SHARP WELL-POSEDNESS AND ILL-POSEDNESS OF THE STATIONARY QUASI-GEOSTROPHIC EQUATION

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ABSTRACT. We consider the stationary problem for the quasi-geostrophic equation on the whole plane and investigate its well-posedness and ill-posedness. In [12], it was shown that the two-dimensional stationary Navier–Stokes equations are ill-posed in the critical Besov spaces $\dot{B}_{p,1}^{\frac{2}{p}-1}(\mathbb{R}^2)$ with $1\leqslant p\leqslant 2$. Although the quasi-geostrophic equation has the same invariant scale structure as the Navier–Stokes equations, we reveal that the quasi-geostrophic equation is well-posed in the scaling critical Besov spaces $\dot{B}_{p,q}^{\frac{2}{p}-1}(\mathbb{R}^2)$ with $(p,q)\in [1,4)\times [1,\infty]$ or (p,q)=(4,2) due to the better properties of the nonlinear structure of the quasi-geostrophic equation compared to that of the Navier–Stokes equations. Moreover, we also prove the optimality for the above range of (p,q) ensuring the well-posedness in the sense that the stationary quasi-geostrophic equation is ill-posed for all the other cases.

1. Introduction

We consider the stationary problem of the quasi-geostrophic equation on the whole plane \mathbb{R}^2 :

$$\begin{cases}
-\Delta\theta + u \cdot \nabla\theta = f, & x \in \mathbb{R}^2, \\
u = \nabla^{\perp}(-\Delta)^{-\frac{1}{2}}\theta, & x \in \mathbb{R}^2,
\end{cases}$$
(1.1)

where $\nabla^{\perp} = (-\partial_{x_2}, \partial_{x_1})$. Here, $\theta = \theta(x)$ and $u = (u_1(x), u_2(x))$ are the unknown potential temperature and velocity field of the fluid, respectively, while f = f(x) is the given external force. The stationary quasi-geostrophic equation has the scaling invariant structure. If θ is a solution to (1.1) for some given external force f, then $\theta_{\lambda}(x) = \lambda \theta(\lambda x)$ also solves (1.1) with $f_{\lambda}(x) = \lambda^3 f(\lambda x)$ for all $\lambda > 0$. We say that the data space D and the solution space S is scaling critical if $||f_{\lambda}||_{D} = ||f||_{D}$ and $||\theta_{\lambda}||_{S} = ||\theta||_{S}$ for all $\lambda > 0$. For instance, the homogeneous Besov spaces $D = \dot{B}_{p,q}^{\frac{2}{p}-3}(\mathbb{R}^2)$ and $S = \dot{B}_{p,q}^{\frac{2}{p}-1}(\mathbb{R}^2)$ $(1 \leq p, q \leq \infty)$ are scaling critical. The aim of this paper is to investigate the well-posedness and ill-posedness of the equation (1.1) in the scaling critical Besov spaces and provide the sharp result in the sense that it completely classifies whether (1.1) is well-posed or ill-posed for all $1 \leq p, q \leq \infty$.

Known results and the position of our study. Before stating the main results precisely, we recall the previous studies related to our work and mention the position of our study. The quasi-geostrophic equation has received much attention for the

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case of the non-stationary initial value problem

$$\begin{cases}
\partial_t \theta + (-\Delta)^{\alpha} \theta + u \cdot \nabla \theta = 0, & t > 0, x \in \mathbb{R}^2, \\
u = \nabla^{\perp} (-\Delta)^{-\frac{1}{2}} \theta, & t \geqslant 0, x \in \mathbb{R}^2, \\
\theta(0, x) = \theta_0(x), & x \in \mathbb{R}^2,
\end{cases} \tag{1.2}$$

where $0 < \alpha \le 1$ and $\theta_0 = \theta_0(x)$ is the given initial datum. The local and global well-posedness for (1.2) are well-known by many researchers. See [6, 7, 16, 27, 28] for the sub-critical case $\alpha > 1/2$, [2, 8, 10, 18, 29] for the critical case $\alpha = 1/2$, and [1, 2, 4, 22] for the super-critical case $0 < \alpha < 1/2$. Here, we remark that for the sub-critical and critical case $\alpha \ge 1/2$, [6, 18] proved the global well-posedness for arbitrary large initial data in the scaling critical framework, whereas the global regularity for the super-critical case remains open. Next, we mention the known results for the stationary quasi-geostrophic equation with the fractional Laplacian:

$$\begin{cases} (-\Delta)^{\alpha}\theta + u \cdot \nabla\theta = f, & x \in \mathbb{R}^2, \\ u = \nabla^{\perp}(-\Delta)^{-\frac{1}{2}}\theta, & x \in \mathbb{R}^2. \end{cases}$$
 (1.3)

There seem less results on (1.3) in comparison with the initial value problem on the non-stationary quasi-geostrophic equation. For (1.3) with $\alpha=1/2$, Dai [9] showed the existence of weak solutions $\theta \in H^{\frac{1}{2}}(\mathbb{R}^2)$ for small data $f \in W^{\frac{1}{2},4}(\mathbb{R}^2) \cap L^{\infty}(\mathbb{R}^2)$ satisfying $\widehat{f}(\xi)=0$ in some low frequency region. Moreover, she removed the condition $\widehat{f}(\xi)=0$ around $\xi=0$ in the case of $1/2<\alpha<1$ and proved the existence of weak solutions $\theta \in H^{\alpha}(\mathbb{R}^2)$ for small data $f \in W^{1-\alpha,4}(\mathbb{R}^2) \cap L^{\frac{4}{2\alpha-1}}(\mathbb{R}^2)$. For the unique existence of strong solutions in the sub-critical case $1/2<\alpha<1$, Hadadifardo–Stefanov [14] proved that for any small external force $f \in L^{\frac{2}{2\alpha-1}}(\mathbb{R}^2)$, there exists a unique small solution $\theta \in \dot{W}^{-2\alpha,\frac{2}{2\alpha-1}}(\mathbb{R}^2)$; note that their framework is scaling critical. Recently, Kozono–Kunstmann–Shimizu [19] considered the generalized quasi-geostrophic equation on \mathbb{R}^n with $n \geqslant 2$:

$$\begin{cases} (-\Delta)^{\alpha}\theta + u \cdot \nabla\theta = f, & x \in \mathbb{R}^n, \\ u = \mathbb{P}S(\nabla(-\Delta)^{-\frac{1}{2}}\theta), & x \in \mathbb{R}^n, \end{cases}$$
 (1.4)

where $\alpha > 0$ and $S \in \mathbb{R}^{n \times n}$ are constant, and $\mathbb{P} = I + \nabla(-\Delta)^{-1}$ div denotes the Helmholtz projection to the divergence free vector fields. In [19], it was shown that if $1/2 < \alpha < 1/2 + n/4$, then for any $1 \leq p < n/(2\alpha - 1)$ and $1 \leq q \leq \infty$, the small solution θ in the critical Besov space $\dot{B}_{p,q}^{\frac{n}{p}+1-2\alpha}(\mathbb{R}^n)$ is constructed for small external force $f \in \dot{B}_{p,q}^{\frac{n}{p}+1-4\alpha}(\mathbb{R}^n)$. Note that (1.1) corresponds to the case n=2 and $\alpha=1$, where [19] does not encompass, and it was conjectured in [19, Remarks (iii) after Theorem 2.3] that (1.1) is not solvable. Indeed, the equation (1.1) has the same scaling structure as for the two-dimensional stationary Navier–Stokes equations, which is the following system with n=2:

$$\begin{cases}
-\Delta u + \mathbb{P}(u \cdot \nabla)u = \mathbb{P}f, & x \in \mathbb{R}^n, \\
\operatorname{div} u = 0, & x \in \mathbb{R}^n
\end{cases}$$
(1.5)

and the first author [12] proved the ill-posedness of (1.5) with n=2 from $\dot{B}_{p,1}^{\frac{2}{p}-3}(\mathbb{R}^2)$ to $\dot{B}_{p,1}^{\frac{2}{p}-1}(\mathbb{R}^2)$ with $1 \leq p \leq 2$, which is the framework that (1.5) is solvable in the

higher dimensional case $n \ge 3$; see [3,17,20,21,24–26] for the details. We notice that the unsolvability of (1.5) with n = 2 comes from fact that the key bilinear estimate

$$\left\| (-\Delta)^{-1} \mathbb{P}(u \cdot \nabla) v \right\|_{\dot{B}^{\frac{2}{p}-1}_{p,q}} \leqslant C \|u\|_{\dot{B}^{\frac{2}{p}-1}_{p,q}} \|v\|_{\dot{B}^{\frac{2}{p}-1}_{p,q}}$$

fails in almost all $1 \leq p, q \leq \infty$; see [11] for the detail. Despite such situation, in the present paper, we reveal that the nonlinear term $u \cdot \nabla \theta$ of the quasi-geostrophic equation (1.1) has a better regularity property than the two-dimensional Navier–Stokes equations and show that (1.1) is well-posed from $\dot{B}_{p,q}^{\frac{2}{p}-3}(\mathbb{R}^2)$ to $\dot{B}_{p,q}^{\frac{2}{p}-1}(\mathbb{R}^2)$ with $(p,q) \in ([1,4) \times [1,\infty]) \cup \{(4,2)\}$. Moreover, we show that the range $(p,q) \in ([1,4) \times [1,\infty]) \cup \{(4,2)\}$ that ensures the well-posed is sharp by showing the ill-posedness for the other case of (p,q).

Reformulation of the problem and the statements of our main results. Following the argument in [17,19], we rewrite (1.1) as

$$\theta = \mathcal{L}f + (-\Delta)^{-1}\operatorname{div}(\theta u), \qquad u = \nabla^{\perp}(-\Delta)^{-\frac{1}{2}}\theta, \tag{1.6}$$

where

$$\mathcal{L}f := (-\Delta)^{-1}f,\tag{1.7}$$

and try to control the product term $(-\Delta)^{-1}\operatorname{div}(\theta_1u_2)$ with $u_2 = \nabla^{\perp}(-\Delta)^{-\frac{1}{2}}\theta_2$ as

$$\|(-\Delta)^{-1}\operatorname{div}(\theta_{1}u_{2})\|_{\dot{B}^{\frac{2}{p}-1}_{p,q}} = \|(-\Delta)^{-1}\operatorname{div}\left(\theta_{1}\nabla^{\perp}(-\Delta)^{-\frac{1}{2}}\theta_{2}\right)\|_{\dot{B}^{\frac{2}{p}-1}_{p,q}}$$

$$\leq C\|\theta_{1}\nabla^{\perp}(-\Delta)^{-\frac{1}{2}}\theta_{2}\|_{\dot{B}^{\frac{2}{p}-2}_{p,q}}.$$

However, we may not proceed the above estimate since the para-product estimate

$$||fg||_{\dot{B}^{\frac{2}{p}-2}_{p,q}} \leqslant C||f||_{\dot{B}^{\frac{2}{p}-1}_{p,q}} ||g||_{\dot{B}^{\frac{2}{p}-1}_{p,q}}$$

fails for all $1 \leq p, q \leq \infty$. Due to the similar circumstance, the two-dimensional Navier–Stokes equations (1.5) with n=2 is ill-posed even in the narrowest critical Besov space and [19] did not treat the case n=2 and $\alpha=1$ in (1.4).

In the present paper, we reveal that, in contrast to the Navier–Stokes equations (1.5), the nonlinear term of the quasi-geostrophic equation (1.1) possesses a better nonlinear structure. To see this, we rewrite the nonlinear term of (1.6) as

$$\mathscr{F}\left[(-\Delta)^{-1}\operatorname{div}(\theta u)\right](\xi) = \mathscr{F}\left[(-\Delta)^{-1}\operatorname{div}\left(\theta\nabla^{\perp}(-\Delta)^{-\frac{1}{2}}\theta\right)\right](\xi)$$
$$= -\int_{\mathbb{R}^2} \frac{\xi \cdot \eta^{\perp}}{|\xi|^2|\eta|} \widehat{\theta}(\xi - \eta)\widehat{\theta}(\eta)d\eta.$$

Here, using

$$-\int_{\mathbb{R}^2} \frac{\xi \cdot \eta^{\perp}}{|\xi|^2 |\eta|} \widehat{\theta}(\xi - \eta) \widehat{\theta}(\eta) d\eta = -\int_{\mathbb{R}^2} \frac{\xi \cdot (\xi - \eta)^{\perp}}{|\xi|^2 |\xi - \eta|} \widehat{\theta}(\eta) \widehat{\theta}(\xi - \eta) d\eta$$
$$= \int_{\mathbb{R}^2} \frac{\xi \cdot \eta^{\perp}}{|\xi|^2 |\xi - \eta|} \widehat{\theta}(\xi - \eta) \widehat{\theta}(\eta) d\eta,$$

we have

$$\mathscr{F}\left[(-\Delta)^{-1}\operatorname{div}(\theta u)\right](\xi) = \frac{1}{2} \int_{\mathbb{R}^2} \frac{\xi \cdot \eta^{\perp}}{|\xi|^2} \left(\frac{1}{|\xi - \eta|} - \frac{1}{|\eta|}\right) \widehat{\theta}(\xi - \eta) \widehat{\theta}(\eta) d\eta$$

$$=\frac{1}{2}\int_{\mathbb{R}^2}\frac{\xi\cdot\eta^{\perp}}{|\xi|^2}\frac{\xi\cdot(2\eta-\xi)}{|\eta|+|\xi-\eta|}\frac{\widehat{\theta}(\xi-\eta)}{|\xi-\eta|}\frac{\widehat{\theta}(\eta)}{|\eta|}d\eta.$$

Therefore, setting

$$\mathcal{B}[\theta_1, \theta_2] := (-\Delta)^{-1} \operatorname{div} \operatorname{div} \mathcal{T} \left[(-\Delta)^{-\frac{1}{2}} \theta_1, \nabla^{\perp} (-\Delta)^{-\frac{1}{2}} \theta_2 \right], \tag{1.8}$$

$$\mathcal{T}[\theta, v] := \mathscr{F}^{-1} \left[\int_{\mathbb{R}^2} \frac{\xi - 2\eta}{2(|\eta| + |\xi - \eta|)} \otimes \left(\widehat{\theta}(\xi - \eta) \widehat{v}(\eta) \right) d\eta \right]$$
(1.9)

for appropriate given functions $\theta_1, \theta_2, \theta: \mathbb{R}^2 \to \mathbb{R}$ and $v: \mathbb{R}^2 \to \mathbb{R}^2$, we have

$$(-\Delta)^{-1}\operatorname{div}(u\theta) = \mathcal{B}[\theta, \theta], \qquad u = \nabla^{\perp}(-\Delta)^{-\frac{1}{2}}\theta.$$

In comparison to the single divergence form $(-\Delta)^{-1} \operatorname{div}(u\theta)$, the rewritten formula $\mathcal{B}[\theta,\theta]$ has the double divergence form, which help us from the difficulty stated above and succeed in closing the nonlinear estimates in some scaling critical Besov spaces; see Lemma 3.1 below. Motivated by this observation, we define the notion of the solution to (1.1) as follows.

Definition 1.1. For $1 \leq p, q \leq \infty$ and $f \in \dot{B}_{p,q}^{\frac{2}{p}-3}(\mathbb{R}^2)$, it is said that $\theta \in \dot{B}_{p,q}^{\frac{2}{p}-1}(\mathbb{R}^2)$ is a solution to (1.1) if it satisfies

$$\theta = \mathcal{L}f + \mathcal{B}[\theta, \theta],$$

where the operators \mathcal{L} and $\mathcal{B}[\cdot,\cdot]$ are defined in (1.7) and (1.8), respectively.

The following theorem is the first main result of this paper that states the well-posedness of (1.1) in the scaling critical Besov spaces.

Theorem 1.2 (Well-posedness of (1.1)). Let $1 \le p < 4$ and $1 \le q \le \infty$, or p = 4 and q = 2. Then, there exist positive constants $\delta_0 = \delta_0(p,q)$ and $\varepsilon_0 = \varepsilon_0(p,q)$ such that if the external force $f \in D_{p,q}(\mathbb{R}^2)$, where

$$D_{p,q}(\mathbb{R}^2) := \left\{ f \in \dot{B}_{p,q}^{\frac{2}{p}-3}(\mathbb{R}^2) \; ; \; \|f\|_{\dot{B}_{p,q}^{\frac{2}{p}-3}} \leqslant \delta_0 \right\},\,$$

then there exists a unique solution $\theta \in S_{p,q}(\mathbb{R}^2)$, where

$$S_{p,q}(\mathbb{R}^2) := \left\{ \theta \in \dot{B}_{p,q}^{\frac{2}{p}-1}(\mathbb{R}^2) \; ; \; \|\theta\|_{\dot{B}_{p,q}^{\frac{2}{p}-1}} \leqslant \varepsilon_0 \right\}.$$

Moreover, the solution map $\Phi: D_{p,q}(\mathbb{R}^2) \ni f \mapsto \theta \in S_{p,q}(\mathbb{R}^2)$ is Lipschitz continuous from $\dot{B}_{p,q}^{\frac{2}{p}-3}(\mathbb{R}^2)$ to $\dot{B}_{p,q}^{\frac{2}{p}-1}(\mathbb{R}^2)$.

Next, we focus on the optimality of the range of (p,q) for well-posedness and state that (1.1) is ill-posed for indexes other than those whose well-posedness is guaranteed by the aforementioned theorem.

Theorem 1.3 (Ill-posedness of (1.1)). Let p > 4 and $1 \le q \le \infty$, or p = 4 and $1 \le q \le \infty$ with $q \ne 2$. Then, (1.1) is ill-posed in the sense that the solution map $f \mapsto \theta$ is discontinuous from $\dot{B}_{p,q}^{\frac{2}{p}-3}(\mathbb{R}^2)$ to $\dot{B}_{p,q}^{\frac{2}{p}-1}(\mathbb{R}^2)$ at the origin. More precisely, the following statements hold true.

(1) For $4 and <math>1 \le q \le \infty$, or p = 4 and $2 < q \le \infty$, there exists a sequence $\{f_N\}_{N=1}^{\infty} \subset \mathcal{S}(\mathbb{R}^2)$ of the external forces satisfying

$$\lim_{N \to \infty} ||f_N||_{\dot{B}_{p,q}^{\frac{2}{p}-3}} = 0,$$

whereas the corresponding sequence $\{\theta_N\}_{N=1}^{\infty} \subset \dot{B}_{p,q}^{\frac{2}{p}-1}(\mathbb{R}^2)$ of solutions fulfills $\liminf_{N\to\infty} \|\theta_N\|_{\dot{B}_{\infty,\infty}^{-1}} > 0.$

(2) For p = 4 and $1 \leq q < 2$, there exists a sequence $\{f_N\}_{N=1}^{\infty} \subset \mathscr{S}(\mathbb{R}^2)$ of the external forces such that

$$\lim_{N \to \infty} \|f_N\|_{\dot{B}_{4,q}^{-\frac{5}{2}}} = 0,$$

while the corresponding sequence $\{\theta_N\}_{N=1}^{\infty} \subset \dot{B}_{4,q}^{-\frac{1}{2}}(\mathbb{R}^2)$ of solutions satisfies

$$\lim_{N\to\infty}\|\theta_N\|_{\dot{B}_{4,q}^{-\frac{1}{2}}}=\infty$$

Remark 1.4. We give some comments on Theorem 1.3. For the case of $(p,q) \in ((4,\infty)\times[1,\infty])\cup(\{4\}\times(2,\infty))$, we actually show that the continuity of the solution map fails in weaker topology $\dot{B}_{p,q}^{\frac{2}{p}-3}(\mathbb{R}^2)\to \dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^2)$, which is similar to [26]. For the case of $(p,q)\in\{4\}\times[1,2)$, we succeed in proving the norm-inflation phenomenon $\|\theta_N\|_{\dot{B}_{4,q}^{-\frac{1}{2}}}\geqslant cN^{\frac{1}{q}-\frac{1}{2}}/(\log N)^2$ that has not been proved for the ill-posedness of the stationary Navier–Stokes equations; see [12,21,24,26].

This paper is organized as follows. In Section 2, we recall the notations and definition of the Besov spaces. In Sections 3 and 4, we provide the proof of Theorems 1.2 and 1.3, respectively.

2. Preliminaries

In this section, we introduce notations and function spaces, which are to be used in this paper. Throughout this paper, we denote by $C \geqslant 1$ and 0 < c < 1 the constants, which may differ in each line. In particular, $C = C(a_1, ..., a_n)$ means that C depends only on $a_1, ..., a_n$. For two non-negative numbers A, B, the relation $A \sim B$ means that there exists positive constant C such that $C^{-1}A \leqslant B \leqslant CA$ holds. Let $\mathscr{S}(\mathbb{R}^2)$ be the set of all Schwartz functions on \mathbb{R}^2 and $\mathscr{S}'(\mathbb{R}^2)$ represents the set of all tempered distributions on \mathbb{R}^2 . We use $L^p(\mathbb{R}^2)$ with $1 \leqslant p \leqslant \infty$ to denote the standard Lebesgue spaces on \mathbb{R}^2 . For $f \in \mathscr{S}(\mathbb{R}^2)$, the Fourier transform and inverse Fourier transform of f are defined as

$$\mathscr{F}[f](\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}^2} e^{-ix\cdot\xi} f(x) dx, \qquad \mathscr{F}^{-1}[f](x) := \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix\cdot\xi} f(\xi) d\xi.$$

Let $\{\phi_j\}_{j\in\mathbb{Z}}\subset\mathscr{S}(\mathbb{R}^2)$ be a dyadic partition of unity satisfying

$$0 \leqslant \widehat{\phi}_0(\xi) \leqslant 1,$$

$$\operatorname{supp} \widehat{\phi}_0 \subset \{\xi \in \mathbb{R}^2 : 2^{-1} \leqslant |\xi| \leqslant 2\},$$

$$\widehat{\phi}_0(\xi) = 1 \quad \text{for all } \xi \in \mathbb{R}^2 \text{ with } \frac{7}{8} \leqslant |\xi| \leqslant \frac{5}{4},$$

and

$$\sum_{j\in\mathbb{Z}}\widehat{\phi_j}(\xi)=1\qquad\text{for all }\xi\in\mathbb{R}^2\setminus\{0\},$$

where $\widehat{\phi}_j(\xi) = \widehat{\phi}_0(2^{-j}\xi)$. We then define the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^2)$ $(1 \leq p, q \leq \infty, s \in \mathbb{R})$ by

$$\dot{B}_{p,q}^{s}(\mathbb{R}^{2}) := \left\{ f \in \mathscr{S}'(\mathbb{R}^{2}) / \mathscr{P}(\mathbb{R}^{2}) \; ; \; \|f\|_{\dot{B}_{p,q}^{s}} < \infty \right\},$$
$$\|f\|_{\dot{B}_{p,q}^{s}} := \left\| \left\{ 2^{sj} \|\phi_{j} * f\|_{L^{p}} \right\}_{j \in \mathbb{Z}} \right\|_{\ell^{q}},$$

where $\mathscr{P}(\mathbb{R}^2)$ denotes the set of all polynomials on \mathbb{R}^2 . It is well-known that if s < 2/p or (s,q) = (2/p,1), then $\dot{B}^s_{p,q}(\mathbb{R}^2)$ is identified as

$$\dot{B}_{p,q}^{s}(\mathbb{R}^{2}) \sim \left\{ f \in \mathscr{S}'(\mathbb{R}^{2}) ; f = \sum_{i \in \mathbb{Z}} \phi_{i} * f \text{ in } \mathscr{S}'(\mathbb{R}^{2}), \quad \|f\|_{\dot{B}_{p,q}^{s}} < \infty \right\}. (2.1)$$

See [23, Theorem 2.31] for the proof of (2.1). We refer to [23] for the basic properties of Besov spaces.

3. Proof of Theorem 1.2

In this section, we present the proof of Theorem 1.2.

Key lemmas. Before starting the proof of Theorem 1.2, we provide lemmas playing key roles in our calculus. The following lemma plays the most crucial role for controlling the nonlinearity of (1.1).

Lemma 3.1. Let $1 \le p < 4$ and $1 \le q \le \infty$, or p = 4 and q = 2. Then, there exists a positive constant C = C(p,q) such that

$$\|\mathcal{B}[f,g]\|_{\dot{B}^{\frac{2}{p}-1}_{p,q}} \leqslant C\|f\|_{\dot{B}^{\frac{2}{p}-1}_{p,q}} \|g\|_{\dot{B}^{\frac{2}{p}-1}_{p,q}}$$

for all $f, g \in \dot{B}^{\frac{2}{p}-1}_{p,q}(\mathbb{R}^2)$.

To prove Lemma 3.1, we prepare a fact for the boundedness of bilinear Fourier multiplier:

Lemma 3.2 ([13]). Let $m \in C^2((\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}^2 \setminus \{0\}))$ satisfy

$$\left|\partial_{\xi}^{\alpha}\partial_{\eta}^{\beta}m(\xi,\eta)\right| \leqslant C_{\alpha,\beta}\left(\left|\xi\right| + \left|\eta\right|\right)^{-\left(\left|\alpha\right| + \left|\beta\right|\right)}, \qquad \xi,\eta \in \mathbb{R}^{2}$$

for $\alpha, \beta \in (\mathbb{N} \cup \{0\})^2$ with $|\alpha|, |\beta| \leq 2$ and for some positive constant $C_{\alpha,\beta}$. Let T_m be a bilinear operator defined by

$$T_m[f,g](x) := \mathscr{F}^{-1} \left[\int_{\mathbb{R}^2} m(\xi - \eta, \eta) \widehat{f}(\xi - \eta) \widehat{g}(\eta) d\eta \right] (x)$$
$$= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2_{\xi} \times \mathbb{R}^2_{\eta}} e^{ix \cdot (\xi + \eta)} m(\xi, \eta) \widehat{f}(\xi) \widehat{g}(\eta) d\xi d\eta.$$

Let p, p_1, p_2 satisfy $1/p = 1/p_1 + 1/p_2$. Assume $1 and <math>1 < p_1, p_2 \le \infty$, or $2 \le p, p_1, p_2 < \infty$. Then, there exists a positive constant $C = C(p, p_1, p_2)$ such that

$$||T_m[f,g]||_{L^p} \leqslant C||f||_{L^{p_1}}||g||_{L^{p_2}}$$

for all $f \in L^{p_1}(\mathbb{R}^2)$ and $g \in L^{p_2}(\mathbb{R}^2)$. Moreover, there holds

$$||T_m[f,g]||_{L^1} \leqslant C||f||_{L^\infty}||g||_{\mathcal{H}^1}$$

for all $f \in L^{\infty}(\mathbb{R}^2)$ and $g \in \mathcal{H}^1(\mathbb{R}^2)$. Here, $\mathcal{H}^1(\mathbb{R}^2)$ denotes the Hardy space on \mathbb{R}^2 with the integrability index 1.

We are in a position to show Lemma 3.1.

Proof. Recalling the Bony decomposition, we decompose $\mathcal{B}[f,g]$ as

$$\mathcal{B}[f,g] = \mathcal{B}_1[f,g] + \mathcal{B}_2[f,g] + \mathcal{B}_3[f,g],$$

where

$$\mathcal{B}_{1}[f,g] := \sum_{k \leqslant \ell-3} \mathcal{B}\left[f_{k},g_{\ell}\right]$$

$$= \frac{1}{2} \sum_{k \leqslant \ell-3} (-\Delta)^{-1} \operatorname{div}\left\{f_{k}\left(\nabla^{\perp}(-\Delta)^{-\frac{1}{2}}g_{\ell}\right) + \left(\nabla^{\perp}(-\Delta)^{-\frac{1}{2}}f_{k}\right)g_{\ell}\right\},$$

$$\mathcal{B}_{2}[f,g] := \mathcal{B}_{1}[g,f],$$

$$\mathcal{B}_{3}[f,g] := \sum_{|k-\ell| \leqslant 2} \mathcal{B}\left[f_{k},g_{\ell}\right].$$

Here, we have used the abbreviation $h_k := \phi_k * h$ for $k \in \mathbb{Z}$ and $h \in \mathscr{S}'(\mathbb{R}^2)/\mathscr{P}(\mathbb{R}^2)$, and also used the following identity

$$\mathcal{B}[f_k, g_\ell] = \frac{1}{2} (-\Delta)^{-1} \operatorname{div} \left\{ f_k \left(\nabla^{\perp} (-\Delta)^{-\frac{1}{2}} g_\ell \right) + \left(\nabla^{\perp} (-\Delta)^{-\frac{1}{2}} f_k \right) g_\ell \right\},\,$$

which is obtained by the similar calculations as in the motivation of the definition of $\mathcal{B}[\cdot,\cdot]$ in Section 1. By the standard paraproduct estimates, there holds

$$\begin{split} &\|\mathcal{B}_{1}[f,g]\|_{\dot{B}^{\frac{2}{p}-1}_{p,q}} \\ &\leqslant C \left\| \left\{ 2^{(\frac{2}{p}-2)j} \sum_{|m|\leqslant 3} \sum_{k\leqslant j+m-3} \|f_{k}\|_{L^{\infty}} \|\nabla^{\perp}(-\Delta)^{-\frac{1}{2}}g_{j+m}\|_{L^{p}} \right\}_{j\in\mathbb{Z}} \right\|_{\ell^{q}} \\ &+ C \left\| \left\{ 2^{(\frac{2}{p}-2)j} \sum_{|m|\leqslant 3} \sum_{k\leqslant j+m-3} \|\nabla^{\perp}(-\Delta)^{-\frac{1}{2}}f_{k}\|_{L^{\infty}} \|g_{j+m}\|_{L^{p}} \right\}_{j\in\mathbb{Z}} \right\|_{\ell^{q}} \\ &\leqslant C \left\| \left\{ \sum_{|m|\leqslant 3} \left\{ \sum_{k\leqslant j+m-3} \left(2^{-k} \|f_{k}\|_{L^{\infty}} \right)^{q} \right\}^{\frac{1}{q}} 2^{(\frac{2}{p}-1)(j+m)} \|g_{j+m}\|_{L^{p}} \right\}_{j\in\mathbb{Z}} \right\|_{\ell^{q}} \\ &\leqslant C \|f\|_{\dot{B}^{-1}_{\infty,q}} \|g\|_{\dot{B}^{\frac{2}{p}-1}_{p,q}} \leqslant C \|f\|_{\dot{B}^{\frac{2}{p}-1}_{p,q}} \|g\|_{\dot{B}^{\frac{2}{p}-1}_{p,q}}. \end{split}$$

Similarly, we have

$$\|\mathcal{B}_2[f,g]\|_{\dot{B}^{\frac{2}{p}-2}_{p,q}} \leqslant C\|f\|_{\dot{B}^{\frac{2}{p}-1}_{p,q}}\|g\|_{\dot{B}^{\frac{2}{p}-1}_{p,q}}.$$

We note that the estimates for $\mathcal{B}_1[f,g]$ and $\mathcal{B}_2[f,g]$ hold for all $1 \leq p,q \leq \infty$. For the estimate of $\mathcal{B}_3[f,g]$, we see that

$$\|\mathcal{B}_{3}[f,g]\|_{\dot{B}^{\frac{2}{p}-2}_{p,q}} \leqslant C \left\| \sum_{|k-\ell| \leqslant 2} \mathcal{T}\left[(-\Delta)^{-\frac{1}{2}} f_{k}, \nabla^{\perp} (-\Delta)^{-\frac{1}{2}} g_{\ell} \right] \right\|_{\dot{B}^{\frac{2}{p}-1}_{p,q}},$$

where $\mathcal{T}[\cdot,\cdot]$ is defined in (1.9). We will apply Lemma 3.2 to control the bilinear operator $\mathcal{T}[\cdot,\cdot]$. For the case of (p,q)=(4,2), it holds

$$\|\mathcal{B}_{3}[f,g]\|_{\dot{B}_{4,2}^{-\frac{1}{2}}} \leqslant C \left\| \sum_{|k-\ell| \leqslant 2} \mathcal{T} \left[(-\Delta)^{-\frac{1}{2}} f_{k}, \nabla^{\perp} (-\Delta)^{-\frac{1}{2}} g_{\ell} \right] \right\|_{L^{2}}$$

$$\leqslant C \sum_{|k-\ell| \leqslant 2} \left\| \mathcal{T} \left[(-\Delta)^{-\frac{1}{2}} f_{k}, \nabla^{\perp} (-\Delta)^{-\frac{1}{2}} g_{\ell} \right] \right\|_{L^{2}}$$

$$\leqslant C \sum_{|k-\ell| \leqslant 2} \left\| (-\Delta)^{-\frac{1}{2}} f_{k} \right\|_{L^{4}} \left\| \nabla^{\perp} (-\Delta)^{-\frac{1}{2}} g_{\ell} \right\|_{L^{4}}$$

$$\leqslant C \sum_{|k-\ell| \leqslant 2} 2^{-k} \|f_{k}\|_{L^{4}} \|g_{\ell}\|_{L^{4}}$$

$$\leqslant C \|f\|_{\dot{B}_{4,2}^{-\frac{1}{2}}} \|g\|_{\dot{B}_{4,2}^{-\frac{1}{2}}}$$

For the case of $2 \leq p < 4$, we see that

$$\|\mathcal{B}_{3}[f,g]\|_{\dot{B}^{\frac{2}{p}-1}_{p,q}} \leqslant C \left\| \sum_{|k-\ell| \leqslant 2} T \left[(-\Delta)^{-\frac{1}{2}} f_{k}, \nabla^{\perp} (-\Delta)^{-\frac{1}{2}} g_{\ell} \right] \right\|_{\dot{B}^{\frac{4}{p}-1}_{\frac{p}{2},q}}$$

$$\leqslant C \left\| \left\{ 2^{(\frac{4}{p}-1)j} \sum_{k \geqslant j-4} \sum_{|k-\ell| \leqslant 2} \left\| T \left[(-\Delta)^{-\frac{1}{2}} f_{k}, \nabla^{\perp} (-\Delta)^{-\frac{1}{2}} g_{\ell} \right] \right\|_{L^{\frac{p}{2}}} \right\}_{j \in \mathbb{Z}} \right\|_{\ell^{q}}$$

$$\leqslant C \left\| \left\{ 2^{(\frac{4}{p}-1)j} \sum_{k \geqslant j-4} \sum_{|k-\ell| \leqslant 2} \left\| (-\Delta)^{-\frac{1}{2}} f_{k} \right\|_{L^{p}} \|g_{\ell}\|_{L^{p}} \right\}_{j \in \mathbb{Z}} \right\|_{\ell^{q}}$$

$$\leqslant C \|f\|_{\dot{B}^{\frac{2}{p}-1}_{p,q}} \|g\|_{\dot{B}^{\frac{2}{p}-1}_{p,q}}$$

For the case of 1 , we have

$$\|\mathcal{B}_{3}[f,g]\|_{\dot{B}^{\frac{2}{p}-1}_{p,q}} \leqslant C \left\| \left\{ 2^{(\frac{2}{p}-1)j} \sum_{k\geqslant j-4} \sum_{|k-\ell|\leqslant 2} \left\| T \left[(-\Delta)^{-\frac{1}{2}} f_{k}, \nabla^{\perp} (-\Delta)^{-\frac{1}{2}} g_{\ell} \right] \right\|_{L^{p}} \right\}_{j\in\mathbb{Z}} \right\|_{\ell^{q}}$$

$$\leqslant C \left\| \left\{ 2^{(\frac{2}{p}-1)j} \sum_{k\geqslant j-4} \sum_{|k-\ell|\leqslant 2} \left\| (-\Delta)^{-\frac{1}{2}} f_{k} \right\|_{L^{\infty}} \|g_{\ell}\|_{L^{p}} \right\}_{j\in\mathbb{Z}} \right\|_{\ell^{q}}$$

$$\leqslant C \|f\|_{\dot{B}^{-1}_{\infty,q}} \|g\|_{\dot{B}^{\frac{2}{p}-1}_{p,q}} \leqslant C \|f\|_{\dot{B}^{\frac{2}{p}-1}_{p,q}} \|g\|_{\dot{B}^{\frac{2}{p}-1}_{p,q}}.$$

For the case of p = 1, it follows that

$$\|\mathcal{B}_{3}[f,g]\|_{\dot{B}^{\frac{2}{p}-1}_{p,q}} \leqslant C \left\| \left\{ 2^{j} \sum_{k \geqslant j-4} \sum_{|k-\ell| \leqslant 2} \left\| T\left[(-\Delta)^{-\frac{1}{2}} f_{k}, \nabla^{\perp} (-\Delta)^{-\frac{1}{2}} g_{\ell} \right] \right\|_{L^{1}} \right\}_{j \in \mathbb{Z}} \right\|_{\ell^{q}}$$

$$\leqslant C \left\| \left\{ 2^{j} \sum_{k \geqslant j-4} \sum_{|k-\ell| \leqslant 2} \left\| (-\Delta)^{-\frac{1}{2}} f_{k} \right\|_{L^{\infty}} \|g_{\ell}\|_{\mathcal{H}^{1}} \right\}_{j \in \mathbb{Z}} \right\|_{\ell^{q}} \\
\leqslant C \|f\|_{\dot{B}_{\infty,q}^{-1}} \|g\|_{\dot{B}_{1,q}^{1}} \leqslant C \|f\|_{\dot{B}_{1,q}^{1}} \|g\|_{\dot{B}_{1,q}^{1}},$$

where we have used

$$\left\| \left\{ 2^{\ell} \|g_{\ell}\|_{\mathcal{H}^{1}} \right\}_{j \in \mathbb{Z}} \right\|_{\ell^{q}} \sim \|g\|_{\dot{B}^{1}_{1,q}},$$

which is implied by

$$||g_{\ell}||_{\mathcal{H}^1} \sim ||g_{\ell}||_{\dot{F}_{1,2}^0} \sim ||g_{\ell-1}||_{L^1} + ||g_{\ell}||_{L^1} + ||g_{\ell+1}||_{L^1}.$$

Here, $\dot{F}_{1,2}^0(\mathbb{R}^2)$ denotes the Lizorkin–Tribel space. Thus, we complete the proof. \square

Proof of Theorem 1.2. Now, we are in a position to present the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $C_0 = C_0(p)$ be a positive constant satisfying

$$\|\mathcal{L}f\|_{\dot{B}^{\frac{2}{p}-1}_{p,q}} \leqslant C_0 \|f\|_{\dot{B}^{\frac{2}{p}-3}_{p,q}}.$$

Let $C = C_1$ be the positive constant appearing in Lemma 3.1. Let $f \in \dot{B}_{p,q}^{\frac{2}{p}-3}(\mathbb{R}^2)$ satisfy

$$||f||_{\dot{B}_{p,q}^{\frac{2}{p}-3}} \leqslant \frac{1}{8C_0C_1}.$$

Let us define the complete metric space (X, d_X) by

$$X := \left\{ \theta \in \dot{B}_{p,q}^{\frac{2}{p}-1}(\mathbb{R}^2) \; ; \; \|\theta\|_{\dot{B}_{p,q}^{\frac{2}{p}-1}} \leqslant \frac{1}{4C_1} . \right\},$$
$$d_X(f,g) := \|f - g\|_{\dot{B}_{p,q}^{\frac{2}{p}-1}}.$$

We consider a map $\Phi[\cdot]$ on X given by

$$\Phi[\theta] := \mathcal{L}f + \mathcal{B}[\theta, \theta], \qquad \theta \in X.$$

For each $\theta \in X$, it follows from Lemma 3.1 that

$$\|\Phi[\theta]\|_{\dot{B}_{p,q}^{\frac{2}{p}-1}} \leqslant C_0 \|f\|_{\dot{B}_{p,q}^{\frac{2}{p}-3}} + C_1 \|\theta\|_{\dot{B}_{p,q}^{\frac{2}{p}-1}}^2$$
$$\leqslant \frac{1}{8C_1} + \frac{1}{16C_1} \leqslant \frac{1}{4C_1},$$

which implies $\Phi[\theta] \in X$. Let $\theta_1, \theta_2 \in X$. Then, since

$$\Phi[\theta_1] - \Phi[\theta_2] = \mathcal{B}[\theta_1 - \theta_2, \theta_1] + \mathcal{B}[\theta_2, \theta_1 - \theta_2],$$

we see that

$$\|\Phi[\theta_{1}] - \Phi[\theta_{2}]\|_{\dot{B}_{p,q}^{\frac{2}{p}-1}} \leqslant C_{1} \left(\|\theta_{1}\|_{\dot{B}_{p,q}^{\frac{2}{p}-1}} + \|\theta_{2}\|_{\dot{B}_{p,q}^{\frac{2}{p}-1}} \right) \|\theta_{1} - \theta_{2}\|_{\dot{B}_{p,q}^{\frac{2}{p}-1}}$$

$$\leqslant \frac{1}{2} \|\theta_{1} - \theta_{2}\|_{\dot{B}_{p,q}^{\frac{2}{p}-1}}.$$

Hence, $\Phi[\cdot]$ is a contraction map on (X, d_X) and thus the Banach fixed point theorem yields the unique existence of $\theta \in X$ satisfying $\theta = \Phi[\theta]$. This completes the proof.

4. Proof of Theorem 1.3

We are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let $0 < \delta \le 1/100$ be a positive constant to be determined depending only on p and q later. In the following, we divide the proof into three steps:

Step 1. $4 and <math>1 \leq q \leq \infty$,

Step 2. p = 4 and $2 < q \leq \infty$,

Step 3. p = 4 and $1 \leq q < 2$.

We mention the outline of the argument in each step. We first define the squeezing sequence $\{f_N\}_{N=1}^{\infty} \subset \mathscr{S}(\mathbb{R}^2)$ of the external forces and then establish the estimates for the first and second iterations

$$\theta_N^{(1)} := \mathcal{L} f_N, \qquad \theta_N^{(2)} := \mathcal{B}[\theta_N^{(1)}, \theta_N^{(1)}],$$

and observe that the $\|\theta_N^{(1)}\|_{\dot{B}_{p,q}^{\frac{2}{p}-1}}$ vanishes as $N \to \infty$ while the norm of $\theta_N^{(2)}$ is bounded from below by a positive quantity that does not tends to 0 as $N \to \infty$. Then, we write the solution $\theta_N = \theta_N^{(1)} + \theta_N^{(2)} + \widetilde{\theta}_N$, where the perturbation $\widetilde{\theta}_N$ should solve

$$\widetilde{\theta}_{N} = \mathcal{B}[\theta_{N}^{(1)}, \theta_{N}^{(2)}] + \mathcal{B}[\theta_{N}^{(2)}, \theta_{N}^{(1)}] + \mathcal{B}[\theta_{N}^{(2)}, \theta_{N}^{(2)}]
+ \mathcal{B}[\theta_{N}^{(1)}, \widetilde{\theta}_{N}] + \mathcal{B}[\widetilde{\theta}_{N}, \theta_{N}^{(1)}] + \mathcal{B}[\theta_{N}^{(2)}, \widetilde{\theta}_{N}] + \mathcal{B}[\widetilde{\theta}_{N}, \theta_{N}^{(2)}] + \mathcal{B}[\widetilde{\theta}_{N}, \widetilde{\theta}_{N}].$$
(4.1)

Solving (4.1) in $\dot{B}_{4,2}^{-\frac{1}{2}}(\mathbb{R}^2)$ similarly as in Theorem 1.2, we find that $\widetilde{\theta}_N$ is relatively smaller than $\theta_N^{(2)}$. Then, combining the estimates for $\theta_N^{(1)}$, $\theta_N^{(2)}$, and $\widetilde{\theta}_N$, we will complete the proof.

Step 1. The case of p > 4 and $1 \leq q \leq \infty$. Let $\chi \in (\mathbb{R}^2)$ be a radial symmetric real valued function satisfying

$$\widehat{\chi}(\xi) = \begin{cases} 1 & (|\xi| \leqslant 1), \\ 0 & (|\xi| \geqslant 2). \end{cases}$$

We define

$$f_N(x) := \delta 2^{\frac{5}{2}N} \chi(x) \cos(2^N x_1), \qquad N \geqslant 100.$$

Then, it follows from

$$\widehat{f_N}(\xi) = \frac{\delta 2^{\frac{5}{2}N}}{2} \left(\widehat{\chi}(\xi + 2^N e_1) + \widehat{\chi}(\xi - 2^N e_1) \right)$$

that supp $\widehat{f_N} \subset \{\xi \in \mathbb{R}^2 ; 2^N - 2 \leqslant |\xi| \leqslant 2^N + 2\}$, which implies

$$||f_N||_{\dot{B}_{p,q}^{\frac{2}{p}-3}} \leqslant C_1 \delta 2^{N(\frac{2}{p}-\frac{1}{2})},$$

$$||\theta_N^{(1)}||_{\dot{B}_{p,q}^{\frac{2}{p}-1}} \leqslant C||f_N||_{\dot{B}_{p,q}^{\frac{2}{p}-3}} \leqslant C_1 \delta 2^{N(\frac{2}{p}-\frac{1}{2})}$$

for some positive constant $C_1 = C_1(\|\phi_0\|_{L^1}, \|\chi\|_{L^p})$. As p > 4, we see that

$$\lim_{N \to \infty} ||f_N||_{\dot{B}_{p,q}^{\frac{2}{p}-3}} = 0.$$

Next, we consider the estimates of the second iteration $\theta_N^{(2)}$. In what follows, we assume $|\xi| \leq 1$. We see that

$$\begin{split} \widehat{\theta_N^{(2)}}(\xi) &= \frac{\delta^2 2^{5N}}{2} \frac{\xi_1 \xi_2}{|\xi|^2} \int_{\mathbb{R}^2} \frac{\eta_1^2}{|\xi - \eta| + |\eta|} \frac{\widehat{\chi}(\xi - \eta + 2^N e_1)}{|\xi - \eta|^3} \frac{\widehat{\chi}(\eta - 2^N e_1)}{|\eta|^3} d\eta \\ &+ \frac{\delta^2 2^{5N}}{2|\xi|^2} \int_{\mathbb{R}^2} \frac{(\xi_2^2 - \xi_1^2) \eta_1 \eta_2 - \xi_1 \xi_2 \eta_2^2}{|\xi - \eta| + |\eta|} \frac{\widehat{\chi}(\xi - \eta + 2^N e_1)}{|\xi - \eta|^3} \frac{\widehat{\chi}(\eta - 2^N e_1)}{|\eta|^3} d\eta \\ &- \frac{\delta^2 2^{5N}}{4} \int_{\mathbb{R}^2} \frac{\xi \cdot \eta^\perp}{|\xi - \eta| + |\eta|} \frac{\widehat{\chi}(\xi - \eta + 2^N e_1)}{|\xi - \eta|^3} \frac{\widehat{\chi}(\eta - 2^N e_1)}{|\eta|^3} d\eta \\ = &: \widehat{I_N^{(1)}}(\xi) + \widehat{I_N^{(2)}}(\xi) + \widehat{I_N^{(3)}}(\xi). \end{split}$$

We consider the estimate of $\widehat{I_N^{(1)}}(\xi)$. We introduce $\psi_n \in \mathscr{S}(\mathbb{R}^2)$ defined by

$$\widehat{\psi_j}(\xi) = 1 - \sum_{k=j-10}^{\infty} \widehat{\phi_k} (\xi - 2^j a), \quad a := \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix},$$

which enable us to focus on the frequency in the region $\{\xi = (\xi_1, \xi_2) ; \xi_1, \xi_2 > 0\}$. Note that ψ_j has the following properties:

$$\psi_j(x) = 2^{2j} \psi_0(2^j x),$$

$$\operatorname{supp} \widehat{\psi_j} \subset \{ \xi \in \mathbb{R}^2 ; |\xi - 2^j a| \leqslant 2^{j-10} \},$$

$$\psi_i = \phi_i * \psi_i$$

for $j \in \mathbb{Z}$. In particular, it follows from the last property that

$$\|\psi_{j} * f\|_{L^{p}} \leqslant C \|\phi_{j} * f\|_{L^{p}}$$

for all $1 \leq p \leq \infty$, $j \in \mathbb{Z}$, and distributions f with $\Delta_j f \in L^p(\mathbb{R}^2)$. For $\eta \in \text{supp}(\widehat{\chi}(\xi - \cdot + 2^N e_1)\widehat{\chi}(\cdot - 2^N e_1))$, we see that $|\xi| \ll |\xi - \eta| \sim |\eta| \sim \eta_1 \sim 2^N$ and $|\eta_2| \lesssim 1$. Thus, we have

$$\widehat{\psi_j}(\xi)\widehat{I_N^{(1)}}(\xi) \sim \delta^2 \widehat{\psi_j}(\xi) \frac{\xi_1 \xi_2}{|\xi|^2} \int_{\mathbb{R}^2} \widehat{\chi}(\xi - \eta + 2^N e_1) \widehat{\chi}(\eta - 2^N e_1) d\eta$$

$$= \delta^2 \widehat{\psi_j}(\xi) \frac{\xi_1 \xi_2}{|\xi|^2} \int_{\mathbb{R}^2} \widehat{\chi}(\xi - \eta') \widehat{\chi}(\eta') d\eta'$$

$$\sim \delta^2 \widehat{\psi_j}(\xi).$$

For the estimate of $\widehat{I_N^{(2)}}(\xi)$, it holds

$$\left| \widehat{I_N^{(2)}}(\xi) \right| \leqslant C\delta^2 2^{5N} \int_{\mathbb{R}^2} \frac{\eta_1 |\eta_2|}{|\xi - \eta| + |\eta|} \frac{\widehat{\chi}(\xi - \eta + 2^N e_1)}{|\xi - \eta|^3} \frac{\widehat{\chi}(\eta - 2^N e_1)}{|\eta|^3} d\eta$$
$$\leqslant C\delta^2 2^{-N} \int_{\mathbb{R}^2} \widehat{\chi}(\xi - \eta + 2^N e_1) \widehat{\chi}(\eta - 2^N e_1) d\eta$$
$$\leqslant C\delta^2 2^{-N}.$$

For the estimate of $\widehat{I_N^{(3)}}(\xi)$, we have

$$\left|\widehat{I_N^{(3)}}(\xi)\right| \leqslant \delta^2 2^{5N} |\xi| \int_{\mathbb{R}^2} \frac{|\eta|}{|\xi - \eta| + |\eta|} \frac{\widehat{\chi}(\xi - \eta + 2^N e_1)}{|\xi - \eta|^3} \frac{\widehat{\chi}(\eta - 2^N e_1)}{|\eta|^3} d\eta$$

$$\leqslant C\delta^2 2^{-N} |\xi| \int_{\mathbb{R}^2} \widehat{\chi}(\xi - \eta + 2^N e_1) \widehat{\chi}(\eta - 2^N e_1) d\eta$$

$$\leqslant C\delta^2 2^{-N}.$$

Hence, we obtain for each $j \leq -1$ that

$$\begin{split} 2^{-j} \left\| \phi_{j} * \theta_{N}^{(2)} \right\|_{L^{\infty}} &\geqslant c2^{-j} \left\| \psi_{j} * I_{N}^{(1)} \right\|_{L^{\infty}} - \sum_{m=2}^{3} 2^{-j} \left\| \phi_{j} * I_{N}^{(m)} \right\|_{L^{\infty}} \\ &\geqslant c \left\| e^{-2^{2j} |\cdot|^{2}} \left(\psi_{j} * I_{N}^{(1)} \right) \right\|_{L^{2}} - C \sum_{m=2}^{3} \left\| \phi_{j} * I_{N}^{(m)} \right\|_{L^{2}} \\ &= c \left\| \frac{1}{4\pi 2^{2j}} \int_{\mathbb{R}^{2}} e^{-\frac{|\xi - \xi'|^{2}}{2^{2j+2}}} \widehat{\psi_{0}}(2^{-j}\xi') \widehat{I_{N}^{(1)}}(\xi') d\xi' \right\|_{L^{2}} - C \sum_{m=2}^{3} \left\| \widehat{\phi_{j}}(\xi) \widehat{I_{N}^{(m)}}(\xi) \right\|_{L^{2}} \\ &\geqslant c\delta^{2} 2^{-2j} \left\| \int_{\mathbb{R}^{2}} e^{-\frac{|\xi - \xi'|^{2}}{2^{2j+2}}} \widehat{\psi_{0}}(2^{-j}\xi') d\xi' \right\|_{L^{2}} - C\delta^{2} 2^{-N} \left\| \widehat{\phi_{0}}(2^{-j}\xi) \right\|_{L^{2}} \\ &= c\delta^{2} \left\| \int_{\mathbb{R}^{2}} e^{-\frac{|\xi - \xi'|^{2}}{4}} \widehat{\psi_{0}}(\xi') d\xi' \right\|_{L^{2}} - C\delta^{2} 2^{j} 2^{-N} \\ &= c\delta^{2} 2^{j} \left\| \int_{\mathbb{R}^{2}} e^{-\frac{|\xi - \xi'|^{2}}{4}} \widehat{\psi_{0}}(\xi') d\xi' \right\|_{L^{2}} - C\delta^{2} 2^{j} 2^{-N} \\ &= c\delta^{2} 2^{j} - C\delta^{2} 2^{j} 2^{-N}. \end{split}$$

If θ_N is a solution to (1.1) with the external force f_N , then the higher iteration part $\widetilde{\theta}_N := \theta_N - \theta_N^{(1)} - \theta_N^{(2)}$ should solve (4.1). From Lemma 3.1 it follows that

$$\begin{split} \left\| \mathcal{B}[\theta_N^{(1)}, \theta_N^{(2)}] + \mathcal{B}[\theta_N^{(2)}, \theta_N^{(1)}] \right\|_{\dot{B}_{4,2}^{-\frac{1}{2}}} &\leqslant C \left\| \theta_N^{(1)} \right\|_{\dot{B}_{4,2}^{-\frac{1}{2}}} \left\| \theta_N^{(2)} \right\|_{\dot{B}_{4,2}^{-\frac{1}{2}}} &\leqslant C \delta^3, \\ \left\| \mathcal{B}[\theta_N^{(2)}, \theta_N^{(2)}] \right\|_{\dot{B}_{4,2}^{-\frac{1}{2}}} &\leqslant C \left\| \theta_N^{(1)} \right\|_{\dot{B}_{4,2}^{-\frac{1}{2}}} &\leqslant C \delta^4, \\ \left\| \mathcal{B}[\theta_N^{(1)}, \theta] + \mathcal{B}[\theta, \theta_N^{(1)}] \right\|_{\dot{B}_{4,2}^{-\frac{1}{2}}} &\leqslant C \left\| \theta_N^{(1)} \right\|_{\dot{B}_{4,2}^{-\frac{1}{2}}} \| \theta \|_{\dot{B}_{4,2}^{-\frac{1}{2}}} &\leqslant C \delta \| \theta \|_{\dot{B}_{4,2}^{-\frac{1}{2}}} \\ \left\| \mathcal{B}[\theta_N^{(2)}, \theta] + \mathcal{B}[\theta, \theta_N^{(2)}] \right\|_{\dot{B}_{4,2}^{-\frac{1}{2}}} &\leqslant C \left\| \theta_N^{(2)} \right\|_{\dot{B}_{4,2}^{-\frac{1}{2}}} \| \theta \|_{\dot{B}_{4,2}^{-\frac{1}{2}}} \\ &\leqslant C \left\| \theta_N^{(1)} \right\|_{\dot{B}_{4,2}^{-\frac{1}{2}}} \| \theta \|_{\dot{B}_{4,2}^{-\frac{1}{2}}} \\ &\leqslant C \delta^2 \| \theta \|_{\dot{B}_{4,2}^{-\frac{1}{2}}}, \\ \| \mathcal{B}[\theta, \theta'] \|_{\dot{B}_{4,2}^{-\frac{1}{2}}} &\leqslant C \| \theta \|_{\dot{B}_{4,2}^{-\frac{1}{2}}} \| \theta' \|_{\dot{B}_{4,2}^{-\frac{1}{2}}} \end{split}$$

for all $\theta, \theta' \in \dot{B}_{4,2}^{-\frac{1}{2}}(\mathbb{R}^2)$. Then, it follows from the similar contraction mapping argument via the suitable small δ as in the proof of Theorem 1.2 that there exists a unique solution $\widetilde{\theta}_N \in \dot{B}_{4,2}^{-\frac{1}{2}}(\mathbb{R}^2)$ to (4.1) satisfying

$$\left\|\widetilde{\theta}_N\right\|_{\dot{B}_{4,2}^{-\frac{1}{2}}} \leqslant C\delta^3$$

with some positive constant C independent of N. Hence, $\theta_N := \theta_N^{(1)} + \theta_N^{(2)} + \widetilde{\theta}_N$ solves (1.1) with the force f_N and satisfies

$$\|\theta_N\|_{\dot{B}_{\infty,\infty}^{-1}} \geqslant \sup_{j \leqslant -1} 2^{-j} \|\phi_j * \theta_N^{(2)}\|_{L^{\infty}} - \|\theta_N^{(1)}\|_{\dot{B}_{\infty,\infty}^{-1}} - C \|\widetilde{\theta}_N\|_{\dot{B}_{4,2}^{-\frac{1}{2}}}$$
$$\geqslant c\delta^2 - C2^{-N} - C\delta2^{-\frac{N}{2}} - C\delta^3.$$

choosing δ sufficiently small, we obtain $\liminf_{N\to\infty} \|\theta_N\|_{\dot{B}_{\infty,\infty}^{-1}} > 0$. Step 2. The case of p=4 and $2< q \leqslant \infty$. Let

$$f_N(x) := \frac{\delta}{\sqrt{\log N}} \sum_{n=10}^N \frac{2^{\frac{5}{2}n^2}}{\sqrt{n}} \chi(x) \cos(2^{n^2} x_1), \qquad N \geqslant 100.$$

Then, we see that

$$||f_N||_{\dot{B}_{4,q}^{-\frac{5}{2}}} \leqslant \frac{C\delta}{\sqrt{\log N}} \left\{ \sum_{n=10}^{N} \frac{1}{n^{\frac{q}{2}}} \right\}^{\frac{1}{q}} \leqslant \begin{cases} \frac{C\delta}{\sqrt{\log N}} & (2 < q \leqslant \infty), \\ C\delta & (q=2) \end{cases}$$

and

$$\left\| \theta_N^{(1)} \right\|_{\dot{B}_{4,q}^{-\frac{1}{2}}} \leqslant C \|f_N\|_{\dot{B}_{4,q}^{-\frac{5}{2}}} \leqslant \begin{cases} \frac{C\delta}{\sqrt{\log N}} & (2 < q \leqslant \infty), \\ C\delta & (q = 2). \end{cases}$$

Similarly to the above calculations as in Step 1, we have

$$\widehat{\theta_N^{(2)}}(\xi) = \sum_{n=M}^N \frac{1}{n \log N} \left[\widehat{I_{n^2}^{(1)}}(\xi) + \widehat{I_{n^2}^{(2)}}(\xi) + \widehat{I_{n^2}^{(3)}}(\xi) \right]$$

$$=: \widehat{J_N^{(1)}}(\xi) + \widehat{J_N^{(2)}}(\xi) + \widehat{J_N^{(3)}}(\xi)$$

for all $\xi \in \mathbb{R}^2$ with $|\xi| \leq 1$. Then, we see that

$$\widehat{\psi_j}(\xi)\widehat{J_N^{(1)}}(\xi) = \frac{1}{\log N} \sum_{n=10}^N \frac{1}{n} \widehat{\psi_j}(\xi) \widehat{I_{n^2}^{(1)}}(\xi) \sim -\delta^2 \widehat{\psi_j}(\xi)$$

and

$$\begin{split} \sum_{m=2}^{3} \left| \widehat{J_N^{(m)}}(\xi) \right| &\leqslant \sum_{m=2}^{3} \sum_{n=10}^{N} \frac{1}{n \log N} \left| \widehat{I_N^{(m)}}(\xi) \right| \\ &\leqslant C \frac{\delta^2}{\log N} \sum_{n=10}^{N} \frac{2^{-n}}{n} \\ &\leqslant C \frac{\delta^2}{\log N}. \end{split}$$

Then, from the similar calculation as above, it holds that

$$2^{-j} \left\| \phi_j * \theta_N^{(2)} \right\|_{L^{\infty}} \geqslant c \left\| e^{-2^{2j} |\cdot|^2} \left(\psi_j * J_N^{(1)} \right) \right\|_{L^2} - C \sum_{m=2}^3 \left\| \phi_j * J_N^{(m)} \right\|_{L^2}$$
$$\geqslant c \delta^2 \left\| \frac{1}{4\pi 2^{2j}} \int_{\mathbb{R}^2} e^{-\frac{|\xi - \xi'|^2}{2^{2j+2}}} \widehat{\psi_0}(2^{-j} \xi') \widehat{J_N^{(1)}}(\xi') d\xi' \right\|_{L^2} - C \sum_{m=2}^3 \left\| \widehat{\phi_j}(\xi) \widehat{J_N^{(m)}}(\xi) \right\|_{L^2}$$

$$\geqslant c\delta^2 2^j - C \frac{\delta^2}{\log N} 2^j$$

for all $j \ge -1$. The rest of the proof is completely same as in Step 1. Step 3. The case of p = 4 and $1 \le q < 2$. Let $N \ge 200$ and let us define

$$f_N(x) = \frac{\delta 2^{\frac{5}{2}N^2}}{N^{\frac{1}{4}}\log N} F_N(x) \cos\left(2^{N^2}x_1\right),$$

where we have set

$$F_N(x) := \sum_{k=10}^{N-50} 2^{-\frac{3}{2}k^2} \phi_{k^2}(x - Rk^2 e_1).$$

Here, we choose R > 0 so large that

$$||F_N||_{L^4} \leqslant CN^{\frac{1}{4}}$$

holds with some constant C independent of N, which is proved by the similar calculation as in [5,15]. We notice that

supp
$$\widehat{f_N} \subset \left\{ \xi \in \mathbb{R}^2 \; ; \; 2^{N^2 - 1} \leqslant |\xi| \leqslant 2^{N^2 + 1} \right\},$$

supp $\widehat{F_N} \subset \left\{ \xi \in \mathbb{R}^2 \; ; \; 2^{99} \leqslant |\xi| \leqslant 2^{(N - 50)^2 + 1} \right\}.$

For the estimate of f_N , it holds

$$||f_N||_{\dot{B}_{4,\sigma}^{-\frac{5}{2}}} \leqslant \left\{ \sum_{j=N^2-1}^{N^2+1} \left(2^{-\frac{5}{2}j} ||\phi_j * f_N||_{L^4} \right)^{\sigma} \right\}^{\frac{1}{\sigma}}$$
$$\leqslant \frac{C\delta}{N^{\frac{1}{4}} \log N} ||F_N||_{L^4} \leqslant \frac{C\delta}{\log N}$$

for all $1 \le \sigma \le \infty$. Similarly as in the previous steps, we focus on the first iteration $\theta_N^{(1)}$ and the second iteration $\theta_N^{(2)}$. For the estimate of $\theta_N^{(1)}$, we have

$$\left\| \theta_N^{(1)} \right\|_{\dot{B}_{4,2}^{-\frac{1}{2}}} \leqslant C \|f_N\|_{\dot{B}_{4,2}^{-\frac{5}{2}}} \leqslant C \frac{\delta}{\log N}. \tag{4.2}$$

We next consider the estimate of the norm of $\theta_N^{(2)}$ in $\dot{B}_{4,q}^{-\frac{1}{2}}(\mathbb{R}^2)$. Similarly to Step 1, we decompose $\theta_N^{(2)}$ as

$$\begin{split} \widehat{\theta_N^{(2)}}(\xi) &= \frac{\delta^2 2^{5N^2}}{2N^{\frac{1}{2}}(\log N)^2} \frac{\xi_1 \xi_2}{|\xi|^2} \int_{\mathbb{R}^2} \frac{\eta_1^2}{|\xi - \eta| + |\eta|} \frac{\widehat{F_N}(\xi - \eta + 2^{N^2}e_1)}{|\xi - \eta|^3} \frac{\widehat{F_N}(\eta - 2^{N^2}e_1)}{|\eta|^3} d\eta \\ &+ \frac{\delta^2 2^{5N^2}}{2N^{\frac{1}{2}}(\log N)^2} \int_{\mathbb{R}^2} \frac{(\xi_2^2 - \xi_1^2)\eta_1\eta_2 - \xi_1 \xi_2 \eta_2^2}{|\xi|^2(|\xi - \eta| + |\eta|)} \frac{\widehat{F_N}(\xi - \eta + 2^{N^2}e_1)}{|\xi - \eta|^3} \frac{\widehat{F_N}(\eta - 2^{N^2}e_1)}{|\eta|^3} d\eta \\ &- \frac{\delta^2 2^{5N^2}}{4N^{\frac{1}{2}}(\log N)^2} \int_{\mathbb{R}^2} \frac{\xi \cdot \eta^\perp}{|\xi - \eta| + |\eta|} \frac{\widehat{F_N}(\xi - \eta + 2^{N^2}e_1)}{|\xi - \eta|^3} \frac{\widehat{F_N}(\eta - 2^{N^2}e_1)}{|\eta|^3} d\eta \\ &=: \widehat{K_N^{(1)}}(\xi) + \widehat{K_N^{(2)}}(\xi) + \widehat{K_N^{(3)}}(\xi) \end{split}$$

for all ξ with $|\xi| \leq 2^{(N-100)^2}$. We first consider the estimate of $K_N^{(1)}$. If

$$\operatorname{supp}\left(\widehat{\psi_{n^2}}(\xi)\widehat{\phi_{k^2}}(\cdot)\widehat{\phi_{\ell^2}}(\xi-\cdot)\right) \neq \varnothing$$

then it should hold $(k,\ell) \in ([10,n-1] \times \{n\}) \cup (\{n\} \times [10,n-1]) \cup \{(k,k) \; ; \; k \geqslant n\},$ which implies

$$\begin{split} \widehat{\psi_{n^2}}(\xi)\widehat{F_N}(\xi - \eta + 2^{N^2}e_1)\widehat{F_N}(\eta - 2^{N^2}e_1) \\ &= \widehat{\psi_{n^2}}(\xi) \sum_{k,\ell=10}^{N-50} \begin{pmatrix} e^{-iRk^2(\xi_1 - \eta_1)}e^{-iR\ell^2\eta_1}2^{-\frac{3}{2}k^2}2^{-\frac{3}{2}n^2} \\ \times \widehat{\phi_{k^2}}(\xi - \eta + 2^{N^2}e_1)\widehat{\phi_{n^2}}(\eta - 2^{N^2}e_1) \end{pmatrix} \\ &= \widehat{\psi_{n^2}}(\xi) \sum_{k=n}^{N-50} e^{-iRk^2\xi_1}2^{-3k^2}\widehat{\phi_{k^2}}(\xi - \eta + 2^{N^2}e_1)\widehat{\phi_{k^2}}(\eta - 2^{N^2}e_1) \\ &+ \widehat{\psi_{n^2}}(\xi) \sum_{\ell=10}^{n-1} \begin{pmatrix} e^{-iRn^2(\xi_1 - \eta_1)}e^{-iR\ell^2\eta_1}2^{-\frac{3}{2}n^2}2^{-\frac{3}{2}\ell^2} \\ \times \widehat{\phi_{n^2}}(\xi - \eta + 2^{N^2}e_1)\widehat{\phi_{\ell^2}}(\eta - 2^{N^2}e_1) \end{pmatrix} \\ &+ \widehat{\psi_{n^2}}(\xi) \sum_{k=10}^{n-1} \begin{pmatrix} e^{-iRk^2(\xi_1 - \eta_1)}e^{-iRn^2\eta_1}2^{-\frac{3}{2}k^2}2^{-\frac{3}{2}n^2} \\ \times \widehat{\phi_{k^2}}(\xi - \eta + 2^{N^2}e_1)\widehat{\phi_{n^2}}(\eta - 2^{N^2}e_1) \end{pmatrix}. \end{split}$$

Thus, we see that

$$\begin{split} \psi_n * K_N^{(1)}(x) \\ &= -\frac{\delta^2 2^{5N^2}}{2N^{\frac{1}{2}}(\log N)^2} 2^{-3n^2} \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i\xi \cdot (x - Rn^2 e_1)} G_{n,n,N}(\xi) d\xi \\ &- \frac{\delta^2 2^{5N^2}}{2N^{\frac{1}{2}}(\log N)^2} \sum_{k=n+1}^{N-50} 2^{-3k^2} \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i\xi \cdot (x - Rk^2 e_1)} G_{n,k,N}(\xi) d\xi \\ &- \frac{\delta^2 2^{5N^2}}{2N^{\frac{1}{2}}(\log N)^2} \sum_{\ell=10}^{n-1} 2^{-\frac{3}{2}n^2} 2^{-\frac{3}{2}\ell^2} \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i\xi \cdot (x - Rk^2 e_1)} H_{n,n,\ell,N}(\xi) d\xi \\ &- \frac{\delta^2 2^{5N^2}}{2N^{\frac{1}{2}}(\log N)^2} \sum_{k=10}^{n-1} 2^{-\frac{3}{2}k^2} 2^{-\frac{3}{2}\ell^2} \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i\xi \cdot (x - Rk^2 e_1)} H_{n,k,n,N}(\xi) d\xi \\ &=: K_{N,n}^{(1,1)}(x) + K_{N,n}^{(1,2)}(x) + K_{N,n}^{(1,3)}(x) + K_{N,n}^{(1,4)}(x), \end{split}$$

where we have set

$$G_{n,k,N}(\xi) := \frac{\xi_1 \xi_2}{|\xi|^2} \widehat{\psi_{n^2}}(\xi) \int_{\mathbb{R}^2} \frac{\eta_1^2}{|\xi - \eta| + |\eta|} \frac{\widehat{\phi_{k^2}}(\xi - \eta + 2^{N^2} e_1)}{|\xi - \eta|^3} \frac{\widehat{\phi_{k^2}}(\eta - 2^{N^2} e_1)}{|\eta|^3} d\eta,$$

$$H_{n,k,\ell,N}(\xi) := \frac{\xi_1 \xi_2}{|\xi|^2} \widehat{\psi_{n^2}}(\xi) \int_{\mathbb{R}^2} e^{iR(k^2 - \ell^2)\eta_1} \frac{\eta_1^2}{|\xi - \eta| + |\eta|} \times \frac{\widehat{\phi_{k^2}}(\xi - \eta + 2^{N^2} e_1)}{|\xi - \eta|^3} \frac{\widehat{\phi_{\ell^2}}(\eta - 2^{N^2} e_1)}{|\eta|^3} d\eta.$$

Let us consider the estimate of $K_N^{(1,1)}(x)$. We remark that $G_{n,k,N}(\xi) \geqslant 0$ holds. Let

$$A_n := \left\{ x \in \mathbb{R}^2 ; |x - Rn^2 e_1| \leqslant \varepsilon 2^{-n^2} \right\},\,$$

where ε is a positive constant to be determined later. Then, we have

$$\left| e^{i\xi \cdot (x - Rn^2 e_1)} - 1 \right| \leqslant |\xi| |x - Rn^2 e_1| \leqslant 2\varepsilon, \qquad x \in A_n, \quad \xi \in \operatorname{supp} \widehat{\psi_n}.$$

and $|A_n| \leq C\varepsilon^2 2^{-2n^2}$. Thus, it holds

$$\begin{aligned} & \left\| K_{N,n}^{(1,1)} \right\|_{L^{4}(A_{n})} \\ & \geqslant c \frac{\delta^{2} 2^{5N^{2}}}{N^{\frac{1}{2}} (\log N)^{2}} 2^{-3n^{2}} |A_{n}|^{\frac{1}{4}} \int_{\mathbb{R}^{2}} G_{n,n,N}(\xi) d\xi \\ & - C \frac{\delta^{2} 2^{5N^{2}}}{N^{\frac{1}{2}} (\log N)^{2}} 2^{-3n^{2}} \left\| \int_{\mathbb{R}^{2}} \left| e^{i\xi \cdot (x - Rn^{2}e_{1})} - 1 \right| G_{n,n,N}(\xi) d\xi \right\|_{L^{4}(A_{n})} \\ & \geqslant c \frac{\delta^{2} 2^{5N^{2}}}{N^{\frac{1}{2}} (\log N)^{2}} 2^{-3n^{2}} \left(\varepsilon^{2} 2^{-2n^{2}} \right)^{\frac{1}{4}} \int_{\mathbb{R}^{2}} G_{n,n,N}(\xi) d\xi \\ & - C \frac{\delta^{2} 2^{5N^{2}}}{N^{\frac{1}{2}} (\log N)^{2}} 2^{-3n^{2}} \int_{\mathbb{R}^{2}} \varepsilon 2^{-n^{2}} G_{n,n,N}(\xi) d\xi \\ & \geqslant \left(c_{*} \varepsilon^{\frac{1}{2}} - C_{*} \varepsilon \right) \frac{\delta^{2} 2^{5N^{2}}}{N^{\frac{1}{2}} (\log N)^{2}} 2^{-3n^{2}} 2^{-\frac{1}{2}n^{2}} \int_{\mathbb{R}^{2}} G_{n,n,N}(\xi) d\xi \end{aligned}$$

with some positive constants c_* and C_* . We choose ε so small that $c_*\varepsilon^{\frac{1}{2}} - C_*\varepsilon > 0$. Here, using

$$G_{n,k,N}(\xi) \sim \widehat{\psi_{n^2}}(\xi) \int_{\mathbb{R}^2} \frac{2^{2N^2}}{2^{N^2}} \frac{\widehat{\phi_{k^2}}(\xi - \eta + 2^{N^2} e_1)}{2^{3N^2}} \frac{\widehat{\phi_{k^2}}(\eta - 2^{N^2} e_1)}{2^{3N^2}} d\eta$$
$$\sim 2^{-5N^2} \widehat{\psi_{n^2}}(\xi) \int_{\mathbb{R}^2} \widehat{\phi_{k^2}}(\xi - \eta) \widehat{\phi_{k^2}}(\eta) d\eta$$
$$\sim 2^{-5N^2} \widehat{\psi_{n^2}}(\xi) \left(\widehat{\phi_{k^2}} * \widehat{\phi_{k^2}}\right) (\xi),$$

we see that

$$\int_{\mathbb{R}^{2}} G_{n,k,N}(\xi) d\xi \sim 2^{-5N^{2}} \int_{\mathbb{R}^{2}} \widehat{\psi_{n^{2}}}(\xi) \left(\widehat{\phi_{k^{2}}} * \widehat{\phi_{k^{2}}} \right) (\xi) d\xi
\sim 2^{-5N^{2}} 2^{2k^{2}} \int_{\mathbb{R}^{2}} \widehat{\psi_{0}} (2^{-n^{2}} \xi) \left(\widehat{\phi_{0}} * \widehat{\phi_{0}} \right) (2^{-k^{2}} \xi) d\xi
\sim 2^{-5N^{2}} 2^{2n^{2}} 2^{2k^{2}}$$

for $k \ge n \ge 100$. Thus, we have

$$\left\| K_{N,n}^{(1,1)} \right\|_{L^4(A_n)} \geqslant c\varepsilon^{\frac{1}{2}} 2^{\frac{1}{2}n^2} \frac{\delta^2}{N^{\frac{1}{2}} (\log N)^2}.$$
 (4.3)

For the estimate of $K_{N,n}^{(1,2)}$, it holds

$$\begin{aligned} \left\| K_{N,n}^{(1,2)} \right\|_{L^{4}(A_{n})} &\leqslant C \frac{\delta^{2} 2^{5N^{2}}}{2N^{\frac{1}{2}} (\log N)^{2}} \sum_{k=n+1}^{N-50} 2^{-3k^{2}} \left(\varepsilon^{2} 2^{-2n^{2}} \right)^{\frac{1}{4}} \int_{\mathbb{R}^{2}} G_{n,k,N}(\xi) d\xi \\ &\leqslant C 2^{\frac{3}{2}n^{2}} \frac{\delta^{2}}{2N^{\frac{1}{2}} (\log N)^{2}} \sum_{k=n+1}^{N-50} 2^{-k^{2}} \\ &\leqslant C 2^{\frac{3}{2}n^{2}} 2^{-(n+1)^{2}} \frac{\delta^{2}}{2N^{\frac{1}{2}} (\log N)^{2}} = C 2^{\frac{1}{2}n^{2}} 2^{-2n-1} \frac{\delta^{2}}{2N^{\frac{1}{2}} (\log N)^{2}}. \end{aligned}$$

$$(4.4)$$

For the estimate of $K_{N,n}^{(1,3)}$, it holds

$$\begin{aligned} \left\| K_{N,n}^{(1,3)} \right\|_{L^{4}(A_{n})} &\leqslant C \frac{\delta^{2} 2^{5N^{2}}}{N^{\frac{1}{2}} (\log N)^{2}} \sum_{\ell=10}^{n-1} 2^{-\frac{3}{2}(k^{2}+\ell^{2})} \left\| \int_{\mathbb{R}^{2}} e^{i(x-Rk^{2}e_{1})\cdot\xi} H_{n,n,\ell,N}(\xi) d\xi \right\|_{L^{4}(A_{n})} \\ &\leqslant C \frac{\delta^{2} 2^{5N^{2}}}{N^{\frac{1}{2}} (\log N)^{2}} \sum_{\ell=10}^{n-1} 2^{-\frac{3}{2}(n^{2}+\ell^{2})} \left(\varepsilon^{2} 2^{-2n^{2}} \right)^{\frac{1}{4}} \int_{\mathbb{R}^{2}} |H_{n,n,\ell,N}(\xi)| d\xi \\ &\leqslant C 2^{-\frac{1}{2}n^{2}} \frac{\delta^{2} 2^{5N^{2}}}{N^{\frac{1}{2}} (\log N)^{2}} \sum_{\ell=10}^{n-1} 2^{-\frac{3}{2}(n^{2}+\ell^{2})} \int_{\mathbb{R}^{2}} |H_{n,n,\ell,N}(\xi)| d\xi \end{aligned}$$

Since we see by $\widehat{\psi_{n^2}}(\xi) \leqslant \widehat{\phi_{n^2}}(\xi)$ that

$$\int_{\mathbb{R}^{2}} |H_{n,n,\ell,N}(\xi)| \, d\xi \leqslant C 2^{-5N^{2}} \int_{\mathbb{R}^{2}} \widehat{\phi_{n^{2}}}(\xi) \int_{\mathbb{R}^{2}} \widehat{\phi_{n^{2}}}(\xi - \eta + 2^{N^{2}} e_{1}) \widehat{\phi_{\ell^{2}}}(\eta - 2^{N^{2}} e_{1}) d\eta d\xi
= C 2^{-5N^{2}} \int_{\mathbb{R}^{2}} \widehat{\phi_{0}}(2^{-n^{2}} \xi) \int_{\mathbb{R}^{2}} \widehat{\phi_{0}}(2^{-n^{2}} \xi - 2^{-n^{2}} \eta) \widehat{\phi_{0}}(2^{-\ell^{2}} \eta) d\eta d\xi
= C 2^{-5N^{2}} 2^{2(n^{2} + \ell^{2})} \int_{\mathbb{R}^{2}} \widehat{\phi_{0}}(\xi) \int_{\mathbb{R}^{2}} \widehat{\phi_{0}}(\xi - 2^{\ell^{2} - n^{2}} \eta) \widehat{\phi_{0}}(\eta) d\eta d\xi
= C 2^{-5N^{2}} 2^{2(n^{2} + \ell^{2})} \int_{\mathbb{R}^{2}} \left(\widehat{\phi_{0}} * \widehat{\phi_{0}} \right) (2^{\ell^{2} - n^{2}} \eta) \widehat{\phi_{0}}(\eta) d\eta d\xi
\leqslant C 2^{-5N^{2}} 2^{2(n^{2} + \ell^{2})}$$

for $\ell \leq n-1$, there holds

$$\left\| K_{N,n}^{(1,3)} \right\|_{L^{4}(A_{n})} \leqslant C 2^{-\frac{1}{2}n^{2}} \frac{\delta^{2}}{N^{\frac{1}{2}} (\log N)^{2}} \sum_{\ell=10}^{n-1} 2^{\frac{1}{2}(n^{2}+\ell^{2})}$$

$$\leqslant C 2^{\frac{1}{2}(n-1)^{2}} \frac{\delta^{2}}{N^{\frac{1}{2}} (\log N)^{2}}.$$

$$(4.5)$$

Similarly, we have

$$\left\| K_{N,n}^{(1,4)} \right\|_{L^4(A_n)} \le C 2^{\frac{1}{2}(n-1)^2} \frac{\delta^2}{N^{\frac{1}{2}}(\log N)^2}.$$
 (4.6)

Combining (4.3), (4.4), (4.5), and (4.6), we have

$$\begin{split} \left\| \psi_{n^{2}} * K_{N}^{(1)} \right\|_{L^{4}} &\geqslant \left\| \psi_{n^{2}} * K_{N}^{(1)} \right\|_{L^{4}(A_{n})} \\ &\geqslant \left\| K_{N,n}^{(1,1)} \right\|_{L^{4}(A_{n})} - \left\| K_{N,n}^{(1,2)} \right\|_{L^{4}(A_{n})} \\ &- \left\| K_{N,n}^{(1,3)} \right\|_{L^{4}(A_{n})} - \left\| K_{N,n}^{(1,4)} \right\|_{L^{4}(A_{n})} \\ &\geqslant \left(c - C2^{-n} \right) 2^{\frac{1}{2}n^{2}} \frac{\delta^{2}}{N^{\frac{1}{2}} (\log N)^{2}}, \end{split}$$

which implies

$$\left\{ \sum_{n=\frac{N}{2}}^{N-100} \left(2^{-\frac{1}{2}n^2} \left\| \psi_{n^2} * K_N^{(1)} \right\|_{L^4} \right)^q \right\}^{\frac{1}{q}} \geqslant c \frac{\delta^2}{(\log N)^2} N^{\frac{1}{q} - \frac{1}{2}}$$

by choosing N sufficiently large. Next, we consider the estimates of $K_N^{(m)}$ (m=2,3), we have

$$\begin{split} \sum_{m=2}^{3} \left| \widehat{\psi_{n^{2}}}(\xi) \widehat{K_{N}^{(m)}}(\xi) \right| &\leqslant C \frac{\delta^{2} 2^{5N^{2}}}{N^{\frac{1}{2}} (\log N)^{2}} \widehat{\psi_{n^{2}}}(\xi) \\ &\times \int_{\mathbb{R}^{2}} \frac{\widehat{F_{N}}(\xi - \eta + 2^{N^{2}} e_{1})}{|\xi - \eta|^{3}} \frac{\widehat{F_{N}}(\eta - 2^{N^{2}} e_{1})}{|\eta|^{3}} d\eta \\ &\leqslant C \frac{\delta^{2} 2^{-N^{2}}}{N^{\frac{1}{2}} (\log N)^{2}} \widehat{\psi_{n^{2}}}(\xi) \int_{\mathbb{R}^{2}} \widehat{F_{N}}(\xi - \eta + 2^{N^{2}} e_{1}) \widehat{F_{N}}(\eta - 2^{N^{2}} e_{1}) d\eta \\ &= C \frac{\delta^{2} 2^{-N^{2}}}{N^{\frac{1}{2}} (\log N)^{2}} \widehat{\psi_{n^{2}}}(\xi) \widehat{(F_{N})^{2}}(\xi), \end{split}$$

which yields

$$\left\{ \sum_{n=\frac{N}{2}}^{N-100} \left(2^{-\frac{1}{2}n^2} \left\| \phi_{n^2} * K_N^{(3)} \right\|_{L^4} \right)^q \right\}^{\frac{1}{q}} \leqslant C \left\{ \sum_{n=\frac{N}{2}}^{N-100} \left\| \widehat{\phi_{n^2}}(\xi) \widehat{K_N^{(3)}}(\xi) \right\|_{L^2}^q \right\}^{\frac{1}{q}} \\
\leqslant C \frac{\delta^2 2^{-N^2}}{N^{\frac{1}{2}} (\log N)^2} \left\{ \sum_{n=\frac{N}{2}}^{N-100} 2^{qn^2} \right\}^{\frac{1}{q}} \left\| F_N \right\|_{L^4}^2 \\
\leqslant C \frac{\delta^2 2^{-200N}}{(\log N)^2}.$$

Hence, we obtain

$$\left\{ \sum_{n=\frac{N}{2}}^{N-100} \left(2^{-\frac{1}{2}n^2} \left\| \psi_{n^2} * \theta_N^{(2)} \right\|_{L^4} \right)^q \right\}^{\frac{1}{q}} \geqslant c \left\{ \sum_{n=\frac{N}{2}}^{N-100} \left(2^{-\frac{1}{2}n^2} \left\| \psi_{n^2} * K_N^{(1)} \right\|_{L^4} \right)^q \right\}^{\frac{1}{q}} \\
- \sum_{m=2}^{3} \left\{ \sum_{n=\frac{N}{2}}^{N-100} \left(2^{-\frac{1}{2}n^2} \left\| \psi_{n^2} * K_N^{(m)} \right\|_{L^4} \right)^q \right\}^{\frac{1}{q}} \\
\geqslant c \frac{\delta^2}{(\log N)^2} N^{\frac{1}{q} - \frac{1}{2}} - C \frac{\delta^2 2^{-200N}}{(\log N)^2} \\
\geqslant c \frac{\delta^2}{(\log N)^2} N^{\frac{1}{q} - \frac{1}{2}}$$

for sufficiently large N. For the higher iteration part, it follows from the same contraction mapping argument as in Step 1 via (4.2) that we may construct the solution $\widetilde{\theta}_N$ to (4.1) with the estimate

$$\left\|\widetilde{\theta}_N\right\|_{\dot{B}_{4/2}^{-\frac{1}{2}}} \leqslant C \frac{\delta^3}{(\log N)^3}$$

provided that the δ is sufficiently small. Hence, the solution $\theta_N = \theta_N^{(1)} + \theta_N^{(2)} + \widetilde{\theta}_N$ satisfies the estimate

$$\begin{split} \|\theta_N\|_{\dot{B}_{4,q}^{-\frac{1}{2}}} &\geqslant \left\{ \sum_{n=\frac{N}{2}}^{N-100} \left(2^{-\frac{1}{2}n^2} \|\phi_{n^2} * \theta_N\|_{L^4} \right)^q \right\}^{\frac{1}{q}} \\ &\geqslant c \left\{ \sum_{n=\frac{N}{2}}^{N-100} \left(2^{-\frac{1}{2}n^2} \|\psi_{n^2} * \theta_N^{(2)}\|_{L^4} \right)^q \right\}^{\frac{1}{q}} - \left\|\theta_N^{(1)}\right\|_{\dot{B}_{4,q}^{-\frac{1}{2}}} - CN^{\frac{1}{q}-\frac{1}{2}} \|\widetilde{\theta}_N\|_{\dot{B}_{4,2}^{-\frac{1}{2}}} \\ &\geqslant c \frac{\delta^2}{(\log N)^2} N^{\frac{1}{q}-\frac{1}{2}} - C \frac{\delta}{\log N} - C \frac{\delta^3}{(\log N)^3} N^{\frac{1}{q}-\frac{1}{2}} \\ &= \frac{N^{\frac{1}{q}-\frac{1}{2}}}{(\log N)^2} \left(c\delta^2 - C\delta N^{\frac{1}{2}-\frac{1}{q}} \log N - \frac{C\delta^3}{\log N} \right) \to \infty \end{split}$$

as $N \to \infty$, which completes the proof.

Data availability.

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflict of interest.

The author has declared no conflicts of interest.

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