

The Fourth-Moment Theorem on Hilbert Spaces

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Abstract

In a work by Bourguin and Campese [Electron. J. Probab. 25, 1–30 (2020)], a fourth-moment theorem for weak convergence to a Gaussian measure on separable Hilbert spaces was proposed. However, a very recent work by Bassetti, Bourguin, Campese, and Peccati [arXiv:2509.13427 (2025)] showed that the distance employed in the former article does not metrize weak convergence on such spaces; consequently, one of the main results therein does not hold as stated.

In this paper, we characterize convergence in distribution to a non-degenerate Gaussian measure on a separable Hilbert space for sequences of multiple Wiener–Itô integrals of fixed order. Assuming convergence of the associated covariance operators in the trace class norm, we prove that weak convergence holds if and only if the fourth weak moments converge to their Gaussian counterparts. A key tool in our approach is a Stein–Malliavin bound for a distance metrizing weak convergence on Hilbert spaces, hence extending the classical real-valued result of Nualart and Peccati [Ann. Probab. 33, 177–193 (2005)].

1 Introduction

The fourth-moment theorem, introduced by Nualart and Peccati in [22], provides a strikingly simple criterion for normal convergence within a fixed Wiener chaos. Let $q \geq 2$ be an integer, and let $F_n = I_q(f_n)$ denote a sequence of multiple Wiener–Itô integrals of order q with respect to a standard Brownian motion, where each kernel $f_n \in L^2(\mathbb{R}_+^q)$ is symmetric and standardized so that it satisfies $\mathbb{E}[F_n^2] = q! \|f_n\|_{L^2(\mathbb{R}_+^q)}^2 = 1$. The theorem states that F_n converges in distribution to a standard normal random variable if and only if the fourth moments converge to those of a standard Gaussian, i.e.,

$$\mathbb{E}[F_n^4] \rightarrow 3, \quad \text{as } n \rightarrow \infty.$$

This remarkable result reduces the challenging problem of weak convergence within a fixed Wiener chaos to a simple condition on the fourth moment. Since its discovery, the fourth-moment theorem has stimulated a wide range of research, focused on establishing quantitative central limit theorems for diverse statistics and under various probability metrics that metrize weak convergence on \mathbb{R} , including, e.g., the total variation and Wasserstein distances.

An important extension of the classical fourth-moment theorem concerns sequences of random variables taking values in a separable Hilbert space \mathcal{H} . In [6], Bourguin and Campese were the first to claim a fourth-moment theorem in such Hilbert spaces. Unfortunately, some of the main results reported in [6] were later shown to be incorrect. This issue was highlighted in the recent work [1], where it was demonstrated that the distance introduced in [6], here denoted by ρ_2 , does,

in fact, not metrize weak convergence. Consequently, as pointed out in [1], the proofs in subsequent works [6, 7, 8, 9, 12, 13] that rely on ρ_2 -bounds to establish weak convergence (or CLTs) are not valid.

While [1] focuses on highlighting that the distance ρ_2 does not metrize weak convergence, we focus here on the validity of the infinite-dimensional fourth-moment theorem of [6]. We point out why the conditions in [6] do not suffice, and propose conditions for a fourth-moment theorem in infinite dimensions. To describe the setting, consider a non-degenerate, centered, \mathcal{H} -valued Gaussian random variable Z and a sequence $\{F_n\}_{n \geq 1}$ of \mathcal{H} -valued multiple Wiener-Itô integrals of fixed order q , which will be discussed more in Section 2 below. Then, Theorem 3.12 in [6] claims that, if

$$\mathbb{E}[\|F_n\|_{\mathcal{H}}^2] \rightarrow \mathbb{E}[\|Z\|_{\mathcal{H}}^2], \quad \mathbb{E}[\|F_n\|_{\mathcal{H}}^4] \rightarrow \mathbb{E}[\|Z\|_{\mathcal{H}}^4], \quad (1.1)$$

as $n \rightarrow \infty$, then $F_n \rightarrow Z$ weakly in \mathcal{H} . However, as demonstrated in the following counterexample, these moment conditions lead to a certain identifiability problem, and are not sufficient to fully characterize the Gaussian limit.

Example 1.1. Let $\{e_k\}_{k \in \mathbb{N}}$ be an orthonormal basis of \mathcal{H} . For $i = 1, 2$, let Z_i be the centered \mathcal{H} -valued Gaussian random variable with covariance operator

$$\mathcal{T}_{Z_i} = e_i \otimes e_i, \quad \text{i.e.,} \quad \mathcal{T}_{Z_i}(h) = \langle h, e_i \rangle_{\mathcal{H}} e_i, \text{ for } h \in \mathcal{H}.$$

Then $\text{Tr}(\mathcal{T}_{Z_i}) = 1$, so that

$$\mathbb{E}\|Z_1\|_{\mathcal{H}}^2 = \mathbb{E}\|Z_2\|_{\mathcal{H}}^2 = 1 \quad \text{and} \quad \mathbb{E}\|Z_1\|_{\mathcal{H}}^4 = \mathbb{E}\|Z_2\|_{\mathcal{H}}^4 = 3.$$

Nevertheless, Z_1 and Z_2 have different laws as \mathcal{H} -valued random variables (their mass lies on the distinct one-dimensional subspaces determined by e_1 and e_2). Hence, the conditions $\mathbb{E}\|F_n\|_{\mathcal{H}}^2 \rightarrow \mathbb{E}\|Z\|_{\mathcal{H}}^2$ and $\mathbb{E}\|F_n\|_{\mathcal{H}}^4 \rightarrow \mathbb{E}\|Z\|_{\mathcal{H}}^4$ do not identify a unique Gaussian limit.¹

Our work addresses this issue by characterizing conditions that are stronger than the conditions in (1.1) and in fact do suffice to prove weak convergence whenever F_n is in a fixed Wiener chaos. To appreciate the setting of our problem, we highlight that there does not exist one canonical way to understand higher order moments of random variables in infinite-dimensional spaces. We identify two conditions. First, we require convergence of the covariance operators in the trace class norm. This implies that the Gaussian limit is unique and avoids the issue that is raised in Example 1.1. The convergence of the covariance operators implies that the second moments of the coordinate processes must converge consistently across all possible orthonormal representations of the Hilbert space. Second, we require that

$$\mathbb{E}[\langle F_n, e_i \rangle_{\mathcal{H}}^4] \rightarrow \mathbb{E}[\langle Z, e_i \rangle_{\mathcal{H}}^4], \quad (1.2)$$

for a fixed orthonormal basis $\{e_i\}_{i \in \mathbb{N}}$ of \mathcal{H} . In the sequel, we show that, given convergence of the covariance operators in the trace class norm, (1.2) is equivalent to convergence of the weak fourth moments.

Our main contribution (Theorem 3.2) is the introduction of a fourth-moment theorem for Hilbert-space valued multiple Wiener integrals of a fixed order. A crucial result is a Stein–Malliavin bound in terms of the trace class norm for a d_2 -distance introduced in [18], that metrizes weak convergence on Hilbert spaces (Theorem 3.1). Along the way, we clarify several aspects of working in

¹Although these random variables are degenerate, we can make them non-degenerate by taking $\tilde{Z}_i = N + Z_i$, where N is a non-degenerate Gaussian random variable that is independent of Z_i , $i = 1, 2$. Then, the same considerations hold.

infinite-dimensional spaces including different notions of differentiation. We also draw connections to the existing literature by relating condition (1.2) to notions of higher order moments of Hilbert space-valued random variables as introduced in [15]. We emphasize that the classical fourth moment Theorem (cf. with items (i) and (ii) of Theorem 5.2.7 in [20]) is a simple corollary of the result obtained herein.

As mentioned above, several works build upon the results in [6] including one of our own works [12]. In light of the issues identified in [1], the conditions for a central limit theorem presented in [12] are insufficient. The results given here build the basis to address the issues that arose in our previous article [12] in future research.

2 Preliminaries

We start by introducing notation as well as recalling several definitions and technical tools used throughout this write-up. In particular, we collect some useful notation in Section 2.1. In Section 2.2, we introduce the two different notions of differentiation of functions in infinite dimensions that are used in this article: the Fréchet and Gross derivatives. A central aspect of our results is using an appropriate distance between probability measures on infinite-dimensional spaces (Section 2.3). We also recall aspects of Stein's method for infinite-dimensional Gaussian approximation in abstract Wiener spaces (Section 2.4). As a final ingredient, we introduce the Malliavin derivative, the semigroup of the Ornstein–Uhlenbeck (OU) process, its associated generator, and the induced pseudo-inverse generator in Section 2.5.

2.1 Notations

Let \mathcal{H} denote a separable Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and norm $\|\cdot\|_{\mathcal{H}}$. Let $\{u_i\}_{i \in \mathbb{N}}$ be an orthonormal basis of a Hilbert space \mathcal{H} . Throughout this section, V and W denote two separable Banach spaces, endowed with respective norms $\|\cdot\|_V$ and $\|\cdot\|_W$. As is customary, we denote by $\mathcal{L}(V : W)$ the space of all bounded linear operators $T : V \rightarrow W$. The space $\mathcal{L}(V : W)$ is equipped with the norm

$$\|T\|_{\text{op}} \doteq \sup_{\|x\|_V \leq 1} \|Tx\|_W,$$

so that $(\mathcal{L}(V : W), \|\cdot\|_{\text{op}})$ is a Banach space. When $V = W$, we write simply $\mathcal{L}(V)$ for the space of bounded linear operators on V . Moreover, we write $\mathcal{L}(V^{\otimes k} : W)$ for the space of multilinear operators from $V^{\otimes k}$ to W . Whenever it benefits the readability, we write $\|T\|_{\text{op}(\mathcal{L}(V:W))}$.

Two closely related families of operators are the Hilbert-Schmidt and the trace class operators. The space of Hilbert-Schmidt operators, denoted by $\text{HS}(\mathcal{H})$, consists of linear operators $T : \mathcal{H} \rightarrow \mathcal{H}$, and is equipped with an inner product and a norm given by

$$\langle T, S \rangle_{\text{HS}(\mathcal{H})} \doteq \sum_{i=1}^{\infty} \langle Tu_i, Su_i \rangle_{\mathcal{H}}, \quad \|T\|_{\text{HS}(\mathcal{H})}^2 \doteq \sum_{i=1}^{\infty} \|Tu_i\|_{\mathcal{H}}^2.$$

The trace class $\mathcal{S}(\mathcal{H})$ consists of operators $T : \mathcal{H} \rightarrow \mathcal{H}$ and carries the norm

$$\|T\|_{\mathcal{S}(\mathcal{H})} = \text{Tr}_{\mathcal{H}}(|T|) \doteq \sum_{i=1}^{\infty} \langle |T|u_i, u_i \rangle_{\mathcal{H}}, \quad |T| \doteq \sqrt{T^*T}. \quad (2.1)$$

If $T \geq 0$ and self-adjoint, then $\|T\|_{\mathcal{S}(\mathcal{H})} = \text{Tr}_{\mathcal{H}}(T) = \sum_{i=1}^{\infty} \langle Tu_i, u_i \rangle_{\mathcal{H}}$. Moreover, the following holds for the three norms defined on $\mathcal{L}(\mathcal{H})$

$$\|T\|_{\text{op}} \leq \|T\|_{\text{HS}(\mathcal{H})} \leq \|T\|_{\mathcal{S}(\mathcal{H})}.$$

We denote by $\text{Lip}_b^1(V)$ the space of uniformly Lipschitz functions $h : V \rightarrow \mathbb{R}$, endowed with the norm $\|h\|_{\text{Lip}_b^1} = \|h\|_{\text{Lip}} + |h(0)|$, and suppose $\|h\|_{\text{Lip}_b^1} \leq 1$, where

$$\|h\|_{\text{Lip}} = \sup_{x \neq y \in V} \frac{|h(x) - h(y)|}{\|x - y\|_V} < \infty.$$

We use x^* to denote an element of the dual space V^* . In this case, since for us V is a Hilbert space, we use the notation $\langle x, x^* \rangle_{V, V^*} \doteq x^*(x)$.

2.2 Fréchet and Gross derivatives

In this section, we define two notions of differentiation in infinite-dimensional spaces, the Fréchet and Gross derivatives. For the presentation here we follow the excellent exposition of [2]. Here, H_0 denotes a Hilbert space that is continuously embedded in V , with associated norm $\|\cdot\|_{H_0}$ and inner product $\langle \cdot, \cdot \rangle_{H_0}$. Later in the subsection, we provide concrete examples for H_0 .

We say that a function $f : V \rightarrow W$ is H_0 -Fréchet differentiable at $x \in V$ if there exists a bounded linear operator $L_x \in \mathcal{L}(H_0 : W)$ such that

$$\lim_{\|h\|_{H_0} \rightarrow 0} \frac{\|f(x+h) - f(x) - L_x h\|_W}{\|h\|_{H_0}} = 0.$$

If it exists, the operator L_x is unique, called the H_0 -Fréchet derivative of f at x , and is denoted by $D_{H_0}f(x) \doteq L_x$. We say that f is H_0 -Fréchet differentiable if it is H_0 -Fréchet differentiable at every point $x \in V$.

Furthermore, we say that f is twice H_0 -Fréchet differentiable at $x \in V$ if (i) f is H_0 -Fréchet differentiable, and (ii) the map $D_{H_0}f : V \rightarrow \mathcal{L}(H_0 : W)$ is also H_0 -Fréchet differentiable. In that case, the unique second-order H_0 -Fréchet derivative of f at x is identified with the bilinear form

$$D_{H_0}^2 f(x) \in \mathcal{L}(H_0^{\otimes 2} : W).$$

Higher order differentiation of f at some $x \in V$ along the subspace H_0 is then defined analogously by induction.

In the sequel, V is often a Hilbert space and $W = \mathbb{R}$. In this case, the Riesz representation theorem says that $D_{H_0}^k f(x)$, as an element of $\mathcal{L}(H_0^k : \mathbb{R})$, can be identified with a unique element in $\mathcal{L}(H_0^{(k-1)} : H_0)$, which we denote by $\nabla_{H_0}^k f(x)$, i.e.,

$$D_{H_0}^k f(x)(h_1, \dots, h_k) = \langle \nabla_{H_0}^k f(x)(h_1, \dots, h_{k-1}), h_k \rangle_{H_0}, \quad h_1, \dots, h_k \in H_0.$$

These relations are clarified and derived from first principles in the recent survey [2]; see Definition 2.10 and Theorem 2.21 therein.

We can now define both Fréchet and Gross derivatives under the unified framework above.

Fréchet differentiability: In the above framework, the usual notion of Fréchet differentiability corresponds to $H_0 = V$. In this case, we write $D_F^k f(x)$ instead of $D_V^k f(x)$ (for $k = 1, 2$) to denote the Fréchet derivative.

Gross differentiability: Let (V, \mathbb{H}, i) be an abstract Wiener Space (see Section 3.9 in [4]), where \mathbb{H} is the Cameron Martin space, and let γ be the induced, centered, and non-degenerate Gaussian measure on V . Then, the Gross derivative corresponds to $H_0 = \mathbb{H}$. Whenever the abstract Wiener space (V, \mathbb{H}, γ) is easily inferred from context, we write as expected $D_{\mathbb{H}}f(x)$ to denote the Gross derivative.

If, for $H_0 = \mathbb{H}$ or $H_0 = \mathcal{H}$, $\nabla_{H_0}^2 f(x)$ is a trace class operator in $\mathcal{L}(H_0 : H_0)$, we define the Laplacian and Gross Laplacian respectively by,

$$\Delta_{\mathcal{H}} f(x) \doteq \text{Tr}_{\mathcal{H}}(\nabla_{\mathcal{H}}^2 f(x)), \quad \Delta_{\mathbb{H}} f(x) \doteq \text{Tr}_{\mathbb{H}}(\nabla_{\mathbb{H}}^2 f(x)). \quad (2.2)$$

Relation Between Fréchet, Gross Differentiability: If f is Fréchet differentiable at $x \in V$, then it is automatically \mathbb{H} -differentiable at x , and the \mathbb{H} -derivative is the restriction of the Fréchet derivative to \mathbb{H} , see Proposition 2.8 in [2]. \mathbb{H} -differentiability does not imply Fréchet differentiability. Indeed, many functionals on Gaussian spaces are not Fréchet differentiable due to irregular behavior in directions orthogonal to \mathbb{H} , but they are \mathbb{H} -differentiable, see Remark 4.1(1) below. If f is twice Fréchet differentiable, then $D_{\mathbb{H}} f(x), x \in \mathcal{H}$, can be viewed as an element in \mathcal{H}^* ; see p. 1241 in [23].

Concept	Our Notation	Section 3.4 in [11]	[23]	Section 2.1 in [2]
Fréchet Derivative ($H_0 = \mathcal{H}$)	D_F	D	'	D
Gross Derivative ($H_0 = \mathbb{H}$)	$D_{\mathbb{H}}$	D_G	D (context-dependent)	$D_{\mathbb{H}}$
Fréchet Gradient ($H_0 = \mathcal{H}$)	∇_F	D	'	∇
Gross Gradient ($H_0 = \mathbb{H}$)	$\nabla_{\mathbb{H}}$	D_G	D (context-dependent)	$\nabla_{\mathbb{H}}$
Laplacian ($H_0 = \mathcal{H}$)	$\text{Tr}_{\mathcal{H}}, \Delta_{\mathcal{H}}$	Tr	Not available	Not available
Gross Laplacian ($H_0 = \mathbb{H}$)	$\text{Tr}_{\mathbb{H}}, \Delta_{\mathbb{H}}$	Tr_G, Δ_G	Tr_G, Δ_H	Not available

Table 1: Notation comparison for Fréchet and Gross derivatives and gradients in commonly referenced works.

2.3 The d_2 -distance

In this section, we state a metric on the space of probability measures on \mathcal{H} that metrizes weak convergence. This metric was introduced by Giné and León in [14]. The recent write-up [1] illuminates the differences between this distance and the metric ρ_2 that was employed in [5].

Recall that, for an (everywhere) Fréchet differentiable operator $h : \mathcal{H} \rightarrow \mathbb{R}$, the k -th (Fréchet) derivative $D_F^k h$ can be identified with a map from \mathcal{H} to $\mathcal{L}(\mathcal{H}^{\otimes k} : \mathbb{R})$. We denote by $\mathcal{C}_b^k(\mathcal{H})$ the space of bounded, \mathbb{R} -valued operators on \mathcal{H} , admitting k Fréchet derivatives, i.e., $h \in \mathcal{C}_b^k(\mathcal{H})$ if

$$\|h\|_{\mathcal{C}_b^k(\mathcal{H})} = \sup_{j=0,\dots,k} \sup_{x \in \mathcal{H}} \|D_F^j h(x)\|_{\text{op}} < \infty,$$

under the convention that $\|D_F^0 h(x)\|_{\mathcal{L}(\mathcal{H}^0 : \mathbb{R})} = \sup_{x \in \mathcal{H}} |h(x)|$. In the display above, the operator norm is taken over the space $\mathcal{L}(\mathcal{H}^{\otimes j} : \mathbb{R})$. Then, the d_j metric on $\mathcal{P}(\mathcal{H})$ (the space of probability measures on \mathcal{H}) is defined as, for $\mu, \nu \in \mathcal{P}(\mathcal{H})$,

$$d_j(\mu, \nu) \doteq \sup_{h \in \mathcal{C}_b^j(\mathcal{H}), \|h\|_{\mathcal{C}_b^j(\mathcal{H})} \leq 1} \left| \int_{\mathcal{H}} h(x)(\mu(dx) - \nu(dx)) \right|, \quad \text{for } j \geq 1.$$

By Theorem 2.4 in [14], for $j \geq 1$, d_j (and in particular, d_2) metrizes weak convergence in $\mathcal{P}(\mathcal{H})$. For two random variables X, Z in \mathcal{H} , we write $d_2(X, Z)$, meaning $d_2(\mathcal{L}(X), \mathcal{L}(Z))$, where $\mathcal{L}(\cdot)$ denotes the law of a random variable.

2.4 Stein's method in abstract Wiener spaces

Let (V, \mathbb{H}, i) be an abstract Wiener Space as in Section 2.2, with an associated Gaussian measure γ_Z on V . We write γ_Z to emphasize that Z here denotes a V -valued random variable with law γ_Z . Let $h : V \rightarrow \mathbb{R}$ be a measurable function in the test class $\text{Lip}_b^1(V)$.

The *Stein operator* associated with γ_Z acts on sufficiently smooth functions $f : V \rightarrow \mathbb{R}$ as

$$\mathcal{A}f(x) \doteq \langle x, D_{\mathbb{H}}f(x) \rangle_{\mathbb{H}} - \Delta_{\mathbb{H}}f(x),$$

where $D_{\mathbb{H}}f(x)$ and $\Delta_{\mathbb{H}}f(x) = \text{Tr}_{\mathbb{H}}(D_{\mathbb{H}}^2f(x))$ were introduced in Section 2.2. For a fixed operator h , the Stein operator gives rise to the so-called *Stein equation* corresponding to h , namely the operator $f_h : V \rightarrow \mathbb{R}$ that solves the following equation

$$\mathcal{A}f_h(x) = h(x) - \int_V h d\gamma_Z, \quad x \in V.$$

When $h \in \text{Lip}_b^1(V)$, existence and uniqueness of the solution f_h to the Stein equation was proved by Shih [23].

In particular, [23] proved that the Stein solution admits the representation

$$f_h(x) = - \int_0^\infty (P_t h(x) - \mathbb{E}[h(Z)]) dt, \quad (2.3)$$

where $\{P_t\}_{t \geq 0}$ is the semigroup associated to the Ornstein-Uhlenbeck process and is defined by Mehler's formula as follows.

$$P_t h(x) \doteq \int_V h \left(e^{-t}x + \sqrt{1 - e^{-2t}}y \right) \gamma_Z(dy), \quad t \geq 0. \quad (2.4)$$

2.5 Malliavin derivative for Hilbert space-valued random variables

We collect in this section some results on the Malliavin derivative, the OU semigroup, its generator, and the induced pseudo-inverse of Hilbert space-valued random variables. We follow here the careful treatment of [24], and we refer to this article for more details. For \mathbb{R} -valued random variables, these results are classical and can be found in, e.g., Chapter 2 in [20].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $W = \{W(h) : h \in \mathfrak{H}\}$ be an isonormal Gaussian process defined on this probability space. Here, \mathfrak{H} is a real separable Hilbert space, equipped with the norm $\|\cdot\|_{\mathfrak{H}}$ and an inner product $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$, and denote by $\{h_i\}_{i \in \mathbb{N}}$ an orthonormal basis of \mathfrak{H} . By definition, W is a centered Gaussian family satisfying

$$\mathbb{E}[W(h)W(g)] = \langle h, g \rangle_{\mathfrak{H}}, \quad h, g \in \mathfrak{H}.$$

An \mathcal{H} -valued random variable F is said to be measurable with respect to W if for every $h \in \mathcal{H}$, the real-valued random variable $\langle F, h \rangle_{\mathcal{H}}$ is measurable with respect to the σ -algebra generated by W .

Any $F \in L^2(\Omega : \mathcal{H})$ that is measurable with respect to W admits a Wiener-Itô chaos expansion

$$F = \mathbb{E}[F] + \sum_{n \geq 1} I_n(f_n), \quad (2.5)$$

where $f_n \in \mathfrak{H}^{\odot n} \otimes \mathcal{H}$ are suitable symmetric kernels, and $I_n(f_n) \in L^2(\Omega : \mathcal{H})$ denotes the \mathcal{H} -valued, n -th multiple Wiener-Itô integral satisfying $\mathbb{E}[I_n(f_n)] = 0$. Here $\mathfrak{H}^{\odot n} \subset \mathfrak{H}^{\otimes n}$ denotes the n -th

symmetric tensor product of \mathfrak{H} ; see Chapter 1 in [21] and Chapter 2 in [20]. For careful definitions of these objects in general Hilbert spaces and proofs of the chaos decomposition claim, see Theorem 4.2 in [12] or Lemma 2.2 in [24] (and note that the definitions therein coincide).

The Malliavin derivative D_M can then be defined in terms of the chaos expansion as follows

$$D_M F \doteq \sum_{n \geq 1} n I_{n-1}(f_n) \in L^2(\Omega : \mathfrak{H} \otimes \mathcal{H}),$$

where F belongs to the domain of D_M defined as

$$\mathbb{D}^{1,2}(\mathcal{H}) \doteq \left\{ F \in L^2(\Omega : \mathcal{H}) \text{ such that } F \text{ is as in (2.5) and } \sum_{n \geq 1} n n! \|f_n\|_{\mathfrak{H}^{\otimes n} \otimes \mathcal{H}}^2 < \infty \right\}.$$

Analogous definitions can be given for the higher order Malliavin derivatives D_M^k , as well as their respective domains $\mathbb{D}^{k,2}(\mathcal{H})$. However, we only use $k = 1$ in the present article.

Cylindrical \mathcal{H} -valued random variables are dense in $\mathbb{D}^{1,2}(\mathcal{H})$ (see, e.g., Section 2 in [18]), ensuring that this definition agrees with the classical derivative on smooth cylindrical functionals.

The divergence operator δ is defined as the adjoint of the Malliavin derivative D_M . More precisely, let $u \in L^2(\Omega : \mathfrak{H} \otimes \mathcal{H})$ be in the domain of δ (denoted $\text{dom}(\delta)$) if there exists a constant C depending on F such that,

$$|\mathbb{E}\langle D_M F, u \rangle_{\mathfrak{H} \otimes \mathcal{H}}| \leq C \|F\|_{L^2(\Omega : \mathcal{H})}$$

for all $f \in \mathbb{D}^{1,2}(\mathcal{H})$. Then, for all $(F, u) \in \mathbb{D}^{1,2}(\mathcal{H}) \times \text{dom}(\delta)$, the random variable $\delta(u) \in L^2(\Omega : \mathcal{H})$ is defined by the duality relation

$$\mathbb{E}[\langle F, \delta(u) \rangle_{\mathcal{H}}] = \mathbb{E}[\langle D_M F, u \rangle_{\mathfrak{H} \otimes \mathcal{H}}] \quad \text{for all } F \in \mathbb{D}^{1,2}(\mathcal{H}); \quad (2.6)$$

see Sections 2.4 and 2.6 in [20] or Section 2.2.3 in [24] for details.

The Ornstein-Uhlenbeck semigroup $\{P_t : t \geq 0\}$ can also be defined to act on Wiener chaos as

$$P_t F \doteq \mathbb{E}[F] + \sum_{n \geq 1} e^{-nt} I_n(f_n), \quad (2.7)$$

whenever F admits the representation in (2.5). It will be clear from context whether we use the definition of the semigroup given in (2.7) (suitable for random variables), or the one in (2.4) (suitable for deterministic functions).

Associated to the semigroup $\{P_t\}$ in (2.7) is its infinitesimal generator L , which acts on random variables F as in (2.5) by

$$L F \doteq \sum_{n \geq 1} (-n) I_n(f_n),$$

with domain

$$\text{dom}(L) \doteq \left\{ F \in L^2(\Omega : \mathcal{H}) \text{ such that } F \text{ is as in (2.5) and } \sum_{n \geq 1} n^2 n! \|f_n\|_{\mathfrak{H}^{\otimes n} \otimes \mathcal{H}}^2 < \infty \right\}.$$

The following identity connects the generator, the adjoint operator, and the Malliavin derivative

$$L = -\delta D_M. \quad (2.8)$$

Finally, for random variables F as in (2.5), the pseudo-inverse L^{-1} of the generator is defined by

$$L^{-1} F \doteq - \sum_{n \geq 1} \frac{1}{n} I_n(f_n), \quad (2.9)$$

so that $LL^{-1}F = F - \mathbb{E}F$.

3 Abstract bound and Fourth-Moment Theorem

Theorem 3.1. *Let Z be a centered, non-degenerate Gaussian random variable on \mathcal{H} with covariance operator \mathcal{I}_Z . Then, for all centered $F \in \mathbb{D}^{1,2}(\mathcal{H})$, it holds that*

$$d_2(F, Z) \leq \frac{1}{2} \left\| \langle D_M F, -D_M L^{-1} F \rangle_{\mathfrak{H}} - \mathcal{I}_Z \right\|_{L^1(\Omega; \mathcal{S}(\mathcal{H}))},$$

where the operator L^{-1} was defined in (2.9).

Proof: Denote by γ_Z the Gaussian measure on \mathcal{H} associated with the random variable Z , and recall that \mathbb{H} is the Hilbert space associated to the abstract Wiener space formulation of γ_Z on \mathcal{H} . Then, we have that

$$\begin{aligned} d_2(F, Z) &= \sup_{h \in \mathcal{C}_b^2(\mathcal{H}), \|h\|_{\mathcal{C}_b^2(\mathcal{H})} \leq 1} |\mathbb{E}(h(F)) - \mathbb{E}(h(Z))| \\ &\leq \sup_{h \in \mathcal{C}_b^2(\mathcal{H}), \|h\|_{\mathcal{C}_b^2(\mathcal{H})} \leq 1} \left| \mathbb{E} \left(\langle F, D_{\mathbb{H}} f_h(F) \rangle_{\mathcal{H}, \mathcal{H}^*} - \Delta_{\mathbb{H}} f_h(F) \right) \right| \end{aligned} \quad (3.1)$$

$$= \sup_{h \in \mathcal{C}_b^2(\mathcal{H}), \|h\|_{\mathcal{C}_b^2(\mathcal{H})} \leq 1} \left| \mathbb{E} \left(\langle F, D_{\mathbb{H}} f_h(F) \rangle_{\mathcal{H}, \mathcal{H}^*} - \text{Tr}_{\mathbb{H}} \nabla_{\mathbb{H}}^2 f_h(F) \right) \right|, \quad (3.2)$$

where f_h was defined in (2.3), and is twice Fréchet differentiable by Lemma 4.1. This also implies that it is twice Gross differentiable; see Proposition 2.8 in [2]. In the calculations above, (3.1) follows by the inclusion $\mathcal{C}_b^2(\mathcal{H}) \subset \text{Lip}_b^1(\mathcal{H})$ and Theorem 4.10 in [23] (but note the adjusted notation here), and (3.2) is due to the definition of the Gross Laplacian; see (2.2). For the first term in (3.2), since f_h is twice Fréchet differentiable, $D_{\mathbb{H}} f_h(F)$ is an element in \mathcal{H}^* and, moreover,

$$\mathbb{E} \left(\langle F, D_{\mathbb{H}} f_h(F) \rangle_{\mathcal{H}, \mathcal{H}^*} \right) = \mathbb{E} \left(\langle F, D_F f_h(F) \rangle_{\mathcal{H}, \mathcal{H}^*} \right); \quad (3.3)$$

see p. 1241 in [23], but note the differences in notation that were highlighted in Table 1. Moreover,

$$\begin{aligned} \mathbb{E} \left(\langle F, D_F f_h(F) \rangle_{\mathcal{H}, \mathcal{H}^*} \right) &= \mathbb{E} \left(\langle LL^{-1} F, \nabla_F f_h(F) \rangle_{\mathcal{H}} \right) \\ &= \mathbb{E} \left(\text{Tr}_{\mathcal{H}} \left(\langle -D_M L^{-1} F, D_M \nabla_F f_h(F) \rangle_{\mathfrak{H}} \right) \right) \end{aligned} \quad (3.4)$$

$$= \mathbb{E} \left(\text{Tr}_{\mathcal{H}} \left(\nabla_F^2 f_h(F) \langle D_M F, -D_M L^{-1} F \rangle_{\mathfrak{H}} \right) \right), \quad (3.5)$$

where (3.4) and (3.5) are respectively due to Lemma 4.2 (iii) and (ii) and since $\nabla_F f_h(F)$ is Fréchet differentiable with bounded second derivative; see Lemma 4.1.

Note further that

$$\text{Tr}_{\mathbb{H}} \left(\nabla_{\mathbb{H}}^2 f_h(F) \right) = \text{Tr}_{\mathcal{H}} \left(\nabla_F^2 f_h(F) \mathcal{I}_Z \right) \quad (3.6)$$

from (3.4.4) in [11].

Combining (3.3), (3.5), and (3.6) yields, for any $h \in \mathcal{C}_b^2(\mathcal{H})$,

$$\mathbb{E} \left(\langle F, \nabla_F f_h(F) \rangle_{\mathcal{H}} - \text{Tr}_{\mathbb{H}} \nabla_{\mathbb{H}}^2 f_h(F) \right) = \mathbb{E} \left(\text{Tr}_{\mathcal{H}} \left(\nabla_F^2 f_h(F) \left(\langle D_M F, -D_M L^{-1} F \rangle_{\mathfrak{H}} - \mathcal{I}_Z \right) \right) \right). \quad (3.7)$$

Since, for $h_1, h_2 \in \mathcal{H}$,

$$\langle \nabla_F^2 f_h(F) h_1, h_2 \rangle_{\mathcal{H}} = D_F^2 f_h(F) [h_1, h_2],$$

then it is easy to see that $\|\nabla_F^2 f_h(F)\|_{\text{op}(\mathcal{L}(\mathcal{H}; \mathcal{H}))} = \|D_F^2 f_h(F)\|_{\text{op}(\mathcal{L}(\mathcal{H}^{\otimes 2}; \mathbb{R}))}$.

Finally, since $\nabla_F^2 f_h(F)$ is a bounded linear operator and $\langle D_M F, -D_M L^{-1} F \rangle_{\mathfrak{H}} - \mathcal{I}_Z$ is of trace class,

$$\begin{aligned} & \mathbb{E} \left(\left| \text{Tr}_{\mathcal{H}} \nabla_F^2 f_h(F) \left(\langle D_M F, -D_M L^{-1} F \rangle_{\mathfrak{H}} - \mathcal{I}_Z \right) \right| \right) \\ & \leq \mathbb{E} \left(\left\| \nabla_F^2 f_h(F) \right\|_{\text{op}(\mathcal{L}(\mathcal{H}; \mathcal{H}))} \text{Tr}_{\mathcal{H}} \left| \langle D_M F, -D_M L^{-1} F \rangle_{\mathfrak{H}} - \mathcal{I}_Z \right| \right) \end{aligned} \quad (3.8)$$

$$\begin{aligned} & \leq \sup_{u \in \mathcal{H}} \left\| D_F^2 f_h(u) \right\|_{\text{op}(\mathcal{L}(\mathcal{H}^{\otimes 2}; \mathbb{R}))} \mathbb{E} \left(\text{Tr}_{\mathcal{H}} \left| \langle D_M F, -D_M L^{-1} F \rangle_{\mathfrak{H}} - \mathcal{I}_Z \right| \right) \\ & \leq \frac{1}{2} \left\| \langle D_M F, -D_M L^{-1} F \rangle_{\mathfrak{H}} - \mathcal{I}_Z \right\|_{L^1(\Omega; \mathcal{S}(\mathcal{H}))}, \end{aligned} \quad (3.9)$$

where (3.8) follows by Theorem 18.11 (e) in [10] and (3.9) is a consequence of (4.1) in Lemma 4.1. The conclusion follows upon combining (3.2), (3.7), and (3.9). \square

For the following theorem, we use the notion of a \mathcal{H} -valued multiple Wiener integrals. For a rigorous construction, we refer to Chapter 1 in [21], Chapter 2 in [20], or [12].

Theorem 3.2 (Infinite-dimensional Fourth-Moment Theorem). *Let Z be a centered, non-degenerate Gaussian random variable on \mathcal{H} with covariance operator \mathcal{I}_Z , and let $\{F_n\}_{n \in \mathbb{N}}$ be a sequence of \mathcal{H} -valued multiple integrals, i.e., $F_n = I_p(f_n)$ for some $p \geq 1$, with respective covariance operators \mathcal{I}_{F_n} and $f_n \in \mathfrak{H}^{\odot p} \otimes \mathcal{H}$. Suppose $\|\mathcal{I}_{F_n} - \mathcal{I}_Z\|_{\mathcal{S}(\mathcal{H})} \rightarrow 0$. Then, as $n \rightarrow \infty$, the following statements are equivalent*

(i) $F_n \xrightarrow{d} Z$.

(ii) $\mathbb{E} [\langle F_n, e_i \rangle_{\mathcal{H}}^4] \rightarrow \mathbb{E} [\langle Z, e_i \rangle_{\mathcal{H}}^4]$ for all orthonormal bases $\{e_i\}_{i \in \mathbb{N}}$ of \mathcal{H} .

(iii) The 4-th weak moments of $\{F_n\}$ converge to the 4-th weak moments of Z , i.e.,

$$\mathbb{E}(x_1^*(F_n) \dots x_4^*(F_n)) \rightarrow \mathbb{E}(x_1^*(Z) \dots x_4^*(Z)),$$

for all $x_1^*, \dots, x_4^* \in \mathcal{H}^*$.

(iv) $\mathbb{E} [\langle F_n, e_i \rangle_{\mathcal{H}}^4] \rightarrow \mathbb{E} [\langle Z, e_i \rangle_{\mathcal{H}}^4]$ for some orthonormal basis $\{e_i\}_{i \in \mathbb{N}}$ of \mathcal{H} .

Proof: The equivalence of (ii) and (iii) is proved in Lemma 4.3 below. That (ii) implies (iv) is immediate. So it remains to show that (i) implies (ii) and that (iv) implies (i).

Proof of (i) implies (ii): Fix an orthonormal basis $\{e_i\}_{i \in \mathbb{N}}$ and note that, since the map $x \mapsto \langle x, e_i \rangle_{\mathcal{H}}, x \in \mathcal{H}$, is an element of \mathcal{H}^* , (i) implies that $\langle F_n, e_i \rangle_{\mathcal{H}} \xrightarrow{d} \langle Z, e_i \rangle_{\mathcal{H}}$ for all $i \geq 1$. By hypercontractivity of the Wiener chaos, for all $k \geq 2$, there is a constant $c(k) > 0$ such that $\mathbb{E} [\langle F_n, e_i \rangle_{\mathcal{H}}^k] \leq c(k) (\mathbb{E} [\langle F_n, e_i \rangle_{\mathcal{H}}^2])^{k/2}$; see Theorem 2.7.2 in [20].

Moreover, $\|\mathcal{I}_{F_n} - \mathcal{I}_Z\|_{\mathcal{S}(\mathcal{H})} \rightarrow 0$ implies that $\mathbb{E} \|F_n\|_{\mathcal{H}}^2 \rightarrow \mathbb{E} \|Z\|_{\mathcal{H}}^2$, which in turn says that $\sup_{n \in \mathbb{N}} \mathbb{E} \|F_n\|_{\mathcal{H}}^2 < \infty$. Then, with $F_{n,i} \doteq \langle F_n, e_i \rangle_{\mathcal{H}}$, for arbitrary $\varepsilon > 0$,

$$\mathbb{E} F_{n,i}^{4+\varepsilon} \leq c(\mathbb{E} F_{n,i}^2)^{\frac{4+\varepsilon}{2}} \leq c \left(\sum_{i=1}^{\infty} \mathbb{E} F_{n,i}^2 \right)^{\frac{4+\varepsilon}{2}} \leq c \left(\sup_{n \in \mathbb{N}} \mathbb{E} \|F_n\|_{\mathcal{H}}^2 \right)^{\frac{4+\varepsilon}{2}} < \infty,$$

which implies that $\{F_{n,i}^4\}_{n \in \mathbb{N}}$ is uniformly integrable. Then, Theorem 3.5 in [3] gives convergence of the moments, namely that $\mathbb{E} [\langle F_n, e_i \rangle_{\mathcal{H}}^4] \rightarrow \mathbb{E} [\langle Z, e_i \rangle_{\mathcal{H}}^4]$, for all $i \geq 1$.

Proof of (iv) implies (i): To prove that (iv) implies (i), we employ Theorem 3.1 so that

$$d_2(F_n, Z) \leq \frac{1}{2} \mathbb{E} \|\Gamma(F_n) - \mathcal{I}_Z\|_{\mathcal{S}(\mathcal{H})}, \quad \Gamma(F) \doteq \langle D_M F, -D_M L^{-1} F \rangle_{\mathfrak{H}}. \quad (3.10)$$

Note that, since F_n is in the p -th chaos, $L^{-1}F_n = -\frac{1}{p}F_n$, hence

$$\Gamma(F_n) = \frac{1}{p} \langle D_M F_n, D_M F_n \rangle_{\mathfrak{H}} \geq 0, \quad \mathbb{E}[\Gamma(F_n)] = \mathcal{I}_{F_n}, \quad (3.11)$$

where the expectation is calculated applying Lemma 4.2 (iii). Thus $\Gamma(F_n)$ is a nonnegative random trace-class operator, with expectation equal to the covariance of F_n .

By triangle inequality, the trace class norm in (3.10) can be further bounded as

$$\mathbb{E} \|\Gamma(F_n) - \mathcal{I}_Z\|_{\mathcal{S}(\mathcal{H})} \leq \mathbb{E} \|\Gamma(F_n) - \mathcal{I}_{F_n}\|_{\mathcal{S}(\mathcal{H})} + \|\mathcal{I}_{F_n} - \mathcal{I}_Z\|_{\mathcal{S}(\mathcal{H})}. \quad (3.12)$$

The second summand in (3.12) vanishes, as $n \rightarrow \infty$, by assumption.

For the first summand of (3.12), fix the orthonormal basis $\{e_i\}_{i \in \mathbb{N}}$ of \mathcal{H} along which the convergences in (iv) hold, and let P_m be the orthogonal projection operator onto $E_m \doteq \text{span}\{e_1, \dots, e_m\}$. To be more precise, $P_m : \mathcal{H} \rightarrow \mathcal{H}$ with $P_m(x) = \sum_{i=1}^m \langle x, e_i \rangle_{\mathcal{H}} e_i$ and $Q_m \doteq I - P_m$. Define $F_n^{(m)} \doteq P_m F_n$ and $Z^{(m)} \doteq P_m Z$. Set $G_n \doteq \Gamma(F_n) - \mathcal{I}_{F_n}$, and for fixed $m \in \mathbb{N}$,

$$\mathbb{E} \|G_n\|_{\mathcal{S}(\mathcal{H})} \leq \mathbb{E} \|P_m G_n P_m\|_{\mathcal{S}(\mathcal{H})} + 2\mathbb{E} \|Q_m G_n P_m\|_{\mathcal{S}(\mathcal{H})} + \mathbb{E} \|Q_m G_n Q_m\|_{\mathcal{S}(\mathcal{H})}, \quad (3.13)$$

where we used that the trace class norm is *-invariant (i.e., $\|A\|_{\mathcal{S}(\mathcal{H})} = \|A^*\|_{\mathcal{S}(\mathcal{H})}$) by Theorem 18.11 (f) in [10]. We consider the three summands on the right-hand side of (3.13) separately.

For the first summand in (3.13), in finite dimension, for any $m \times m$ matrix M , we have

$$\|M\|_{\mathcal{S}(\mathcal{H})} \leq \sqrt{m} \|M\|_{\text{HS}(\mathcal{H})}.$$

Then, by Cauchy-Schwarz,

$$\mathbb{E} \|P_m G_n P_m\|_{\mathcal{S}(\mathcal{H})} \leq \sqrt{m} \sqrt{\mathbb{E} \|P_m G_n P_m\|_{\text{HS}(\mathcal{H})}^2} = \sqrt{m} \sqrt{\sum_{i,j=1}^m \mathbb{E} \langle G_n e_i, e_j \rangle_{\mathcal{H}}^2}. \quad (3.14)$$

Moreover, by denoting $F_{n,i} \doteq \langle F_n, e_i \rangle_{\mathcal{H}}$ and $f_{n,i} \doteq \langle f_n, e_i \rangle_{\mathcal{H}}$, the following bound holds

$$\begin{aligned} \sum_{i,j=1}^m \mathbb{E} \langle G_n e_i, e_j \rangle_{\mathcal{H}}^2 &= \sum_{i,j=1}^m \mathbb{E} (\langle \Gamma(F_n) e_i, e_j \rangle_{\mathcal{H}} - \langle \mathcal{I}_{F_n} e_i, e_j \rangle_{\mathcal{H}})^2 \\ &\leq \sum_{i,j=1}^m (\langle \mathcal{I}_{F_n} e_i, e_j \rangle_{\mathcal{H}} - \mathbb{E} F_{n,i} F_{n,j})^2 \\ &\quad + \frac{1}{2} \sum_{i,j=1}^m \sum_{r=1}^{p-1} c_p(r)^2 (\|f_{n,i} \otimes_{p-r} f_{n,i}\|_{\mathfrak{H}^{\otimes 2r}}^2 + \|f_{n,j} \otimes_{p-r} f_{n,j}\|_{\mathfrak{H}^{\otimes 2r}}^2) \end{aligned} \quad (3.15)$$

$$\leq \frac{1}{2} \sum_{i,j=1}^m \left(\mathbb{E} [F_{n,i}^4] - 3 (\mathbb{E} [F_{n,i}^2])^2 + \mathbb{E} [F_{n,j}^4] - 3 (\mathbb{E} [F_{n,j}^2])^2 \right) \sum_{r=1}^{p-1} \binom{2r}{r}, \quad (3.16)$$

where

$$c_p(r) \doteq p(r-1)! \binom{p-1}{r-1}^2 \sqrt{(2p-2r)!}.$$

Here (3.15) follows from (6.2.1) of Lemma 6.2.1 in [20], with $\alpha = \langle \mathcal{I}_{F_n} e_i, e_j \rangle_{\mathcal{H}}$, $p = q$, and $f = f_{n,i}$, $g = f_{n,j}$. The inequality (3.16) follows from (6.2.6) in [20] since $\langle \mathcal{I}_{F_n} e_i, e_j \rangle_{\mathcal{H}} = \mathbb{E} F_{n,i} F_{n,j}$.

Note also that, for a fixed unit vector $e \in \mathcal{H}$,

$$|\mathbb{E}\langle F_n, e \rangle_{\mathcal{H}}^2 - \mathbb{E}\langle Z, e \rangle_{\mathcal{H}}^2| = |\langle (\mathcal{J}_{F_n} - \mathcal{J}_Z)e, e \rangle_{\mathcal{H}}| \leq \|\mathcal{J}_{F_n} - \mathcal{J}_Z\|_{\text{op}} \leq \|\mathcal{J}_{F_n} - \mathcal{J}_Z\|_{\mathcal{S}(\mathcal{H})} \rightarrow 0.$$

Namely, the condition $\|\mathcal{J}_{F_n} - \mathcal{J}_Z\|_{\mathcal{S}(\mathcal{H})} \rightarrow 0$ implies the convergence $\mathbb{E}\langle F_n, u \rangle_{\mathcal{H}}^2 \rightarrow \mathbb{E}\langle Z, u \rangle_{\mathcal{H}}^2$, for all $u \in \mathcal{H}$. By assumption, we have $\mathbb{E}F_{n,i}^4 \rightarrow \mathbb{E}\langle Z, e_i \rangle_{\mathcal{H}}^4$, for all $i \in \mathbb{N}$, as $n \rightarrow \infty$, i.e., we have convergence of fourth and second moments along the basis $\{e_i\}_{i \in \mathbb{N}}$. Then, by combining (3.14), (3.16), and the convergence of moments along the basis $\{e_i\}_{i \in \mathbb{N}}$, we get

$$\mathbb{E}\|P_m G_n P_m\|_{\mathcal{S}(\mathcal{H})} \rightarrow 0, \quad (3.17)$$

as $n \rightarrow \infty$, for all fixed $m \in \mathbb{N}$.

For the third term of (3.13),

$$\mathbb{E}\|Q_m G_n Q_m\|_{\mathcal{S}(\mathcal{H})} \leq \mathbb{E}\|Q_m \Gamma(F_n) Q_m\|_{\mathcal{S}(\mathcal{H})} + \|Q_m \mathbb{E} \Gamma(F_n) Q_m\|_{\mathcal{S}(\mathcal{H})} = 2\|Q_m \mathbb{E} \Gamma(F_n) Q_m\|_{\mathcal{S}(\mathcal{H})},$$

where the equality follows, since by (3.11) and Tonelli's theorem

$$\mathbb{E}\|Q_m \Gamma(F_n) Q_m\|_{\mathcal{S}(\mathcal{H})} = \sum_{i=1}^{\infty} \langle Q_m \mathbb{E} \Gamma(F_n) Q_m e_i, e_i \rangle_{\mathcal{H}} = \|Q_m \mathbb{E} \Gamma(F_n) Q_m\|_{\mathcal{S}(\mathcal{H})}.$$

Then, for fixed m and as $n \rightarrow \infty$,

$$\|Q_m \mathbb{E} \Gamma(F_n) Q_m\|_{\mathcal{S}(\mathcal{H})} = \sum_{i=1}^{\infty} \langle Q_m \mathbb{E} \Gamma(F_n) Q_m e_i, e_i \rangle_{\mathcal{H}} = \sum_{i=m+1}^{\infty} \langle \mathcal{J}_{F_n} e_i, e_i \rangle_{\mathcal{H}} \rightarrow \sum_{i=m+1}^{\infty} \langle \mathcal{J}_Z e_i, e_i \rangle_{\mathcal{H}}, \quad (3.18)$$

where the convergence is due to

$$\|Q_m (\mathcal{J}_{F_n} - \mathcal{J}_Z) Q_m\|_{\mathcal{S}(\mathcal{H})} \leq \|Q_m\|_{\text{op}}^2 \|\mathcal{J}_{F_n} - \mathcal{J}_Z\|_{\mathcal{S}(\mathcal{H})} = \|\mathcal{J}_{F_n} - \mathcal{J}_Z\|_{\mathcal{S}(\mathcal{H})} \rightarrow 0, \quad (3.19)$$

as $n \rightarrow \infty$; the inequality follows by Theorem 18.11 (g) in [10] and the second relation is due to $\|Q_m\|_{\text{op}} = 1$. Finally, the tail in (3.18) vanishes as $m \rightarrow \infty$ since $\sum_{i=1}^{\infty} \langle \mathcal{J}_Z e_i, e_i \rangle_{\mathcal{H}} = \|\mathcal{J}_Z\|_{\mathcal{S}(\mathcal{H})} = \mathbb{E}\|Z\|_{\mathcal{H}}^2 < \infty$.

For the second summand in (3.13), i.e., the cross terms, we get

$$\mathbb{E}\|Q_m G_n P_m\|_{\mathcal{S}(\mathcal{H})} \leq \mathbb{E}\|Q_m \Gamma(F_n) P_m\|_{\mathcal{S}(\mathcal{H})} + \|Q_m \mathcal{J}_{F_n} P_m\|_{\mathcal{S}(\mathcal{H})}. \quad (3.20)$$

For the second summand on the right hand side of (3.20), with explanations given below,

$$\begin{aligned} \|P_m \mathcal{J}_{F_n} Q_m\|_{\mathcal{S}(\mathcal{H})} &= \|(P_m \mathcal{J}_{F_n}^{1/2})(\mathcal{J}_{F_n}^{1/2} Q_m)\|_{\mathcal{S}(\mathcal{H})} \\ &\leq \|P_m \mathcal{J}_{F_n}^{1/2}\|_{\text{HS}(\mathcal{H})} \|\mathcal{J}_{F_n}^{1/2} Q_m\|_{\text{HS}(\mathcal{H})} \end{aligned} \quad (3.21)$$

$$= \| |P_m \mathcal{J}_{F_n}^{1/2}|^2 \|_{\mathcal{S}(\mathcal{H})}^{\frac{1}{2}} \| |\mathcal{J}_{F_n}^{1/2} Q_m|^2 \|_{\mathcal{S}(\mathcal{H})}^{\frac{1}{2}}, \quad (3.22)$$

where we refer to the proof of Proposition 18.8 ((d) implies (a)) in [10] for (3.21) and (3.22) follows by Definition 18.4 of [10] and recalling that $|A|^2 = A^* A$ with A^* denoting the adjoint operator of A from (2.1). Furthermore, by assumption and (3.19) (one can argue similarly for the truncated part since $\|P_m\|_{\text{op}} = 1$),

$$\| |P_m \mathcal{J}_{F_n}^{1/2}|^2 \|_{\mathcal{S}(\mathcal{H})} = \sum_{i=1}^m \langle \mathcal{J}_{F_n} e_i, e_i \rangle_{\mathcal{H}} \rightarrow \sum_{i=1}^m \langle \mathcal{J}_Z e_i, e_i \rangle_{\mathcal{H}},$$

$$\|\mathcal{J}_{F_n}^{1/2} Q_m\|_{\mathcal{S}(\mathcal{H})}^2 = \sum_{i=m+1}^{\infty} \langle \mathcal{J}_{F_n} e_i, e_i \rangle_{\mathcal{H}} \rightarrow \sum_{i=m+1}^{\infty} \langle \mathcal{J}_Z e_i, e_i \rangle_{\mathcal{H}},$$

so together with (3.22), we get, for fixed m ,

$$\begin{aligned} \lim_{n \rightarrow \infty} \|P_m \mathcal{J}_{F_n} Q_m\|_{\mathcal{S}(\mathcal{H})} &\leq \left(\sum_{i=1}^m \langle \mathcal{J}_Z e_i, e_i \rangle_{\mathcal{H}} \right)^{1/2} \left(\sum_{i=m+1}^{\infty} \langle \mathcal{J}_Z e_i, e_i \rangle_{\mathcal{H}} \right)^{1/2} \\ &\leq \|\mathcal{J}_Z\|_{\mathcal{S}(\mathcal{H})}^{1/2} \left(\sum_{i=m+1}^{\infty} \langle \mathcal{J}_Z e_i, e_i \rangle_{\mathcal{H}} \right)^{1/2}, \end{aligned} \quad (3.23)$$

which converges to 0 as $m \rightarrow \infty$. Similarly, for the first summand in (3.20), and since $\Gamma(F_n)$ is nonnegative by (3.11), we get

$$\begin{aligned} \mathbb{E} \|Q_m \Gamma(F_n) P_m\|_{\mathcal{S}(\mathcal{H})} &= \mathbb{E} \|(P_m \Gamma(F_n)^{1/2})(\Gamma(F_n)^{1/2} Q_m)\|_{\mathcal{S}(\mathcal{H})} \\ &\leq \mathbb{E} \|P_m \Gamma(F_n)^{1/2}\|_{\text{HS}(\mathcal{H})} \|\Gamma(F_n)^{1/2} Q_m\|_{\text{HS}(\mathcal{H})} \\ &= \mathbb{E} \| |P_m \Gamma(F_n)^{1/2}|^2 \|_{\mathcal{S}(\mathcal{H})}^{\frac{1}{2}} \| |\Gamma(F_n)^{1/2} Q_m|^2 \|_{\mathcal{S}(\mathcal{H})}^{\frac{1}{2}} \\ &\leq \left(\mathbb{E} \| |P_m \Gamma(F_n)^{1/2}|^2 \|_{\mathcal{S}(\mathcal{H})} \right)^{\frac{1}{2}} \mathbb{E} \left(\| |\Gamma(F_n)^{1/2} Q_m|^2 \|_{\mathcal{S}(\mathcal{H})} \right)^{\frac{1}{2}}, \end{aligned} \quad (3.24)$$

where the last line follows from Cauchy-Schwarz. Furthermore,

$$\begin{aligned} \mathbb{E} \| |P_m \Gamma(F_n)^{1/2}|^2 \|_{\mathcal{S}(\mathcal{H})} &= \sum_{i=1}^m \langle \mathcal{J}_{F_n} e_i, e_i \rangle_{\mathcal{H}} \rightarrow \sum_{i=1}^m \langle \mathcal{J}_Z e_i, e_i \rangle_{\mathcal{H}}, \\ \mathbb{E} \| |\Gamma(F_n)^{1/2} Q_m|^2 \|_{\mathcal{S}(\mathcal{H})} &= \sum_{i=m+1}^{\infty} \langle \mathcal{J}_{F_n} e_i, e_i \rangle_{\mathcal{H}} \rightarrow \sum_{i=m+1}^{\infty} \langle \mathcal{J}_Z e_i, e_i \rangle_{\mathcal{H}}, \end{aligned}$$

so together with (3.24), we get, for fixed m ,

$$\lim_{n \rightarrow \infty} \mathbb{E} \|Q_m \Gamma(F_n) P_m\|_{\mathcal{S}(\mathcal{H})} \leq \|\mathcal{J}_Z\|_{\mathcal{S}(\mathcal{H})}^{1/2} \left(\sum_{i=m+1}^{\infty} \langle \mathcal{J}_Z e_i, e_i \rangle_{\mathcal{H}} \right)^{1/2}. \quad (3.25)$$

Therefore, combining (3.10), (3.12), (3.13), (3.17), (3.18), (3.20), (3.23) and (3.25), letting $n \rightarrow \infty$ and then $m \rightarrow \infty$, we obtain $d_2(F_n, Z) \rightarrow 0$, i.e., $F_n \xrightarrow{d} Z$, as $n \rightarrow \infty$. \square

Remark 3.1. 1. The result of Theorem 3.2 is consistent with the multivariate fourth-moment theorem [20, Theorem 6.2.3] in the following way: The fourth moment condition in the infinite-dimensional setting is only needed for a fixed orthonormal basis. Likewise, in the multivariate setting, the fourth-moment convergence only needs to be satisfied for each component of the sequence $\{F_n\}$.

However, on the level of second moments, we require convergence of the covariance operators in trace class norm. In the finite-dimensional setting the variances and cross-covariances need to converge to those of the limiting variable across any fixed basis. While in finite dimensions, trace class convergence and convergence of variances and cross-covariances are equivalent, the behavior in infinite dimensions is more nuanced. In particular, in infinite dimensions, convergence of cross-covariances even along all orthonormal bases does not suffice to imply convergence of the covariance operators in trace class norm.

We emphasize that this comparison is based on the multivariate fourth-moment theorem (Theorem 6.2.3 in [20]) assuming that the chaos order is the same across dimensions.

2. Note that condition (iv) of Theorem 3.2 can alternatively be written as $\mathbb{E} [\langle F_n, u \rangle_{\mathcal{H}}^4] \rightarrow \mathbb{E} [\langle Z, u \rangle_{\mathcal{H}}^4]$, as $n \rightarrow \infty$, for every unit vector $u \in \mathcal{H}$.
3. Note that the condition $\|\mathcal{T}_{F_n} - \mathcal{T}_Z\|_{\mathcal{S}(\mathcal{H})} \rightarrow 0$ required herein is stronger than the assumption $\text{Tr}(S_n - S) \rightarrow 0$ required in [5], or its equivalent formulation $\mathbb{E}\|F_n\|_{\mathcal{H}}^2 \rightarrow \mathbb{E}\|Z\|_{\mathcal{H}}^2$. This stronger assumption is needed due to the identifiability issue illustrated through Example 1.1.

4 Auxiliary Lemmas

Lemma 4.1. *Let $h \in \mathcal{C}_b^j(\mathcal{H})$ and suppose $\|h\|_{\mathcal{C}_b^j(\mathcal{H})} \leq 1$, $j = 1, 2$. Then, the (Stein solution) operator f_h defined in (2.3) admits two Fréchet derivatives. Moreover,*

$$\sup_{x \in \mathcal{H}} \|D_F^j f_h(x)\|_{\text{op}(\mathcal{L}(\mathcal{H}^{\otimes j}; \mathbb{R}))} \leq \frac{1}{j}, \quad j = 1, 2. \quad (4.1)$$

Proof: We begin by conjecturing the form of the Fréchet derivative $D_F f_h(x)$. We then verify, using the definition, that this is indeed the Fréchet derivative of f_h whenever $h \in \mathcal{C}_b^2(\mathcal{H})$.

First, set $g_u : \mathcal{H} \rightarrow \mathcal{H}$, $g_u(x) = e^{-u}x + \sqrt{1 - e^{-2u}}y$, for $u \geq 0$ and fixed $y \in \mathcal{H}$. Then, $D_F g_u(x)[z] = e^{-u}z$, i.e., for all $u \geq 0$, this is the multiplication operator associated to the constant function e^{-u} . Since h is Fréchet differentiable and by the chain rule for Fréchet derivatives (see p. 337 in [17]), we get

$$\begin{aligned} D_F(h(g_u(x)))[z] &= ((D_F h)(g_u(x)))[D_F g_u(x)[z]] \\ &= ((D_F h)(g_u(x)))[e^{-u}z] = e^{-u}((D_F h)(g_u(x)))[z]. \end{aligned}$$

The last identity follows since $D_F h(x)[z]$ is linear in z .

By exchanging integration and Fréchet differentiation, this leads us to defining the candidate derivative

$$D_F f_h(x)[z] \doteq - \int_0^\infty e^{-u} \int_{\mathcal{H}} (D_F h)(e^{-u}x + \sqrt{1 - e^{-2u}}y)[z] \gamma_Z(dy) du, \quad (4.2)$$

We formally verify (4.2) by using the definition of the Fréchet derivative and showing

$$\lim_{\|z\|_{\mathcal{H}} \rightarrow 0} \frac{|f_h(x+z) - f_h(x) - D_F f_h(x)[z]|}{\|z\|_{\mathcal{H}}} = 0. \quad (4.3)$$

First, observe that

$$\begin{aligned} &|f_h(x+z) - f_h(x) - D_F f_h(x)[z]| \\ &\leq \int_0^\infty \int_{\mathcal{H}} \left| h(e^{-u}(x+z) + \sqrt{1 - e^{-2u}}y) - h(e^{-u}x + \sqrt{1 - e^{-2u}}y) \right. \\ &\quad \left. - (D_F h)(e^{-u}x + \sqrt{1 - e^{-2u}}y)[e^{-u}z] \right| \gamma_Z(dy) du. \end{aligned} \quad (4.4)$$

By the definition of the Fréchet differentiability of h (cf. (4.3) with h replacing f_h , $e^{-u}x + \sqrt{1 - e^{-2u}}y$ replacing x , and $e^{-u}z$ replacing z), it follows that, $\gamma_Z \otimes du$ a.s.,

$$\lim_{\|z\|_{\mathcal{H}} \rightarrow 0} \frac{1}{\|z\|_{\mathcal{H}}} \left(\left| h(e^{-u}(x+z) + \sqrt{1-e^{-2u}}y) - h(e^{-u}x + \sqrt{1-e^{-2u}}y) - (D_F h)(e^{-u}x + \sqrt{1-e^{-2u}}y)[e^{-u}z] \right| \right) = 0. \quad (4.5)$$

Recalling the notation $g_u : \mathcal{H} \rightarrow \mathcal{H}$, $g_u(x) = e^{-u}x + \sqrt{1-e^{-2u}}y$ to rewrite (4.5), note that

$$\begin{aligned} & |h(g_u(x+z)) - h(g_u(x)) - (D_F h)(g_u(x))[e^{-u}z]| \\ & \leq |h(g_u(x+z)) - h(g_u(x))| + |(D_F h)(g_u(x))[e^{-u}z]| \\ & \leq \|h\|_{\text{Lip}} \|e^{-u}z\|_{\mathcal{H}} + \sup_{x \in \mathcal{H}} \|(D_F h)(x)\|_{\text{op}(\mathcal{L}(\mathcal{H}:\mathbb{R}))} \|e^{-u}z\|_{\mathcal{H}} \\ & \leq e^{-u} \left(\|h\|_{\text{Lip}} \|z\|_{\mathcal{H}} + \sup_{x \in \mathcal{H}} \|(D_F h)(x)\|_{\text{op}(\mathcal{L}(\mathcal{H}:\mathbb{R}))} \|z\|_{\mathcal{H}} \right) \leq 2e^{-u} \|z\|_{\mathcal{H}}. \end{aligned}$$

The second inequality follows by the fact that h is Lipschitz since $h \in \mathcal{C}_b^1(\mathcal{H})$.

The quantity in the last line is integrable with regard to $du \otimes \gamma_Z$. Consequently, by combining (4.4), (4.5), and the dominated convergence theorem, (4.3) follows. Thus f_h is Fréchet differentiable with derivative (4.2).

Then the bound in (4.1) for $j = 1$ follows immediately from the form of $D_F f_h(x)$ in (4.2) and the respective bound for h since

$$\begin{aligned} \sup_{x \in \mathcal{H}} \|D_F f_h(x)\|_{\text{op}(\mathcal{L}(\mathcal{H}:\mathbb{R}))} & \leq \int_0^\infty e^{-u} \int_{\mathcal{H}} \sup_{x \in \mathcal{H}} \|(D_F h)(e^{-u}x + \sqrt{1-e^{-2u}}y)\|_{\text{op}(\mathcal{L}(\mathcal{H}:\mathbb{R}))} \gamma_Z(dy) du \\ & \leq \sup_{x \in \mathcal{H}} \|D_F h(x)\|_{\text{op}(\mathcal{L}(\mathcal{H}:\mathbb{R}))} \leq 1. \end{aligned}$$

Similarly, the candidate for the second derivative is given by

$$D_F^2 f_h(x)[z_1, z_2] \doteq - \int_0^\infty e^{-2u} \int_{\mathcal{H}} (D_F^2 h)(e^{-u}x + \sqrt{1-e^{-2u}}y)[z_1, z_2] \gamma_Z(dy) du, \quad z_1, z_2 \in \mathcal{H}.$$

Given $h \in \mathcal{C}_b^2(\mathcal{H})$ and $\|h\|_{\mathcal{C}_b^2(\mathcal{H})} \leq 1$, the same arguments as for the first derivative then show that f_h is twice Fréchet differentiable with second derivative $D_F^2 f_h$, as well as the bound in (4.1) for $j = 2$. \square

Remark 4.1. 1. We note that, in fact, $h \in \mathcal{C}_b^2(\mathcal{H})$ is a strong condition. For example, in important separable Banach spaces such as the space of continuous trajectories $\mathcal{H} = C([0, 1])$, there are no non-trivial, Fréchet differentiable functions with bounded support; see the discussion in Section 5.4 of [4]. Despite this, it turns out that the class $\mathcal{C}_b^2(\mathcal{H})$ is broad enough to metrize weak convergence whenever \mathcal{H} is a Hilbert space; see [1].

2. In the statement above, we do not claim that $f_h \in \mathcal{C}_b^2(\mathcal{H})$. This is because, the claim that $\sup_{x \in \mathcal{H}} |f_h(x)| < \infty$ is not immediate, even if $h \in \mathcal{C}_b^2(\mathcal{H})$. Nonetheless, it turns out that it is not required for the aims of this article.

Lemma 4.2. *Let $F, G \in \mathbb{D}^{1,2}(\mathcal{H}) \subset L^2(\Omega : \mathcal{H})$, and set $\Gamma(F, G) = \langle D_M F, D_M G \rangle_{\mathfrak{H}}$. Then,*

(i) $\Gamma(\cdot, \cdot)$ is bilinear, almost surely positive (i.e., $\langle \Gamma(F, F)u, u \rangle_{\mathcal{H}} \geq 0$), and symmetric such that for every $u, v \in \mathcal{H}$,

$$\langle \Gamma(F, G)u, v \rangle_{\mathcal{H}} = \langle u, \Gamma(F, G)v \rangle_{\mathcal{H}}.$$

(ii) Let $\varphi, \psi : \mathcal{H} \rightarrow \mathcal{H}$ be continuously Fréchet-differentiable. Then, for $F, G \in \mathbb{D}^{1,2}(\mathcal{H})$,

$$\Gamma(\varphi(F), \psi(G)) = \nabla_F \psi(G) \Gamma(F, G) (\nabla_F \varphi(F))^*.$$

(iii) For all $F, G \in \text{dom}(-L)$,

$$\mathbb{E}[\text{Tr}_{\mathcal{H}} \Gamma(F, G)] = \mathbb{E}[\langle D_M F, D_M G \rangle_{\mathfrak{H} \otimes \mathcal{H}}] = -\mathbb{E}\langle LF, G \rangle_{\mathcal{H}} = -\mathbb{E}\langle F, LG \rangle_{\mathcal{H}}.$$

Proof: *Proof of (i):* Bilinearity and a.s. positivity are immediate. For symmetry, fix $u, v \in \mathcal{H}$. Then, it follows from

$$\begin{aligned} \langle \Gamma(F, G)u, v \rangle_{\mathcal{H}} &= \langle \langle D_M F, D_M G \rangle_{\mathfrak{H}} u, v \rangle_{\mathcal{H}} \\ &= \langle \langle D_M F, u \rangle_{\mathcal{H}}, \langle v, D_M G \rangle_{\mathcal{H}} \rangle_{\mathfrak{H}} \\ &= \langle \langle v, D_M G \rangle_{\mathcal{H}}, \langle D_M F, u \rangle_{\mathcal{H}} \rangle_{\mathfrak{H}} = \langle \Gamma(G, F)v, u \rangle_{\mathcal{H}}, \end{aligned}$$

by symmetry of the inner products $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{H}}$.

Proof of (ii): By (4.10) of Lemma 4.7. in [16], the Malliavin chain rule for \mathcal{H} -valued random variables gives

$$D_M(\varphi(F)) = \nabla_F \varphi(F) D_M F \in \mathfrak{H} \otimes \mathcal{H}, \quad D_M(\psi(G)) = \nabla_F \psi(G) D_M G \in \mathfrak{H} \otimes \mathcal{H},$$

where $\nabla_F \varphi(F)$ and $\nabla_F \psi(G)$ are elements of $\mathcal{L}(\mathcal{H} : \mathcal{H})$. Then,

$$\begin{aligned} \langle \Gamma(\varphi(F), \psi(G))u, v \rangle_{\mathcal{H}} &= \langle \langle D_M \varphi(F), u \rangle_{\mathcal{H}}, \langle D_M \psi(G), v \rangle_{\mathcal{H}} \rangle_{\mathfrak{H}} \\ &= \langle \langle \nabla_F \varphi(F) D_M F, u \rangle_{\mathcal{H}}, \langle \nabla_F \psi(G) D_M G, v \rangle_{\mathcal{H}} \rangle_{\mathfrak{H}} \\ &= \langle \langle D_M F, (\nabla_F \varphi(F))^* u \rangle_{\mathcal{H}}, \langle D_M G, (\nabla_F \psi(G))^* v \rangle_{\mathcal{H}} \rangle_{\mathfrak{H}} \\ &= \langle \Gamma(F, G) (\nabla_F \varphi(F))^* u, (\nabla_F \psi(G))^* v \rangle_{\mathcal{H}} \\ &= \langle \nabla_F \psi(G) \Gamma(F, G) (\nabla_F \varphi(F))^* u, v \rangle_{\mathcal{H}} \end{aligned}$$

which yields

$$\Gamma(\varphi(F), \psi(G)) = \nabla_F \psi(G) \Gamma(F, G) (\nabla_F \varphi(F))^*.$$

Proof of (iii): Using that δ is the adjoint of D_M (2.6), we have that

$$\mathbb{E}\langle D_M F, D_M G \rangle_{\mathfrak{H} \otimes \mathcal{H}} = \mathbb{E}\langle \delta D_M F, G \rangle_{\mathcal{H}}. \quad (4.6)$$

By Lemma 2.4. in [24] and given (2.8), we also have $\delta D_M F = -LF$. This combined with (4.6), then gives

$$\mathbb{E}\langle D_M F, D_M G \rangle_{\mathfrak{H} \otimes \mathcal{H}} = \mathbb{E}\langle \delta D_M F, G \rangle_{\mathcal{H}} = -\mathbb{E}\langle LF, G \rangle_{\mathcal{H}}. \quad (4.7)$$

To identify the left-hand side with $\mathbb{E}[\text{Tr}_{\mathcal{H}} \Gamma(F, G)]$, fix an orthonormal basis $\{e_j\}_{j \in \mathbb{N}}$ of \mathcal{H} and note that

$$\text{Tr}_{\mathcal{H}} \Gamma(F, G) = \sum_{j=1}^{\infty} \langle \Gamma(F, G)e_j, e_j \rangle_{\mathcal{H}} = \sum_{j=1}^{\infty} \langle \langle D_M F, e_j \rangle_{\mathcal{H}}, \langle D_M G, e_j \rangle_{\mathcal{H}} \rangle_{\mathfrak{H}} = \langle D_M F, D_M G \rangle_{\mathfrak{H} \otimes \mathcal{H}},$$

where we used Parseval's identity. Hence, using (4.7), we get

$$\mathbb{E}[\text{Tr}_{\mathcal{H}} \Gamma(F, G)] = \mathbb{E}[\langle D_M F, D_M G \rangle_{\mathfrak{H} \otimes \mathcal{H}}] = -\mathbb{E}\langle LF, G \rangle_{\mathcal{H}}.$$

Symmetry of L gives also $-\mathbb{E}\langle F, LG \rangle_{\mathcal{H}}$. □

For the lemma below, we refer to [15] for notions of higher order moments of Hilbert space-valued random variables. In particular, we mention that, if Z is an \mathcal{H} -valued random variable such that $\mathbb{E}\|Z\|_{\mathcal{H}}^k < \infty$, then the weak k -th moment defined by $\mathbb{E}(x_1^*(Z) \dots x_k^*(Z)) \in \mathbb{R}$ exists, where $k \in \mathbb{N}$ and $x_1^*, \dots, x_k^* \in \mathcal{H}^*$.

Lemma 4.3. *Let $k = 2, 4$ and $\{F_n\}_{n \in \mathbb{N}}$ be a sequence of centered, \mathcal{H} -valued random variables and Z be a centered, \mathcal{H} -valued random variable such that $\mathbb{E}\|Z\|_{\mathcal{H}}^k < \infty$ and $\mathbb{E}\|F_n\|_{\mathcal{H}}^k < \infty$. The following are equivalent:*

(i) *the k -th weak moments of $\{F_n\}$ converge to the k -th weak moments of Z , i.e., $\mathbb{E}(x_1^*(F_n) \dots x_k^*(F_n)) \rightarrow \mathbb{E}(x_1^*(Z) \dots x_k^*(Z))$, for all $x_1^*, \dots, x_k^* \in \mathcal{H}^*$.*

(ii) *$\mathbb{E}[\langle F_n, e_i \rangle_{\mathcal{H}}^k] \rightarrow \mathbb{E}[\langle Z, e_i \rangle_{\mathcal{H}}^k]$, and for all orthonormal bases $\{e_i\}_{i \in \mathbb{N}}$.*

Proof: By the Riesz representation theorem, every $x^* \in \mathcal{H}^*$ can be identified with an element $x \in \mathcal{H}$ so that $x^*(h) = \langle h, x \rangle_{\mathcal{H}}$ for all $h \in \mathcal{H}$. Define the quadratic and quartic forms

$$Q_n(u) \doteq \mathbb{E}[\langle F_n, u \rangle_{\mathcal{H}}^2], \quad R_n(u) \doteq \mathbb{E}[\langle F_n, u \rangle_{\mathcal{H}}^4],$$

and similarly Q, R with F_n replaced by Z .

(i) \Rightarrow (ii): Fix an orthonormal basis $\{e_i\}_{i \in \mathbb{N}}$ and choose $x_1^* = \dots = x_k^* = \langle \cdot, e_i \rangle_{\mathcal{H}}$. Then by (i),

$$\mathbb{E}[\langle F_n, e_i \rangle_{\mathcal{H}}^k] \rightarrow \mathbb{E}[\langle Z, e_i \rangle_{\mathcal{H}}^k], \quad k = 2, 4,$$

which is exactly (ii).

(ii) \Rightarrow (i): Identify as above $x_k^* = x_k \in \mathcal{H}, k = 2, 4$. First, Condition (ii) for all bases means that for any unit vector $u \in \mathcal{H}$ we can choose a basis containing u , and hence

$$Q_n(u) \rightarrow Q(u), \quad R_n(u) \rightarrow R(u). \quad (4.8)$$

Furthermore, any element $u \in \mathcal{H}$ can be written as a scalar times a unit vector $u = \|u\|_{\mathcal{H}} \frac{u}{\|u\|_{\mathcal{H}}}$. Then the same convergences (4.8) hold for any $u \in \mathcal{H}$.

For $k = 2$, for any $x_1^*, x_2^* \in \mathcal{H}^*$,

$$\mathbb{E}x_1^*(F_n)x_2^*(F_n) = \mathbb{E}\langle F_n, x_1 \rangle_{\mathcal{H}}\langle F_n, x_2 \rangle_{\mathcal{H}} = \frac{1}{4}(Q_n(x_1 + x_2) - Q_n(x_1 - x_2)).$$

Since $Q_n \rightarrow Q$ pointwise, we obtain

$$\mathbb{E}\langle F_n, x_1 \rangle_{\mathcal{H}}\langle F_n, x_2 \rangle_{\mathcal{H}} \rightarrow \mathbb{E}\langle Z, x_1 \rangle_{\mathcal{H}}\langle Z, x_2 \rangle_{\mathcal{H}} = \mathbb{E}x_1^*(Z)x_2^*(Z).$$

For $k = 4$, let $x_i^* \in \mathcal{H}^*, i = 1, 2, 3, 4$ and denote their respective representations $x_i \in \mathcal{H}, i = 1, 2, 3, 4$. Then,

$$M_n^{(4)}(x_1, x_2, x_3, x_4) \doteq \mathbb{E}\left[\prod_{j=1}^4 \langle F_n, x_j \rangle_{\mathcal{H}}\right].$$

Using the general polarization identity as in equation (3) of [19], we obtain

$$M_n^{(4)}(x_1, x_2, x_3, x_4) = \frac{1}{4!} \sum_{r=1}^4 (-1)^{4-r} \sum_{\substack{S \subset \{1,2,3,4\} \\ |S|=r}} R_n\left(\sum_{j \in S} x_j\right).$$

Since $R_n(\cdot) \rightarrow R(\cdot)$ pointwise on \mathcal{H} , we may pass to the limit to get

$$M_n^{(4)}(x_1, x_2, x_3, x_4) \rightarrow \frac{1}{4!} \sum_{r=1}^4 (-1)^{4-r} \sum_{\substack{S \subset \{1,2,3,4\} \\ |S|=r}} R\left(\sum_{j \in S} x_j\right) = \mathbb{E}\left[\prod_{j=1}^4 \langle Z, x_j \rangle_{\mathcal{H}}\right],$$

which concludes the proof. \square

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