# QUADRATIC EMBEDDING CONSTANTS OF CARTESIAN PRODUCTS AND JOINS OF GRAPHS

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ABSTRACT. The quadratic embedding constant (QEC) of a finite, simple, connected graph originated from the classical work of Schoenberg [Ann. of Math., 1935] and [Trans. Amer. Math. Soc., 1938] on Euclidean distance geometry. In this article, we study the QEC of graphs in terms of two graph operations: the Cartesian product and the join of graphs. We derive a general formula for the QEC of the join of an arbitrary graph with a regular graph and with a complete multipartite graph. We then provide quadratic embedding constants for the Cartesian product of an arbitrary graph G with a complete graph and with a complete bipartite graph in terms of QEC(G).

#### 1. Introduction

All graphs in this paper are assumed to be finite, simple, and unweighted. Given a graph G = (V, E), we define C(V): the space of all  $\mathbb{R}$ -valued functions on V, e: the constant function in C(V) taking value 1, J: matrix with all entries one and  $\langle \cdot, \cdot \rangle$ : the canonical inner product on  $C(V) \cong \mathbb{R}^{|V|}$ ,  $A_G$ : the adjacency matrix of G whose (v, w) entry is 1 if v is adjacent to w, and 0 otherwise, for all  $v, w \in V$ . If G is connected, we define  $D_G$ : the distance matrix of G with the (v, w) entry given by the length of a shortest path connecting  $v \neq w \in V$ , and  $(D_G)_{vv} = 0 \ \forall v \in V$ . For a positive integer n, we define  $[n] := \{1, \ldots, n\}$ . Finally, for a square matrix M,  $\sigma(M)$  denotes the set of all eigenvalues of M and  $\sigma_0(M) := \{\lambda \in \sigma(M) : \lambda \text{ has an associated eigenvector } x \text{ with } \langle e, x \rangle = 0\}$ .

The goal of this article is to study the quadratic embedding constant of a connected graph G using the graph operations, the Cartesian product and the join of graphs. A connected graph G = (V, E) is quadratically embeddable into a Euclidean space  $\mathcal{H}$  if there exists a map  $\psi: V \to \mathcal{H}$  such that  $d(i,j) = \|\psi(i) - \psi(j)\|^2$  for all  $i,j \in V$ , where  $\|\cdot\|$  denotes the norm of  $\mathcal{H}$ . Such a map  $\psi$  from V into  $\mathcal{H}$  is called a quadratic embedding of G and if G admits a quadratic embedding, we say that G is of QE class. Graphs of QE class have numerous applications in several branches of mathematics, including quantum probability and non-commutative harmonic analysis [2, 3, 5, 6, 7, 15, 16]. In 1935, Schoenberg [19] gave a striking characterization of the QE class in terms of the conditional negative definiteness. A symmetric matrix  $A \in \mathbb{R}^{n \times n}$  is said to be conditionally negative definite if  $f^T A f \leq 0$  for all  $f \in \mathbb{R}^n$  with  $\langle e, f \rangle = 0$ . Schoenberg showed that a connected graph G is of QE class if and only if  $D_G$  is conditionally negative definite. Inspired by this result, Obata–Zakiyyah [17] introduced the notion of quadratic embedding constant of G, denoted and defined as follows.

QEC(G) := 
$$\max\{\langle f, D_G f \rangle; f \in C(V), \langle f, f \rangle = 1, \langle e, f \rangle = 0\}.$$

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In particular, Schoenberg's result states that a connected graph G belongs to the QE class if and only if  $QEC(G) \leq 0$ . Since then, the QEC of graphs has been studied by several authors [1, 8, 10, 11, 12, 13, 17]. For instance, the quadratic embedding constants of some well-known classes of graphs, including the complete graph, cycle, path, and complete multipartite graph are studied in [10, 13, 17]. In [4], we provided tight lower and upper bounds for the QEC of a tree, and showed that it lies between a path and a star for all trees.

Determining the quadratic embedding constant for a more general class of graphs is a challenging problem. A well-studied question involving the QEC asks to study the QEC of a connected graph G by factorizing it into smaller graphs using a graph operation. In the literature, QEC of graphs is explored using several graph operations, including Cartesian product [17], star product [1, 11], lexicographic product [8], graph joins [8], and cluster of graphs [4]. In [14], Obata provided the QEC formula for the wheel graph  $K_1+C_n$ . In [8], Lou-Obata-Huang extended this result by giving a formula for QEC( $G_1 + G_2$ ), when  $G_1$  and  $G_2$  are regular graphs. Recently, Młotkowski-Obata [9] gave a formula for the quadratic embedding constant of the join of  $\overline{K}_m$  and an arbitrary graph G. Our first main result extends their work by replacing  $\overline{K}_m$  with an arbitrary regular graph. We provide a formula for QEC( $G_1 + G_2$ ), when  $G_1$  is a regular graph and  $G_2$  is an arbitrary graph. Note that if  $G_1 + G_2$  is complete then QEC( $G_1 + G_2$ ) = -1.

**Theorem 1.1.** Let  $G_1$  and  $G_2$  be two disjoint graphs on  $m \ge 1$  and  $n \ge 1$  vertices respectively such that  $G_1 + G_2$  is not complete. Suppose  $G_1$  is an r-regular graph. Then

QEC
$$(G_1 + G_2) = -2 - \left(\bigcup_{i=1}^{4} (\Lambda_i(G_1 + G_2) \cap (-\infty, -1))\right),$$

where

$$\Lambda_1(G_1 + G_2) := \{r - m\} \cap (\sigma(A_{G_1}) \cup \sigma(A_{G_2} - J)),$$

$$\Lambda_2(G_1 + G_2) := \{r - 2m\} \cap \sigma(A_{G_2}),$$

$$\Lambda_3(G_1 + G_2) := (\sigma_0(A_{G_1}) \cup \sigma_0(A_{G_2})) \setminus \{r - m, r - 2m\},\$$

$$\Lambda_4(G_1 + G_2) := \{ \lambda \in \mathbb{R} \setminus (\{r - m, r - 2m\} \cup \sigma(A_{G_1}) \cup \sigma(A_{G_2})) : m = (\lambda + 2m - r) \langle e, (A_{G_2} - \lambda I)^{-1} e \rangle \}.$$

In the literature, the quadratic embedding constant of a graph join  $G_1 + G_2$  is studied only when either  $G_1$  or  $G_2$  is regular [8, 9, 14]. So, it is natural to ask about the case in which neither of the graphs is regular. Our next main result addresses this question. In particular, we obtain the quadratic embedding constant of the join of a complete multipartite graph and an arbitrary graph G. To state our next main result, we need to define a real-valued function.

For distinct positive integers  $m_{i_1}, m_{i_2}, \ldots, m_{i_q}$  and any positive integers  $a_1, a_2, \ldots, a_q$ , define the function  $P : \mathbb{R} \setminus \{-m_{i_1}, -m_{i_2}, \ldots, -m_{i_q}\} \to \mathbb{R}$  via

$$P(\lambda) := 1 + \sum_{p=1}^{q} \frac{a_p m_{i_p}}{\lambda + m_{i_p}}.$$
 (1.1)

Notice that all the zeros of P are negative real numbers. To see this, it suffices to show that the polynomial  $f(\lambda) = \prod_{p=1}^{q} (\lambda + m_{i_p}) + \sum_{p=1}^{q} a_p m_{i_p} \prod_{\substack{s=1 \ s \neq p}}^{q} (\lambda + m_{i_s})$  has q negative real zeros. Without loss

of generality assume that  $m_{i_1} > m_{i_2} > \cdots > m_{i_q}$ . By a simple calculation one can verify that  $f(-m_{i_j})f(-m_{i_{j+1}}) < 0$  for all  $j \in [q-1]$ . Thus each interval  $(-m_{i_j}, -m_{i_{j+1}}), 1 \leq j \leq q-1$  contains a root of f. Since f is a real polynomial with positive coefficients, all q roots of f are negative real numbers.

**Theorem 1.2.** Let G be a graph with  $n \geq 1$  vertices and  $K_{m_1,m_2,...,m_k}$  be a complete multipartite graph such that  $K_{m_1,m_2,...,m_k} + G$  is not complete. Suppose  $m_{i_1}, m_{i_2}, ..., m_{i_q}$  are distinct with  $m_{i_p}$  appears  $a_p$  times for all  $p \in [q]$ . Let P be the function defined in (1.1) and  $\lambda_1, \lambda_2, ..., \lambda_q$  be the zeros of P. Then

$$QEC(K_{m_1,\dots,m_k}+G) = -2 - \min\left(\bigcup_{i=1}^{4} (\Lambda_i(K_{m_1,\dots,m_k}+G) \cap (-\infty,-1)) \cup \{-m_{i_p} : p \in [q], a_p \ge 2, m_{i_p} \ne 1\}\right),$$

where

$$\Lambda_1(K_{m_1,\dots,m_k} + G) := \{-m_{i_p} : p \in [q], a_p = 1\} \cap \sigma(A_G - J),$$

$$\Lambda_2(K_{m_1,\dots,m_k}+G) := \{\lambda_p : p \in [q]\} \cap \sigma(A_G),$$

$$\Lambda_3(K_{m_1,\ldots,m_k} + G) := \sigma_0(A_G) \setminus \{-m_{i_n}, \lambda_p : p \in [q]\},$$

$$\Lambda_4(K_{m_1,\ldots,m_k}+G) := \{\lambda \in \mathbb{R} \setminus (\{-m_{i_p},\lambda_p : p \in [q]\} \cup \sigma(A_G)) : P(\lambda)\langle e, (A_G-\lambda I)^{-1}e \rangle = P(\lambda)-1\}.$$

If q=1 in the above theorem i.e,  $m_1=m_2=\cdots=m_k=m$ , then  $K_{m,m,\dots,m}$  is a regular graph of degree (k-1)m and -m is an eigenvalue of  $A_{K_{m,m,\dots,m}}$ . A straightforward calculation shows that Theorem 1.2 can be derived from Theorem 1.1.

We next turn our attention to the Cartesian product of graphs. In [17], Obata–Zakiyyah showed that the quadratic embedding constant of the Cartesian product of two graphs of QE class is zero. It is natural to study the case when at least one of the graphs in the Cartesian product does not belong to the QE class. In Corollary 3.5, we show that the Cartesian product of two connected graphs  $G_1$  and  $G_2$  is of QE class if and only if both  $G_1$  and  $G_2$  are of QE class. Thus, if a graph G is of QE class, then  $K_m \times G$  is of QE class and QEC( $K_m \times G$ ) = 0. In our next main result, for a connected graph G of non-QE class, we obtain QEC( $K_m \times G$ ) in terms of QEC(G).

**Theorem 1.3.** Let G be a connected graph of non-QE class with n vertices and  $K_m$  be a complete graph. Then  $QEC(K_m \times G) = m QEC(G)$ .

In our final main result, we derive  $QEC(K_{m,n} \times G)$  in terms of QEC(G) when either G or  $K_{m,n}$  is not of QE class.

**Theorem 1.4.** Let G be a connected graph with l vertices and  $K_{m,n}$  be a complete bipartite graph.

- (i) If  $K_{m,n}$  is of QE class and G is of non-QE class, then  $QEC(K_{m,n} \times G) = (m+n) QEC(G)$ .
- (ii) If  $K_{m,n}$  is of non-QE class, then  $QEC(K_{m,n} \times G) = \max\{(m+n) QEC(G), l \frac{n(m-2)+m(n-2)}{m+n} \}$ .

**Organization of the paper:** The remaining sections are devoted to proving our main results above. In Section 2, we recap the definition of graph join, some preliminary results, and prove our first two main results, Theorems 1.1 and 1.2. In the final section, we first recall the Cartesian product of graphs and the quadratic embedding constant of a complete bipartite graph, and then prove Theorems 1.3 and 1.4.

## 2. Quadratic embedding constant and Graph Join

We begin by proving Theorems 1.1 and 1.2 – quadratic embedding constant of the join of two graphs with at least one of them is not regular. The proof requires some preliminary results. The first result gives a sharp lower bound for the quadratic embedding constant of a connected graph.

**Proposition 2.1.** [8] Let G be a connected graph on n vertices. Then  $QEC(G) \ge -1$  and equality holds if and only if G is a complete graph  $K_n$ .

The following result provides a handy technique for computing the quadratic embedding constant of a connected graph using the Lagrange multipliers.

**Proposition 2.2.** [14, Section 3] Let  $M \in \mathbb{R}^{l \times l}$   $(l \geq 3)$  be a symmetric matrix and S(M) be the set of all stationary points  $(f, \lambda, \mu) \in \mathbb{R}^l \times \mathbb{R} \times \mathbb{R}$  of

$$\psi(f,\lambda,\mu) = \langle f, Mf \rangle - \lambda(\langle f, f \rangle - 1) - \mu \langle e, f \rangle,$$

or equivalently,  $(f, \lambda, \mu) \in \mathbb{R}^l \times \mathbb{R} \times \mathbb{R}$  satisfies the system of the following three equations

$$(M - \lambda I)f = \frac{\mu}{2} e, \quad \langle f, f \rangle = 1 \quad and \quad \langle e, f \rangle = 0.$$
 (2.1)

Then S(M) is nonempty. Moreover,

- (i)  $\max\{\langle f, Mf \rangle; f \in \mathbb{R}^l, \langle f, f \rangle = 1, \langle e, f \rangle = 0\} = \max\{\lambda : (f, \lambda, \mu) \in \mathcal{S}(M)\}.$
- (ii)  $\min\{\langle f, Mf \rangle; f \in \mathbb{R}^l, \langle f, f \rangle = 1, \langle e, f \rangle = 0\} = \min\{\lambda : (f, \lambda, \mu) \in \mathcal{S}(M)\}.$

To prove Theorems 1.1 and 1.2, we now recall the definition of the join of two disjoint graphs. Note that two graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  is disjoint if  $V_1 \cap V_2 = \emptyset$ .

**Definition 2.3.** Let  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  be two disjoint graphs. Then the join of  $G_1$  and  $G_2$ , denoted by  $G_1 + G_2$ , is a graph with vertex set  $V = V_1 \cup V_2$  and edge set  $E = E_1 \cup E_2 \cup \{(u_i, v_j); u_i \in V_1, v_j \in V_2\}$ .

**Example 2.4.** Example of a join graph  $P_4 + C_3$ :

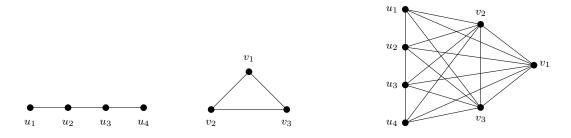


FIGURE 1.  $P_4$ ,  $C_3$  and  $P_4 + C_3$  (Left to right)

In 2017, Obata [14] derived an interesting formula for the quadratic embedding constant of a graph join using its adjacency matrix.

**Proposition 2.5.** [14, Proposition 2.1.] Let  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  be two disjoint graphs and let  $V = V_1 \cup V_2$ . Then

$$QEC(G_1 + G_2) = -2 - \min\{\langle f, A_{G_1 + G_2} f \rangle : f \in C(V), \langle f, f \rangle = 1, \langle e, f \rangle = 0\}.$$

Let  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  be two disjoint graphs with  $|V_1| = m \ge 1$  and  $|V_2| = n \ge 1$ . Since a connected graph on two vertices is the complete graph  $K_2$ , we assume  $m + n \ge 3$ . Note that a graph join  $G_1 + G_2$  is a connected graph, and its diameter is either one or two. Thus one can write the adjacency matrix of  $G_1 + G_2$  as the block matrix  $A_{G_1+G_2} = \begin{pmatrix} A_{G_1} & J \\ J & A_{G_2} \end{pmatrix}$ . For the adjacency matrix  $A_{G_1+G_2}$ , the equation (2.1) reduces to the following system of equations:

$$(A_{G_1} - J - \lambda I)x = \frac{\mu}{2} e, (2.2)$$

$$(A_{G_2} - J - \lambda I)y = \frac{\mu}{2} e, (2.3)$$

$$\langle e, x \rangle + \langle e, y \rangle = 0, \tag{2.4}$$

$$\langle x, x \rangle + \langle y, y \rangle = 1, \tag{2.5}$$

where  $x \in C(V_1) \cong \mathbb{R}^m$  and  $y \in C(V_2) \cong \mathbb{R}^n$ . Let  $\mathbb{S}(A_{G_1+G_2})$  be the set of all solutions  $(x, y, \lambda, \mu) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$  of the system of equations (2.2)–(2.5). Then all the stationary points  $\mathcal{S}(A_{G_1+G_2})$  of the function

$$\psi(x, y, \lambda, \mu) = \left\langle \begin{pmatrix} x \\ y \end{pmatrix}, A_{G_1 + G_2} \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle - \lambda(\langle x, x \rangle + \langle y, y \rangle - 1) - \mu(\langle e, x \rangle + \langle e, y \rangle)$$

coincides with the points in  $S(A_{G_1+G_2})$ . By Propositions 2.5 and 2.2, one can conclude that

$$QEC(G_1 + G_2) = -2 - \min\{\lambda : (x, y, \lambda, \mu) \in \mathbb{S}(A_{G_1 + G_2})\}.$$
(2.6)

Define  $\Lambda(G_1+G_2):=\{\lambda:(x,y,\lambda,\mu)\in\mathbb{S}(A_{G_1+G_2})\}$ . Then, we can rewrite (2.6) as

$$QEC(G_1 + G_2) = -2 - \min \Lambda(G_1 + G_2). \tag{2.7}$$

With these basic ingredients, we now prove Theorem 1.1.

Proof of Theorem 1.1. Note that a graph join  $G_1 + G_2$  is complete if and only if  $G_1$  and  $G_2$  both are complete graphs. Since  $G_1 + G_2$  is not complete, by Proposition 2.1,  $QEC(G_1 + G_2) > -1$  and using (2.6), it is sufficient to consider all  $\lambda < -1$  such that  $(x, y, \lambda, \mu) \in \mathbb{S}(A_{G_1+G_2})$ . By (2.7),  $QEC(G_1 + G_2) = -2 - \min(\Lambda(G_1 + G_2) \cap (-\infty, -1))$ . Thus, to complete the proof, it suffices to show that

$$\Lambda(G_1 + G_2) \cap (-\infty, -1) = \bigcup_{i=1}^{4} (\Lambda_i(G_1 + G_2) \cap (-\infty, -1)).$$

We now split the proof into several lemmas.

**Lemma 2.6.** Let  $\lambda = r - m$ . Then  $\lambda$  appears in the solution of (2.2)-(2.5) if and only if  $\lambda \in \sigma(A_{G_1}) \cup \sigma(A_{G_2} - J)$ .

*Proof.* Suppose  $(x, y, r - m, \mu)$  is a solution of (2.2)-(2.5). Since  $G_1$  is a r-regular graph,  $A_{G_1}e = re$  and the equation (2.2) implies  $\mu = 0$ . From equations (2.2) and (2.3) we have

$$(A_{G_1} - (r - m)I) x = \langle e, x \rangle e,$$
  

$$(A_{G_2} - J - (r - m)I)y = 0.$$

Notice that, by (2.5), either x or y is nonzero. Thus, if  $\langle e, x \rangle = 0$ , then from the above equations, one can conclude that  $r - m \in \sigma(A_{G_1}) \cup \sigma(A_{G_2} - J)$ . Let  $\langle e, x \rangle \neq 0$ . Then  $\langle e, y \rangle \neq 0$  from (2.4). Thus  $y \neq 0$  and  $r - m \in \sigma(A_{G_2} - J)$ .

To prove the converse, suppose  $r - m \in \sigma(A_{G_1}) \cup \sigma(A_{G_2} - J)$ . We now consider two cases.

Case I.  $r - m \in \sigma(A_{G_2} - J)$ : Then there exists  $0 \neq y_0 \in \mathbb{R}^n$  such that

$$(A_{G_2} - J + (m - r)I)y_0 = 0.$$

Since  $G_1$  is a r-regular graph on m vertices, r-m is an eigenvalue of  $A_{G_1}-J$  and the corresponding eigenvector is  $\alpha e$  for any  $\alpha \in \mathbb{R}$ . Thus  $(x = \alpha e, y = \beta y_0, \lambda = r - m, \mu = 0)$  is a solution of (2.2)-(2.5), where  $\alpha, \beta \in \mathbb{R}$  are chosen appropriately from equations (2.4)-(2.5) in terms of  $\langle e, y_0 \rangle$  and  $\langle y_0, y_0 \rangle$ .

Case II.  $r - m \in \sigma(A_{G_1})$ : Then there exists  $0 \neq x_0 \in \mathbb{R}^m$  such that

$$(A_{G_1} + (m-r)I)x_0 = 0.$$

Since  $r \neq r - m$  and e is an eigenvector of  $A_{G_1}$  corresponding to the eigenvalue r,  $\langle e, x_0 \rangle = 0$ . Thus,  $(x = \frac{x_0}{\|x_0\|_2}, y = 0, \lambda = r - m, \mu = 0)$  satisfies equations (2.2)-(2.5).

We next consider  $\lambda \neq r - m$ . To proceed further, we rewrite the equations (2.2) and (2.3) using relation (2.4). By a straightforward calculation, we have

$$\langle e, x \rangle = -\langle e, y \rangle = \frac{m}{r - m - \lambda} \frac{\mu}{2},$$
 (2.8)

$$(A_{G_1} - \lambda I)x = Jx + \frac{\mu}{2}e = \langle e, x \rangle e + \frac{\mu}{2}e = \frac{m}{r - m - \lambda} \frac{\mu}{2}e + \frac{\mu}{2}e = \frac{r - \lambda}{r - m - \lambda} \frac{\mu}{2}e,$$
 (2.9)

and

$$(A_{G_2} - \lambda I)y = Jy + \frac{\mu}{2}e = \langle e, y \rangle e + \frac{\mu}{2}e = -\frac{m}{r - m - \lambda} \frac{\mu}{2}e + \frac{\mu}{2}e = \frac{r - 2m - \lambda}{r - m - \lambda} \frac{\mu}{2}e.$$
 (2.10)

**Lemma 2.7.** Let  $\lambda = r - 2m$ . Then  $\lambda$  appears in the solution of (2.2)-(2.5) if and only if  $\lambda \in \sigma(A_{G_2})$ .

*Proof.* Suppose  $(x, y, r-2m, \mu) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$  is a solutions of (2.2)-(2.5). Then  $(x, y, r-2m, \mu)$  satisfies (2.8)-(2.10). From (2.10), we get

$$(A_{G_2} + (2m - r)I) y = 0.$$

We next show that  $y \neq 0$ . By Gershgorin's theorem, r is the eigenvalue of  $A_{G_1}$  with the largest absolute value. Since (2m-r) > r,  $\{A_{G_1} + (2m-r)I\}$  is an invertible matrix. If y = 0, then  $\mu = 0$  by (2.8) and x = 0 by (2.9), which contradicts  $\langle x, x \rangle + \langle y, y \rangle = 1$ . Thus  $y \neq 0$  and  $r - 2m \in \sigma(A_{G_2})$ .

Conversely, suppose that  $r-2m \in \sigma(A_{G_2})$ . Then there exists  $0 \neq y_0 \in \mathbb{R}^n$  such that

$$(A_{G_2} + (2m - r)I)y_0 = 0.$$

Let  $x = \frac{1}{m} \frac{\mu}{2} e$  and  $y = cy_0$ , where  $\mu, c \in \mathbb{R}$ . Then  $(x, y, r - 2m, \mu)$  satisfies (2.9) and (2.10) for any  $\mu, c \in \mathbb{R}$ . Now, choose c and  $\mu$  such that

$$\frac{\mu}{2} + c \langle e, y_0 \rangle = 0,$$

$$\frac{1}{m} \left(\frac{\mu}{2}\right)^2 + c^2 \langle y_0, y_0 \rangle = 1.$$

Then,  $(x = \frac{1}{m} \frac{\mu}{2} e, y = cy_0, \lambda = r - 2m, \mu)$  is a solution of equations (2.2)-(2.5).

**Lemma 2.8.** Let  $\lambda \in (\sigma(A_{G_1}) \cup \sigma(A_{G_2})) \setminus \{r - m, r - 2m\}$ . Then  $\lambda$  appears in the solution of (2.2)-(2.5) if and only if  $\lambda \in \sigma_0(A_{G_1}) \cup \sigma_0(A_{G_2})$ .

*Proof.* Suppose  $\lambda \in \sigma_0(A_{G_1}) \cup \sigma_0(A_{G_2})$ . If  $\lambda \in \sigma_0(A_{G_1})$ , then there exists  $x_0 \neq 0$  such that

$$A_{G_1}x_0 = \lambda x_0$$
 and  $\langle e, x_0 \rangle = 0$ .

Then  $(x = \frac{x_0}{\|x_0\|_2}, y = 0, \lambda, \mu = 0)$  is a solution of (2.2)-(2.5). Similarly, if  $\lambda \in \sigma_0(A_{G_2})$ , there exists  $y_0 \neq 0$  such that  $(x = 0, y = \frac{y_0}{\|y_0\|_2}, \lambda, \mu = 0)$  satisfies (2.2)-(2.5).

To prove the converse, let  $(x, y, \lambda, \mu) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$  be a solution of the system of equations (2.2)-(2.5). Then either x or y is non-zero. We now split the remaining part of the proof into two cases.

Case I.  $\mu = 0$ : By equations (2.8)–(2.10)), we have  $\langle e, x \rangle = \langle e, y \rangle = 0$  and

$$(A_{G_1} - \lambda I)x = 0, \ (A_{G_2} - \lambda I)y = 0$$

Since both x and y can not be zero at the same time,  $\lambda \in \sigma_0(A_{G_1}) \cup \sigma_0(A_{G_2})$ .

Case II.  $\mu \neq 0$ : Let  $\lambda \in \sigma(A_{G_2})$ . Since  $\lambda \neq r-m, r-2m$ , by (2.10),  $y \neq 0$  and it is a solution of the equation (2.10). This implies  $\operatorname{rank}(A_{G_2} - \lambda I, e) = \operatorname{rank}(A_{G_2} - \lambda I)$ . Thus  $\operatorname{rank}(A_{G_2} - \lambda I, e) < n$ , since  $A_{G_2} \in \mathbb{R}^{n \times n}$  and  $\lambda \in \sigma(A_{G_2})$ . Hence there exists a  $z \in \mathbb{R}^n$  with  $\langle z, z \rangle = 1$  such that  $z^T(A_{G_2} - \lambda I, e) = 0$ . Since  $(A_{G_2} - \lambda I)$  is a symmetric matrix,

$$(A_{G_2} - \lambda I)z = 0$$
 and  $\langle e, z \rangle = 0$ .

Thus  $\lambda \in \sigma_0(A_{G_2})$ .

Since  $\lambda < -1$ , a similar argument shows that if  $\lambda \in \sigma(A_{G_1})$ , then  $\lambda \in \sigma_0(A_{G_1})$ . This concludes the proof.

**Lemma 2.9.** Let  $\lambda \in \mathbb{R} \setminus (\{r-m, r-2m\} \cup \sigma(A_{G_1}) \cup \sigma(A_{G_2}))$ . Then  $\lambda$  appears in the solution of (2.2)-(2.5) if and only if

$$m - (\lambda + 2m - r)\langle e, (A_{G_2} - \lambda I)^{-1} e \rangle = 0.$$
 (2.11)

*Proof.* Suppose  $\lambda$  appears in the solutions of (2.2)-(2.5). Then there exist  $x \in \mathbb{R}^m$ ,  $y \in \mathbb{R}^n$  and  $\mu \in \mathbb{R}$  such that  $(x, y, \lambda, \mu)$  satisfies (2.8)-(2.10) and (2.5). Since  $\lambda \notin \sigma(A_{G_2})$ ,  $y = \frac{r-2m-\lambda}{r-m-\lambda} \frac{\mu}{2} (A_{G_2} - \lambda I)^{-1} e$  by equation (2.10). Thus, from (2.4) and (2.8) we have

$$\frac{m}{r-m-\lambda}\frac{\mu}{2} + \left\langle e, \frac{r-2m-\lambda}{r-m-\lambda}\frac{\mu}{2}(A_{G_2} - \lambda I)^{-1}e \right\rangle = 0$$

$$\Rightarrow \frac{\mu}{2}\left(m - (\lambda + 2m - r)\langle e, (A_{G_2} - \lambda I)^{-1}e\rangle\right) = 0$$

If  $\mu = 0$ , then x = y = 0 by (2.9) and (2.10) which contradicts (2.5). Thus

$$m - (\lambda + 2m - r)\langle e, (A_{G_2} - \lambda I)^{-1} e \rangle = 0.$$

Conversely, suppose  $\lambda$  satisfies (2.11). Take

$$x = \frac{\mu}{2} \frac{1}{r - m - \lambda} e$$
 and  $y = \frac{\mu}{2} \frac{r - 2m - \lambda}{r - m - \lambda} (A_{G_2} - \lambda I)^{-1} e$ 

such that  $\mu \in \mathbb{R}$  satisfies the following equation

$$\left(\frac{\mu}{2}\right)^2 \left( \left(\frac{1}{r - m - \lambda}\right)^2 m + \left(\frac{r - 2m - \lambda}{r - m - \lambda}\right)^2 \| (A_{G_2} - \lambda I)^{-1} e \|_2^2 \right) = 1.$$

Then  $(x, y, \lambda, \mu)$  fulfills the equations (2.2)-(2.5)).

Using the above four lemmas, one can verify that

$$\Lambda(G_1 + G_2) \cap (-\infty, -1) = \bigcup_{i=1}^{4} (\Lambda_i(G_1 + G_2) \cap (-\infty, -1)).$$

This completes the proof.

We next prove Theorem 1.2 – we derive quadratic embedding constant of  $K_{m_1,m_2,...,m_k}+G$ , where G is a graph of order n. The adjacency matrix of  $K_{m_1,m_2,...,m_k}$  is a matrix of order  $m_1+m_2+\cdots+m_k$  of the form

$$A_{K_{m_1,m_2,...,m_k}} = \begin{pmatrix} 0 & J & J & \dots & J \\ J & 0 & J & \dots & J \\ J & J & 0 & \dots & J \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J & J & J & \dots & 0 \end{pmatrix}.$$

For  $A_{K_{m_1,m_2,...,m_k}+G}$ , the system of equations (2.2)-(2.5) reduces to

$$(J+\lambda I)x^{i} = -\frac{\mu}{2} e \text{ for } i \in [k]$$
 (2.12)

$$(A_G - J - \lambda I)y = \frac{\mu}{2} e, \qquad (2.13)$$

$$\sum_{i=1}^{k} \langle e, x^i \rangle + \langle e, y \rangle = 0, \tag{2.14}$$

$$\sum_{i=1}^{k} \langle x^i, x^i \rangle + \langle y, y \rangle = 1, \tag{2.15}$$

where  $x = (x^{1T}, x^{2T}, \dots, x^{kT})^T$  and  $x^i \in \mathbb{R}^{m_i}$  for all  $i \in [k]$  and  $y \in \mathbb{R}^n$ .

Proof of Theorem 1.2. Since  $K_{m_1,m_2,...,m_k}+G$  is not a complete graph, by equation (2.7) and Proposition 2.1,  $\text{QEC}(K_{m_1,m_2,...,m_k}+G)=-2-\min\left(\Lambda(K_{m_1,m_2,...,m_k}+G)\cap(-\infty,-1)\right)$ . To complete the proof, it is sufficient to show that

$$\Lambda(K_{m_1,m_2,\dots,m_k}+G)\cap(-\infty,-1)=\bigcup_{i=1}^4(\Lambda_i(K_{m_1,m_2,\dots,m_k}+G)\cap(-\infty,-1))\cup\{-m_{i_p}:p\in[q],a_p\geq 2,m_{i_p}\neq 1\}.$$

Without loss of generality, assume that  $m_1 \geq m_2 \geq \cdots \geq m_k$ . We first claim that if  $a_p \geq 2$  then  $-m_{i_p} \in \Lambda(K_{m_1,m_2,\ldots,m_k} + G)$ ; in other words,  $-m_{i_p}$  appears in the solution of (2.12)-(2.15). Let  $a_p \geq 2$  for some  $p \in [q]$  and consider two possible cases.

Case I.  $a_p$  is even: Choose  $\mu = 0$ , y = 0 and  $x = (x^{1T}, x^{2T}, \dots, x^{kT})^T \in \mathbb{R}^{m_1 + m_2 + \dots + m_k}$  such that

$$x^{j} = \begin{cases} \frac{1}{\sqrt{a_{p}m_{i_{p}}}}e, & \text{if } \sum_{g=1}^{p-1}a_{g} + 1 \leq j \leq \sum_{g=1}^{p-1}a_{g} + \frac{a_{p}}{2}, \\ -\frac{1}{\sqrt{a_{p}m_{i_{p}}}}e, & \text{if } \sum_{g=1}^{p-1}a_{g} + \frac{a_{p}}{2} + 1 \leq j \leq \sum_{g=1}^{p-1}a_{g} + a_{p}, \\ 0, & \text{otherwise.} \end{cases}$$

Then,  $(x^1, \ldots, x^k, y, -m_{i_p}, \mu)$  is a solution of (2.12)-(2.15).

Case II.  $a_p$  is odd: Take  $\mu = 0$ , y = 0 and

$$x^{j} = \begin{cases} \frac{1}{\sqrt{(a_{p}-1)m_{i_{p}}}}e, & \text{if } \sum_{g=1}^{p-1}a_{g}+1 \leq j \leq \sum_{g=1}^{p-1}a_{g}+\frac{a_{p}-1}{2}, \\ -\frac{1}{\sqrt{(a_{p}-1)m_{i_{p}}}}e, & \text{if } \sum_{g=1}^{p-1}a_{g}+\frac{a_{p}-1}{2}+1 \leq j \leq \sum_{g=1}^{p-1}a_{g}+(a_{p}-1), \\ 0, & \text{otherwise.} \end{cases}$$

Then  $(x^1, \ldots, x^k, -m_{i_p}, \mu)$  is a solution of (2.12)-(2.15).

To classify all other elements of  $\Lambda(K_{m_1,m_2,...,m_k}+G)\cap(-\infty,-1)$ , we now split the rest of the proof into several lemmas.

**Lemma 2.10.** Let  $\lambda = -m_{i_p}$  for some  $p \in [q]$  such that  $a_p = 1$ . Then  $\lambda$  appears in the solutions of (2.12)-(2.15) if and only if  $\lambda \in \sigma(A_G - J)$ .

*Proof.* Suppose  $(x, y, -m_{i_p}, \mu)$  is a solution of (2.12)-(2.15), where  $x = (x^{1T}, x^{2T}, \dots, x^{kT})^T \in \mathbb{R}^{m_1 + \dots + m_k}, y \in \mathbb{R}^n, \mu \in \mathbb{R}$ . From (2.12), one can derive

$$\mu = 0, \ x^{i_p} = ce \text{ for arbitrary } c \in \mathbb{R}, \ x^j = 0 \text{ for all } j \in [k] \text{ and } j \neq i_p.$$
 (2.16)

Thus the equations (2.13)-(2.15) reduces to the following

$$(A_G - J + m_{i_p}I)y = 0$$
,  $cm_{i_p} + \langle e, y \rangle = 0$ ,  $c^2m_{i_p} + \langle y, y \rangle = 1$ . (2.17)

If y = 0, then c = 0 from the second identity of (2.17) and hence the third identity does not hold. Thus  $y \neq 0$  and  $-m_{i_p} \in \sigma(A_G - J)$ .

Conversely, suppose  $-m_{i_p} \in \sigma(A_G - J)$  for some  $p \in [q]$  with  $a_p = 1$ . Then, there exists an  $0 \neq y_0 \in \mathbb{R}^n$  such that  $(A_G - J + m_{i_p} I)y_0 = 0$ . Define  $x := (x^{1^T}, x^{2^T}, \dots, x^{k^T})^T$  such that  $x^j \in \mathbb{R}^{m_j}$  and

$$x^j := \begin{cases} ce, & \text{if } j = i_p, \\ 0, & \text{Otherwise.} \end{cases}$$

By choosing appropriate values of  $\gamma$  and c, one can show that  $(x, y = \gamma y_0, -m_{i_p}, 0)$  is a solution of (2.12)-(2.15).

Since  $\lambda < -1$ , before we proceed to the next lemma, note that for  $\lambda \neq -m_1, -m_2, \ldots, -m_k$ , equation (2.12) implies that

$$x^{i} = -\frac{1}{\lambda + m_{i}} \frac{\mu}{2} e, \quad i \in [k]$$
 (2.18)

and equations (2.13), (2.14) implies that

$$(A_G - \lambda I)y = \frac{\mu}{2}e + Jy = \frac{\mu}{2}e + (-\sum_{i=1}^k \langle e, x^i \rangle)e = \left(1 + \sum_{p=1}^q \frac{a_p m_{i_p}}{\lambda + m_{i_p}}\right) \frac{\mu}{2}e = P(\lambda) \frac{\mu}{2}e,$$
 (2.19)

where P is defined in (1.1).

**Lemma 2.11.** Let  $\lambda = \lambda_i$  for some  $p \in [q]$ . Then  $\lambda$  appears in the solutions of (2.12)-(2.15) if and only if  $\lambda \in \sigma(A_G)$ .

*Proof.* Let  $\lambda = \lambda_i$  for some  $p \in [q]$ . Since  $\lambda_i$  is zero of P,

$$P(\lambda) = 1 + \sum_{p=1}^{q} \frac{a_p m_{i_p}}{\lambda + m_{i_p}} = 0.$$

Suppose  $\lambda$  appears in the solutions of (2.12)-(2.15). Then there exist  $x = (x^{1^T}, x^{2^T}, \dots, x^{k^T})^T \in \mathbb{R}^{m_1 + \dots + m_k}, y \in \mathbb{R}^n, \mu \in \mathbb{R}$  such that  $(x, y, \lambda, \mu)$  satisfies (2.18), (2.19), (2.14) and (2.15). By (2.19),  $(A_G - \lambda I)y = 0$ . We next show that  $y \neq 0$ . If not, then from (2.18) and (2.14), we have

$$-\frac{\mu}{2} \sum_{p=1}^{q} \frac{a_p m_{i_p}}{\lambda + m_{i_p}} = 0.$$

This implies  $\mu = 0$  and so  $x^j = 0$  for all  $j \in [k]$ , which contradicts (2.15). Thus  $y \neq 0$  and from (2.19),  $\lambda \in \sigma(A_G)$ .

Conversely, suppose  $\lambda \in \sigma(A_G)$ . Then there exists an  $0 \neq y_0 \in \mathbb{R}^n$  such that

$$(A_G - \lambda I)y_0 = 0.$$

Choose  $x^j = -\frac{1}{\lambda + m_j} \frac{\mu}{2} e$  for  $j \in [k]$  and  $c, \mu$  such that

$$-\frac{\mu}{2} \left( \sum_{p=1}^{q} \frac{a_p m_{i_p}}{\lambda + m_{i_p}} \right) + c \langle e, y_0 \rangle = 0,$$

$$\left(\frac{\mu}{2}\right)^2 \left(\sum_{p=1}^q \frac{a_p m_{i_p}}{(\lambda + m_{i_p})^2}\right) + c^2 \langle y_0, y_0 \rangle = 1.$$

Then  $(x, y, \lambda, \mu)$  is a solution of (2.12)-(2.15), where  $x = (x^{1^T}, x^{2^T}, \dots, x^{k^T})^T \in \mathbb{R}^{m_1 + \dots + m_k}, y = cy_0 \in \mathbb{R}^n$ .

**Lemma 2.12.** Let  $\lambda \in \sigma(A_G) \setminus \{-m_{i_p}, \lambda_p : p \in [q]\}$ . Then  $\lambda$  appears in the solutions of (2.12)-(2.15) if and only if  $\lambda \in \sigma_0(A_G)$ .

*Proof.* Suppose  $\lambda \in \sigma_0(A_G)$ . Then there exists  $0 \neq y \in \mathbb{R}^n$  such that

$$A_G y = \lambda y$$
 and  $\langle e, y \rangle = 0$ .

Then  $(x^1 = 0, x^2 = 0, \dots, x^k = 0, \frac{y}{\|y\|_2}, \lambda, \mu = 0)$  is a solution of (2.12)-(2.15).

To prove the converse, let  $(x_1, \ldots, x_k, y, \lambda, \mu) \in \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_k} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$  be a solution of the system of equations (2.12)-(2.15). To show  $\lambda \in \sigma_0(A_G)$ , we now consider two cases.

Case I.  $\mu = 0$ : By equations (2.18), (2.19), we have

$$x^{i} = 0 \text{ for } i \in [k]$$

$$A_{G}y = \lambda y.$$

From (2.14) and (2.15), we have  $\langle e, y \rangle = 0$  and  $y \neq 0$ . Thus  $\lambda \in \sigma_0(A_G)$ .

Case II.  $\mu \neq 0$ : Since  $\lambda$  is not a zero of P and y is a solution of the equation (2.19),  $\operatorname{rank}(A_G - \lambda I, e) = \operatorname{rank}(A_G - \lambda I)$ . Thus  $\operatorname{rank}(A_G - \lambda I, e) < n$ , since  $A_G \in \mathbb{R}^{n \times n}$  and  $\lambda \in \sigma(A_G)$ . Hence there exists a  $z \in \mathbb{R}^n$  with  $\langle z, z \rangle = 1$  such that  $z^T(A_G - \lambda I, e) = 0$ . Since  $(A_G - \lambda I)$  is a symmetric matrix,

$$(A_G - \lambda I)z = 0$$
 and  $\langle e, z \rangle = 0$ .

Thus  $\lambda \in \sigma_0(A_G)$  and concludes the proof.

**Lemma 2.13.** Let  $\lambda \in \mathbb{R} \setminus (\{-m_{i_p}, \lambda_p : p \in [q]\} \cup \sigma(A_G))$ . Then  $\lambda$  appears in the solutions of (2.12)-(2.15) if and only if  $\lambda$  is the solutions of

$$P(\lambda)\langle e, (A_G - \lambda I)^{-1}e \rangle - (P(\lambda) - 1) = 0$$
(2.20)

*Proof.* Suppose  $\lambda$  is in the solutions of (2.12)-(2.15). Then, there exist  $(x^1, \ldots, x^k, y, \lambda, \mu)$  that satisfy (2.18), (2.19), (2.14) and (2.15). From (2.19), we obtain

$$y = P(\lambda) \frac{\mu}{2} (A_G - \lambda I)^{-1} e.$$
 (2.21)

Applying (2.18), (2.21) in (2.14), we have

$$\frac{\mu}{2} \left( P(\lambda) \langle e, (A_G - \lambda I)^{-1} e \rangle - (P(\lambda) - 1) \right) = 0.$$

If  $\mu = 0$ , then from (2.18),  $x^i = 0$  for all  $i \in [k]$  and from (2.21), y = 0 which contradicts (2.15). Thus  $\mu \neq 0$  and  $\lambda$  satisfies (2.20).

To prove the converse, assume that  $\lambda$  satisfies (2.20). Take  $x^i$  for  $i \in [k]$  and y as in (2.18) and (2.21) respectively, and  $\mu$  so that equation (2.15) holds. Then  $(x^1, \dots, x^k, y, \lambda, \mu)$  is a solution of (2.12)-(2.15).

Using the above four lemmas, one can conclude that

$$\Lambda(K_{m_1,m_2,\dots,m_k}+G)\cap(-\infty,-1)=\bigcup_{i=1}^4(\Lambda_i(K_{m_1,m_2,\dots,m_k}+G)\cap(-\infty,-1))\cup\{-m_{i_p}:p\in[q],a_p\geq 2,m_{i_p}\neq 1\}.$$

This completes the proof.

### 3. Quadratic embedding constant and Cartesian Product of Graphs

In this section, we prove Theorems 1.3 and 1.4 – for a arbitrary connected graph G, we derive a formula for the quadratic embedding constant of the Cartesian product of a complete graph  $K_m$  and G as well as QEC of the Cartesian product of a complete bipartite graph  $K_{m,n}$  and G in terms of QEC(G). We first recall the definition of the Cartesian product of two graphs.

**Definition 3.1.** Let  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  be two connected graphs. The Cartesian of graphs  $G_1$  and  $G_2$ , denoted by  $G_1 \times G_2$ , is a graph with vertex set  $V = V_1 \times V_2$  and two vertices  $(u_i, v_k), (u_j, v_l) \in V$  are adjacent in  $G_1 \times G_2$  if and only if either  $u_i \sim u_j$  and  $v_k = v_l$ , or  $u_i = u_j$  and  $v_k \sim v_l$ .

## **Example 3.2.** Example of the Cartesian product of $P_3$ and $C_3$ :

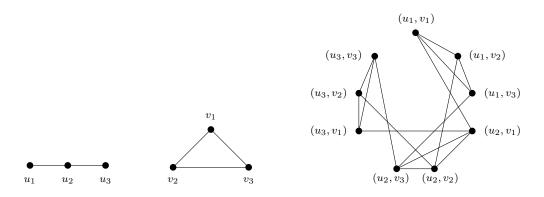


FIGURE 2.  $P_3$ ,  $C_3$  and  $P_3 \times C_3$  (Left to right)

Let  $G_1$  and  $G_2$  be two connected graphs with vertex set  $V_1 = \{u_1, u_2, \ldots, u_m\}$  and  $V_2 = \{v_1, v_2, \ldots, v_n\}$ , respectively. Suppose that  $M_k = \{(u_k, v_1), (u_k, v_2), \ldots, (u_k, v_n)\}$  for all  $k \in [m]$ . Then the vertex set of  $G_1 \times G_2$  is  $\bigcup_{k=1}^m M_k$ . Then the distance matrix  $D_{G_1 \times G_2}$  is an  $m \times m$  block matrix whose each block is of order n. Let  $(u_i, v_p)$  and  $(u_j, v_q)$  be two arbitrary elements from  $M_i$  and  $M_j$ , respectively. Let  $u_i \sim u_{i_1} \sim u_{i_2} \sim \cdots \sim u_{i_l} \sim u_j$  and  $v_p \sim v_{p_1} \sim v_{p_2} \sim \cdots \sim v_{p_r} \sim v_q$  be two paths in  $G_1$  and  $G_2$ , respectively. Then

$$(u_i, v_p) \sim (u_{i_1}, v_p) \sim \cdots \sim (u_{i_l}, v_p) \sim (u_j, v_p) \sim (u_j, v_{p_1}) \sim \cdots \sim (u_j, v_{p_r}) \sim (u_j, v_q)$$

is a path from  $(u_i, v_p)$  to  $(u_j, v_q)$  in  $G_1 \times G_2$  and thus  $G_1 \times G_2$  is also a connected graph. Also note that any path between  $(u_i, v_p)$  and  $(u_j, v_q)$  in  $G_1 \times G_2$  gives us a path between  $u_i$  and  $u_j$  in  $G_1$ , and a path between  $v_p$  and  $v_q$  in  $G_2$ . This implies

$$d_{G_1 \times G_2}((u_i, v_p), (u_j, v_q)) = d_{G_1}(u_i, u_j) + d_{G_2}(v_p, v_q).$$
(3.1)

Thus the ij-th block of  $D_{G_1\times G_2}$  formed by rows  $M_i$  and columns  $M_j$  is  $d_{G_1}(u_i,u_j)$   $J+D_{G_2}$ , and

$$D_{G_1 \times G_2} = \begin{pmatrix} D_{G_2} & D_{G_2} + d_{G_1}(u_1, u_2)J & \dots & D_{G_2} + d_{G_1}(u_1, u_m)J \\ D_{G_2} + d_{G_1}(u_2, u_1)J & D_{G_2} & \dots & D_{G_2} + d_{G_1}(u_2, u_m)J \\ \vdots & & \vdots & \ddots & \vdots \\ D_{G_2} + d_{G_1}(u_m, u_1)J & D_{G_2} + d_{G_1}(u_m, u_2)J & \dots & D_{G_2} \end{pmatrix}.$$

The first result of this section gives us a relation between  $QEC(G_1 \times G_2)$  and  $QEC(G_1)$ ,  $QEC(G_2)$ .

**Proposition 3.3.** Let  $G_1$  and  $G_2$  be two connected graphs. Then  $QEC(G_1) \leq QEC(G_1 \times G_2)$  and  $QEC(G_2) \leq QEC(G_1 \times G_2)$ .

*Proof.* Note that the subgraph induced by the vertices  $\{(u_1, v_1), (u_1, v_2), \dots, (u_1, v_n)\}$  is the graph  $G_2$  and using (3.1),  $d_{G_1 \times G_2}((u_1, v_p), (u_1, v_q)) = d_{G_2}(v_p, v_q)$ . Thus  $G_2$  is isometrically embedded in

 $G_1 \times G_2$ . By [17, Theorem 3.1],  $\text{QEC}(G_2) \leq \text{QEC}(G_1 \times G_2)$ . Similarly one can show that  $G_1$  is isometrically embedded in  $G_1 \times G_2$  and  $\text{QEC}(G_1) \leq \text{QEC}(G_1 \times G_2)$ .

We now recall an interesting result that gives us the quadratic embedding constant of the Cartesian product of two arbitrary graphs of QE class.

**Theorem 3.4.** [17, Theorem 3.3] Let  $G_1$  and  $G_2$  be two non-trivial graphs of QE class. Then,  $QEC(G_1 \times G_2) = 0$ .

As an immediate consequence of the above two results, we conclude the following.

Corollary 3.5. Let  $G_1$  and  $G_2$  be two connected graphs. Then  $G_1 \times G_2$  is of QE class if and only if both  $G_1$  and  $G_2$  are of QE class.

*Proof.* If  $G_1$  and  $G_2$  are of QE class, then by Theorem 3.4,  $G_1 \times G_2$  is of QE class. To prove the converse, let either  $G_1$  or  $G_2$  be of the non-QE class. By Proposition 3.3,  $0 < \text{QEC}(G_1 \times G_2)$ , a contradiction since  $G_1 \times G_2$  is a graph of the QE class. Thus  $G_1$  and  $G_2$  are of QE class.

For a graph G of QE class, by Theorem 3.4, QEC $(K_m \times G) = 0$ . We now obtain a formula of QEC $(K_m \times G)$  for an arbitrary connected graph G of non-QE class.

*Proof of Theorem 1.3.* Let V be the vertex set of  $K_m \times G$ . Note that

$$D_{K_m \times G} = \begin{pmatrix} D_G & D_G + J & \dots & D_G + J \\ D_G + J & D_G & \dots & D_G + J \\ \vdots & \vdots & \ddots & \vdots \\ D_G + J & D_G + J & \dots & D_G \end{pmatrix}.$$

Let  $\mathcal{S}(D_{K_m \times G})$  be the set of all  $(f, \lambda, \mu) \in (C(V) \cong \mathbb{R}^{mn}) \times \mathbb{R} \times \mathbb{R}$  satisfying

$$(D_{K_m \times G} - \lambda I)f = \frac{\mu}{2} e, \tag{3.2}$$

$$\langle f, f \rangle = 1, \tag{3.3}$$

$$\langle e, f \rangle = 0. \tag{3.4}$$

By Proposition 2.2, QEC $(K_m \times G) = \max\{\lambda : (f, \lambda, \mu) \in \mathcal{S}(D_{K_m \times G})\}$ . Suppose  $f = (x^{1^T}, x^{2^T}, \dots, x^{m^T})^T$ , where  $x^i \in \mathbb{R}^n$  for all  $i \in [m]$ . Then, the equation (3.2) gives us a system of m equations. Subtracting the (i+1)-th equation from the i-th equation, we get

$$(J + \lambda I)(x^i - x^{i+1}) = 0, \quad \forall \quad i \in [m-1].$$
 (3.5)

Since G is a connected graph of non-QE class, by Proposition 3.3, it is enough to consider  $0 < \lambda$ . From (3.5), we have  $x^i = x^{i+1}$  for all  $i \in [m-1]$  and equation (3.4) gives us  $\langle e, x^1 \rangle = 0$ . Thus the equations (3.2)-(3.4) reduces to

$$(mD_G - \lambda I)x^1 = \frac{\mu}{2} e, \tag{3.6}$$

$$m\langle x^1, x^1 \rangle = 1, (3.7)$$

$$\langle e, x^1 \rangle = 0. (3.8)$$

Notice that  $QEC(G) = \max\{a : (x, a, \eta) \in \mathcal{S}(D_G)\}$ , where  $\mathcal{S}(D_G)$  is the set of all stationary points  $(x, a, \eta) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$  satisfying

$$(D_G - aI)x = \frac{\eta}{2} e, \quad \langle x, x \rangle = 1, \quad \langle e, x \rangle = 0.$$
 (3.9)

From equations (3.6)-(3.9), the following relations

$$\lambda = ma$$
,  $x^1 = \frac{1}{\sqrt{m}} x$ ,  $\mu = \sqrt{m} \eta$ ,

gives us a one to one correspondence between  $\mathcal{S}(D_{K_m \times G})$  and  $\mathcal{S}(D_G)$ . Thus

QEC
$$(K_m \times G) = \max\{\lambda : ((x^{1T}, x^{2T}, \dots, x^{mT})^T, \lambda, \mu) \in \mathcal{S}(D_{K_m \times G})\}$$
  
=  $\max\{ma : (x, a, \eta) \in \mathcal{S}(D_G)\}$   
=  $m \text{ QEC}(G)$ .

We next derive the quadratic embedding constant of the Cartesian product of  $K_{m,n}$  and an arbitrary connected graph G. To proceed, we require a preliminary result for the quadratic embedding constant of a complete bipartite graph  $K_{m,n}$ .

**Theorem 3.6.** [17, Theorem 2.8] Let  $K_{m,n}$  be a complete bipartite graph on (m+n) vertices. Then  $QEC(K_{m,n}) = \frac{m(n-2)+n(m-2)}{m+n}$  for all  $m \ge 1$ ,  $n \ge 1$ .

Moreover,  $K_{m,n}$  is of QE class if and only if one of the following holds:

- (i) Either m = 1 or n = 1.
- (ii) m = n = 2.

Let G be a connected graph on l vertices and  $K_{m,n}$  be a complete bipartite graph on m+n vertices. Without loss of generality, assume that  $m \geq n$ . For m = n = 1,  $K_{1,1} = K_2$  and by Theorem 1.3,  $\text{QEC}(K_{1,1} \times G) = 2 \, \text{QEC}(G)$  for any connected graph G of non-QE class. Let m > 1. Then, all possibilities of m and n can be expressed in three disjoint sets: (a) m > 1, n = 1, (b) m = n = 2 and (c) m > 2,  $n \geq 2$ . Note that the distance matrix of  $K_{m,n} \times G$  is

$$D_{K_{m,n}\times G} = \begin{pmatrix} D_G & D_G + 2J & \dots & D_G + 2J & D_G + J & D_G + J & \dots & D_G + J \\ D_G + 2J & D_G & \dots & D_G + 2J & D_G + J & D_G + J & \dots & D_G + J \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ D_G + 2J & D_G + 2J & \dots & D_G & D_G + J & D_G + J & \dots & D_G + J \\ D_G + J & D_G + J & \dots & D_G + J & D_G & D_G + 2J & \dots & D_G + 2J \\ D_G + J & D_G + J & \dots & D_G + J & D_G + 2J & D_G & \dots & D_G + 2J \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \ddots & \vdots \\ D_G + J & D_G + J & \dots & D_G + J & D_G + 2J & D_G + 2J & \dots & D_G \end{pmatrix}.$$

Now, QEC $(K_{m,n} \times G) = \max\{\lambda : (f,\lambda,\mu) \in \mathcal{S}(D_{K_{m,n}\times G})\}$ , where  $\mathcal{S}(D_{K_{m,n}\times G})$  is the set of all stationary points  $(f,\lambda,\mu) \in \mathbb{R}^{(m+n)l} \times \mathbb{R} \times \mathbb{R}$  satisfying

$$(D_{K_{m,n}\times G} - \lambda I)f = \frac{\mu}{2} e, \tag{3.10}$$

$$\langle f, f \rangle = 1, \tag{3.11}$$

$$\langle e, f \rangle = 0. \tag{3.12}$$

Notice that the equation (3.10) gives us a system of m+n equations. Suppose that  $f=(x^{1^T},x^{2^T},\ldots,x^{m+n^T})^T$ , where  $x^i \in \mathbb{R}^l$  for all  $i \in [m+n]$ . Then, subtracting the (i+1)-th equation from the i-th equation, we get the following relations for different cases.

Case I. For m = n = 2 and  $m > 2, n \ge 2$ ,

$$(J + \frac{\lambda}{2}I)(x^i - x^{i+1}) = 0, \ \forall \ 1 \le i \le m - 1 \ \& \ m + 1 \le i \le m + n - 1$$
 (3.13)

and subtracting the (m + 1)-th equation from the m-th equation, we have

$$\sum_{p=1}^{m-1} Jx^p - Jx^m + Jx^{m+1} - \sum_{q=m+2}^{m+n} Jx^q + \lambda(x^{m+1} - x^m) = 0.$$
 (3.14)

**Case II.** For m > 1, n = 1,

$$(J + \frac{\lambda}{2}I)(x^i - x^{i+1}) = 0, \ \forall \ 1 \le i \le m - 1$$
(3.15)

and subtracting the (m+1)-th equation from the m-th equation, we have

$$\sum_{p=1}^{m-1} Jx^p - Jx^m + Jx^{m+1} + \lambda(x^{m+1} - x^m) = 0.$$
(3.16)

We are interested in the Cartesian product  $K_{m,n} \times G$  where either  $K_{m,n}$  or G is of non-QE class. Otherwise, the quadratic embedding constant is zero by Theorem 3.4. Thus, it suffices to consider  $\lambda > 0$ , and from (3.13), (3.15),

$$x^{i} = x^{i+1}, \forall \ 1 \le i \le m-1 \ \& \ m+1 \le i \le m+n-1$$
 (3.17)

and

$$x^{i} = x^{i+1}, \forall \ 1 \le i \le m-1, \tag{3.18}$$

respectively. Also, recall that  $QEC(G) = \max\{a : (x, a, \eta) \in \mathcal{S}(D_G)\}\$ , where  $\mathcal{S}(D_G)$  is the set of all stationary points  $(x, a, \eta) \in \mathbb{R}^l \times \mathbb{R} \times \mathbb{R}$  satisfying

$$(D_G - aI)x = \frac{\eta}{2} e, \quad \langle x, x \rangle = 1, \quad \langle e, x \rangle = 0.$$
 (3.19)

Keeping these relations in hand, we now prove Theorem 1.4.

Proof of Theorem 1.4 (i). Without loss of generality assume that  $m \ge n$ . By Theorem 3.6,  $K_{m,n}$  is of QE class if and only if either m = n = 2 or  $m \ge 1, n = 1$ . Thus, we consider two cases.

Case I. m = n = 2: From (3.14) and (3.17),  $x^1 = x^2 = x^3 = x^4$ . This reduces the equations (3.10)-(3.12) as follows.

$$(4D_G - \lambda I)x^1 = \frac{\mu}{2} e, \quad 4\langle x^1, x^1 \rangle = 1, \quad \langle e, x^1 \rangle = 0.$$
 (3.20)

Using (3.19) and (3.20), we have

QEC
$$(K_{2,2} \times G) = \max\{\lambda : ((x^{1T}, x^{2T}, x^{3T}, x^{4T})^T, \lambda, \mu) \in \mathcal{S}(D_{K_{2,2} \times G})\}$$
  
=  $\max\{4a : (x, a, \eta) \in \mathcal{S}(D_G)\}$   
=  $4 \text{ QEC}(G)$ .

Case II.  $m \ge 1, n = 1$ : For m = n = 1, by Theorem 1.3,  $QEC(K_{1,1} \times G) = 2QEC(G)$ . Suppose m > 1, n = 1. By equation (3.18),  $x^i = x^{i+1}$  for all  $i \in [m-1]$  and equation (3.16) implies that

$$(m-2)Jx^{m} + Jx^{m+1} + \lambda(x^{m+1} - x^{m}) = 0. (3.21)$$

Also, using (3.12), we get

$$mJx^m + Jx^{m+1} = 0. (3.22)$$

Multiplying equations (3.21) and (3.22) by m+1 and (m-2)+1 respectively, and then computing their difference, we get

$${J + \frac{\lambda}{2}(m+1)I}(x^m - x^{m+1}) = 0.$$

Since  $\lambda > 0$ ,  $x^m = x^{m+1}$ . So, equations (3.10)-(3.12) reduce to

$$\{(m+1)D_G - \lambda I\}x^1 = \frac{\mu}{2} e, \quad (m+1)\langle x^1, x^1 \rangle = 1, \quad \langle e, x^1 \rangle = 0,$$

and using (3.19) we have

QEC
$$(K_{m,1} \times G) = \max\{\lambda : ((x^{1T}, x^{2T}, \dots, x^{m+1T})^T, \lambda, \mu) \in \mathcal{S}(D_{K_{m,1} \times G})\}$$
  
=  $\max\{(m+1)a : (x, a, \eta) \in \mathcal{S}(D_G)\}$   
=  $(m+1)$  QEC $(G)$ .

Proof of Theorem 1.4 (ii). By (3.17),  $x^i = x^{i+1}$  for all  $1 \le i \le m-1$  &  $m+1 \le i \le m+n-1$  and from (3.14), we have

$$(m-2)Jx^{m} - (n-2)Jx^{m+1} + \lambda(x^{m+1} - x^{m}) = 0.$$
(3.23)

From (3.12), we have

$$mJx^m + nJx^{m+1} = 0. (3.24)$$

Now, multiplying (3.23) and (3.24) by m + n and (n - 2) - (m - 2) respectively and adding them, we get

$$\{J - \lambda \frac{m+n}{m(n-2) + n(m-2)}I\}(x^m - x^{m+1}) = 0.$$
(3.25)

We now divide the remaining proof into two cases.

Case I. Suppose  $\lambda \neq l \frac{n(m-2)+m(n-2)}{m+n}$ . Then, from (3.25),  $x^m = x^{m+1}$ . Again, equations (3.10)-(3.12) reduce to

$$\{(m+n)D_G - \lambda I\}x^1 = \frac{\mu}{2} e, \quad (m+n)\langle x^1, x^1 \rangle = 1, \quad \langle e, x^1 \rangle = 0,$$

and from (3.19), we have

QEC
$$(K_{m,n} \times G) = \max\{\lambda : ((x^{1^T}, x^{2^T}, \dots, x^{m+n^T})^T, \lambda, \mu) \in \mathcal{S}(D_{K_{m,n} \times G})\}$$
  
=  $\max\{(m+n)a : (x, a, \eta) \in \mathcal{S}(D_G)\}$   
=  $(m+n)$  QEC $(G)$ .

Case II. Suppose  $\lambda = l \frac{n(m-2) + m(n-2)}{m+n}$ . Take

$$x^{1} = x^{2} = \dots = x^{m} = -\frac{\sqrt{n}}{\sqrt{ml(m+n)}} e, \ x^{m+1} = x^{m+2} = \dots = x^{m+n} = \frac{\sqrt{m}}{\sqrt{nl(m+n)}} e,$$

and

$$\mu = \left(\frac{n-m}{m+n}\right)\sqrt{\frac{mnl}{m+n}}.$$

By a simple calculation, one can see that the equations (3.10)-(3.12) are satisfied by the above choice of  $((x^{1T}, x^{2T}, \dots, x^{m+nT})^T, \lambda, \mu) \in \mathbb{R}^{(m+n)l} \times \mathbb{R} \times \mathbb{R}$ . Hence,

$$QEC(K_{m,n} \times G) = \max\{(m+n) QEC(G), l \frac{n(m-2) + m(n-2)}{m+n} \}.$$

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# REFERENCES

- [1] E.T. Baskoro, N. Obata, Determining finite connected graphs along the quadratic embedding constants of paths. Electron. J. Graph Theory Appl. 9, 539-560 (2021).
- [2] M. Bożejko, Positive-definite kernels, length functions on groups and non-commutative von Neumann inequality. Studia Math. 95, 107-118 (1989).
- [3] M. Bożejko, T. Januszkiewicz and R.J. Spatzier, *Infinite Coxeter groups do not have Kazhdan's property*. J. Operator Theory **19**, 63-67 (1988).
- [4] P.N. Choudhury and R. Nandi, Quadratic embedding constants of graphs: Bounds and distance spectra, Linear Algebra Appl. 680, 108-125 (2024).
- [5] R.L. Graham and P.M. Winkler, On isometric embedding of graphs. Trans. Amer. Math. Soc. 288, 527-536 (1985).
- [6] U. Haagerup, An example of a nonnuclear C\*-algebra which has the metric approximation property. Invent. Math. 50, 289-293 (1979).
- [7] A. Hora and N. Obata, Quantum Probability and Spectral Analysis of Graphs, Springer, 2007.
- [8] Z. Lou, N. Obata and Q. Huang, Quadratic embedding constants of graph joins. Graphs Combin. 38 (2022), no. 5, Paper No. 161, 22 pp.
- [9] W. Młotkowski and N. Obata, Quadratic embedding constants of fan graphs and graph joins, Linear Algebra Appl. 709, 58-91 (2025).

- [10] W. Młotkowski, Quadratic embedding constants of path graphs. Linear Algebra Appl. 644, 95-107 (2022).
- [11] W. Młotkowski and N. Obata, On quadratic embedding constants of star product graphs. Hokkaido Math. J. 49, 129-163 (2020).
- [12] N. Obata, Primary non-QE graphs on six vertices. Interdiscip. Inform. Sci. 29, 141-156 (2023).
- [13] N. Obata, Complete multipartite graphs of non-QE class. Electron. J. Graph Theory Appl. 11, 511-527 (2023).
- [14] N. Obata, Quadratic embedding constants of wheel graphs. Interdiscip. Inform. Sci. 23, 171-174 (2017).
- [15] N. Obata, Markov product of positive definite kernels and applications to Q-matrices of graph products. Colloq. Math. 122, 177-184 (2011).
- [16] N. Obata, Positive Q-matrices of graphs. Studia Math. 179, 81-97 (2007).
- [17] N. Obata, A.Y. Zakiyyah, Distance matrices and quadratic embedding of graphs. Electron. J. Graph Theory Appl. 6, 37-60 (2018).
- [18] I.J. Schoenberg, Metric spaces and positive definite functions. Trans. Amer. Math. Soc. 44, 522-536 (1938).
- [19] I.J. Schoenberg, Remarks to Maurice Frechet's article "Sur la definition axiomatique d'une classe d'espace distancis vectoriellement applicable sur l'espace de Hilbert". Ann. of Math. 36, 724-732 (1935).
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