$$

Now (4.24) follows from this and Lemma S.6(i-iii) in [2] that does work with $\gamma=0$ the same in our case. Therefore, we deduce that $\delta \mathbb{E}|U_{3,\delta,N}|<\eta$ for sufficiently small δ (depending on N and thus on η). Together with (4.23) and since η was arbitrary, we obtain $\delta U_{3,\delta} \stackrel{L^1}{\to} 0$.

Proof of Theorem 3.4. Consider the error decomposition (3.3). The term $\delta^{-1}(\mathcal{I}_{\delta})^{-1}\mathcal{R}_{\delta} \stackrel{\mathbb{P}}{\to} 0$ as $\delta \to 0$ by Lemmas 4.9, 4.10 and 4.11, so there is no asymptotic bias from the nonlinearity. For the other term, we write

$$\delta^{-1}(\mathcal{I}_{\delta})^{-1}\mathcal{M}_{\delta} = \frac{\mathcal{M}_{\delta}}{(\mathcal{I}_{\delta})^{1/2}} \cdot \frac{1}{(\delta^{2}\mathcal{I}_{\delta})^{1/2}}.$$
 (4.25)

Proposition 3(i)-(iii) in [2] provides

$$\frac{\mathcal{I}_{\delta}}{\mathbb{E}\bar{\mathcal{I}}_{\delta}} = \frac{\bar{\mathcal{I}}_{\delta}}{\mathbb{E}\bar{\mathcal{I}}_{\delta}} + \frac{\delta^{2}o_{\mathbb{P}}(\delta^{-2})}{\delta^{2}\mathbb{E}\bar{\mathcal{I}}_{\delta}} \stackrel{\mathbb{P}}{\to} 1,$$

as well as

$$\delta^2 \mathbb{E} \bar{\mathcal{I}}_\delta \stackrel{\mathbb{P}}{\to} \frac{T \|K'\|_{L^2(\mathbb{R})}^2}{2\vartheta \|K\|_{L^2(\mathbb{R})}^2}.$$

Therefore, the first factor in (4.25) converges to N(0,1) in distribution by the standard continuous martingale central limit theorem (e.g., [26], Theorem 1.19, or [27], Theorem 5.5.4), while the second factor assembles the stated asymptotic variance. The result follows by applying Slutsky's lemma.

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