# PERPETUITIES WITH LIGHT TAILS AND THE LOCAL DEPENDENCE MEASURE

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ABSTRACT. This work investigates the tail behavior of solutions to the affine sto-chastic fixed-point equation of the form  $X \stackrel{d}{=} AX + B$ , where X and (A,B) are independent. Focusing on the light-tail regime, following [Burdzy et al. (2022), Ann. Appl. Probab.] we introduce a local dependence measure along with an associated Legendre-type transform. These tools allow us to effectively describe the logarithmic right-tail asymptotics of the solution X.

Moreover, we extend our analysis to a related recursive sequence  $X_n = A_n X_{n-1} + B_n$ , where  $(A_n, B_n)_n$  are i.i.d. copies of (A, B). For this sequence, we construct deterministic scaling  $(f_n)_n$  such that  $\limsup_{n\to\infty} X_n/f_n$  is a.s. positive and finite, with its non-random explicit value provided.

#### 1. Introduction

We study the tail behavior of a solution X to the stochastic fixed-point equation

(1.1) 
$$X \stackrel{d}{=} AX + B$$
,  $X$  and  $(A, B)$  are independent.

When such a solution exists, it is referred to as a perpetuity. to Our primary focus is on the light-tail regime. Throughout the paper, we assume that A and B are almost surely nonnegative and nonzero with positive probability, thus excluding trivial cases. For comprehensive background material, we refer to [3].

The influence of the joint distribution of (A, B) on the tail behavior of X is notably diverse. When conditions ensure a light-tailed solution (see [13, 17, 16, 21, 5]), the analysis of (1.1) differs fundamentally from that in the heavy-tailed setting (cf. [18, 19, 20, 12, 15, 9]). Roughly, the distinction between the light- and heavy-tailed cases hinges upon whether  $\mathbb{P}(A \leq 1) = 1$  with light-tailed B or  $\mathbb{P}(A > 1) > 0$ . In particular, for light-tailed distributions, the dependence structure between A and B significantly affects the tail asymptotics. Conversely, for heavy-tailed solutions, marginal distributions predominantly determine asymptotics, with dependence structure influencing only multiplicative constants (see e.g. [11, 19, 8]).

Our methodology builds upon the framework established by [5], which analyzed the left-tail behavior (near  $0^+$ ) of X. Following these ideas, we introduce a local dependence measure (LDM) along with its associated Legendre-type transform. These tools

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form the cornerstone of our analysis and effectively describe the logarithmic right-tail asymptotics of X.

Specifically, we assume the following limit exists for all y > 0: (a LDM)

$$g(y) = \lim_{t \to \infty} \frac{\log \mathbb{P}(Aty + B > t)}{\log \mathbb{P}(B > t)},$$

where the function  $t \mapsto -\log \mathbb{P}(B > t)$  is regularly varying with index  $\rho > 0$ . Our first main result (Theorem 6.1) shows that if a solution to (1.1) exists and the function g satisfies a technical condition (termed admissibility), then

$$\lim_{t \to \infty} \frac{\log \mathbb{P}(X > t)}{\log \mathbb{P}(B > t)} = \lambda^*,$$

where  $\lambda^*$  is the unique nonzero fixed point of the function  $\phi_{\rho}(\lambda) = \inf_{y>0} \{y^{\rho}\lambda + g(y)\}$ . Moreover, we show that  $\lambda^* = \inf_{y \in (0,1)} \left\{ \frac{g(y)}{1-y^{\rho}} \right\}$ .

It is natural, given (1.1), to consider the recursive sequence

$$(1.2) X_n = A_n X_{n-1} + B_n, \quad n \ge 1,$$

where  $(A_n, B_n)_{n\geq 1}$  are i.i.d. copies of (A, B), and  $X_0$  is independent of this sequence. This sequence forms a Markov chain whose stationary distribution solves (1.1). Our second main result (Theorem 7.1) identifies an upper envelope for  $(X_n)_{n\geq 0}$  when  $X_0 = 0$ . We construct a deterministic scaling sequence  $(f_n)_n$  such that

$$\limsup_{n \to \infty} \frac{X_n}{f_n}$$

is almost surely positive and finite, with its deterministic explicit value provided. To the best of our knowledge, this constitutes the first characterization of the upper envelope for a sequence defined by (1.2). The proof extends techniques from [5], which derived a lower envelope under related but distinct assumptions.

Our study is inspired by several previous investigations into the tail behavior of X. A common strategy in earlier work (cf. [15, 14, 23, 4]) involves bounding the tail probability of X from above and below as

$$\mathbb{P}(X_n > t) \le \mathbb{P}(X > t) \le \mathbb{P}(X'_n > t),$$

where both  $(X_n)_{n\geq 0}$  and  $(X'_n)_{n\geq 0}$  satisfy (1.2) for some special  $X_0$  and  $X'_0$ . Thus, understanding  $\mathbb{P}(X>t)$  effectively reduces to analyzing the tail asymptotics of  $\mathbb{P}(AY+B>t)$ , with Y independent of (A,B) but not necessarily satisfying (1.1).

1.1. Relation to the literature. We note that similar ideas to the definition of LDM have appeared in related literature. In [4, Theorem 1], for instance, it is assumed that there exists a finite function f such that

(1.3) 
$$f(y) = \lim_{t \to \infty} \frac{\mathbb{P}(Ay + B > t)}{\mathbb{P}(B > t)}.$$

If A and B are independent, A has a finite moment generating function, and  $t \mapsto \mathbb{P}(B > \log t)$  is regularly varying with index  $-\alpha \leq 0$ , then by the Breiman Lemma (see e.g. [10]) one obtains explicit form of  $f(y) = \mathbb{E}[\exp(\alpha y A)]$ . However, if A and B are not

independent, yet (1.3) holds, the form of f may differ (see [4, Remark 2.3]). Under (1.3) with  $\mathbb{P}(B > t) \sim a t^c e^{-bt}$ ,  $\mathbb{E}[e^{bB} \mathbf{1}_{A=1}] < 1$  and some technical assumptions, [4] established that

$$\lim_{t\to\infty}\frac{\mathbb{P}(X>t)}{\mathbb{P}(B>t)}=\frac{\mathbb{E}[f(X)]}{1-\mathbb{E}[e^{bB}\mathbf{1}_{A=1}]}.$$

A closely related scenario is presented in [8], where it is assumed that  $t \mapsto \mathbb{P}(A > t)$  is regularly varying with index  $-\alpha < 0$ , along with the conditions  $\mathbb{E}[A^{\alpha}] < 1$  and  $\limsup_{t \to \infty} \mathbb{P}(B > t)/\mathbb{P}(A > t) < \infty$ . In this setting, f is defined as

$$f(y) = \lim_{t \to \infty} \frac{\mathbb{P}(Ay + B > t)}{\mathbb{P}(A > t)},$$

and, under further technical conditions on the distribution of A, the tail asymptotics of X are described by

$$\lim_{t\to\infty}\frac{\mathbb{P}(X>t)}{\mathbb{P}(A>t)}=\frac{\mathbb{E}[f(X)]}{1-\mathbb{E}[A^{\alpha}]}.$$

In the special case of independence between A and B, explicit computation of f becomes possible.

Thus, our contribution can be viewed as an advancement of this line of research, relaxing independence assumptions and thereby enabling analysis of a broader class of models.

Additionally, logarithmic asymptotics of the tail behavior of X were considered in [22], under the assumption that the functions

$$t \mapsto -\log \mathbb{P}(1/(1-A) > t)$$
 and  $t \mapsto -\log \mathbb{P}(B > t)$ 

are regularly varying (or when either A or B is bounded), again requiring independence of A and B. For general dependent A and B, however, only a lower bound on the tail was provided. As noted, the light tails heavily depend on the dependence structure of (A, B). This relation seems to be captured quite well by a function h, which was defined in [22] by

$$h(t) = \inf_{s \ge 1} \left\{ -s \log \mathbb{P}\left(\frac{1}{1 - A} > s, B > \frac{t}{s}\right) \right\}, \qquad t > 0.$$

There it was shown (see [22, Theorem 5.1]) that

$$\liminf_{t \to \infty} \frac{\log \mathbb{P}(X > t)}{h(t)} \ge -c$$

with an explicit constant c depending on the index of regular variation of h. In Section 8 we present an example demonstrating that this lower bound is generally not optimal.

In contrast, our paper addresses this gap by deriving precise logarithmic asymptotics within a general framework built upon the local dependence measure. Our results extend and complement previous findings in light-tailed scenarios where the tail behavior of X is predominantly governed by the distribution of B rather than A.

In [5] a less common problem in the theory of (1.1) was considered. If A and B are nonnegative, then X, if exists, is also nonnegative. Thus, one may investigate the tail

behavior of X near the left endpoint of the support. One of the main results in [5] establishes that if the limit

$$g(y) = \lim_{\varepsilon \to 0^+} \frac{\log \mathbb{P}(A\varepsilon y + B < \varepsilon)}{\log \mathbb{P}(B < \varepsilon)}$$

exists for  $y \in [0, \infty)$ , where  $\varepsilon \mapsto -\log \mathbb{P}(B < \varepsilon)$  is a regularly varying function with index  $\rho < 0$  at 0, then—under some technical assumptions (analogous to the admissibility condition in the present paper)—one has

$$\lim_{\varepsilon \to 0^+} \frac{\log \mathbb{P}(X < \varepsilon)}{\log \mathbb{P}(B < \varepsilon)} = \inf_{y > 1} \left\{ \frac{y^{\rho}}{y^{\rho} - 1} g(y) \right\}.$$

Another result in [5] concerns the lower envelope of the sequence (1.2) with  $X_0 = 0$ ; specifically, there exists an explicit deterministic scaling sequence  $(h_n)_n$  such that

$$\liminf_{n \to \infty} \frac{X_n}{h_n}$$

is deterministic, positive and finite.

There are clear analogies between the statements and proofs in the present paper and those in [5]. However, aside from Lemmas 5.6 and 7.5, which are taken directly from [5], all other results in our paper have independent proofs. Notably, our analysis reveals quantitatively different behavior for  $\rho \in (0,1]$  and  $\rho > 1$  (see, e.g., Theorem 3.11), a phenomenon that was not observed in [5]. While [5] applied the left-tail results to a Fleming–Viot-type process, in Section 8 we illustrate our findings using a new family of distributions.

We also note that the assumptions in the present paper and in [5] are compatible. Consequently, one can readily construct a distribution of (A, B) such that both the left and right tail behaviors of X satisfying (1.1) are asymptotically available and the sequence  $X_n = A_n X_{n-1} + B_n$  exhibits explicit lower and upper envelopes. For example, assume that the random pair (A, B) is positively quadrant dependent (see Section 3.1) and that for  $\rho, \sigma > 0$  one has

$$\lim_{t\to\infty}\frac{-\log\mathbb{P}(B>t)}{t^\rho}=\lambda_+\qquad\text{and}\qquad\lim_{\varepsilon\to0^+}\frac{-\log\mathbb{P}(B<\varepsilon)}{\varepsilon^{-\sigma}}=\lambda_-$$

together with

$$a_{-} = \operatorname{ess\,inf}(A) \ge 0$$
 and  $a_{+} = \operatorname{ess\,sup}(A) < 1$ .

Under these conditions, since  $\mathbb{E}[\log A] < 0$ ,  $\mathbb{E}[\max\{\log B, 0\}] < \infty$ , Theorems 6.1 and 3.11 in this paper together with [5, Theorem 4.1 and Proposition 5.4] yield

$$\lim_{t\to\infty}\frac{-\log\mathbb{P}(X>t)}{t^\rho}=c_+\qquad\text{and}\qquad\lim_{\varepsilon\to0^+}\frac{-\log\mathbb{P}(X<\varepsilon)}{\varepsilon^{-\sigma}}=c_-,$$

where

$$c_{+} = \lambda_{+} \begin{cases} \left(1 - a_{+}^{\rho/(\rho - 1)}\right)^{\rho - 1}, & \text{if } \rho > 1, \\ 1, & \text{if } \rho \in (0, 1), \end{cases} \quad \text{and} \quad c_{-} = \lambda_{-} \left(1 - a_{-}^{\sigma/(1 + \sigma)}\right)^{-(1 + \sigma)}.$$

Moreover, if  $X_0 = 0$ , then by Theorem 7.1 and [5, Theorem 6.1],

$$\limsup_{n\to\infty}\frac{X_n}{(\log n)^{1/\rho}}=c_+^{-1/\rho}\qquad\text{and}\qquad \liminf_{n\to\infty}\frac{X_n}{(\log n)^{-1/\sigma}}=c_-^{1/\sigma}.$$

1.2. Organization of the paper. The remainder of the paper is organized as follows. In the next section, we give a short review of the theory of regularly varying functions. Section 3 introduces the local dependence measure and its Legendre-type transform, and establishes their key properties. We show that when A and B are independent (or more generally, positively quadrant dependent), the LDM admits an explicit representation. In Section 4, we derive the asymptotics of  $t \mapsto \log \mathbb{P}(AX + B > t)$  under independence between X and (A, B), where X does not necessarily satisfy (1.1). This is a result that is instrumental in controlling the asymptotics of  $\mathbb{P}(X_n > t)$ . Section 5 discusses the existence, uniqueness, and basic properties of solutions to (1.1). Section 6 is devoted to our first main result, the precise logarithmic tail asymptotics of the solution to (1.1). In Section 7 we prove our second main result: the identification of the upper envelope for the sequence  $(X_n)_{n\geq 0}$  with  $X_0 = 0$ . Finally, in Section 8 we describe a family of distributions of pairs (A, B) for which the LDM is computable and provide its general form. We use our earlier results to demonstrate that the lower bound in [22] is not optimal in general.

## 2. Regular variation

We write  $f(x) \sim g(x)$  if  $\lim_{x\to\infty} f(x)/g(x) = 1$ . A measurable function  $f: (0,\infty) \to (0,\infty)$  is said to be regularly varying of index  $\rho \in \mathbb{R}$  if and only if  $f(\lambda x) \sim \lambda^{\rho} f(x)$  for all  $\lambda > 0$ .

The class of regularly varying functions of index  $\rho \in \mathbb{R}$  is denoted by  $\mathcal{R}^{\rho}$ . In particular, elements of the class  $\mathcal{R}^{0}$  are called slowly varying functions.

We gather all the necessary properties of regularly varying functions in the lemma below.

**Lemma 2.1.** Let  $f \in \mathbb{R}^{\rho}$  with  $\rho > 0$ . Then:

- (i)  $f(x) = x^{\rho}\ell(x)$  for some slowly varying function  $\ell \in \mathbb{R}^0$ .
- (ii)  $\lim_{x\to\infty} f(x) = \infty$ .
- (iii) For any C > 1 and  $\delta > 0$ , there exists K such that for  $x, y \geq K$ ,

$$\frac{f(y)}{f(x)} \le C \max \left\{ \left(\frac{y}{x}\right)^{\rho+\delta}, \left(\frac{y}{x}\right)^{\rho-\delta} \right\}.$$

(iv) There exists  $q \in \mathbb{R}^{1/\rho}$  such that

$$f(g(x)) \sim g(f(x)) \sim x$$

and g is determined uniquely to within asymptotic equivalence  $\sim$ .

(v) There exists  $\tilde{f} \in \mathbb{R}^{\rho}$  such that  $\tilde{f}$  continuous, strictly increasing and  $\tilde{f}(x) \sim f(x)$ .

*Proof.* (i) follows directly from the definition of  $\mathcal{R}^{\rho}$ . (ii) is proved in [2, Proposition 1.5.1], (iii) is known as the Potter bounds, [2, Theorem 1.5.6]. (iv) is proved in [2, Theorem 1.5.12]. For (v), fix X > 0 so large that f is locally bounded on  $[X, \infty)$  and define  $\tilde{f}(x) = \rho \int_X^x t^{-1} f(t) dt$  for  $x \in (X, \infty)$ . Extend the definition of  $\tilde{f}$  to (0, X] in a

way that it is continuous and strictly increasing on  $(0, \infty)$ . By Karamata's Theorem, [2, Theorem 1.5.11], we have  $\tilde{f}(x) \sim f(x)$ .

#### 3. Exponential decay and local dependence measure

**Definition 3.1.** Let f be a function defined on a neighborhood of infinity such that  $f(t) \to \infty$  as  $t \to \infty$ . We say that a nonnegative random variable X has an exponential-f-decay tail if

$$\lim_{t \to \infty} \frac{-\log \mathbb{P}(X > t)}{f(t)} = \lambda,$$

where  $\lambda \in [0, \infty]$ . We call such a random variable an  $\mathrm{ED}_f(\lambda)$ -random variable.

We note that if X is a bounded random variable, then it is  $\mathrm{ED}_f(\infty)$  for any f.

**Definition 3.2.** Let (A, B) be a pair of nonnegative random variables, and let  $f \in \mathcal{R}^{\rho}$  with  $\rho > 0$ . A function  $g : [0, \infty) \to [0, \infty]$  is said to be the local dependence measure of (A, B) (denoted  $\text{LDM}_f^{\rho}$ ), if the limit

$$g(y) = \lim_{t \to \infty} \frac{-\log \mathbb{P}(Aty + B > t)}{f(t)}, \quad y \in [0, \infty)$$

exists.

In general, for an arbitrary pair (A, B) and a function f, the existence of g on  $[0, \infty)$  is not guaranteed. Nevertheless, we will show that g(y) = 0 for  $y > a_+^{-1}$ , where

$$a_{+} = \operatorname{ess\,sup}(A).$$

Furthermore, if A and B are positively quadrant dependent (PQD) and  $g(0) < \infty$ , then g exists on  $[0, \infty)$  and its explicit form can be derived—it depends only on  $a_+$  and g(0) ( see Section 3.1 for details). Another family of distributions for which an explicit expression for g is available is presented in Section 8.

Assume that  $\tilde{f} \in \mathcal{R}^{\rho}$  is such that  $\tilde{f}$  continuous, strictly increasing and  $\tilde{f}(x) \sim f(x)$ . Clearly, we have  $\mathrm{ED}_{f}(\lambda) = \mathrm{ED}_{\tilde{f}}(\lambda)$  and  $\mathrm{LDM}_{f}^{\rho} = \mathrm{LDM}_{\tilde{f}}^{\rho}$ . In view of Lemma 2.1, without loss of generality, we will assume from now on that every regularly varying function f is continuous and strictly increasing.

**Lemma 3.3.** Assume that (A, B) are nonnegative random variables and  $f \in \mathbb{R}^{\rho}$  with  $\rho > 0$ . Suppose g(0) exists and  $a_{+} > 0$ . Then:

(i) For every  $y > a_+^{-1}$ ,

$$\lim_{t \to \infty} \frac{-\log \mathbb{P}(Aty + B > t)}{f(t)} = 0.$$

(ii) For every  $y \in [0, a_{+}^{-1})$ ,

$$\liminf_{t \to \infty} \frac{-\log \mathbb{P}(Aty + B > t)}{f(t)} \ge g(0)(1 - a_+ y)^{\rho}.$$

(iii) If  $y > x \ge 0$  and both g(x) and g(y) exists, then  $g(y) \le g(x)$ ,

*Proof.* (i) Assume  $y > a_+^{-1}$ . Clearly,  $\mathbb{P}(A > 1/y) > 0$ . Notice that  $\{A > 1/y\} = \{Aty > t\} \subset \{Aty + B > t\}$  for t > 0 and therefore,

$$0 \le \liminf_{t \to \infty} \frac{-\log \mathbb{P}(Aty + B > t)}{f(t)}$$

$$\le \limsup_{t \to \infty} \frac{-\log \mathbb{P}(Aty + B > t)}{f(t)} \le \lim_{t \to \infty} \frac{-\log \mathbb{P}(A > 1/y)}{f(t)} = 0.$$

(ii) Since  $\mathbb{P}(A \leq a_+) = 1$ , we have

$$\mathbb{P}(Aty + B > t) \le \mathbb{P}(a_+ty + B > t) = \mathbb{P}(B > t(1 - a_+y)).$$

Thus, for  $y \in [0, 1/a_+)$ ,

$$\lim_{t \to \infty} \frac{-\log \mathbb{P}(Aty + B > t)}{f(t)} \ge \lim_{t \to \infty} \frac{-\log \mathbb{P}(B > t(1 - a_{+}y))}{f(t)}$$

$$= \lim_{t \to \infty} \frac{-\log \mathbb{P}(B > t(1 - a_{+}y))}{f(t(1 - a_{+}y))} \cdot \frac{f(t(1 - a_{+}y))}{f(t)}$$

$$= g(0)(1 - a_{+}y)^{\rho}.$$

(iii) Let  $y > x \ge 0$ . We note that since A is a nonnegative random variable, for all t > 0, we have the inclusion  $\{Atx + B > t\} \subset \{Aty + B > t\}$ . By the monotonicity of  $\mathbb P$  and the assumption that both g(x) and g(y) are well defined we get that  $g(y) \le g(x)$ .

**Remark 3.4.** Lemma 3.3 implies several key properties of the local dependence measure g:

- On the interval  $(a_+^{-1}, \infty)$ , the limit always exists and equals 0.
- If g is well defined on  $[0, a_+^{-1})$ , then it is bounded below by  $g(0)(1 a_+ y)^{\rho}$  on this interval.
- If g exists on the entire  $[0, \infty)$ , then it is a nonincreasing function.

In Section 3.1, we show that the lower bound is attained for positively quadrant dependent A and B.

**Definition 3.5.** For a function  $g:(0,\infty)\to [0,\infty]$ , let  $\phi_{\rho}:[0,\infty)\to [0,\infty]$  be a Legendre-type transform defined by

$$\phi_{\rho}(\lambda) = \inf_{y>0} \{y^{\rho}\lambda + g(y)\}, \qquad \lambda \in [0, \infty).$$

From this point forward, unless explicitly stated otherwise, we assume that g exists on  $[0, \infty)$ .

**Lemma 3.6.** Let g be the  $LDM_f^{\rho}$  for the nonnegative random variables (A, B) such that  $a_+ = \operatorname{ess\,sup}(A) > 0$ , where  $f \in \mathcal{R}^{\rho}$  and  $\rho > 0$ . Let  $\phi_{\rho}$  be its Legendre-type transform. Then:

- (i)  $\phi_{\rho}$  is finite, nondecreasing, concave.
- (ii)  $\phi_{\rho}(0) = \phi_{\rho}(0^{+}) = 0.$
- (iii)  $\phi_{\rho}$  is continuous.

- (iv)  $\sup_{y>0} \{g(y)\} \ge \phi_{\rho}(\lambda)$ .
- (v) If there exists  $\lambda > 0$  such that  $\phi_{\rho}(\lambda) > \lambda$ , then  $\phi_{\rho}(\lambda) = \lambda$  for at most one  $\lambda > 0$ .
- (vi) With  $\gamma = g(0)$  and  $a_+ = \operatorname{ess\,sup}(A) > 0$ ,

$$\lambda a^{-\rho} \ge \phi_{\rho}(\lambda) \ge \begin{cases} \left( \gamma^{1/(1-\rho)} + a_{+}^{\rho/(\rho-1)} \lambda^{1/(1-\rho)} \right)^{1-\rho}, & \text{if } \rho > 1, \\ \min\left\{ \gamma, \lambda a_{+}^{-\rho} \right\}, & \text{if } 0 < \rho \le 1. \end{cases}$$

*Proof.* (i) Finiteness follows from Lemma 3.3 (i), monotonicity is obvious,  $\phi_{\rho}$  as the point-wise infimum of a family of affine functions is concave.

- (ii) We have  $\phi_{\rho}(0) = \inf_{y>0} \{g(y)\} = 0$  by Lemma 3.3 (i). Moreover, by the definition of  $\phi_{\rho}$  we get  $\phi_{\rho}(\lambda) \leq y^{\rho}\lambda + g(y)$  for any y > 0. Thus, by taking y sufficiently large (recall Lemma 3.3 (i)) and letting  $\lambda \downarrow 0^+$ , we obtain  $\phi_{\rho}(0^+) \leq 0$ .
- (iii) From (ii) we get continuity in 0. Continuity on  $(0, \infty)$  is a consequence of the fact that  $\phi_{\rho}$  is concave, which was stated in (i).
- (iv) By the definition of  $\phi_{\rho}(\lambda)$ , we have  $\phi_{\rho}(\lambda) \leq y^{\rho}\lambda + g(y)$  for all y > 0. Letting  $y \downarrow 0^+$  (noting that g is a nonincreasing function, so its limit as  $y \to 0^+$  exists, although it may be infinite), we obtain  $\phi_{\rho}(\lambda) \leq g(0^+) = \sup_{y>0} \{g(y)\}.$
- (v) Define  $\psi(\lambda) = \phi_{\rho}(\lambda) \lambda$ . Then  $\psi$  is concave, continuous, and  $\psi(0) = 0$ . Suppose there exist  $0 < \lambda_1 < \lambda_2$  with  $\phi_{\rho}(\lambda_1) = \lambda_1$  and  $\phi_{\rho}(\lambda_2) = \lambda_2$ , so that  $\psi(0) = \psi(\lambda_1) = \psi(\lambda_2) = 0$ . By concavity,  $\psi \equiv 0$  on  $[0, \lambda_2]$  and  $\psi(\lambda) \leq 0$  for  $\lambda > \lambda_2$ . Now, if some  $\lambda_0$  satisfies  $\phi_{\rho}(\lambda_0) > \lambda_0$  (i.e.,  $\psi(\lambda_0) > 0$ ), then we obtain a contradiction. Hence, the equation  $\phi_{\rho}(\lambda) = \lambda$  can have at most one solution for  $\lambda > 0$ .
- (vi) The upper and lower bounds on  $\phi_{\rho}(\lambda)$  are based on properties (i) and (ii) from Lemma 3.3. The derivation of these bounds follows from the proof of Theorem 3.11.

**Definition 3.7.** We say that a LDM<sub>f</sub><sup> $\rho$ </sup> function g is admissible if either: there exists  $\lambda > 0$  such that  $\phi_{\rho}(\lambda) > \lambda$  or g(0) = 0 (in which case  $\phi_{\rho} \equiv 0$ ).

The notion of admissibility is introduced to exclude those distributions of (A, B) and choices of the function f for which our analytical framework fails (see Lemma 6.3). To illustrate, consider the following example. Assume that the nonnegative random variables A and B are independent,  $a_+ = \operatorname{ess\,sup}(A) \in (0,1]$  and  $f(t) = -\log \mathbb{P}(B > t)$  is regularly varying with index  $-\rho < 0$ . Then, by Theorem 3.11, g exists and is given by  $g(y) = \max\{1 - a_+ y, 0\}^{\rho}$  for  $y \ge 0$ . In this case, one can verify that the admissibility condition holds.

In contrast, consider the same model but with the choice  $f(t) = t^{\sigma}$  for some  $\sigma \in (0, \rho)$ . In this setting, the LDM is given by

$$g(y) = \lim_{t \to \infty} \frac{-\log \mathbb{P}(Aty + B > t)}{t^{\sigma}} = \begin{cases} \infty, & \text{if } y < a_{+}^{-1}, \\ 0, & \text{if } y > a_{+}^{-1}, \end{cases}$$

for any  $\sigma \in (0, \rho)$ . A direct calculation shows that the corresponding Legendre-type transform satisfies  $\phi_{\sigma}(\lambda) = \lambda$  for all  $\lambda > 0$ , thereby violating the condition  $\phi_{\sigma}(\lambda) > \lambda$ . Consequently, this LDM is non-admissible.

**Definition 3.8.** For a LDM<sub>f</sub> function g, we define

(3.1) 
$$\lambda^* = \inf_{y \in (0,1)} \left\{ \frac{g(y)}{1 - y^{\rho}} \right\}.$$

Lemma 3.9.

- (i) Assume  $c \geq 0$ . Then,  $\phi_{\rho}(c) \geq c$  if and only if  $c \leq \lambda^*$ .
- (ii) If  $\lambda^* < \infty$ , then  $\phi_{\rho}(\lambda^*) = \lambda^*$ .
- (iii) With  $\gamma = g(0)$  and  $a_+ = \operatorname{ess\,sup}(A) > 0$ ,

$$\lambda^* \ge \begin{cases} \gamma \left( 1 - a_+^{\rho/(\rho - 1)} \right)^{\rho - 1}, & \text{if } \rho > 1, \\ \gamma, & \text{if } 0 < \rho \le 1. \end{cases}$$

(iv)  $\lambda^* = \infty \iff g(0) = \infty$ .

*Proof.* (i) We have

$$\phi_{\rho}(c) \ge c \iff \inf_{y>0} \left\{ y^{\rho}c + g(y) \right\} \ge c \iff \forall y > 0 \ y^{\rho}c + g(y) \ge c$$
$$\iff \forall y \in (0,1) \ y^{\rho}c + g(y) \ge c \iff c \le \inf_{y \in (0,1)} \left\{ \frac{g(y)}{1 - y^{\rho}} \right\}.$$

- (ii) Let  $(\lambda_n)_n$  be a sequence such that  $\lambda_n \downarrow \lambda^*$ . Then, by (i) we have  $\phi_\rho(\lambda_n) < \lambda_n$  and by continuity of  $\phi_\rho$ , we obtain  $\phi_\rho(\lambda^*) \leq \lambda^*$ . Setting  $c = \lambda^*$  in (i), we obtain reversed bound.
- (iii) The lower bound on  $\lambda^*$  quickly follows from the lower bound on g in Lemma 3.3 (ii).
- (iv) If  $g(0) = \infty$ , then by (iii) we obtain  $\lambda^* \geq \infty$ . If  $\lambda^* = \infty$ , then by definition of  $\lambda^*$ , we have for any  $y \in (0,1)$ ,  $g(y) \geq (1-y^\rho)\lambda^*$ . Since g is nonincreasing,  $g(0) = \infty$ .

# 3.1. Positive quadrant dependence.

**Definition 3.10.** We say that random variables A and B are positively quadrant dependent (PQD for short) if, for all  $a, b \in \mathbb{R}$ ,

$$\mathbb{P}(A>a,B>b) \geq \mathbb{P}(A>a)\mathbb{P}(B>b).$$

It turns out that for a PQD pair (A, B), the LDM always exists  $[0, \infty)$  and, moreover, has explicit form.

Recall that  $a_+ = \operatorname{ess\,sup}(A) = \inf \{ x \in \mathbb{R} \colon \mathbb{P}(A > x) = 0 \}.$ 

**Theorem 3.11.** Assume that nonnegative random variables A and B are PQD. Suppose that  $\gamma := g(0)$  exists and is finite for some  $f \in \mathbb{R}^{\rho}$  with  $\rho > 0$ . Then, the local dependence measure g of (A, B) exists on  $[0, \infty)$ . Suppose  $a_+ > 0$ .

(i) For 
$$y \geq 0$$
,

$$g(y) = \gamma \max\{1 - a_+ y, 0\}^{\rho}.$$

(ii) If  $a_+ \in (0,1]$ , then for  $\lambda \geq 0$ ,

$$\phi_{\rho}(\lambda) = \begin{cases} \left( \gamma^{1/(1-\rho)} + a_{+}^{\rho/(\rho-1)} \lambda^{1/(1-\rho)} \right)^{1-\rho}, & \text{if } \rho > 1, \\ \min\left\{ \gamma, \lambda a_{+}^{-\rho} \right\}, & \text{if } 0 < \rho \le 1, \end{cases}$$

and

$$\lambda^* = \begin{cases} \gamma \left( 1 - a_+^{\rho/(\rho - 1)} \right)^{\rho - 1}, & \text{if } \rho > 1, \\ \gamma, & \text{if } 0 < \rho \le 1. \end{cases}$$

*Proof.* (i) By Lemma 3.3 (ii), for  $y \in [0, a_+^{-1})$  we have

$$\liminf_{t \to \infty} \frac{-\log \mathbb{P}(Aty + B > t)}{f(t)} \ge \gamma (1 - a_+ y)^{\rho}.$$

Since we assume that g(0) is finite, the above inequality extends to  $y = a_+^{-1}$ .

We now show that for  $y \in [0, a_+^{-1}]$ , the quantity  $\gamma(1-a_+y)^{\rho}$  is an upper bound for the superior limit. Fix  $\delta \in (0, a_+)$ . Then, if  $0 \le y \le a_+^{-1} < (a_+ - \delta)^{-1}$ , we obtain

$$\mathbb{P}(Aty + B > t) \ge \mathbb{P}(Aty + B > t, A > a_{+} - \delta) \ge \mathbb{P}((a_{+} - \delta)ty + B > t, A > a_{+} - \delta)$$
  
 
$$\ge \mathbb{P}(A > a_{+} - \delta) \mathbb{P}(B > t(1 - (a_{+} - \delta)y)),$$

where we have used the fact that A and B are PQD. Therefore,

$$\limsup_{t \to \infty} \frac{-\log \mathbb{P}(Aty + B > t)}{f(t)} \le \lim_{t \to \infty} \frac{-\log \mathbb{P}(A > a_+ - \delta) - \log \mathbb{P}(B > t(1 - (a_+ - \delta)y))}{f(t)}$$

Since  $f(t) \to \infty$  as  $t \to \infty$ , and from the fact that  $\mathbb{P}(A > a_+ - \delta) > 0$ , we obtain

$$\limsup_{t \to \infty} \frac{-\log \mathbb{P}(Aty + B > t)}{f(t)}$$

$$\leq \lim_{t \to \infty} \frac{-\log \mathbb{P}(B > t(1 - (a_{+} - \delta)y))}{f(t(1 - (a_{+} - \delta)y))} \cdot \frac{f(t(1 - (a_{+} - \delta)y))}{f(t)} = \gamma(1 - (a_{+} - \delta)y)^{\rho}.$$

By letting  $\delta \downarrow 0^+$ , we conclude that for  $0 \le y \le a_+^{-1}$ ,

$$\gamma (1 - a_{+}y)^{\rho} \leq \liminf_{t \to \infty} \frac{-\log \mathbb{P}(Aty + B > t)}{f(t)}$$

$$\leq \limsup_{t \to \infty} \frac{-\log \mathbb{P}(Aty + B > t)}{f(t)} \leq \gamma (1 - a_{+}y)^{\rho}.$$

Thus, g exists on  $[0, a_+^{-1}]$  and is equal to  $\gamma(1-a_+y)^{\rho}$  on this interval. Finally, by Remark 3.4, for  $y > a_+^{-1}$  we have that g(y) exists and equals 0. This completes the proof of (i).

(ii) Since g(y) = 0 for  $y \ge a_+^{-1}$ , we have

$$\phi_{\rho}(\lambda) = \inf_{y>0} \left\{ \lambda y^{\rho} + g(y) \right\} = \inf_{y \in (0, a_{+}^{-1})} \left\{ \lambda y^{\rho} + g(y) \right\} = \inf_{y \in (0, a_{+}^{-1})} \left\{ h_{1}(y) \right\},$$

where  $h_1: (0, a_+^{-1}) \to \mathbb{R}$  is defined by  $h_1(y) = \lambda y^{\rho} + \gamma (1 - a_+ y)^{\rho}$ .

If  $0 < \rho < 1$ , then  $h_1$  is increasing on (0, K) and decreasing on  $(K, a_+^{-1})$ , where  $K = (a_+ + (\frac{a_+ \gamma}{\lambda})^{1/(1-\rho)})^{-1}$ . Therefore,

$$\inf_{(0,a_{-}^{-1})} \{h_1(y)\} = \min \{h_1(0^+), h_1((a_{+}^{-1})^-)\} = \min \{\gamma, \lambda a_{+}^{-\rho}\}.$$

The case  $\rho \in (0,1)$  easily extends to  $\rho = 1$ .

If  $\rho > 1$ , then  $h_1$  is decreasing on (0, K) and increasing on  $(K, a_+^{-1})$ , where K is as before. Therefore,

$$\inf_{(0,a_+^{-1})} \{h_1(y)\} = h_1(K) = \frac{\lambda \gamma}{\left(\lambda^{\frac{1}{\rho-1}} + a_+^{\frac{\rho}{\rho-1}} \gamma^{\frac{1}{\rho-1}}\right)^{\rho-1}} = \left(\gamma^{1/(1-\rho)} + a_+^{\rho/(\rho-1)} \lambda^{1/(1-\rho)}\right)^{1-\rho}.$$

Moreover, it is easy to see that  $h_1(K) \leq \lambda a_+^{-\rho}$ .

We have

$$\lambda^* = \inf_{y \in (0,1)} \left\{ \frac{g(y)}{1 - y^{\rho}} \right\} = \inf_{y \in (0,1)} \left\{ h_2(y) \right\},\,$$

where  $h_2: (0,1) \to \mathbb{R}$  is defined by  $h_2(y) = \gamma(1-a_+y)^{\rho}/(1-y^{\rho})$ . If  $\rho \in (0,1]$ , then  $h_2$  is increasing on (0,1) and therefore

$$\inf_{y \in (0,1)} \{ h_2(y) \} = h_2(0^+) = \gamma.$$

If  $\rho > 1$ , then  $h_2$  is increasing on  $(a_+^{1/(\rho-1)}, 1)$  and decreasing on  $(0, a_+^{1/(\rho-1)})$ . Therefore,

$$\inf_{y \in (0,1)} \left\{ h_2(y) \right\} = h_2 \left( a_+^{1/(\rho - 1)} \right) = \gamma \left( 1 - a_+^{\rho/(\rho - 1)} \right)^{\rho - 1}.$$

4. Tails of AX + B

Recall our standing assumption that  $\mathbb{P}(A > 0) > 0$ . In this section, X is assumed to be independent of (A, B) but does not necessarily satisfy (1.1).

**Theorem 4.1.** Let g be the  $\mathrm{LDM}_f^{\rho}$  for the nonnegative random variables (A, B), where  $f \in \mathbb{R}^{\rho}$  with  $\rho > 0$ . Suppose X is  $\mathrm{ED}_f(\lambda)$  with  $\lambda \in [0, \infty)$ . Then AX + B is  $\mathrm{ED}_f(\phi_{\rho}(\lambda))$ .

*Proof.* First, we prove that

(4.1) 
$$\limsup_{t \to \infty} \frac{-\log \mathbb{P}(AX + B > t)}{f(t)} \le \phi_{\rho}(\lambda).$$

For any y > 0 and t > 0, we have

$$\mathbb{P}(AX + B > t) \ge \mathbb{P}(AX + B > t, X > ty) \ge \mathbb{P}(Aty + B > t, X > ty)$$
$$= \mathbb{P}(Aty + B > t) \mathbb{P}(X > ty).$$

Thus,

$$\limsup_{t \to \infty} \frac{-\log \mathbb{P}(AX + B > t)}{f(t)} \le \limsup_{t \to \infty} \frac{-\log \mathbb{P}(Aty + B > t) - \log \mathbb{P}(X > ty)}{f(t)}$$

$$\leq g(y) + \limsup_{t \to \infty} \frac{-\log \mathbb{P}(X > ty)}{f(ty)} \cdot \frac{f(ty)}{f(t)} = g(y) + \lambda y^{\rho}.$$

By taking  $\inf_{y>0}$  on both sides, we obtain (4.1).

Next, we establish the lower bound,

(4.2) 
$$\liminf_{t \to \infty} \frac{-\log \mathbb{P}(AX + B > t)}{f(t)} \ge \phi_{\rho}(\lambda).$$

If  $\lambda = 0$ , then by Lemma 3.6 (ii), we have  $\phi_{\rho}(\lambda) = 0$ , and the above inequality is trivial. Now suppose  $\lambda > 0$ . Since  $\phi_{\rho}$  is finite, there exists a > 0 such that  $\lambda a^{\rho} \geq \phi_{\rho}(\lambda)$ . We note that

(4.3) 
$$\lim_{t \to \infty} \frac{-\log \mathbb{P}(AX + B > t, X > ta)}{f(t)} \ge \lim_{t \to \infty} \frac{-\log \mathbb{P}(X > ta)}{f(t)}$$
$$= \lim_{t \to \infty} \frac{-\log \mathbb{P}(X > ta)}{f(ta)} \cdot \frac{f(ta)}{f(t)} = \lambda a^{\rho} \ge \phi_{\rho}(\lambda).$$

Moreover, by Lemma 3.6 (iv), we have  $\sup_{y>0} \{g(y)\} \ge \phi_{\rho}(\lambda)$ . Fix  $\eta > 0$ . There exists  $b \in (0, a)$  such that  $g(b) \ge \phi_{\rho}(\lambda) - \eta/2$ . We have

$$\lim_{t \to \infty} \frac{-\log \mathbb{P}(AX + B > t, X \le tb)}{f(t)} \ge \lim_{t \to \infty} \frac{-\log \mathbb{P}(Atb + B > t)}{f(t)} = g(b)$$

$$\ge \phi_{\rho}(\lambda) - \frac{\eta}{2}.$$

Combining (4.3) and (4.4), we deduce that there exists M > 0 such that for all t > M,

$$\mathbb{P}(AX + B > t, X > ta) + \mathbb{P}(AX + B > t, X \le tb) \le 2\exp(-f(t)(\phi_{\rho}(\lambda) - \eta)).$$

Now, fix y, h > 0 such that h < y. Then,

(4.5) 
$$\mathbb{P}(AX + B > t, t(y - h) < X \le ty) \le \mathbb{P}(Aty + B > t, t(y - h) < X)$$

$$= \mathbb{P}(Aty + B > t) \mathbb{P}(X > t(y - h)).$$

Since

$$\lim_{t\to\infty}\frac{-\log\mathbb{P}(X>t(y-h))}{f(t)}=\lim_{t\to\infty}\frac{-\log\mathbb{P}(X>t(y-h))}{f(t(y-h))}\cdot\frac{f(t(y-h))}{f(t)}=\lambda(y-h)^\rho,$$

for sufficiently large t, we obtain

$$\mathbb{P}(X > t(y - h)) \le \exp(-f(t)(\lambda(y - h)^{\rho} - \eta)).$$

If  $g(y) < \infty$ , then, from the definition of g, we conclude that

$$\exists M > 0 \text{ such that } \forall t > M \quad \mathbb{P}(Aty + B > t) \leq \exp(-f(t)(g(y) - \eta)).$$

Thus, from (4.5), we get for sufficiently large t,

(4.6) 
$$\mathbb{P}(AX + B > t, \ t(y - h) < X \le ty) \le \exp(-f(t)(g(y) + \lambda(y - h)^{\rho} - 2\eta)).$$

If  $g(y) = \infty$ , then for any G > 0 there exists N > 0 such that for all t > N, we obtain a bound

(4.7) 
$$\mathbb{P}(AX + B > t, \ t(y - h) < X \le ty) \le \exp(-f(t)(G + \lambda(y - h)^{\rho})).$$

We now consider two cases: (i)  $\rho \ge 1$  and (ii)  $\rho \in (0,1)$ .

(i) By the convexity of  $(0,\infty) \ni x \mapsto x^{\rho}$ , we obtain

$$\lambda y^{\rho} - \lambda (y - h)^{\rho} \le \lambda h \rho y^{\rho - 1}.$$

Using (4.6), the definition of  $\phi_{\rho}$ , and the above inequality, we conclude that if  $g(y) < \infty$ , then for all y > h > 0 and sufficiently large t,

$$\mathbb{P}(AX + B > t, t(y - h) < X \le ty) \le \exp(-f(t)(g(y) + \lambda y^{\rho} - \lambda y^{\rho} + \lambda(y - h)^{\rho} - 2\eta))$$
  
$$\le \exp(-f(t)(\phi_{\rho}(\lambda) - \lambda h\rho y^{\rho-1} - 2\eta)).$$

The same bound holds when  $g(y) = \infty$  by choosing G sufficiently large in (4.7).

Fix  $n \in \mathbb{N}$ . Let  $h_0 = 0$ ,  $h_k - h_{k-1} = \frac{a-b}{n} = h$  for  $k = 1, \ldots, n$ . Then for sufficiently large t,

$$\mathbb{P}(AX + B > t, tb < X \le ta) = \sum_{k=1}^{n} \mathbb{P}(AX + B > t, t(a - h_k) < X \le t(a - h_{k-1}))$$

$$\le \sum_{k=1}^{n} \exp(-f(t)(\phi_{\rho}(\lambda) - \lambda h\rho(a - h_{k-1})^{\rho-1} - 2\eta))$$

$$\le n \exp(-f(t)(\phi_{\rho}(\lambda) - \lambda h\rho a^{\rho-1} - 2\eta)).$$

Thus, for sufficiently large t,

$$\mathbb{P}(AX + B > t) = \mathbb{P}(AX + B > t, X \in t(b, a]) + \mathbb{P}(AX + B > t, X \notin t(b, a])$$

$$\leq n \exp(-f(t)(\phi_{\rho}(\lambda) - \lambda h \rho a^{\rho - 1} - 2\eta)) + 2 \exp(-f(t)(\phi_{\rho}(\lambda) - \eta))$$

$$\leq (n + 2) \exp(-f(t)(\phi_{\rho}(\lambda) - \lambda h \rho a^{\rho - 1} - 2\eta)).$$

Therefore, for sufficiently large t,

$$\frac{-\log \mathbb{P}(AX + B > t)}{f(t)} \ge \frac{-\log(n+2)}{f(t)} + \phi_{\rho}(\lambda) - \lambda h \rho a^{\rho-1} - 2\eta.$$

Taking  $\liminf_{t\to\infty}$  on both sides, we get

$$\liminf_{t \to \infty} \frac{-\log \mathbb{P}(AX + B > t)}{f(t)} \ge \phi_{\rho}(\lambda) - \lambda h \rho a^{\rho - 1} - 2\eta.$$

By letting  $n \uparrow \infty$  (recall that h = (a - b)/n) and then  $\eta \downarrow 0^+$ , we obtain (4.2).

(ii) For  $\rho \in (0,1)$ , we have  $y^{\rho} - (y-h)^{\rho} \le h^{\rho}$  for y > h > 0. Similarly to case (i), for all y > h > 0 and sufficiently large t, we get

$$\mathbb{P}(AX + B > t, t(y - h) < X \le ty) \le \exp(-f(t)(g(y) + \lambda y^{\rho} - \lambda y^{\rho} + \lambda(y - h)^{\rho} - 2\eta))$$
  
$$\le \exp(-f(t)(\phi_{\rho}(\lambda) - \lambda h^{\rho} - 2\eta))$$

and this bound also holds when  $g(y) = \infty$ . Proceeding as in (i), we obtain that for sufficiently large t,

$$\mathbb{P}(AX + B > t) \le (n+2)\exp(-f(t)(\phi_{\rho}(\lambda) - \lambda h^{\rho} - 2\eta)).$$

The remaining part of the proof is analogous.

5. The equation 
$$X \stackrel{d}{=} AX + B$$

The stochastic fixed-point equation (1.1) associated with the affine recursion  $X_n = A_n X_{n-1} + B_n$  has been extensively studied in the literature. We summarize key results on the existence and uniqueness of the solution below, [3].

**Theorem 5.1.** Assume that  $\mathbb{P}(A \geq 0, B \geq 0) = 1$ . If

$$\mathbb{E}[\log A] < 0$$
 and  $\mathbb{E}[\max\{\log B, 0\}] < \infty$ ,

then there exists a unique solution X to

$$X \stackrel{d}{=} AX + B$$
,  $(A, B)$  and  $X$  are independent.

Moreover, X is given by the a.s. convergent series representation

$$X \stackrel{d}{=} \sum_{n=1}^{\infty} B_n \prod_{k=1}^{n-1} A_k,$$

where  $(A_n, B_n)_n$  are independent copies of (A, B).

Additionally, if  $X_n = A_n X_{n-1} + B_n$  for n = 1, 2, ..., where  $X_0$  is independent of  $(A_n, B_n)_n$ , then  $X_n$  converges in distribution to X.

We say that a real random variable Y is stochastically majorized by Z and we write  $Y \leq_{st} Z$  if  $\mathbb{P}(Y \leq t) \geq \mathbb{P}(Z \leq t)$  for all  $t \in \mathbb{R}$ .

**Lemma 5.2.** Let  $(A_n, B_n)_{n\geq 1}$  be a sequence of independent copies of a generic pair (A, B). Let  $X_0$  be independent of  $(A_n, B_n)_{n\geq 1}$  and define  $X_n = A_n X_{n-1} + B_n$ . Assume that A > 0 a.s.

- (i) If  $X_1 \geq_{st} X_0$ , then  $X_n \geq_{st} X_{n-1}$  for all  $n \geq 1$ .
- (ii) If  $X_1 \leq_{st} X_0$ , then  $X_n \leq_{st} X_{n-1}$  for all  $n \geq 1$ .

*Proof.* (i) We proceed by induction, assume that  $X_n \ge_{st} X_{n-1}$  for  $n \ge 1$ . Then, since  $A_{n+1} \ge 0$  a.s. and  $X_n$  is independent of  $(A_{n+1}, B_{n+1})$ , we obtain

$$X_{n+1} \stackrel{d}{=} A_{n+1}X_n + B_{n+1} \ge_{st} A_{n+1}X_{n-1} + B_{n+1} \stackrel{d}{=} X_n.$$

Point (ii) is proved in the same way.

**Corollary 5.3.** If  $X_n$  converges in distribution to X, then  $X_1 \geq_{st} X_0$  implies that  $X \geq_{st} X_n$  for all  $n \in \mathbb{N}$  and if  $X_1 \leq_{st} X_0$ , then  $X \leq_{st} X_n$  for all  $n \in \mathbb{N}$ .

Lemma 5.4. If  $A \ge 0$  a.s., then

$$\operatorname{ess\,sup}(X) = \operatorname{ess\,sup}\left(\frac{B}{1-A} \mid A < 1\right).$$

*Proof.* Denote  $x_+ = \operatorname{ess\,sup}(X)$  and

$$x_0 = \operatorname{ess\,sup}\left(\frac{B}{1-A} \mid A < 1\right).$$

Let  $g_{a,b}(t) = a t + b$ . With

$$G(A,B) = \{g_{(a_1,b_1)} \circ \ldots \circ g_{(a_n,b_n)} \colon (a_i,b_i) \in \text{supp}(A,B), i = 1,\ldots,n, n \ge 1\}$$

by [3, Proposition 2.5.3], we have

$$\operatorname{supp}(X) = \overline{\left\{\frac{b}{1-a} \colon g_{a,b} \in G(A,B), a < 1\right\}}.$$

Thus,

$$\overline{\left\{\frac{b}{1-a}\colon (a,b)\in \operatorname{supp}(A,B), a<1\right\}}\subset \operatorname{supp}(X)$$

and therefore  $x_{+} \geq x_{0}$ . If  $x_{0} = \infty$ , then  $x_{+} = \infty$  and there is nothing to prove.

Assume that  $x_0 < \infty$ . By [3, Lemma 2.5.1], for every  $(a, b) \in \text{supp}(A, B)$  we have

$$a \operatorname{supp}(X) + b \subset \operatorname{supp}(X)$$

and therefore  $ax_+ + b \le x_+$  for all  $(a, b) \in \text{supp}(A, B)$ . Thus,  $Ax_+ + B \le x_+$  a.s. and

$$B\mathbf{1}_{A\geq 1} \leq x_{+}(1-A)\mathbf{1}_{A\geq 1} \leq x_{0}(1-A)\mathbf{1}_{A\geq 1}$$
 a.s.,

where the latter inequality follows from the fact that  $x_{+} \geq x_{0}$ .

By definition of  $x_0$  we have

$$B\mathbf{1}_{A<1} \le x_0(1-A)\mathbf{1}_{A<1}$$
, a.s.

and therefore  $B \leq x_0(1-A)$  a.s. Thus,

$$X \stackrel{d}{=} \sum_{k=1}^{\infty} A_1 \dots A_{k-1} B_k \le x_0 \sum_{k=1}^{\infty} A_1 \dots A_{k-1} (1 - A_k) = x_0,$$
 a.s.,

which implies that  $\mathbb{P}(X \leq x_0) = 1$ , i.e.,  $x_0 \geq x_+$ .

We have established above that the right endpoint of the support of X coincides with the right endpoint of the conditional distribution of B/(1-A) given A < 1. Note that if this endpoint is finite, the asymptotic behavior of  $t \mapsto \mathbb{P}(X > t)$  as  $t \to \infty$  becomes trivial, and this case should thus be excluded from further analysis. It turns out that suitable conditions involving g(1) or  $\lambda^*$  are sufficient to ensure this exclusion, but only under the assumption that  $\mathbb{P}(A \in [0,1)) = 1$ . In what follows, we explain that restricting our analysis to such distributions does not lead to any significant loss of generality.

By [13, Theorem 4.1], if  $\mathbb{P}(A > 1) > 0$ , then the tail of X is at least of power-law type:

$$\liminf_{t \to \infty} \frac{\log \mathbb{P}(X > t)}{\log t} > -\infty.$$

Since this behavior lies outside the light-tail regime we consider, we henceforth restrict to the case where  $A \in [0,1]$  a.s. Moreover, when  $\mathbb{P}(A=1) > 0$ , and under suitable nondegeneracy conditions, [1, Theorem 1.7] shows that the moment generating function  $\mathbb{E}[\exp(tX)]$  is finite if and only if

$$\mathbb{E}[e^{tB}\mathbf{1}_{A=1}] < 1.$$

In addition, [7, Lemma 5] establishes that

$$\liminf_{t \to \infty} \frac{-\log \mathbb{P}(X > t)}{t} = \sup\{t \in \mathbb{R} \colon \mathbb{E}[e^{tX}] < \infty\} = \sup\{t \in \mathbb{R} \colon \mathbb{E}[e^{tB}\mathbf{1}_{A=1}] < 1\} =: t_0.$$

If  $t_0 < \infty$ , the analysis for the case  $\mathbb{P}(A = 1) > 0$  is complete. Under  $\mathbb{P}(A = 1) > 0$ , we have  $t_0 = \infty$  if and only if  $\mathbb{P}(B = 0 \mid A = 1) = 1$ . In this degenerate scenario, one can show that

$$X \stackrel{d}{=} \tilde{A}X + \tilde{B}$$
,  $X$  and  $(\tilde{A}, \tilde{B})$  are independent,

where  $(\tilde{A}, \tilde{B}) \stackrel{d}{=} (A, B) \mid \{A < 1\}$ . Indeed, for a bounded continuous function f we have

$$\mathbb{E}[f(X)] = \mathbb{E}[f(AX + B)] = \mathbb{E}[f(AX + B)\mathbf{1}_{A=1}] + \mathbb{E}[f(AX + B)\mathbf{1}_{A<1}]$$
$$= \mathbb{E}[f(X)] \,\mathbb{P}(A=1) + \mathbb{E}[f(\tilde{A}X + \tilde{B})] \,\mathbb{P}(A<1),$$

which immediately implies that  $\mathbb{E}[f(X)] = E[f(\tilde{A}X + \tilde{B})].$ 

Therefore, to avoid these solved cases, we eventually assume that  $\mathbb{P}(A \in [0,1)) = 1$ .

**Lemma 5.5.** Let g be a local dependence measure of (A, B) with  $\mathbb{P}(A \in [0, 1), B \ge 0) = 1$  for  $f \in \mathbb{R}^{\rho}$ ,  $\rho > 0$ . Then:

- (i) If  $g(1) < \infty$ , then ess  $\sup \left( \frac{B}{1-A} \mid A < 1 \right) = \infty$ .
- (ii) If  $\lambda^* < \infty$ , then  $g(1) < \infty$ .

*Proof.* (i) First we notice that since A < 1 a.s., then  $\operatorname{ess\,sup}\left(\frac{B}{1-A} \mid A < 1\right) = \operatorname{ess\,sup}\left(\frac{B}{1-A}\right)$ . Moreover,

$$\lim_{t \to \infty} \frac{-\log \mathbb{P}\left(\frac{B}{1-A} > t\right)}{f(t)} = \lim_{t \to \infty} \frac{-\log \mathbb{P}(At + B > t)}{f(t)} = g(1) < \infty.$$

Then for sufficiently large t we get that  $\mathbb{P}(\frac{B}{1-A} > t) > 0$ , which proves the assertion.

(ii) Condition  $\lambda^* < \infty$  implies that there exists  $y_0 \in (0,1)$  such that  $g(y_0) < \infty$ . Since g is nonincreasing, we have  $g(1) \leq g(y_0) < \infty$ .

We will also need the following result, which was proved in [5, Lemma 6.10].

**Lemma 5.6.** Suppose that  $X \stackrel{d}{=} AX + B$ , where X and (A, B) are independent. For any bounded, uniformly continuous function f on  $\mathbb{R}$  and any increasing positive integer sequence  $(n_k)_k$ , a.s.,

$$\limsup_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} f(X_{n_k}) \ge \mathbb{E}[f(X)].$$

6. Tails of 
$$X \stackrel{d}{=} AX + B$$

Throughout this section, we assume that  $\mathbb{P}(A \in [0,1)) = 1$  (and thus  $\mathbb{E}[\log A] < 0$ ) and  $\mathbb{E}[\max\{\log B, 0\}] < \infty$ , which implies that there exists a unique solution X to

$$X \stackrel{d}{=} AX + B$$
, X and  $(A, B)$  are independent.

Recall that our standing assumption is that both A and B are nonzero. The main result in this section is as follows.

**Theorem 6.1.** Let g be an admissible  $LDM_f^{\rho}$  for the nonnegative random variables (A, B), where  $f \in \mathcal{R}^{\rho}$  and  $\rho > 0$ . Then X is  $ED_f(\lambda^*)$ , where  $\lambda^* \in [0, \infty]$  is defined in (3.1).

The proof of Theorem 6.1 will rely on several lemmas.

**Lemma 6.2.** If there exists  $\kappa > 0$  such that  $\kappa < \phi_{\rho}(\kappa)$ , then there exists a nonnegative  $\mathrm{ED}_f(\kappa)$ -random variable  $Z_{\kappa}$  such that

(6.1) 
$$Z_{\kappa} \geq_{st} AZ_{\kappa} + B, \qquad Z_{\kappa} \text{ and } (A, B) \text{ are independent.}$$

*Proof.* Let  $Z_0$  be a nonnegative random variable independent of (A, B), with the distribution defined by (recall that, without loss of generality, we assumed that f is strictly increasing and continuous)

$$\mathbb{P}(Z_0 > t) = \exp(-\kappa f(t)), \quad t > 0.$$

Clearly,  $Z_0$  is  $\mathrm{ED}_f(\kappa)$ .

Thus, by Theorem 4.1, we have

$$\lim_{t \to \infty} \frac{-\log \mathbb{P}(AZ_0 + B > t)}{f(t)} = \phi_{\rho}(\kappa) > \kappa = \lim_{t \to \infty} \frac{-\log \mathbb{P}(Z_0 > t)}{f(t)},$$

which implies that there exists M > 0 such that

(6.2) 
$$\forall t \ge M \ \mathbb{P}(Z_0 > t) \ge \mathbb{P}(AZ_0 + B > t).$$

Let  $Z_{\kappa}$  be a random variable, independent of (A, B), with the distribution defined by

$$\mathbb{P}(Z_{\kappa} \in \cdot) = \mathbb{P}(Z_0 \in \cdot \mid Z_0 > M).$$

For  $t \geq M$ , we obtain

$$\mathbb{P}(AZ_{\kappa} + B > t) = \mathbb{P}(AZ_{0} + B > t \mid Z_{0} > M) \leq \frac{\mathbb{P}(AZ_{0} + B > t)}{\mathbb{P}(Z_{0} > M)} 
\stackrel{(6.2)}{\leq} \frac{\mathbb{P}(Z_{0} > t)}{\mathbb{P}(Z_{0} > M)} = \mathbb{P}(Z_{0} > t \mid Z_{0} > M) = \mathbb{P}(Z_{\kappa} > t),$$

while for t < M, we have  $\mathbb{P}(Z_{\kappa} > t) = 1 \ge \mathbb{P}(AZ_{\kappa} + B > t)$ . Thus, (6.1) holds true. For  $t \ge M$ , we have as  $t \to \infty$ ,

$$\frac{-\log \mathbb{P}(Z_{\kappa} > t)}{f(t)} = \frac{-\log \mathbb{P}(Z_0 > t \mid Z_0 > M)}{f(t)} = \frac{-\log \mathbb{P}(Z_0 > t) + \log \mathbb{P}(Z_0 > M)}{f(t)} \to \kappa.$$

**Lemma 6.3.** Under the assumptions of Theorem 6.1,

$$\liminf_{t \to \infty} \left( -f(t)^{-1} \log \mathbb{P}(X > t) \right) \ge \lambda^*.$$

*Proof.* The inequality holds trivially in the case  $\lambda^* = 0$  and in the case where X has a bounded support (so that ess  $\sup \left(\frac{B}{1-A} \mid A < 1\right) < \infty$ , recall Lemma 5.4), in which we have  $\lambda^* = \infty$ .

Assume that  $\lambda^* \in (0, \infty]$  and ess  $\sup \left(\frac{B}{1-A} \mid A < 1\right) = \infty$ , which, by Lemma 5.4, implies that  $\mathbb{P}(X > t) > 0$  for any  $t \in \mathbb{R}$ . Moreover, since  $\lambda^* > 0$ , we must have

g(0) > 0. The admissibility of g ensures that there exists  $\kappa > 0$  such that  $\kappa < \phi_{\rho}(\kappa)$ . Let  $Z_{\kappa}$  be a random variable whose distribution is constructed in Lemma 6.2. Let  $X_0 = Z_{\kappa}$ . From (6.1), we obtain  $X_1 \leq_{st} X_0$ , which, by Corollary 5.3, implies

$$\forall t \in \mathbb{R} \quad \frac{-\log \mathbb{P}(X > t)}{f(t)} \ge \frac{-\log \mathbb{P}(X_0 > t)}{f(t)}.$$

Taking  $\liminf_{t\to\infty}$  on both sides, we get

$$\liminf_{t\to\infty}\frac{-\log\mathbb{P}(X>t)}{f(t)}\geq \liminf_{t\to\infty}\frac{-\log\mathbb{P}(X_0>t)}{f(t)}=\kappa.$$

Therefore,

$$\liminf_{t \to \infty} \frac{-\log \mathbb{P}(X > t)}{f(t)} \ge \sup \left\{ \kappa > 0 \colon \phi_{\rho}(\kappa) > \kappa \right\}.$$

By Lemma 3.9, we have

$$\{c \ge 0 \colon \phi_{\rho}(c) \ge c\} = \begin{cases} [0, \lambda^*], & \lambda^* < \infty, \\ [0, \infty), & \lambda^* = \infty. \end{cases}$$

However, by Lemma 3.6 (v), the set  $\{c \ge 0 : \phi_{\rho}(c) = c\}$  contains at most two elements. Therefore,

$$\sup \{\kappa > 0 \colon \phi_{\rho}(\kappa) > \kappa\} = \sup \{c \ge 0 \colon \phi_{\rho}(c) \ge c\} = \lambda^*.$$

Lemma 6.4. Under the assumptions of Theorem 6.1, if additionally

$$s = \limsup_{t \to \infty} \left( -f(t)^{-1} \log \mathbb{P}(X > t) \right) < \infty,$$

then  $s < \lambda^*$ .

*Proof.* For all t > 0 and y > 0, we have

(6.3) 
$$\mathbb{P}(X > t) = \mathbb{P}(AX + B > t) \ge \mathbb{P}(AX + B > t, X > ty)$$
$$> \mathbb{P}(Aty + B > t, X > ty) = \mathbb{P}(Aty + B > t)\mathbb{P}(X > ty).$$

Thus, for y > 0,

$$s \leq \limsup_{t \to \infty} \frac{-\log \mathbb{P}(Aty + B > t) - \log \mathbb{P}(X > ty)}{f(t)}$$
  
$$\leq \limsup_{t \to \infty} \frac{-\log \mathbb{P}(Aty + B > t)}{f(t)} + \limsup_{t \to \infty} \frac{-\log \mathbb{P}(X > ty)}{f(ty)} \frac{f(ty)}{f(t)} = g(y) + s y^{\rho}.$$

Now, using the assumption that  $s < \infty$ , we obtain  $s(1 - y^{\rho}) \le g(y)$ . Therefore

$$s \le \inf_{y \in (0,1)} \left\{ \frac{g(y)}{1 - y^{\rho}} \right\} = \lambda^*.$$

**Lemma 6.5.** Under the assumptions of Theorem 6.1, if additionally  $\lambda^* < \infty$ , then  $\limsup_{t \to \infty} (-f(t)^{-1} \log \mathbb{P}(X > t)) < \infty$ .

*Proof.* The assumption  $\lambda^* < \infty$  implies that there exists  $y \in (0,1)$  such that  $g(y) < \infty$ . Fix  $\eta > 0$ . By the definition of g, we conclude that

$$\exists M > 0 \ \forall t \ge M \quad \frac{-\log \mathbb{P}(Aty + B > t)}{f(t)} \le g(y) + \eta.$$

Using (6.3), we obtain for t > 0 and  $k = 0, 1, \ldots$ 

$$-\log \mathbb{P}(X > ty^k) + \log \mathbb{P}(X > ty^{k+1}) \le -\log \mathbb{P}(Aty^{k+1} + B > ty^k)$$
  
$$\le f(ty^k)(g(y) + \eta),$$

provided  $ty^k \geq M$ . Summing these inequalities for  $k = 0, \dots, n \in \mathbb{N}$ , we get

(6.4) 
$$-\log \mathbb{P}(X > t) + \log \mathbb{P}(X > ty^{n+1}) \le (g(y) + \eta) \sum_{k=0}^{n} f(ty^{k}),$$

provided  $ty^k \ge M$  for k = 0, ..., n. Set  $n = n_t = \lfloor \log(\frac{M}{t})/\log(y) \rfloor$ . It is easy to verify that

$$(6.5) ty^{n_t} \ge M \ge ty^{n_t+1}.$$

Thus, (6.4) implies that

$$\limsup_{t \to \infty} \left\{ \frac{-\log \mathbb{P}(X > t)}{f(t)} + \frac{\log \mathbb{P}(X > ty^{n_t + 1})}{f(t)} \right\} \le (g(y) + \eta) \limsup_{t \to \infty} \sum_{k=0}^{n_t} \frac{f(ty^k)}{f(t)}.$$

Using Potter bounds, Lemma 2.1 (iv), for C=2 and  $\delta=\rho/2$ , we conclude that for sufficiently large t,

$$\frac{f(ty^k)}{f(t)} \le 2y^{k\rho/2}.$$

Since  $y \in (0,1)$ , the series  $\sum_{k=0}^{\infty} \frac{f(ty^k)}{f(t)}$  is finite as  $t \to \infty$ . Finally, we observe that

$$\limsup_{t \to \infty} \frac{-\log \mathbb{P}(X > t)}{f(t)} + \liminf_{t \to \infty} \frac{\log \mathbb{P}(X > ty^{n_t + 1})}{f(t)} \\
\leq \limsup_{t \to \infty} \left\{ \frac{-\log \mathbb{P}(X > t)}{f(t)} + \frac{\log \mathbb{P}(X > ty^{n_t + 1})}{f(t)} \right\} < \infty.$$

By (6.5), we obtain  $\mathbb{P}(X > ty^{n_t+1}) \ge \mathbb{P}(X > M)$ . Since  $\lambda^* < \infty$ , by Lemma 5.5, we have ess  $\sup \left(\frac{B}{1-A} \mid A < 1\right) = \infty$ , which, by Lemma 5.4, implies that  $\operatorname{ess\,sup}(X) = \infty$ . Therefore  $\mathbb{P}(X > M) > 0$  and the second term on the left-hand side above is 0. This concludes the proof.

Proof of Theorem 6.1. If  $\lambda^* = \infty$ , then X is  $\mathrm{ED}_f(\lambda^*)$  by Lemma 6.3. If  $\lambda^* < \infty$ , then Lemma 6.3 gives the lower bound, which, by Lemma 6.5, is the same as the upper bound of Lemma 6.4.

## 7. Upper envelope for $(X_n)_n$

Similarly as in the previous section, we assume that (A, B) are nonzero nonnegative random variables such that  $\mathbb{P}(A \in [0, 1)) = 1$  and  $\mathbb{E}[\max\{\log B, 0\}] < \infty$  and that X is the unique solution X to (1.1). Let  $(A_n, B_n)_{n\geq 1}$  be a sequence of independent copies of (A, B). We set  $X_0 = 0$  and consider the sequence  $X_n = A_n X_{n-1} + B_n$  for  $n \geq 1$ .

In view of Lemma 2.1 (v), without loss of generality, we assume that f is continuous and strictly increasing so that  $f^{-1}$  is well defined.

**Theorem 7.1.** Let g be an admissible  $LDM_f^{\rho}$  for (A, B), where  $f \in \mathcal{R}^{\rho}$  and  $\rho > 0$ . Assume that  $\lambda^* \in (0, \infty)$ . Then, almost surely

$$\limsup_{n \to \infty} \frac{X_n}{f^{-1}(\log n)} = (\lambda^*)^{-1/\rho}.$$

All lemmas in this section implicitly make the same assumptions as those in Theorem 7.1.

Lemma 7.2. Almost surely, we have

$$\limsup_{n \to \infty} \frac{X_n}{f^{-1}(\log n)} \le (\lambda^*)^{-1/\rho}.$$

*Proof.* Fix  $\epsilon > 0$ . There exists  $\delta \in (0,1)$  such that  $\gamma := (1-\delta)(1+\epsilon) > 1$ . By Theorem 6.1, X is  $\mathrm{ED}_f(\lambda^*)$ . Since  $\lambda^* < \infty$ , there exists M > 0 such that

$$\forall t \ge M \ \mathbb{P}(X > t) \le e^{-\lambda^* f(t)(1-\delta)}$$

Define

$$t_n = f^{-1}\left(\frac{(1+\epsilon)\log n}{\lambda^*}\right), \quad n \in \mathbb{N}.$$

Since  $B_1 = X_1 \geq_{st} X_0 = 0$ , Corollary 5.3 implies that for all  $n \in \mathbb{N}$ , it holds that  $X \geq_{st} X_n$ . As  $t_n \to \infty$ , for sufficiently large n, we obtain

$$\mathbb{P}(X_n > t_n) \le \mathbb{P}(X > t_n) \le e^{-(1+\epsilon)(1-\delta)\log n} = n^{-\gamma}.$$

Since  $\gamma > 1$ , we conclude that

$$\sum_{n=1}^{\infty} \mathbb{P}\left(X_n > t_n\right) < \infty.$$

By the Borel-Cantelli Lemma, we get that almost surely,

$$\limsup_{n \to \infty} \frac{X_n}{f^{-1}\left(\frac{(1+\epsilon)\log n}{\lambda^*}\right)} \le 1.$$

By Lemma 2.1 (iv),  $f^{-1}$  is regularly varying with index  $\frac{1}{\rho}$ , and therefore,

$$f^{-1}\left(\frac{(1+\epsilon)\log n}{\lambda^*}\right) \sim \left(\frac{1+\epsilon}{\lambda^*}\right)^{\frac{1}{\rho}} f^{-1}(\log n).$$

Hence, almost surely,

$$\limsup_{n \to \infty} \frac{X_n}{f^{-1}(\log n)} \le \left(\frac{1+\epsilon}{\lambda^*}\right)^{\frac{1}{\rho}}.$$

By letting  $\epsilon \to 0^+$ , we obtain the assertion.

In the following lemma, we assume that  $X_0$  is arbitrary but independent of  $(A_n, B_n)_n$ .

**Lemma 7.3.** For any  $\delta > 0$ , there exist  $y_* \in (0,1)$  and  $\tilde{M} > 0$  such that if  $ty_*^{n-1} \geq \tilde{M}$ , then, a.s.,

$$P(X_n > t \mid X_0) \ge \mathbf{1}_{(ty_*^n,\infty)}(X_0) \exp(-(1+\delta)\lambda^* f(t)).$$

Before proving the above result, we first present a simple lemma.

**Lemma 7.4.** For all  $n \ge 1$ , y > 0 and t > 0 we have, a.s.,

$$\mathbb{P}(X_n > t \mid X_0) \ge \mathbf{1}_{(y^n t, \infty)}(X_0) \prod_{k=0}^{n-1} \mathbb{P}(t \, y^k A y + B > t \, y^k).$$

*Proof.* Notice that

$$\mathbb{P}(X_n > t \mid X_0) \ge \mathbb{P}(A_n X_{n-1} + B_n > t, X_{n-1} > ty \mid X_0)$$
  
 
$$\ge \mathbb{P}(t A_n y + B_n > t, X_{n-1} > ty \mid X_0)$$
  
 
$$= \mathbb{P}(t Ay + B > t) \mathbb{P}(X_{n-1} > ty \mid X_0).$$

We obtain the assertion by iterating the above inequality.

Now we are ready to present the proof of Lemma 7.3.

Proof of Lemma 7.3. Fix  $\alpha > 0$  and let  $y_* \in (0,1)$  be such that

(7.1) 
$$\lambda^* \le \frac{g(y_*)}{1 - y_*^{\rho}} \le \lambda^* (1 + \alpha).$$

Such  $y_*$  exists as a consequence of the definition of  $\lambda^* = \inf_{y \in (0,1)} \{g(y)/(1-y^{\rho})\}$  and the fact that  $\lambda^* > 0$ . Clearly  $g(y^*) > 0$ . By Lemma 7.4, we have, a.s.,

$$\mathbb{P}(X_n > t \mid X_0) \ge \mathbf{1}_{(y_*^n t, \infty)}(X_0) \prod_{k=0}^{n-1} \mathbb{P}\left(t y_*^k A y + B > t y_*^k\right).$$

By the definition of q, there exists M > 0 such that

$$\forall t \ge M \ \mathbb{P}(Aty_* + B > t) \ge \exp(-(1 + \alpha)g(y_*)f(t)).$$

Thus, if  $ty_*^k \ge M$  for  $k = 0, \dots, n-1$ , (which is equivalent to  $ty_*^{n-1} \ge M$ ), we obtain

(7.2) 
$$\mathbb{P}(X_n > t \mid X_0) \ge \mathbf{1}_{(y_*^n t, \infty)}(X_0) \exp\left(-(1+\alpha)g(y_*) \sum_{k=0}^{n-1} f(ty_*^k)\right).$$

Fix  $\eta$  in  $(0, \rho)$ . By Lemma 2.1 (iii), there exists  $M_1 > 0$  such that for all  $z \in (0, 1]$  and for all t > 0 such that  $tz \ge M_1$ , it holds that

$$\frac{f(tz)}{f(t)} \le (1+\alpha)z^{\rho-\eta}$$

Assume that  $ty_*^{n-1} \ge M_1$ . Then for k = 0, ..., n-1, we have  $ty_*^k \ge ty_*^{n-1} \ge M_1$ . Therefore,

(7.3) 
$$f(ty_*^k) \le (1+\alpha)y_*^{k(\rho-\eta)}f(t), \qquad k = 0, ..., n-1.$$

Notice that for all  $n \in \mathbb{N}$  and sufficiently small  $\eta$ , we have

(7.4) 
$$\frac{1 - y_*^{n(\rho - \eta)}}{1 - y_*^{\rho - \eta}} \le \frac{1}{1 - y_*^{\rho - \eta}} \le \frac{1 + \alpha}{1 - y_*^{\rho}},$$

where the first inequality above follows from the fact that  $y_* \in (0,1)$  and the latter is true for sufficiently small  $\eta$ . Take  $\eta \in (0,\rho)$  satisfying the condition above.

Define  $\tilde{M} = \max\{M_1, M\}$  and assume  $ty_*^{n-1} \ge \tilde{M}$ . Then, by (7.2), (7.3), (7.4) and (7.1), we obtain

$$\mathbb{P}(X_{n} > t \mid X_{0}) \stackrel{(7.3)}{\geq} \mathbf{1}_{(y_{*}^{n}t,\infty)}(X_{0}) \exp\left(-(1+\alpha)^{2}g(y_{*}) \sum_{k=0}^{n-1} y_{*}^{k(\rho-\eta)}f(t)\right) \\
= \mathbf{1}_{(y_{*}^{n}t,\infty)}(X_{0}) \exp\left(-(1+\alpha)^{2}g(y_{*}) \frac{1-y_{*}^{n(\rho-\eta)}}{1-y_{*}^{\rho-\eta}}f(t)\right) \\
\stackrel{(7.4)}{\geq} \mathbf{1}_{(y_{*}^{n}t,\infty)}(X_{0}) \exp\left(-(1+\alpha)^{3}g(y_{*}) \frac{1}{1-y_{*}^{\rho}}f(t)\right) \\
\stackrel{(7.1)}{\geq} \mathbf{1}_{(y_{*}^{n}t,\infty)}(X_{0}) \exp\left(-(1+\alpha)^{4}\lambda^{*}f(t)\right).$$

Since, for all  $\delta > 0$ , it holds that  $1 + \delta$  is of the form  $(1 + \alpha)^4$  for some  $\alpha > 0$ , this concludes the proof.

**Lemma 7.5.** Let  $\tilde{M}, \lambda^*, \epsilon > 0$  and  $y_* \in (0,1)$ . Suppose that f is regularly varying with index  $\rho > 0$ . There exists a strictly increasing sequence  $(k_n)_{n=1}^{\infty}$  of positive integers and a positive constant  $c > \tilde{M}y_*$  such that, for  $n \ge 1$ ,

(7.5) 
$$c \ge f^{-1} \left( \frac{\log k_{n+1}}{\lambda^* (1+\epsilon)} \right) y_*^{k_{n+1}-k_n} > \tilde{M} y_*.$$

Moreover, for any  $\gamma \in (0,1)$ , there exists K > 0 such that for all  $n \ge 1$ ,

$$(7.6) k_n^{\gamma} \le Kn.$$

*Proof.* Denote H(t) = f(1/t). By definition, f is regularly varying (at  $\infty$ ) with index  $\rho > 0$  if and only if H is regularly varying at 0 with index  $-\rho$ , [2, page 8]. Since  $f^{-1}(t) = 1/H^{-1}(t)$ , (7.5) can be rewritten as

$$H^{-1}\left(\frac{\log k_{n+1}}{\lambda^*(1+\epsilon)}\right)\left(\frac{1}{y_*}\right)^{k_{n+1}-k_n-1} < \frac{1}{\tilde{M}} \quad \text{and} \quad H^{-1}\left(\frac{\log k_{n+1}}{\lambda^*(1+\epsilon)}\right)\left(\frac{1}{y_*}\right)^{k_{n+1}-k_n} \ge \frac{1}{c},$$

where  $c^{-1} < (\tilde{M}y_*)^{-1}$ . The existence of a sequence  $(k_n)_{n=1}^{\infty}$  satisfying the two above conditions and (7.6) was established in [5, Lemma A.1].

We will use the following version of the Borel–Cantelli lemma, which can be found in [6, Theorem 5.1.2].

**Lemma 7.6.** Suppose that  $(\mathcal{F}_n)_{n\geq 0}$  is a filtration such that  $\mathcal{F}_0 = \{\emptyset, \Omega\}$ . Assume that  $A_n \in \mathcal{F}_n$  for all  $n \geq 0$ . Then,

$$\limsup_{n \to \infty} A_n = \left\{ \sum_{n=1}^{\infty} \mathbb{P}(A_n \mid \mathcal{F}_{n-1}) = \infty \right\}.$$

**Lemma 7.7** (Kronecker's Lemma). Assume that  $a_n \uparrow \infty$ . If  $\sum_{n=1}^{\infty} x_n/a_n$  converges, then  $\lim_{n\to\infty} a_n^{-1} \sum_{m=1}^n x_m = 0$ . Equivalently, if  $\lim_{n\to\infty} a_n^{-1} \sum_{m=1}^n x_m \neq 0$ , then  $\sum_{n=1}^{\infty} x_n/a_n$  does not converge.

Lemma 7.8. Almost surely, we have

$$\limsup_{n \to \infty} \frac{X_n}{f^{-1}(\log n)} \ge (\lambda^*)^{-1/\rho}.$$

*Proof.* Fix  $\epsilon > 0$  and define

$$t_n = f^{-1} \left( \frac{\log n}{\lambda^* (1 + \epsilon)} \right).$$

Let  $(\mathcal{F}_n)_{n\geq 0}$  denote the natural filtration of the sequence  $(X_n)_{n\geq 0}$ . By Lemma 7.6, it suffices to show that for a strictly increasing sequence of positive integers  $(k_n)_{n\geq 1}$ , we have, a.s.,

$$\sum_{n=1}^{\infty} \mathbb{P}(X_{k_n} > t_{k_n} \mid \mathcal{F}_{k_{n-1}}) = \infty,$$

which implies that  $\mathbb{P}(\limsup_{n\to\infty} A_{k_n}) = 1$ . Since  $\limsup_{n\to\infty} A_n \supset \limsup_{n\to\infty} A_{k_n}$ , this would prove the assertion.

Since  $(X_{k_n})_{n\geq 1}$  is a Markov chain, we have

$$\mathbb{P}(X_{k_{n+1}} > t_{k_{n+1}} \mid \mathcal{F}_{k_n}) = \mathbb{P}(X_{k_{n+1}} > t_{k_{n+1}} \mid X_{k_n}).$$

By Lemma 7.3, applied to the sequence  $(Y_n)_{n\in\mathbb{N}\cup\{0\}}$  defined by  $Y_i=X_{k_n+i}$ , we obtain that for any  $\delta>0$ , there exist  $y_*\in(0,1)$  and  $\tilde{M}>0$  such that

(7.7) 
$$\mathbb{P}(X_{k_{n+1}} > t \mid X_{k_n}) \ge \mathbf{1}_{(tu_*^{k_{n+1}-k_n}.\infty)}(X_{k_n}) \exp(-(1+\delta)\lambda^* f(t)),$$

provided  $ty_*^{k_{n+1}-k_n-1} \ge \tilde{M}$ .

We use the sequence  $(k_n)_{n\geq 1}$  from Lemma 7.5. The condition  $t_{k_{n+1}}y_*^{k_{n+1}-k_n-1}\geq \tilde{M}$  is satisfied as a consequence of the lower bound in (7.5). Therefore, using (7.7) and the upper bound in (7.5), we obtain, a.s.,

$$\mathbb{P}(X_{k_{n+1}} > t_{k_{n+1}} \mid X_{k_n}) \ge \mathbf{1}_{(c,\infty)}(X_{k_n}) \exp\left(-\frac{1+\delta}{1+\epsilon} \log k_{n+1}\right) = \mathbf{1}_{(c,\infty)}(X_{k_n}) \frac{1}{k_{n+1}^{\gamma}},$$

where  $\gamma := (1+\delta)/(1+\epsilon)$ . By decreasing  $\delta$  if necessary, we ensure that  $\gamma < 1$ . Our goal is to show that  $\sum_{n\geq 1} \mathbf{1}_{(c,\infty)}(X_{k_n}) k_{n+1}^{-\gamma}$  diverges a.s. By Lemma 7.7, applied to  $a_n = k_{n+1}^{\gamma}$  and  $x_n = \mathbf{1}_{(c,\infty)}(X_{k_n})$ , to meet this goal, it suffices to show that, a.s.,

(7.8) 
$$\limsup_{m \to \infty} \frac{1}{k_{m+1}^{\gamma}} \sum_{n=1}^{m} \mathbf{1}_{(c,\infty)}(X_{k_n}) > 0.$$

From (7.6), we have  $k_n^{\gamma} \leq Kn$  for some K > 0 and all  $n \geq 1$ . Therefore, a.s.,

$$\limsup_{m \to \infty} \frac{1}{k_{m+1}^{\gamma}} \sum_{n=1}^{m} \mathbf{1}_{(c,\infty)}(X_{k_n}) \ge \limsup_{m \to \infty} \frac{K^{-1}}{m+1} \sum_{n=1}^{m} \mathbf{1}_{(c,\infty)}(X_{k_n}).$$

Let  $f_c: \mathbb{R} \to \mathbb{R}$  be defined by  $f_c(x) = (x-c)/c\mathbf{1}_{[c,2c]}(x) + \mathbf{1}_{(2c,\infty)}(x)$ . Since  $f_c$  is bounded and uniformly continuous on  $\mathbb{R}$ , and since  $\mathbf{1}_{(c,\infty)} \ge f_c \ge \mathbf{1}_{(2c,\infty)}$ , by Lemma 5.6 we have

$$\limsup_{m \to \infty} \frac{1}{m+1} \sum_{n=1}^{m} \mathbf{1}_{(c,\infty)}(X_{k_n}) \ge \limsup_{m \to \infty} \frac{1}{m+1} \sum_{n=1}^{m} f_c(X_{k_n}) \ge \mathbb{E}[f_c(X)] \ge \mathbb{P}(X > 2c).$$

Since  $\mathbb{P}(X > t) > 0$  for any  $t \in \mathbb{R}$  (recall that under  $\lambda^* < \infty$ , by Lemmas 5.4 and 5.5, we have ess  $\sup(X) = \infty$ ), (7.8) follows. Therefore, by Lemma 7.6, for every  $\epsilon > 0$ , we have almost surely

$$\limsup_{n \to \infty} \frac{X_n}{f^{-1}(\frac{\log n}{\lambda^*(1+\epsilon)})} \ge 1.$$

Proceeding as in the proof of Lemma 7.2, we obtain

$$\limsup_{n \to \infty} \frac{X_n}{f^{-1}(\log n)} \ge \left(\frac{1}{\lambda^*(1+\epsilon)}\right)^{\frac{1}{\rho}}.$$

The proof is completed by letting  $\epsilon \to 0^+$ .

Proof of Theorem 7.1. This follows directly from Lemmas 7.2 and 7.8.  $\square$ 

#### 8. Example

In this section, we present a family of distributions of (A, B), which admit an explicit LDM, but is not PQD. Thus, allowing us to illustrate our results on explicit example outside PQD pairs. This family is indexed by a continuous, nondecreasing function  $\alpha \colon [0,1) \mapsto [1,\infty)$ .

Let  $\rho > 0$ , and consider a random vector (A, B) whose law consists of a unique atom and an absolutely continuous component. Specifically, assume that  $\mathbb{P}(A = 0, B = 1) = 1 - e^{-1}$ , and that

$$\mathbb{P}(A > a, B > b) = \exp(-\alpha(a)\beta(b)), \quad a \in [0, 1), b \ge 1,$$

with

$$\beta(b) = b^{\rho}$$
 and  $\operatorname{ess\,sup}(A) = 1$ .

Since  $e^{-1} = \mathbb{P}(A > 0, B > 1) = \exp(-\alpha(0)\beta(1))$ , we deduce that  $\alpha(0) = 1$ . In addition, one may verify that  $\alpha(1^-) = \infty$ , that  $\alpha(a) = -\log \mathbb{P}(A > a)$  for  $a \ge 0$  and that  $\log \mathbb{P}(B > b) = -b^{\rho}$  for  $b \ge 1$ .

Our goal is to compute the LDM<sub>f</sub><sup> $\rho$ </sup> function for (A, B) with  $f(t) = -\log \mathbb{P}(B > t) = t^{\rho}$ . Note that A and B are not positively quadrant dependent, so that the results of Section 3.1 do not apply.

By Lemma 3.3 (i), we have g(y) = 0 for y > 1. Fix  $y \in [0,1)$  and consider t > 1/(1-y). Then,

$$\mathbb{P}(Aty + B > t) = \int_0^1 \int_{t(1-ya)}^\infty \frac{\partial^2 \exp(-\alpha(a)\beta(b))}{\partial a \, \partial b} \, \mathrm{d}b \, \mathrm{d}a$$

$$= -\int_0^1 \frac{\partial \exp(-\alpha(a)\beta(b))}{\partial a} \Big|_{b=t(1-ya)} \, \mathrm{d}a$$

$$= t^\rho \int_0^1 (1 - ya)^\rho e^{-t^\rho (1-ya)^\rho \alpha(a)} \alpha'(a) \, \mathrm{d}a$$

$$= t^\rho \int_1^\infty (1 - y\alpha^{-1}(u))^\rho e^{-t^\rho (1-y\alpha^{-1}(u))^\rho u} \, \mathrm{d}u,$$

where the substitution  $u = \alpha(a)$  was used (with  $\alpha^{-1}$  denoting the generalized inverse of  $\alpha$ ). Since  $(1 - y)^{\rho} \leq (1 - y\alpha^{-1}(u))^{\rho} \leq 1$ , we obtain

$$(1-y)^{\rho} t^{\rho} \int_{1}^{\infty} e^{-t^{\rho} (1-y\alpha^{-1}(u))^{\rho} u} du \le \mathbb{P}(Aty + B > t) \le t^{\rho} \int_{1}^{\infty} e^{-t^{\rho} (1-y\alpha^{-1}(u))^{\rho} u} du.$$

By the Laplace method, it follows that g exists and equals for  $y \in (0,1)$ ,

$$g(y) = \lim_{t \to \infty} \frac{-\log \mathbb{P}(Aty + B > t)}{t^{\rho}} = \inf_{u \in [1, \infty)} \{ (1 - y\alpha^{-1}(u))^{\rho} u \} = \inf_{a \in [0, 1)} \{ (1 - ya)^{\gamma} \alpha(a) \}.$$

We obtain  $g(0) = \alpha(0) = 1$  and

$$\begin{split} \phi_{\rho}(\lambda) &= \inf_{y>0} \left\{ y^{\rho} \lambda + g(y) \right\} \\ &= \begin{cases} \min\{1, \lambda\}, & \rho \in (0, 1], \\ \min\left\{ \inf_{a \in [0, 1)} \left\{ \left( \alpha(a)^{1/(1-\rho)} + a^{\rho/(\rho-1)} \lambda^{1/(1-\rho)} \right)^{1-\rho} \right\}, \lambda \right\}, & \rho > 1. \end{cases} \end{split}$$

Moreover, one can easily show that

$$\begin{split} \lambda^* &= \inf_{y \in (0,1)} \left\{ \frac{g(y)}{1-y^\rho} \right\} = \inf_{a \in [0,1)} \inf_{y \in (0,1)} \left\{ \frac{\alpha(a)(1-ya)^\rho}{1-y^\rho} \right\} \\ &= \left\{ \inf_{a \in [0,1)} \left\{ \alpha(a) \right\} = 1, & \rho \in (0,1], \\ \inf_{a \in [0,1)} \left\{ \alpha(a)(1-a^{\rho/(\rho-1)})^{\rho-1} \right\}, & \rho > 1. \end{split}$$

Consider the case  $\rho > 1$ . Then, if X is a solution to (1.1), by Theorem 6.1, we obtain

(8.1) 
$$\lim_{t \to \infty} \frac{\log \mathbb{P}(X > t)}{t^{\rho}} = -\inf_{a \in [0,1)} \left\{ \alpha(a) \left( 1 - a^{\rho/(\rho - 1)} \right)^{\rho - 1} \right\}.$$

We are going to relate this result to the findings of [22]. In [22, Theorem 5.1], it was shown that

(8.2) 
$$\liminf_{t \to \infty} \frac{\log \mathbb{P}(X > t)}{h(t)} \ge -\left(s\left(1 - \left(1 - \frac{1}{s}\right)^{\gamma/(\gamma - 1)}\right)\right)^{\gamma - 1},$$

where

$$h(t) = \inf_{s \ge 1} \left\{ -s \log \mathbb{P}\left(A > 1 - \frac{1}{s}, B > \frac{t}{s}\right) \right\},\,$$

 $\gamma$  is the index of regular variation of h and  $s = \lim_{t \to \infty} \sigma(t)$ , where  $\sigma$  is any function satisfying

$$h(t) = -\sigma(t) \log \mathbb{P}\left(A > 1 - \frac{1}{\sigma(t)}, B > \frac{t}{\sigma(t)}\right) + o(1), \quad t \to \infty.$$

In our setting, for t > 1 one can write

$$h(t) = \inf_{s \in [1,t]} \left\{ s \, \alpha \left( 1 - \frac{1}{s} \right) \beta \left( \frac{t}{s} \right) \right\} = t^{\rho} \inf_{s \in [1,t]} \left\{ s^{1-\rho} \, \alpha \left( 1 - \frac{1}{s} \right) \right\}.$$

Now, we consider two cases:

(a) 
$$\alpha(a) = (1-a)^{1-\rho}$$
 with  $\rho > 1$ ,

(b) 
$$\alpha(a) = \exp(a/(1-a))$$
 with  $\rho > 2$ .

In case (a) one immediately obtains  $h(t) = t^{\rho}$  for t > 1. In this situation, one can take  $\sigma(t) = \min\{s, t\}$  for any  $s \ge 1$ . Consequently,  $\gamma = \rho$  and (8.2) becomes (after taking  $\sup_{s>1}$  of both sides)

$$\liminf_{t \to \infty} \frac{\log \mathbb{P}(X > t)}{t^{\rho}} \ge -\inf_{s \ge 1} \left\{ \left[ s \left( 1 - \left( 1 - \frac{1}{s} \right)^{\rho/(\rho - 1)} \right) \right]^{\rho - 1} \right\}.$$

A direct calculation shows that this lower bound agrees with the expression in (8.1). Case (b). Then,

$$h(t) = t^{\rho} \inf_{s \in [1,t]} \left\{ s^{1-\rho} e^{s-1} \right\}.$$

For  $t > \rho - 1 > 1$ , the infimum is attained at  $s = \rho - 1$ , yielding

$$h(t) = t^{\rho}(\rho - 1)^{1-\rho}e^{\rho - 2}.$$

Again, one obtains  $\gamma = \rho$ , and by taking  $\sigma(t) = \rho - 1 = s$ , (8.2) gives

$$\begin{split} \lim \inf_{t \to \infty} \frac{\log \mathbb{P}(X > t)}{t^{\rho}} &= \lim \inf_{t \to \infty} \frac{\log \mathbb{P}(X > t)}{h(t)} \cdot \frac{h(t)}{t^{\rho}} \\ &\geq -\left( (\rho - 1) \left( 1 - \left( 1 - \frac{1}{\rho - 1} \right)^{\rho/(\rho - 1)} \right) \right)^{\rho - 1} \cdot (\rho - 1)^{1 - \rho} e^{\rho - 2} \\ &= -e^{a/(1 - a)} \left( 1 - a^{\rho/(\rho - 1)} \right)^{\rho - 1} \big|_{a = 1 - (\rho - 1)^{-1}}. \end{split}$$

On the other hand, by (8.1), we have

$$\lim_{t \to \infty} \frac{\log \mathbb{P}(X > t)}{t^{\rho}} = -\inf_{a \in [0,1)} \left\{ e^{a/(1-a)} \left( 1 - a^{\rho/(\rho-1)} \right)^{\rho-1} \right\}$$

$$> -e^{a/(1-a)} \left( 1 - a^{\rho/(\rho-1)} \right)^{\rho-1} \Big|_{a=1-(\rho-1)^{-1}}.$$

Thus, the lower bound provided by Theorem 5.1 in [22] is, in general, not optimal.

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