Unbiased simulation of Asian options

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Abstract

We provide an extension of the unbiased simulation method for SDEs developed in Henry-Labordère et al. [Ann Appl Probab. 27:6 (2017) 1-37] to a class of path-dependent dynamics, pertaining for Asian options. In our setting, both the payoff and the SDE's coefficients depend on the (weighted) average of the process or, more precisely, on the integral of the solution to the SDE against a continuous function with bounded variations. In particular, this applies to the numerical resolution of the class of path-dependent PDEs whose regularity, in the sens of Dupire, is studied in Bouchard and Tan [Ann. I.H.P., to appear].

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1 Introduction

Let $d \geq 1$, T > 0 and W be a d-dimensional Brownian motion, $\mu : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{S}^d$ be the drift and diffusion coefficients, where \mathbb{S}^d denotes the collection of all $d \times d$ dimensional matrices. We consider the process X defined as the solution to the path-dependent SDE

$$X_{t} = x_{0} + \int_{0}^{t} \mu(s, X_{s}, I_{s}) ds + \int_{0}^{t} \sigma(s, X_{s}, I_{s}) dW_{s}, \text{ with } I_{t} := \int_{0}^{t} X_{s} dA_{s}, \quad (1)$$

where W is a d-dimensional Brownian motion and $A:[0,T] \longrightarrow \mathbb{R}$ is a (deterministic) continuous function with finite variations. Our main objective is to provide a Monte-Carlo simulation method to estimate the expected value

$$V_0 := \mathbb{E}[g(X_T, I_T)], \tag{2}$$

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for some function $g: \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}$.

The numerical simulation of the SDE (1) is usually based on a discrete time approximation scheme (e.g. the Euler scheme), so that the corresponding Monte Carlo estimator for V_0 in (2) suffers from a discretization error, see e.g. the seminal work of Talay and Tubaro [21], as well as Kloeden and Platen [18] and Graham and Talay [14] for an overview of various discrete time approximation approaches.

For a special class of SDEs, including one-dimensional homogeneous SDEs with constant volatility coefficient, exact simulation methods of the marginal distribution of X_T are available, see e.g. Beskos and Roberts [3], Beskos, Papaspiliopoulos and Roberts [4], Jourdain and Sbai [17], etc.

One can also get rid of the discretization error in the Monte Carlo estimator by appealing to the so-called unbiased simulation methods. One possibility is to apply a random level in the multilevel Monte Carlo method of Giles [13] in the case where (X, A) is an Itô process, see e.g. Rhee and Glynn [19]. In the uniformly elliptic case, i.e. $A \equiv 0$ and $g(X_T, I_T) = g(X_T)$, another one consists in multiplying the terminal payoff by a suitable random variable \mathcal{M}_T , also called Malliavin weight, so as to compensate exactly the expectation biais induced by the use of an Euler scheme \widehat{X} in place of X. The definition of \mathcal{M}_T is based on the parametrix method for PDEs, which ensures that

$$V_0 = \mathbb{E}[g(\widehat{X}_T)\mathcal{M}_T]. \tag{3}$$

Since \widehat{X} and \mathcal{M}_T can be simulated exactly, this provides an unbiased MC estimator for V_0 , see e.g. Bally and Kohatsu-Higa [2], Andersson and Kohatsu-Higa [1], Chen, Frikha and Li [6], as well as Henry-Labordère, Tan and Touzi [16] and Doumbia, Oudjane and Warin [8].

In this paper, we extend the approach of [8, 16] to the path-dependent setting (1)-(2). Even when A is absolutely continuous, e.g. $A_t = t$, $(X_t, I_t)_{t \geq 0}$ is a degenerate diffusion process and the estimator proposed in [16] cannot apply. The reason is that it relies on the Markovian representation of V_0 as a function of (X_0, I_0) . Rewriting V_0 as the time 0 value of a path-dependent functional, and building on the approach developed in [5] to study the (Dupire's) regularity of the path-depend PDE associated to (2), we consider a pretty general situation in which A does not need to be absolutely continuous. We are able to find the weight function \mathcal{M}_T associated to the Dupire's vertical derivatives of the associated value function, and bound the second moment of the estimator so that the Monte Carlo estimation error can be controlled in the usual way. This requires a structural condition relating the Hölder regularity of the coefficients with the regularity of A in the spirit of [5], see Assumption 3.1 and condition (33) below.

The paper is organized as follows. In Section 2, we explain how to construct our unbiased representation for V_0 in (2), under general, abstract, integrability conditions. Then, in Section 3, we provide sufficient conditions that ensure the (square) integrability of our estimator. Finally, in Section 4, we provide some numerical examples in

the context of the pricing of Asian options. Technical lemmas are left to Appendix A.

2 Unbiased estimator for path-dependent SDEs

Let us consider the path-dependent SDE in (1)

$$dX_t = \mu(t, X_t, I_t)dt + \sigma(t, X_t, I_t)dW_t, \text{ with } I_t := \int_0^t X_s dA_s, \tag{4}$$

where $A:[0,T] \longrightarrow \mathbb{R}$ is a given continuous function with finite variations, and W is a Brownian motion. Our objective is to estimate

$$V_0 := \mathbb{E}[g(X_T, I_T)], \tag{5}$$

in which $(\mu, \sigma, g) : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}^d \times \mathbb{S}^d \times \mathbb{R}$ are measurable.

We assume that the volatility function σ is non-degenerate, and that the above path-dependent SDE (4) has a unique weak solution. See Assumption 2.1 below.

Assumption 2.1. (i) The coefficient functions μ and σ are both continuous and have at most linear growth, the volatility coefficient σ is non-degenerate in the sense that there exists some constant $\varepsilon_0 > 0$ such that

$$\sigma \sigma^{\top}(t, x, \bar{x}) \geq \varepsilon_0 I_d, \text{ for all } (t, x, \bar{x}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d.$$

Further, the path-dependent SDE (4) has a unique weak solution, for any initial condition $(t, x, \bar{x}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$.

(ii) The function g is continuous and has at most polynomial growth. Moreover, the $map\ u: [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}$ defined by

$$u(t, x, \bar{x}) := \mathbb{E}\left[g(X_T, I_T) \middle| X_t = x, I_t = \bar{x}\right]$$
(6)

admits first and second order derivatives $D_x u(t, x, \bar{x})$ and $D_{xx}^2 u(t, x, \bar{x})$ with respect to its second argument, that are continuous and have at most polynomial growth on $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d$.

Remark 2.2. The existence of the derivatives $D_x u(t, x, \bar{x})$ and $D_{xx}^2 u(t, x, \bar{x})$ can be obtained by working along the lines of [5]. However, they will not enjoy a uniform polynomial growth unless g is smooth enough. This condition can however easily be removed by adding an approximation argument in our proofs, see Remark 2.6 and Corollary 3.8 below. Importantly, no differentiability assumption is made with respect to the second space variable \bar{x} (which would require all coefficients to be smooth).

2.1 The representation formula

Let us first explain how to construct the representation formula for the expected value V_0 of the functional of (X_T, I_T) .

Let $\rho: \mathbb{R} \longrightarrow \mathbb{R}_+$ be the density function of some distribution supported on \mathbb{R}_+ , and $F: \mathbb{R} \longrightarrow [0,1]$ be the corresponding c.d.f. (cumulative distribution function). We assume that $\rho > 0$ on [0,T] and that F(T) < 1. We consider a sequence $(\tau_k)_{k \geq 1}$ of i.i.d. random variables following the distribution with c.d.f. F, independent of W. We then define the random time grid $(T_k)_{k \geq 1}$ by $T_0 := 0$ and

$$T_k := (T_{k-1} + \tau_k) \wedge T. \quad k \ge 1, \tag{7}$$

Set

$$\Delta T_{k+1} := T_{k+1} - T_k, \quad k \ge 0, \quad N_T := \max\{k \ge 0 : T_k < T\}$$

so that N_T denotes the number of points of the grid $(T_k)_{k\geq 0}$ belonging to [0,T).

We then define the process $(\widehat{X}, \widehat{I})$ as the Euler scheme of (4) on the random time grid $(T_k)_{k\geq 0}$, that is

$$\widehat{X}_0 := x_0, \quad \widehat{I}_0 = 0,$$

and then, for each $k = 0, 1, \dots, N_T$,

$$\widehat{X}_t := \widehat{X}_{T_k} + \mu(T_k, \widehat{X}_{T_k}, \widehat{I}_{T_k})(t - T_k) + \sigma(T_k, \widehat{X}_{T_k}, \widehat{I}_{T_k})(W_t - W_{T_k}), \quad t \in (T_k, T_{k+1}],$$

and

$$\widehat{I}_t := \widehat{I}_{T_k} + \int_{T_k}^t \widehat{X}_s dA_s, \quad t \in (T_k, T_{k+1}].$$
 (8)

Equivalently, given $(\widehat{X}_{T_k}, \widehat{I}_{T_k})$, the couple $(\widehat{X}_{T_{k+1}}, \widehat{I}_{T_{k+1}})$ is defined by

$$\widehat{X}_{T_{k+1}} := \widehat{X}_{T_k} + \mu(T_k, \widehat{X}_{T_k}, \widehat{I}_{T_k}) (T_{k+1} - T_k) + \sigma(T_k, \widehat{X}_{T_k}, \widehat{I}_{T_k}) (W_{T_{k+1}} - W_{T_k}), \quad (9)$$

and

$$\widehat{I}_{T_{k+1}} = \widehat{I}_{T_k} + \widehat{X}_{T_k} \left(A_{T_{k+1}} - A_{T_k} \right) + \mu(T_k, \widehat{X}_{T_k}, \widehat{I}_{T_k}) \int_{T_k}^{T_{k+1}} (A_{T_{k+1}} - A_s) ds
+ \sigma(T_k, \widehat{X}_{T_k}, \widehat{I}_{T_k}) \int_{T_k}^{T_{k+1}} (A_{T_{k+1}} - A_s) dW_s \quad (10)$$

$$= \widehat{I}_{T_k} + \widehat{X}_{T_{k+1}} \left(A_{T_{k+1}} - A_{T_k} \right) - \mu(T_k, \widehat{X}_{T_k}, \widehat{I}_{T_k}) \int_{T_k}^{T_{k+1}} (A_s - A_{T_k}) ds
- \sigma(T_k, \widehat{X}_{T_k}, \widehat{I}_{T_k}) \int_{T_k}^{T_{k+1}} (A_s - A_{T_k}) dW_s. \quad (11)$$

In particular, given (T_k, T_{k+1}) , the conditional distribution of the random vector

$$\left(W_{T_{k+1}} - W_{T_k}, \int_{T_k}^{T_{k+1}} (A_s - A_{T_k}) dW_s\right)$$

is a Gaussian vector, so that the sequence $(\widehat{X}_{T_k}, \widehat{I}_{T_k})_{k\geq 0}$ can be simulated explicitly.

From now on let us define, for each $k \geq 0$,

$$\mu_{T_k} := \mu \big(T_k, \widehat{X}_{T_k}, \widehat{I}_{T_k} \big), \quad \sigma_{T_k} := \sigma \big(T_k, \widehat{X}_{T_k}, \widehat{I}_{T_k} \big), \quad a_{T_k} := \sigma_{T_k} \sigma_{T_k}^\top,$$

and

$$D_x u_{T_k} := D_x u(T_k, \widehat{X}_{T_k}, \widehat{I}_{T_k}), \quad D_{xx}^2 u_{T_k} := D_{xx}^2 u(T_k, \widehat{X}_{T_k}, \widehat{I}_{T_k}),$$

as well as

$$\overline{A}_{k,k+1} := \frac{1}{\Delta T_{k+1}} \int_{T_k}^{T_{k+1}} A_r dr, \quad m_{k,k+1}^1 := \frac{1}{\Delta T_{k+1}} \int_{T_k}^{T_{k+1}} (A_r - A_{T_k}) dr,$$

$$m_{k,k+1}^2 := \frac{1}{\Delta T_{k+1}} \int_{T_k}^{T_{k+1}} \left(A_r - \overline{A}_{k,k+1} \right)^2 dr,$$

and

$$\tilde{m}_{k,k+1}^2 := \frac{1}{\Delta T_{k+1}} \int_{T_k}^{T_{k+1}} (A_r - A_{T_k})^2 dr.$$

We also set, for each $k \geq 1$,

$$\widehat{\mathcal{W}}_k^1 := \left(\mu_{T_k} - \mu_{T_{k-1}}\right) \cdot M_{k+1},\tag{12}$$

and

$$\widehat{\mathcal{W}}_{k}^{2} := \frac{1}{2} \text{Tr} \left[\left(a_{T_{k}} - a_{T_{k-1}} \right) \left(M_{k+1} M_{k+1}^{\top} - \frac{1}{\Delta T_{k+1}} \frac{\tilde{m}_{k,k+1}^{2} a_{T_{k}}^{-1}}{m_{k,k+1}^{2} a_{T_{k}}^{-1}} \right) \right], \tag{13}$$

where

$$M_{k+1} := \frac{1}{\Delta T_{k+1} m_{k,k+1}^2} \left(\sigma_{T_k}^{\top} \right)^{-1} \int_{T_k}^{T_{k+1}} \left(\tilde{m}_{k,k+1}^2 - m_{k,k+1}^1 \left(A_s - A_{T_k} \right) \right) dW_s. \tag{14}$$

Finally, for each $n \geq 1$, using $D_x u$ and $D_{xx}^2 u$ defined in Assumption 2.1 and the fact that $\Delta T_{N_T+1} = T - T_{N_T}$ by construction, we introduce

$$\widehat{\psi}_{n} := \mathbf{1}_{\{N_{T} \leq n-1\}} \frac{g(\widehat{X}_{T}, \widehat{I}_{T}) - g(\widehat{X}_{T_{N_{T}}}, \widehat{I}_{T_{N_{T}}}) \mathbf{1}_{\{N_{T} > 0\}}}{1 - F(\Delta T_{N_{T}+1})} \prod_{k=1}^{N_{T}} \left(\frac{\widehat{W}_{k}^{1} + \widehat{W}_{k}^{2}}{\rho(\Delta T_{k})}\right) \\
+ \mathbf{1}_{\{N_{T} \geq n\}} \frac{\left(b_{T_{n}} - b_{T_{n-1}}\right) \cdot D_{x} u_{T_{n}} + \frac{1}{2} \text{Tr}\left[\left(a_{T_{n}} - a_{T_{n-1}}\right) D_{xx}^{2} u_{T_{n}}\right]}{\rho(\Delta T_{n})} \prod_{k=1}^{n-1} \left(\frac{\widehat{W}_{k}^{1} + \widehat{W}_{k}^{2}}{\rho(\Delta T_{k})}\right) \tag{15}$$

with the convention that the product over an empty set equals 1, and then define the representation random variable $\hat{\psi}$ by

$$\widehat{\psi} := \frac{g(\widehat{X}_T, \widehat{I}_T) - g(\widehat{X}_{T_{N_T}}, \widehat{I}_{T_{N_T}}) \mathbf{1}_{\{N_T > 0\}}}{1 - F(\Delta T_{N_T + 1})} \prod_{k=1}^{N_T} \frac{\widehat{\mathcal{W}}_k^1 + \widehat{\mathcal{W}}_k^2}{\rho(\Delta T_k)}.$$
(16)

Recalling that $T_{N_T} < T$ and $T_{N_T+1} = T$ so that $\Delta W_{N_T+1} = W_T - W_{T_{N_T}}$, we introduce $(\widetilde{X}_T, \widetilde{I}_T)$ by reversing the sign of the Brownian increment on $[T_{N_T}, T]$ only, i.e.

$$\widetilde{X}_T := \widehat{X}_{T_{N_T}} + \mu(T_{N_T}, \widehat{X}_{T_{N_T}}, \widehat{I}_{T_{N_T}}) (T - T_{N_T}) - \sigma(T_{N_T}, \widehat{X}_{T_{N_T}}, \widehat{I}_{T_{N_T}}) (W_T - W_{T_{N_T}}),$$

and

$$\begin{split} \widetilde{I}_T := \widehat{I}_{T_{N_T}} + \widetilde{X}_T \big(A_T - A_{T_{N_T}} \big) - \mu(T_{N_T}, \widehat{X}_{T_{N_T}}, \widehat{I}_{T_{N_T}}) \int_{T_{N_T}}^T \big(A_s - A_{T_{N_T}} \big) ds \\ + \sigma(T_{N_T}, \widehat{X}_{T_{N_T}}, \widehat{I}_{T_{N_T}}) \int_{T_{N_T}}^T \big(A_s - A_{T_{N_T}} \big) dW_s. \end{split}$$

We then define

$$\begin{split} \widetilde{\psi}_{n} &:= \mathbf{1}_{\{N_{T} \leq n-1\}} \, \frac{g\big(\widetilde{X}_{T}, \widetilde{I}_{T}\big) - g\big(\widehat{X}_{T_{N_{T}}}, \widehat{I}_{T_{N_{T}}}\big) \mathbf{1}_{\{N_{T} > 0\}} \Big(\prod_{k=1}^{N_{T}-1} \frac{\widehat{\mathcal{W}}_{k}^{1} + \widehat{\mathcal{W}}_{k}^{2}}{\rho(\Delta T_{k})} \Big) \frac{\widehat{\mathcal{W}}_{N_{T}}^{2} - \widehat{\mathcal{W}}_{N_{T}}^{1}}{\rho(\Delta T_{N_{T}})} \\ &+ \mathbf{1}_{\{N_{T} \geq n\}} \, \frac{\big(b_{T_{n}} - b_{T_{n-1}}\big) \cdot D_{x} u_{T_{n}} + \frac{1}{2} \mathrm{Tr} \big[\big(a_{T_{n}} - a_{T_{n-1}}\big) D_{xx}^{2} u_{T_{n}} \big]}{\rho(\Delta T_{n})} \, \prod_{k=1}^{n-1} \Big(\, \frac{\widehat{\mathcal{W}}_{k}^{1} + \widehat{\mathcal{W}}_{k}^{2}}{\rho(\Delta T_{k})} \Big), \end{split}$$

as well as

$$\widetilde{\psi} := \frac{g(\widetilde{X}_T, \widetilde{I}_T) - g(\widehat{X}_{T_{N_T}}, \widehat{I}_{T_{N_T}}) \mathbf{1}_{\{N_T > 0\}}}{1 - F(\Delta T_{N_T + 1})} \left(\prod_{k=1}^{N_T - 1} \frac{\widehat{\mathcal{W}}_k^1 + \widehat{\mathcal{W}}_k^2}{\rho(\Delta T_k)} \right) \frac{\widehat{\mathcal{W}}_{N_T}^2 - \widehat{\mathcal{W}}_{N_T}^1}{\rho(\Delta T_{N_T})}.$$

Finally, let

$$\psi_n := \frac{\widehat{\psi}_n + \widetilde{\psi}_n}{2}, \quad n \ge 1, \quad \text{and} \quad \psi := \frac{\widehat{\psi} + \widetilde{\psi}}{2}.$$
(17)

Theorem 2.3. Let Assumption 2.1 hold true. Suppose in addition that ψ_n (resp. $\widehat{\psi}_n$, $\widetilde{\psi}_n$) is integrable for each $n \geq 1$, then

$$\mathbb{E}[\psi_n] = V_0 \quad (resp. \ \mathbb{E}[\widehat{\psi}_n] = V_0, \ \mathbb{E}[\widetilde{\psi}_n] = V_0), \quad for \ each \ n \geq 1.$$

Further assume that the sequence $(\psi_n)_{n\geq 1}$ (resp. $(\widehat{\psi}_n)_{n\geq 1}$, $(\widetilde{\psi}_n)_{n\geq 1}$) is uniformly integrable, then the random variable ψ (resp. $\widehat{\psi}$, $\widetilde{\psi}$) is also integrable and

$$\mathbb{E}[\psi] = V_0 \quad (resp. \ \mathbb{E}[\widehat{\psi}] = V_0, \ \mathbb{E}[\widetilde{\psi}] = V_0). \tag{18}$$

- Remark 2.4. (i) Note that the random variables $(\psi_n)_{n\geq 1}$ depend on the value function u which is exactly the value we aim to estimate. This sequence is only used to approximate ψ in the proof of (18). The random variable ψ does not depend on u and can be simulated exactly, so that it provides an unbiased square integrable Monte Carlo estimator for V_0 by (18), as soon as it is square integrable, see Section 3.
- (ii) By the symmetry of the distribution of the Brownian motion, the random variable $\widetilde{\psi}$ can be considered as a form of an antithetic variable of $\widehat{\psi}$. They have the same distribution. The reason for considering ψ rather than $\widehat{\psi}$ is that its integrability is easier to handle, see more discussions in Remark 3.4.

Before turning to the proof of (18), let us also discuss Assumption 2.1.

Remark 2.5. The non-degeneracy condition on σ in Assumption 2.1 is crucial, see in particular (14).

Remark 2.6. The existence, continuity and growth condition on D_xu and D_{xx}^2u in Assumption 2.1 is only used to define the random variables $(\widehat{\psi}_n)_{n\geq 1}$ in (15).

- (i) When $A_t = t$ for all $t \in [0,T]$, and (μ,σ) are bounded and continuous, and satisfy a technical Hölder condition, it is proved in Francesco and Pascucci [12] that $D_x u$ and $D_{xx}^2 u$ exist and are continuous. The extension to a class of general continuous functions A with finite variations was obtained by Bouchard and Tan [5]. However, a uniform polynomial growth can not be expected unless the coefficients (including the terminal payoff function) are smooth.
- (ii) The formulation of $\widehat{\psi}$ does not depend on $(D_x u, D_{xx}^2 u)$. In particular, if there exists a sequence of coefficients $(\mu^m, \sigma^m, g^m)_{m\geq 1}$ such that each corresponding value function u^m satisfies Assumption 2.1, and $(\mu^m, \sigma^m, g^m) \longrightarrow (\mu, \sigma, g)$ pointwisely, one can easily deduce that the corresponding estimators $\widehat{\psi}^m \longrightarrow \widehat{\psi}$ pointwisely. If one can check in addition that $(\widehat{\psi}^m)_{m\geq 1}$ is uniformly integrable, then (18) still holds, see Corollary 3.8 below.

2.2 Proof of Theorem 2.3

The proof of Theorem 2.3 is based on the path-dependent PDE satisfied by the value function u associated to V_0 in (6). Namely, let $\mathbb{D}([0,T])$ denote de Skorokhod space of all \mathbb{R}^d -valued càdlàg paths on [0,T] and define

$$\bar{u}(t,\omega) := \mathbb{E}\left[g(X_T, I_T) \middle| X_{t\wedge \cdot} = \omega_{t\wedge \cdot}\right], (t,\omega) \in [0,T] \times \mathbb{D}([0,T]). \tag{19}$$

Then,

$$\bar{u}(t,\omega) = u(t,\omega_t,I_t(\omega)), \text{ with } I_t(\omega) := \int_0^t \omega_s dA_s, \text{ for all } (t,\omega) \in [0,T] \times \mathbb{D}([0,T]).$$

Let us also recall the definition of Dupire's horizontal and vertical derivatives of the path-dependent functional $\bar{u}:[0,T]\times\mathbb{D}([0,T])\longrightarrow\mathbb{R}$, see [9]:

$$\partial_t \bar{u}(t,\omega) := \lim_{\varepsilon \searrow 0} \frac{\bar{u}(t+\varepsilon,\omega_{t\wedge \cdot}) - u(t,\omega)}{\varepsilon},$$

and

$$\partial_{\omega}\bar{u}(t,\omega) := \lim_{y \to 0} \frac{\bar{u}(t,\omega + y\mathbf{1}_{\{\cdot \ge t\}}) - u(t,\omega)}{y}.$$
 (20)

Similarly, one can defined the second order vertical derivative $\partial_{\omega\omega}^2 \bar{u}$, whenever these quantities are well-defined, which is the case under Assumption 2.1 since

$$\partial_{\omega}\bar{u}(t,\omega) = D_x u(t,x,\bar{x})$$
 and $\partial^2_{\omega\omega}\bar{u}(t,\omega) = D^2_{xx} u(t,x,\bar{x}),$

with

$$(x,\bar{x}) := \left(\omega_t, \int_0^t \omega_s dA_s\right).$$

Then, it follows from Cont and Fournié [7] that \bar{u} satisfies the linear path-dependent PDE

$$\partial_t \bar{u}(t,\omega) + \mu(t,\omega_t, I_t(\omega)) \cdot \partial_\omega \bar{u}(t,\omega) + \frac{1}{2} \text{Tr} \left[\sigma \sigma^\top(t,\omega_t, I_t(\omega)) \partial^2_{\omega\omega} \bar{u}(t,\omega) \right] = 0, \quad (21)$$

at any $(t, \omega) \in [0, T) \times \mathbb{D}([0, T])$, with terminal condition $\bar{u}(T, \omega) = g(\omega_T, I_T(\omega)), \omega \in \mathbb{D}([0, T])$.

For ease off notations, let us also define for all s < t

$$\overline{A}_{s,t} := \frac{1}{t-s} \int_s^t A_r dr, \quad m_{s,t}^1 := \frac{1}{t-s} \int_s^t (A_r - A_s) dr,$$

and

$$m_{s,t}^2 := \frac{1}{t-s} \int_0^t \left(A_r - \overline{A}_{s,t} \right)^2 dr, \quad \tilde{m}_{s,t}^2 := \frac{1}{t-s} \int_s^t (A_r - A_s)^2 dr.$$

Proof of Theorem 2.3. By symmetry of the Brownian motion, $\widetilde{\psi}_n$ and $\widetilde{\psi}$ have the same distribution as $\widehat{\psi}_n$ and $\widehat{\psi}$. We therefore only consider $\widehat{\psi}_n$ and $\widehat{\psi}$ in the following.

(i) Let us first consider the case n=1. Recall that $V_0 = \bar{u}(0,\omega)$ where \bar{u} satisfies the PPDE (21) and $\omega_0 = x_0$. Given fixed constants $b_0 \in \mathbb{R}^d$ and $\sigma_0 \in \mathbb{S}^d$, we can rewrite the PPDE (21) as

$$\partial_t \bar{u}(t,\omega) + b_0 \cdot \partial_\omega \bar{u}(t,\omega) + \frac{1}{2} \text{Tr} \left[\sigma_0 \sigma_0^\top \partial_{\omega\omega}^2 \bar{u}(t,\omega) \right] + \bar{f}(t,\omega) = 0,$$

with

$$\bar{f}(t,\omega) := f(t,\omega_t, I_t(\omega)),$$

and

$$f(t, x, \bar{x}) := \left(\mu(t, x, \bar{x}) - b_0\right) \cdot D_x u(t, x, \bar{x}) + \frac{1}{2} \operatorname{Tr} \left[\left(\sigma \sigma^\top(t, x, \bar{x}) - \sigma_0 \sigma_0^\top\right) D_x^2 u(t, x, \bar{x}) \right]$$

Then, by the Feynman-Kac's formula in [7],

$$\bar{u}(t,\omega) = u(t,\omega_t, I_t(\omega)) = \mathbb{E}\Big[g\big(\overline{X}_T^{t,\omega}, I_T(\overline{X}^{t,\omega})\big) + \int_t^T f\big(\overline{X}_s^{t,\omega}, I_s(\overline{X}^{t,\omega})\big)ds\Big], \quad (22)$$

where

$$\overline{X}_s^{t,\omega} = \omega_s \mathbf{1}_{\{s < t\}} + \left(\omega_t + b_0(s-t) + \sigma_0(W_s - W_t)\right) \mathbf{1}_{\{s \ge t\}}.$$

Notice that b_0 and σ_0 can be chosen arbitrary. By letting t=0, $\omega_0=x_0$, $b_0=\mu(0,x_0,0)$ and $\sigma_0=\sigma(0,x_0,0)$, and recalling that $T_1=\tau_1\wedge T$ where τ_1 is a random variable independent of W with density function ρ and distribution function F, it follows that

$$V_{0} = \mathbb{E}\left[g(\overline{X}_{T}^{0,x_{0}}, I_{T}(\overline{X}^{0,x_{0}})) + \int_{t}^{T} f(s, \overline{X}_{s}^{0,x_{0}}, I_{s}(\overline{X}^{0,x_{0}})) ds\right]$$

$$= \mathbb{E}\left[\frac{g(\widehat{X}_{T}, I_{T}(\widehat{X}))}{1 - F(T)} \mathbf{1}_{\{T_{1} \geq T\}} + \frac{f(T_{1}, \widehat{X}_{T_{1}}, I_{T_{1}}(\widehat{X}))}{\rho(T_{1})} \mathbf{1}_{\{T_{1} < T\}}\right] = \mathbb{E}\left[\widehat{\psi}_{1}\right].$$

(ii) Next, define for s < t and $\omega \in \mathbb{D}([0,T])$

$$M_{s,t}^{\omega} := \frac{1}{(t-s)m_{s,t}^2} (\sigma_s^{\omega \top})^{-1} \int_s^t \left(\tilde{m}_{s,t}^2 - m_{s,t}^1 (A_u - A_s) \right) dW_u$$
 (23)

with

$$\sigma_s^{\omega} := \sigma(s, \omega_s, I_s(\omega)).$$

Also define

$$f_{s,t}^{\omega}(x,\bar{x}) := (\mu(t,x,\bar{x}) - \mu(s,\omega_s,I_s(\omega)) \cdot D_x u(t,x,\bar{x})$$

$$+ \frac{1}{2} \text{Tr} \left[\left(\sigma \sigma^{\top}(t,x,\bar{x}) - \sigma \sigma^{\top}(s,\omega_s,I_s(\omega)) D_x^2 u(t,x,\bar{x}) \right].$$

In view of (22) and Lemma A.1 (see also (8)), one has, for any n,

$$\partial_{\omega} \bar{u}(t,\omega) = D_{x} u(t,\omega_{t}, I_{t}(\omega))$$

$$= \mathbb{E} \Big[g \big(\overline{X}_{T}^{t,\omega}, I_{T}(\overline{X}^{t,\omega}) \big) M_{t,T}^{\omega} + \int_{t}^{T} f_{t,s}^{\omega} \big(\overline{X}_{s}^{t,\omega}, I_{s}(\overline{X}^{t,\omega}) \big) M_{t,s}^{\omega} ds \Big]$$

$$= \mathbb{E} \Big[\Big(g \big(\overline{X}_{T}^{t,\omega}, I_{T}(\overline{X}^{t,\omega}) \big) - g(\omega_{t}, I_{t}(\omega)) \Big) M_{t,T}^{\omega} + \int_{t}^{T} f_{t,s}^{\omega} \big(\overline{X}_{s}^{t,\omega}, I_{s}(\overline{X}^{t,\omega}) \big) M_{t,s}^{\omega} ds \Big]$$

$$= \mathbb{E} \Big[\frac{g \big(\overline{X}_{T}^{t,\omega}, I_{T}(\overline{X}^{t,\omega}) \big) - g(\omega_{t}, I_{t}(\omega))}{1 - F(T - t)} M_{t,T}^{\omega} \mathbf{1}_{\{t + \tau_{n} \geq T\}} + \frac{f_{t,t + \tau_{n}}^{\omega} \big(\overline{X}_{t + \tau_{n}}^{t,\omega}, I_{t + \tau_{n}}(\overline{X}^{t,\omega}) \big)}{\rho(\tau_{n})} M_{t,t + \tau_{n}}^{\omega} \mathbf{1}_{\{t + \tau_{n} < T\}} \Big], \tag{24}$$

in which we used that $\mathbb{E}[M_{t,T}^{\omega}] = 0$ and that τ_n is independent of W and has ρ for density. Similarly, for any n,

$$\partial_{\omega\omega}^{2} \bar{u}(t,\omega) = D_{xx}^{2} u(t,\omega_{t}, I_{t}(\omega))$$

$$= \mathbb{E}\Big[\Big(g(\overline{X}_{T}^{t,\omega}, I_{T}(\overline{X}^{t,\omega})) - g(\omega_{t}, I_{t}(\omega))\Big)\Big(M_{t,T}^{\omega} M_{t,T}^{\omega\top} - \frac{1}{T-t} \frac{\tilde{m}_{t,T}^{2}}{m_{t,T}^{2}} (\sigma_{t}^{\omega} \sigma_{t}^{\omega\top}))^{-1}\Big)$$

$$+ \int_{t}^{T} f_{t,s}^{\omega} (\overline{X}_{s}^{t,\omega}, I_{s}(\overline{X}^{t,\omega}))\Big(M_{t,s}^{\omega} M_{t,s}^{\omega\top} - \frac{1}{t-s} \frac{\tilde{m}_{t,s}^{2}}{m_{s,t}^{2}} (\sigma_{t}^{\omega} \sigma_{t}^{\omega\top})^{-1}\Big) ds\Big]$$

$$= \mathbb{E}\Big[\frac{g(\overline{X}_{T}^{t,\omega}, I_{T}(\overline{X}^{t,\omega})) - g(\omega_{t}, I_{t}(\omega))}{1-F(T-t)} \Big(M_{t,T}^{\omega} M_{t,T}^{\omega\top} - \frac{1}{T-t} \frac{\tilde{m}_{t,T}^{2}}{m_{t,T}^{2}} (\sigma_{t}^{\omega} \sigma_{t}^{\omega\top})^{-1}\Big) \mathbf{1}_{\{t+\tau_{n} \geq T\}}$$

$$+ \frac{f_{t,t+\tau_{n}}^{\omega} (\overline{X}_{t+\tau_{n}}^{t,\omega}, I_{t+\tau_{n}}(\overline{X}^{t,\omega}))}{\rho(\tau_{n})} \Big(M_{t,t+\tau_{n}}^{\omega} M_{t,t+\tau_{n}}^{\omega\top} - \frac{1}{\tau_{n}} \frac{\tilde{m}_{t,t+\tau_{n}}^{2}}{m_{t,t+\tau_{n}}^{2}} (\sigma_{t}^{\omega} \sigma_{t}^{\omega\top})^{-1}\Big) \mathbf{1}_{\{t+\tau_{n} < T\}}\Big].$$

$$(25)$$

(iii) Take now $n \geq 2$ and observe that $\widehat{\psi}_{n-1}$ involves $\partial_{\omega} \bar{u}$ and $\partial^2_{\omega\omega} \bar{u}$ (or equivalently $D_x u$ and $D^2_{xx} u$) through the definition of f. By setting $t = T_{n-1}$, $\omega_{T_{n-1} \wedge \cdot \cdot} = \widehat{X}_{T_{n-1} \wedge \cdot}$ in (24) and (25), and then plugging the representation formula of $\partial_{\omega} \bar{u}(T_{n-1}, \widehat{X})$ and $\partial^2_{\omega\omega} \bar{u}(T_{n-1}, \widehat{X})$ into $\widehat{\psi}_{n-1}$ in (15), it follows that

$$\mathbb{E}[\widehat{\psi}_{n-1}] = \mathbb{E}[\widehat{\psi}_n].$$

Since $V_0 = \mathbb{E}[\widehat{\psi}_1]$ by (i), it follows that $V_0 = \mathbb{E}[\widehat{\psi}_n]$ for all $n \geq 1$ by induction.

(iv) Finally, we notice that

$$\lim_{n \to \infty} \widehat{\psi}_n = \widehat{\psi}, \text{ a.s.}$$

Since we assumed that $(\widehat{\psi}_n)_{n\geq 1}$ is uniformly integrable, it follows that $\widehat{\psi}$ is integrable and satisfies

$$V_0 = \lim_{n \to \infty} \mathbb{E}[\widehat{\psi}_n] = \mathbb{E}[\lim_{n \to \infty} \widehat{\psi}_n] = \mathbb{E}[\widehat{\psi}].$$

3 Integrability under structural conditions on the coefficients

Let us provide here some sufficient conditions to ensure the (square) integrability conditions of ψ , required in Theorem 2.3.

3.1 An upper bound of the estimators

Recall that A is a (deterministic) function with finite variation and let |A| be the corresponding total variation process. We now impose a structural condition relating the path-regularity of |A| and the regularity of the coefficients of the diffusion process X, in the spirit of [5].

Assumption 3.1. There are constants L > 0 and $\alpha_1, \alpha_2 \in (0,1]$, together with modulus of continuity $\varpi_1(\cdot)$ and $\varpi_2(\cdot)$, such that

$$\varpi_i(C(|A|_t - |A|_s)) \le L \ C \ (t-s)^{\alpha_i/2}, \text{ for all } 0 \le s \le t \le T, \ C \ge 1, \ i = 1, 2.$$
 (26)

(i) The functions $(\mu, \sigma) : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{S}^d$ are uniformly bounded, and for all (t, x, \bar{x}) , $(s, y, \bar{y}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\left| \left(\mu, \sigma \right) (t, x, \bar{x}) - \left(\mu, \sigma \right) (s, y, \bar{y}) \right| \leq L \left(|t - s|^{\alpha_1/2} + \left| x - y \right|^{\alpha_1} + \varpi_1 \left(|\bar{x} - \bar{y}| \right) \right). \tag{27}$$

(ii) For all $(x, \bar{x}), (y, \bar{y}) \in \mathbb{R}^d \times \mathbb{R}^d$,

$$\left| g(x,\bar{x}) - g(y,\bar{y}) \right| \leq L \left(\left| x - y \right|^{\alpha_2} + \varpi_2 \left(\left| \bar{x} - \bar{y} \right| \right) \right). \tag{28}$$

(iii) For all (x, \bar{x}) , $(\Delta x, \Delta \bar{x}) \in \mathbb{R}^d \times \mathbb{R}^d$

$$\left| g\left(x + \Delta x, \bar{x} + \Delta \bar{x} \right) + g\left(x - \Delta x, \bar{x} + \Delta \bar{x} \right) - 2g\left(x, \bar{x} \right) \right| \leq L\left(\left| \Delta x \right|^{2\alpha_2} + \varpi_2\left(\left| \Delta \bar{x} \right| \right)^2 \right). \tag{29}$$

Remark 3.2. (i) Assume that there is some constant $\alpha_0 \geq \frac{\alpha_1}{2}$ such that A satisfies $|A|_t - |A|_s \leq (t-s)^{\alpha_0}$ for all $0 \leq s \leq t \leq T$, and $(\mu, \sigma)(t, x, \bar{x})$ is $\frac{\alpha_1}{2}$ -Hölder in t, α_1 -Hölder in x and $\frac{\alpha_1}{2\alpha_0}$ -Hölder in \bar{x} , then (27) holds true with $\varpi_1(s) := s^{\alpha_1/(2\alpha_0)}$ for all $s \geq 0$.

Similarly, if $g(x,\bar{x})$ is α_2 -Hölder in x and $\frac{\alpha_2}{2\alpha_0}$ -Hölder in \bar{x} , then (28) holds true with $\varpi_2(s) := s^{\alpha_1/(2\alpha_0)}$ for all $s \geq 0$.

(ii) The condition (29) is equivalent to

$$\left| g(x, \bar{x} + \Delta \bar{x}) - g(x, \bar{x}) \right| \leq L \varpi_2 (|\Delta \bar{x}|)^2,$$

and

$$\left|g(x+\Delta x,\bar{x})+g(x-\Delta x,\bar{x})-2g(x,\bar{x})\right| \leq L\left|\Delta x\right|^{2\alpha_2}.$$

Let $\alpha_2 \in (\frac{1}{2}, 1]$, $(A_t)_{t \in [0,T]}$ satisfies $|A|_t - |A|_s \leq (t-s)^{\alpha_0}$ for some $\alpha_0 \geq \alpha_2$. Assume that $g(x, \bar{x})$ is continuous differentiable in x, $D_x g(x, \bar{x})$ is $(2\alpha_2 - 1)$ -Hölder in x, and $g(x, \bar{x})$ is $\frac{\alpha_2}{\alpha_0}$ -Hölder in \bar{x} , then (29) holds true with $\varpi_2(s) := s^{\alpha_2/(2\alpha_0)}$ for all $s \geq 0$.

Under this condition, each moments of $|\psi_n|$ can be controlled by the norm of the simpler quantity, which does not depend on (μ, σ) anymore,

$$\phi_{n}^{C} := \mathbf{1}_{\{N_{T} \leq n-1\}} \frac{C\Delta T_{N_{T}+1}^{\alpha_{2}}}{1 - F(\Delta T_{N_{T}+1})} \prod_{k=1}^{N_{T}} \left(\frac{C\Delta T_{k}^{\alpha_{1}/2}}{\rho(\Delta T_{k})} \frac{1}{\Delta T_{k+1}} \frac{\tilde{m}_{k,k+1}^{2}}{m_{k,k+1}^{2}} \right)$$

$$+ \mathbf{1}_{\{N_{T} \geq n\}} \frac{C\Delta T_{n}^{\alpha_{1}/2}}{\rho(\Delta T_{n})} \prod_{k=1}^{n-1} \left(\frac{C\Delta T_{k}^{\alpha_{1}/2}}{\rho(\Delta T_{k})} \frac{1}{\Delta T_{k+1}} \frac{\tilde{m}_{k,k+1}^{2}}{m_{k,k+1}^{2}} \right)$$

and moments of $|\psi|$ can be controlled through

$$\phi^{C} := \lim_{n \to \infty} \phi_{n}^{C} = \frac{C\Delta T_{N_{T}+1}^{\alpha_{2}}}{1 - F(\Delta T_{N_{T}+1})} \prod_{k=1}^{N_{T}} \left(\frac{C\Delta T_{k}^{\alpha_{1}/2}}{\rho(\Delta T_{k})} \frac{1}{\Delta T_{k+1}} \frac{\tilde{m}_{k,k+1}^{2}}{m_{k,k+1}^{2}} \right),$$

for a suitable constants C. To state this result, we need to introduce the sub- σ -field

$$\mathcal{G}_{\mathbb{T}} := \sigma(\tau_k : k > 1).$$

Proposition 3.3. Let Assumptions 2.1 and 3.1 hold true. Then, for each $p \ge 1$, there exist a constant $C_{1,p} > 0$ which depends only on $p \ge 1$, $\varepsilon_0 > 0$ in Assumption 2.1, L > 0 in Assumption 3.1 and the L^{∞} -norm of $(D_x u, D_{xx}^2 u, g)$, such that

$$\mathbb{E}\Big[\big| \psi_n \big|^p \ \Big| \mathcal{G}_{\mathbb{T}} \Big] \ \leq \ \big| \phi_n^{C_{1,p}} \big|^p, \ a.s., \ \textit{ for all } n \geq 1,$$

and a constant $C_{2,p} > 0$ which depends only on the constants $p \ge 1$, $\varepsilon_0 > 0$ in Assumption 2.1 and L > 0 in Assumption 3.1, such that

$$\mathbb{E}\left[\left|\psi\right|^{p} \left|\mathcal{G}_{\mathbb{T}}\right] \leq \left|\phi^{C_{2,p}}\right|^{p}, \ a.s. \tag{30}$$

Proof. First, we notice that, for some constant C > 0 depending only on the constant L > 0 in Assumption 3.1 and $\varepsilon_0 > 0$ in Assumption 2.1,

$$\left|\widehat{\mathcal{W}}_{k}^{1}\right| \leq C\left(\Delta T_{k}^{\alpha_{1}/2} + \left|\widehat{X}_{T_{k}} - \widehat{X}_{T_{k-1}}\right|^{\alpha_{1}} + \varpi_{1}\left(\left|\widehat{I}_{T_{k}} - \widehat{I}_{T_{k-1}}\right|\right)\right) |M_{k+1}|.$$

and

$$\left|\widehat{\mathcal{W}}_{k}^{2}\right| \leq C\left(\Delta T_{k}^{\alpha_{1}/2} + \left|\widehat{X}_{T_{k}} - \widehat{X}_{T_{k-1}}\right|^{\alpha_{1}} + \varpi_{1}\left(\left|\widehat{I}_{T_{k}} - \widehat{I}_{T_{k-1}}\right|\right)\right)\left(\left|M_{k+1}\right|^{2} + \frac{1}{\Delta T_{k+1}}\frac{\tilde{m}_{k,k+1}^{2}}{m_{k,k+1}^{2}}\right).$$

Further, there is some constant C > 0 such that

$$|g(\widehat{X}_{T},\widehat{I}_{T}) - g(\widetilde{X}_{T},\widetilde{I}_{T})|$$

$$\leq C\left(\left|\Delta T_{N_{T}+1}\right|^{\alpha_{2}} + \left|\sigma_{T_{N_{T}}}\Delta W_{N_{T}+1}\right|^{\alpha_{2}} + \varpi_{2}\left(\left|\widehat{I}_{T} - \widehat{I}_{T_{N_{T}}}\right|\right) + \varpi_{2}\left(\left|\widetilde{I}_{T} - \widehat{I}_{T_{N_{T}}}\right|\right)\right),$$

and

$$\begin{aligned} & \left| g(\overline{X}_T, \overline{I}_T) + g(\widetilde{X}_T, \widetilde{I}_T) - 2g(\overline{X}_{T_{N_T}}, \overline{I}_{T_{N_T}}) \right| \\ & \leq & C \left(\left| \Delta T_{N_T+1} \right|^{2\alpha_2} + \left| \sigma_{T_{N_T}} \Delta W_{N_T+1} \right|^{2\alpha_2} + \varpi_2 \left(\left| \widehat{I}_T - \widehat{I}_{T_{N_T}} \right| \right)^2 + \varpi_2 \left(\left| \widetilde{I}_T - \widehat{I}_{T_{N_T}} \right| \right)^2 \right). \end{aligned}$$

Notice that, for any q > 0, there exists $C_q > 0$ such that

$$\mathbb{E}\left[\left|\Delta W_{k}\right|^{2q} + \left|\widehat{X}_{T_{k}} - \widehat{X}_{T_{k-1}}\right|^{2q} \middle| \mathcal{G}_{\mathbb{T}}\right] \leq C_{q} \Delta T_{k}^{q},$$

and by (26),

$$\mathbb{E}\Big[\varpi_i\big(\widehat{I}_{T_k} - \widehat{I}_{T_{k-1}}\big)^q \Big| \mathcal{G}_{\mathbb{T}}\Big] \ \leq \ C \mathbb{E}\Big[\max_{0 \leq t \leq T} |X_t|^q\Big] \Delta T_k^{\alpha_i q/2} \ \leq \ C_q \Delta T_k^{\alpha_i q/2}, \ i = 1, 2.$$

Together with Lemma A.2 and the definition of ψ_n and ψ in and above (17), it follows that, for any $p \geq 1$,

$$\mathbb{E}\left[\left|\psi_{n}\right|^{p} \middle| \mathcal{G}_{\mathbb{T}}\right] \leq \left|\phi_{n}^{C_{1,p}}\right|^{p}, \text{ a.s.,}$$

and

$$\mathbb{E}\left[\left|\psi\right|^{p} \middle| \mathcal{G}_{\mathbb{T}}\right] \leq \left|\phi^{C_{2,p}}\right|^{p}, \text{ a.s.}$$

for a constant $C_{1,p} > 0$ depending only on ε_0 , L and the L^{∞} norm of $D_x u$, $D_{xx}^2 u$ and g, and a constant $C_{2,p} > 0$ depending only on ε_0 and L.

Remark 3.4. The estimator ψ in (17) has a better integrability property than the estimator $\hat{\psi}$ in (16) in general. Let us consider for example the case where

$$A_t = t$$
, for all $t \in [0, T]$, so that $\frac{\tilde{m}_{k,k+1}^2}{m_{k,k+1}^2} = 4$ (see Lemma A.2).

For $\widehat{\psi}$, one can naturally apply (28) to obtain the bound

$$\mathbb{E}\Big[\big| \big(g(\widehat{X}_T, \widehat{I}_T) - g(\widehat{X}_{T_{N_T}}, \widehat{I}_{T_{N_T}}) \big) \widehat{\mathcal{W}}_{N_T}^2 \big| \ \Big| \ \mathcal{G}_{\mathbb{T}} \Big] \ \leq \ C \Delta T_{N_T + 1}^{-1 + \alpha_2/2},$$

where the r.h.s. term is generally not square integrable for $\alpha_2 \in (0,1]$ (see also Lemma A.3 below).

However, for the estimator ψ , one can apply (29) to obtain the bound

$$\mathbb{E}\Big[\big| \big(g(\widehat{X}_T, \widehat{I}_T) + g(\widetilde{X}_T, \widetilde{I}_T) - 2g(\widehat{X}_{T_{N_T}}, \widehat{I}_{T_{N_T}}) \big) \widehat{\mathcal{W}}_{N_T}^2 \big| \ \Big| \ \mathcal{G}_{\mathbb{T}} \Big] \ \leq \ C \Delta T_{N_T + 1}^{-1 + \alpha_2},$$

where the r.h.s. term becomes square integrable when $\alpha_2 \in (\frac{1}{2}, 1]$.

3.2 A sufficient condition for square integrability

We provide here a sufficient condition to ensure the (square) integrability of ϕ_n^C and ϕ^C . Recall that T_k is defined in (7) with the sequence of i.i.d. random variables $(\tau_k)_{k\geq 1}$, where τ_1 has the density function ρ .

Theorem 3.5. Let Assumptions 2.1 and 3.1 hold true. Suppose that the density function ρ satisfies, for some constants $C_1 > 0$, $C_2 > 0$ and $\kappa_1 > 0$, $\kappa_2 > 0$,

$$C_1 s^{\kappa_1 - 1} \le \rho(s) \le C_2 s^{\kappa_2 - 1}, \text{ for all } s \in [0, T].$$
 (31)

Suppose further that there exists $C_3 > 0$ and $\beta \in (0,1]$ such that

$$m_{s,t}^2 > 0$$
, $\frac{\tilde{m}_{s,t}^2}{m_{s,t}^2} \le C_3 (t-s)^{-\beta}$, for all $s < t$. (32)

Assume in addition that, for some $p \geq 1$,

$$\kappa_2 + p\left(\frac{\alpha_1}{2} - \kappa_1 - \beta\right) > 0 \quad and \quad p(\alpha_2 - \beta - 1) + 1 > 0. \tag{33}$$

Then for any constant C > 0, the sequence $(|\phi_n^C|^p)_{n \ge 1}$ is uniformly integrable.

Consequently, if (33) holds for p = 1, then ψ satisfies the representation (18). If (33) holds for p = 2, then ψ is in addition square integrable.

Remark 3.6. (i) Let ρ be the density of the Gamma distribution, with parameters $\kappa > 0$ and $\theta > 0$, i.e.

$$\rho(s) = \frac{s^{\kappa - 1} e^{-s/\theta}}{\Gamma(\kappa)\theta^{\kappa}}, \text{ for all } s > 0,$$

then it satisfies (31) with $\kappa_1 = \kappa_2 = \kappa$.

(ii) When $A_t = t$ for all $t \in [0, T]$, it follows by Lemma A.2 that

$$\frac{\tilde{m}_{s,t}^2}{m_{s,t}^2} = 4, \quad for \ all \ s < t,$$

so that (32) holds true with $\beta = 0$. Let us also refer to [5, Example 2.4] for more examples of A satisfying (32) with explicit constant $\beta \geq 0$.

Remark 3.7. Let $A_t = t$ for all $t \in [0,T]$, so that (32) holds true with $\beta = 0$. Assume that for some constant $\alpha \in (0,1]$, the functions $(\mu,\sigma)(t,x,\bar{x})$ are both α -Hölder in (t,x,\bar{x}) , $g(x,\bar{x})$ is α -Hölder in \bar{x} , and $D_x g(x,\bar{x})$ is α -Hölder in x. Then Assumption 3.1 holds with $\alpha_1 = \alpha_2 = \alpha$.

- (i) When p=1, the condition (33) holds true as soon as coefficient κ_1, κ_2 in (31) satisfies $\kappa_1 \kappa_2 < \alpha/2$.
- (ii) When p=2, the condition (33) holds true as soon as $\alpha > \frac{1}{2} \vee (2\kappa_1 \kappa_2)$.

Proof of Theorem 3.5. By changing the constant C > 0 from line from line, it follows by (31) and (32) that

$$\begin{split} \phi_{n}^{C} &\leq \mathbf{1}_{\{N_{T} \leq n-1\}} \ C\Delta T_{N_{T}+1}^{\alpha_{2}} \prod_{k=1}^{N_{T}} \left(C\Delta T_{k}^{\frac{\alpha_{1}}{2}-\kappa_{1}+1} \ \Delta T_{k+1}^{-(1+\beta)} \right) \\ &+ \mathbf{1}_{\{N_{T} \geq n\}} \ C\Delta T_{n}^{\frac{\alpha_{1}}{2}-\kappa_{1}+1} \ \prod_{k=1}^{n-1} \left(C\Delta T_{k}^{\frac{\alpha_{1}}{2}-\kappa_{1}+1} \ \Delta T_{k+1}^{-(1+\beta)} \right) \\ &\leq \mathbf{1}_{\{N_{T} \leq n-1\}} \ C\Delta T_{N_{T}+1}^{\alpha_{2}-(1+\beta)} \prod_{k=1}^{N_{T}} \left(C\Delta T_{k}^{\frac{\alpha_{1}}{2}-\kappa_{1}-\beta} \right) + \mathbf{1}_{\{N_{T} \geq n\}} \ \prod_{k=1}^{n} \left(C\Delta T_{k}^{\frac{\alpha_{1}}{2}-\kappa_{1}-\beta} \right). \end{split}$$

Therefore, for any $p \ge 1$ and constant C > 0 big enough, when (33) holds true, or equivalently,

$$\theta := p(\kappa_1 + \beta - \frac{\alpha_1}{2}) < \kappa_2 \text{ and } \eta := 1 + p(\alpha_2 - \beta - 1) > 0,$$

it follows by Lemma A.3 that

$$\mathbb{E}\Big[\sup_{n\geq 1} \left|\phi_n^C\right|^p\Big] \leq \mathbb{E}\Big[C^{p+1}\Delta T_{N_T+1}^{\eta-1}\prod_{k=1}^{N_T} \left(C^{p+1}\Delta T_k^{-\theta}\right)\Big] < \infty.$$

Consequently, when (33) holds for some $p \ge 1$, the sequence $(|\phi_n^C|^p)_{n\ge 1}$ is uniformly integrable for any constant C > 0.

Consequently, it follows by Theorem 2.3 and Proposition 3.3 that ψ provides the representation result in (18). When (33) holds for p=2, then by Proposition 3.3, ψ is further square integrable.

Finally, in view of Remark 2.6.(iii) on the smoothness assumption of the value function, we provide here an approximation result.

Corollary 3.8. Let $(\mu^m, \sigma^m, g^m)_{m\geq 1}$ be a sequence of coefficient functions such that $(\mu^m, \sigma^m, g^m) \longrightarrow (\mu, \sigma, g)$ pointwisely. Suppose that $(\mu^m, \sigma^m, g^m)_{m\geq 1}$ and (μ, σ, g) all satisfy Assumption 2.1.(i) with the same parameters $\varepsilon_0 > 0$ and the same linear growth, and all satisfy Assumption 3.1 with uniform parameters $\alpha_1, \alpha_2 \in (0,1]$ and L > 0. Assume in addition that, for each $m \geq 1$, the coefficient function (μ^m, σ^m, g^m) induces a value function u^m satisfying the smoothness conditions in Assumption 2.1.(ii). Then, when condition (33) holds with p = 2, one has

$$\mathbb{E}[\psi^2] < \infty$$
, and $\mathbb{E}[\psi] = \mathbb{E}[g(X_T, I_T)] =: V_0$.

Proof. First, for each $m \geq 1$, let us denote by (X^m, I^m) the (weak) solution of the SDE (4) with coefficient functions (μ^m, σ^m) , by ψ^m the corresponding representation random variable as defined in (16). Then it follows by Theorem 3.5 that

$$\mathbb{E}[\psi^m] = \mathbb{E}[g_m(X_T^m, I_T^m)], \text{ for each } m \ge 1.$$

Next, by the standard stability results for SDEs, together with the uniqueness of the weak solution (X, I) to the SDE (4) with coefficient (μ, σ) , it follows that

 $(X_T^m, I_T^m) \longrightarrow (X_T, I_T)$ weakly. Further, since $(\mu^m, \sigma^m, g^m) \longrightarrow (\mu, \sigma, g)$ pointwisely, one further obtains that

$$\psi^m \longrightarrow \psi$$
, and $g_m(X_T^m, I_T^m) \longrightarrow g(X_T, I_T)$, weakly.

Finally, using the growth condition on g^m , and the upper bound result in (30) with uniform constant $C_{2,p}$ for all $m \geq 1$, it is easy to deduce that $(\psi^m)_{m\geq 1}$ and $(g_m(X_T^m, I_T^m))_{m\geq 1}$ are both uniformly integrable, and that ψ is square integrable. It follows then

$$\mathbb{E}[\psi] = \mathbb{E}[g(X_T, I_T)].$$

The case with constant volatility coefficient function When $\sigma(\cdot) \equiv \sigma_0$ for some non-degenerate matrix $\sigma_0 \in \mathbb{S}_d$, it is easy to see that $\widehat{\mathcal{W}}_k^2 = 0$ for all $k \geq 1$. In view of Remark 3.4 and Theorem 3.5, one can also consider the variable $\widehat{\psi}$ as a Monte Carlo unbiased estimator. In this case, one can drop the conditions in Assumption 3.1.(iii) for the (square) integrability analysis of the estimator.

Since the arguments are almost the same as those in Proposition 3.3 and Theorem 3.5, let us directly provide the representation result without the proof.

Proposition 3.9. Let Assumptions 2.1 and 3.1.(i)-(ii) hold true. Then for each $p \ge 1$, there exists a constant $C_p > 0$ such that

$$\mathbb{E}\Big[|\widehat{\psi}|^p \Big| \mathcal{G}_{\mathbb{T}} \Big] \leq \left| \frac{C_p \Delta T_{N_T+1}^{\alpha_2/2}}{1 - F(\Delta T_{N_T+1})} \prod_{k=1}^{N_T} \left(\frac{C_p \Delta T_k^{\alpha_1/2}}{\rho(\Delta T_k)} \sqrt{\frac{1}{\Delta T_{k+1}} \frac{\widetilde{m}_{k,k+1}^2}{m_{k,k+1}^2}} \right) \right|^p.$$

Further, assume that (31) holds with parameters $\kappa_1 > 0$, $\kappa_2 > 0$, and (32) holds with parameter $\beta \in (0,1]$, and for $p \geq 1$,

$$2\kappa_2 + p(1 + \alpha_1 - 2\kappa_1 - \beta) > 0$$
 and $p(\alpha_2 - \beta - 1) + 2 > 0$.

Then $|\widehat{\psi}|^p$ is integrable and hence $\widehat{\psi}$ satisfies the representation (18).

4 Numerical examples

For numerical examples, we consider the Asian option pricing problem in two different financial models. The first is the Bachelier model, where the underlying risky asset follows the dynamic with constant volatility coefficient:

$$dX_t = rX_t dt + \sigma dW_t. (34)$$

The second model is the Black-Scholes model, where the risky asset follows the dynamic

$$dX_t = rX_t dt + \sigma X_t dW_t. (35)$$

In above, r > 0 is the interest rate, and W is a Brownian motion (under the risk-neutral probability). Let us consider the Asian option with payoff

$$(I_T - K)_+$$
, where $I_T := \int_0^T X_t dt$,

so that the corresponding no-arbitrage price is given by

$$V_0 := \mathbb{E}\Big[e^{-rT}\big(I_T - K\big)_+\Big].$$

Under the Bachelier model (34), one can compute directly that

$$I_T = \int_0^T X_t dt = \frac{1}{r} \Big((e^{rT} - 1) X_0 + \sigma \int_0^T (e^{r(T-t)} - 1) dW_s \Big),$$

and then deduce the reference value:

$$V_0 = \Sigma \left(\frac{1}{\sqrt{2\pi}} e^{-(K-\mu)^2/(2\Sigma^2)} - \frac{K-\mu}{\Sigma} \Phi\left(-\frac{K-\mu}{\Sigma}\right) \right),$$

with

$$\mu := \frac{1}{r} (e^{rT} - 1) X_0, \quad \Sigma^2 := \frac{\sigma^2}{r^2} (\frac{1}{2r} (e^{2rT} - 1) - \frac{2}{r} (e^{rT} - 1) + T).$$

Under the Black-Scholes model (35), we use the dimension reduction PDE technique of Rogers and Shi [20] to compute a reference value of V_0 . To solve the dimension reduced PDE (2.4) in [20] with variable (t, ξ) , we apply the implicit finite difference method with discretization parameters $\Delta \xi = \frac{1}{400}$ and $\Delta t = \frac{1}{8000}$.

For our numerical implementation of the unbiased Monte Carlo simulation method, the discrete time grid $(T_k)_{k\geq 1}$ are obtained by simulating the i.i.d. random variables $(\tau_k)_{k\geq 1}$ of density function:

$$\rho(t) := \frac{\kappa}{(2T)^{\kappa}} t^{\kappa - 1} \mathbf{1}_{\{t \in [0, 2T]\}},$$

with parameter $\kappa = 0.35$.

The numerical simulation results are presented in Tables 1 and 2. We observe that the unbiased Monte Carlo simulation method works much better (with smaller Monte Carlo error) under the Bachelier model with has constant volatility function (c.f. Table 1) than that under the Black-Scholes model (c.f. Table 2). Theoretically, one can observe that the Malliavin weight term $\widehat{\mathcal{W}}_k^2$ in (13) has a higher order variance than $\widehat{\mathcal{W}}_k^1$ in (12) in general. Under the Bachelier model (34), the volatility function is constant, so that the term $\widehat{\mathcal{W}}_k^2$ disappears and hence the corresponding estimator has smaller variance.

In the Black-Scholes model, the constant σ is the Lipschitz constant of the volatility function. With smaller σ , the absolute value of $\widehat{\mathcal{W}}_k^2$ in (13) is also smaller so that it has smaller variance (recall that it always has expectation value 0). Of course, for the unbiased Monte Carlo simulation method, one can always increase the number of simulations N to reduce the Monte Carlo error of the estimation.

	Reference value	Unbiased MC mean	$\operatorname{Std}/\sqrt{N}$
$\sigma = 0.05$	2.7182	2.7159	0.0022
$\sigma = 0.1$	3.6470	3.6439	0.0034
$\sigma = 0.15$	4.6960	4.6921	0.0046
$\sigma = 0.2$	5.7781	5.7733	0.0058

Table 1: Bachelier model: r = 0.05, $X_0 = 100$, K = 100, T = 1, number of simulation $N = 10^7$.

	PDE reference value	Unbiased MC mean	$\operatorname{Std}/\sqrt{N}$
$\sigma = 0.05$	2.7495	2.7127	0.0031
$\sigma = 0.1$	3.6755	3.6356	0.0108
$\sigma = 0.15$	4.7120	4.7054	0.0474
$\sigma = 0.2$	5.7833	5.9329	0.2595

Table 2: Black-Scholes model: r = 0.05, $X_0 = 100$, K = 100, T = 1, number of simulation $N = 10^8$.

A Technical lemmas

Recall that, for s < t, we define

$$\overline{A}_{s,t} := \frac{1}{t-s} \int_s^t A_r dr, \quad m_{s,t}^1 := \frac{1}{t-s} \int_s^t (A_r - A_s) dr,$$

and

$$m_{s,t}^2 := \frac{1}{t-s} \int_0^t \left(A_r - \overline{A}_{s,t} \right)^2 dr, \quad \ \tilde{m}_{s,t}^2 := \frac{1}{t-s} \int_s^t (A_r - A_s)^2 dr,$$

and then

$$\Sigma_{s,t}^A := \begin{bmatrix} (t-s) & \int_s^t (A_t - A_r) dr \\ \int_s^t (A_t - A_r) dr & \int_s^t (A_t - A_r)^2 dr \end{bmatrix},$$

so that

$$Det(\Sigma_{s,t}^{A}) = (t-s) \int_{s}^{t} (A_{t} - A_{r})^{2} dr - \left(\int_{s}^{t} (A_{t} - A_{r}) dr \right)^{2}
= (t-s) \int_{s}^{t} (A_{r} - A_{s})^{2} dr - \left(\int_{s}^{t} (A_{r} - A_{s}) dr \right)^{2} = (t-s)^{2} m_{s,t}^{2}.$$

Further, let us fix a matrix $\sigma_0 \in \mathbb{S}^d$ and a *d*-dimensional standard Brownian motion W, we define, for all $x, \bar{x} \in \mathbb{R}^d$,

$$\Phi(s,t,x,\bar{x},W) := \begin{bmatrix} I_d & 0 \\ (A_t - A_s)I_d & I_d \end{bmatrix} \begin{bmatrix} x \\ \bar{x} \end{bmatrix} + \sigma_0 \int_s^t \begin{bmatrix} I_d \\ (A_t - A_u)I_d \end{bmatrix} dW_u,$$

and (recall (23))

$$M_{s,t} := \frac{1}{(t-s)m_{s,t}^2} (\sigma_0^\top)^{-1} \int_s^t \left(\tilde{m}_{s,t}^2 - m_{s,t}^1 (A_u - A_s) \right) dW_u.$$
 (36)

Lemma A.1. Let $g: \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}$ be a function with polynomial growth, then

$$D_x \mathbb{E}\left[g\big(\Phi(s,t,x,\bar{x},W_{\cdot})\big)\right] = \mathbb{E}\left[g\big(\Phi(s,t,x,\bar{x},W_{\cdot})\big)M_{s,t}\right],$$

and

$$D_{xx}^2 \mathbb{E}\Big[g\big(\Phi(s,t,x,\bar{x},W_{\cdot})\big)\Big] = \mathbb{E}\Big[g\big(\Phi(s,t,x,\bar{x},W_{\cdot})\big)\Big(M_{s,t}M_{s,t}^{\top} - \frac{1}{t-s}\frac{\tilde{m}_{s,t}^2}{m_{s,t}^2}\big(\sigma_0\sigma_0^{\top}\big)^{-1}\big)\Big].$$

Proof. Let us denote the 2d-dimensional Gaussian vector

$$Z := \sigma_0 \int_s^t \begin{bmatrix} I_d \\ (A_t - A_u)I_d \end{bmatrix} dW_u.$$

Then

$$D_x \mathbb{E}\Big[g\big(\Phi(s,t,x,\bar{x},W.)\big)\Big] = \begin{bmatrix} I_d \\ (A_t - A_s)I_d \end{bmatrix}^\top \mathbb{E}\Big[g\big(\Phi(s,t,x,\bar{x},W.)\big) \text{Var}\big[Z\big]^{-1}Z\Big],$$

and

$$D_{xx}^{2} \mathbb{E} \Big[g \big(\Phi(s, t, x, \bar{x}, W_{\cdot}) \big) \Big] = \begin{bmatrix} I_{d} \\ (A_{t} - A_{s})I_{d} \end{bmatrix}^{\top}$$

$$\mathbb{E} \Big[g \big(\Phi(s, t, x, \bar{x}, W_{\cdot}) \big) \Big(\operatorname{Var}[Z]^{-1} Z \, \left(\operatorname{Var}[Z]^{-1} Z \right)^{\top} - \operatorname{Var}[Z]^{-1} \Big) \Big] \begin{bmatrix} I_{d} \\ (A_{t} - A_{s})I_{d} \end{bmatrix}.$$

By direct computation, one has $\operatorname{Var}[Z] = \sigma_0 \sigma_0^{\top} \Sigma_{s,t}^A$, so that

$$\begin{bmatrix} I_d \\ (A_t - A_s)I_d \end{bmatrix}^\top \operatorname{Var}[Z]^{-1}Z = (\sigma_0 \sigma_0^\top)^{-1} \sigma_0 \int_s^t \begin{bmatrix} 1 \\ A_t - A_s \end{bmatrix}^\top (\Sigma_{s,t}^A)^{-1} \begin{bmatrix} 1 \\ A_t - A_u \end{bmatrix} dW_u.$$

Notice that

$$(\Sigma_{s,t}^{A})^{-1} = \frac{1}{\text{Det}(\Sigma_{s,t}^{A})} \begin{bmatrix} \int_{s}^{t} (A_{t} - A_{u})^{2} du & -\int_{s}^{t} (A_{t} - A_{u}) du \\ -\int_{s}^{t} (A_{t} - A_{u}) du & (t - s) \end{bmatrix},$$

it follows by direct computation that

$$\begin{bmatrix} I_d \\ (A_t - A_s)I_d \end{bmatrix}^{\top} \operatorname{Var}[Z]^{-1} Z = M_{s,t},$$

with $M_{s,t}$ being defined in (36).

Similarly, one has

$$\begin{bmatrix} I_d \\ (A_t - A_s)I_d \end{bmatrix}^{\top} \left(\operatorname{Var}[Z]^{-1} Z \left(\operatorname{Var}[Z]^{-1} Z \right)^{\top} - \operatorname{Var}[Z]^{-1} \right) \right] \begin{bmatrix} I_d \\ (A_t - A_s)I_d \end{bmatrix}$$

$$= M_{s,t} M_{s,t}^{\top} - \frac{1}{t - s} \frac{\tilde{m}_{s,t}^2}{m_{s,t}^2} \left(\sigma_0 \sigma_0^{\top} \right)^{-1},$$

which concludes the proof.

Lemma A.2. Let s < t and $M_{s,t}$ be defined in (36). Then $M_{s,t}$ is a Gaussian vector, i.e.

$$M_{s,t} \sim N\left(0, \frac{1}{t-s} \frac{\tilde{m}_{s,t}^2}{m_{s,t}^2} (\sigma_0 \sigma_0^\top)^{-1}\right).$$

In particular, when $A_t = t$ for all $t \in [0, T]$,

$$\tilde{m}_{s,t}^2 = \frac{(t-s)^2}{3}, \quad m_{s,t}^2 = \frac{(t-s)^2}{12} \quad \text{so that} \quad \frac{1}{t-s} \frac{\tilde{m}_{s,t}^2}{m_{s,t}^2} \left(\sigma_0 \sigma_0^\top\right)^{-1} = \frac{4}{t-s} \left(\sigma_0 \sigma_0^\top\right)^{-1}.$$

Proof. First, it is clear that $M_{s,t}$ is a Gaussian random vector with expectation 0, and

$$\operatorname{Var}[M_{s,t}] = \frac{1}{(t-s)^2 (m_{s,t}^2)^2} (\sigma_0 \sigma_0^\top)^{-1} \int_s^t (\tilde{m}_{s,t}^2 - m_{s,t}^1 (A_u - A_s))^2 du$$
$$= \frac{(t-s)\tilde{m}_{s,t}^2 m_{s,t}^2}{(t-s)^2 (m_{s,t}^2)^2} (\sigma_0 \sigma_0^\top)^{-1} = \frac{1}{t-s} \frac{\tilde{m}_{s,t}^2}{m_{s,t}^2} (\sigma_0 \sigma_0^\top)^{-1}.$$

Next, when $A_t = t$ for all $t \in [0, T]$, it follows by direct computation that

$$\tilde{m}_{s,t}^2 = \frac{(t-s)^2}{3}, \quad m_{s,t}^2 = \frac{(t-s)^2}{12} \text{ so that } \frac{1}{t-s} \frac{\tilde{m}_{s,t}^2}{m_{s,t}^2} \left(\sigma_0 \sigma_0^\top\right)^{-1} = \frac{4}{t-s} \left(\sigma_0 \sigma_0^\top\right)^{-1}.$$

Let T_k is defined by (7) with the i.i.d. sequence of random variables $(\tau_k)_{k\geq 1}$, where τ_1 has the density function $\rho(s)$.

Lemma A.3. Assume that the density function $\rho(s)$ satisfies, for some constant $\widetilde{C} > 0$,

$$\rho(s) \le \widetilde{C}s^{\kappa-1}, \text{ for all } s \in [0,T].$$

Then for all constants C > 0, $\eta > 0$ and $\theta < \kappa$, it holds that

$$\mathbb{E}\Big[\Delta T_{N_T+1}^{\eta-1} \prod_{k=1}^{N_T} \left(C \Delta T_k^{-\theta} \right) \Big] < \infty.$$

Proof. First, for each $n \geq 0$, we denote $S_n := \{0 < s_1 < \cdots < s_n\}$ and $ds := ds_1 \cdots ds_n$, and then compute that

$$\mathbb{E}\Big[\big(T - T_{N_T}\big)^{\eta - 1} \prod_{k=1}^{N_T} \Big(C\Delta T_k^{-\theta}\Big) \mathbf{1}_{\{N_T = n\}}\Big]$$

$$= \int_{S_n} ds \int_{T - s_n}^{\infty} \rho(t) dt \ (T - s_n)^{\eta - 1} \prod_{k=1}^{n} \Big(C(s_k - s_{k-1})^{-\theta} \rho(s_k - s_{k-1})\Big)$$

$$\leq (C\widetilde{C})^n \int_{S_n} (T - s_n)^{\eta - 1} \Big(\prod_{k=1}^{n} (s_k - s_{k-1})^{\kappa - \theta - 1}\Big) ds$$

$$= (C\widetilde{C})^n T^{\eta + n(\kappa - \theta) - 1} \frac{\Gamma(\eta) \Gamma^n(\kappa - \theta)}{\Gamma(\eta + n(\kappa - \theta))}.$$

Then it follows that, for some constant K > 0,

$$\mathbb{E}\Big[(T - T_{N_T})^{\eta - 1} \prod_{k=1}^{N_T} \left(C \Delta T_k^{-\theta} \right) \Big] \le \sum_{n=0}^{\infty} \frac{K^n}{\Gamma(\eta + n(\kappa - \theta))} < \infty.$$

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