On the differentiability of the local time of the $(1+\beta)$-stable super-Brownian motion

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Abstract

We consider the local time of the $(1+\beta)$-stable super-Brownian motion with $0 < \beta < 1$. It is shown by Mytnik and Perkins (*Ann. Probab.*, 31(3), 1413 – 1440, (2003)) that the local time, denoted by $L(t,x)$, is jointly continuous in $d=1$ while $L(t,x)$ is locally unbounded in x in $d \geq 2$ where it exists. This paper shows that the local time is continuously differentiable in the spatial parameter x in $d=1$. Moreover, we give a representation of the spatial derivative of the local time, denoted by $\frac{\partial}{\partial x} L(t,x)$, and further prove that the derivative is locally γ-Hölder continuous in x with any index $\gamma \in (0, \frac{\beta}{1+\beta})$.

1 Introduction

1.1 Background and main results

Let $M_F = M_F(\mathbb{R}^d)$ be the space of finite measures on $(\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d))$ equipped with the topology of weak convergence of measures. For any measure μ and function ϕ, we write

$$\langle \mu, \phi \rangle = \mu(\phi) = \int_{\mathbb{R}^d} \phi(x)\mu(dx),$$

whenever the integral exists. Let $\beta \in (0,1)$. A super-Brownian motion $X = (X_t, t \geq 0)$ with $(1+\beta)$-stable branching mechanism is an M_F-valued strong Markov process, defined on some complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, such that for any non-negative bounded Borel function ϕ on $\mathbb{R}^d$,

$$\mathbb{E}_{X_0}\Big(\exp\left(-X_t(\phi)\right)\Big) = \exp\left(-X_0(V_t^\phi)\right). \tag{1.1}$$

Here we use $\mathbb{P}_{X_0}$ to denote the law of X starting from $X_0 \in M_F$, $\mathbb{E}_{X_0}$ the associated expectation, and $V_t(x) = V_t^\phi(x)$ is the unique solution of the mild form of

$$\frac{\partial V_t(x)}{\partial t} = \frac{\Delta}{2} V_t(x) - V_t^{1+\beta}(x), \quad V_0(x) = \phi(x).$$

By letting P_t be the semigroup of the standard d-dimensional Brownian motion, we may rewrite the above as

$$V_t = P_t \phi - \int_0^t P_s(V_{t-s}^{1+\beta}) ds. \tag{1.2}$$

We refer the reader to Dawson [1], Section 6, for more information on the above model.

Define the weighted occupation measure Y_t of X by

$$Y_t(\cdot) := \int_0^t X_s(\cdot)ds, \quad \forall t \geq 0. \tag{1.3}$$

Fleischmann [3] proves that for any $t > 0$, Y_t is absolutely continuous if and only if $d < 2 + 2/\beta$, in which case Y_t admits a density function, $L(t,x)$, called the **local time** of the super-Brownian motion. In particular, for any measurable function ϕ on $\mathbb{R}^d$, we have

$$Y_t(\phi) = \int_{\mathbb{R}^d} \phi(x)L(t,x)dx. \tag{1.4}$$

For the case of $\beta = 1$, Sugitani [10] proves the existence of a jointly continuous version of the local time $L(t,x)$ under suitable conditions on X_0 in dimensions $d \leq 3$. Moreover, when $d = 1$, he further shows that $L(t,x)$ is differentiable with respect to x while its spatial derivative, $\frac{\partial}{\partial x}L(t,x)$, is also jointly continuous, provided that X_0 is atomless.

In contrast, when $0 < \beta < 1$, the function $x \mapsto L(t,x)$ is continuous only in $d = 1$; in higher dimensions $d \geq 2$, $x \mapsto L(t,x)$ is locally unbounded. We state the following results from Theorem 1.3 of [7]. For any function $f : \mathbb{R}^d \to \mathbb{R}$ and open set $B \subset \mathbb{R}^d$, we define $\|f\|_B$ to be the essential supremum (with respect to Lebesgue measure) of f on B.

Theorem A (Mytnik-Perkins [7]). *Let $X_0 \in M_F(\mathbb{R}^d)$ and $0 < \beta < 1$.*
(i) For $d = 1$, there exists a jointly continuous version of $L(t,x)$ in $\mathbb{R}_+ \times \mathbb{R}$.
(ii) For $2 \leq d < 2 + 2/\beta$, with $\mathbb{P}_{X_0}$-probability one,

$$\|L(t,\cdot)\|_U = \infty,$$

whenever $\int_U L(t,x)dx > 0$ for any open set $U \subset \mathbb{R}^d$ and $t > 0$.

Now that $x \mapsto L(t,x)$ is continuous only in $d = 1$, we are interested in whether $x \mapsto L(t,x)$ is differentiable. From now on, by slightly abusing the notation, we let $L(t,x)$ be the jointly continuous version from Theorem A (i).

Unlike the $\beta = 1$ case in Sugitani [10], the high moment calculations of the local time $L(t,x)$ do not exist anymore for our case. So new ideas and methods are needed to prove the differentiability of $L(t,x)$. It is well known that our $(1 + \beta)$-stable super-Brownian motion is the unique in law solution to some **martingale problem** (see, e.g., Dawson [1], Section 6). Let $C_b^2(\mathbb{R}^d)$ denote the space of bounded continuous functions whose derivatives of order less than 3 are also bounded continuous. For any $\phi \in C_b^2(\mathbb{R}^d)$, we have

$$M_t(\phi) = X_t(\phi) - X_0(\phi) - \int_0^t X_s\left(\frac{\Delta}{2}\phi\right)ds$$

is an $\mathcal{F}_t$-martingale. Note that $M_t(\phi)$ is a purely discontinuous martingale; see Lemma 2.4 below for more detailed descriptions of this martingale. Using this martingale problem, Mytnik and Xiang [8] and [11] establish the Tanaka formula for our local time $L(t,x)$. To state their results, we let

$$p_s(x) = (2\pi s)^{-1/2}e^{-\frac{x^2}{2s}}, \quad \forall s > 0, x \in \mathbb{R}$$

be the density function of the one-dimensional standard Brownian motion. For any $\lambda > 0$, we define

$$G_\lambda(x) := \int_0^\infty e^{-\lambda s}p_s(x)ds, \quad \forall x \in \mathbb{R}. \tag{1.5}$$

We will show in Lemma 2.2 below that the above G^λ has a simpler form given by

$$G^\lambda(x) = \frac{1}{\sqrt{2\lambda}} e^{-\sqrt{2\lambda}|x|}, \quad \forall x \in \mathbb{R}. \tag{1.6}$$

Set

$$G^x_\lambda(y) := G_\lambda(y - x), \quad \forall x, y \in \mathbb{R}.$$

The Tanaka formula of the local time from [8] and [11] gives that

$$L(t, x) = X_0(G^x_\lambda) - X_t(G^x_\lambda) + \lambda \int_0^t X_s(G^x_\lambda)ds + M_t(G^x_\lambda). \tag{1.7}$$

The definition for $M_t(G^x_\lambda)$ will be made rigorous below in Lemma 2.4.

For any $t, \lambda > 0$ and $x \in \mathbb{R}$, we define

$$Z^\lambda(t, x) := L(t, x) - X_0(G^x_\lambda). \tag{1.8}$$

Then (1.7) implies that

$$Z^\lambda(t, x) = -X_t(G^x_\lambda) + \lambda \int_0^t X_s(G^x_\lambda)ds + M_t(G^x_\lambda). \tag{1.9}$$

To study the differentiability of $Z^\lambda(t, x)$, or $L(t, x)$, the above suggests that we need to investigate the differentiability of G^λ. Let $\mathrm{sgn}(x) = x/|x|$ if $x \neq 0$ and $\mathrm{sgn}(0) = 0$. For any $\lambda > 0$, define

$$g_\lambda(x) := -\mathrm{sgn}(x)e^{-\sqrt{2\lambda}|x|}, \quad \text{and} \quad g^x_\lambda(y) := g_\lambda(y - x), \quad \forall x, y \in \mathbb{R}. \tag{1.10}$$

Let $D_x f(x)$ (resp. $D^+_x f(x)$, $D^-_x f(x)$) denote the derivative (resp. right derivative, left derivative) of $f(x)$. One can easily check by (1.6) that

$$(D_x G_\lambda)(x) = g_\lambda(x), \quad \forall x \neq 0, \tag{1.11}$$
$$(D^+_x G_\lambda)(0) = -1, \quad (D^-_x G_\lambda)(0) = 1.$$

The lemma below tells us that the differentiability of $Z^\lambda(t, x)$ is closely related to that of the local time $L(t, x)$.

Lemma 1.1. *Let $X_0 \in M_F$ and $\lambda > 0$. We have $x \mapsto X_0(G^x_\lambda)$ is continuous on $\mathbb{R}$. Moreover,*

$$(D^+_x X_0(G^x_\lambda))(x) = X_0(g^x_\lambda) - X_0(\{x\}), \quad \forall x \in \mathbb{R};$$
$$(D^-_x X_0(G^x_\lambda))(x) = X_0(g^x_\lambda) + X_0(\{x\}), \quad \forall x \in \mathbb{R}. \tag{1.12}$$

In particular, if $X_0 \in M_F(\mathbb{R})$ is atomless, then $D_x X_0(G^x_\lambda) = X_0(g^x_\lambda)$ and $x \mapsto X_0(g^x_\lambda)$ is continuous on $\mathbb{R}$.

The proof of Lemma 1.1 is elementary; it is deferred to Appendix A.

Now we state our main results. For any function $f : \mathbb{R} \to \mathbb{R}$ and $\gamma \in (0, 1]$, we say f is locally Hölder continuous with index γ if for any $K > 0$, there exists some constant $C_K > 0$ such that

$$|f(x) - f(y)| \le C_K |x - y|^\gamma, \quad \forall x, y \in [-K, K].$$

When $\gamma = 1$, we say f is locally Lipschitz continuous.

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Theorem 1.2. *Let $d = 1$ and $0 < \beta < 1$. For any $X_0 \in M_F(\mathbb{R})$, $t > 0$ and $\lambda > 0$, with $\mathbb{P}_{X_0}$-probability one, the following (i) and (ii) hold:*

(i) *The function $x \mapsto Z^\lambda(t, x)$ is differentiable on $\mathbb{R}$.*

(ii) *The function $x \mapsto \frac{\partial}{\partial x} Z^\lambda(t, x)$ is locally Hölder continuous with any index $\gamma \in (0, \frac{\beta}{1+\beta})$.*

In particular, if X_0 is atomless, then $\frac{\partial}{\partial x} L(t, x)$ exists and is continuous on $\mathbb{R}$, given by

$$\frac{\partial}{\partial x} L(t, x) = X_0(g_\lambda^x) - X_t(g_\lambda^x) + \lambda \int_0^t X_s(g_\lambda^x) ds + M_t(g_\lambda^x). \tag{1.13}$$

Remark 1.3. *We do not discuss the differentiability of $L(t, x)$ in t as we know that $L(t, x) = \int_0^t X(s, x) ds$ where $X(s, x)$ is the density function of X_s in $d = 1$, hence the differentiability is clear. Meanwhile, Theorem 1.2 of [7] states that $X(t, x)$ is locally unbounded in time t.*

1.2 Proof of the main theorem

The proof of Theorem 1.2 follows by "differentiating" the Tanaka formula (1.9) for $Z^\lambda(t, x)$, with the help of the lemma below.

Lemma 1.4. *For a continuous stochastic process $\{\Gamma_x, x \in \mathbb{R}\}$ such that $\mathbb{E}|\Gamma_x| < \infty$ for all $x \in \mathbb{R}$, if*

$$\lim_{h \to 0} \frac{1}{h} \mathbb{E} \left| \Gamma_{x+h} - \Gamma_x \right| = 0, \quad \forall x \in \mathbb{R}, \tag{1.14}$$

then with probability one,

$$\Gamma_x = \Gamma_0, \quad \forall x \in \mathbb{R}.$$

The proof of Lemma 1.4 deviates from our discussions on the local time, so it is deferred to Appendix B.

Fix any $\lambda > 0$. Recall from (1.8) that $Z^\lambda(t, x) = L(t, x) - X_0(G_\lambda^x)$. Combine Theorem A and Lemma 1.1 to see that for any $t > 0$,

$$x \mapsto Z^\lambda(t, x) \text{ is continuous on } \mathbb{R} \text{ a.s.} \tag{1.15}$$

Recall the Tanaka formula for $Z^\lambda(t, x)$ from (1.9). Recall g_λ from (1.10). Define

$$H^\lambda(t, x) := -X_t(g_\lambda^x) + \lambda \int_0^t X_s(g_\lambda^x) ds + M_t(g_\lambda^x), \quad \forall t > 0, x \in \mathbb{R}. \tag{1.16}$$

Combining (1.9) and (1.16), one may observe that (1.11) suggests $\frac{\partial}{\partial x} Z^\lambda(t, x) = H^\lambda(t, x)$. The following results will then confirm this observation.

Proposition 1.5. *Let $d = 1$ and $0 < \beta < 1$. Fix any $X_0 \in M_F(\mathbb{R})$ and $t, \lambda > 0$.*
(i)

$$\lim_{h \to 0} \mathbb{E}_{X_0} \left| \frac{Z^\lambda(t, x + h) - Z^\lambda(t, x)}{h} - H^\lambda(t, x) \right| = 0, \quad \forall x \in \mathbb{R}.$$

(ii) *With $\mathbb{P}_{X_0}$-probability one, the function $x \mapsto H^\lambda(t, x)$ is locally Hölder continuous with any index $\gamma \in (0, \frac{\beta}{1+\beta})$. Moreover, for any $K > 0$ and $q \in (1, 1 + \beta)$, there is some constant $C = C(K, q, X_0, t, \lambda) > 0$ such that*

$$\mathbb{E}_{X_0} \left| H^\lambda(t, x) - H^\lambda(t, y) \right| \leq C|x - y|^{\frac{1}{q}}, \quad \forall -K \leq x, y \leq K. \tag{1.17}$$

Remark 1.6. *Since high moment calculations of $H^\lambda(t,x)$ fail, one cannot simply apply Kolmogorov's continuity criterion to obtain the continuity of $x \mapsto H^\lambda(t,x)$. Instead, we carefully study jumps of the martingale measure and use Lemma 3.3 below to prove Proposition 1.5 (ii).*

Given Proposition 1.5, we are ready to finish the proof of our main theorem.

Proof of Theorem 1.2 assuming Proposition 1.5. Fix any $t > 0$ and $\lambda > 0$. For any $x \in \mathbb{R}$, we define

$$W^\lambda(t,x) := Z^\lambda(t,x) - Z^\lambda(t,0) - \int_0^x H^\lambda(t,z)dz \tag{1.18}$$

so that $W^\lambda(t,0) = 0$. Since $z \mapsto H^\lambda(t,z)$ is a.s. continuous by Proposition 1.5, the integral in (1.18) is well-defined. By (1.15), we get that $x \mapsto W^\lambda(t,x)$ is a.s. continuous. Next, we claim that

$$\lim_{h \to 0} \frac{1}{h} \mathbb{E}_{X_0} \Big| W^\lambda(t,x+h) - W^\lambda(t,x) \Big| = 0, \quad \forall x \in \mathbb{R}. \tag{1.19}$$

Apply Lemma 1.4 with the above to see that $\mathbb{P}_{X_0}$-a.s.,

$$\forall x \in \mathbb{R}, \quad W^\lambda(t,x) = W^\lambda(t,0) = 0.$$

That is, with $\mathbb{P}_{X_0}$-probability one,

$$\forall x \in \mathbb{R}, \quad Z^\lambda(t,x) = Z^\lambda(t,0) + \int_0^x H^\lambda(t,z)dz.$$

We conclude that $x \mapsto Z^\lambda(t,x)$ is differentible, whose derivative is given by

$$D_x Z^\lambda(t,x) = H^\lambda(t,x).$$

In view of Proposition 1.5 (ii), we further get that $D_x Z^\lambda(t,x)$ is locally Hölder continuous with any index $\gamma \in (0, \frac{\beta}{1+\beta})$. If X_0 is atomless, by Lemma 1.1, we have $D_x X_0(G_\lambda^x) = X_0(g_\lambda^x)$ and $x \mapsto X_0(g_\lambda^x)$ is continuous. Hence

$$D_x L(t,x) = D_x Z^\lambda(t,x) + X_0(g_\lambda^x) = H^\lambda(t,x) + X_0(g_\lambda^x).$$

Plugging in $H^\lambda(t,x)$ from (1.16) to conclude that (1.13) holds.

It remains to prove (1.19).

Fix $x \in \mathbb{R}$. Without loss of generality, we assume $0 < h < 1$. Using (1.18), one gets that

$$\frac{1}{h} \mathbb{E}_{X_0} \Big| W^\lambda(t,x+h) - W^\lambda(t,x) \Big| = \frac{1}{h} \mathbb{E}_{X_0} \Big| Z^\lambda(t,x+h) - Z^\lambda(t,x) - \int_x^{x+h} H^\lambda(t,z)dz \Big|$$

$$\leq \mathbb{E}_{X_0} \left| \frac{Z^\lambda(t,x+h) - Z^\lambda(t,x)}{h} - H^\lambda(t,x) \right| + \mathbb{E}_{X_0} \left| \frac{1}{h} \int_x^{x+h} H^\lambda(t,z)dz - H^\lambda(t,x) \right|. \tag{1.20}$$

Apply Proposition 1.5 (i) to see that

$$\mathbb{E}_{X_0} \left| \frac{Z^\lambda(t,x+h) - Z^\lambda(t,x)}{h} - H^\lambda(t,x) \right| \to 0 \text{ as } h \downarrow 0. \tag{1.21}$$

Next, for the second expectation in (1.20), we have

$$\mathbb{E}_{X_0} \Big| \frac{1}{h} \int_x^{x+h} H^\lambda(t,z)dz - H^\lambda(t,x) \Big| \leq \frac{1}{h} \int_x^{x+h} \mathbb{E}_{X_0} |H^\lambda(t,z) - H^\lambda(t,x)|dz$$

$$\leq \sup_{x \leq z \leq x+h} \mathbb{E}_{X_0} |H^\lambda(t,z) - H^\lambda(t,x)|. \tag{1.22}$$

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Let $K > 0$ be large such that $-K < x < z < x + h < x + 1 < K$. Pick any $q \in (1, 1 + \beta)$. One may apply Proposition 1.5 (ii) to get that

$$\sup_{x \leq z \leq x+h} \mathbb{E}_{X_0} |H^\lambda(t,z) - H^\lambda(t,x)| \leq C \sup_{x \leq z \leq x+h} |x - z|^{\frac{1}{q}} = C h^{\frac{1}{q}} \to 0 \ \text{ as } \ h \downarrow 0. \tag{1.23}$$

Combine (1.20)-(1.23) to conclude that

$$\frac{1}{h} \mathbb{E}_{X_0} \left| W^\lambda(t, x + h) - W^\lambda(t, x) \right| \to 0 \ \text{ as } \ h \downarrow 0. \tag{1.24}$$

The proof is now complete. ∎

The rest of the paper is devoted to the proofs of Proposition 1.5 and some technical lemmas.

Organization of the paper. In Section 2, we present the moment calculations regarding $X_t(\phi), Y_t(\phi)$ as well as the martingale term $M_t(\phi)$, and give the proof of Proposition 1.5 (i). Next, in Section 3 we study the continuity of $H^\lambda(t,x)$ and prove Proposition 1.5 (ii). The Appendix contains proof of some auxiliary lemmas.

Convention on constants. A constant of the form $C(a, b, \dots)$ means that this constant depends on parameters $a, b, \dots$. Constants whose values are unimportant and may change from line to line are denoted $C, c, c_d, \dots$. Constants whose values will be referred to later and appear initially in say, Lemma i.j are denoted $c_{i.j}$.

2 Moment calculations

In this section and the next, we fix the initial measure $X_0 \in M_F(\mathbb{R})$. To simplify notation, we write $\mathbb{P}$ for $\mathbb{P}_{X_0}$ and $\mathbb{E}$ for $\mathbb{E}_{X_0}$. This section aims to prove Proposition 1.5 (i).

Fix $\lambda > 0$. For any $t > 0$ and $x \in \mathbb{R}$, by using $Z^\lambda(t,x)$ from (1.9) and $H^\lambda(t,x)$ from (1.16), one can check that

$$\mathbb{E} \left| \frac{Z^\lambda(t, x + h) - Z^\lambda(t, x)}{h} - H^\lambda(t, x) \right| \leq \mathbb{E} \left\langle X_t, \left| \frac{G_\lambda^{x+h} - G_\lambda^x}{h} - g_\lambda^x \right| \right\rangle$$
$$+ \lambda \mathbb{E} \left\langle Y_t, \left| \frac{G_\lambda^{x+h} - G_\lambda^x}{h} - g_\lambda^x \right| \right\rangle + \mathbb{E} \left| \frac{1}{h} \left(M_t(G_\lambda^{x+h}) - M_t(G_\lambda^x) \right) - M_t(g_\lambda^x) \right|, \tag{2.1}$$

where we also use Y_t from (1.3). It suffices to prove that the three expectations on the right-hand side above converge to 0 as $h \to 0$.

2.1 Convergence of the first two expectations

We first deal with the first two expectations on the right-hand side of (2.1). This subsection will give the proof of the following lemma.

Lemma 2.1. *For any $t > 0$ and $x \in \mathbb{R}$, we have*

$$(i) \ \lim_{h \to 0} \mathbb{E} \left[\left\langle X_t, \left| \frac{G_\lambda^{x+h} - G_\lambda^x}{h} - g_\lambda^x \right| \right\rangle \right] = 0.$$

$$(ii) \ \lim_{h \to 0} \mathbb{E} \left[\left\langle Y_t, \left| \frac{G_\lambda^{x+h} - G_\lambda^x}{h} - g_\lambda^x \right| \right\rangle \right] = 0.$$

To prove the above lemma, we recall from (1.5) that

$$G_\lambda(x) = \int_0^\infty e^{-\lambda s} p_s(x) ds = \int_0^\infty e^{-\lambda s} \frac{1}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}} ds, \quad \forall x \in \mathbb{R}. \tag{2.2}$$

Lemma 2.2. *For any $\lambda > 0$ and $x \in \mathbb{R}$, we let $G_\lambda(x)$ be as in (2.2). Then*

$$G_\lambda(x) = \frac{1}{\sqrt{2\lambda}} e^{-\sqrt{2\lambda}|x|}, \quad \forall x \in \mathbb{R}, \qquad (2.3)$$

and

$$|G_\lambda(x) - G_\lambda(y)| \leq |x - y|, \quad \forall x, y \in \mathbb{R}. \qquad (2.4)$$

Proof. See Appendix C. ∎

Next, recall $g_\lambda(x)$ from (1.10). It is easy to see that

$$\frac{d}{dx} G_\lambda(x) = g_\lambda(x) = -\mathrm{sgn}(x) e^{-\sqrt{2\lambda}|x|}, \quad \forall x \neq 0.$$

Hence

$$\lim_{h \to 0} \left| \frac{G_\lambda^{x+h}(y) - G_\lambda^x(y)}{h} - g_\lambda^x(y) \right| = 0, \quad \forall y \neq x. \qquad (2.5)$$

Moreover, by (2.4), one can easily check that

$$\left| \frac{G_\lambda^{x+h}(y) - G_\lambda^x(y)}{h} - g_\lambda^x(y) \right| \leq 2, \quad \forall x, y \in \mathbb{R}, \ 0 < |h| < 1. \qquad (2.6)$$

For any function ϕ, the mean measure of $\langle X_t, \phi \rangle$ gives that

$$\mathbb{E}\langle X_t, \phi \rangle = \langle X_0, P_t \phi \rangle, \qquad (2.7)$$

which follows easily from the Laplace functional (1.1) (see also, e.g., the Green's function representation (1.9) in [4]). By the definition of Y_t, we immediately get that

$$\mathbb{E}\langle Y_t, \phi \rangle = \int_0^t \langle X_0, P_s \phi \rangle ds. \qquad (2.8)$$

Using (2.7) and (2.8), we are ready to give the proof of Lemma 2.1.

Proof of Lemma 2.1. (i) Fix $t > 0$ and $x \in \mathbb{R}$. By (2.7), one gets that

$$\mathbb{E}\left[\left\langle X_t, \left| \frac{G_\lambda^{x+h} - G_\lambda^x}{h} - g_\lambda^x \right| \right\rangle \right] = \int_{\mathbb{R}} X_0(dz) \int p_t(z - y) \left| \frac{G_\lambda^{x+h}(y) - G_\lambda^x(y)}{h} - g_\lambda^x(y) \right| dy.$$

In view of (2.5) and (2.6), one may apply the Dominated Convergence Theorem to see that

$$\lim_{h \to 0} \mathbb{E}\left[\left\langle X_t, \left| \frac{G_\lambda^{x+h} - G_\lambda^x}{h} - g_\lambda^x \right| \right\rangle \right] = 0.$$

(ii) Fix $t > 0$ and $x \in \mathbb{R}$. By (2.8), one gets that

$$\mathbb{E}\left[\left\langle Y_t, \left| \frac{G_\lambda^{x+h} - G_\lambda^x}{h} - g_\lambda^x \right| \right\rangle \right] = \int_{\mathbb{R}} X_0(dz) \int_0^t ds \int p_s(z - y) \left| \frac{G_\lambda^{x+h}(y) - G_\lambda^x(y)}{h} - g_\lambda^x(y) \right| dy.$$

In view of (2.5) and (2.6), again, one may apply the Dominated Convergence Theorem to see that

$$\lim_{h \to 0} \mathbb{E}\left[\left\langle Y_t, \left| \frac{G_\lambda^{x+h} - G_\lambda^x}{h} - g_\lambda^x \right| \right\rangle \right] = 0.$$

∎

2.2 Martingale decomposition

In this subsection, we will prove the following result for the martingale term on the right-hand side of (2.1). Then the proof of Proposition 1.5 (i) is completed in view of (2.1) and Lemmas 2.1.

Lemma 2.3. *For any $t > 0$ and $x \in \mathbb{R}$, we have*

$$\lim_{h \to 0} \mathbb{E} \left| \frac{1}{h} \left(M_t(G_\lambda^{x+h}) - M_t(G_\lambda^x) \right) - M_t(g_\lambda^x) \right| = 0. \tag{2.9}$$

To do so, we first recall some facts about our $(1 + \beta)$ super-Brownian motion. By (1.2), we have X is a superprocess with the branching mechanism given by $\Psi(u) = u^{1+\beta}$, which can also be written as

$$\Psi(u) = \int_0^\infty (e^{-ur} - 1 + ur)\nu(dr).$$

Here

$$\nu(dr) = \frac{\beta(\beta + 1)}{\Gamma(1 - \beta)} r^{-2-\beta} dr =: c_\beta r^{-2-\beta} dr,$$

with Γ denoting the Gamma function. By Section 6.2.2. of [1], we have $t \mapsto X_t$ is a.s. discontinuous. Denote by $\Delta X_t := X_t - X_{t-}$ the jumps of X at time t. Then $\Delta X_t = m(t)\delta_{x(t)}$ with the set of the jump time being dense in $[0, \zeta)$, where $\zeta = \inf\{t \geq 0 : X_t(1) = 0\}$ is the extinction time of the superprocess.

The following lemma gives the martingale decomposition of X (see Section 6 of [1], or Lemma 1.6 of [4]).

Lemma 2.4. *(Martingale decomposition of X) Let $d \geq 1$ and $X_0 \in M_F(\mathbb{R}^d)$.*

(i) The jumps of X are of the form $\Delta X_s = r\delta_x$ with jump time $s > 0$, size $r > 0$ and location $x \in \mathbb{R}^d$. More precisely, there exists a Poisson random measure $N(ds, dx, dr)$ on $(0, \infty) \times \mathbb{R}^d \times (0, \infty)$ describing the jumps $r\delta_x$ of X at times s at sites x of size r.

(ii) The compensator $\hat{N}$ of N is given by

$$\hat{N}(ds, dx, dr) = c_\beta \, ds \, X_s(dx) \, r^{-2-\beta} \, dr,$$

that is, $\tilde{N} := N - \hat{N}$ is a martingale measure on $(0, \infty) \times \mathbb{R}^d \times (0, \infty)$.

(iii) For all non-negative $\phi \in C_b^2(\mathbb{R}^d)$ and $t > 0$, the martingale decomposition of X is

$$\langle X_t, \phi \rangle = \langle X_0, \phi \rangle + \frac{1}{2} \int_0^t \langle X_s, \Delta\phi \rangle \, ds + M_t(\phi), \tag{2.10}$$

with the purely discontinuous martingale

$$M_t(\phi) := \int_{(0,t] \times \mathbb{R}^d} \phi(x) \, M(ds, dx) := \int_{(0,t] \times \mathbb{R}^d \times (0,\infty)} \phi(x) \, r \tilde{N}(ds, dx, dr). \tag{2.11}$$

The definition of $M_t(\phi)$ in (2.11) holds for a much larger class of functions ϕ. For any $\mu \in M_F(\mathbb{R})$ and $s > 0$, we let

$$\mu p_s(x) = \int_{\mathbb{R}} p_s(x - y)\mu(dy).$$

For any $p \geq 1$, we define

$$\mathcal{L}_{loc}^p := \mathcal{L}_{loc}^p(\mathbb{R}^+ \times \mathbb{R}, \, X_0 p_s(x) ds dx)$$

to be the space of equivalence classes of measurable funcitons $\psi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ such that

$$\int_0^T ds \int_{\mathbb{R}} |\psi(s,x)|^p \mu p_s(x) dx < \infty, \quad \forall T > 0.$$

Lemma 1.7 of [4] implies that for any $p \in (1+\beta, 2)$ and $\psi \in \mathcal{L}_{loc}^p$, we have

$$t \mapsto M_t(\psi) := \int_{(0,t] \times \mathbb{R}} \psi(s,x) M(ds, dx)$$

is well-defined. Moreover, by using the proof of Lemma 3.1 of [6] (see also Lemma 2.6 of [4]), we get the following moment bounds regarding the above martingale measure below.

Lemma 2.5. *For any $p \in (1+\beta, 2)$, $q \in (1, 1+\beta)$ and $T > 0$, there exists some constant $C = C(\beta, p, q, T) > 0$ such that for any $\psi \in \mathcal{L}_{loc}^p$,*

$$\mathbb{E}\Big[\sup_{0 \leq t \leq T} |M_t(\psi)|^q \Big] \leq C \int_0^T ds \int_{\mathbb{R}} |\psi(s,y)|^q \, X_0 p_s(y) dy$$
$$+ C \Big[\int_0^T ds \int_{\mathbb{R}} |\psi(s,y)|^p X_0 p_s(y) dy \Big]^{q/p}.$$

Recall G_λ from (2.3) and g_λ from (1.10). Since $\|G_\lambda\|_\infty \leq 1$ and $\|g_\lambda\|_\infty \leq 1$, one gets that $G_\lambda, g_\lambda \in \mathcal{L}_{loc}^p$ for any $p \geq 1$. It follows that $M_t(G_\lambda^x)$ and $M_t(g_\lambda^x)$ are both well-defined. By the linearity, for any $h \neq 0$ and $x \in \mathbb{R}$, we have

$$\frac{1}{h}\Big(M_t(G_\lambda^{x+h}) - M_t(G_\lambda^x) \Big) - M_t(g_\lambda^x) = M_t\Big(\frac{G_\lambda^{x+h} - G_\lambda^x}{h} - g_\lambda^x \Big).$$

Apply Lemma 2.5 and the above to see that for any $q \in (1, 1+\beta)$ and $T > 0$,

$$\mathbb{E}\Big[\sup_{0 \leq t \leq T} \Big| \frac{1}{h}(M_t(G_\lambda^{x+h}) - M_t(G_\lambda^x)) - M_t(g_\lambda^x) \Big|^q \Big]$$
$$\leq C \int_0^T ds \int_{\mathbb{R}} \Big| \frac{G_\lambda^{x+h}(y) - G_\lambda^x(y)}{h} - g_\lambda^x(y) \Big|^q X_0 p_s(y) dy$$
$$+ C \Big[\int_0^T ds \int_{\mathbb{R}} \Big| \frac{G_\lambda^{x+h}(y) - G_\lambda^x(y)}{h} - g_\lambda^x(y) \Big|^p X_0 p_s(y) dy \Big]^{q/p}. \qquad (2.12)$$

By using (2.5) and (2.6), we apply Dominated Convergence Theorem to see that

$$\lim_{h \to 0} \int_0^T ds \int_{\mathbb{R}} \Big| \frac{G_\lambda^{x+h}(y) - G_\lambda^x(y)}{h} - g_\lambda^x(y) \Big|^q X_0 p_s(y) dy = 0,$$

and

$$\lim_{h \to 0} \int_0^T ds \int_{\mathbb{R}} \Big| \frac{G_\lambda^{x+h}(y) - G_\lambda^x(y)}{h} - g_\lambda^x(y) \Big|^p X_0 p_s(y) dy = 0.$$

Using in the above in (2.12) to see that

$$\sup_{0 \leq t \leq T} \Big| \frac{1}{h}(M_t(G_\lambda^{x+h}) - M_t(G_\lambda^x)) - M_t(g_\lambda^x) \Big| \to 0 \text{ in } L^q.$$

In particular, the above implies that

$$\lim_{h \to 0} \mathbb{E}\Big[\sup_{0 \leq t \leq T} \Big| \frac{1}{h}(M_t(G_\lambda^{x+h}) - M_t(G_\lambda^x)) - M_t(g_\lambda^x) \Big|^q \Big] = 0.$$

In particular, by Hölder's inequality, we conclude that for any $t > 0$,

$$\lim_{h \to 0} \mathbb{E}\Big| \frac{1}{h}\Big(M_t(G_\lambda^{x+h}) - M_t(G_\lambda^x) \Big) - M_t(g_\lambda^x) \Big| = 0,$$

as required.

3 Continuity of the derivative

In this section, we will consider the continuity of $x \mapsto H^\lambda(t, x)$ defined in (1.13), and give the proof of Proposition 1.5 (ii). We first show that $\langle X_t, g_\lambda^x \rangle$, $\int_0^t \langle X_s, g_\lambda^x \rangle ds$ and $M_t(g_\lambda^x)$ all admit a continuous version in $x \in \mathbb{R}$.

Lemma 3.1. *Fix $t > 0$. With probability one, $x \mapsto \langle X_t, g_\lambda^x \rangle$ and $x \mapsto \int_0^t \langle X_s, g_\lambda^x \rangle ds$ are both locally Lipschitz continuous on $\mathbb{R}$.*

Proof. Fix $t > 0$ and $K > 0$. Pick any $x_1, x_2 \in [-K, K]$ such that $x_1 < x_2$. We note that when $d = 1$, Theorem 1.2 of [7] implies that for our superprocess X_t, there is a continuous function $X(t, x)$ such that for any Borel function ϕ,

$$\langle X_t, \phi \rangle = \int_{\mathbb{R}} X(t, x) \phi(x) dx.$$

Recall g_λ^x from (1.10). It follows that

$$|X_t(g_\lambda^{x_1}) - X_t(g_\lambda^{x_2})| \leq \int_{\mathbb{R}} |g_\lambda(y - x_1) - g_\lambda(y - x_2)| X(t, y) dy. \tag{3.1}$$

Notice that if $xy > 0$, then $\text{sgn}(x) = \text{sgn}(y)$. Hence

$$|g_\lambda(x) - g_\lambda(y)| = \left| e^{-\sqrt{2\lambda}|x|} - e^{-\sqrt{2\lambda}|y|} \right| \leq \sqrt{2\lambda}|x - y|,$$

where the last inequality uses the mean value theorem. Hence

$$|g_\lambda(y - x_1) - g_\lambda(y - x_2)| \leq \sqrt{2\lambda}|x_1 - x_2|, \quad \forall y \in \mathbb{R} \setminus [x_1, x_2]. \tag{3.2}$$

Using the above to see that (3.1) is bounded by

$$|X_t(g_\lambda^{x_1}) - X_t(g_\lambda^{x_2})| \leq \sqrt{2\lambda}|x_1 - x_2| \int_{\mathbb{R} \setminus [x_1, x_2]} X(t, y) dy + 2 \int_{y \in [x_1, x_2]} X(t, y) dy$$

$$\leq \left(\sqrt{2\lambda} X_t(1) + 2 \sup_{y \in [-K, K]} X(t, y) \right) |x_1 - x_2|. \tag{3.3}$$

Since $X(t, x)$ is continuous a.s., we get that $\sup_{y \in [-K, K]} X(t, y) < \infty$ a.s. Note that $\mathbb{E}[X_t(1)] = X_0(1)$ by (2.7), hence $X_t(1) < \infty$ a.s. We conclude that (3.3) implies $x \mapsto \langle X_t, g_\lambda^x \rangle$ is locally Lipschitz continuous on $\mathbb{R}$.

Turning to $\int_0^t \langle X_s, g_\lambda^x \rangle ds = \langle Y_t, g_\lambda^x \rangle$, we use (1.4) to see that

$$|Y_t(g_\lambda^{x_1}) - Y_t(g_\lambda^{x_2})| \leq \int_{\mathbb{R}} |g_\lambda(y - x_1) - g_\lambda(y - x_2)| L(t, y) dy. \tag{3.4}$$

The rest follows similarly to $\langle X_t, g_\lambda^x \rangle$. We omit the details. ∎

Next, we calculate the moment bounds for $M_t(g_\lambda^x)$.

Lemma 3.2. *For any $t > 0$, $q \in (1, 1 + \beta)$ and $K > 0$, there exists some constant $C = C(t, q, X_0(1), K) > 0$ such that*

$$\mathbb{E}|M_t(g_\lambda^{x_1}) - M_t(g_\lambda^{x_2})|^q \leq C|x_1 - x_2|, \quad \forall x_1, x_2 \in [-K, K]. \tag{3.5}$$

Proof. For any $x_1 < x_2$, we have

$$M_t(g_\lambda^{x_1}) - M_t(g_\lambda^{x_2}) = \int_0^t \int_{\mathbb{R}} (g_\lambda^{x_1}(y) - g_\lambda^{x_2}(y))(1_{y<x_1} + 1_{y>x_2})M(ds,dy)$$

$$- \int_0^t \int_{y \in [x_1,x_2]} (g_\lambda^{x_2}(y) - g_\lambda^{x_1}(y))M(ds,dy)$$

$$=: I_t(x_1,x_2) - Z_t(x_1,x_2). \tag{3.6}$$

Let $p = (3+\beta)/2 \in (1+\beta, 2)$ and $q \in (1, 1+\beta)$. By applying Lemma 2.5, we obtain that there exists some constant $C = C(\beta, p, q, t) > 0$ such that

$$\mathbb{E}[|I_t(x_1,x_2)|^q] \le C \int_0^t ds \int_{\mathbb{R}} |g_\lambda^{x_1}(y) - g_\lambda^{x_2}(y)|^q (1_{y<x_1} + 1_{y>x_2})X_0 p_s(y)dy$$

$$+ C\Big[\int_0^t ds \int_{\mathbb{R}} |g_\lambda^{x_1}(y) - g_\lambda^{x_2}(y)|^p (1_{y<x_1} + 1_{y>x_2})X_0 p_s(y)dy\Big]^{q/p}. \tag{3.7}$$

Using (3.2), we see that the first term on the right-hand side above is bounded by

$$C(2\lambda)^{q/2}|x_1 - x_2|^q \int_0^t ds \int_{\mathbb{R}} X_0 p_s(y)dy = C(2\lambda)^{q/2}|x_1 - x_2|^q t X_0(1).$$

Similarly, the second term on the right-hand side of (3.7) is bounded by

$$C\Big[(2\lambda)^{p/2}|x_1 - x_2|^p t X_0(1)\Big]^{q/p}.$$

It follows that (3.7) becomes

$$\mathbb{E}[|I_t(x_1,x_2)|^q] \le C_{t,\lambda,p,q,X_0(1)}|x_1 - x_2|^q. \tag{3.8}$$

Turning to $Z_t(x_1, x_2)$, similar to (3.7), we obtain

$$\mathbb{E}[|Z_t(x_1,x_2)|^q] \le C \int_0^t ds \int_{y \in [x_1,x_2]} |g_\lambda^{x_1}(y) - g_\lambda^{x_2}(y)|^q X_0 p_s(y)dy$$

$$+ C\Big[\int_0^t ds \int_{y \in [x_1,x_2]} |g_\lambda^{x_1}(y) - g_\lambda^{x_2}(y)|^p X_0 p_s(y)dy\Big]^{q/p}. \tag{3.9}$$

Using the bound $|g_\lambda^{x_1}(y) - g_\lambda^{x_2}(y)| \le 2$, we get that the first term on the right-hand side above is bounded by

$$C \int_0^t ds \int_{y \in [x_1,x_2]} 2^q dy \int_{\mathbb{R}} p_s(y-z)X_0(dz) \le CX_0(1)|x_1 - x_2| \int_0^t \frac{1}{\sqrt{s}}ds$$

$$\le CX_0(1)t^{1/2}|x_1 - x_2|.$$

Similarly, the second term on the right-hand side of (3.9) is bounded by

$$C\Big[CX_0(1)t^{1/2}|x_1 - x_2|\Big]^{q/p}.$$

It follows that (3.9) becomes

$$\mathbb{E}[|Z_t(x_1,x_2)|^q] \le C_{t,\lambda,p,q,X_0(1)}(|x_1 - x_2|^{q/p} + |x_1 - x_2|). \tag{3.10}$$

Combine (3.6), (3.8) and (3.10) to see that

$$\mathbb{E}|M_t(g_\lambda^{x_1}) - M_t(g_\lambda^{x_2})|^q \le C(|x_1 - x_2|^q + |x_1 - x_2|^{q/p} + |x_1 - x_2|) \le C|x_1 - x_2|,$$

where the last inequality uses $|x_1 - x_2| \le 2K$. $\blacksquare$

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The moment bounds in (3.5) do not satisfy the condition for Kolmogorov's continuity criterion. In fact, the moment calculations for $Z_t(x_1, x_2)$ as in (3.10) will not give us the desired bounds. To overcome this difficulty, we will use the following lemma, adapted from Theorem III.5.6 of Gihman and Skorohod [5], to show that $M_t(g_\lambda^x)$ has a locally Hölder continuous version.

Lemma 3.3. *Let $K > 0$. Consider a random process $(\xi_x)_{x \in [-K, K]}$ taking values in $\mathbb{R}$. If there exist a non-negative, monotonically non-decreasing function $g(h)$ and a function $q(r, h)$ such that for any $r > 0$ and $h > 0$,*

$$\mathbb{P}\{|\xi(x+h) - \xi(x)| > rg(h)\} \leq q(r, h), \quad \forall x \in [-K, K-h] \ \text{ and }$$

$$G = \sum_{n=0}^{\infty} g(2^{-n}K) < \infty, \quad Q(r) = \sum_{n=1}^{\infty} 2^n q(r, 2^{-n}K) < \infty, \tag{3.11}$$

then the process $(\xi_x)_{x \in [-K, K]}$ has a version $(\tilde{\xi}_x)_{x \in [-K, K]}$ with continuous sample paths. If we further assume that

$$Q(0, r) = \sum_{n=0}^{\infty} 2^n q(r, 2^{-n}K) \to 0 \quad as \ r \to \infty$$

for some integer $m > 0$, then, with probability one, there exists a constant $C = C(\omega)$ such that

$$\sup_{|x' - x''| < \delta} |\tilde{\xi}(x') - \tilde{\xi}(x'')| \leq C \cdot G\left(\left[\log_2 \frac{T}{2\delta}\right]\right), \quad \forall \delta > 0,$$

where $G(m) = \sum_{n=0}^{\infty} g(2^{-n}K)$.

Before moving to the proof of the continuity of $x \mapsto M_t(g_\lambda^x)$, we state some preliminary results.

3.1 Some preliminaries

The main difficulty comes from analyzing $Z_t(x_1, x_2)$ as (3.6). We introduce the following lemma from Lemma 2.15 of [4] that identifies the stochastic integral against the martingale measure M with a time-changed, spectrally positive $(1 + \beta)$-stable process.

Lemma 3.4 ([4]). *Suppose $p \in (1 + \beta, 2)$ and let $\psi \in \mathcal{L}_{loc}^p$ with $\psi \geq 0$. There exists a spectrally positive $(1 + \beta)$-stable process $\{L_s : s \geq 0\}$ such that*

$$\int_{(0,t] \times \mathbb{R}} \psi(s, y) M(ds, dy) = L_{T(t)}, \quad t \geq 0,$$

where $T(t) := \int_0^t ds \int_{\mathbb{R}} (\psi(s, y))^{1+\beta} X_s(dy)$.

Let $x_1 < x_2 \in \mathbb{R}$. Recall $Z_t(x_1, x_2)$ from (3.6). Set

$$\psi_0(y) := (g_\lambda(y - x_2) - g_\lambda(y - x_1)) \cdot 1_{\{x_1 \leq y \leq x_2\}} \tag{3.12}$$

so that $Z_t(x_1, x_2) = M_t(\psi_0)$. One may check that ψ_0 satisfies the assumption of Lemma 3.4. Hence there exists a spectrally positive $(1 + \beta)$-stable process $\{L_s : s \geq 0\}$ such that

$$Z_t(x_1, x_2) = M_t(\psi_0) = L_{T(t)}, \tag{3.13}$$

where

$$T(t) := \int_0^t ds \int_{x_1}^{x_2} \left(g_\lambda(y - x_2) - g_\lambda(y - x_1)\right)^{1+\beta} X_s(dy). \tag{3.14}$$

Lemma 3.5. *For any $t > 0$, $K > 0$ and $\varepsilon > 0$, there is some positive constant $c_{3.5} = c_{3.5}(\varepsilon, t, K, X_0(1))$ such that for $T(t)$ as in (3.14), we have*

$$\mathbb{P}\left(T(t) \leq c_{3.5}|x_1 - x_2|\right) \geq 1 - \varepsilon, \quad \forall -K \leq x_1 < x_2 \leq K. \tag{3.15}$$

Proof. Fix $t > 0$, $K > 0$ and $\varepsilon > 0$. Let $-K \leq x_1 < x_2 \leq K$. Apply (1.4) to see that

$$T(t) = \int_{x_1}^{x_2} (g_\lambda(z - x_2) - g_\lambda(z - x_1))^{1+\beta} L(t, z) dz.$$

Use $\|g_\lambda\|_\infty \leq 1$ to bound the above by

$$T(t) \leq 2^{1+\beta} \int_{x_1}^{x_2} L(t, z) dz \leq 2^{1+\beta}|x_1 - x_2| \cdot \sup_{|z| \leq K} L(t, z), \tag{3.16}$$

Since $L(t, z)$ is jointly continuous a.s. by Theorem A, we get

$$\mathbb{P}\left(\sup_{|z| \leq K} L(t, z) < \infty\right) = 1.$$

Hence for any $\varepsilon > 0$, there exists a constant $C > 0$ such that

$$\mathbb{P}\left(\sup_{|z| \leq K} L(t, z) \leq C\right) \geq 1 - \varepsilon. \tag{3.17}$$

The proof is complete in view of (3.16) and (3.17). ■

Next, we study the jumps of our superprocess X in the closed interval $[x_1, x_2]$. Recall from Lemma 2.4 that all the jumps of X, ΔX_s, is given by $\Delta X_s = r\delta_x$ for some $r > 0$ and $x \in \mathbb{R}$. Let

$$\Delta X_s^{[x_1, x_2]} := \Delta X_s([x_1, x_2]) = X_s([x_1, x_2]) - X_{s-}([x_1, x_2]) \tag{3.18}$$

denote the jump size of X in the closed interval $[x_1, x_2]$ at time $s > 0$.

Lemma 3.6. *For any $t > 0$, $K > 0$ and $\varepsilon > 0$, there exists some positive constant $c_{3.6} = c_{3.6}(\varepsilon, t, K, X_0(1))$ such that for all $-K \leq x_1 < x_2 \leq K$,*

$$\mathbb{P}\left(\exists s < t, \Delta X_s^{[x_1, x_2]} \geq c_{3.6}|x_1 - x_2|^{\frac{1}{1+\beta}}\right) \leq \varepsilon. \tag{3.19}$$

Proof. Fix $t > 0$, $K > 0$ and $\varepsilon > 0$. Let $-K \leq x_1 < x_2 \leq K$. For any $b > 0$, we set

$$y_b = b|x_1 - x_2|^{\frac{1}{1+\beta}}. \tag{3.20}$$

Recall the random point measure $N(ds, dx, dr)$ in Lemma 2.4. Define

$$Y_0 := N\left([0, 2^{-1}t] \times [x_1, x_2] \times (y_b, \infty)\right),$$
$$Y_n := N\left([(1 - 2^{-n})t, (1 - 2^{-n-1})t] \times [x_1, x_2] \times (y_b, \infty)\right), \quad n \geq 1,$$

One can check by definition that

$$\mathbb{P}\left(\exists s < t, \Delta X_s^{[x_1, x_2]} \geq b|x_1 - x_2|^{\frac{1}{1+\beta}}\right) \leq \mathbb{P}\left(\sum_{n=0}^{\infty} Y_n \geq 1\right) \leq \sum_{n=0}^{\infty} \mathbb{E}Y_n. \tag{3.21}$$

In the last inequality, we used Markov's inequality. From the formula for the compensator $\hat{N}$ of N in Lemma 2.4, we obtain that for each $n \geq 0$,

$$\mathbb{E}[Y_n] = c_\beta \int_{(1-2^{-n})t}^{(1-2^{-n-1})t} ds \cdot \mathbb{E}\left(X_s([x_1, x_2])\right) \int_{y_b}^{\infty} r^{-2-\beta} dr.$$

Apply (2.7) to see that

$$\mathbb{E}\left(X_s([x_1, x_2])\right) = \int_{\mathbb{R}} X_0(dz) \int_{\mathbb{R}} p_s(z - y) 1_{y \in [x_1, x_2]} dy \leq C s^{-1/2} X_0(1) |x_1 - x_2|.$$

Plugging in the above to get that

$$\mathbb{E}[Y_n] \leq C X_0(1) |x_1 - x_2| \cdot y_b^{-1-\beta} \int_{(1-2^{-n})t}^{(1-2^{-n-1})t} s^{-1/2} ds.$$

Sum $\mathbb{E}[Y_n]$ for all $n \geq 0$ to obtain

$$\sum_{n=0}^{\infty} \mathbb{E} Y_n \leq C X_0(1) |x_1 - x_2| \cdot y_b^{-1-\beta} \int_0^t s^{-1/2} ds$$

$$\leq C t^{1/2} X_0(1) b^{-(1+\beta)}, \tag{3.22}$$

where the last inequality uses y_b from (3.20). Let $b > 0$ be large enough such that

$$C t^{1/2} X_0(1) b^{-(1+\beta)} \leq \varepsilon. \tag{3.23}$$

Combine (3.21), (3.22) and (3.23) to conclude that

$$\mathbb{P}\left(\exists s < t, \ \Delta X_s^{[x_1, x_2]} \geq b |x_1 - x_2|^{\frac{1}{1+\beta}}\right) \leq \sum_{n=0}^{\infty} \mathbb{E} Y_n \leq \varepsilon,$$

as required. ∎

The final ingredient is Lemma 2.3 from [4]. Recall $\{L_s : s \geq 0\}$ is a spectrally positive $(1 + \beta)$-stable process starting from $L_0 = 0$.

Lemma 3.7. *Denote by $\Delta L_s := L_s - L_{s-} > 0$ the jumps of L. There exists some absolute constant $C > 0$ such that for any $t > 0$ and $x, y > 0$, we have*

$$\mathbb{P}\left(\sup_{0 \leq u \leq t} L_u \cdot 1\left\{\sup_{0 \leq v \leq u} \Delta L_v \leq y\right\} \geq x\right) \leq \left(\frac{Ct}{xy^\beta}\right)^{x/y}.$$

and

$$\mathbb{P}\left(\inf_{0 \leq u \leq t} L_u < -x\right) \leq \exp\left(-\frac{C x^{(1+\beta)/\beta}}{t^{1/\beta}}\right).$$

We will now proceed to the proof of Proposition 1.5 (ii).

3.2 Proof of Proposition 1.5 (ii)

To complete the proof of Proposition 1.5 (ii), we will show that $x \mapsto M_t(g_\lambda^x)$ has a locally Hölder continuous version with any index $0 < \gamma < \frac{\beta}{1+\beta}$, using Lemma 3.3. We will do this in three steps.

Step 1. Fix $\gamma \in (0, \frac{\beta}{1+\beta})$. Let $\varepsilon > 0$, $K > 0$ and $t > 0$. Fix $x_1, x_2 \in \mathbb{R}$ such that

$$-K \leq x_1 < x_2 \leq K.$$

Pick $q \in (1, 1+\beta)$ such that

$$0 < \gamma < 1 - \frac{1}{q} < \frac{\beta}{1+\beta}. \tag{3.24}$$

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Recall $T(t)$ from (3.14) and $\Delta X_s^{[x_1,x_2]}$ from (3.18). Define the event A^ε by

$$A^\varepsilon := \{\Delta X_s^{[x_1,x_2]} < c_{3.6}|x_1 - x_2|^{\frac{1}{1+\beta}} \text{ for all } s < t\} \cap \{T(t) \le c_{3.5}|x_1 - x_2|\}. \tag{3.25}$$

By Lemmas 3.5 and 3.6, we have that

$$\mathbb{P}(A^\varepsilon) \ge 1 - 2\varepsilon. \tag{3.26}$$

Denote

$$M_t(g_\lambda^{x,\varepsilon}) := M_t(g_\lambda^x) \cdot 1_{A^\varepsilon}. \tag{3.27}$$

For any $r > 0$, we apply (3.6) to see that

$$
\begin{aligned}
&\mathbb{P}(|M_t(g_\lambda^{x_1,\varepsilon}) - M_t(g_\lambda^{x_2,\varepsilon})| \ge 3r|x_1 - x_2|^\gamma) \\
&\le \mathbb{P}(|I_t(x_1, x_2)| \ge r|x_1 - x_2|^\gamma) + \mathbb{P}(|Z_t(x_1, x_2) \cdot 1_{A^\varepsilon}| \ge r|x_1 - x_2|^\gamma) \\
&\le Cr^{-q}|x_1 - x_2|^{q(1-\gamma)} + \mathbb{P}(|Z_t(x_1, x_2)1_{A^\varepsilon}| \ge r|x_1 - x_2|^\gamma),
\end{aligned}
\tag{3.28}
$$

where the last inequality uses the Markov inequality and (3.8) with our $q \in (1, 1+\beta)$. It remains to bound the last probability above.

Recall (3.13) to see that

$$
\begin{aligned}
&\mathbb{P}(|Z_t(x_1, x_2) \cdot 1_{A^\varepsilon}| \ge r|x_1 - x_2|^\gamma) \\
&\le \mathbb{P}(L_{T(t)} \le -r|x_1 - x_2|^\gamma, A^\varepsilon) + \mathbb{P}(L_{T(t)} \ge r|x_1 - x_2|^\gamma, A^\varepsilon) =: I_1 + I_2.
\end{aligned}
\tag{3.29}
$$

Using the definition of A^ε, we get that

$$I_1 \le \mathbb{P}\left(\inf_{u \le c_{3.5}|x_1-x_2|} L_u \le -r|x_1 - x_2|^\gamma\right) \le e^{-Cr^{(1+\beta)/\beta}|x_1-x_2|^{\gamma(1+\frac{1}{\beta})-\frac{1}{\beta}}}, \tag{3.30}$$

where the last inequality follows by applying Lemma 3.7.

Turning to I_2, by the very definition of $M(ds, dy)$ (see Lemma 2.4 or Théorème 7 in [2]), we know that the jumps of $M(ds, dy)$ are the same as those of X_s. Hence on the event A^ε, the jump sizes of $M(ds, dy)$ with locations in $[x_1, x_2]$ do not exceed that of X in $[x_1, x_2]$, which are bounded above by

$$\Delta X_s^{[x_1,x_2]} < c_{3.6}|x_1 - x_2|^{\frac{1}{1+\beta}}.$$

Since $Z_t(x_1, x_2) = M_t(\psi_0)$ with ψ_0 as in (3.12), one may conclude from the above that the jumps of the process $u \mapsto \int_{(0,u]\times\mathbb{R}} \psi_0(y)M(ds, dy)$ are bounded by

$$c_{3.6}|x_1 - x_2|^{\frac{1}{1+\beta}} \cdot \|\psi_0\|_\infty \le 2c_{3.6}|x_1 - x_2|^{\frac{1}{1+\beta}}.$$

Returning to $L_{T(t)}$ as in (3.13), the above implies that on A^ε,

$$\Delta L_v \le 2c_{3.6}|x_1 - x_2|^{\frac{1}{1+\beta}}, \quad \forall v \le T(t).$$

It follows that

$$
\begin{aligned}
I_2 &\le \mathbb{P}(L_{T(t)} \ge r|x_1 - x_2|^\gamma, \sup_{v \le T(t)} \Delta L_v \le c_{3.6}|x_1 - x_2|^{\frac{1}{1+\beta}}, A^\varepsilon) \\
&\le \mathbb{P}\left(\sup_{0 \le u \le c_{3.5}|x_1-x_2|} L_u \cdot 1\left\{\sup_{0\le v \le u} \Delta L_u \le c_{3.6}|x_1 - x_2|^{\frac{1}{1+\beta}}\right\} \ge r|x_1 - x_2|^\gamma\right) \\
&\le \left(\frac{C \cdot c_{3.5}|x_1 - x_2|}{r|x_1 - x_2|^\gamma \cdot (c_{3.6}|x_1 - x_2|^{\frac{1}{1+\beta}})^\beta}\right)^{c_{3.6}^{-1}r|x_1-x_2|^{\gamma-\frac{1}{1+\beta}}} \\
&\le (Cr^{-1}|x_1 - x_2|^{\frac{1}{1+\beta}-\gamma})^{Cr|x_1-x_2|^{\gamma-\frac{1}{1+\beta}}},
\end{aligned}
\tag{3.31}
$$

where the last inequality follows by applying Lemma 3.7.

Combine (3.28)-(3.31) to conclude that

$$\mathbb{P}(|M_t(g_\lambda^{x_1,\varepsilon})-M_t(g_\lambda^{x_2,\varepsilon})| \geq 3r|x_1-x_2|^\gamma) \leq Cr^{-q}|x_1-x_2|^{q(1-\gamma)}$$
$$+ e^{-Cr^{(1+\beta)/\beta}|x_1-x_2|^{\gamma(1+\frac{1}{\beta})-\frac{1}{\beta}}} + (Cr^{-1}|x_1-x_2|^{\frac{1}{1+\beta}-\gamma})Cr|x_1-x_2|^{\gamma-\frac{1}{1+\beta}}. \quad (3.32)$$

Step 2. By comparing (3.32) with the conditions of Lemma 3.3, we let $g(h) = 3h^\gamma$ and

$$q(r,h) = Cr^{-q}h^{q(1-\gamma)} + e^{-Cr^{(1+\beta)/\beta}h^{\gamma(1+\frac{1}{\beta})-\frac{1}{\beta}}} + (Cr^{-1}h^{\frac{1}{1+\beta}-\gamma})Crh^{\gamma-\frac{1}{1+\beta}}.$$

To apply Lemma 3.3 to the process $x \mapsto M_t(g_\lambda^{x,\varepsilon})$ for $x \in [-K, K]$, we will verifty that

$$G = \sum_{n=0}^\infty g(2^{-n}K) < \infty, \quad Q(r) = \sum_{n=0}^\infty 2^n q(r, 2^{-n}K) < \infty, \ \forall r > 0$$

and

$$Q(0, r) = \sum_{n=0}^\infty 2^n q(r, 2^{-n}K) \to 0 \quad \text{as } r \to \infty.$$

- Condition on G: Easy to notice that

$$G = \sum_{n=0}^\infty 3(2^{-n}K)^\gamma = 3K^\gamma \sum_{n=0}^\infty (2^{-\gamma})^n < \infty.$$

- Condition on $Q(r)$: For any $r > 0$ and $n \geq 0$, we have

$$2^n q(r, 2^{-n}K) = 2^n Cr^{-q}(2^{-n}K)^{q(1-\gamma)} + 2^n e^{-Cr^{(1+\beta)/\beta}(2^{-n}K)^{\gamma(1+\frac{1}{\beta})-\frac{1}{\beta}}}$$
$$+ 2^n (Cr^{-1}(2^{-n}K)^{\frac{1}{1+\beta}-\gamma})Cr(2^{-n}K)^{\gamma-\frac{1}{1+\beta}}.$$

It follows that

$$Q(r) = Cr^{-q}K^{q(1-\gamma)}\sum_{n=0}^\infty 2^{n(1-q(1-\gamma))} + \sum_{n=0}^\infty 2^n e^{-Cr^{(1+\beta)/\beta}K^{\gamma(1+\frac{1}{\beta})-\frac{1}{\beta}}2^{n(\frac{1}{\beta}-\gamma(1+\frac{1}{\beta}))}}$$
$$+ \sum_{n=0}^\infty 2^n (Cr^{-1}K^{\frac{1}{1+\beta}-\gamma}2^{-n(\frac{1}{1+\beta}-\gamma)})CrK^{\gamma-\frac{1}{1+\beta}}2^{n(\frac{1}{1+\beta}-\gamma)} =: Q_A(r) + Q_B(r) + Q_C(r).$$

$$(3.33)$$

Recall from (3.24) to see that

$$\gamma < 1 - \frac{1}{q} \iff 1 - q(1-\gamma) < 0.$$

Hence

$$Q_A(r) = Cr^{-q}K^{q(1-\gamma)}\sum_{n=0}^\infty 2^{n(1-q(1-\gamma))} < \infty. \quad (3.34)$$

Next, again we use (3.24) to see that

$$\gamma < \frac{\beta}{1+\beta} \iff \frac{1}{\beta} - \gamma(1+\frac{1}{\beta}) > 0. \quad (3.35)$$

It follows that $Q_B(r) < \infty$.

Thirdly, by (3.35) and $\beta \in (0,1)$, we get that

$$\frac{1}{1+\beta} - \gamma > 0.$$

Hence, one may easily check that $Q_C(r) < \infty$. Hence we conclude $Q(r) < \infty$ in view of (3.33).

- Condition on $Q(0,r)$: We will show that $Q(0,r) \to 0$ as $r \to \infty$. It suffices to show that all three summations in (3.33) converge to 0 as $r \to \infty$.

  The first term $Q_A(r)$ follows easily in view of (3.34). For the second term $Q_B(r)$, we note that for each $n \geq 0$, the summand converges to 0 as $r \to \infty$. In the meantime, the summand is bounded by

  $$2^n e^{-CK^{\gamma(1+\frac{1}{\beta})-\frac{1}{\beta}} 2^{n(\frac{1}{\beta}-\gamma(1+\frac{1}{\beta}))}}, \quad \forall n \geq 0, \, r \geq 1.$$

  The above is summable for $n \geq 0$. The Dominated Convergence Theorem concludes that $Q_B(r) \to 0$ is $r \to \infty$.

  Finally, for the third term $Q_C(r)$, we note that when $r \geq 1$ is large, we get

  $$Cr^{-1}K^{\frac{1}{1+\beta}-\gamma} 2^{-n(\frac{1}{1+\beta}-\gamma)} \leq \frac{1}{2}, \quad \forall n \geq 0.$$

  Hence, the summand is bounded by

  $$2^n \cdot 2^{-CK^{\gamma-\frac{1}{1+\beta}} 2^{n(\frac{1}{1+\beta}-\gamma)}}.$$

  The above is summable for $n \geq 0$. In the meantime, it is clear that for each $n \geq 0$, the summand converges to 0 as $r \to \infty$. The Dominated Convergence Theorem concludes that $Q_C(r) \to 0$ is $r \to \infty$.

Now that all the conditions for Lemma 3.3 are satisfied, we may conclude that $x \mapsto M_t(g_\lambda^{x,\varepsilon})$ has a continuous version for $x \in [-K, K]$, and in particular, with probability one, there exists a constant $C = C(\omega)$ such that

$$\sup_{|x_1-x_2|<\delta} |M_t(g_\lambda^{x_1,\varepsilon}) - M_t(g_\lambda^{x_2,\varepsilon})| \leq C(\omega)G\left(\left[\log_2 \frac{K}{2\delta}\right]\right), \quad \forall \delta > 0, \tag{3.36}$$

where

$$G(m) = \sum_{n=m}^{\infty} g(2^{-n}K).$$

Plugging in $g(h) = 3h^\gamma$, one can check that there is some absolute constant $C > 0$ such that

$$G\left(\left[\log_2 \frac{K}{2\delta}\right]\right) = \sum_{n=[\log_2 \frac{K}{2\delta}]}^{\infty} 3(2^{-n}K)^\gamma \leq C\delta^\gamma.$$

Hence we deduce from (3.36) that

$$\sup_{|x_1-x_2|<\delta} |M_t(g_\lambda^{x_1,\varepsilon}) - M_t(g_\lambda^{x_2,\varepsilon})| \leq C(\omega)\delta^\gamma, \quad \forall \delta > 0.$$

Hence for any $\delta > 0$, if $\delta/2 < |x_1 - x_2| < \delta$, the above implies that

$$|M_t(g_\lambda^{x_1,\varepsilon}) - M_t(g_\lambda^{x_2,\varepsilon})| \leq C(\omega)(2|x_1-x_2|)^\gamma \leq C(\omega)|x_1-x_2|^\gamma. \tag{3.37}$$

Step 3. Using (3.37), we will show that $M_t(g_\lambda^x)$ also admits a version which is locally γ-Hölder continuous. For any $J > 0$, we use (3.27) to see that

$$\mathbb{P}\Big(\sup_{x_1,x_2\in[-K,K],x_1\neq x_2} \frac{|M_t(g_\lambda^{x_1}) - M_t(g_\lambda^{x_2})|}{|x_1-x_2|^\gamma} > J\Big)$$

$$\leq \mathbb{P}\Big(\sup_{x_1,x_2\in[-K,K],x_1\neq x_2} \frac{|M_t(g_\lambda^{x_1,\varepsilon}) - M_t(g_\lambda^{x_2,\varepsilon})|}{|x_1-x_2|^\gamma} > J\Big) + \mathbb{P}(A^{\varepsilon,c}),$$

where $A^{\varepsilon,c}$ denotes the complement of A^ε. Notice that (3.37) gives that

$$\limsup_{J\to\infty} \mathbb{P}\Big(\sup_{x_1,x_2\in[-K,K],x_1\neq x_2} \frac{|M_t(g_\lambda^{x_1,\varepsilon}) - M_t(g_\lambda^{x_2,\varepsilon})|}{|x_1-x_2|^\gamma} > J\Big) = 0.$$

Hence, together with (3.26), we obtain

$$\limsup_{J\to\infty} \mathbb{P}\Big(\sup_{x_1,x_2\in[-K,K],x_1\neq x_2} \frac{|M_t(g_\lambda^{x_1}) - M_t(g_\lambda^{x_2})|}{|x_1-x_2|^\gamma} > J\Big) \leq 2\varepsilon.$$

Since ε can be arbitrarily small, the above immediately implies that

$$\mathbb{P}\Big(\sup_{x_1,x_2\in[-K,K],x_1\neq x_2} \frac{|M_t(g_\lambda^{x_1}) - M_t(g_\lambda^{x_2})|}{|x_1-x_2|^\gamma} < \infty\Big) = 1. \tag{3.38}$$

Hence we prove the desired local Hölder continuity of $x \mapsto M_t(g_\lambda^x)$ with any index $\gamma < \frac{\beta}{1+\beta}$.

With the three steps above, we are ready to finish the proof of Proposition 1.5 (ii).

Proof of Proposition 1.5 (ii). The Hölder continuity of $x \mapsto H^\lambda(t,x)$ is immediate in view of (1.16), Lemma 3.1 and (3.38). It remains to prove (1.17).

Let $q \in (1, 1+\beta)$ and $K > 0$. For any $-K \leq x_1 < x_2 \leq K$, by Lemma 3.2 and Hölder's inequality, we have

$$\mathbb{E}|M_t(g_\lambda^{x_1}) - M_t(g_\lambda^{x_2})| \leq (\mathbb{E}|M_t(g_\lambda^{x_1}) - M_t(g_\lambda^{x_2})|^q)^{1/q} \leq C|x_1-x_2|^{1/q}. \tag{3.39}$$

Next, by using (3.2), we get that

$$\mathbb{E}|X_t(g_\lambda^{x_1}) - X_t(g_\lambda^{x_2})| \leq \sqrt{2\lambda}|x_1-x_2| \cdot \mathbb{E}[X_t(1)] + \mathbb{E}\left[\int_\mathbb{R} 2\cdot 1_{y\in[x_1,x_2]} X_t(dy)\right].$$

Now apply the mean measure formula from (2.7) to see that $\mathbb{E}[X_t(1)] = X_0(1)$ and

$$\mathbb{E}\left[\int_\mathbb{R} 2\cdot 1_{y\in[x_1,x_2]} X_t(dy)\right] = 2\int_\mathbb{R} X_0(dy) \int_{y\in[x_1,x_2]} p_t(y,z)dz \leq Ct^{-1/2}|x_1-x_2|.$$

It follows that

$$\mathbb{E}|X_t(g_\lambda^{x_1}) - X_t(g_\lambda^{x_2})| \leq (Ct^{-1/2} + \sqrt{2\lambda}X_0(1))|x_1-x_2|. \tag{3.40}$$

Similarly, one may use (2.8) and (3.2) to obtain

$$\mathbb{E}|Y_t(g_\lambda^{x_1}) - Y_t(g_\lambda^{x_2})| \leq \sqrt{2\lambda}|x_1-x_2| \cdot \mathbb{E}[Y_t(1)] + \mathbb{E}\left[\int_\mathbb{R} 2\cdot 1_{y\in[x_1,x_2]} Y_t(dy)\right]$$

$$\leq \sqrt{2\lambda}t X_0(1)\cdot|x_1-x_2| + Ct^{1/2}|x_1-x_2|. \tag{3.41}$$

The proof of (1.17) follows by (3.39)-(3.41) and $|x_1-x_2| \leq 2K$. The proof of Proposition 1.5 (ii) is complete. ∎

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A   Proof of Lemma 1.1

Let $X_0 \in M_F$ and $\lambda > 0$. By using (2.4), it is immediate that for any $x_1, x_2 \in \mathbb{R}$,

$$|X_0(G_\lambda^{x_1}) - X_0(G_\lambda^{x_2})| \leq CX_0(1)|x_1 - x_2|. \tag{A.1}$$

Hence $x \mapsto X_0(G_\lambda^x)$ is continuous on $\mathbb{R}$. Next, fix $x \in \mathbb{R}$ and let $h > 0$. Then we have

$$\frac{1}{h}|X_0(G_\lambda^{x+h}) - X_0(G_\lambda^x)| = \Big| \int_{z \neq x} \frac{1}{h}(G_\lambda(z - x - h) - G_\lambda(z - x))X_0(dz)$$
$$+ \int_{\{x\}} \frac{1}{h}(G_\lambda(z - x - h) - G_\lambda(z - x))X_0(dz)\Big| \tag{A.2}$$

Using (1.11), (A.1) and Dominated Convergence Theorem, we may let $h \downarrow 0$ to see that

$$(D_x^+ X_0(G_\lambda^x))(x) = \lim_{h\downarrow 0} \frac{1}{h}|X_0(G_\lambda^{x+h}) - X_0(G_\lambda^x)| = X_0(g_\lambda^x) - X_0(\{x\}).$$

The proof of $(D_x^- X_0(G_\lambda^x))(x)$ follows similarly.

Hence if $X_0 \in M_F(\mathbb{R})$ is atomless, (1.12) readily implies

$$D_x X_0(G_\lambda^x) = X_0(g_\lambda^x). \tag{A.3}$$

Moreover, for any $x \in \mathbb{R}$ and $h > 0$, by using (3.2), we get that

$$|X_0(g_\lambda^{x+h}) - X_0(g_\lambda^x)| \le \sqrt{2\lambda} X_0(1) h + \int_\mathbb{R} 2 \cdot 1_{z \in [x,x+h]} X_0(dz). \tag{A.4}$$

Since $X_0(\{x\}) = 0$, if we let $h \downarrow 0$, we have

$$\int_\mathbb{R} 2 \cdot 1_{z \in [x,x+h]} X_0(dz) \to 0. \tag{A.5}$$

Hence

$$\lim_{h\downarrow 0} |X_0(g_\lambda^{x+h}) - X_0(g_\lambda^x)| = 0. \tag{A.6}$$

The case for $h \uparrow 0$ is similar. We omit the details.

B   Proof of Lemma 1.4

Proof. Define $\widetilde{\Gamma}_x := \Gamma_x - \Gamma_0$ for each $x \in \mathbb{R}$. We claim that

$$\frac{d}{dx} \mathbb{E}|\widetilde{\Gamma}_x| = 0, \quad \forall x \in \mathbb{R}. \tag{B.1}$$

Given the above, we get that for any fixed $x \in \mathbb{R}$,

$$\mathbb{E}|\widetilde{\Gamma}_x| = \mathbb{E}|\widetilde{\Gamma}_0| = 0,$$

thus giving

$$\widetilde{\Gamma}_x = 0 \text{ a.s.} \implies \Gamma_x = \Gamma_0 \text{ a.s.}$$

Apply Fubini's theorem to see that the above implies that with probability one, $\Gamma_x = \Gamma_0$ for almost all $x \in \mathbb{R}$. Since $x \mapsto \Gamma_x$ is continuous on $\mathbb{R}$ a.s., the desired conclusion follows.

It remains to prove (B.1). Notice that

$$\lim_{h\to 0} \frac{1}{|h|} \left| \mathbb{E}|\widetilde{\Gamma}_{x+h}| - \mathbb{E}|\widetilde{\Gamma}_x| \right| \le \lim_{h\to 0} \frac{1}{|h|} \mathbb{E} \left| |\widetilde{\Gamma}_{x+h}| - |\widetilde{\Gamma}_x| \right| \le \lim_{h\to 0} \frac{1}{|h|} \mathbb{E} \left| \widetilde{\Gamma}_{x+h} - \widetilde{\Gamma}_x \right| = 0,$$

where the second inequality uses the triangle inequality, and the last equality follows from the assumption (1.14). Therefore, we conclude that

$$\frac{d}{dx} \mathbb{E}|\widetilde{\Gamma}_x| = \lim_{h\to 0} \frac{1}{h} \left(\mathbb{E}|\widetilde{\Gamma}_{x+h}| - \mathbb{E}|\widetilde{\Gamma}_x| \right) = 0, \quad \forall x \in \mathbb{R},$$

as required. ∎

C   Proof of Lemma 2.2

Proof. Recall that

$$G_\lambda(x) = (2\pi)^{-1/2} \int_0^\infty s^{-1/2} e^{-\lambda s - x^2/(2s)} ds. \tag{C.1}$$

We will use the integral representation for the Bessel function $K_\nu(z)$ (See, e.g., 10.32 in[9])

$$\int_0^\infty t^{\nu-1} e^{-at - \frac{b}{t}} \, dt = 2\Big(\frac{b}{a}\Big)^{\nu/2} K_\nu(2\sqrt{ab}), \tag{C.2}$$

where $K_\nu(x)$ is the modified Bessel function of the second kind.

By letting $\nu = 1/2$ and using the formula $K_{1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}$ to (C.2), we obtain

$$\int_0^\infty t^{-1/2} e^{-at - \frac{b}{t}} \, dt = \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}}. \tag{C.3}$$

Let $a = \lambda$ and $b = \frac{x^2}{2}$ in (C.3), then

$$\int_0^\infty t^{-1/2} e^{-\lambda t - \frac{x^2}{2t}} \, dt = \sqrt{\frac{\pi}{\lambda}} e^{-\sqrt{2\lambda}|x|}.$$

Substitute it into (C.1) to see that

$$G_\lambda(x) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi}{\lambda}} e^{-\sqrt{2\lambda}|x|} = \frac{1}{\sqrt{2\lambda}} e^{-\sqrt{2\lambda}|x|}.$$

Next, by the mean-value theorem, for any $x, y \in \mathbb{R}$

$$|G_\lambda(x) - G_\lambda(y)| = \frac{1}{\sqrt{2\lambda}} \left| e^{-\sqrt{2\lambda}|x|} - e^{-\sqrt{2\lambda}|y|} \right|$$

$$\leq \frac{1}{\sqrt{2\lambda}} \sqrt{2\lambda} \cdot \left| |x| - |y| \right| \leq |x - y|.$$

The proof is now complete. ∎