GROMOV HYPERBOLICITY III: IMPROVED GEOMETRIC CHARACTERIZATION IN EUCLIDEAN SPACES AND BEYOND

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Dedicated to Professor Pekka Koskela on the occasion of his 65th birthday

ABSTRACT. This is the third article of a series of our recent works, addressing an open question of Bonk-Heinonen-Koskela [3], to study the relationship between (inner) uniformality and Gromov hyperbolicity in infinite dimensional spaces. Our main focus of this paper is to establish improved geometric characterization of Gromov hyperbolicity.

More precisely, we develop an elementary measure-independent approach to establish the geometric characterization of Gromov hyperbolicity for general proper Euclidean subdomains, which addresses a conjecture of Bonk-Heinonen-Koskela [Asterisque 2001] for unbounded Euclidean subdomains. Our main results not only improve the corresponding result of Balogh-Buckley [Invent. Math. 2003], but also clean up the relationship between the two geometric conditions, ball separation condition and Gehring-Hayman inequality, that used to characterize Gromov hyperbolicity. We also provide a negative answer to an open problem of Balogh-Buckley by constructing an Euclidean domain with ball separation property but fails to satisfy the Gehring-Hayman inequality. Furthermore, we prove that ball separation condition, together with an LLC-2 condition, implies inner uniformality and thus the Gehring-Hayman inequality.

As a consequence of our new approach, we are able to prove such a geometric characterization of Gromov hyperbolicity in the fairly general setting of metric spaces (without measures), which substantially improves the main result of Koskela-Lammi-Manojlović [Ann. Sci. Éc. Norm. Supér. 2014]. In particular, we not only provide a new purely metric proof of the main reuslt of Balogh-Buckley and Koskela-Lammi-Manojlović, but also derive explicit dependence of various involved constants, which improves all the previous known results.

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1. Instruction

1.1. **Background.** Recall the classical uniformization theorem states that the class of simply connected proper domains in $\mathbb{R}^2$ can arise as conformal images of the unit disk $\mathbb{D} \subset \mathbb{R}^2$. Searching for a suitable higher dimensional Euclidean space or even abstract metric space extension of such a beautiful theory, we may formally formulate the uniformization problems in the form

$$\{\text{a class of good domains in } \mathbb{R}^n \text{ or } X\} = \mathcal{F}\left(\{\text{a class of model domains in } \mathbb{R}^n \text{ or } X\}\right), \quad \text{(UP)}$$

where $\mathcal{F}$ consists of homeomorphisms with good geometric properties (such as conformal maps).

In a seminal work [4], Bonk, Heinonen and Koskela have successfully developed a rich uniformization theory for (UP). To record their theory, we need to recall a couple of basic definitions. The first notion is the class of (inner) uniform domains in a general metric space.

Definition 1.1. A domain Ω in a metric space $X = (X, d)$ is called *c-uniform*, $c \geq 1$, if each pair of points z_1, z_2 in Ω can be joined by a rectifiable curve γ in Ω satisfying

(1) $\min_{j=1,2}\{\ell_d(\gamma[z_j, z])\} \leq c\, d_\Omega(z)$ for all $z \in \gamma$, and
(2) $\ell_d(\gamma) \leq c\, d(z_1, z_2)$,

where $\ell_d(\gamma)$ denotes the arc-length of γ with respect to the metric d, $\gamma[z_j, z]$ the subcurve of γ between z_j and z, and $d_\Omega(z) := d(z, \partial\Omega)$. In a c-uniform domain Ω, any curve $\gamma \subset \Omega$, which satisfies conditions (1) and (2) above, is called a *c-uniform curve* or a *double c-cone curve*.

If the condition (2) in Definition 1.1 is replaced by the weaker inequality

$$\ell_d(\gamma) \leq c\, \sigma_\Omega(z_1, z_2), \tag{1.1}$$

where σ_Ω is the *inner distance* in Ω defined by

$$\sigma_\Omega(z_1, z_2) = \inf\{\ell(\alpha) : \ \alpha \subset \Omega \text{ is a rectifiable curve joining } z_1 \text{ and } z_2\},$$

then Ω is said to be *c-inner uniform* and the corresponding curve γ is called a *c-inner uniform curve*. When the context is clear, we often drop the subscript Ω from σ_Ω and simply write σ.

If Ω only satisfies the condition (1) in Definition 1.1, then it is said to be a *c-John domain*, and the corresponding curve γ is called a *c-John curve* or a *c-cone curve*.

The class of John domains was initially introduced by F. John in his study of elasticity [24] and the name was coined by Martio and Sarvas in [26], where they also introduced the class of uniform domains. These classes of domains are central in modern geometric function theory in $\mathbb{R}^n$ or more general metric spaces and have wide connections with many other mathematical subjects related to analysis and geometry; see for instance [7, 8, 9, 14, 10, 17, 18, 19, 20, 22, 27].

The second notion is the class of Gromov hyperbolic spaces introduced by M. Gromov in his celebrated work [12]. Recall that

Definition 1.2. A geodesic metric space $X = (X, d)$ is called *δ-Gromov hyperbolic*, $\delta > 0$, if each side of a geodesic triangle in X lies in the δ-neighborhood of the other two sides.

Gromov hyperbolicity is a large-scale property, which generalizes the metric properties of classical hyperbolic geometry and of trees, and it turns out to be very useful in geometric group theory and metric geometry [3, 5, 6, 13]. For more about Gromov hyperbolicity and its connection to geometric function theory, see for instance [1, 2, 4, 23, 30].

Definition 1.3. Let (X, d) be a metric space and k the quasihyperbolic distance induced by d (see Section 2.1 below for precise definition). A domain $\Omega \subsetneq (X, d)$ is called δ-Gromov hyperbolic if the metric space (Ω, k) is δ-Gromov hyperbolic.

Now, we are able to present the uniformization theory of Bonk-Heinonen-Koskela [4]: on the left hand side of (UP), one takes the class of Gromov hyperbolic domains, while on the right hand side, one uses the well-known class of (inner) uniform domains. Then Bonk, Heinonen and Koskela have established in their main result [4, Theorem 1.1] a rather general uniformization theory for Gromov hyperbolic spaces:

$$\{\text{some class of Gromov hyperbolic spaces}\} = \mathcal{F}\left(\{\text{bounded locally compact uniform spaces}\}\right),$$

where $\mathcal{F}$ is a class of good homeomorphisms (i.e. quasiisometries). In their second main result [4, Theorem 1.11], Bonk, Heinonen and Koskela proved that inner uniform domains in $\mathbb{R}^n$ are Gromov hyperbolic, which provides a large class of (nontrivial) examples of Gromov hyperbolic domains in Euclidean spaces. It is well known that each quasiconformal image of a Gromov hyperbolic domain is again Gromov hyperbolic [11]. In particular, all simply connected domains in $\mathbb{R}^2$ are Gromov hyperbolic (by the Riemann mapping theorem). Furthermore, it was shown that a Gromov hyperbolic domain Ω in $\overline{\mathbb{R}}^n$ equipped with the spherical metric d_s necessarily satisfies both *the Gehring-Hayman inequality* and *the ball separation property*.

Motivated by the theory of quasiconformal mappings in infinite dimensional Banach spaces [28, 29], they asked the following interesting challenging open problem in [4, Page 5]:

Question (Bonk-Heinonen-Koskela): *Is there a general relationship between uniformity and Gromov hyperbolicity that would cover infinite-dimensional situations as well.*

Addressing the above open problem, in our earlier works [16, 15], we have established the dimension-free Gehring-Hayman inequality and dimension-free inner uniform estimates for quasi-geodesics in infinite dimensional Banach spaces. In this third paper, we shall employ some of the techniques developed in our previous works [16, 15], to establish improved geometric characterization of Gromov hyperbolicity via the Gehring-Hayman inequality and ball separation property. For later references, we recall these two fundamental concepts here.

Definition 1.4. A domain Ω in a metric space (X, d) is said to satisfy the *C-Gehring-Hayman inequality*, $C > 0$, if for all $x, y \in \Omega$ and each quasihyperbolic geodesic $\gamma_{xy} \subset \Omega$ with end points

x and y, it holds

$$\ell_d(\gamma_{xy}) \leq C\sigma_\Omega(x,y). \tag{1.2}$$

To emphasize the distance d, we also say that (Ω, d) satisfies the C-Gehring-Hayman inequality.

Definition 1.5. A domain Ω in a metric space (X,d) is said to satisfy the *C-ball separation condition*, $C > 0$, if for each quasihyperbolic geodesic $\gamma_{xy} \subset \Omega$, every $z \in \gamma_{xy}$ and every curve $\gamma \subset \Omega$ joining x and y, it holds

$$B_\sigma\left(z, Cd_\Omega(z)\right) \cap \gamma \neq \emptyset. \tag{1.3}$$

To emphasize the distance σ, we also say that (Ω, σ) satisfies the C-ball separation condition.

These two geometric properties are potentially much easier to verify than the more complicated Gromov hyperbolicity and thus it is natural to ask *whether these two properties actually fully characterize Gromov hyperbolicity.* For the reverse implication, Bonk, Heinonen and Koskela made the following conjecture:

Conjecture ([4, Page 75]): If $\Omega \subset (\overline{\mathbb{R}}^n, d_s)$ satisfies both the Gehring-Hayman inequality and the ball separation condition, then (Ω, k) is Gromov hyperbolic.

In another fundamental work, built upon the seminal works [4] and [20], Balogh and Buckley [2, Theorem 0.1] have verified affirmatively the above conjecture of Bonk-Heinonen-Koskela. As a direction application of their main result [2, Theorem 0.1], they obtained the following interesting geometric characterization of Gromov hyperbolicity for *bounded Euclidean domains* $\Omega \subset \mathbb{R}^n$; see the comments right before statement of [2, Theorem 0.1].

Theorem A (Balogh-Buckley, [2])**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then the following two conditions are equivalent.

(1) (Ω, k) is δ-Gromov hyperbolic.

(2) Ω satisfies both the C-Gehring-Hayman inequality and the C-ball separation condition.

Moreover, the constants δ and C depend only on each other and on the dimension n and $\mathrm{diam}(\Omega)$.

Based on Theorem A, it is nature to ask

Question B. Does the conjecture of Bonk-Heinonen-Koskela hold for general proper Euclidean subdomains?

Another fundamental problem is the relationship between the Gehring-Hayman inequality and the ball separation condition. In [2], Balogh and Buckley constructed a planar domain $\Omega \subset \mathbb{R}^2$ which satsifies the Gehring-Hayman inequality, but fails to have the ball separation property, showing that the Gehring-Hayman inequality alone does not imply ball separation property. Then they asked the reverse implication, which has become a fundamental open problem in the field.

Question C ([2, Page 272])**.** Does ball separation property alone implies the Gehring-Hayman inequality for Euclidean domains?

Notice that all the three conditions in Theorem A (or more generally Theorem 1.6 below) are only based on purely metric concepts and thus it is natural to ask for an extension of such a beautiful characterization to abstract metric spaces. Such an extension was initially given in [2], which relies essentially on the existence of suitable Poincaré inequality (or being Loewner) for the underlying space (see [2, the last paragraph on page 265]). As was pointed out by Koskela, Lammi and Manojlović [25], supporting an abstract Poincaré inequality (or being Loewner) is a somewhat restrictive assumption for the underlying space. In their main result [25, Theorem 1.2], they have successfully obtained a suitable extension of Theorem A to locally compact Q-*regular* length spaces that are addtionally *annularly quasiconvex*. More precisely, they proved the following result.

Theorem D ([25, Theorem 1.2])**.** Let $Q > 1$ and let (X, d, μ) be a (Q, C_0)-regular metric measures space with (X, d) a locally compact and annularly C_1-quasiconvex length space. Let $\Omega \subset X$ be a bounded proper subdomain. Then the following conditions are equivalent.

(1) (Ω, k) is δ-Gromov hyperbolic.
(2) Ω satisfies both the C-Gehring-Hayman inequality and the C-ball separation condition.

Moreover, the constants δ and C depend only on each other and on Q, C_0, C_1 and $\mathrm{diam}(\Omega)$.

Recall that given $Q \geq 1$ and $C \geq 1$, we say that a metric measure space $X = (X, d, \mu)$ is *(Ahlfors)* (Q, C)*-regular* if for each $x \in X$ and $0 < r \leq \mathrm{diam}_d(X)$,

$$C^{-1} r^Q \leq \mu(B_d(x, r)) \leq C r^Q.$$

Theorem D improves the corresponding (metric space) result of Balogh-Buckley by weakening the requirement on underlying space, from supporting an abstract Poincaré inequality (or being Loewner) to the weaker geometric annular quasiconvexity. But the assumption on the existence of a Q-regularity measure is somewhat unnature as all the three conditions are purely metric. It is thus natural to ask

Question E. Does the geometric characterization of Gromov hyperbolicty hold in general metric spaces without a reference measure?

Note that the involved constants in Theorem D depend not only on each other and the dimension Q, but also on $\mathrm{diam}(\Omega)$ and *the constants associated to annular quasiconvexity.* One would wonder whether these latter dependences on $\mathrm{diam}(\Omega)$, C_0 and C_1 are really necessary. These basic questions are direct motivations of the current paper.

1.2. **Main results.** In our first main result, we provide an affirmative answer to Question B.

Theorem 1.6. *Let $\Omega \subset \mathbb{R}^n$ be a proper subdomain. Then the following conclusions hold.*

(1) *If (Ω, k) is δ-Gromov hyperbolic, then Ω satisfies the C-Gehring-Hayman inequality and the C-ball separation condition with $C = e^{(9\tau)^{n+1}}$ and $\tau = e^{(4\delta)^{192n(1+\delta)}}$.*
(2) *If Ω satisfies both the C-Gehring-Hayman inequality and the C-ball separation condition, then (Ω, k) is δ-Gromov hyperbolic with $\delta = 50C^6(3 + C)^2$.*

Theorem 1.6 improves Theorem A in the following two aspects.

- It removes the extra dependence on $\mathrm{diam}(\Omega)$, thus in particular removes the a priori boundedness requirement for Ω.
- It provides an explicit dependence of involved constants in terms of given data.

Our second main result shows that the answer to Question C is negative.

Theorem 1.7. *For each $n \geq 2$, there exists a proper subdomain $\Omega \subset \mathbb{R}^n$ such that Ω satisfies ball separation condition, but fails to have the Gehring-Hayman inequality.*

Theorem 1.7 implies that the two geometric properties appearing in the characterization of Gromov hyperbolicity in Theorem 1.6 are optimal in the sense that none of the two conditions is redundant. On the other hand, in the positive direction, we prove in our third main result that the ball separation property, together with an LLC-2 condition, do imply the Gehring-Hayman inequality. Recall that a domain Ω in a metric space $X = (X, d)$ is c_0-LLC-2 if each pair of points $a, b \in \Omega\backslash\overline{B(x,r)}$ can be joined in $\Omega\backslash\overline{B(x,r/c_0)}$.

Theorem 1.8. *Let $\Omega \subset \mathbb{R}^n$ be a proper subdomain. If Ω is c_0-LLC-2 with the C-ball separation property, then it is C_1-inner uniform with $C_1 = e^{(C_2)^{3+4C_2^2}}$ and $C_2 = (210c_0C)^{2n^2+5n}$. In particular, Ω satisfies the C_1-Gehring-Hayman inequality.*

As was briefly pointed out in the previous section, all the known proofs for Theorem A (or weaker forms of Theorem 1.6) depend heavily on the uniformization theory of Bonk-Heinonen-Koskela [4], in particular on the Lebesgue measure and integration in $\mathbb{R}^n$. The dependence on the dimension n comes from the Ahlfors n-regularity of the Lebesgue n-measure in $\mathbb{R}^n$. In this paper, we shall provide an elementary measure-independent proof of Theorem 1.6 (and thus in particular a new proof of Theorem A), using only the metric doubling property of $\mathbb{R}^n$ (equipped with the standard Euclidean distance). Recall that

Definition 1.9. A metric space $X = (X, d)$ is called Q-*doubling*, if for each ball $B(x, r)$, every $r/2$-separated subset of $B(x, r)$ has at most Q points.

A simple volume comparison implies that $\mathbb{R}^n$ (equipped with the standard Euclidean distance) is 2^n-doubling, according to Definition 1.9.

As an application of our general new approach, we are actually able to prove Theorem 1.6 in the setting of locally compact Q-doubling metric spaces. Our next main result provides an affirmative answer to Question E. It can be viewed as a natural extension of the Euclidean characterization of Gromov hyperbolicity (Theorem 1.6) to abstract metric spaces.

Theorem 1.10. *Fix $Q > 1$. Let $X = (X, d)$ be a locally compact Q-doubling length space and $\Omega \subset X$ a proper subdomain. Then the following conclusions hold.*

(1) *If (Ω, k) is δ-Gromov hyperbolic, then Ω satisfies the C-Gehring-Hayman inequality and the C-ball separation condition with $C = e^{5\tau \cdot Q^{\log_2 9\tau}}$ and $\tau = e^{(2\delta Q)^{192(1+\delta)}}$.*
(2) *If Ω satisfies both the C-Gehring-Hayman inequality and the C-ball separation condition, then (Ω, k) is δ-Gromov hyperbolic with $\delta = 50C^6(3 + C)^2$.*

Theorem 1.10 improves Theorem D in the following aspects.

- There is no measure involved in the statement of Theorem 1.10.
- It removes the annular quasiconvexity assumption on X.
- The constants δ and C depend only on each other and on Q, not additionally on $\operatorname{diam}(\Omega)$ or the quasiconvexity constant in Theorem D. In particular, Ω is not necessarily bounded.
- The dependence of constants is explicit (although not optimal) in terms of given data.

As one easily observed from the statement of conclusion (2) in Theorem 1.10, the doubling constant Q does not appear in the Gromov hyperbolicity coefficient δ. This indicates that the Q-doubling assumption for X might not be necessary. This is indeed the case and we state it as a separate theorem below.

Theorem 1.11. *Let $X = (X, d)$ be a metric space and $\Omega \subset X$ a proper subdomain such that (Ω, k) is geodesic. If Ω satisfies both the C-Gehring-Hayman inequality and the C-ball separation condition, then (Ω, k) is δ-Gromov hyperbolic with $\delta = 50C^6(1 + C)^2$.*

Observe that if X is quasiconvex, then the identity map $id : (\Omega, d) \to (\Omega, \sigma_\Omega)$ is a homeomorphism, and so by [4, Proposition 2.8], (Ω, k) is proper and geodesic. This shows that

Theorem 1.11 holds for many infinite dimensional spaces, such as all Banach spaces. In the finite dimensional case, we know from [2, Theorem 2.4 and Theorem 6.1] that if (X,d) is a minimally nice metric space, which satisfies both the C-Gehring-Hayman inequality and the C-ball separation condition, then (Ω,k) is δ-Gromov hyperbolic with $\delta=\delta(C)$. However, the proof there was less direct: one first proves that the C-ball separation condition implies another C_1-separation condition with $C_1=C_1(C)$, and then use the new C_1-separation condition and C-Gehring-Hayman inequality to derive the Gromov hyperbolicity of (Ω,k). In this paper, we shall provide an elementary and direct proof of Theorem 1.11 in Section 4.

Theorem 1.8 also admits a similar extension in Q-doubling metric spaces.

Theorem 1.12. *Let $Q>1$ and let $X=(X,d)$ be a Q-doubling metric space. If $\Omega\subset X$ is c_0-LLC-2 with the C-ball separation property, then it is C_1-inner uniform with $C_1=e^{(C_2)^{3+4C_2^2}}$ and $C_2=(42c_0C)^2\cdot Q^{(210c_0C)^2(\log_2^{2Q}+1)}$. In particular, Ω satisfies the C_1-Gehring-Hayman inequality.*

Theorems 1.10, 1.11 and 1.12 give rather complete extension of the Euclidean geometric characterization of Gromov hyperbolicity to abstract metric spaces. Motivated by the theory of quasiconformal mappings in infinite dimensional Banach spaces, it would be interesting to know whether Theorems 1.10 and 1.12 indeed hold without dimension restrictions. This was already asked twenty years ago by Väisälä, and Bonk-Heinonen-Koskela; see [4, last paragraph on page 5]. As it is fundamental for the geometric characterization of Gromov hyperbolicity in general metric setting, we reformulate it here.

Question F. Can we remove the Q-doubling assumption in Theorems 1.10 and 1.12?

Note that an affirmative answer to Question F provide not only a satisfied solution of Question of Bonk-Heinonen-Koskela stated in the previous section, but also an open problem of Väisälä. We expect the techniques developed in this paper and our previous works [16, 15] would be very useful in the study of Question F. In our future work, we shall continue working on this question.

1.3. **Outline of the proof.**

In this section, we briefly outline the proofs of our main results.

On the proof of Theorem 1.10. As we have already commented in the previous section, all the known proofs of (weaker forms of) Theorem 1.6 or Theorem 1.10 depend on the uniformization theory of Bonk-Heinonen-Koskela [4], which relies crucially on the Lebesgue measure and integration, and thus a measure is necessary. Our starting point is to find a new purely metric (and thus measure-independent) proof of Theorem 1.10.

For the proof of Theorem 1.10 (1), it consists of two main steps. In the first step, we show that the Gromov hyperbolicity of (Ω,k) implies that (Ω,σ) satisfies the ball separation property (i.e. Theorem 4.1). Then in the second step, we use the ball separation property, together with the Gromov hyperbolicity of (Ω,k), to prove the Gehring-Hayman inequality for (Ω,d) (i.e. Theorem 5.3).

In either steps, we shall apply nontrivial contradiction arguments. In the first step, suppose the ball separation condition fails, then for a quasihyperbolic geodesic γ and another curve α with the same end points, we may find a point $x_{0,0}\in\gamma$ so that for some large $\tau\gg1$, it holds

$$\sigma(x_{0,0},\alpha)>\tau d_\Omega(x_{0,0}). \tag{*}$$

Then the major effort is paid to find two sequences of points on γ and α with prescribed control on the quasihyperbolic distances between successive points. Then contradiction occurs when comparing the quasihyperbolic distance between a final pair of points if τ in (*) is too large.

More precisely, as (Ω, k) is δ-Gromov hyperbolic, we may find sequences of points on γ and α as in the following claim (see Claim 4.2):

Claim A: There exists a positive integer $N < [\frac{\ell_k(\alpha_{xy})}{3}] + 1$ such that

(1) for each positive integer $\varsigma \in \{1, \cdots, N\}$, there are $y^1_{0,\varsigma} \in \alpha_{xy}$ and $y^2_{0,\varsigma} \in \alpha_{xy}[y^1_{0,\varsigma}, y]$; for each $\varsigma \in \{2, \cdots, N\}$, there exists $y^1_{0,\varsigma} \in \alpha_{xy}[y^1_{0,\varsigma-1}, y]$ with $k_\Omega(y^1_{0,\varsigma-1}, y^1_{0,\varsigma}) \geq 1 + 3\delta$.

(2) for each $\varsigma \in \{1, \cdots, N\}$ and $\gamma_{y^1_{0,\varsigma} y^2_{0,\varsigma}} \in \Lambda_{y^1_{0,\varsigma} y^2_{0,\varsigma}}(\Omega)$, there exists $x_{0,\varsigma} \in \gamma_{y^1_{0,\varsigma} y^2_{0,\varsigma}}$ with
$$\sigma(x_{0,\varsigma}, \alpha[y^1_{0,\varsigma}, y^2_{0,\varsigma}]) > \tau d_\Omega(x_{0,\varsigma}).$$

(3) for each $\varsigma \in \{1, \cdots, N-1\}$, $k_\Omega(y^1_{0,\varsigma}, y^2_{0,\varsigma}) \geq 1 + 3\delta$ and $k_\Omega(y^1_{0,N}, y^2_{0,N}) < 1 + 3\delta$.

Suppose **Claim A** holds. Then
$$\begin{aligned} k_\Omega(y^1_{0,N}, y^2_{0,N}) \geq k_\Omega(y^1_{0,N}, x_{0,N}) &\overset{(2.1)}{\geq} \log\left(1 + \frac{\ell(\gamma_{y^1_{0,N} y^2_{0,N}}[y^1_{0,N}, x_{0,N}])}{\min\{d_\Omega(y^1_{0,N}), d_\Omega(x_{0,N})\}}\right) \\ &\geq \log\left(1 + \frac{\sigma(y^1_{0,N}, x_{0,N})}{d_\Omega(x_{0,N})}\right) \geq \log\frac{\sigma(x_{0,N}, \alpha[y^1_{0,N}, y^2_{0,N}])}{d_\Omega(x_{0,N})} > \log\tau, \end{aligned}$$
which clearly contradicts with **Claim A** (3) as τ is big.

The key ingredient towards the proof of **Claim A** is the construction of point sequences given by Lemma 4.12 (or the iterated version Lemma 4.13. A novel point in our proof of this key lemma is to introduce two new family of curves (see Definitions 4.2 and 4.5), on which there are certain special points with controlled quasihyperbolic distances. Some basic properties for these curve families in Gromov hyperbolic domains are developed in Section 4.1.

In the second step, we prove the Gehring-Hayman inequality. At the level of idea, our proof for this step is largely inspired by our recent work [16] (but of course technically very different), where we proved the Gehring-Hayman inequality for certain special domains in general Banach spaces. A similarity in both proofs is to first derive a weaker version of Gehring-Hayman inequality (see Theorem 5.2 and Lemma 5.5 below and compare it with [16, Theorem 3.3]). Similar to (the spirit of) [16, Sections 5.2 - 5.4], we construct two special partitions of λ and α in Proposition 5.6 to run a contradiction argument. It is worth pointing out that, because the underlying space X is doubling, our proof of Proposition 5.6 is actually technically simpler than the corresponding constructions used in [16, Sections 5.2 - 5.4].

On the proof of Theorem 1.11. For the proof of Theorem 1.10 (2), i.e. Theorem 1.11, we again use a contradiction argument.

To be more precise, fix an arbitrary geodesic triangle $\Delta_{x_1x_2x_3}$ in Ω and a point $x_0 \in \gamma_{x_1x_2}$. Suppose on the contrary that
$$k_\Omega(x_0, \gamma_{x_1x_3} \cup \gamma_{x_2x_3}) > 50C^6(3 + C)^2.$$

Select $y_0 \in \gamma_{x_1x_3} \cup \gamma_{x_2x_3}$ such that
$$\sigma(x_0, y_0) = \inf_{y \in \gamma_{x_1x_3} \cup \gamma_{x_2x_3}} \sigma(x_0, y).$$

Then the key step is to find points with good control on geodesic triangles (see Claim 3.1): there exist two points $z_1 \in \gamma_{x_1x_2}[x_1, x_0]$ and $z_2 \in \gamma_{x_1x_2}[x_0, x_2]$ such that

$$\sigma(y_0, z_1) \leq \frac{3}{5(3+C)^2}\sigma(x_0, y_0) \quad \text{and} \quad \sigma(y_0, z_2) \leq \frac{3}{5(3+C)^2}\sigma(x_0, y_0).$$

Once we were able to prove the above claim, then it follows

$$\begin{aligned}\ell(\gamma_{x_1x_2}[z_1, z_2]) &\geq \sigma(z_1, x_0) + \sigma(z_2, x_0)\\ &\geq \sigma(y_0, x_0) - \sigma(y_0, z_1) + \sigma(y_0, x_0) - \sigma(y_0, z_2)\\ &\geq \left(2 - \frac{6}{5(3+C)^2}\right)\sigma(y_0, x_0).\end{aligned}$$

On the other hand, the C-Gehring-Hayman inequality gives

$$\ell(\gamma_{x_1x_2}[z_1, z_2]) \leq C\sigma(z_1, z_2) \leq C(\sigma(z_1, y_0) + \sigma(y_0, z_2)) \leq \frac{6C}{5(3+C)^2}\sigma(y_0, x_0),$$

which clearly contradicts with the previous estimate.

A novel point for the proof of above claim is that it is simply based on repeated application of the ball separation condition and Gehring-Hayman inequality. Comparing with the proofs of [2, Theorem 2.4 and Theorem 6.1], our proof here is more elementary and direct.

On the proof of Theorem 1.12. The proof of Theorem 1.12 essentially reduces to the verification of Gehring-Hayman inequality and thus it is in spirit close to the argument in Section 5 or [16]. A major technical difference here is to prove the following length comparison result (see Lemma 6.2) in an LLC-2 domain with ball separation property.

Claim B: Suppose that x, y and z are three distinct points in Ω, and $\min\{k_\Omega(x, y), k_\Omega(x, z)\} \geq 1$. Fix $\gamma_{xy} \in \Lambda_{xy}(\Omega)$ and $\gamma_{xz} \in \Lambda_{xz}(\Omega)$. If $k_\Omega(y, z) \leq 1$, then

$$\max\{\ell(\gamma_{xy}), \ell(\gamma_{xz})\} < e^{9\kappa_0\kappa_2} \min\{\ell(\gamma_{xy}), \ell(\gamma_{xz})\},$$

where κ_0 and κ_2 are two constants depending explicitly on given data.

Suppose **Claim B** holds. Then Theorem 1.12 follows from an easy contradiction argument. The idea for proof of **Claim B** is inspired by [16, Section 2]. Indeed, a variant of this result was already proved in [16, Lemma 2.27], for Gromov hyperbolic domains. In our case, we make use of the LLC-2 property of Ω, instead of the Gehring-Hayman inequality assumed in [16, Lemma 2.27].

Structure. The structure of this paper is as follows. In Section 2, we recall some basic facts about quasihyperbolic distance, Gromov hyperbolic spaces and doubling metric spaces. In Section 3, we prove Theorem 1.11 and thus also Theorem 1.10 (2). The proof of Theorem 1.10 (1) is divided into two sections: Sections 4 and 5. In Section 4, we prove that Gromov hyperbolicity implies the ball separation condition, while in Section 5, we prove that Gromov hyperbolicity, together with ball separation condition, implies the Gehring-Hayman inequality. Section 6 contains the proof of Theorem 1.12. In the final section, Section 7, we present detailed constructions of the domain in Theorem 1.7.

Notations. Throughout this paper, the metric space X is always assumed to be a length space. For each proper subdomain $\Omega \subset X$, we use $\Lambda_{xy}(\Omega)$ to stand for the set of all quasihyperbolic geodesics in Ω with end points x and y, and use γ_{xy} to denote some quasihyperbolic geodesic in Λ_{xy}. Meanwhile, also we use $\Gamma_{xy}(\Omega)$ to stand for the set of all curves in Ω with end points x and y.

2. Preliminaries

2.1. **Quasihyperbolic distance.** We start this section by recalling the definition of quasihyperbolic metric, which was initially introduced by Gehring and Palka [9] for a domain in $\mathbb{R}^n$ and then has been extensively studied in [8]. The *quasihyperbolic length* of a rectifiable arc γ in a proper domain $\Omega \subsetneq (X, d)$ is defined as

$$\ell_k(\gamma) := \int_\gamma \frac{|dz|}{d_\Omega(z)}.$$

For any z_1, z_2 in Ω, the *quasihyperbolic distance* $k_\Omega(z_1, z_2)$ between z_1 and z_2 is set to be

$$k_\Omega(z_1, z_2) = \inf_\gamma \{\ell_k(\gamma)\},$$

where the infimum is taken over all rectifiable arcs γ joining z_1 and z_2 in Ω.

An arc γ from z_1 to z_2 is called a *quasihyperbolic geodesic* if $\ell_k(\gamma) = k_\Omega(z_1, z_2)$. Clearly, each subarc of a quasihyperbolic geodesic is a quasihyperbolic geodesic.

For any z_1, z_2 in Ω, let $\gamma \subset \Omega$ be an arc with end points z_1 and z_2. Then we have the following elementary estimates (see for instance [28, Section 2])

$$\ell_k(\gamma) \geq \log\Big(1 + \frac{\ell(\gamma)}{\min\{d_\Omega(z_1), d_\Omega(z_2)\}}\Big) \tag{2.1}$$

and

$$\begin{aligned} k_\Omega(z_1, z_2) &\geq \log\Big(1 + \frac{\sigma_\Omega(z_1, z_2)}{\min\{d_\Omega(z_1), d_\Omega(z_2)\}}\Big) \\ &\geq \log\Big(1 + \frac{|z_1 - z_2|}{\min\{d_\Omega(z_1), d_\Omega(z_2)\}}\Big) \geq \Big|\log \frac{d_\Omega(z_2)}{d_\Omega(z_1)}\Big|. \end{aligned} \tag{2.2}$$

Next we will give the quasihyperbolic distance between two points when they are very close.

Lemma 2.1. *Let (X, d) be a length space and Ω a proper domain of X. Suppose that there are a constant $a > 1$ and two points x_1 and x_2 in Ω such that $d(x_1, x_2) \leq a^{-1} d_\Omega(x_1)$. Then*

$$k_\Omega(x_1, x_2) \leq \frac{9a}{10(a-1)} \frac{d(x_1, x_2)}{d_\Omega(x_1)} \leq \frac{10}{9}(a-1)^{-1}$$

and

$$\ell([x_1, x_2]) \leq \frac{10a}{9(a-1)} e^{\frac{10}{9}(a-1)^{-1}} d(x_1, x_2).$$

Proof. Since (X, d) is a length space, for each $\varepsilon \in (0, (9a+1)^{-1})$, there exists some curve $\alpha = \alpha_\varepsilon$ in X connecting x_1 and x_2 such that

$$\ell(\alpha) \leq \big(1 + (a-1)\varepsilon\big) d(x_1, x_2). \tag{2.3}$$

We claim that $\alpha \subset \Omega$. Indeed, if not, then there exists some point $z \in \alpha \cap \partial\Omega$ and thus it follows from (2.3) and the assumption $d(x_1, x_2) \leq a^{-1} d_\Omega(x_1)$ that

$$d_\Omega(x_1) \leq d(z, x_1) \leq \ell(\alpha) \leq \big(1 + (a-1)\varepsilon\big) d(x_1, x_2) \leq \frac{\big(1 + (a-1)\varepsilon\big)}{a} d_\Omega(x_1).$$

It follows that $\varepsilon \geq 1$, which clearly contradicts with our choice of ε.

Let $x \in \alpha$. Since $d(x_1, x_2) \le a^{-1} d_\Omega(x_1)$, (2.3) gives

$$d_\Omega(x) \ge d_\Omega(x_1) - \ell(\alpha) \ge a^{-1}(a-1)(1-\varepsilon) d_\Omega(x_1).$$

Then it follows from the above estimate and our assumption $d(x_1, x_2) \le a^{-1} d_\Omega(x_1)$ that

$$\begin{aligned} \log\Big(1 + \frac{\ell([x_1, x_2])}{d_\Omega(x_1)}\Big) &\overset{(2.1)}{\le} k_\Omega(x_1, x_2) \le \int_\alpha \frac{|dx|}{d_\Omega(x)} \\ &\le \int_\alpha \frac{a|dx|}{(a-1)(1-\varepsilon) d_\Omega(x_1)} \overset{(2.3)}{\le} \frac{a\big(1 + (a-1)\varepsilon\big)}{(a-1)(1-\varepsilon)} \cdot \frac{d(x_1, x_2)}{d_\Omega(x_1)} \\ &\le \frac{10a}{9(a-1)} \cdot \frac{d(x_1, x_2)}{d_\Omega(x_1)} \le \frac{10}{9}(a-1)^{-1}. \end{aligned} \tag{2.4}$$

Applying the elementary inequality, $t \le (1+t)\log(1+t)$ when $t > 0$, with $t = \frac{\ell([x_1, x_2])}{d_\Omega(x_1)}$ and using (2.4) lead to

$$\frac{\ell([x_1, x_2])}{d_\Omega(x_1)} \le e^{\frac{10}{9}(a-1)^{-1}} \log\Big(1 + \frac{\ell([x_1, x_2])}{d_\Omega(x_1)}\Big).$$

This, together with the last second estimate in (2.4), gives

$$\ell([x_1, x_2]) \le \frac{10a}{9(a-1)} e^{\frac{10}{9}(a-1)^{-1}} d(x_1, x_2).$$

The proof of lemma is thus complete. □

2.2. **Some elementary estimates.** The following simple estimate in Gromov hyperbolic domains will be frequently used in our later proofs.

Lemma 2.2. *Suppose that Ω is δ-Gromov hyperbolic, and $x, y, z \in \Omega$ are distinct points. For each $w \in \gamma_{xy}$, if $k_\Omega(w, y) \ge 2\delta + k_\Omega(y, z)$, then there must exist some point $u \in \gamma_{xz}$ such that $k_\Omega(w, u) \le \delta$.*

Proof. Since Ω is δ-Gromov hyperbolic, there exists some $u \in \gamma_{xz} \cup \gamma_{yz}$ such that

$$k_\Omega(w, u) \le \delta.$$

If $u \in \gamma_{yz}$, then

$$k_\Omega(y, u) \ge k_\Omega(w, y) - k_\Omega(w, u) \ge k_\Omega(y, z) + \delta \ge k_\Omega(y, u) + \delta,$$

which is impossible. Thus $u \in \gamma_{xz}$ and the proof of lemma is complete. □

We shall also need the following elementary estimate for Q-doubling metric spaces.

Lemma 2.3 ([21, Lemma 4.1.11])**.** *Let $X = (X, d)$ be a Q-doubling metric space with constant Q and $\Omega \subset X$ a domain. Fix $R > 0$ and $a \ge 1$ and let $r = \frac{R}{a}$. Then for any $x \in \Omega$, the ball $\mathbb{B}(x, R)$ contains at most b balls with radius r such that they are disjoint from each other, where $b \le Q^{[\log_2 a]}$. Here and hereafter, $[\cdot]$ means the greatest integer part.*

The following result will be needed in our later proof of Theorem 1.7.

Theorem 2.4 ([4, Theorem 2.10])**.** *A quasihyperbolic geodesic in an A-uniform space is a B-uniform curve with $B = 128A^4 e^{(4A)^6}$.*

3. Proof of Theorem 1.11

We assume that Ω satisfies both C-Gehring-Hayman inequality and the C-ball separation condition. Then we will prove that (Ω, k) is δ-Gromov hyperbolic with $\delta = 50C^6(3+C)^2$.

For any x_1, x_2 and x_3 in Ω, let $\gamma_{x_1x_2} \in \Lambda_{x_1x_2}(\Omega)$, $\gamma_{x_1x_3} \in \Lambda_{x_1x_3}(\Omega)$ and $\gamma_{x_2x_3} \in \Lambda_{x_2x_3}(\Omega)$. Then we will prove the geodesic triangle $\Delta_{x_1x_2x_3}$ has the δ-thin property.

Fix an arbitrary point $x_0 \in \gamma_{x_1x_2}$ and select $y_0 \in \gamma_{x_1x_3} \cup \gamma_{x_2x_3}$ such that

$$\sigma(x_0, y_0) = \inf_{y \in \gamma_{x_1x_3} \cup \gamma_{x_2x_3}} \sigma(x_0, y).$$

Without loss of generality, we may assume that $y_0 \in \gamma_{x_1x_3}$.

Since (Ω, σ) satisfies the C-ball separation condition, we have

$$B_\sigma\big(x_0, Cd_\Omega(x_0)\big) \cap \big(\gamma_{x_1x_3} \cup \gamma_{x_2x_3}\big) \neq \emptyset$$

and so

$$\sigma(x_0, y_0) \leq Cd_\Omega(x_0). \tag{3.1}$$

We shall use a contradiction argument to prove

$$k_\Omega(x_0, \gamma_{x_1x_3} \cup \gamma_{x_2x_3}) \leq 50C^6(3+C)^2. \tag{3.2}$$

To this end, suppose on the contrary that

$$k_\Omega(x_0, \gamma_{x_1x_3} \cup \gamma_{x_2x_3}) > 50C^6(3+C)^2. \tag{3.3}$$

We divide the proof into a few steps. In the first step, we shall prove the following assertion.

Step 1. For each $\gamma_{x_0y_0} \in \Lambda_{x_0y_0}$ and $x \in \gamma_{x_0y_0}$, we have

$$\sigma(y_0, x) \leq 2C^3(1+C)d_\Omega(x). \tag{3.4}$$

Since (Ω, d) satisfies the C-Gehring-Hayman inequality, it holds

$$\sigma(y_0, x) \leq \ell(\gamma_{x_0y_0}) \leq C\sigma(x_0, y_0). \tag{3.5}$$

Let α be an $d_\Omega(x)$-geodesic connecting x_0 and y_0 in Ω, that is,

$$\ell(\alpha) \leq \sigma(x_0, y_0) + d_\Omega(x).$$

Since (Ω, σ) satisfies the C-ball separation condition, we have

$$B_\sigma\big(x, Cd_\Omega(x)\big) \cap \alpha \neq \emptyset$$

and so there exists some point $y \in \alpha$ such that

$$\sigma(x, y) \leq Cd_\Omega(x). \tag{3.6}$$

Applying the C-ball separation condition again, we obtain that there exist two points $y_1 \in \gamma_{x_1x_3}[x_1, y_0] \cup \gamma_{x_1x_2}[x_1, x_0]$ and $y_2 \in \gamma_{x_1x_3}[y_0, x_3] \cup \gamma_{x_2x_3} \cup \gamma_{x_1x_2}[x_0, x_2]$ such that

$$\max\left\{\sigma\left(x, y_1\right), \sigma\left(x, y_2\right)\right\} \leq Cd_\Omega(x). \tag{3.7}$$

Note that $\gamma_{x_1x_3} = \gamma_{x_1x_3}[x_1, y_0] \cup \gamma_{x_1x_3}[y_0, x_3]$. In below, we present the proof of (3.4) in two cases.

Case 1-1. Either $y_1 \in \gamma_{x_1x_3}[x_1, y_0]$ or $y_2 \in (\gamma_{x_1x_3}[y_0, x_3] \cup \gamma_{x_2x_3})$ holds.

In this case, we take $z = y_1$ if $y_1 \in \gamma_{x_1x_3}[x_1, y_0]$, and $z = y_2$ otherwise. Then we have

$$\begin{aligned}\sigma(x_0, y) + \sigma(y, y_0) \leq \ell(\alpha) &\leq \sigma(x_0, y_0) + d_\Omega(x) \leq \sigma(x_0, y) + \sigma(y, z) + d_\Omega(x)\\ &\leq \sigma(x_0, y) + \sigma(x, y) + \sigma(x, z) + d_\Omega(x),\end{aligned}$$

which implies

$$\sigma(y_0, y) \leq \sigma(x, y) + \sigma(x, z) + d_\Omega(x) \overset{(3.6)+(3.7)}{\leq} 2Cd_\Omega(x) + d_\Omega(x).$$

Consequently, we have

$$\sigma(y_0, x) \leq \sigma(y_0, y) + \sigma(x, y) \overset{(3.6)}{\leq} 3Cd_\Omega(x) + d_\Omega(x).$$

This gives (3.4).

Case 1-2. $y_1 \in \gamma_{x_1x_2}[x_1, x_0]$ and $y_2 \in \gamma_{x_1x_2}[x_0, x_2]$.

In this case, note that

$$\begin{aligned}\ell(\gamma_{x_1x_2}[y_1, y_2]) &\geq \sigma(x_0, x) - \sigma(x, y_1) + \sigma(x_0, x) - \sigma(x, y_2)\\ &\overset{(3.7)}{\geq} 2\sigma(x, x_0) - 2Cd_\Omega(x).\end{aligned}$$

Since (Ω, d) satisfies the C-Gehring-Hayman inequality, we further have

$$\ell(\gamma_{x_1x_2}[y_1, y_2]) \leq C\sigma(y_1, y_2) \leq C\left(\sigma(x, y_1) + \sigma(x, y_2)\right) \overset{(3.7)}{\leq} 2C^2 d_\Omega(x).$$

Combining the above two estimates gives

$$\sigma(x_0, x) \leq C(1 + C)d_\Omega(x). \tag{3.8}$$

If $\sigma(x_0, x) \leq \frac{1}{2}d_\Omega(x_0)$, then $d_\Omega(x) \geq d_\Omega(x_0) - \sigma(x_0, x) \geq \frac{1}{2}d_\Omega(x_0)$, and we have

$$\sigma(y_0, x) \overset{(3.5)}{\leq} C\sigma(x_0, y_0) \overset{(3.1)}{\leq} C^2 d_\Omega(x_0) \leq 2C^2 d_\Omega(x),$$

which gives (3.4).

If $\sigma(x_0, x) > \frac{1}{2}d_\Omega(x_0)$, then we have

$$\sigma(y_0, x) \overset{(3.5)}{\leq} C\sigma(x_0, y_0) \overset{(3.1)}{\leq} C^2 d_\Omega(x_0) \leq 2C^2\sigma(x_0, x) \overset{(3.8)}{\leq} 2C^3(1 + C)d_\Omega(x).$$

In either cases, the proof of (3.4) is complete.

In the second step, we shall prove the following estimate.

Step 2. $d_\Omega(y_0) \leq \frac{1}{16(3+C)^4}\sigma(x_0, y_0)$.

Suppose on the contrary that the above estimate fails, that is,

$$d_\Omega(y_0) > \frac{1}{16(3 + C)^4}\sigma(x_0, y_0).$$

Then we may obtain a contraction to (3.3) as follows. Since (Ω, d) satisfies the C-Gehring-Hayman inequality, we have

$$\ell(\gamma_{x_0y_0}[y_0, x]) \leq C\sigma(y_0, x) \overset{(3.4)}{\leq} 2C^4(1 + C)d_\Omega(x). \tag{3.9}$$

This gives, in particular, that for each $w \in \gamma_{x_0y_0}$ and $x \in \gamma_{x_0y_0}[w, x_0]$, it holds

$$\ell(\gamma_{x_0y_0}[w, x]) \leq \ell(\gamma_{x_0y_0}[y_0, x]) \leq 2C^4(1 + C)d_\Omega(x).$$

Then we know from [15, Lemma 2.1] that for each $w \in \gamma_{x_0y_0}$, it holds

$$k_\Omega(w, x_0) \leq 8C^4(1 + C)\log\left(1 + \frac{\ell(\gamma_{x_0y_0})}{d_\Omega(w)}\right) \overset{(3.5)}{\leq} 8C^4(1 + C)\log\left(1 + \frac{C\sigma(x_0, y_0)}{d_\Omega(w)}\right). \tag{3.10}$$

Since $d_\Omega(y_0) > \frac{1}{16(3+C)^4}\sigma(x_0, y_0)$, we may apply (3.10) with $w = y_0$ to obtain

$$k_\Omega(x_0, y_0) < 40C^4(3 + C) \log 16(3 + C) < 50C^6(3 + C)^2,$$

which clearly contradicts with (3.3).

Then we get the desired estimate in Step 2, which is:

$$d_\Omega(y_0) \le \frac{1}{16(3 + C)^4}\sigma(x_0, y_0). \tag{3.11}$$

Under this condition, we shall prove the following claim.

Claim 3.1. *There exist two points $z_1 \in \gamma_{x_1x_2}[x_1, x_0]$ and $z_2 \in \gamma_{x_1x_2}[x_0, x_2]$ such that*

$$\sigma(y_0, z_1) \le \frac{3}{5(3 + C)^2}\sigma(x_0, y_0) \quad \textit{and} \quad \sigma(y_0, z_2) \le \frac{3}{5(3 + C)^2}\sigma(x_0, y_0). \tag{3.12}$$

We shall only prove the existence of z_1 in Claim 3.1, as the proof of other case is similar. Select $w_1 \in \gamma_{x_0y_0}$ so that

$$\sigma(y_0, w_1) = \frac{1}{8(3 + C)^4}\sigma(x_0, y_0). \tag{3.13}$$

If $\sigma(y_0, x_1) \le 2\sigma(y_0, w_1)$, then for $z_1 = x_1$, (3.13) gives

$$\sigma(y_0, z_1) = \sigma(y_0, x_1) \le \frac{1}{4(3 + C)^4}\sigma(x_0, y_0) \le \frac{3}{5(3 + C)^2}\sigma(x_0, y_0).$$

In this case, the proof of Claim 3.1 is complete.

Thus, we may assume $\sigma(y_0, x_1) > 2\sigma(y_0, w_1)$. In this case, let $u_1 \in \gamma_{x_1x_3}[y_0, x_1]$ be the last point along the direction from y_0 to x_1 with

$$\sigma(y_0, u_1) = 2\sigma(y_0, w_1) = \frac{1}{4(3 + C)^4}\sigma(x_0, y_0). \tag{3.14}$$

For each $u \in \gamma_{u_1w_1}$, we infer from the C-Gehring-Hayman inequality that

$$\sigma(u_1, u) \le \ell(\gamma_{u_1w_1}) \le C\sigma(u_1, w_1) \le C\left(\sigma(u_1, y_0) + \sigma(y_0, w_1)\right) \overset{(3.13)+(3.14)}{\le} \frac{3\sigma(x_0, y_0)}{8(3 + C)^3}. \tag{3.15}$$

By the C-ball separation condition, for each $u \in \gamma_{u_1w_1}$, there exists some point $v \in \gamma_{x_1x_2}[x_1, x_0] \cup \gamma_{x_1x_3}[x_1, u_1] \cup \gamma_{y_0x_0}[w_1, x_0]$ such that

$$\sigma(u, v) \le Cd_\Omega(u). \tag{3.16}$$

Case 2-1. $v \in \gamma_{x_1x_2}[x_1, x_0]$.

In this case, note first that

$$\begin{aligned}
\sigma(y_0, v) &\le \sigma(y_0, u_1) + \sigma(u_1, u) + \sigma(u, v) \overset{(3.16)}{\le} \sigma(y_0, u_1) + \sigma(u_1, u) + Cd_\Omega(u)\\
&\le \sigma(y_0, u_1) + \sigma(u_1, u) + C(d_\Omega(u_1) + \sigma(u, u_1))\\
&\le \sigma(y_0, u_1) + (1 + C)\sigma(u_1, u) + C(d_\Omega(y_0) + \sigma(y_0, u_1))\\
&\overset{(3.14)}{=} \frac{1 + C}{4(3 + C)^4}\sigma(x_0, y_0) + (1 + C)\sigma(u_1, u) + Cd_\Omega(y_0).
\end{aligned}$$

On the other hand, by the C-Gehring-Hayman inequality, we have

$$\sigma(u_1, u) \le \ell(\gamma_{u_1w_1}) \le C\sigma(u_1, w_1) \le C\big(\sigma(u_1, y_0)+\sigma(y_0, w_1)\big) \overset{(3.14)}{=} 3C\sigma(y_0, w_1) = \frac{3}{8(3 + C)^3}\sigma(x_0, y_0).$$

The previous two estimates, together with (3.11), give

$$\sigma(y_0, v) \le \frac{6C^2 + 29C + 7}{16(3+C)^4}\sigma(x_0, y_0) < \frac{3}{5(3+C)^2}\sigma(x_0, y_0).$$

Claim 3.1 follows from the above estimate by taking $v = z_1$.

Case 2-2. $v \in \gamma_{x_1x_3}[x_1, u_1] \cup \gamma_{y_0x_0}[w_1, x_0]$.

In this case, we claim that the following estimate holds:

$$\sigma\left(u_1, u\right) < 12C^4(3+C)^3 d_\Omega(u). \tag{3.17}$$

Case 2-2-1. $v \in \gamma_{x_0y_0}[w_1, x_0]$.

In this case, by the C-Gehring-Hayman inequality, we have

$$\sigma(y_0, v) \ge C^{-1}\ell(\gamma_{y_0x_0}[y_0, v]) \ge C^{-1}\sigma(y_0, w_1) \overset{(3.13)}{>} \frac{1}{8(3+C)^5}\sigma(x_0, y_0)$$

and so

$$\begin{aligned}\sigma(u_1, u) &\overset{(3.15)}{\le} \frac{3}{8(3+C)^3}\sigma(x_0, y_0) \le 3(3+C)^2\sigma(y_0, v) \overset{(3.4)}{<} 6C^3(3+C)^3 d_\Omega(v)\\ &\le 6C^3(3+C)^3\left(d_\Omega(u) + \sigma(u, v)\right) \overset{(3.16)}{\le} 6C^4(3+C)^3 d_\Omega(u).\end{aligned}$$

This proves (3.17).

Case 2-2-2. $v \in \gamma_{x_1x_3}[x_1, u_1]$.

Let α be an $d_\Omega(u)$-geodesic connecting u_1 and y_0 in Ω, that is, $\ell(\alpha) \le \sigma(u_1, y_0) + d_\Omega(u)$. Then by the C-ball separation condition, there exists some point $v_1 \in \gamma_{y_0x_0}[w_1, y_0] \cup \alpha$ such that

$$\sigma(u, v_1) \le C d_\Omega(u). \tag{3.18}$$

Moreover, we have

$$\begin{aligned}\frac{1}{4(3+C)^4}\sigma(x_0, y_0) &\overset{(3.14)}{=} \sigma(y_0, u_1) \le \sigma(y_0, v) \le \sigma(y_0, u) + \sigma(u, v)\\ &\overset{(3.16)}{\le} \sigma(y_0, u) + C d_\Omega(u).\end{aligned} \tag{3.19}$$

We first consider the case $v_1 \in \gamma_{x_0y_0}[y_0, w_1]$.

- If $\sigma(y_0, v_1) \ge \frac{1}{16(3+C)^5}\sigma(x_0, y_0)$, then we may argue as in Case 2-2-1 to derive (3.17).
- If $\sigma(y_0, v_1) < \frac{1}{16(3+C)^5}\sigma(x_0, y_0)$, then

$$\sigma(y_0, u) \le \sigma(y_0, v_1) + \sigma(u, v_1) \overset{(3.18)}{<} \frac{1}{16(3+C)^5}\sigma(x_0, y_0) + C d_\Omega(u), \tag{3.20}$$

which, together with (3.15) and (3.19), shows that

$$\sigma(u_1, u) < 3C(3+C) d_\Omega(u).$$

Next, we consider the case $v_1 \in \alpha$. In this case, we have

$$\begin{aligned}\sigma(y_0,u) &\leq \sigma(y_0,v_1)+\sigma(u,v_1) \overset{(3.18)}{\leq} \ell(\alpha)-\ell(\alpha[u_1,v_1])+Cd_\Omega(u)\\ &\leq \sigma(y_0,u_1)+d_\Omega(u)-\sigma(u_1,u)+\sigma(u,v_1)+Cd_\Omega(u)\\ &\overset{(3.18)}{\leq} \sigma(y_0,u_1)+d_\Omega(u)-\sigma(u_1,u)+2Cd_\Omega(u)\\ &\overset{(3.19)}{\leq} \sigma(y_0,u)+d_\Omega(u)-\sigma(u_1,u)+3Cd_\Omega(u).\end{aligned}$$

Thus it follows

$$\sigma(u_1,u) \leq 3Cd_\Omega(u)+d_\Omega(u)=(1+3C)d_\Omega(u).$$

The above estimate gives (3.17).

Note that (3.10) and C-Gehring-Hayman inequality imply

$$k_\Omega(w_1,x_0) \leq 8C^4(1+C)\log\Big(1+\frac{\ell(\gamma_{x_0y_0}[w_1,x_0])}{d_\Omega(w_1)}\Big) \leq 8C^4(1+C)\log\Big(1+\frac{C\sigma(y_0,x_0)}{d_\Omega(w_1)}\Big).$$

By (3.4) and (3.13), we have

$$d_\Omega(w_1) \geq \frac{\sigma(y_0,w_1)}{2C^3(1+C)}=\frac{\sigma(x_0,y_0)}{16C^3(3+C)^4}.$$

The above two estimates yield

$$k_\Omega(w_1,x_0) \leq 65C^4(1+C)\log 2(3+C),$$

which together with (3.3) implies

$$k_\Omega(u_1,w_1) \geq k_\Omega(x_0,u_1)-k_\Omega(w_1,x_0) > 49C^6(3+C)^2. \tag{3.21}$$

On the other hand, by [15, Lemma 2.1], (3.15) and (3.17), we have

$$k_\Omega(u_1,w_1) \leq 48C^4(3+C)^3\log\left(1+\frac{\ell(\gamma_{u_1w_1})}{d_\Omega(u_1)}\right) \leq 48C^4(3+C)^2\log\Big(1+\frac{3\sigma(x_0,y_0)}{8(3+C)^3d_\Omega(u_1)}\Big).$$

This, combining with (3.21), implies that

$$d_\Omega(u_1) < \frac{1}{48(3+C)^8}\sigma(x_0,y_0). \tag{3.22}$$

Since (Ω,σ) satisfies the C-ball separation condition, there exists some point $w \in \gamma_{x_1x_2}[x_1,x_0]\cup \gamma_{y_0x_0}$ such that

$$\sigma(u_1,w) \leq Cd_\Omega(u_1),$$

and so by (3.22),

$$\sigma(u_1,w) \leq \frac{C}{48(3+C)^8}\sigma(x_0,y_0).$$

Then we have

$$\sigma(y_0,w) \geq \sigma(u_1,y_0)-\sigma(u_1,w) \overset{(3.14)}{>} \frac{1}{5(3+C)^4}\sigma(x_0,y_0).$$

If $w \in \gamma_{y_0x_0}$, then the above estimate gives

$$\frac{1}{10C^3(3+C)^5}\sigma(x_0,y_0) < \frac{1}{2C^3(1+C)}\sigma(y_0,w) \overset{(3.4)}{\leq} d_\Omega(w) \leq d_\Omega(u_1)+\sigma(u_1,w) \leq (1+C)d_\Omega(u_1),$$

which clearly contradicts with (3.22). Thus, $w \in \gamma_{x_1x_2}[x_1,x_0]$, and we take $z_1=w$. Then the argument used in Case 2-1 shows $\sigma(y_0,z_1) \leq \frac{3}{5(3+C)^2}\sigma(x_0,y_0)$. This completes the proof of Claim 3.1.

Step 3. Contradiction.

By (3.12), we have

$$\begin{aligned}\ell(\gamma_{x_1x_2}[z_1,z_2]) &\geq \sigma(z_1,x_0)+\sigma(z_2,x_0)\\ &\geq \sigma(y_0,x_0)-\sigma(y_0,z_1)+\sigma(y_0,x_0)-\sigma(y_0,z_2)\\ &\geq \left(2-\frac{6}{5(3+C)^2}\right)\sigma(y_0,x_0).\end{aligned}$$

On the other hand, the C-Gehring-Hayman inequality gives

$$\ell(\gamma_{x_1x_2}[z_1,z_2]) \leq C\sigma(z_1,z_2) \leq C(\sigma(z_1,y_0)+\sigma(y_0,z_2)) \leq \frac{6C}{5(3+C)^2}\sigma(y_0,x_0),$$

which clearly contradicts with the previous estimate. This indicates that (3.3) can not hold and thus the proof of Theorem 1.11 is complete.

4. Gromov hyperbolicity implies the ball separation condition

In this and the next section, we shall prove Theorem 1.10 and thus we assume throughout these two sections that (X,d) is a Q-doubling metric space and (Ω,k) is δ-Gromov hyperbolic with $\delta=C\geq 8$. Our strategy is to first show that Gromov hyperbolicity of (Ω,k) implies that (Ω,σ) satisfies the ball separation condition. Then, in the next section, we shall use the ball separation property of (Ω,σ), together with Gromov hyperbolicity of (Ω,k), to prove that (Ω,d) satisfies the Gehring-Hayman inequality.

In this section, we shall prove the following theorem.

Theorem 4.1. *Suppose (Ω,k) is C-Gromov hyperbolic. Then (Ω,σ) satisfies the τ-ball separation condition with $\tau=e^{(2CQ)^{192(1+C)}}$.*

For simplicity of our exposition, we make the following conventions: Any consecutive points on γ_{xy} or α are always taken along the direction from x to y, and a partition of a sub-curve of γ_{xy} or α means a finite sequence of consecutive points on this sub-curve with its endpoints included. γ_{xzy} means z is a point on γ_{xy}.

4.1. **Two special classes of curves.** In this section, we use the notation $\gamma_{xx_0y}\in\Lambda_{xy}(\Omega)$ to denote a curve $\gamma_{xy}\in\Lambda_{xy}(\Omega)$ with $x_0\in\gamma_{xy}$. An important technical step towards the proof is to introduce two new classes of curves and derive a couples of fundamental lemmas about quasihyperbolic distances related to points on these curves.

Definition 4.2. Given $\gamma_{xx_0y}\in\Lambda_{xy}(\Omega)$, $\alpha\in\Gamma_{xy}(\Omega)$ and $\theta>0$, the class $P_\alpha^{\gamma_{xx_0y}}(\theta)$ consists of all curves γ in Ω with the following properties:

(1) γ connects x to some point $z\in\alpha$ with $\gamma\in\Lambda_{xz}(\Omega)$;
(2) there exists $z_0\in\gamma$ with $k_\Omega(x_0,z_0)=k_\Omega(x_0,\gamma)\geq\theta$;
(3) for each $w\in\alpha(z,y]$ and for each $\gamma_{xw}\in\Lambda_{xw}(\Omega)$, it holds $k_\Omega(x_0,\gamma_{xw})<\theta$.

To emphasize the points z_0 and z on γ, we shall write $\gamma=\gamma_{xz_0z}$ for a general curve in $P_\alpha^{\gamma_{xx_0y}}(\theta)$.

The following result shows that for each $\gamma_{xx_0y}\in\Lambda_{xy}(\Omega)$, $\alpha\in\Gamma_{xy}(\Omega)$ and $\theta>1+C$, if $k_\Omega(x,x_0)>1+\theta$, then the class $P_\alpha^{\gamma_{xx_0y}}(\theta)\neq\emptyset$.

Lemma 4.3. *For each $\gamma_{xx_0y}\in\Lambda_{xy}(\Omega)$, $\alpha\in\Gamma_{xy}(\Omega)$ and $\theta>1+C$, if $k_\Omega(x,x_0)>1+\theta$, then there exist a point $z\in\alpha$ and a curve $\gamma=\gamma_{xz_0z}\in P_\alpha^{\gamma_{xx_0y}}(\theta)$.*

Proof. Take $v_1 \in \alpha$ with $k_\Omega(v_1, y) \le 1$, and then fix $\gamma_{xv_1} \in \Lambda_{xv_1}(\Omega)$ and $\gamma_{v_1y} \in \Lambda_{v_1y}(\Omega)$. Since (Ω, k) is C-Gromov hyperbolic, there exists some point $v_2 \in \gamma_{xv_1} \cup \gamma_{v_1y}$ such that

$$k_\Omega(x_0, v_2) \le C.$$

It follows from the above estimate and triangle's inequality

$$\begin{aligned} k_\Omega(x_0, \gamma_{xv_1}) &\le \max\{k_\Omega(x_0, v_2), k_\Omega(v_1, v_2) + k_\Omega(x_0, v_2)\} \\ &\le k_\Omega(v_1, y) + k_\Omega(x_0, v_2) \le 1 + C. \end{aligned} \tag{4.1}$$

Next, select $u_1 \in \alpha$ with $k_\Omega(u_1, x) \le 1$ and $\gamma_{xu_1} \in \Lambda_{xu_1}(\Omega)$. Since $k_\Omega(x, x_0) > 1 + \theta$, we obtain from our choice of u_1 that

$$k_\Omega(x_0, \gamma_{xu_1}) \ge k_\Omega(x_0, x) - k_\Omega(x, u_1) > \theta > 1 + C.$$

Finally, based on the previous estimate and (4.1), we may choose z to be the last point on α along the direction from x to y such that there exists some $\gamma = \gamma_{xz} \in \Lambda_{xz}(\Omega)$ with $k_\Omega(x_0, \gamma) \ge \theta$. Then it is easy to see that $\gamma \in P_\alpha^{\gamma_{xx_0y}}(\theta)$. Thus the proof of Lemma 4.3 is complete. □

Lemma 4.4. *For each $\gamma_{xx_0y} \in \Lambda_{xy}(\Omega)$, $\alpha \in \Gamma_{xy}(\Omega)$ and any $\gamma_{xz_0z} \in P_\alpha^{\gamma_{xx_0y}}(\theta)$, it holds*

$$\theta \le k_\Omega(x_0, z_0) < 1 + \theta + C.$$

Proof. Select $w_1 \in \alpha[z, y]$ such that

$$k_\Omega(w_1, z) \le 1. \tag{4.2}$$

Since $\gamma_{xz_0z} \in P_\alpha^{\gamma_{xx_0y}}(\theta)$ and $w_1 \in \alpha[z, y]$, by Definition 4.2(3), for each $\gamma_{xw_1} \in \Lambda_{xw_1}(\Omega)$, there exists some point $y_0 \in \gamma_{xw_1}$ such that

$$k_\Omega(x_0, y_0) < \theta. \tag{4.3}$$

Since (Ω, k) is C-Gromov hyperbolic, there exists some point $v_1 \in \gamma_{xz} \cup \gamma_{zw_1}$ such that

$$k_\Omega(y_0, v_1) \le C.$$

On the other hand, by triangle's inequality and the above estimate, we have

$$k_\Omega(y_0, \gamma_{xz}) \le \le k_\Omega(w_1, z) + k_\Omega(y_0, v_1) \overset{(4.2)}{\le} 1 + C.$$

Then it follows from the above estimate that

$$\theta \overset{\text{Definition 4.2(2)}}{\le} k_\Omega(x_0, z_0) = k_\Omega(x_0, \gamma_{xz}) \le k_\Omega(x_0, y_0) + k_\Omega(y_0, \gamma_{xz}) \overset{(4.3)}{<} 1 + \theta + C.$$

The proof is thus complete. □

Definition 4.5. For each given $\gamma_{xx_0y} \in \Lambda_{xy}(\Omega)$, $\alpha \in \Gamma_{xy}(\Omega)$, $z \in \alpha$ and $w \in \alpha[z, w]$, the class $O_{\alpha[z,w]}^{\gamma_{xx_0y}}(\theta_2)$ with $\theta_2 > 0$, consists of all curves $\gamma \in \Lambda_{zw}(\Omega)$ so that there exists $y_0 \in \gamma$ with

$$k_\Omega(x_0, y_0) = k_\Omega(x_0, \gamma) \le \theta_2.$$

To emphasize the points z, y_0 and w on γ, we shall write $\gamma = \gamma_{zy_0w}$ for a general curve in $O_{\alpha[z,w]}^{\gamma_{xx_0y}}(\theta_2)$.

The following lemmas is very critical in the proof of Theorem 4.1.

Lemma 4.6. *Fix $\gamma_{xz_1y} \in \Lambda_{xy}(\Omega)$, $\alpha \in \Gamma_{xy}(\Omega)$, $\gamma_{xz_{1,1}y_1} \in P_\alpha^{\gamma_{xz_1y}}(\theta)$, and $z_2 \in \gamma_{xy}[z_1, y]$ and $y_2 \in \alpha[y_1, y]$ with $\gamma_{xz_{1,2}y_2} \in P_\alpha^{\gamma_{xz_2y}}(\theta)$. Suppose $\theta > 2C$, $\gamma_{y_1z_{2,1}y_2} \in O_{\alpha[y_1,y_2]}^{\gamma_{xz_1y}}(2C)$ and $\gamma_{y_1z_{1,3}y_3} \in P_{\alpha[y_1,y_2]}^{\gamma_{y_1z_3y_2}}(\theta)$ for some $z_3 \in \gamma_{y_1y_2}[y_1, z_{2,1}]$. If $k_\Omega(z_{1,1}, z_3) \geq 2(1+\theta+4C)$, then there exists some $z_{2,3} \in \gamma_{xy_1}[y_1, z_{1,1}]$ such that*

$$\theta - C \leq k_\Omega(z_{2,3}, \gamma_{y_1y_3}) < 1+\theta+2C \quad \textit{and} \quad k_\Omega(z_{2,3}, z_{1,3}) < 1+\theta+2C.$$

Proof. It follows from Lemma 4.4 and $\gamma_{y_1z_{2,1}y_2} \in O_{\alpha[y_1,y_2]}^{\gamma_{xz_1y}}(2C)$ that

$$k_\Omega(z_{1,1}, z_{2,1}) \leq k_\Omega(z_{1,1}, z_1) + k_\Omega(z_1, z_{2,1}) < 1+\theta+3C.$$

This together with our assumption $k_\Omega(z_{1,1}, z_3) \geq 2(1+\theta+4C)$ gives

$$k_\Omega(z_3, z_{2,1}) \geq k_\Omega(z_{1,1}, z_3) - k_\Omega(z_{1,1}, z_{2,1}) > 1+\theta+5C > 2C + k_\Omega(z_{1,1}, z_{2,1}). \tag{4.4}$$

Since $z_{2,1} \in \gamma_{y_1y_2}[y_2, z_3]$, by (4.4), we may apply Lemma 2.2 (with $x = y_1, y = z_{2,1}, z = z_{1,1}$ and $w = z_3$) to find a point $z_{2,3} \in \gamma_{xy_1}[y_1, z_{1,1}]$ such that

$$k_\Omega(z_3, z_{2,3}) \leq C.$$

On the other hand, by Lemma 4.4, we have

$$\theta \leq k_\Omega(z_3, z_{1,3}) = k_\Omega(z_3, \gamma_{y_1y_3}) < 1+\theta+C.$$

Combining the above two estimates gives

$$1+\theta+2C > k_\Omega(z_3, \gamma_{y_1y_3}) + k_\Omega(z_3, z_{2,3}) \geq k_\Omega(z_{2,3}, \gamma_{y_1y_3}) \geq k_\Omega(z_3, \gamma_{y_1y_3}) - k_\Omega(z_3, z_{2,3}) \geq \theta - C$$

and

$$k_\Omega(z_{2,3}, z_{1,3}) \leq k_\Omega(z_3, z_{1,3}) + k_\Omega(z_3, z_{2,3}) < 1+\theta+2C.$$

The proof of Lemma 4.6 is complete. □

Lemma 4.7. *Suppose that $\gamma_{xz_{1,1}y_1} \in P_\alpha^{\gamma_{xz_1y}}(3C)$, $y_2 \in \alpha[y_1, y]$ and $\gamma_{y_1z_{2,1}y_2} \in O_{\alpha[y_1,y_2]}^{\gamma_{xz_1y}}(2C)$. For $y_3 \in \alpha[y_1, y_2]$, fix $\gamma_{y_1y_3} \in \Lambda_{y_1y_3}(\Omega)$ and $\gamma_{xy_3} \in \Lambda_{xy_3}(\Omega)$. If $k_\Omega(z_{2,1}, z_3) > 22C$ and $2C \leq k_\Omega(z_3, \gamma_{y_1y_3}) \leq 7C$ for some $z_3 \in \gamma_{y_1y_2}[y_1, z_{2,1}]$, then $2C \leq k_\Omega(z_1, \gamma_{xy_3}) \leq 3C$.*

Proof. First, we prove

$$k_\Omega(z_1, \gamma_{y_1y_3}) > 19C. \tag{4.5}$$

Since $\gamma_{y_1z_{2,1}y_2} \in O_{\alpha[y_1,y_2]}^{\gamma_{xz_1y}}(2C)$, we know

$$k_\Omega(z_1, z_{2,1}) \leq 2C. \tag{4.6}$$

Select $z_{1,3} \in \gamma_{y_1y_3}$ such that

$$k_\Omega(z_{2,1}, \gamma_{y_1y_3}) = k_\Omega(z_{2,1}, z_{1,3}). \tag{4.7}$$

Since (Ω, k) is C-Gromov hyperbolic and $k_\Omega(z_3, \gamma_{y_1y_3}[y_1, z_{1,3}]) \geq k_\Omega(z_3, \gamma_{y_1y_3}) \geq 2C$, we infer that there exists some point $z_{3,1} \in \gamma_{z_{2,1}z_{1,3}}$ ($\gamma_{z_{2,1}z_{1,3}} \in \Lambda_{z_{2,1}z_{1,3}}$) such that

$$k_\Omega(z_3, z_{3,1}) \leq C,$$

which, together with the assumption $k_\Omega(z_{2,1}, z_3) > 22C$, shows that

$$k_\Omega(z_{2,1}, z_{1,3}) \geq k_\Omega(z_{2,1}, z_{3,1}) \geq k_\Omega(z_{2,1}, z_3) - k_\Omega(z_3, z_{3,1}) > 21C. \tag{4.8}$$

Combining this with (4.6) and (4.7), we obtain

$$k_\Omega(z_1, \gamma_{y_1y_3}) \geq k_\Omega(z_{2,1}, z_{1,3}) - k_\Omega(z_1, z_{2,1}) > 19C,$$

which gives (4.5).

Since $\gamma_{xz_{1,1}y_1} \in P_\alpha^{\gamma_{xz_1y}}(3C)$, Lemma 4.4 shows

$$k_\Omega(z_1, z_{1,1}) < 5C, \tag{4.9}$$

from which we obtain

$$k_\Omega(z_{1,1}, \gamma_{y_1y_3}) \geq k_\Omega(z_1, \gamma_{y_1y_3}) - k_\Omega(z_1, z_{1,1}) \overset{(4.5)}{>} 14C.$$

Next we shall find a point $w_1 \in \gamma_{y_1y_2}[z_3, z_{2,1}]$ with

$$2C < k_\Omega(w_1, \gamma_{y_1y_3}) \leq 7C. \tag{4.10}$$

If $k_\Omega(z_3, \gamma_{y_1y_3}) > 2C$, then we take $w_1 = z_3$. On the other hand, if $k_\Omega(z_3, \gamma_{y_1y_3}) = 2C$, then we take $w_{1,1} \in \gamma_{y_1y_2}[z_3, z_{2,1}]$ with

$$k_\Omega(z_3, w_{1,1}) = 5C. \tag{4.11}$$

We claim

$$k_\Omega(w_{1,1}, \gamma_{y_1y_3}) \geq 3C. \tag{4.12}$$

Otherwise, there exists some point $w_{1,2} \in \gamma_{y_1y_3}$ such that

$$k_\Omega(w_{1,1}, w_{1,2}) < 3C.$$

Then

$$k_\Omega(z_3, w_{1,1}) \geq 2C + k_\Omega(w_{1,1}, w_{1,2}).$$

Applying Lemma 2.2 (with $x = y_1, y = w_{1,1}, z = w_{1,2}$ and $w = z_3$), we know that $k_\Omega(z_3, \gamma_{y_1y_3}) \leq C$, this contradicts with the assumption in this lemma. Thus, by (4.12), we let $w_1 \in \gamma_{y_1y_2}[z_3, w_{1,1}]$ with $k_\Omega(w_1, \gamma_{y_1y_3}) > 2C$, which, together with (4.11), shows (4.10) holds.

Note that by (4.6) and (4.9), we have

$$k_\Omega(z_{1,1}, z_{2,1}) \leq k_\Omega(z_1, z_{1,1}) + k_\Omega(z_1, z_{2,1}) \leq 7C, \tag{4.13}$$

and so it follows from our assumption that

$$k_\Omega(w_1, z_{2,1}) \geq k_\Omega(z_{2,1}, z_3) - k_\Omega(z_3, w_1) \geq 17C \geq 10C + k_\Omega(z_{1,1}, z_{2,1}). \tag{4.14}$$

Applying Lemma 2.2 (with $x = y_1, y = z_{2,1}, z = z_{1,1}$ and $w = w_1$), we infer that there is some point $z_{2,3} \in \gamma_{xy_1}[z_{1,1}, y_1]$ with

$$k_\Omega(w_1, z_{2,3}) \leq C. \tag{4.15}$$

We obtain from (4.10) and (4.15) that

$$k_\Omega(z_{2,3}, \gamma_{y_1y_3}) \geq k_\Omega(w_1, \gamma_{y_1y_3}) - k_\Omega(w_1, z_{2,3}) > C. \tag{4.16}$$

Select $z_{3,3} \in \gamma_{y_1y_3}$ such that

$$k_\Omega(z_{2,3}, z_{3,3}) = k_\Omega(z_{2,3}, \gamma_{y_1y_3}).$$

Then

$$C \overset{(4.16)}{<} k_\Omega(z_{2,3}, z_{3,3}) \leq k_\Omega(z_{2,3}, w_1) + k_\Omega(w_1, \gamma_{y_1y_3}) \overset{(4.10)\,and\,(4.15)}{\leq} 8C. \tag{4.17}$$

Since (Ω, k) is C-Gromov hyperbolic, (4.16) implies that there exists some point $z_{4,3} \in \gamma_{xy_3}$ such that

$$k_\Omega(z_{2,3}, z_{4,3}) \leq C. \tag{4.18}$$

Then we have

$$k_\Omega(z_{3,3}, z_{4,3}) \leq k_\Omega(z_{2,3}, z_{3,3}) + k_\Omega(z_{2,3}, z_{4,3}) \overset{(4.17)+(4.18)}{\leq} 9C. \tag{4.19}$$

For each $z \in \gamma_{xy_1}[z_{2,3}, y_1]$, then we have

$$k_\Omega(z_1, z) \geq k_\Omega(z_{1,1}, z) - k_\Omega(z_1, z_{1,1}) \geq k_\Omega(z_{1,1}, z_{2,3}) - k_\Omega(z_1, z_{1,1})$$
$$\overset{(4.9)}{\geq} k_\Omega(z_{2,1}, w_1) - k_\Omega(z_{2,3}, w_1) - k_\Omega(z_{1,1}, z_{2,1}) - 5C,$$

which, together with (4.14), (4.13) and (4.15), yields

$$k_\Omega(z_1, z) > 4C. \tag{4.20}$$

Choose $z_{5,3} \in \gamma_{xy_3}$ such that $k_\Omega(z_1, z_{5,3}) = k_\Omega(z_1, \gamma_{xy_3})$. Since $\gamma_{xz_{1,1}y_1} \in P_\alpha^{\gamma_{xz_1y}}(3C)$, Definition 4.2 (3) implies that

$$k_\Omega(z_1, \gamma_{xy_3}) = k_\Omega(z_1, z_{5,3}) \leq 3C. \tag{4.21}$$

Next, we prove the lower bound on $k_\Omega(z_1, \gamma_{xy_3})$. To this end, we consider two separate cases.

Case I: $z_{5,3} \in \gamma_{xy_3}[x, z_{4,3}]$.

In this case, since (Ω, k) is C-Gromov hyperbolic, we may find some point $z_{6,3} \in \gamma_{xy_1}[x, z_{2,3}] \cup \gamma_{z_{2,3}z_{4,3}}$ with

$$k_\Omega(z_{5,3}, z_{6,3}) \leq C.$$

Then, by (4.21) and the above estimate, we have

$$k_\Omega(z_1, z_{6,3}) \leq k_\Omega(z_1, z_{5,3}) + k_\Omega(z_{5,3}, z_{6,3}) \leq 4C. \tag{4.22}$$

This implies that $z_{6.3} \in \gamma_{xy_1}[x, z_{2,3}]$. Indeed, if not, then by (4.20),

$$k_\Omega(z_1, z_{6,3}) \geq 5C,$$

which contradicts with (4.22). Then it follows from Lemma 4.4 and the above estimate that

$$k_\Omega(z_1, z_{5,3}) \geq k_\Omega(z_1, z_{6,3}) - k_\Omega(z_{5,3}, z_{6,3}) \geq k_\Omega(z_1, \gamma_{xy_1}) - k_\Omega(z_{5,3}, z_{6,3}) \geq 2C. \tag{4.23}$$

Case II: $z_{5,3} \in \gamma_{xy_3}[z_{4,3}, y_3]$.

In this case, since (Ω, k) is C-Gromov hyperbolic, there exists some $z_{7,3} \in \gamma_{xy_1} \cup \gamma_{y_1y_3}$ such that

$$k_\Omega(z_{5,3}, z_{7,3}) \leq C. \tag{4.24}$$

It follows from (4.21) and (4.24) that

$$k_\Omega(z_1, z_{7,3}) \leq k_\Omega(z_1, z_{5,3}) + k_\Omega(z_{5,3}, z_{7,3}) \leq 4C,$$

and so (4.5) yields $z_{7,3} \notin \gamma_{y_1y_3}$. Moreover, (4.20) yields $z_{7,3} \notin \gamma_{xy_1}[z_{2,3}, y_1]$. Thus, $z_{7,3} \in \gamma_{xy_1}[x, z_{2,3}]$. A similar discussion as in (4.23) shows that

$$k_\Omega(z_1, z_{5,3}) \geq 2C.$$

The proof is thus complete. □

Definition 4.8. Fix $\gamma_{xx_1y} \in \Lambda_{xy}(\Omega)$ and $\alpha \in \Gamma_{xy}(\Omega)$. For $z \in \alpha$ and $\gamma_{xz} \in \Lambda_{xz}(\Omega)$, if there exists $z_1 \in \gamma_{xz}$ such that $2C \leq k_\Omega(x_1, \gamma_{xz_1z}) = k_\Omega(x_1, z_1) \leq 7C$, then we write $\gamma_{xz_1z} \in Q_\alpha^{\gamma_{xx_1y}}$.

Lemma 4.9. *Fix $\gamma_{xx_1y} \in \Lambda_{xy}(\Omega)$ and $\alpha \in \Gamma_{xy}(\Omega)$. If $\gamma_{xz_1z} \in P_\alpha^{\gamma_{xx_1y}}(3C)$, then $\gamma_{xz_1z} \in Q_\alpha^{\gamma_{xx_1y}}$.*

Proof. This follows directly from Lemma 4.4. □

Lemma 4.10. *Fix $\gamma_{xx_1y} \in \Lambda_{xy}(\Omega)$, $\alpha \in \Gamma_{xy}(\Omega)$ and $\gamma_{xz_1z} \in Q_\alpha^{\gamma_{xx_1y}}$. We assume that for some $y_1 \in \gamma_{xy}[x_1, y]$ and $w \in \alpha[z, y]$, $\gamma_{xw_1w} \in Q_\alpha^{\gamma_{xy_1y}}$. If $k_\Omega(x_1, y_1) \geq 30C$, then the following results hold:*

(1) *for any $\gamma_{zw} \in \Lambda_{zw}(\Omega)$, there exist some point $w_0 \in \gamma_{zw}$ such that $\gamma_{zw_0w} \in O_{\alpha[z,w]}^{\gamma_{xx_1y}}(2C)$.*
(2) *for each $u \in \gamma_{xw_1w}[w_1, w]$ and $v \in \gamma_{xz_1z}[z_1, z]$, $k_\Omega(u, v) \geq 3C$.*

Proof. First of all, fix any $\gamma_{zw} \in \Lambda_{zw}(\Omega)$. Since $\gamma_{xw_1w} \in Q_\alpha^{\gamma_{xy_1y}}$, $k_\Omega(w_1, y_1) \leq 7C$ and thus

$$k_\Omega(x_1, y_1) \geq 30C \geq 23C + k_\Omega(w_1, y_1),$$

which, together with Lemma 2.2 (with $x = x$, $w = x_1$, $y = y_1$ and $z = w_1$), yields that there exists some point $y_2 \in \gamma_{xw}[x, w_1]$ such that

$$k_\Omega(x_1, y_2) \leq C. \tag{4.25}$$

and again, by (Ω, k) being C-Gromov hyperbolic, there exists some point $w_2 \in \gamma_{wy}$ ($\gamma_{wy} \in \Lambda_{wy}(\Omega)$) such that

$$k_\Omega(y_1, w_2) \leq C. \tag{4.26}$$

Hence

$$k_\Omega(w_1, w_2) \leq k_\Omega(w_1, y_1) + k_\Omega(y_1, w_2) \leq 8C \tag{4.27}$$

and

$$k_\Omega(w_1, y_2) \geq k_\Omega(x_1, y_1) - k_\Omega(x_1, y_2) - k_\Omega(y_1, w_1) \geq 21C. \tag{4.28}$$

Now we prove Lemma 4.10 (1). Since $\gamma_{xz_1z} \in Q_\alpha^{\gamma_{xx_1y}}$, $k_\Omega(x_1, z_1) \geq 2C$, and there exists some point $x_2 \in \gamma_{zy}$ ($\gamma_{zy} \in \Lambda_{zy}(\Omega)$) such that

$$k_\Omega(x_1, x_2) \leq C. \tag{4.29}$$

It follows from " (Ω, k) being C-Gromov hyperbolic" that there exists some point $w_0 \in \gamma_{zw} \cup \gamma_{wy}$ such that

$$k_\Omega(x_2, w_0) \leq C. \tag{4.30}$$

Then

$$k_\Omega(x_1, w_0) \leq k_\Omega(x_1, x_2) + k_\Omega(x_2, w_0) \overset{(4.29) \text{ and } (4.30)}{\leq} 2C. \tag{4.31}$$

We claim $w_0 \in \gamma_{zw}$. Otherwise,

$$k_\Omega(w_0, w_2) \geq k_\Omega(x_1, y_1) - k_\Omega(x_1, w_0) - k_\Omega(y_1, w_2) \overset{(4.26) \text{ and } (4.31)}{\geq} 17C. \tag{4.32}$$

If $w_0 \in \gamma_{wy}[w, w_2]$, then (4.27) and (4.32) imply that

$$k_\Omega(w_0, w_2) \geq 9C + k_\Omega(w_1, w_2).$$

Applying Lemma 2.2 (with $x = w, y = w_2$, $z = w_1$ and $w = w_0$), we obtain that there exists some $w_3 \in \gamma_{xw}[w, w_1]$ such that

$$k_\Omega(w_0, w_3) \leq C,$$

which, together with (4.25), (4.29) and (4.30), shows

$$k_\Omega(y_2, w_3) \leq k_\Omega(y_2, x_1) + k_\Omega(x_1, x_2) + k_\Omega(x_2, w_0) + k_\Omega(w_0, w_3) \leq 4C,$$

which, contradicts with

$$k_\Omega(y_2, w_3) \geq k_\Omega(w_1, y_2) \overset{(4.28)}{\geq} 21C.$$

If $w_0 \in \gamma_{wy}[w_2, y]$, then a similar discuss shows we will get a contradiction. Hence we get $w_0 \in \gamma_{zw}$. (4.31) implies the proof of Lemma 4.10 (1) is finished.

Next we prove Lemma 4.10 (2). Since $k_\Omega(x_1,y_1)\geq 30C$, we get from (4.25) and the assumption $\gamma_{xw_1w}\in Q_\alpha^{\gamma_{xy_1y}}$ that for each $u\in\gamma_{xw_1w}[w_1,w]$, it holds

$$k_\Omega(u,w_2)\geq k_\Omega(w_2,w_1)\geq k_\Omega(x_1,y_1)-k_\Omega(x_1,w_2)-k_\Omega(y_1,w_1)\geq 22C. \tag{4.33}$$

Then, by (4.30),

$$k_\Omega(u,w_2)\geq 22C\geq 21C+k_\Omega(w_2,w_0).$$

Applying Lemma 2.2 (with $x=w, y=w_2$ and $z=w_0$), we obtain that there exists some point $u_1\in\gamma_{xw}[w_0,w]$ such that

$$k_\Omega(u,u_1)\leq C.$$

Then, by (4.30) and (4.33),

$$k_\Omega(u_1,w_0)\geq k_\Omega(u,w_2)-k_\Omega(u,u_1)-k_\Omega(w_2,w_0)\geq 20C. \tag{4.34}$$

Since $\gamma_{xz_1z}\in Q_\alpha^{\gamma_{xx_1y}}$, it follows from Lemma 4.10 (1) that

$$k_\Omega(z_1,w_0)\leq k_\Omega(x_1,z_1)+k_\Omega(x_1,w_0)\leq 9C.$$

If $k_\Omega(z_1,v)\geq 11C$, then

$$k_\Omega(z_1,v)\geq 2C+k_\Omega(z_1,w_0).$$

Applying Lemma 2.2 (with $x=z$, $y=z_1$ and $z=w_0$), we obtain that there exists some point $v_1\in\gamma_{zw}[z,w_0]$ such that

$$k_\Omega(v,v_1)\leq C,$$

which, together with (4.34), shows that

$$k_\Omega(u,v)\geq k_\Omega(u_1,v_1)-k_\Omega(v,v_1)-k_\Omega(u,u_1)\geq k_\Omega(u_1,w_0)-2C\geq 18C.$$

If $k_\Omega(z_1,v)<11C$, then we obtain from (4.25) and (4.33) that

$$\begin{aligned}k_\Omega(u,v)&\geq k_\Omega(u,w_2)-k_\Omega(w_2,z_1)-k_\Omega(z_1,v)\geq 11C-k_\Omega(w_2,z_1)\\&\geq 11C-k_\Omega(w_2,x_1)-k_\Omega(x_1,z_1)\geq 3C.\end{aligned}$$

This completes the proof of Lemma 4.10 (2). □

Lemma 4.11. *Fix $\gamma_{xz_1y}\in\Lambda_{xy}(\Omega)$ and $\alpha\in\Gamma_{xy}(\Omega)$. Suppose $y_1\in\alpha$, $y_2\in\alpha[y_1,y]$ and $y_3\in\alpha[y_1,y_2]$. Then the following two assertions hold.*

(1) *If $\gamma_{xz_{1,1}y_1}\in P_\alpha^{\gamma_{xz_1y}}(3C)$ and $\gamma_{y_1z_{1,2}y_2}\in O_{\alpha[y_1,y_2]}^{\gamma_{xz_1y}}(2C)$, then for each $z\in\gamma_{xy_1}[y_1,z_{1,1}]$ with $k_\Omega(z_{1,1},z)\geq 10C$,*

$$k_\Omega(z,\gamma_{y_1y_2}[z_{1,2},y_1])\leq C.$$

(2) *If $\gamma_{xz_{1,1}y_1}\in Q_\alpha^{\gamma_{xz_1y}}$, $\gamma_{xz_{2,1}y_3}\in Q_\alpha^{\gamma_{xz_1y}}$ and $k_\Omega(z_1,\gamma_{y_1y_3})\geq 11C$, then for each $z\in\gamma_{y_1y_3}$,*

$$k_\Omega(z,\gamma_{xy_1}[y_1,z_{1,1}]\cup\gamma_{xy_3}[z_{2,1},y_3])\leq C.$$

Proof. (1). Since $\gamma_{xz_{1,1}y_1}\in P_\alpha^{\gamma_{xz_1y}}(3C)$, Lemma 4.4 gives $k_\Omega(z_1,z_{1,1})<5C$. Since $\gamma_{y_1z_{1,2}y_2}\in O_{\alpha[y_1y_2]}^{\gamma_{xz_1y}}(2C)$, we obtain

$$k_\Omega(z_{1,1},z_{1,2})\leq k_\Omega(z_1,z_{1,1})+k_\Omega(z_1,z_{1,2})<7C,$$

and thus

$$k_\Omega(z_{1,1},z)\geq 10C\geq 3C+k_\Omega(z_{1,1},z_{1,2}).$$

Applying Lemma 2.2 (with $x = y_1$, $y = z_{1,1}$, $z = z_{1,2}$ and $w = z$), we infer that there exists some point $w \in \gamma_{y_1y_2}[z_{1,2}, y_1]$ such that

$$k_\Omega(z, w) \le C,$$

from which Lemma 4.11(1) follows.

(2). Since (Ω, k) is C-Gromov hyperbolic, for each $z \in \gamma_{y_1y_3}$, there exists some $w \in \gamma_{xy_1} \cup \gamma_{xy_3}$ such that

$$k_\Omega(z, w) \le C. \tag{4.35}$$

We shall prove

$$w \in \gamma_{xy_3}[z_{2,1}, y_3] \cup \gamma_{xy_1}[z_{1,1}, y_1] \tag{4.36}$$

via a contradiction argument.

Suppose (4.36) fails. Then $w \in \gamma_{xy_3}[x, z_{2,1}] \cup \gamma_{xy_1}[x, z_{1,1}]$. As the discuss is similar for $w \in \gamma_{xy_1}[x, z_{1,1}]$, without loss of generality, we may assume $w \in \gamma_{xy_3}[x, z_{2,1}]$. Since $\gamma_{xz_{2,1}y_3} \in Q_\alpha^{\gamma_{xz_1y}}$, $k_\Omega(z_1, z_{2,1}) \le 7C$ and thus by triangle's inequality and (4.35), we have

$$\begin{aligned} k_\Omega(z_{2,1}, w) &\ge k_\Omega(z_{2,1}, z) - k_\Omega(z, w) \ge k_\Omega(z_1, z) - k_\Omega(z_{2,1}, z_1) - C \\ &\ge k_\Omega(z_1, \gamma_{y_1y_3}) - 8C \ge 3C \ge 2C + k_\Omega(z, w). \end{aligned}$$

Thus we may apply Lemma 2.2 (with $x = y_3$, $w = z_{2,1}$, $y = w$ and $z = z$) to obtain a point $u \in \gamma_{y_1y_3}[z, y_3]$ with

$$k_\Omega(z_{2,1}, u) \le C.$$

Consequently, we get

$$k_\Omega(z_1, \gamma_{y_1y_3}) \le k_\Omega(z_1, z_{2,1}) + k_\Omega(z_{2,1}, u) \le 8C,$$

which clearly contradicts with the assumption $k_\Omega(z_1, \gamma_{y_1y_3}) \ge 11C$. Thus (4.36) holds and the proof is complete. □

4.2. **Proof of Theorem 4.1.** In this section, we shall prove Theorem 4.1 via a contradiction argument. For notational simplicity, we write $\gamma = \gamma_{xy}$ ($\gamma_{xy} \in \Lambda_{xy}$) and $\alpha \in \Gamma_{xy}$.

Suppose on the contrary that there exists some point $x_{0,0} \in \gamma$ such that

$$\sigma(x_{0,0}, \alpha) > \tau d_\Omega(x_{0,0}), \tag{4.37}$$

where $\tau = e^{(2CQ)^{192Q(1+C)^3}}$. Set $N_1 = [\frac{1}{32C^3} \log \tau]$. Then

$$e^{32CN_1} \le \tau^{\frac{1}{C^2}} < \tau. \tag{4.38}$$

Let $x_{1,0} = y_{1,0} = x$, $x_{1,N_1+1} = x_{0,0}$, $M_0 = 2N_1$ and $M_1 = [e^{-4-32C} \cdot e^{\frac{1}{2}\log_Q \frac{N_1}{4}}]$.

The following lemma will be crucial for the proof of Theorem 4.1.

Lemma 4.12. *For each $p \in \{1, \cdots, N_1\}$, there exists a sequence of successive points $x_{1,p} \in \gamma[x_{1,p-1}, x_{0,0}]$ along the direction from $x_{1,p-1}$ to $x_{0,0}$ such that $\sigma(x_{0,0}, x_{1,p}) \le e^{32C(1+N_1)} d_\Omega(x_{0,0})$. Moreover, the following conclusions hold.*

1. *for each $p \in \{0, \cdots, N_1\}$, $k_\Omega(x_{1,p}, x_{1,p+1}) > 30C$;*
2. *for each $p \in \{1, \cdots, N_1 + 1\}$, there exist $y_{1,p} \in \alpha[y_{1,p-1}, y]$, $\gamma_{xy_{1,p}} \in \Lambda_{xy_{1,p}}(\Omega)$ and a point $z_{1,p} \in \gamma_{xy_{1,p}}$ such that $\gamma_{xz_{1,p}y_{1,p}} \in P_\alpha^{\gamma_{xx_{1,p}y}}(3C)$, and for each $p \in \{1, \cdots, N_1\}$ and $\gamma_{y_{1,p}y_{1,p+1}} \in \Lambda_{y_{1,p}y_{1,p+1}}(\Omega)$, there exists a point $w_{1,p} \in \gamma_{y_{1,p}y_{1,p+1}}$ such that $\gamma_{y_{1,p}w_{1,p}y_{1,p+1}} \in O^{\gamma_{xx_{1,p}y}}_{\alpha[y_{1,p}, y_{1,p+1}]}(2C)$;*
3. *there exists some integer $N_{1,1} > \frac{N_1}{2}$ such that*

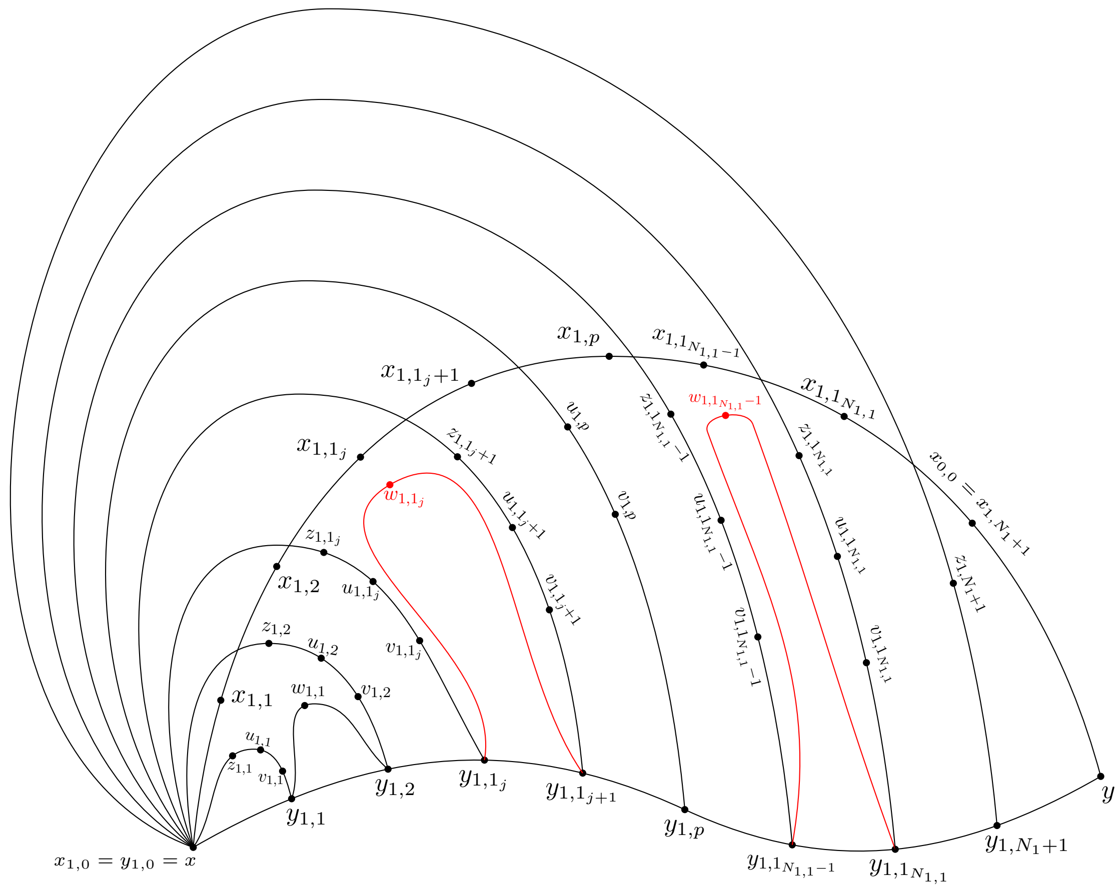

FIGURE 1. Illustration for the proof of Lemma 4.12

- *there exists a sequence of integers* $\{1_j\}_{j=1}^{N_{1,1}} \subset \{1, \cdots, N_1 - 1\}$ *with* $1_j < 1_{j+1}$ *for each* $j \in \{1, \cdots, N_{1,1} - 1\}$.
- *there exists a sequence* $\{x_{2,jM_1-(t-1)}\}_{j,t}$, *indexed with* $j \in \{1, \cdots, N_{1,1}\}$ *and* $t \in \{1, \cdots, M_1\}$, *of successive points on* $\gamma_{y_{1,1_j} y_{1,1_j+1}}[y_{1,1_j}, w_{1,1_j}]$ *along the direction from* $y_{1,1_j}$ *to* $w_{1,1_j}$ *so that*

$$d_\Omega(x_{2,jM_1-(t-1)}) \leq r_1 \quad \textit{with } r_1 = e^{-\frac{1}{2}\log_Q \frac{N_1}{4}} \cdot e^{32C(N_1+2)+2} d_\Omega(x_{0,0}).$$

  Moreover, it holds

$$\frac{3}{4} e^{32C(N_1+2)} d_\Omega(x_{0,0}) < \sigma(x_{0,0}, x_{2,jM_1-(t-1)}) < \frac{3}{4} e^{32C(N_1+2)+2} d_\Omega(x_{0,0}).$$

- *for each* $j \in \{1, \cdots, N_{1,1}\}$ *and* $t \in \{2, \cdots, M_1\}$,

$$k_\Omega(x_{2,jM_1-(t-1)}, x_{2,jM_1-(t-2)}) > 30C.$$

Proof. (1) For each $p \in \{1, \cdots, N_1\}$, select $x_{1,p} \in \gamma[x, x_{0,0}]$ such that

$$\sigma(x_{0,0}, x_{1,p}) = e^{32C(N_1+1-p)} d_\Omega(x_{0,0}). \tag{4.39}$$

As $x_{1,N_1+1} = x_{0,0}$, it follows from (4.39) that

$$d_\Omega(x_{1,p}) \leq d_\Omega(x_{0,0}) + \sigma(x_{0,0}, x_{1,p}) = (1 + e^{32C(N_1+1-p)}) d_\Omega(x_{0,0}). \tag{4.40}$$

For each $p \in \{1, \dots, N_1\}$, we have

$$\sigma(x_{1,p}, x_{1,p+1}) \geq \sigma(x_{1,p}, x_{0,0}) - \sigma(x_{1,p+1}, x_{0,0}) \overset{(4.39)}{=} e^{32C(N_1-p)}(e^{32C}-1)d_\Omega(x_{0,0})$$

and so

$$\begin{aligned} k_\Omega(x_{1,p}, x_{1,p+1}) &\overset{(2.2)}{\geq} \log\left(1 + \frac{\sigma(x_{1,p}, x_{1,p+1})}{d_\Omega(x_{1,p+1})}\right) \\ &\overset{(4.37)+(4.40)}{\geq} \log\left(1 + \frac{e^{32C(N_1-p)}(e^{32C}-1)}{1+e^{32C(N_1-p)}}\right) > 30C. \end{aligned} \tag{4.41}$$

This proves (1).

(2) For each $p \in \{1, \dots, N_1+1\}$, we note that

$$\sigma(x, x_{1,p}) \geq \sigma(x, x_{0,0}) - \sigma(x_{0,0}, x_{1,p}) \overset{(4.37)+(4.39)}{\geq} \tau d_\Omega(x_{0,0}) - e^{32C(N_1+1-p)}d_\Omega(x_{0,0})$$

and so

$$k_\Omega(x, x_{1,p}) \overset{(2.2)}{\geq} \log\left(1 + \frac{\sigma(x, x_{1,p})}{d_\Omega(x_{1,p})}\right) \overset{(4.40)}{\geq} \log\left(1 + \frac{\tau - e^{32C(N_1+1-p)}}{1+e^{32C(N_1+1-p)}}\right) > \frac{1}{2}\log\tau > 1+3C. \tag{4.42}$$

Then Lemma 4.3 and (4.42) imply that for each $p \in \{1, \dots, N_1+1\}$, there exist $y_{1,p} \in \alpha[y_{1,p-1}, y]$, $\gamma_{xy_{1,p}} \in \Lambda_{xy_{1,p}}(\Omega)$ and $z_{1,p} \in \gamma_{xy_{1,p}}$ such that

$$\gamma_{xz_{1,p}y_{1,p}} \in P_\alpha^{\gamma_{xx_{1,p}y}}(3C). \tag{4.43}$$

Moreover, for each $p \in \{1, \dots, N_1\}$, by (4.41) and (4.43), we may apply Lemma 4.10 (1) (with $z = y_{1,p}$ and $w = y_{1,p+1}$) to infer that for each $\gamma_{y_{1,p}y_{1,p+1}} \in \Lambda_{y_{1,p}y_{1,p+1}}(\Omega)$, there exists $w_{1,p} \in \gamma_{y_{1,p}y_{1,p+1}}$ such that

$$\gamma_{y_{1,p}w_{1,p}y_{1,p+1}} \in O_{\alpha[y_{1,p},y_{1,p+1}]}^{\gamma_{xx_{1,p}y}}(2C). \tag{4.44}$$

This proves (2).

(3) For each $p \in \{1, \dots, N_1\}$, it follows from Lemma 4.4 and (2.2) that

$$\log\left(1 + \frac{\sigma(x_{1,p}, z_{1,p})}{d_\Omega(x_{1,p})}\right) \leq k_\Omega(x_{1,p}, z_{1,p}) < 5C,$$

and so

$$\sigma(x_{1,p}, z_{1,p}) \leq (e^{5C}-1)d_\Omega(x_{1,p}) \overset{(4.40)}{\leq} (e^{5C}-1)(1+e^{32C(N_1+1-p)})d_\Omega(x_{0,0}).$$

It follows from the above estimate and (4.39) that

$$\sigma(x_{0,0}, z_{1,p}) \leq \sigma(x_{0,0}, x_{1,p}) + \sigma(x_{1,p}, z_{1,p}) \leq e^{5C}(1+e^{32C(N_1+1-p)})d_\Omega(x_{0,0}). \tag{4.45}$$

Based on (4.37) and (4.45), we may choose

$$u_{1,p} \in \gamma_{xz_{1,p}y_{1,p}}[z_{1,p}, y_{1,p}] \cap S(x_{0,0}, e^{32C(N_1+2)}d_\Omega(x_{0,0})) \tag{4.46}$$

and then let

$$v_{1,p} \in \gamma_{xz_{1,p}y_{1,p}}[u_{1,p}, y_{1,p}] \cap S(x_{0,0}, e^{32C(N_1+2)+1}d_\Omega(x_{0,0})) \tag{4.47}$$

be the first point along the direction from $u_{1,p}$ to $y_{1,p}$.

Next, we claim that there exists an integer

$$N_{1,1} > \frac{N_1}{2} \tag{4.48}$$

such that

- there is a sequence of integers $\{1_j\}_{j=1}^{N_{1,1}} \subset \{1, \cdots, N_1 - 1\}$ with $1_j < 1_{j+1}$ for each $j \in \{1, \cdots, N_{1,1} - 1\}$.
- for each $j \in \{1, \ldots, N_{1,1}\}$ and each $u \in \gamma_{xz_{1,1_j}y_{1,1_j}}[u_{1,1_j}, v_{1,1_j}]$,

$$d_\Omega(u) \le r_1 = e^{-\frac{1}{2}\log_Q \frac{N_1}{4}} \cdot e^{32C(N_1+2)+2} d_\Omega(x_{0,0}). \tag{4.49}$$

Indeed, suppose on the contrary that our claim fails. Then there exist $N_{1,2} > \frac{N_1}{3}$ integers $\rho_1, \ldots, \rho_{N_{1,2}}$ in $\{1, \ldots, N_1\}$ such that

- for each $t \in \{1, \ldots, N_{1,2} - 1\}$, $\rho_t < \rho_{t+1}$.
- for each $t \in \{1, \ldots, N_{1,2}\}$, there exists a point $u^1_{1,\rho_t} \in \gamma_{xz_{1,\rho_t}y_{1,\rho_t}}[u_{1,\rho_t}, v_{1,\rho_t}]$ so that

$$d_\Omega(u^1_{1,\rho_t}) > r_1. \tag{4.50}$$

Let $B_{0,0} = \mathbb{B}\big(x_{0,0}, e^{32C(N_1+2)+2} d_\Omega(x_{0,0})\big)$. For each $t \in \{1, \ldots, N_{1,2}\}$, we take

$$B_{\rho_t} = \mathbb{B}\left(u^1_{1,\rho_t}, \frac{1}{3} r_1\right).$$

Then, by the choice of u_{1,ρ_t} and v_{1,ρ_t} in (4.46) and (4.47), for each $u \in \overline{B_{\rho_t}}$, we have

$$\begin{aligned}\sigma(x_{0,0}, u) &\le \sigma(x_{0,0}, u^1_{1,\rho_t}) + \sigma(u, u^1_{1,\rho_t}) \le e^{32C(N_1+2)+1} d_\Omega(x_{0,0}) + \frac{1}{3} r_1 \\ &< e^{32C(N_1+2)+2} d_\Omega(x_{0,0}),\end{aligned}$$

and so

$$\overline{B_{\rho_t}} \subset B_{0,0}.$$

If all these balls are disjoint, then applying Lemma 2.3 with $R = e^{32C(N_1+2)+2} d_\Omega(x_{0,0})$ and $r = \frac{1}{3} e^{-\frac{1}{2}\log_Q \frac{N_1}{4}} R$ gives

$$N_{1,2} < \frac{N_1}{3},$$

which is a contradiction.

In the other case, there exist two integers $q_1 < q_2 \in \{1, \ldots, N_{1,2}\}$ such that $B_{\rho_{q_1}} \cap B_{\rho_{q_2}} \ne \emptyset$. It follows that

$$d(u_{1,\rho_{q_1}}, u_{1,\rho_{q_2}}) \le \frac{2}{3} r_1 \overset{(4.50)}{\le} \frac{2}{3} \min\left\{d_\Omega(u_{1,\rho_{q_1}}), d_\Omega(u_{1,\rho_{q_2}})\right\},$$

and thus by Lemma 2.1, we have

$$k_\Omega(u_{1,\rho_{q_1}}, u_{1,\rho_{q_2}}) \le \frac{20}{9} < 3C. \tag{4.51}$$

Note that by (4.46), we have $u_{1,\rho_{q_1}} \in \gamma_{xz_{1,\rho_{q_1}}y_{1,\rho_{q_1}}}[z_{1,\rho_{q_1}}, y_{1,\rho_{q_1}}]$ and $u_{1,\rho_{q_2}} \in \gamma_{xz_{1,\rho_{q_2}}y_{1,\rho_{q_2}}}[z_{1,\rho_{q_2}}, y_{1,\rho_{q_2}}]$. Then it follows from Lemma 4.12 (1), (2) and Lemma 4.10 that

$$k_\Omega(u_{1,\rho_{q_1}}, u_{1,\rho_{q_2}}) \ge 3C,$$

which clearly contradicts with (4.51). The proof of claim is thus complete.

Now, for each $j \in \{1, \ldots, N_{1,1}\}$, we let $x^1_{1,j} = u_{1,1_j}$, $y^1_{1,j} = y_{1,1_j}$ and $y^2_{1,j} = y_{1,1_j+1}$. For each $t \in \{2, \cdots, M_1\}$, select $x^t_{1,j} \in \gamma_{y_{1,1_j}z_{1,1_j}y_{1,1_j-1}}$ such that

$$\sigma(x^t_{1,j}, x^{t-1}_{1,j}) = e^{32C} r_1. \tag{4.52}$$

Then, we have

$$k_\Omega(x^t_{1,j}, x^{t-1}_{1,j}) \overset{(2.1)}{\ge} \log\left(1 + \frac{\sigma(x^t_{1,j}, x^{t-1}_{1,j})}{d_\Omega(x^{t-1}_{1,j})}\right) \overset{(4.49)}{\ge} \log\left(1 + \frac{\sigma(x^t_{1,j}, x^{t-1}_{1,j})}{r_1}\right) \overset{(4.52)}{>} 32C. \tag{4.53}$$

Note that by (4.45) and (4.46), we have

$$\sigma(x^1_{1,j}, z_{1,1_j}) \geq \sigma(x_{0,0}, x^1_{1,j}) - \sigma(x_{0,0}, z_{1,1_j}) > e^{32C(N_1+2)-1} d_\Omega(x_{0,0}),$$

from which it follows

$$k_\Omega(x^1_{1,j}, z_{1,1_j}) \overset{(2.1)}{\geq} \log\left(1 + \frac{\sigma(x^1_{1,j}, z_{1,1_j})}{d_\Omega(x^1_{1,j})}\right) \overset{(4.49)}{>} \frac{1}{2}\log_Q \frac{N_1}{4} - 3 \geq 10C.$$

This, together with Lemma 4.11 (1) and Lemma 4.12 (2), shows that there exists some point $x^{1,1}_{1,j} \in \gamma_{y^1_{1,j} w_{1,1_j} y^2_{1,j}}[y^1_{1,j}, w_{1,1_j}]$ such that

$$k_\Omega(x^1_{1,j}, x^{1,1}_{1,j}) \leq C. \tag{4.54}$$

Next, we observe the following iteration: For each $t \in \{1, \cdots, M_1 - 1\}$, if there exists some point $x^{1,t}_{1,j} \in \gamma_{y^1_{1,j} w_{1,1_j} y^2_{1,j}}[y^1_{1,j}, w_{1,1_j}]$ such that

$$k_\Omega(x^t_{1,j}, x^{1,t}_{1,j}) \leq C,$$

then there exists some point $x^{1,t+1}_{1,j} \in \gamma_{y^1_{1,j} w_{1,1_j} y^2_{1,j}}[y^1_{1,j}, x^{1,t}_{1,j}]$ such that

$$k_\Omega(x^{t+1}_{1,j}, x^{1,t+1}_{1,j}) \leq C.$$

Indeed, note that

$$k_\Omega(x^{t+1}_{1,j}, x^t_{1,j}) \overset{(4.53)}{>} 32C \geq 31C + k_\Omega(x^t_{1,j}, x^{1,t}_{1,j}).$$

The claim follows directly from Lemma 2.2 (with $x = y^1_{1,j}, y = x^t_{1,j}, z = x^{1,t}_{1,j}$ and $w = x^{t+1}_{1,j}$).

The above observation, together with (4.54), implies that for each $j \in \{1, \ldots, N_{1,1}\}$ and $t \in \{1, \cdots, M_1\}$, there exists some point $x^{1,t+1}_{1,j} \in \gamma_{y^1_{1,j} w_{1,1_j} y^2_{1,j}}[y^1_{1,j}, x^{1,t}_{1,j}]$ such that

$$k_\Omega(x^{t+1}_{1,j}, x^{1,t+1}_{1,j}) \leq C. \tag{4.55}$$

Note that

$$\begin{aligned}\frac{3}{4}e^{32C(N_1+2)} d_\Omega(x_{0,0}) &\overset{(4.46)+(4.52)}{<} \sigma(x_{0,0}, x^t_{1,j}) - \sigma(x^t_{1,j}, x^{1,t}_{1,j}) \leq \sigma(x_{0,0}, x^{1,t}_{1,j}) \\ &\leq \sigma(x_{0,0}, x^t_{1,j}) + \sigma(x^t_{1,j}, x^{1,t}_{1,j}) \overset{(4.46)+(4.52)}{<} \frac{3}{4}e^{32C(N_1+2)+2} d_\Omega(x_{0,0}).\end{aligned} \tag{4.56}$$

and that

$$\max\left\{\log\frac{d_\Omega(x^{1,t+1}_{1,j})}{d_\Omega(x^{t+1}_{1,j})}, \log\frac{\sigma(x^{1,t}_{1,j}, x^{1,t+1}_{1,j})}{d_\Omega(x^{t+1}_{1,j})}\right\} \overset{(2.1)}{\leq} k_\Omega(x^{t+1}_{1,j}, x^{1,t+1}_{1,j}) \overset{(4.55)}{\leq} C.$$

It follows from the above estimate and (4.49) that

$$\max\{\sigma(x^{1,t}_{1,j}, x^{1,t+1}_{1,j}), d_\Omega(x^{1,t+1}_{1,j})\} \leq e^C r_1. \tag{4.57}$$

For each $j \in \{1, \ldots, N_{1,1}\}$ and $t \in \{1, \cdots, M_1 - 1\}$, we have

$$k_\Omega(x^{1,t}_{1,j}, x^{1,t+1}_{1,j}) \geq k_\Omega(x^t_{1,j}, x^{t+1}_{1,j}) - k_\Omega(x^t_{1,j}, x^{1,t}_{1,j}) - k_\Omega(x^{t+1}_{1,j}, x^{1,t+1}_{1,j}) \overset{(4.53)+(4.55)}{>} 30C. \tag{4.58}$$

Finally, for each $j \in \{1, \ldots, N_{1,1}\}$ and $t \in \{1, \cdots, M_1\}$, if we take $x_{2,jM_1-(t-1)} = x^{1,t}_{1,j}$, then it follows from (4.56) $\sim$ (4.58), that Lemma 4.12 (3) holds. The proof is thus complete. □

For each $i \in \{2, \cdots, M_0\}$, set $r_{i-1} = e^{-\frac{1}{2}\log_Q \frac{N_{i-1}}{4}} \cdot e^{20C(N_1+2)+2i-2} d_\Omega(x_{0,0})$ and $N_i = (M_1 - 1)N_{i-1,1}$. Then an iteration of Lemma 4.12 gives the following more general result.

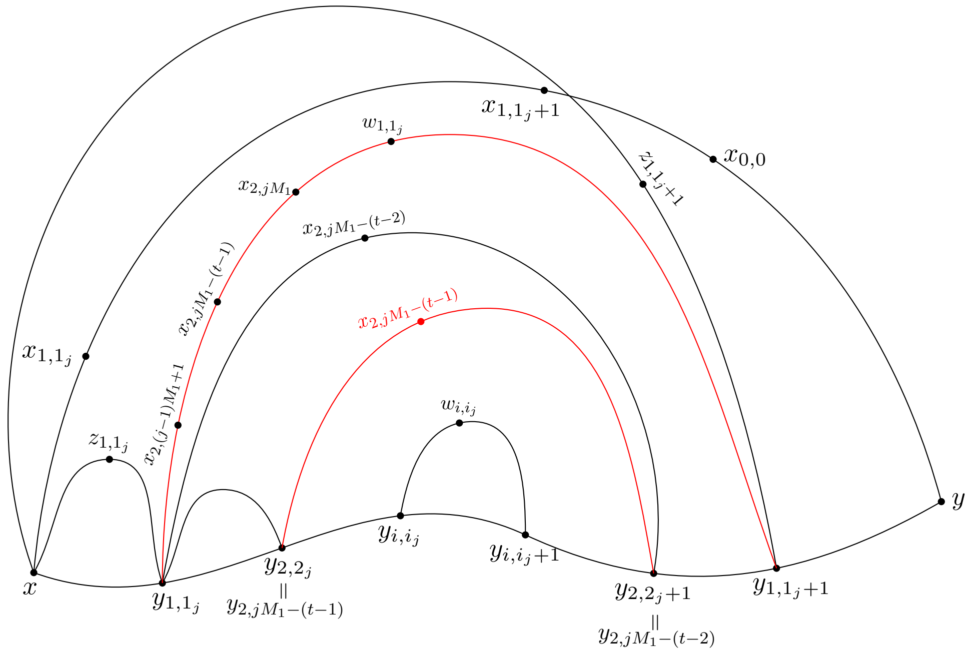

FIGURE 2. Illustration for the proof of Lemma 4.13

Lemma 4.13. *Given r_i and N_i as above, there exists some integer $N_{i-1,1} > \frac{N_{i-1}}{2}$ such that the following conclusions hold:*

(1) *for each $i \in \{3, \cdots, M_0\}$, there exists $\{(i-1)_j\}_{j=1}^{N_{i-1,1}} \subset \bigcup_{t=2}^{M_1} \bigcup_{p=1}^{N_{i-2,1}} \{pM_1 - t + 1\}$, a sequence of integers, with $(i-1)_j < (i-1)_{j+1}$ for each $j \in \{1, \cdots, N_{i-1,1} - 1\}$.*

(2) *for each $i \in \{3, \cdots, M_0\}$, there exists a sequence $x_{i,jM_1-(t-1)}$, indexed with $j \in \{1, \cdots, N_{i-1,1}\}$ and $t \in \{1, \cdots, M_1\}$, of successive points on $\gamma_{y_{i-1,(i-1)_j} y_{i-1,(i-1)_j+1}} \in \Lambda_{y_{i-1,(i-1)_j} y_{i-1,(i-1)_j+1}}(\Omega)$ so that*

$$d_\Omega(x_{i,jM_1-t+1}) \le r_{i-1}.$$

Moreover, it holds

$$\frac{3}{4} e^{32C(N_1+2)+2(i-2)} d_\Omega(x_{0,0}) < \sigma(x_{0,0}, x_{i,jM_1-t+1}) < \frac{3}{4} e^{32C(N_1+2)+2(i-1)} d_\Omega(x_{0,0});$$

(3) *for each $i \in \{3, \cdots, M_0\}$, $j \in \{1, \cdots, N_{i-1,1}\}$ and $t \in \{2, \cdots, M_1\}$,*

$$k_\Omega(x_{i,jM_1-t+1}, x_{i,jM_1-t+2}) > 30C.$$

Moreover, there exist $y_{i,jM_1-t+1} \in \alpha[y_{i-1,(i-1)_j}, y_{i-1,(i-1)_j+1}]$, $\gamma_{y_{i-1,(i-1)_j} y_{i,jM_1-t+1}}$ from $\Lambda_{y_{i-1,(i-1)_j} y_{i,jM_1-t+1}}(\Omega)$ and $z_{i,jM_1-t+1} \in \gamma_{y_{i-1,(i-1)_j} y_{i,jM_1-t+1}}$ such that

$$\gamma_{y_{i-1,(i-1)_j} z_{i,jM_1-t+1} y_{i,jM_1-t+1}} \in P_{\alpha[y_{i-1,(i-1)_j}, y_{i-1,(i-1)_j+1}]}^{\gamma_{y_{i-1,(i-1)_j} x_{i,jM_1-t+1} y_{i-1,(i-1)_j+1}}}(3C),$$

and for each $\gamma_{y_{i,jM_1-t+1} y_{i,jM_1-t+2}} \in \Lambda_{y_{i,jM_1-t+1} y_{i,jM_1-t+2}}(\Omega)$, there exists a point $w_{i,jM_1-t+1} \in \gamma_{y_{i,jM_1-t+1} y_{i,jM_1-t+2}}$ such that

$$\gamma_{y_{i,jM_1-t+1} w_{i,jM_1-t+1} y_{i,jM_1-t+2}} \in O_{\alpha[y_{i,jM_1-t+1}, y_{i,jM_1-t+2}]}^{\gamma_{y_{i-1,(i-1)_j} x_{i,jM_1-t+1} y_{i-1,(i-1)_j+1}}}(2C).$$

Proof. We shall prove the following iteration claim:

Claim 4.1. *If Lemma 4.13 holds for all $i \le k$, then it holds for $i = k+1$.*

Suppose we were able to prove the above claim. Then as Lemma 4.12 implies that Lemma 4.13 holds for the case $i = k = 2$, the result follows immediately from Claim 4.1. Thus it suffices to prove Claim 4.1 for $k \ge 3$.

To this end, for each $s \in \{1, \cdots, N_{k-1,1}\}$ and $t \in \{1, \cdots, M_1\}$, Lemma 4.13 (3) yields

$$\log\left(1 + \frac{\sigma(z_{k,sM_1-t+1}, x_{k,sM_1-t+1})}{d_\Omega(x_{k,sM_1-t+1})}\right) \overset{(2.2)}{\le} k_\Omega(z_{k,sM_1-t+1}, x_{k,sM_1-t+1}) \le 3C,$$

and so

$$\sigma(z_{k,sM_1-t+1}, x_{k,sM_1-t+1}) \le (e^{3C} - 1) d_\Omega(x_{k,sM_1-t+1}) \le (e^{3C} - 1) r_{k-1}.$$

This, together with Lemma 4.13 (2), shows

$$\sigma(x_{0,0}, z_{k,sM_1-t+1}) < \sigma(x_{0,0}, x_{k,sM_1-t+1}) + \sigma(z_{k,sM_1-t+1}, x_{k,sM_1-t+1}) < e^{32C(N_1+2)+2(k-1)} d_\Omega(x_{0,0}).$$

Thus we may take

$$u^t_{k,s} \in \gamma_{y_{k-1,(k-1)_s} z_{k,sM_1-t+1} y_{k,sM_1-t+1}}[z_{k,sM_1-t+1}, y_{k,sM_1-t+1}] \cap S(x_{0,0}, e^{32C(N_1+2)+2k-2} d_\Omega(x_{0,0})) \tag{4.59}$$

and

$$v^t_{k,s} \in \gamma_{y_{k-1,(k-1)_s} z_{k,sM_1-t+1} y_{k,sM_1-t+1}}[y_{k,sM_1-t+1}, u^t_{k,s}] \cap S(x_{0,0}, e^{32C(N_1+2)+2k-1} d_\Omega(x_{0,0})) \tag{4.60}$$

be the first point along the direction from u_{k,sM_1-t+1} to y_{k,sM_1-t+1}. Then we define $u_{k,sM_1-t+1} = u^t_{k,s}$ and $v_{k,sM_1-t+1} = v^t_{k,s}$.

As in the proof of Lemma 4.12, we next claim that there exists an integer

$$N_{k,1} > \frac{N_k}{2} \tag{4.61}$$

such that

- there is a sequence of integers $\{k_{1,j}\}_{j=1}^{N_{k,1}} \subset \{1, \cdots, N_k\}$ with $k_{1,j} < k_{1,j+1}$ for each $j \in \{1, \ldots, N_{k,1} - 1\}$.
- for each $j \in \{1, \ldots, N_{k,1}\}$, there exists some $s_j \in \{1, \cdots, N_{k-1,1}\}$ so that for each $u \in \gamma_{y_{k-1,(k-1)_{s_j}} z_{k,k_j} y_{k,k_j}}[u_{k,k_j}, y_{k,k_j}]$,

$$d_\Omega(u) \le r_k = e^{-\frac{1}{2}\log_Q \frac{N_k}{4}} \cdot e^{32C(N_1+2)+2k} d_\Omega(x_{0,0}). \tag{4.62}$$

Indeed, suppose on the contrary that our claim fails. Then there exist $N_{k,2} \ge \frac{N_k}{3}$ integers $k_{1,1}, \ldots, k_{1,N_{k,2}}$ in $\{1, \ldots, N_k\}$ such that

- for each $s \in \{1, \ldots, N_{k,2} - 1\}$, $k_{1,s} < k_{1,s+1}$.
- for each $s \in \{1, \ldots, N_{k,2}\}$, there exist an integer $t_s \in \{1, \cdots, N_{k-1,1}\}$ and some point $u^0_{k,k_{1,s}} \in \gamma_{y_{k-1,(k-1)_{t_s}} z_{k,k_{1,s}} y_{k,k_{1,s}}}[u_{k,k_{1,s}}, y_{k,k_{1,s}}]$ such that

$$d_\Omega(u^0_{k,k_{1,s}}) > r_k. \tag{4.63}$$

Let $B_{0,k} = \mathbb{B}(x_{0,0}, e^{32C(N_1+2)+2k} d_\Omega(x_{0,0}))$. For each $s \in \{1, \ldots, N_{k,2}\}$, we take

$$B_{k_{1,s}} = \mathbb{B}\left(u^0_{k,k_{1,s}}, \frac{1}{3} r_k\right).$$

Then by the choice of $u^t_{k,s}$ and $v^t_{k,s}$ in (4.59) and (4.60), for each $u \in \overline{B_{k_{1,s}}}$, we have

$$\sigma(x_{0,0}, u) \le \sigma(x_{0,0}, u^0_{k,k_{1,s}}) + \sigma(u, u^0_{k,k_{1,s}}) \le e^{32C(N_1+2)+2k-1} d_\Omega(x_{0,0}) + \frac{1}{3} r_k$$

$$< e^{32C(N_1+2)+2k} d_\Omega(x_{0,0}),$$

and so

$$\overline{B_{k_{1,s}}} \subset B_{0,k}.$$

If all these balls are disjoint, then applying Lemma 2.3 with $R_k = e^{20C(N_1+2)+2k} d_\Omega(x_{0,0})$ and $r = \frac{1}{3} e^{-\frac{1}{2}\log_Q \frac{N_k}{4}} R_k$ gives

$$N_{k,2} < \frac{N_k}{3},$$

which is a contradiction.

In the other case, there exist two integers $s_1 < s_2 \in \{1, \ldots, N_{k,2}\}$ such that $B_{k_{1,s_1}} \cap B_{k_{1,s_2}} \neq \emptyset$. It follows that

$$d(u^0_{k,k_{1,s_1}}, u^0_{k,k_{1,s_2}}) \leq \frac{2}{3} r_k \overset{(4.63)}{<} \frac{2}{3} \min\{d_\Omega(u^0_{k,k_{1,s_1}}), d_\Omega(u^0_{1,k_{1,s_2}})\},$$

and thus by Lemma 2.1, we have

$$k_\Omega(u^0_{k,k_{1,s_1}}, u^0_{k,k_{1,s_2}}) < \frac{20}{9}. \tag{4.64}$$

For each $i \in \{1, \cdots, k-1\}$, $j \in \{1, \cdots, N_{i,1}\}$ and $t \in \{2, \cdots, M_1\}$, note that $x_{i+1,jM_1} \in \gamma_{y_{i,i_j} y_{i,i_j+1}}[x_{i+1,jM_1}, y_{i,i_j+1}]$, and thus it follows from our induction assumption and Lemma 4.13 (3) that

$$k_\Omega(w_{i,i_j}, x_{i+1,jM_1-t+1}) > k_\Omega(x_{i+1,jM_1}, x_{i+1,jM_1-t+1}) > 30C. \tag{4.65}$$

Moreover, by Lemmas 4.7 and Lemma 4.13 (3), there exist $q \in \{1, \cdots, k-1\}$, $j \in \{1, \cdots, N_{q,1}\}$ and $t_1 < t_2 \in \{2, \cdots, M_1\}$ such that

$$\gamma_1 := \gamma_{y_{q,q_j} v_{q+1,jM_1-t_1+1} y_{k-1,(k-1)_{1,s_1}}} \in Q^{\gamma_{y_{q,q_j} x_{q+1,jM_1-t_1+1} y_{q,q_j+1}}}_{\alpha[y_{q,q_j} y_{q,q_j+1}]},$$

$$\gamma_2 := \gamma_{y_{q,q_j} v^1_{q+1,jM_1-t_1+1} y_{k,k_{1,s_1}}} \in Q^{\gamma_{y_{q,q_j} x_{q+1,jM_1-t_1+1} y_{q,q_j+1}}}_{\alpha[y_{q,q_j} y_{q,q_j+1}]},$$

$$\gamma_3 := \gamma_{y_{q,q_j} v_{q+1,jM_1-t_2+1} y_{k-1,(k-1)_{1,s_2}}} \in Q^{\gamma_{y_{q,q_j} x_{q+1,jM_1-t_1+1} y_{q,q_j+1}}}_{\alpha[y_{q,q_j} y_{q,q_j+1}]},$$

and

$$\gamma_4 := \gamma_{y_{q,q_j} v^1_{q+1,jM_1-t_2+1} y_{k,k_{1,s_2}}} \in Q^{\gamma_{y_{q,q_j} x_{q+1,jM_1-t_2+1} y_{q,q_j+1}}}_{\alpha[y_{q,q_j} y_{q,q_j+1}]}.$$

Then, combining Lemma 4.11 with (4.65), we infer that there exist two points

$$v^0_{k,k_{1,s_1}} \in \gamma_1[v_{q+1,jM_1-t_1+1}, y_{k-1,(k-1)_{1,s_1}}] \cup \gamma_2[v^1_{q+1,jM_1-t_1+1}, y_{k,k_{1,s_1}}]$$

and

$$v^0_{k,k_{1,s_2}} \in \gamma_3[v_{q+1,jM_1-t_2+1}, y_{k-1,(k-1)_{1,s_2}}] \cup \gamma_4[v^1_{q+1,jM_1-t_2+1}, y_{k,k_{1,s_2}}]$$

such that

$$k_\Omega(u^0_{k,k_{1,s_1}}, v^0_{k,k_{1,s_1}}) \leq C \quad \text{and} \quad k_\Omega(u^0_{k,k_{1,s_2}}, v^0_{k,k_{1,s_2}}) \leq C.$$

This, together with (4.64), shows that

$$k_\Omega(v^0_{k,k_{1,s_1}}, v^0_{k,k_{1,s_2}}) \leq k_\Omega(u^0_{k,k_{1,s_1}}, v^0_{k,k_{1,s_1}}) + k_\Omega(u^0_{k,k_{1,s_2}}, v^0_{k,k_{1,s_2}}) + k_\Omega(u^0_{k,k_{1,s_1}}, u^0_{k,k_{1,s_2}}) < 3C,$$

which, together with Lemma 4.13 (3), clearly contradicts with Lemma 4.10 (2). The proof of claim is thus complete.

Let $r_k = e^{-\frac{1}{2}\log_Q \frac{N_k}{4}} R_k$. For each $j \in \{1, \ldots, N_{k,1}\}$, let $x^1_{k,j} = u_{k,k_j}$ and for each $t \in \{2, \cdots, M_1\}$, select $x^t_{k,j} \in \gamma_{y_{k-1,(k-1)_{s_j}} z_{k,k_j} y_{k,k_j}}$ such that

$$\sigma(x^t_{k,k_j}, x^{t-1}_{k,k_j}) = e^{32C} r_k. \tag{4.66}$$

Then, we have

$$k_\Omega(x^t_{k,k_j}, x^{t-1}_{k,k_j}) \overset{(2.1)}{\geq} \log\left(1+\frac{\sigma(x^t_{k,k_j}, x^{t-1}_{k,k_j})}{d_\Omega(x^{t-1}_{k,k_j})}\right) \overset{(4.62)}{\geq} \log\left(1+\frac{\sigma(x^t_{k,k_j}, x^{t-1}_{k,k_j})}{r_k}\right) \overset{(4.66)}{>} 32C. \tag{4.67}$$

Furthermore, by Lemma 4.11 (1), there exists some point $x^{1,1}_{k,k_j} \in \gamma_{y_{k,k_j} w_{k,k_j} y_{k,k_j+1}}[y_{k,k_j},\, w_{k,k_j}]$ such that

$$k_\Omega(x^1_{k,k_j}, x^{1,1}_{k,k_j}) \leq C. \tag{4.68}$$

Based on (4.68), we may argue similarly as in (4.55) to conclude that for each $t \in \{2, \cdots, M_1\}$, there exists some point $x^{1,t}_{k,k_j} \in \gamma_{y_{k,k_j} w_{k,k_j} y_{k,k_j+1}}[y_{k,k_j},\, x^{1,t-1}_{k,k_j}]$ with

$$k_\Omega(x^t_{k,k_j}, x^{1,t}_{k,k_j}) \leq C. \tag{4.69}$$

Finally, for each $j \in \{1, \ldots, N_{k,1}\}$ and $t \in \{1, \cdots, M_1\}$, if we take $x_{k+1, jM_1-(t-1)} = x^{1,t}_{k,j}$, then a similar discussion as in the proof of (4.56) $\sim$ (4.58) in Lemma 4.12 (3) (using (4.66) $\sim$ (4.69)) shows that Claim 4.1 holds for the case $i = k+1$. This completes the proof of Claim 4.1 and thus also Lemma 4.13. $\square$

Proof of Theorem 4.1. Towards a contradiction, suppose (4.37) holds and we shall prove the following claim.

Claim 4.2. *There exists a positive integer $N < [\frac{\ell_k(\alpha_{xy})}{3}] + 1$ such that*

1. *for each positive integer $\varsigma \in \{1, \cdots, N\}$, there are $y^1_{0,\varsigma} \in \alpha_{xy}$ and $y^2_{0,\varsigma} \in \alpha_{xy}[y^1_{0,\varsigma}, y]$; for each $\varsigma \in \{2, \cdots, N\}$, there exists $y^1_{0,\varsigma} \in \alpha_{xy}[y^1_{0,\varsigma-1}, y]$ with $k_\Omega(y^1_{0,\varsigma-1}, y^1_{0,\varsigma}) \geq 1 + 3C$.*
2. *for each $\varsigma \in \{1, \cdots, N\}$ and $\gamma_{y^1_{0,\varsigma} y^2_{0,\varsigma}} \in \Lambda_{y^1_{0,\varsigma} y^2_{0,\varsigma}}(\Omega)$, there exists $x_{0,\varsigma} \in \gamma_{y^1_{0,\varsigma} y^2_{0,\varsigma}}$ with*
$$\sigma(x_{0,\varsigma}, \alpha[y^1_{0,\varsigma}, y^2_{0,\varsigma}]) > \tau d_\Omega(x_{0,\varsigma}).$$
3. *for each $\varsigma \in \{1, \cdots, N-1\}$, $k_\Omega(y^1_{0,\varsigma}, y^2_{0,\varsigma}) \geq 1 + 3C$ and $k_\Omega(y^1_{0,N}, y^2_{0,N}) < 1 + 3C$.*

Suppose the above claim holds. Then

$$\begin{aligned} k_\Omega(y^1_{0,N}, y^2_{0,N}) \geq k_\Omega(y^1_{0,N}, x_{0,N}) &\overset{(2.1)}{\geq} \log\left(1 + \frac{\ell(\gamma_{y^1_{0,N} y^2_{0,N}}[y^1_{0,N}, x_{0,N}])}{\min\{d_\Omega(y^1_{0,N}), d_\Omega(x_{0,N})\}}\right) \\ &\geq \log\left(1 + \frac{\sigma(y^1_{0,N}, x_{0,N})}{d_\Omega(x_{0,N})}\right) \geq \log\frac{\sigma(x_{0,N}, \alpha[y^1_{0,N}, y^2_{0,N}])}{d_\Omega(x_{0,N})} > \log\tau, \end{aligned}$$

which clearly contradicts with Claim 4.2 (3).

Thus, it remains to prove the claim. For the case $\varsigma = 1$, we infer from Lemma 4.13 (with $i = M_0$) that for each $j \in \{1 \cdots, N_{M_0-1,1}\}$ and $t \in \{1, \cdots, M\}$, there exists some point

$$x_{M_0, jM_1-t+1} \in \gamma_{y_{M_0-1,(M_0-1)_j} y_{M_0-1,(M_0-1)_j+1}}$$

with $y_{M_0-1,(M_0-1)_j}$ and $y_{M_0-1,(M_0-1)_j+1}$ given by Lemma 4.13 (3), so that

$$\sigma(x_{0,0}, x_{M_0, jM_1-t+1}) < \frac{5}{4} e^{32C(N_1+2)+2M_0+1} d_\Omega(x_{0,0}) < \tau^{\frac{1}{C}} d_\Omega(x_{0,0}).$$

For each $z \in \alpha[y_{M_0-1,(M_0-1)_j}, y_{M_0-1,(M_0-1)_j+1}]$, it follows from the above estimate and triangle's inequality that

$$\sigma(x_{M_0, jM_1-t+1}, z) \geq \sigma(x_{0,0}, z) - \sigma(x_{0,0}, x_{M_0, jM_1-t+1}) \overset{(4.37)}{\geq} (\tau - \tau^{\frac{1}{C}}) d_\Omega(x_{0,0}).$$

On the other hand, by Lemma 4.13, we have

$$d_\Omega(x_{M_0,jM_1-t+1}) \le r_{M_0-1} = e^{-\frac{1}{2}\log_Q \frac{N_{M_0-1}}{4}} \cdot e^{32C(N_1+2)+2M_0-2} d_\Omega(x_{0,0}),$$

and

$$N_{M_0-1} \ge \frac{(M_1-1)^{M_0-2}}{2^{M_0-2}} \cdot N_1 > e^{128QC^3N_1}.$$

Thus it follows

$$d_\Omega(x_{M_0,jM_1-t+1}) \le \tau^{-1} d_\Omega(x_{0,0}).$$

Combining the above two estimates gives

$$\sigma(x_{M_0,jM_1-t+1}, \alpha[y_{M_0-1,(M_0-1)_j}, y_{M_0-1,(M_0-1)_j+1}]) > \tau d_\Omega(x_{M_0,jM_1-t+1}).$$

Moreover, by Lemma 4.12, we know that $y_{M_0-1,(M_0-1)_j} \in \alpha[y_{1,1}, y]$. This together with (4.43) shows that there exists $\zeta_{1,1} \in \gamma_{xy_{M_0-1,(M_0-1)_j}}$ such that

$$k_\Omega(x, \zeta_{1,1}) < 3C.$$

Hence, by (4.42), we have

$$k_\Omega(x, y_{M_0-1,(M_0-1)_j}) \ge k_\Omega(x, x_{1,1}) - k_\Omega(x, \zeta_{1,1}) > \frac{1}{2}\log\tau - 3C > 1 + 3C.$$

Set $y^1_{0,1} = y_{M_0-1,(M_0-1)_j} \in \alpha_{xy}$, $y^2_{0,1} = y_{M_0-1,(M_0-1)_j+1} \in \alpha_{xy}[y^1_{0,1}, y]$ and $x_{0,1} = x_{M_0,jM_1-t+1} \in \gamma_{y^1_{0,1}y^2_{0,1}}$. Then Claim 4.2 holds for the case $\varsigma = 1$.

If Claim 4.2 holds as $\varsigma = k$ for each $k \in \{1, \cdots, N-1\}$, then by replacing $x = y^1_{0,k}$, $y = y^2_{0,k}$ and $z_{0,k} = x_{0,0}$, Claim 4.2 holds for $\varsigma = k+1$. This completes the proof of Claim 4.2 and thus also the proof of Theorem 4.1.

□

5. Gromov hyperbolicity implies the Gehring-Hayman inequality

In this section, we shall complete the proof of Theorem 1.10: Gromov hyperbolicity implies the Gehring-Hayman inequality. The main result of this section states that (Ω, d) satisfies the Gehring-Hayman inequality if (Ω, σ) has ball separation condition and (Ω, k) is Gromov hyperbolic. More precisely, we have the following result.

Theorem 5.1. *Suppose (Ω, k) is C-Gromov hyperbolic and (Ω, σ) satisfies τ-ball separation condition. Then (Ω, d) satisfies the C_1-Gehring-Hayman inequality with $C_1 = \tau_1^2$, where $\tau_1 = e^{5\tau\cdot Q^{\log_2 9\tau}}$.*

Note that if (Ω, k) is C-Gromov hyperbolic, then we know from Theorem 4.1 that (Ω, σ) satisfies the τ-ball separation condition with $\tau = e^{(2CQ)^{192(1+C)}}$. Thus, Theorem 1.10 follows immediately from Theorem 5.1.

5.1. **A version of diameter Gehring-Hayman inequality.** In this section, we let $G_{xy}(\Omega)$ to denote the collection of curves $\alpha \in \Gamma_{xy}(\Omega)$ such that

$$\ell(\alpha) \le \sigma(x, y) + e^{-\tau_1} \min\{\sigma(x, y), d_\Omega(x), d_\Omega(y)\}.$$

For the proof of Theorem 5.1, we first prove the following weaker version of diameter Gehring-Hayman inequality.

Theorem 5.2. *For any $x_1, x_2 \in \Omega$, let $\gamma_{x_1x_2} \in \Lambda_{x_1x_2}(\Omega)$ and $\alpha_{x_1x_1} \in G_{x_1x_2}(\Omega)$. Then*
$$\operatorname{diam}_\sigma(\gamma_{x_1x_2}) \leq \tau_1 \ell(\alpha_{x_1x_2}).$$

Proof. Let $\gamma_{x_1x_2}$ and $\alpha_{x_1x_2}$ be given as in the theorem. We may assume
$$\operatorname{diam}_\sigma(\gamma_{x_1x_2}) > \tau_1 \ell(\alpha_{x_1x_2}) = e^{5\tau \cdot Q^{\log_2 9\tau}} \ell(\alpha_{x_1x_2}). \tag{5.1}$$
Then it follows from Lemma 2.1 that for each $z \in \alpha_{x_1x_2}$, we have
$$\sigma(x_1, x_2) \geq \frac{1}{2} d_\Omega(z). \tag{5.2}$$

Let $z_1 = y_1 = x_1$ and $m = [Q^{\log_2 9\tau}]$. For each $i \in \{2, \ldots, m\}$, we take $y_i \in \gamma_{x_1x_2}[x_1, x_0]$ with
$$\sigma(y_1, y_i) = e^{5\tau(i-1)} \sigma(x_1, x_2), \tag{5.3}$$
and so (2.2) and (5.2) imply that
$$k_\Omega(y_1, y_i) \geq \log\left(1 + \frac{\sigma(y_1, y_i)}{d_\Omega(y_1)}\right) > 1 + 3C.$$
Hence it follows from Lemma 4.3 that for each $i \in \{2, \ldots, m\}$, there exists some point $z_i \in \alpha_{z_{i-1}x_2}$ and $y_{i,1} \in \gamma_{z_{i-1}z_i}$ such that $\gamma_{y_1y_{i,1}z_i} \in P_\alpha^{\gamma_{x_1y_ix_2}}(3C)$ and thus by Lemma 4.4, we have
$$k_\Omega(y_{i,1}, y_i) < 5C. \tag{5.4}$$

For each $i \in \{1 \ldots, m\}$, it holds
$$d_\Omega(y_i) \leq d_\Omega(y_1) + \sigma(y_1, y_i) \overset{(5.2)+(5.3)}{<} (2 + e^{5\tau(i-1)}) \sigma(x_1, x_2).$$
Moreover, for each $i \in \{2, \cdots, m-1\}$, by (2.2) and (5.3), we have
$$k_\Omega(y_i, y_{i+1}) \geq \log\left(1 + \frac{\sigma(y_i, y_{i+1})}{d_\Omega(y_i)}\right) \geq \log\left(1 + \frac{\sigma(y_1, y_{i+1}) - \sigma(y_1, y_i)}{d_\Omega(y_i)}\right) > 4\tau > 30C. \tag{5.5}$$
Therefore, it follows from Lemma 4.10 (2) that
$$k_\Omega(\gamma_{y_1z_i}[z_i, y_{i,1}], \gamma_{y_1z_{i+1}}[z_{i+1}, y_{i+1,1}]) \geq 3C. \tag{5.6}$$

Next we claim that for each $i \in \{2, \cdots, m\}$, it holds
$$\sigma(y_{i,1}, z_i) \geq (e^C - 2) \sigma(x_1, x_2). \tag{5.7}$$

Indeed, by (2.2) and (5.4), we have
$$\max\left\{\log \frac{\sigma(y_i, y_{i,1})}{d_\Omega(y_i)}, \log \frac{d_\Omega(y_i)}{d_\Omega(y_{i,1})}\right\} \leq k_\Omega(y_{i,1}, y_i) < 5C,$$
and so it follows
$$\sigma(y_i, y_{i,1}) \leq e^{5C} d_\Omega(y_i) \ \text{ and } \ d_\Omega(y_i) \leq e^{5C} d_\Omega(y_{i,1}). \tag{5.8}$$

If $\sigma(y_1, y_i) \leq e^{5\tau - 6C} d_\Omega(y_i)$, then by (5.8), it holds
$$\sigma(y_1, y_i) \leq e^{5\tau - C} d_\Omega(y_{i,1}),$$
and hence
$$\sigma(y_{i,1}, z_i) \geq d_\Omega(y_{i,1}) - d_\Omega(z_i) \overset{(5.2)+(5.3)}{\geq} (e^C - 2) \sigma(x_1, x_2).$$

If $\sigma(y_1, y_i) > e^{5\tau - 6C} d_\Omega(y_i)$, then (5.8) yields that
$$\sigma(y_i, y_{i,1}) \leq e^{11C - 5\tau} \sigma(y_1, y_i)$$

which, together with (5.3), shows that

$$\begin{aligned}\sigma(y_{i,1}, z_i) &\geq \sigma(y_1, y_i) - \sigma(y_i, y_{i,1}) - \sigma(y_1, z_i)\\ &\geq \left(1 - e^{11C-5\tau}\right)\sigma(y_1, y_i) - \ell(\alpha_{x_1x_2})\\ &> (e^{5\tau} - e^{11C} - 2)\sigma(x_1, x_2).\end{aligned}$$

In either cases, we have proved (5.7).

For each $i \in \{2, \dots, m-1\}$, by (5.7), we may select $y_{i,2} \in \gamma_{y_1z_i}[z_i, y_{i,1}]$ such that

$$\sigma(z_i, y_{i,2}) = 3\ell(\alpha_{x_1x_2}). \tag{5.9}$$

Since (Ω, σ) satisfies the τ-ball separation condition, we have

$$\tau d_\Omega(y_{i,2}) \geq \sigma(y_{i,2}, \alpha_{x_1x_2}[y_1, z_i]) \geq \sigma(y_{i,2}, z_i) - \ell(\alpha_{x_1x_2}) \geq 2\ell(\alpha_{x_1x_2}),$$

which gives

$$d_\Omega(y_{i,2}) \geq \frac{2}{\tau}\ell(\alpha_{x_1x_2}). \tag{5.10}$$

For each $i \in \{2, \dots, m\}$, set $B_i = \mathbb{B}(y_{i,2}, \frac{1}{2\tau}\ell(\alpha_{x_1x_2}))$. Then $B_i \in \Omega$ by (5.10). For each $y \in \overline{B_i}$, it follows from (5.9) and (5.10) that

$$\sigma(y_1, y) \leq \sigma(y_1, z_i) + \sigma(y_{i,2}, z_i) + \frac{1}{2\tau}\ell(\alpha_{x_1x_2}) \leq (4 + (2\tau)^{-1})\ell(\alpha_{x_1x_2}).$$

Hence, we have

$$\bigcup_{i=2}^{m} \overline{B_i} \subset \mathbb{B}\left(y_1, (4 + (2\tau)^{-1})\ell(\alpha_{x_1x_2})\right),$$

which, together with Lemma 2.3 and the fact that $m = [Q^{\log_2 9\tau}]$, implies that there exist two integers $s_1 < s_2 \in \{2, \cdots, m\}$ such that

$$B_{s_1} \cap B_{s_2} \neq \emptyset,$$

and so

$$d_\Omega(y_{s_1,2}) \overset{(5.10)}{\geq} \frac{2}{\tau}\ell(\alpha_{x_1x_2}) \geq 2\sigma(y_{s_1,2}, y_{s_2,2}).$$

This, together with Lemma 2.1, yields

$$k_\Omega(y_{s_1,2}, y_{s_1,2}) \leq \frac{10}{9},$$

which clearly contradicts with (5.6). The proof of Theorem 5.2 is thus complete. $\square$

5.2. **Proof of Theorem 5.1.** Based on Theorem 5.2, we are ready to prove the following Theorem 5.3, which implies Theorem 5.1.

Theorem 5.3. *Let $\gamma_{xy} \in \Lambda_{xy}(\Omega)$ and $\alpha_{xy} \in G_{xy}(\Omega)$. Then $\ell(\gamma_{xy}) \leq \tau_1^2\ell(\alpha_{xy})$.*

To prove Theorem 5.3, we need a couple of auxiliary results.

Lemma 5.4. *For any pair of points $x, y \in \Omega$ with*

$$d(x, y) \geq \frac{1}{2}\max\{d_\Omega(x), d_\Omega(y)\},$$

and $\beta \in \Lambda_{xy}(\Omega)$, there exists a finite sequence of balls $\{B_i\}_{i=1}^{k_1}$ in Ω such that

(1) *for each $i \in \{1, \dots, k_1\}$, $B_i = \mathbb{B}(y_i, r_i)$ with $r_i = \frac{1}{4}d_\Omega(y_i)$, where $y_0 = x$, $y_i \in \beta[y_{i-1}, y]$, but $y_i \notin B_{i-1}$.*

(2) *$y_{k_1+1} = y \in B_{k_1}$ (Possibly, $y_{k_1+1} = y_{k_1}$).*
(3) *if $k_1 \geq 3$, then $B_i \cap B_j = \emptyset$ for any pair $\{i, j\} \subset \{1, \ldots, k_1\}$ with $|j - i| > 1$.*
(4) *for each $i \in \{1, \ldots, k_1 - 1\}$, we have*
  (a) *$B_i \cap B_{i+1} \neq \emptyset$.*
  (b) *$\ell(\beta[y_i, y_{i+1}]) \leq \frac{11}{9} d_\Omega(y_i) \leq \frac{44}{9} d(y_i, y_{i+1})$.*
  (c) *for all z, $w \in \beta[y_i, y_{i+1}]$,*

$$\frac{9}{20} d_\Omega(z) \leq d_\Omega(w) \leq \frac{20}{9} d_\Omega(z).$$

(5) *for each $i \in \{1, \ldots, k_1 - 1\}$, $\log \frac{5}{4} \leq k_\Omega(y_i, y_{i+1}) \leq \frac{20}{27}$.*
(6) *$\ell(\beta[y_{k_1}, y_{k_1+1}]) \leq \frac{9}{20} d_\Omega(y_{k_1})$ and $k_\Omega(y_{k_1}, y_{k_1+1}) \leq \frac{10}{27}$.*

Proof. The proof consists of a few steps.

Step 1. Construct an initial sequence $\{x_j\}_{j=1}^{k_0}$ of points on β.

In this step, we shall prove that there exists a finite sequence $\{x_j\}_{j=1}^{k_0} \subset \beta$ with $x_1 = x$ such that

$$d(x_j, x_{j+1}) = \frac{1}{4} d_\Omega(x_j) \ \text{ and } \ y \in \mathbb{B}\left(x_{k_0}, \frac{1}{4} d_\Omega(x_{k_0})\right).$$

To this end, set $x_1 = x$, and let x_2 be the last point on β, along the direction from x to y, which intersects with the sphere $\mathbb{S}(x, \frac{1}{4} d_\Omega(x))$.

If $y \in \mathbb{B}(x_2, \frac{1}{4} d_\Omega(x_2))$, then we take $k_0 = 2$. Otherwise, let x_3 be the last point on $\beta[x_2, y]$, along the direction from x_2 to y, which intersects with the sphere $\mathbb{S}(x_2, \frac{1}{4} d_\Omega(x_2))$.

Repeating this procedure for finitely many times, we eventually found a point $x_{k_0} \in \beta[x_{k_0-1}, y]$ such that $y \in \mathbb{B}(x_{k_0}, \frac{1}{4} d_\Omega(x_{k_0}))$. It is possible that $x_{k_0} = y$.

Step 2. Refine the initial sequence $\{x_j\}_{j=1}^{k_0}$.

Let $\{x_j\}_{j=1}^{k_0}$ be the sequence constructed in Step 1. For each $i \in \{1, \ldots, k_0\}$, set

$$B_{1,j} = \mathbb{B}\left(x_j, \frac{1}{4} d_\Omega(x_j)\right).$$

We are going to select the desired sequence of balls from $\{B_{1,j}\}_{j=1}^{k_0}$.

(1) Set $B_1 = B_{1,1}$ and $y_1 = x_1$. Then we define

$$r_1 = \max\{r : \ r \in \{2, \ldots, k_0\} \ \text{ and } \ B_1 \cap B_{1,r} \neq \emptyset\}.$$

(2) Set $B_2 = B_{1,r_1}$ and $y_2 = x_{r_1}$.
- If $r_1 = k_0$, then we found a sequence of balls $\{B_i\}_{i=1}^{k_1}$ with $k_1 = 2$.
- If $r_1 < k_0$, then define

$$r_2 = \max\{r : \ r \in \{r_1 + 1, \ldots, k_0\} \ \text{ and } \ B_2 \cap B_{1,r} \neq \emptyset\}.$$

(3) Set $B_3 = B_{1,r_2}$ and $y_3 = x_{r_2}$.
- If $r_2 = k_0$, then we found a sequence of balls $\{B_i\}_{i=1}^{k_1}$ with $k_1 = 3$.
- If $r_2 < k_0$, then define

$$r_3 = \max\{r : \ r \in \{r_2 + 1, \ldots, k_0\} \ \text{ and } \ B_3 \cap B_{1,r} \neq \emptyset\}.$$

(4) Set $B_4 = B_{1,r_3}$ and $y_4 = x_{r_3}$ $\cdots$

Repeating this procedure, we find an integer $k_1 \leq k_0$ such that

$$\max\{r:\ r \in \{r_{k_1-1}+1, \ldots, k_0\} \text{ and } B_{k_1-1} \cap B_{1,r} \neq \emptyset\} = k_0.$$

Then set $B_{k_1} = B_{1,k_0}$, $y_{k_1} = x_{k_0}$ and $y_{k_1+1} = y$. It is possible that $y_{k_1+1} = y_{k_1}$.

In this way, we have found a sequence of balls $\{B_i\}_{i=1}^{k_1}$ with $k_1 \geq 2$.

Step 3. Verify all the listed properties.

In this step, we shall prove that the selected sequence $\{B_i\}_{i=1}^{k_1}$ and point y_{k_1+1} satisfy all listed properties. That $\{B_i\}_{j=1}^{k_1}$ satisfies the properties (1) $\sim$ (3) and (4a) is clear from the construction.

For each $i \in \{1, \ldots, k_1 - 1\}$, by property (3), we have

$$\mathbb{S}\left(y_i, \frac{1}{4}d_\Omega(y_i)\right) \bigcap \mathbb{B}\left(y_{i+1}, \frac{1}{4}d_\Omega(y_{i+1})\right) \neq \emptyset.$$

Select $y_{1,i} \in \mathbb{S}\left(y_i, \frac{1}{4}d_\Omega(y_i)\right) \bigcap \mathbb{B}\left(y_{i+1}, \frac{1}{4}d_\Omega(y_{i+1})\right)$. Applying Lemma 2.1 with $a = 4$, we obtain

$$k_\Omega(y_i, y_{1,i}) \leq \frac{10}{27} \quad \text{and} \quad k_\Omega(y_{1,i}, y_{i+1}) \leq \frac{10}{27},$$

which implies

$$k_\Omega(y_i, y_{i+1}) \leq \frac{20}{27}. \tag{5.11}$$

By (2.2), for any pair of points z, $w \in \beta[y_i, y_{i+1}]$, we have

$$\max\left\{\log\left(1 + \frac{\ell(\beta[y_i, y_{i+1}])}{d_\Omega(y_i)}\right), \left|\log\frac{d_\Omega(w)}{d_\Omega(z)}\right|, \left|\log\frac{d_\Omega(z)}{d_\Omega(w)}\right|\right\} \leq k_\Omega(y_i, y_{i+1}),$$

and thus it follows from (5.11) and the above estimate that

$$\frac{9}{20}d_\Omega(z) \leq d_\Omega(w) \leq \frac{20}{9}d_\Omega(z) \text{ and } \ell(\beta[y_i, y_{i+1}]) \leq \frac{11}{9}d_\Omega(y_i). \tag{5.12}$$

Moreover, the fact that $y_{i+1} \notin \mathbb{B}(y_i, \frac{1}{4}d_\Omega(y_i))$ for $i \in \{1, \ldots, k_1 - 1\}$ implies

$$d(y_i, y_{i+1}) \geq \frac{1}{4}d_\Omega(y_i), \tag{5.13}$$

which, together with (2.1), shows that

$$k_\Omega(y_i, y_{i+1}) \geq \log\left(1 + \frac{d(y_i, y_{i+1})}{d_\Omega(y_i)}\right) > \log\frac{5}{4}. \tag{5.14}$$

Since $y_{k_1+1} = y \in B_{k_1}$, Lemma 2.1 with $a = 4$ gives

$$k_\Omega(y_{k_1}, y_{k_1+1}) \leq \frac{10}{27}. \tag{5.15}$$

A similar argument as in (5.12), applying (2.2) and (5.15), gives

$$\ell(\beta[y_{k_1}, y_{k_1+1}]) \leq \frac{9}{20}d_\Omega(y_{k_1}). \tag{5.16}$$

Now, we conclude from (5.11) $\sim$ (5.16) that all the remaining properties listed in Lemma 5.4 hold. The proof is thus complete. $\square$

Lemma 5.5. *Let $\beta \in \Lambda_{xy}(\Omega)$. Suppose for some $x_0 \in \beta$, $d_\Omega(x_0) = \sup_{w\in\beta}\{d_\Omega(w)\}$. If there exists a constant $\mu_1 \geq 1$ such that for any $z \in \beta$, we have*

$$d(x,z) \leq \mu_1 d_\Omega(z), \tag{5.17}$$

then for $\lambda = \frac{11}{9}e^2\mu_1\big([Q^{\log_2 4e^2\mu_1(1+\mu_1)}]+1\big)$, it holds

$$\ell(\beta) \leq \lambda d_\Omega(x_0) \quad \text{and} \quad k_\Omega(x,y) \leq \lambda \log\frac{3d_\Omega(x_0)}{d_\Omega(x)}.$$

Proof. If $d(x,y) < \frac{1}{2}\max\{d_\Omega(x), d_\Omega(y)\}$, then Lemma 2.1 with $a = 2$ implies

$$k_\Omega(x,y) \leq \frac{10}{9} \leq \lambda \log\frac{3d_\Omega(x_0)}{d_\Omega(x)}. \tag{5.18}$$

This, together with (2.2), gives

$$\log\Big(1+\frac{\ell(\beta)}{d_\Omega(x)}\Big) \leq k_\Omega(x,y) \leq \frac{10}{9},$$

from which it follows that

$$\ell(\beta) \leq \big(e^{\frac{10}{9}}-1\big)d_\Omega(x) \leq \big(e^{\frac{10}{9}}-1\big)d_\Omega(x) < \lambda d_\Omega(x_0). \tag{5.19}$$

Thus, in the following proof, we may assume that

$$d(x,y) \geq \frac{1}{2}\max\{d_\Omega(x), d_\Omega(y)\}. \tag{5.20}$$

By Lemma 5.4, there exist finite sequences of balls $\big\{B_i = \mathbb{B}(y_i, r_i)\big\}_{i=1}^{k_1}$ and points $\{y_i\}_{i=1}^{k_1+1}$ in Ω which satisfy all the properties listed in Lemma 5.4. Select $w_0 \in \beta$ such that

$$d(x,w_0) = \sup_{z\in\beta}\{d(x,z)\}.$$

Then there is an integer k_2, depending on x and w_0, such that

$$e^{k_2-1}d_\Omega(x) \leq d(x,w_0) < e^{k_2}d_\Omega(x). \tag{5.21}$$

In the following, we consider two cases.

Case 1. $k_2 \leq \log(3\mu_1)$.

Let

$$\mathcal{B} = \mathbb{B}(x, 4\mu_1 d_\Omega(x)).$$

Then for each $i \in \{1,\dots,k_1\}$, we have

$$d(x,y_i) \leq d(x,w_0) \overset{(5.21)}{\leq} e^{k_2}d_\Omega(x) \leq 3\mu_1 d_\Omega(x),$$

and then,

$$4r_i = d_\Omega(y_i) \leq d_\Omega(x) + d(x,y_i) \leq (1+3\mu_1)d_\Omega(x).$$

It follows that for each $i \in \{1,\dots,k_1\}$, $d(y_i,x) + r_i < 4\mu_1 d_\Omega(x)$ and so

$$B_i \subset \mathcal{B}. \tag{5.22}$$

Let $a = 16\mu_1(1+\mu_1)$ and $R = 4\mu_1 d_\Omega(x)$. Since for each $i \in \{1,\dots,k_1\}$,

$$d_\Omega(x) \leq d(x,y_i) + d_\Omega(y_i) \overset{(5.17)}{\leq} (1+\mu_1)d_\Omega(y_i), \tag{5.23}$$

we get from the above estimate

$$\frac{R}{a} = \frac{d_\Omega(x)}{4(1+\mu_1)} \le \frac{1}{4} d_\Omega(y_i).$$

Based on this, (5.22) and Lemma 5.4 (3), we may apply Lemma 2.3 to conclude

$$k_1 \le [Q^{\log_2 16\mu_1(1+\mu_1)}] + 1.$$

Then it follows from the above estimate and Lemma 5.4 (5) that

$$k_\Omega(x,y) = \sum_{i=1}^{k_1} k_\Omega(y_i, y_{i+1}) \le \frac{20}{27}\left([Q^{\log_2 16\mu_1(1+\mu_1)}] + 1\right). \tag{5.24}$$

Moreover, since for all $i \in \{1, \ldots, k_1\}$, $d_\Omega(y_i) \le d_\Omega(x_0)$, we obtain

$$\begin{aligned} \ell(\beta) = \sum_{i=1}^{k_1} \ell(\beta[y_i, y_{i+1}]) &\overset{\text{Lemma 5.4}}{\le} \frac{11}{9}\sum_{i=1}^{k_1} d_\Omega(y_i) \\ &\le \frac{11}{9}\left([Q^{\log_2 16\mu_1(1+\mu_1)}] + 1\right) d_\Omega(x_0). \end{aligned} \tag{5.25}$$

In this case, we have proved Lemma 5.5.

Case 2. $k_2 > \log(3\mu_1)$.

For each $p \in \{1, \ldots, k_2 + 1\}$, set

$$\mathcal{B}_p = \mathbb{B}(x, e^p d_\Omega(x)).$$

Then (5.21) implies that

$$\beta \subset \mathcal{B}_{k_2}. \tag{5.26}$$

For each $z \in \mathcal{B}_p \cap \beta$, we observe that

$$\mathbb{B}\left(z, \frac{1}{4} d_\Omega(z)\right) \subset \mathcal{B}_{p+1}. \tag{5.27}$$

Indeed, for any $w \in \mathbb{B}\big(z, \frac{1}{4} d_\Omega(z)\big)$, we have

$$d(x,w) \le d(x,z) + \frac{1}{4} d_\Omega(z) \le \frac{5}{4} d(x,z) + \frac{1}{4} d_\Omega(x) \le \frac{1}{4}(1 + 5e^p) d_\Omega(x),$$

which gives (5.27).

It follows from (5.26) and (5.27) that

$$\bigcup_{i=1}^{k_1} B_i \subset \mathcal{B}_{k_2+1}.$$

For $p \in \{1, \ldots, k_2\}$, let

$$\beta_p = \beta \cap (\mathcal{B}_p \backslash \mathcal{B}_{p-1}), \quad \text{where } \mathcal{B}_0 = \emptyset,$$

and Define

$$\beta_p = \begin{cases} 0, & \text{if } \beta_p \cap \{y_i\}_{i=1}^{k_1} = \emptyset, \\ \text{card}\big\{\beta_p \cap \{y_i\}_{i=1}^{k_1}\big\}, & \text{otherwise.} \end{cases}$$

When $\beta_p \cap \{y_i\}_{i=1}^{k_1} \neq \emptyset$, let

$$\{y_{p,j}\}_{j=1}^{t_p} = \beta_p \cap \{y_i\}_{i=1}^{k_1}$$

and denote by $B_{p,j}$ the ball in $\{B_i\}_{i=1}^{k_1}$ with center $y_{p,j}$. Then it follows from (5.27) that

$$\bigcup_{j=1}^{t_p} B_{p,j} \subset \mathcal{B}_{p+1}.$$

Next, we claim that for all $p \in \{1, \cdots, k_2\}$, it holds

$$t_p \leq [Q^{\log_2 4e^2(1+\mu_1)}] + 1. \tag{5.28}$$

Indeed, when $p = 1$ and $t_1 \geq 1$, we set $R_1 = e^2 d_\Omega(x)$ and $a_1 = 4(1+\mu_1)e^2$. Then for each $j \in \{1, \ldots, t_1\}$, it holds

$$\frac{R_1}{a_1} = \frac{d_\Omega(x)}{4(1+\mu_1)} \overset{(5.23)}{\leq} \frac{1}{4} d_\Omega(y_{1,j}),$$

we know from Lemma 2.3 and Lemma 5.4 (3) that

$$t_1 \leq \left[Q^{\log_2 4e^2(1+\mu_1)}\right] + 1.$$

When $p \in \{2, \ldots, k_2\}$ and $t_p \geq 1$, note that for any $u \in \beta_p$, we have

$$d_\Omega(u) \overset{(5.17)}{\geq} \frac{1}{\mu_1} d(x,u) \geq \frac{1}{\mu_1} e^{p-1} d_\Omega(x). \tag{5.29}$$

Set $R_p = e^{p+1} d_\Omega(x)$ and $a_p = 4\mu_1 e^2$. Then for each $j \in \{1, \ldots, t_p\}$, it holds

$$\frac{R_p}{a_p} = \frac{e^{p-1} d_\Omega(x)}{4\mu_1} \overset{(5.29)}{\leq} \frac{1}{4} d_\Omega(y_{p,j}),$$

and thus, the desired estimate (5.28) follows from Lemma 2.3 and Lemma 5.4 (3).

Since for each $p \in \{1, \ldots, k_2\}$ and each $j \in \{1, \ldots, t_p\}$, $y_{p,j} \in \beta_p \subset \mathcal{B}_p \backslash \mathcal{B}_{p-1}$, and so

$$d_\Omega(y_{p,j}) \leq d(x, y_{p,j}) + d_\Omega(x) \leq (1+e^p) d_\Omega(x). \tag{5.30}$$

This, together with assertions (4b) and (6) of Lemma 5.4, implies that

$$\ell(\beta) \;=\; \sum_{i=1}^{k_1} \ell(\beta[y_i, y_{i+1}]) \leq \frac{11}{9} \sum_{i=1}^{k_1} d_\Omega(y_i) \leq \frac{11}{9} \sum_{p=1}^{k_2} \sum_{j=1}^{t_p} d_\Omega(y_{p,j}).$$

Since

$$\sum_{j=1}^{t_p} d_\Omega(y_{p,j}) \overset{(5.30)}{\leq} (1+e^p) t_p d_\Omega(x) \overset{(5.28)}{\leq} \big([Q^{\log_2 4e^2(1+\mu_1)}] + 1\big)(1+e^p) d_\Omega(x),$$

and

$$d(x, w_0) \overset{(5.17)}{\leq} \mu_1 d_\Omega(w_0) \leq \mu_1 d_\Omega(x_0), \tag{5.31}$$

we get from the estimate of $\ell(\beta)$ that

$$\begin{aligned}
\ell(\beta) &\leq \frac{11}{9}\big([Q^{\log_2 4e^2(1+\mu_1)}] + 1\big) e^{k_2+1} d_\Omega(x) \\
&\overset{(5.21)}{\leq} \frac{11}{9} e^2 \big([Q^{\log_2 4e^2(1+\mu_1)}] + 1\big) d(x, w_0) \\
&\overset{(5.31)}{\leq} \frac{11}{9} e^2 \mu_1 \big([Q^{\log_2 4e^2(1+\mu_1)}] + 1\big) d_\Omega(x_0) = \lambda d_\Omega(x_0).
\end{aligned}$$

On the other hand, note that

$$\begin{aligned}k_1 = \sum_{p=1}^{k_2} t_p &\overset{(5.28)}{\leq} \big([Q^{\log_2 4e^2(1+\mu_1)}]+1\big)k_2\\ &\overset{(5.21)}{\leq} \big([Q^{\log_2 4e^2(1+\mu_1)}]+1\big)\Big(1+\log\frac{d(x,w_0)}{d_\Omega(x)}\Big)\\ &\overset{(5.31)}{\leq} 2\mu_1\big([Q^{\log_2 4e^2(1+\mu_1)}]+1\big)\log\frac{3d_\Omega(x_0)}{d_\Omega(x)}.\end{aligned}$$

This, together with Lemma 5.4(5), gives

$$k_\Omega(x,y) \leq \sum_{i=1}^{k_1} k_\Omega(y_i,y_{i+1}) \leq \frac{40}{27}\mu_1\big([Q^{\log_2 4e^2(1+\mu_1)}]+1\big)\log\frac{3d_\Omega(x_0)}{d_\Omega(x)} \leq \lambda\log\frac{3d_\Omega(x_0)}{d_\Omega(x)}.$$

The proof is thus complete. □

For notational simplicity, we write $\gamma=\gamma_{xy}$ and $\alpha=\alpha_{xy}$. Suppose on the contrary that

$$\ell(\gamma) > \tau_1^2\ell(\alpha). \tag{5.32}$$

Then we have

$$d(x,y) \geq \frac{3}{4}\max\{d_\Omega(x), d_\Omega(y)\}. \tag{5.33}$$

Indeed, if not, then Lemma 2.1 with $a=\frac{4}{3}$ gives

$$\ell(\gamma) \leq \frac{40}{9}e^{\frac{10}{3}}d(x,y) \leq \tau_1^2\ell(\alpha),$$

which contradicts (5.32).

By (5.33) and Lemma 5.4, there exist a finite sequence of balls $\big\{B_\rho=\mathbb{B}(x_\rho,r_\rho)\big\}_{\rho=1}^{t}$ in Ω and a finite sequence of points $\{x_\rho\}_{\rho=1}^{t+1}$ in γ which satisfy all assertions in Lemma 5.4. Here, $x_1=x$, $x_{t+1}=y\in B_t$ (with the possibility that $x_{t+1}=x_t$). Then $\{x_\rho\}_{\rho=1}^{t+1}$ forms a partition of γ. Furthermore, it follows from Lemma 5.4 (4b) that

$$\ell(\gamma)=\sum_{\rho=1}^{t}\ell(\gamma[x_\rho,x_{\rho+1}]) \leq \frac{11}{9}\sum_{\rho=1}^{t} d_\Omega(x_\rho). \tag{5.34}$$

Let $x_0\in\gamma_{xy}$ be such that

$$d_\Omega(x_0)=\sup_{z\in\gamma_{xy}}\{d_\Omega(z)\}.$$

By the assumption in Theorem 5.3, we have

$$\sigma(x,x_0) \leq \mathrm{diam}_\sigma(\gamma) \leq \tau_1\sigma(x,y). \tag{5.35}$$

To layer the elements in the partition $\{x_\rho\}_{\rho=1}^{t+1}$, we set

$$S_0=\max_{1\leq\rho\leq t+1}\{d_\Omega(x_\rho)\} \quad\text{and}\quad T_0=\min_{1\leq\rho\leq t+1}\{d_\Omega(x_\rho)\}. \tag{5.36}$$

Then there must exist an integer $t_2\geq 0$ be such that

$$2^{t_2}T_0 \leq S_0 < 2^{t_2+1}T_0. \tag{5.37}$$

For each $i\in\{0,\dots,t_2\}$, we define the ith layer A_i of the partition as

$$A_i=\big\{u_j^i\in\{x_1,\dots,x_{t+1}\}:\ 2^iT_0\leq d_\Omega(u_j^i)<2^{i+1}T_0\big\}, \tag{5.38}$$

and then set $q_i = \mathrm{Card}\{A_i\}$, with the usual convention that $q_i = 0$ if $A_i = \emptyset$. We define

$$\lambda_1 = [Q^{\log_2 56\tau}] + 1 \quad \text{and} \quad \lambda_2 = 4[Q^{\log_2 56\tau}] + 25[Q^{\log_2 56\tau}] + 36. \tag{5.39}$$

Case 1: $\max\big\{q_i : \ i \in \{0, 1, \dots, t_2\}\big\} \le \lambda_2$.

In this case, we have

$$\begin{aligned}\ell(\gamma) &\overset{(5.34)}{\le} \frac{11}{9}\sum_{i=0}^{t_2}\sum_{j=0}^{q_i} d_\Omega(u_{i,j}) \le \frac{11}{9}\lambda_2\sum_{i=0}^{t_2} 2^{i+1}T_0 \overset{(5.37)+(5.38)}{\le} \frac{44}{9}\lambda_2 d_\Omega(x_0)\\ &\le \frac{44}{9}\left(\sigma(x, x_0) + d_\Omega(x)\right) \overset{(5.33)+(5.35)}{\le} \frac{44}{9}\left(\Big(\tau_1 + \frac{4}{3}\Big)\ell(\alpha)\right) < \tau_1^2\ell(\alpha),\end{aligned}$$

which clearly contradicts with (5.32).

Case 2: $\max\big\{q_i : \ i \in \{0, 1, \dots, t_2\}\big\} > \lambda_2$.

In this case, we need the following result, whose proof will be postponed to Section 5.3.

Proposition 5.6. *Suppose that* $\max\big\{q_i : \ i \in \{0, 1, \dots, t_2\}\big\} > \lambda_2$. *Then there are partitions* $P_\gamma = \{u_j\}_{j=0}^{s_0+1} \subset \gamma$ *and* $P_\alpha = \{w_j\}_{j=0}^{s_0+1} \subset \alpha$, *where* $u_0 = w_0 = x$ *and* $u_{s_0+1} = w_{s_0+1} = y$, *such that the following conclusions hold:*

(1) $s_0 \ge \lambda_1$.

(2) *For each* $j \in \{0, 1, \dots, s_0\}$ *and every* $i \in \{0, \dots, t_2\}$,
$$\mathrm{Card}\big\{\gamma[u_j, u_{j+1}] \cap A_i\big\} \le \lambda_2.$$

(3) *For each* $j \in \{1, \dots, s_0 - 1\}$,
$$\sigma(u_j, u_{j+1}) \ge 3\tau \max\{d_\Omega(u_j), d_\Omega(u_{j+1})\}$$
and
$$\sigma(w_j, w_{j+1}) \ge \frac{3}{2}\tau \max\{d_\Omega(u_j), d_\Omega(u_{j+1})\}.$$

(4) *For each* $j \in \{1, \dots, s_0\}$,
$$w_j \in \alpha[w_{j-1}, w_{j+1}] \quad \textit{and} \quad \sigma(u_j, w_j) \le \tau d_\Omega(u_j).$$

(5) $\sigma(w_0, w_1) \ge 3\tau d_\Omega(u_1)$ *and* $\sigma(w_{s_0}, w_{s_0+1}) \ge 3\tau d_\Omega(u_{s_0})$.

Here and hereafter, the elements of a sequence of points on γ_{xy} or α_{xy} are consecutively listed along the direction from x to y.

With the aid of Proposition 5.6, we are ready to complete the proof of Theorem 5.3.

Proof of Theorem 5.3. Since $\max\big\{q_i : \ i \in \{0, 1, \dots, t_2\}\big\} > \lambda_2$, by Proposition 5.6, there are partitions $P_\gamma = \{u_j\}_{j=0}^{s_0+1} \subset \gamma$ and $P_\alpha = \{w_j\}_{j=0}^{s_0+1} \subset \alpha$ such that all the conclusions of Proposition 5.6 are satisfied, where $v_0 = w_0 = x$ and $v_{s_0+1} = w_{s_0+1} = y$.

Next, we shall prove that for each $j \in \{0, 1, \dots, s_0\}$, it holds

$$\ell(\gamma[u_j, u_{j+1}]) \le \frac{88}{27\tau}\lambda_2(3\tau\tau_1 +)\ell(\alpha[w_j, w_{j+1}]). \tag{5.40}$$

For each $j \in \{0, 1, \dots, s_0\}$, Theorem 5.2 implies that

$$\mathrm{diam}_\sigma(\gamma_{xy}[u_j, u_{j+1}]) \le \tau_1\sigma(u_j, u_{j+1}). \tag{5.41}$$

Set

$$d_\Omega(v_j) = \max\left\{d_\Omega(u) : u \in \gamma[u_j, u_{j+1}] \bigcap \Big(\bigcup_{i=0}^{t_2} A_i\Big)\right\}.$$

Then there is an integer $t_3 \geq 0$ such that

$$2^{t_3}\, T_0 \leq d_\Omega(v_j) < 2^{t_3+1}\, T_0, \tag{5.42}$$

where T_0 is given by (5.36). By (5.37), $t_3 \leq t_2 + 1$.

Claim: For each $j \in \{0, 1, \dots, s_0\}$,

$$\ell(\gamma_{xy}[u_j, u_{j+1}]) \leq \frac{44}{9}\lambda_2 d_\Omega(v_j).$$

For the proof of this claim, let

$$\{v_{j,s}\}_{s=0}^{p} = \gamma_{xy}[u_j, u_{j+1}] \cap \{x_\rho\}_{\rho=1}^{t+1},$$

where $v_{j,0} = u_j$ and $v_{j,p+1} = u_{j+1}$. Set

$$A_j^i = \left\{u_{j,m}^i \in \{v_{j,0}, \dots, v_{j,p} :\ 2^i\, T_0 \leq d_\Omega(u_{j,m}^i) < 2^{i+1}\, T_0\right\} \tag{5.43}$$

and $\mathrm{Card}\{A_j^i\} = q_{j,i}$. Then by Proposition 5.6 (2), $q_{j,i} \leq \lambda_2$ and thus,

$$\begin{aligned}\ell(\gamma[u_j, u_{j+1}]) = \sum_{s=0}^{p} \ell(\gamma[v_{j,s}, v_{j,s+1}]) &\overset{\text{Lemma 5.4(4b)}}{\leq} \frac{11}{9}\sum_{s=0}^{p} d_\Omega(v_{j,s}) \leq \frac{11}{9}\sum_{i=0}^{t_3}\sum_{m=0}^{q_{j,i}} d_\Omega(u_{j:i,m})\\ &\overset{(5.43)}{\leq} \frac{11}{9}\lambda_2 \sum_{i=0}^{t_3} 2^{i+1} T_0 \overset{(5.42)}{\leq} \frac{44}{9}\lambda_2 d_\Omega(v_j).\end{aligned}$$

This completes the proof of Claim.

Since $v_j \in \gamma[u_j, u_{j+1}]$, we have

$$\begin{aligned} d_\Omega(v_j) &\leq \min\left\{\sigma(u_j, v_j) + d_\Omega(u_j), \sigma(u_{j+1}, v_j) + d_\Omega(u_{j+1})\right\}\\ &\leq \mathrm{diam}_\sigma(u_j, u_{j+1}) + \min\left\{d_\Omega(u_j), d_\Omega(u_{j+1})\right\}.\end{aligned}$$

This, combining with the above claim and (5.41), gives

$$\ell(\gamma[u_j, u_{j+1}]) \leq \frac{44}{9}\lambda_2\big(\tau_1\sigma(u_j, u_{j+1}) + \min\{d_\Omega(u_j), d_\Omega(u_{j+1})\}\big). \tag{5.44}$$

Case A: $j \in \{1, \dots, s_0 - 1\}$.

By Assertions (3) and (4) of Proposition 5.6, we have

$$\sigma(w_j, w_{j+1}) \geq \sigma(u_j, u_{j+1}) - \sigma(u_j, w_j) - \sigma(u_{j+1}, w_{j+1}) \geq \frac{1}{3}\sigma(u_j, u_{j+1}),$$

which, together with Proposition 5.6 (3) and (5), shows that

$$\ell(\gamma[u_j, u_{j+1}]) \overset{(5.44)}{\leq} \frac{44}{9}\lambda_2\Big(3\tau_1 + \frac{2}{\tau}\Big)\sigma(w_j, w_{j+1}) < \frac{88}{27\tau}\lambda_2(3\tau\tau_1 + 1)\ell(\alpha[w_j, w_{j+1}]).$$

Case B: $j \in \{0, s_0\}$.

We only consider the case $j = 0$, as the proof for the other case is similar.
If $d_\Omega(u_1) \leq \frac{1}{2\tau}\sigma(u_0, u_1)$, then

$$d_\Omega(u_0) \leq d_\Omega(u_1) + \sigma(u_0, u_1) \leq \frac{2\tau + 1}{2\tau}\sigma(u_0, u_1).$$

Since $w_0 = u_0 = x$, it follows from the assumption $d_\Omega(u_1) \le \frac{1}{2\tau}\sigma(u_0,u_1)$ and Proposition 5.6 (4) that

$$\sigma(w_0,w_1) \ge \sigma(u_0,u_1) - \sigma(u_1,w_1) \ge \frac{1}{2}\sigma(u_0,u_1).$$

This gives

$$d_\Omega(u_1) \le \frac{1}{\tau}\sigma(w_0,w_1),$$

and thus, we have

$$\ell(\gamma[u_0,u_1]) \overset{(5.44)}{\le} \frac{44}{9}\lambda_2\Big(2\tau_1 + \frac{1}{\tau}\Big)\sigma(w_0,w_1) \le \frac{44}{9\tau}\lambda_2(2\tau\tau_1+1)\ell(\alpha[w_0,w_1]).$$

If $d_\Omega(u_1) > \frac{1}{2\tau}\sigma(u_0,u_1)$, then again by Proposition 5.6 (5) and (5.44), we obtain

$$\ell(\gamma[u_0,u_1]) \le \frac{44}{9}\lambda_2(2\tau\tau_1+1)d_\Omega(u_1) \le \frac{44}{27\tau}\lambda_2(2\tau\tau_1+1)\ell(\alpha[w_0,w_1]).$$

In either cases, we have proved (5.40).

Now, it follows from (5.40) that

$$\ell(\gamma) = \sum_{j=1}^{s_0+1}\ell(\gamma[u_{j-1},u_j]) \le \frac{88}{27\tau}\lambda_2(3\tau\tau_1+1)\ell(\alpha) < \tau_1^2\ell(\alpha),$$

which again contradicts with (5.32). Thus, the proof of Theorem 5.3 is complete. □

5.3. **Proof of Proposition 5.6.** In this section, we present the proof of Proposition 5.6. It requies a couple of auxilliary lemmas. Recall in below the choices for λ_1 and λ_2 in (5.39).

Lemma 5.7. *For each $i \in \{0,\dots,t_2\}$ and any $u \in \gamma$, it holds*

$$\mathrm{Card}\{\mathbb{B}(u, 3\cdot 2^{i+2}\tau T_0) \cap A_i\} \le \lambda_1.$$

Proof. Let $v \in A_i \cap \mathbb{B}(u, 3\cdot 2^{i+2}\tau T_0)$. Then for any $w \in \overline{\mathbb{B}}(v, 4^{-1}d_\Omega(v))$, by (5.38), we have

$$d(w,u) \le d(w,v) + d(v,u) < (6\tau + 4^{-1})2^{i+1}T_0 < 7\cdot 2^{i+1}\tau T_0.$$

This implies that

$$\overline{\mathbb{B}}(v, 4^{-1}d_\Omega(v)) \subset \mathbb{B}(u, 7\cdot 2^{i+1}\tau T_0).$$

Based on Lemma 5.4 (3) and (5.38), the conclusion follows from Lemma 2.3 applied with $R = 7\cdot 2^{i+1}\tau T_0$, $a = 56\tau$ and $r = \frac{R}{a} = 2^{i-2}T_0$. □

The next lemma gives a useful partition of the ith layer of γ.

Lemma 5.8. *For $v_1 \in \gamma$ and $v_2 \in \gamma[v_1, x_{t+1}]$, suppose that there exists some $i \in \{0,\dots,t_2\}$ such that*

(i) *the set $B_i = \gamma[v_1,v_2] \cap A_i = \{u_r\}_{r=1}^{q}$ with $q > \lambda_2$.*
(ii) *there are $w_1 \in \alpha$ and $w_2 \in \alpha[w_1, y]$ such that for any $u \in A_i$,*

$$\max\{\sigma(v_1,w_1), \sigma(v_2,w_2)\} \le \tau d_\Omega(u).$$

Then there exists a partition $\{u_{p_j}\}_{j=1}^{\iota} \subset B_i \cap \gamma[v_1,v_2]$ such that

(1) *$\iota \ge \lambda_1$.*

(2) *for each* $j \in \{0, 1, \dots, \iota\}$,

$$\mathrm{Card}\big\{\gamma[u_{p_j}, u_{p_{j+1}}] \cap B_i\big\} \le \lambda_1 + 1,$$

where $u_{p_0} = v_1$ *and* $u_{p_{\iota+1}} = v_2$.

(3) $\min\{\sigma(v_1, u_{p_1}), \sigma(v_2, u_{p_\iota})\} \ge 6\tau d_\Omega(u_{p_1})$.

(4) $\min\{\sigma(u_{p_j}, u_{p_{j+1}}) : \ j \in \{1, \dots, \iota - 1\}\} \ge 6\tau d_\Omega(u_{p_j})$.

(5) *for each* $j \in \{1, \dots, \iota\}$ *and* $\alpha_{v_2 w_2} \in \Gamma_{v_2 w_2}$,

$$\sigma(u_{p_j}, \alpha_{v_2 w_2}) > \tau d_\Omega(u_{p_j}).$$

Proof. Since $B_i \subset A_i$, Lemma 5.7 gives

$$\mathrm{Card}\big\{\mathbb{B}(v_1, 3\cdot 2^{i+2}\tau T_0) \cap B_i\big\} \le \lambda_1.$$

It follows that there are at least $k_1 \ge \lambda_2 - \lambda_1$ points $\{u_r\}_{r=1}^{k_1}$ in B_i, which are not contained in $\mathbb{B}(v_1, 3\cdot 2^{i+2}\tau T_0)$. Let

$$q_1 := \min\{r \in \{1, \cdots, k_1\} : u_r \notin B(v_1, 3\cdot 2^{i+2}\tau T_0)\}.$$

Then we have

$$\mathrm{Card}\big\{\gamma[v_1, u_{q_1}] \cap B_i\big\} \le \lambda_1 + 1,$$

and

$$\sigma(v_1, u_{q_1}) \ge d(v_1, u_{q_1}) \ge 3\cdot 2^{i+2}\tau T_0 \overset{(5.38)}{\ge} 6\tau d_\Omega(u_{q_1}).$$

Applying Lemma 5.7 again, we obtain $\mathrm{Card}\big\{\mathbb{B}(u_{q_1}, 3\cdot 2^{i+2}\tau T_0) \cap B_i\big\} \le \lambda_1$ and thus there are at least $k_2 \ge \lambda_2 - 2\lambda_1 - 1$ points $\{u_r\}_{r=s}^{k_2+s-1}$ in B_i with $s > q_1$, which are not contained in $\mathbb{B}(u_{q_1}, 3\cdot 2^{i+2}\tau T_0)$. Let

$$q_2 := \min\{r \in \{s, \cdots, k_2 + s - 1\} : u_r \notin B(u_{q_1}, 3\cdot 2^{i+2}\tau T_0)\}.$$

Then

$$\mathrm{Card}\big\{\gamma[u_{q_1}, u_{q_2}] \cap B_i\big\} \le \lambda_1 + 1,$$

and

$$\sigma(u_{q_1}, u_{q_2}) \ge d(u_{q_1}, u_{q_2}) \ge 3\cdot 2^{i+2}\tau T_0 \overset{(5.38)}{\ge} 6\tau d_\Omega(u_{q_1}).$$

Repeating this procedure, we may find a finite sequence of points $\{u_{q_s}\}_{s=1}^{\iota_1} \subset B_i$ such that

$$\iota_1 \ge \frac{q}{\lambda_1 + 1} \ge 4[Q^{\log_2 56\tau}] + 8 \quad \text{and} \quad \mathrm{Card}\big\{\gamma[u_{p_{\iota_1}+1}, v_2] \cap B_i\big\} \le \lambda_1 + 1.$$

Applying Lemma 5.7 once again, we infer that

$$\mathrm{Card}\big\{\mathbb{B}(v_2, 3\cdot 2^{i+2}\tau T_0) \cap B_i\big\} \le \lambda_1 \quad \text{and} \quad \mathrm{Card}\big\{\mathbb{B}(v_2, 3\cdot 2^{i+2}\tau T_0) \cap \{u_{q_r}\}_{r=1}^{\iota_1}\big\} \le \lambda_1.$$

This shows that there are ι points in $\{u_{q_r}\}_{r=1}^{\iota_1}$, which are not contained in $\mathbb{B}(v_2, 3\cdot 2^{i+2}\tau T_0)$. Denote these ι points by $\{u_{p_j}\}_{j=1}^{\iota}$. Then it follows from the preceeding construction that

(1) $\iota \ge \iota_1 - \lambda_1 - 1 \ge 2[Q^{\log_2 56\tau}] + 5 \ge \lambda_1$.

(2) for each $j \in \{0, 1, \dots, \iota\}$,

$$\mathrm{Card}\big\{\gamma[u_{p_j}, u_{p_{j+1}}] \cap B_i\big\} \le \lambda_1 + 1,$$

where $u_{p_0} = v_1$ and $u_{p_{\iota+1}} = v_2$.

(3) $\min\{\sigma(v_1, u_{p_1}), \sigma(v_2, u_{p_\iota})\} \ge 6\tau d_\Omega(u_{p_1})$.

(4) $\min\{\sigma(u_{p_j}, u_{p_{j+1}}) : \ j \in \{1, \dots, \iota - 1\}\} \ge 6\tau d_\Omega(u_{p_j})$.

To finish the proof of lemma, it remains to verity the last statement (5). Suppose on the contrary that there exists $j \in \{1, \dots, \iota\}$ such that

$$\sigma(u_{p_j}, \alpha_{v_2 w_2}) \le \tau d_\Omega(u_{p_j}).$$

On the one hand, we know from the assumption (ii) of lemma that

$$\begin{aligned}\sigma(u_{p_j}, v_2) &\le \sigma(u_{p_j}, w_2) + \sigma(w_2, v_2) \le \sigma(u_{p_j}, \alpha_{v_2 w_2}) + \ell(\alpha_{v_2 w_2}) + \sigma(w_2, v_2)\\ &\le \tau d_\Omega(u_{p_j}) + (2+\tau_1^{-1})\sigma(w_2, v_2) < 4\tau d_\Omega(u_{p_j}).\end{aligned}$$

On the other hand, for each $j \in \{1, \dots, \iota\}$,

$$\mathbb{B}(v_2, 3\cdot 2^{i+2}\tau T_0) \cap \{u_{p_j}\}_{j=1}^{\iota} = \emptyset,$$

and thus

$$\sigma(u_{p_j}, v_2) \ge d(u_{p_j}, v_2) \ge 3\cdot 2^{i+2}\tau T_0 \overset{(5.38)}{\ge} 6\tau d_\Omega(u_{p_j}).$$

This is a contradiction and hence the proof of lemma is complete. □

Corresponding to the partition of γ in Lemma 5.8, we have an associated partition on α.

Lemma 5.9. *Under the assumptions of Lemma 5.8, there exists a partition $\{w_{p_j}\}_{j=1}^{\iota} \subset \alpha[w_1, w_2]$ such that*

(1) *for each $j \in \{1, \dots, \iota\}$,*

$$w_{p_j} \in \alpha[w_{p_{j-1}}, w_{p_{j+1}}] \quad \textit{and} \quad \sigma(u_{p_j}, w_{p_j}) \le \tau d_\Omega(u_{p_j}),$$

where $w_{p_0} = w_1$ and $w_{p_{\iota+1}} = w_2$.

(2) *for each $j \in \{0, 1, \dots, \iota\}$,*

$$\sigma(w_{p_j}, w_{p_{j+1}}) \ge 3\tau d_\Omega(u_{p_1}).$$

Proof. (1) Let $\{u_{p_j}\}_{j=1}^{\iota} \subset \gamma[v_1, v_2]$ be the sequence obtained in Lemma 5.8. Fix $\alpha_{v_1 w_1} \in \Gamma_{v_1 w_1}$ and $\alpha_{v_2 w_2} \in \Gamma_{v_2 w_2}$.

Consider the point u_{p_1}. Since (Ω, σ) satisfies the τ-ball separation condition, there exists a point $w_{p_1} \in \alpha_{v_1 w_1} \cup \alpha[w_1, w_2] \cup \alpha_{v_2 w_2}$ such that

$$\sigma(u_{p_1}, w_{p_1}) \le \tau d_\Omega(u_{p_1}). \tag{5.45}$$

It follows from Lemma 5.8 (5) that $w_{p_1} \in \alpha_{v_1 w_1} \cup \alpha[w_1, w_2]$.

Suppose that $w_{p_1} \in \alpha_{v_1 w_1}$. Since by assumption $\sigma(v_1, w_1) \le \tau d_\Omega(u_{p_1})$, we obtain that

$$\sigma(v_1, u_{p_1}) \le \sigma(v_1, w_{p_1}) + \sigma(w_{p_1}, u_{p_1}) \overset{(5.45)}{\le} \ell(\alpha_{v_1 w_1}) + \tau d_\Omega(u_{p_1}) \le (2+\tau_1^{-1})\tau d_\Omega(u_{p_1}),$$

which contradicts with Lemma 5.8 (3). This shows that $w_{p_1} \in \alpha[w_1, w_2]$.

Fix $\alpha_{u_{p_1} w_{p_1}} \in \Gamma_{u_{p_1} w_{p_1}}$, $\alpha_{u_{p_2} w_{p_2}} \in \Gamma_{u_{p_2} w_{p_2}}$ and consider the point u_{p_2}. Since (Ω, σ) satisfies the τ-ball separation condition, there exists a point $w_{p_2} \in \alpha_{u_{p_1} w_{p_1}} \cup \alpha[w_{p_1}, w_2] \cup \alpha_{v_2 w_2}$ such that

$$\sigma(u_{p_2}, w_{p_2}) \le \tau d_\Omega(u_{p_2}). \tag{5.46}$$

It follows from Lemma 5.8 (5) that $w_{p_2} \in \alpha_{u_{p_1} w_{p_1}} \cup \alpha_{xy}[w_{p_1}, w_2]$.

Suppose that $w_{p_2} \in \alpha_{u_{p_1} w_{p_1}}$. Then we have

$$\begin{aligned}\sigma(u_{p_1}, u_{p_2}) &\le \sigma(u_{p_1}, w_{p_2}) + \sigma(w_{p_2}, u_{p_2}) \overset{(5.46)}{\le} \ell(\alpha_{u_{p_1} w_{p_1}}) + \tau d_\Omega(u_{p_2})\\ &\le (1+\tau_1^{-1})\sigma(u_{p_1}, w_{p_1}) + \tau d_\Omega(u_{p_2}) \overset{(5.38)+(5.45)}{\le} (3+2\tau_1^{-1})\tau d_\Omega(u_{p_2}),\end{aligned}$$

which contradicts with Lemma 5.8 (4). This shows that $w_{p_2} \in \alpha[w_{p_1}, w_2]$.

Repeating this procedure, we may find a finite sequence of points $\{w_{p_j}\}_{j=1}^{\iota} \subset \alpha[w_1, w_2]$ which satisfies the first statement of the lemma.

(2) Since by assumption

$$\max\{\sigma(v_1, w_1), \sigma(v_2, w_2)\} \leq \tau d_\Omega(u_{p_1}),$$

we obtain from Lemmas 5.8 (3) and 5.9 (1) that

$$\sigma(w_1, w_{p_1}) \geq \sigma(v_1, u_{p_1}) - \sigma(v_1, w_1) - \sigma(u_{p_1}, w_{p_1}) \geq 4\tau d_\Omega(u_{p_1})$$

and from Lemma 5.8 (3) and Lemma 5.9 (1) that

$$\sigma(w_2, w_{p_\iota}) \geq \sigma(v_2, u_{p_\iota}) - \sigma(u_{p_\iota}, w_{p_\iota}) - \sigma(v_2, w_2) \overset{(5.38)}{\geq} 3\tau d_\Omega(u_{p_1}).$$

Moreover, for each $j \in \{2, \dots, \iota\}$, it follows from Lemma 5.8 (4) and Lemma 5.9 (1) that

$$\sigma(w_{p_{j-1}}, w_{p_j}) \geq \sigma(u_{p_{j-1}}, u_{p_j}) - \sigma(u_{p_j}, w_{p_j}) - \sigma(u_{p_{j-1}}, w_{p_{j-1}}) \overset{(5.38)}{\geq} 3\tau d_\Omega(u_{p_1}).$$

This proves the second statement, and hence, the proof of lemma is complete. □

Now, we are ready to prove Proposition 5.6.

Proof of Proposition 5.6. Set

$$r_1 = \min\big\{i \in \{0, 1, \dots, t_2\} : \ q_i > \lambda_2\big\},$$

$v_1 = w_1 = x$ and $v_2 = w_2 = y$. Then by Lemmas 5.8 and 5.9, there are partitions $\{u_{\nu_1}^{r_1}\}_{\nu_1=0}^{\iota_1+1} \subset \gamma$ and $\{w_{\nu_1}^{r_1}\}_{\nu_1=0}^{\iota_1+1} \subset \alpha$ such that all the conclusions in Lemmas 5.8 and 5.9 are satsified, where $u_{r_1,0} = x$, $u_{r_1,\nu_1+1} = y$, $w_0^{r_1} = x$, $w_{\nu_1+1}^{r_1} = y$ and $\{u_{\nu_1}^{r_1}\}_{\nu_1=1}^{\iota_1} \subset A_{r_1}$. Then, we have

$$\gamma = \bigcup_{\nu_1=0}^{\iota_1} \gamma[u_{\nu_1}^{r_1}, u_{\nu_1+1}^{r_1}] \quad \text{and} \quad \alpha = \bigcup_{\nu_1=0}^{\iota_1} \alpha[w_{\nu_1}^{r_1}, w_{\nu_1+1}^{r_1}].$$

If for each $\nu_1 \in \{0, 1, \dots, \iota_1\}$ and for all $i \in \{r_1, \dots, t_2\}$,

$$\mathrm{Card}\big\{\gamma[u_{\nu_1}^{r_1}, u_{\nu_1+1}^{r_1}] \cap A_i\big\} \leq \lambda_2,$$

then for all $i \in \{0, 1, \dots, t_2\}$, we have

$$\mathrm{Card}\big\{\gamma[u_{\nu_1}^{r_1}, u_{\nu_1+1}^{r_1}] \cap A_i\big\} \leq \lambda_2.$$

Let

$$P_\gamma^{r_1} = \{u_{\nu_1}^{r_1}\}_{\nu_1=0}^{\iota_1+1} \quad \text{and} \quad P_\alpha^{r_1} = \{w_{\nu_1}^{r_1}\}_{\nu_1=0}^{\iota_1+1}.$$

If there are some $\nu_1 \in \{0, 1, \dots, \iota_1\}$ and an $i \in \{r_1 + 1, \dots, t_2\}$ such that

$$\mathrm{Card}\big\{\gamma[u_{\nu_1}^{r_1}, u_{\nu_1+1}^{r_1}] \cap A_i\big\} > \lambda_2,$$

then we set

$$r_2 = \min\big\{i \in \{r_1 + 1, \dots, t_2\} : \ q_i > \lambda_2\big\}.$$

Clearly,

$$r_2 > r_1. \tag{5.47}$$

Assume that there are $K \geq 1$ sub-curves $\gamma[u_{\nu_{1,1}}^{r_1},\ u_{\nu_{1,1}+1}^{r_1}]$, $\cdots$, $\gamma[u_{\nu_{1,K}}^{r_1},\ u_{\nu_{1,K}+1}^{r_1}]$ of γ such that for each $k \in \{1, \dots, K\}$,

$$\mathrm{Card}\big\{\gamma[u_{\nu_{1,k}}^{r_1}, u_{\nu_{1,k}+1}^{r_1}] \cap A_{r_2}\big\} > \lambda_2,$$

and for any other sub-curves $\gamma[u^{r_1}_{\nu_1}, u^{r_1}_{\nu_1+1}]$ of γ, it holds

$$\mathrm{Card}\big\{\gamma[u^{r_1}_{\nu_1}, u^{r_1}_{\nu_1+1}] \cap A_{r_2}\big\} \le \lambda_2.$$

For each $\nu_1 \in \{0, 1, \dots, \iota_1\}$ and $u \in A_{r_2}$, by the choice of $w^{r_1}_{\nu_1}$, we have

$$\sigma(u^{r_1}_{\nu_1}, w^{r_1}_{\nu_1}) \le \tau d_\Omega(u^{r_1}_{\nu_1}) \overset{(5.38)}{\le} 2^{r_1+1}\tau T_0 \overset{(5.47)}{\le} 2^{r_2}\tau T_0 \overset{(5.38)}{\le} \tau d_\Omega(u).$$

It follows again from Lemmas 5.8 and 5.9 that for each $k \in \{1, \dots, K\}$, there exist partitions of $\gamma[u^{r_1}_{\nu_{1,k}}, u^{r_1}_{\nu_{1,k}+1}]$ and $\alpha[w^{r_1}_{\nu_{1,k}}, w^{r_1}_{\nu_{1,k}+1}]$, respectively, such that all the conclusions in Lemmas 5.8 and 5.9 are satisfied.

Through the implementation of this partition, we get a subdivision of γ (resp. α), for which we denote by

$$P^{r_1,r_2}_\gamma = \{u^{r_1,r_2}_{\nu_2}\}^{\iota_2+1}_{\nu_2=0} \qquad (\text{resp. } P^{r_1,r_2}_\alpha = \{w^{r_1,r_2}_{\nu_2}\}^{\iota_2+1}_{\nu_2=0}),$$

where ι_2 is a constant with $\iota_2 \ge \iota_1 \ge \lambda_1$. Clearly, we have

$$\gamma = \bigcup_{\nu_2=0}^{\iota_2} \gamma[u^{r_1,r_2}_{\nu_2}, u^{r_1,r_2}_{\nu_2+1}], \quad \alpha = \bigcup_{\nu_2=0}^{\iota_2} \alpha[w^{r_1,r_2}_{\nu_2}, w^{r_1,r_2}_{\nu_2+1}],$$

and for each $\nu_2 \in \{0, 1, \dots, \iota_2\}$ and for any $i \le r_2$, it holds

$$\mathrm{Card}\big\{\gamma[u^{r_1,r_2}_{\nu_2}, u^{r_1,r_2}_{\nu_2+1}] \cap A_i\big\} \le \lambda_2.$$

Repeating the procedure as above k_0 times, we thus obtain the following partition of γ and α:

$$P^{r_1,r_2,\dots,r_{k_0}}_\gamma = \{u^{r_1,r_2,\dots,r_{k_0}}_{\nu_{k_0}}\}^{\iota_{k_0}+1}_{\nu_{k_0}=0} \quad \text{and} \quad P^{r_1,r_2,\dots,r_{k_0}}_\alpha = \{w^{r_1,r_2,\dots,r_{k_0}}_{\nu_{k_0}}\}^{\iota_{k_0}+1}_{\nu_{k_0}=0}.$$

where ι_{k_0} is a constant with $\iota_{k_0} \ge \lambda_1$, such that these two partitions satisfy all the conclusions in Lemmas 5.8 and 5.9. It follows that

$$\gamma = \bigcup_{\nu_{k_0}=0}^{\iota_{k_0}} \gamma[u^{r_1,r_2,\dots,r_{k_0}}_{\nu_{k_0}}, u^{r_1,r_2,\dots,r_{k_0}}_{\nu_{k_0}+1}] \quad \text{and} \quad \alpha = \bigcup_{\nu_{k_0}=0}^{\iota_{k_0}} \alpha[w^{r_1,r_2,\dots,r_{k_0}}_{\nu_{k_0}}, w^{r_1,r_2,\dots,r_{k_0}}_{\nu_{k_0}+1}].$$

Furthermore, for each $\nu_{k_0} \in \{0, 1, \dots, \iota_{k_0}\}$ and any $i \in \{0, 1, \dots, t_2\}$, it holds

$$\mathrm{Card}\big\{\gamma[u^{r_1,r_2,\dots,r_{k_0}}_{\nu_{k_0}}, u^{r_1,r_2,\dots,r_{k_0}}_{\nu_{k_0}+1}] \cap A_i\big\} \le \lambda_2.$$

This proves the first two assertions in Proposition 5.6. The last three assertions are direct consequences of that from Lemmas 5.8 and 5.9. Thus the proof of Proposition 5.6 is complete. □

6. Ball separation condition and LLC-2 imply Gehring-Hayman inequality

In this section, we shall prove that

i) For $x, y \in \Omega$, $\gamma_{xy} \in \Lambda_{xy}(\Omega)$ and $\alpha_{xy} \in \Gamma_{xy}(\Omega)$, it holds

$$\ell(\gamma_{xy}) \le e^{19\kappa_0\kappa_1\kappa_2}\ell(\alpha_{xy}). \tag{6.1}$$

ii) For each $z \in \gamma_{xy}$, it holds

$$\min\{\ell(\gamma_{xy}[x,z]), \ell(\gamma_{xy}[y,z])\} \le \kappa_0 d_\Omega(z), \tag{6.2}$$

where $\kappa_0 = (14ec_0C)^2([Q^{2\log_2 70ec_0C}] + 1)$, $\kappa_1 = \kappa_0([Q^{\log_2 15\kappa_0}] + 1)$ and $\kappa_2 = \kappa_1^{4\kappa_1^2+1}$.

Then it follows from i) and ii) that for each $x, y \in \Omega$, each $\gamma_{xy} \in \Lambda_{xy}(\Omega)$ is a C-inner uniform curve and in particular Ω is C-inner uniform with $C = e^{19\kappa_0\kappa_1\kappa_2}$.

In the first step, we prove statement ii).

Lemma 6.1. *Let $x_1, x_2 \in \Omega$ and $\gamma = \gamma_{x_1x_2} \in \Lambda_{x_1x_2}(\Omega)$. Then for each $x \in \gamma$, we have*

$$\min\{\ell(\gamma[x_1, x]), \ell(\gamma[x_2, x])\} \le \kappa_0 d_\Omega(x).$$

Proof. Select $x_0 \in \gamma_{x_1x_2}$ such that

$$\min\{\mathrm{diam}(\gamma[x_1, x_0]), \mathrm{diam}(\gamma[x_2, x_0])\} \ge \frac{1}{3}\mathrm{diam}(\gamma), \tag{6.3}$$

Our aim is prove that for each $i \in \{1, 2\}$ and $x \in \gamma[x_i, x_0]$,

$$\mathrm{diam}(\gamma[x_i, x]) \le 10c_0Cd_\Omega(x). \tag{6.4}$$

Without loss of generality, we only consider the case $x \in \gamma[x_1, x_0]$ and shall prove the stronger statement that

$$\mathrm{diam}(\gamma[x_1, x]) \le 10c_0Cd_\Omega(x). \tag{6.5}$$

Let $z_1 \in \gamma([x_1, x])$ $z_1 \in \Omega\backslash\mathbf{B}\left(x, \frac{1}{9}\mathrm{diam}(\gamma[x_1, x])\right)$. It follows from $\mathrm{diam}(\gamma[x_2, x]) \ge \frac{1}{3}\mathrm{diam}(\gamma)$ that there exists $z_2 \in \gamma[x_2, x]$ that belongs to $\Omega\backslash\mathbf{B}\left(x, \frac{1}{9}\mathrm{diam}(\gamma[x_1, x])\right)$.

Since Ω is c_0-LLC-2, there exists some curve

$$\beta \subset \Omega\backslash\mathbf{B}\left(x, \frac{1}{9c_0}\mathrm{diam}(\gamma[x_1, x])\right)$$

joining z_1 and z_2, and since (Ω, σ) satisfies the C-ball separation condition, we have

$$\mathbf{B}_\sigma(x, Cd_\Omega(x)) \cap \beta \ne \emptyset,$$

from which it follows

$$\mathrm{diam}(\gamma[x_1, x]) \le 9c_0Cd_\Omega(x).$$

This establishes (6.5) and thus completes the proof of (6.4).

Our aim is to prove that for each $i = 1, 2$ and $x \in \gamma[x_i, x_0]$, we have

$$\ell(\gamma[x_i, x]) \le (14ec_0C)^2([Q^{2\log_2 70ec_0C}] + 1)d_\Omega(x). \tag{6.6}$$

Without loss of generality, by (6.4), we only consider the case $x \in \gamma[x_1, x_0]$. Take $y_1 \in \gamma[x_1, x]$ such that

$$d_\Omega(y_1) = \sup_{y\in\gamma[x_1,x]}\{d_\Omega(y)\}.$$

Then we have

$$d_\Omega(y_1) \le d_\Omega(x) + d(y_1, x) \le d_\Omega(x) + \mathrm{diam}(\gamma[x_1, x]) \overset{(6.5)}{\le} (1 + 10c_0C)d_\Omega(x). \tag{6.7}$$

Note that by (6.5), for each $z \in \gamma[x_1, x])$, it holds

$$d(x_1, z) \le \mathrm{diam}(\gamma[x_1, z]) \le 10c_0Cd_\Omega(z).$$

Thus we infer from Lemma 5.5 (with $\beta = \gamma[x_1, x]$, $\mu_1 = 10c_0C$ and $x_0 = y_1$) that

$$\ell(\gamma[x_1, x]) \le 13e^2c_0C([Q^{2\log_2 70ec_0C}] + 1)d_\Omega(y_1) \overset{(6.7)}{\le} (14ec_0C)^2([Q^{2\log_2 70ec_0C}] + 1)d_\Omega(x).$$

This establishes (6.6) as desired. The proof of Lemma (6.1) is thus complete. ☐

For the proof of statement i), we need the following technical lemma. A version of this lemma was proved in [16, Lemma 2.27] for Gromov hyperbolic domains.

Lemma 6.2. *Suppose that x, y and z are three distinct points in Ω, and $\min\{k_\Omega(x,y), k_\Omega(x,z)\} \geq 1$. Fix $\gamma_{xy} \in \Lambda_{xy}(\Omega)$ and $\gamma_{xz} \in \Lambda_{xz}(\Omega)$. If $k_\Omega(y,z) \leq 1$, then*

$$\max\{\ell(\gamma_{xy}), \ell(\gamma_{xz})\} < e^{9\kappa_0\kappa_2} \min\{\ell(\gamma_{xy}), \ell(\gamma_{xz})\}.$$

Proof. It follows from Lemma 2.1 and the assumption "$\min\{k_\Omega(x,y), k_\Omega(x,z)\} \geq 1$" that

$$d(x,y) \geq \frac{9}{19} \max\{d_\Omega(x), d_\Omega(y)\} \ \text{ and } \ d(x,z) \geq \frac{9}{19} \max\{d_\Omega(x), d_\Omega(z)\}. \tag{6.8}$$

Without loss of generality, we may assume that

$$\min\{\ell(\gamma_{xy}), \ell(\gamma_{xz})\} = \ell(\gamma_{xy}).$$

Note that by (2.2), we have

$$\max\left\{\log\left(1 + \frac{\sigma(y,z)}{\min\{d_\Omega(y), d_\Omega(z)\}}\right),\ \log\frac{\max\{d_\Omega(y), d_\Omega(z)\}}{\min\{d_\Omega(y), d_\Omega(z)\}}\right\} \leq k_\Omega(y,z) \leq 1.$$

It follows that

$$\sigma(y,z) \leq (e-1)\min\{d_\Omega(y), d_\Omega(z)\} \tag{6.9}$$

and

$$\max\{d_\Omega(y), d_\Omega(z)\} \leq e\min\{d_\Omega(y), d_\Omega(z)\}. \tag{6.10}$$

Take $y_0 \in \gamma_{xy}$ and $z_0 \in \gamma_{xz}$ such that

$$\ell(\gamma_{xy}[x,y_0]) = \frac{1}{2}\ell(\gamma_{xy}) \ \text{ and } \ \ell(\gamma_{xz}[z,z_0]) = \frac{1}{2}\ell(\gamma_{xz}). \tag{6.11}$$

We consider the following two cases.

Case I: $\ell(\gamma_{xy}) \leq e^{\kappa_2}\min\{d_\Omega(y), d_\Omega(x)\}$.

In this case, we have

$$\log\left(1 + \frac{\ell(\gamma_{xz})}{d_\Omega(z)}\right) \overset{(2.2)}{\leq} k_\Omega(x,z) \leq k_\Omega(x,y) + k(y,z) \leq k_\Omega(x,y_0) + k_\Omega(y,y_0) + 1. \tag{6.12}$$

Note that by Lemma 6.1 and our choice of y_0, it holds

$$\ell(\gamma_{xy}[x,y_0]) \leq \kappa_0 d_\Omega(y_0) \quad \text{and} \quad \ell(\gamma_{xy}[y,y_0]) \leq \kappa_0 d_\Omega(y_0).$$

Thus it follows from [15, Lemma 2.1] that

$$k(x,y_0) \leq 4\kappa_0 \log\left(1 + \frac{\ell(\gamma_{xy}[x,y_0])}{d_\Omega(x)}\right)$$

and that

$$k(y,y_0) \leq 4\kappa_0 \log\left(1 + \frac{\ell(\gamma_{xy}[y,y_0])}{d_\Omega(y)}\right).$$

Substituting these estimates in (6.12), we obtain that

$$\begin{aligned}\log\left(1 + \frac{\ell(\gamma_{xz})}{\min\{d_\Omega(z), d_\Omega(x)\}}\right) &\leq 4\kappa_0\left(\log\left(1 + \frac{\ell(\gamma_{xy}[x,y_0])}{d_\Omega(x)}\right) + \log\left(1 + \frac{\ell(\gamma_{xy}[y_1,y_0])}{d_\Omega(y)}\right)\right) + 1 \\ &\leq 8\kappa_0\kappa_2 \overset{(6.8)}{<} \frac{20\kappa_0\kappa_2\ell(\gamma_{xy})}{\min\{d_\Omega(x), d_\Omega(y)\}} \overset{(6.10)}{<} \frac{20e\kappa_0\kappa_2\ell(\gamma_{xy})}{\min\{d_\Omega(x), d_\Omega(z)\}}.\end{aligned}$$

Since $\log(1+t) \geq a_1 t$ for $t \leq a_1^{-1} - 1$ and $a_1 > 0$, it follows from the above inequality that

$$\frac{e^{-8\kappa_0\kappa_2}\ell(\gamma_{xz})}{\min\{d_\Omega(x), d_\Omega(z)\}} \leq \log\Big(1 + \frac{\ell(\gamma_{xz})}{\min\{d_\Omega(z), d_\Omega(x)\}}\Big) \leq 20e\kappa_0\kappa_2 \frac{\ell(\gamma_{xy})}{\min\{d_\Omega(x), d_\Omega(z)\}}$$

and so

$$\ell(\gamma_{xz}) \leq 20\kappa_0\kappa_2 e^{1+8\kappa_0\kappa_2}\ell(\gamma_{xy}) \leq e^{9\kappa_0\kappa_2}\min\{\ell(\gamma_{xy}), \ell(\gamma_{xz})\}. \tag{6.13}$$

This completes the proof for this case.

Case II: $\ell(\gamma_{xy}) > e^{\kappa_2}\min\{d_\Omega(x), d_\Omega(y)\}$.

In this case, select $m_1 \geq [\kappa_2] - 1$ such that

$$e^{m_1+1}d_\Omega(y) < \ell(\gamma_{xy}) \leq e^{m_1+2}d_\Omega(y).$$

No loss of generality, by (6.10), we may assume that

$$\min\{d_\Omega(x), d_\Omega(y)\} = d_\Omega(y),$$

since the proof for the other case is similar.

Set $y_1 = y$ and $z_1 = z$. Choose $y_{m_1+1} \in \gamma_{xy_1}[y_1, y_0]$, $y_{1,m_1+1} \in \gamma_{xy_1}[x, y_0]$, $z_{m_1+1} \in \gamma_{xz_1}[z_1, z_0]$ and $z_{1,m_1+1} \in \gamma_{xz_1}[x, z_0]$ such that

$$\begin{aligned}\ell(\gamma_{xy_1}[y_1, y_{m_1+1}]) = \ell(\gamma_{xz_1}[z_1, z_{m_1+1}]) = \ell(\gamma_{xy_1}[x, y_{1,m_1+1}]) = \ell(\gamma_{xz_1}[x, z_{1,m_1+1}])\\ = e^{m_1}d_\Omega(y_1).\end{aligned} \tag{6.14}$$

Next, we claim that

Claim 6.1. $\max\{k_\Omega(y_{m_1+1}, z_{m_1+1}), k_\Omega(y_{1,m_1+1}, z_{1,m_1+1})\} \leq 2\kappa_2$.

Suppose on the contrary that

$$\max\{k_\Omega(y_{m_1+1}, z_{m_1+1}), k_\Omega(y_{1,m_1+1}, z_{1,m_1+1})\} > 2\kappa_2.$$

Without loss of generality, we may assume that

$$k_\Omega(y_{m_1+1}, z_{m_1+1}) > 2\kappa_2, \tag{6.15}$$

as the proof for the other case is similar.

For each $i \in \{2, ..., m_1+1\}$, let $y_i \in \gamma_{xy_1}[y_1, y_0]$ and $z_i \in \gamma_{xz_1}[z_1, y_0]$ be such that

$$\ell(\gamma_{xy_1}[y_1, y_i]) = \ell(\gamma_{xz_1}[z_1, z_i]) = e^{i-1}d_\Omega(y_1). \tag{6.16}$$

Then, for each $i \in \{1, ..., m_1\}$, we have

$$\ell(\gamma_{xy_1}[y_{i+1}, y_i]) = \ell(\gamma_{xy_1}[y_1, y_{i+1}]) - \ell(\gamma_{xy_1}[y_1, y_i]) \overset{(6.16)}{=} (e-1)e^{i-1}d_\Omega(y_1).$$

By Lemma 6.1 and our choice of y_{i+1} in (6.16), it holds

$$\ell(\gamma_{xy_1}[y_1, y_{i+1}]) = e^i d_\Omega(y_1) \leq \kappa_0 d_\Omega(y_{i+1}). \tag{6.17}$$

Thus, combining the above two estimates gives

$$\ell(\gamma_{xy_1}[y_{i+1}, y_i]) \leq (e-1)\kappa_0 d_\Omega(y_i).$$

For each $i \in \{1, ..., m_1\}$, it follows from the above inequality and [15, Lemma 2.1] that

$$k_\Omega(y_i, y_{i+1}) \leq 4\kappa_0 \log\Big(1 + \frac{\ell(\gamma_{xy_1}[y_i, y_{i+1}])}{d_\Omega(y_i)}\Big) \leq 4\kappa_0\log(1 + e\kappa_0). \tag{6.18}$$

Similarly, one can prove that

$$k_\Omega(z_i, z_{i+1}) \leq 4\kappa_0\log(1 + e\kappa_0). \tag{6.19}$$

Based on (6.15), we take $t \in \{2, \cdots, m_1 + 1\}$ to be the smallest integer such that

$$k_\Omega(y_t, z_t) \geq \kappa_2.$$

Then, it follows from (6.18) and (6.19) that

$$\kappa_2 \leq k_\Omega(y_t, z_t) \leq \sum_{i=1}^{t-1} \left(k_\Omega(y_i, y_{i+1}) + k_\Omega(z_i, z_{i+1})\right) + k_\Omega(y, z) \leq 8(t-1)\kappa_0 \log(1 + e\kappa_0) + 1.$$

Thus, we obtain

$$t > \frac{\kappa_2}{8\kappa_0 \log(1 + e\kappa_0)}. \tag{6.20}$$

Now, let $s_1 = t$, and based on (6.20), for each $i \in \{2, \cdots, [\kappa_1^2]\}$, set $s_i = s_{i-1} - [\kappa_1^{2i}]$. For each $i \in \{2, \cdots, [\kappa_1^2]\}$, by our choices of κ_0, κ_1 and κ_2, we have

$$\ell(\gamma_{xy_1}[y_1, y_{s_i}]) \geq d_\Omega(y_{s_i}) - d_\Omega(y_1) \overset{(6.17)}{\geq} (1 - \kappa_0 e^{1-s_i}) d_\Omega(y_{s_i}) > \frac{1}{2} d_\Omega(y_{s_i}).$$

Then

$$\frac{1}{2} e^{[\kappa_1^{2i}]} d_\Omega(y_{s_i}) \leq e^{[\kappa_1^{2i}]} \ell(\gamma_{xy_1}[y_1, y_{s_i}]) \overset{(6.16)}{=} \ell(\gamma_{xy_1}[y_1, y_{s_{i-1}}]) \overset{\text{Lemma 6.1}}{\leq} \kappa_0 d_\Omega(y_{s_{i-1}}), \tag{6.21}$$

and

$$d_\Omega(y_{s_{i-1}}) \leq d_\Omega(y_1) + \ell(\gamma_{xy_1}[y_1, y_{s_{i-1}}]) \overset{(6.16)}{<} 2e^{[\kappa_1^{2i}]} \ell(\gamma_{xy_1}[y_1, y_{s_i}]) \overset{\text{Lemma 6.1}}{\leq} 2\kappa_0 e^{[\kappa_1^{2i}]} d_\Omega(y_{s_i}).$$

It follows from these two estimates that

$$\frac{1}{2\kappa_0} e^{[\kappa_1^{2i}]} d_\Omega(y_{s_i}) \leq d_\Omega(y_{s_{i-1}}) \leq 2\kappa_0 e^{[\kappa_1^{2i}]} d_\Omega(y_{s_i}). \tag{6.22}$$

Similarly, one can prove

$$\frac{1}{2} d_\Omega(y_{s_i}) \leq \ell(\gamma_{xy_1}[y_1, y_{s_i}]) = \ell(\gamma_{xz_1}[z_1, z_{s_i}]) \leq \kappa_0 d_\Omega(z_{s_i}). \tag{6.23}$$

For each $i \in \{2, \cdots, [\kappa_1^2]\}$, let $x_{0,i}$ bisect the length of $\gamma_{y_{s_i} z_{s_i}}$ ($\gamma_{y_{s_i} z_{s_i}} \in \Lambda_{y_{s_i} z_{s_i}}$). Then we have

$$\begin{aligned}
k_\Omega(y_{s_i}, z_{s_i}) &\geq k_\Omega(y_t, z_t) - k_\Omega(y_t, y_{s_i}) - k_\Omega(z_t, z_{s_i}) \\
&\geq k_\Omega(y_t, z_t) - \sum_{l=s_i}^{t-1} \left(k_\Omega(y_{l+1}, y_l) + k_\Omega(z_{l+1}, z_l)\right) \\
&\overset{(6.18)+(6.19)}{\geq} \kappa_2 - 8\kappa_0(t - s_i - 1) \log(1 + e\kappa_0) \\
&> \kappa_2 - \kappa_1^{2[\kappa_1^2]+2},
\end{aligned}$$

where in the last inequality, we used the fact that $t - s_i = \sum_{l=2}^{i} \kappa_1^{2l} < 2\kappa_1^{2i}$.

On the other hand, it follows from Lemma 6.1, [15, Lemma 2.1] and (6.23) that

$$\begin{aligned}
k_\Omega(y_{s_i}, z_{s_i}) &= k_\Omega(y_{s_i}, x_{0,i}) + k_\Omega(x_{0,i}, z_{s_i}) \\
&\leq 4\kappa_0 \left(\log\left(1 + \frac{\ell(\gamma_{y_{s_i} z_{s_i}})}{2d_\Omega(y_{s_i})}\right) + \log\left(1 + \frac{\ell(\gamma_{y_{s_i} z_{s_i}})}{2d_\Omega(z_{s_i})}\right) \right) \\
&\leq 4\kappa_0 \left(\log(1 + \frac{\ell(\gamma_{y_{s_i} z_{s_i}})}{2d_\Omega(y_{s_i})}) + \log\left(1 + \frac{\kappa_0 \ell(\gamma_{y_{s_i} z_{s_i}})}{d_\Omega(y_{s_i})}\right) \right).
\end{aligned}$$

Combining the above two estimates with Lemma 6.1 gives

$$\ell(\gamma_{y_{s_i}z_{s_i}}) > e^{\frac{\kappa_2}{9\kappa_0}} d_\Omega(y_{s_i}) \geq \frac{1}{\kappa_0} e^{\frac{\kappa_2}{9\kappa_0}} \ell(\gamma_{xy_1}[y_1, y_{s_i}]) \geq \frac{1}{\kappa_0} e^{\frac{\kappa_2}{9\kappa_0} - [\kappa_1^{2i}]} \ell(\gamma_{xy_1}[y_1, y_{s_1}]). \tag{6.24}$$

For each $i \in \{2, \cdots, [\kappa_1^2]\}$, by (6.24), we may take $y_{1,i} = \gamma_{y_{s_i}z_{s_i}} \cap \mathbb{S}(y_1, \ell(\gamma_{xy_1}[y_1, y_{s_1}]))$ along the direction from y_{s_i} to z_{s_i}, and $z_{1,i} = \gamma_{y_{s_i}z_{s_i}} \cap \mathbb{S}(y_1, \ell(\gamma_{xy_1}[y_1, y_{s_1}]))$ along the direction from z_{s_i} to y_{s_i}. Then it holds

$$\begin{aligned} \ell(\gamma_{y_{s_i}z_{s_i}}[y_{s_i}, y_{1,i}]) &\leq \ell(\gamma_{xy_1}[y_1, y_{s_i}]) + \ell(\gamma_{xy_1}[y_1, y_{s_1}]) \\ &\overset{s_i<s_1}{<} 2\ell(\gamma_{xy_1}[y_1, y_{s_1}]) \overset{(6.24)}{<} \frac{1}{2}\ell(\gamma_{y_{s_i}z_{s_i}}). \end{aligned} \tag{6.25}$$

Similarly, one can prove

$$\ell(\gamma_{y_{s_i}z_{s_i}}[z_{s_i}, z_{1,i}]) < 2\ell(\gamma_{xy_1}[y_1, y_{s_1}]) < \frac{1}{2}\ell(\gamma_{y_{s_i}z_{s_i}}). \tag{6.26}$$

Note that

$$k_\Omega(z_{s_1}, z_{s_{i-1}}) \leq \sum_{l=s_{i-1}}^{s_1-1} k_\Omega(z_{l+1}, z_l) \overset{(6.19)}{\leq} 4\kappa_0\kappa_1^{2(i-1)} \log(1+e\kappa_0) \tag{6.27}$$

and

$$k_\Omega(y_{s_1}, y_{s_{i-1}}) \leq \sum_{l=s_{i-1}}^{s_1-1} k_\Omega(y_{l+1}, y_l) \overset{(6.18)}{\leq} 4\kappa_0\kappa_1^{2(i-1)} \log(1+e\kappa_0) \tag{6.28}$$

Thus, we obtain from Lemma 6.1, [15, Lemma 2.1] and (6.25) that

$$\begin{aligned} k_\Omega(y_{s_{i-1}}, y_{1,i-1}) &\leq 4\kappa_0 \log\left(1 + \frac{\ell(\gamma_{y_{s_{i-1}}z_{s_{i-1}}}[y_{s_{i-1}}, y_{1,i-1}])}{d_\Omega(y_{s_{i-1}})}\right) \\ &\leq 4\kappa_0 \log\left(1 + \frac{\ell(\gamma_{xy_1}[y_1, y_{s_1}])}{d_\Omega(y_{s_{i-1}})}\right) < 8\kappa_0\kappa_1^{2(i-1)}. \end{aligned}$$

On the other hand, it follows from (2.1) and Lemma 6.1 that

$$\begin{aligned} k_\Omega(y_{s_i}, y_{1,i}) &\geq \log\frac{\ell(\gamma_{y_{s_i}z_{s_i}}[y_{s_i}, y_{1,i}])}{d_\Omega(y_{s_i})} \geq \log\frac{\kappa_0(\ell(\gamma_{xy_1}[y_1, y_{s_1}]) - \ell(\gamma_{xy_1}[y_1, y_{s_i}]))}{\ell(\gamma_{xy_1}[y_1, y_{s_i}])} \\ &> \kappa_1^{2i}. \end{aligned}$$

The above two inequalities yield that

$$k_\Omega(y_{s_i}, y_{1,i}) > k_\Omega(y_{s_{i-1}}, y_{1,i-1}) + (\kappa_1 - 9\kappa_0)\kappa_1^{2i-1}. \tag{6.29}$$

A similar discussion as in (6.29) shows that

$$k_\Omega(z_{s_i}, z_{1,i}) > k_\Omega(z_{s_{i-1}}, z_{1,i-1}) + (\kappa_1 - 9\kappa_0)\kappa_1^{2i-1}. \tag{6.30}$$

Let $B_0 = \mathbb{B}(y_1, 3\ell(\gamma_{xy_1}[y_1, y_{s_1}]))$. For each $t \in \{1, \ldots, [\kappa_1^2]\}$, we take

$$B_t = \mathbb{B}\left(y_{1,i}, \frac{1}{2\kappa_0}\ell(\gamma_{xy_1}[y_1, y_{s_1}])\right) \text{ and } B_{1,t} = \mathbb{B}\left(z_{1,i}, \frac{1}{2\kappa_0}\ell(\gamma_{xy_1}[y_1, y_{s_1}])\right)$$

For each $u \in (\overline{B_t} \cup \overline{B_{1,t}})$, by (6.25) and (6.26),

$$\sigma(y_1, u) < 3\ell(\gamma_{xy_1}[y_1, y_{s_1}]),$$

and so

$$(\overline{B_t} \cup \overline{B_{1,t}}) \subset B_0.$$

Then applying Lemma 2.3 with $R = 3\ell(\gamma_{xy_1}[y_1, y_{s_1}])$ and $r = \frac{1}{6\kappa_0}R$ gives that there must exist $p < q \in \{2, \cdots, [\kappa_1^2]\}$ such that

$$\mathbb{B}(y_{1,p}, r) \cap \mathbb{B}(y_{1,q}, r) \neq \emptyset \text{ and } \mathbb{B}(z_{1,p}, r) \cap \mathbb{B}(z_{1,q}, r) \neq \emptyset,$$

and so by Lemma 2.1,

$$\max\{k_\Omega(y_{1,p}, y_{1,q}), k_\Omega(z_{1,p}, z_{1,q})\} < \frac{20}{9},$$

which, together with (6.29) and (6.30) , yields

$$\begin{aligned} k_\Omega(y_{s_q}, z_{s_q}) &> k_\Omega(y_{s_p}, y_{1,p}) + k_\Omega(y_{1,q}, z_{1,q}) + k_\Omega(z_{1,p}, z_{s_p}) + 2(\kappa_1 - 9\kappa_0)\kappa_1^{2q-1} \\ &> k_\Omega(y_{s_p}, y_{1,p}) + k_\Omega(y_{1,p}, z_{1,p}) + k_\Omega(z_{1,p}, z_{s_p}) + 2(\kappa_1 - 9\kappa_0)\kappa_1^{2q-1} - \frac{40}{9} \\ &= k_\Omega(y_{s_p}, z_{s_p}) + 2(\kappa_1 - 9\kappa_0)\kappa_1^{2q-1} - \frac{40}{9}, \end{aligned}$$

and so (6.27) shows that

$$\begin{aligned} k_\Omega(y_{s_q}, z_{s_q}) &> k_\Omega(y_{s_p}, z_{s_p}) + k_\Omega(y_{s_1}, y_{s_p}) + k_\Omega(z_{s_1}, z_{s_p}) \\ &+ 2(\kappa_1 - 9\kappa_0)\kappa_1^{2q-1} - 8\kappa_0\kappa_1^{2(p-1)}\log(1 + e\kappa_0) - \frac{40}{9} \\ &> k_\Omega(y_{s_1}, z_{s_1}) = k_\Omega(y_t, z_t), \end{aligned}$$

which contradicts with the construction of y_t. Hence Claim 6.1 holds.

Then by Claim 6.1,

$$\begin{aligned} k_\Omega(z_{1,m_1+1}, z_{m_1+1}) &\le k_\Omega(y_{m_1+1}, y_{1,m_1+1}) + k_\Omega(y_{1,m_1+1}, z_{1,m_1+1}) + k_\Omega(y_{m_1+1}, z_{m_1+1}) \\ &\le k_\Omega(y_{m_1+1}, y_{1,m_1+1}) + 4\kappa_2. \end{aligned} \tag{6.31}$$

Also we know from Lemma 6.1 and [15, Lemma 2.1] that

$$\begin{aligned} k_\Omega(y_{m_1+1}, y_{1,m_1+1}) &= k_\Omega(y_{m_1+1}, y_0) + k_\Omega(y_{1,m_1+1}, y_0) \\ &\le 4\kappa_0 \log\left(1 + \frac{\ell(\gamma_{xy_1}[y_{m_1+1}, y_0])}{d_\Omega(y_{m_1+1})}\right) + 4\kappa_0 \log\left(1 + \frac{\ell(\gamma_{xy_1}[y_{1,m_1+1}, y_0])}{d_\Omega(y_{1,m_1+1})}\right) \\ &< 8\kappa_0 \log(1 + e^2\kappa_0), \end{aligned}$$

which, together with (2.1) and (6.31), shows that

$$\log\left(1 + \frac{\ell(\gamma_{xz_1}[z_{1,m_1+1}, z_{m_1+1}])}{d_\Omega(z_{m_1+1})}\right) \le k_\Omega(z_{1,m_1+1}, z_{m_1+1}) < 5\kappa_2.$$

Hence, by (6.10) and (6.14),

$$\begin{aligned} \ell(\gamma_{xz_1}[z_{1,m_1+1}, z_{m_1+1}]) &< e^{5\kappa_2} d_\Omega(z_{m_1+1}) \\ &< e^{5\kappa_2}(d_\Omega(z_1) + \ell(\gamma_{xy_1}[y_1, y_{m_1+1}])) < e^{5\kappa_2}\ell(\gamma_{xy}). \end{aligned}$$

Finally, we get from (6.14) that

$$\begin{aligned} \ell(\gamma_{xz_1}) &= \ell(\gamma_{xz_1}[x, z_{1,m_1+1}]) + \ell(\gamma_{xz_1}[z_{1,m_1+1}, z_{m_1+1}]) + \ell(\gamma_{xz_1}[y_1, z_{m_1+1}]) \\ &< e^{5\kappa_2}\ell(\gamma_{xy}). \end{aligned}$$

The proof of Lemma 6.2 is thus complete. □

Proof of Theorem 1.12. We shall prove (6.1), via a contradiction argument, that for any $\gamma_{xy} \in \Lambda_{xy}(\Omega)$, it holds

$$\ell(\gamma_{xy}) \leq e^{19\kappa_0\kappa_1\kappa_2}\ell(\alpha_{xy}),$$

where $\alpha_{xy} \in \Gamma_{xy}(\Omega)$.

Suppose on the contrary that there exists some $\gamma_0 \in \Lambda_{xy}$ such that

$$\ell(\gamma_0) > e^{19\kappa_0\kappa_1\kappa_2}\ell(\alpha_{xy}).$$

Then it follows from Lemma 6.2 that for any $\gamma \in \Lambda_{xy}$,

$$\ell(\gamma) \geq e^{-9\kappa_0\kappa_2}\ell(\gamma_0) > e^{10\kappa_0\kappa_1\kappa_2}\ell(\alpha_{xy}). \tag{6.32}$$

Let $x_0 = x$ and let x_1 be the last point on α along the direction from x to y such that for some $\gamma_1 \in \Lambda_{xx_1}(\Omega)$, it holds

$$\ell(\gamma_1) \leq \ell(\alpha_{xy}).$$

Based on Lemma 6.2 and (6.32), we may repeat the above procedure to find $[\kappa_1] - 1$ successive points $x_i \in \alpha$ such that for each $i \in \{1, \cdots, [\kappa_1] - 1\}$, x_i is the last point in $\alpha[x_{i-1}, y]$ along the direction from x_{i-1} to y which satisfies for some $\gamma_i \in \Lambda_{xx_i}(\Omega)$, it holds

$$\ell(\gamma_i) \leq e^{10\kappa_0\kappa_2(i-1)}\ell(\alpha_{xy}). \tag{6.33}$$

For $i \in \{2, \cdots, [\kappa_1] - 1\}$, by (6.33), we may take $x_{0,i} \in \gamma_i$ to be the first point along the direction from x_i to x such that

$$\ell(\gamma_i[x_i, x_{0,i}]) = \ell(\alpha_{xy}). \tag{6.34}$$

Then, it follows from Lemma 6.1 and (6.33) that

$$\min\{\ell(\gamma_i[x_i, x_{0,i}]), \ell(\gamma_i[x, x_{0,i}])\} = \ell(\gamma_i[x_i, x_{0,i}]) = \ell(\alpha_{xy}) \leq \kappa_0 d_\Omega(x_{0,i}). \tag{6.35}$$

Let $B_{0,0} = \mathbb{B}(x, 3\ell(\alpha_{xy}))$. For each $i \in \{1, \ldots, [\kappa_1] - 1\}$, set

$$B_{1,i} = \mathbb{B}\left(x_{0,i}, \frac{1}{5\kappa_0}\ell(\alpha_{xy})\right).$$

Then (6.35) implies that $\overline{B_{1,i}} \subset \mathbb{B}(x_{0,i}, \frac{1}{2}d_\Omega(x_{0,i})) \subset \Omega$. For each $u \in \overline{B_{1,i}}$, we have

$$d(x, u) \leq d(x, x_{0,i}) + d(x_{0,i}, u) \leq \sigma(x, x_i) + d(x_i, x_{0,i}) + d(x_{0,i}, u) \overset{(6.34)}{\leq} (2 + \frac{1}{5\kappa_0})\ell(\alpha),$$

and so

$$\overline{B_{1,i}} \subset B_{0,0}.$$

Then, by Lemma 2.3, there exist two indices $s < t \in \{2, \cdots, [\kappa_1] - 1\}$ such that

$$\overline{B_{1,s}} \cap \overline{B_{1,t}} \neq \emptyset$$

and thus

$$d(x_{0,t}, x_{0,s}) \leq \frac{2}{5\kappa_0}\ell(\alpha) \overset{(6.35)}{\leq} \frac{2}{5}d_\Omega(x_{0,t}).$$

Applying Lemma 2.1 with $a = \frac{5}{2}$ gives

$$k_\Omega(x_{0,s}, x_{0,t}) \leq \frac{20}{27} < 1,$$

which, together with Lemma 6.2, shows that

$$\ell(\gamma_t[x, x_{0,t}]) \leq e^{9\kappa_0\kappa_2}\ell(\gamma_s[x, x_{0,s}]).$$

It follows from the above estimate and (6.34) that

$$\ell(\gamma_t) = \ell(\gamma_t[x, x_{0,t}]) + \ell(\gamma_t[x_t, x_{0,t}]) \le e^{9\kappa_0\kappa_2}\ell(\gamma_s[x, x_{0,s}]) + \ell(\gamma_s[x_s, x_{0,s}]) \le e^{9\kappa_0\kappa_2}\ell(\gamma_s).$$

On the other hand, since $s < t$, Lemma 6.2 and (6.33) imply

$$\ell(\gamma_t) \ge e^{10\kappa_0\kappa_2(t-1)-9\kappa_0\kappa_2}\ell(\alpha_{xy}) \ge e^{9\kappa_0\kappa_2}e^{10\kappa_0\kappa_2(s-1)}\ell(\alpha_{xy}) > e^{9\kappa_0\kappa_2}\ell(\gamma_s).$$

The above two estimates clearly contradicts with each other. Consequently, it holds

$$\ell(\gamma) \le e^{19\kappa_0\kappa_1\kappa_2}\ell(\alpha).$$

The proof of Theorem 1.12 is thus complete. □

7. Ball separation condition alone does not imply Gehring-Hayman inequality

For simplicity, we shall only present the example of Theorem 1.7 in $\mathbb{R}^2$. It will be clear from our construction that it extends to all dimensions with straightforward modifications.

7.1. **Construction of the domains.** First we let $n > 5$ be an integer, $x_0 = (8, 0)$ and $z_0 = (-8, 0)$.

Step 1. The construction of points sequence $\{x_i\}_{i=1}^{2n+1}$, $\{x_{1,i}\}_{i=1}^{2n+1}$ and $\{x_{2,i}\}_{i=1}^{2n+1}$.

Let $x_1 = 8(\cos\frac{11\pi}{24}, \sin\frac{11\pi}{24})$ and $x_{1,1} = 8^{\frac{3}{2}}(\cos\frac{\pi}{24}, \sin\frac{\pi}{24})$. For each $i \in \{1, \cdots, n\}$, we take $x_{2i} = 3\cdot 8^{(\frac{3}{2})^i}(\cos\frac{11\pi}{24}, \sin\frac{11\pi}{24})$, $x_{1,2i} = (8^{(\frac{3}{2})^{1+i}}\cdot\cos\frac{\pi}{24}, 3\cdot 8^{(\frac{3}{2})^i}\sin\frac{11\pi}{24} + 8^{(\frac{3}{2})^{1+i}}\cdot\sin\frac{\pi}{24})$, $x_{2i+1} = 6\cdot 8^{(\frac{3}{2})^i}(\cos\frac{11\pi}{24}, \sin\frac{11\pi}{24})$ and $x_{1,2i+1} = (\frac{3\sqrt{3}}{2}\cdot 8^{(\frac{3}{2})^{1+i}}, 6\cdot 8^{(\frac{3}{2})^i}\sin\frac{11\pi}{24} + \frac{3}{2}\cdot 8^{(\frac{3}{2})^{1+i}})$, and let $x_{2,2i+1} = (\frac{3\sqrt{3}}{2}\cdot 8^{(\frac{3}{2})^{1+i}}, \frac{3\sqrt{3}}{2}\cdot 8^{(\frac{3}{2})^{1+i}}\cdot\tan\frac{\pi}{24})$.

Step 2. The construction of points sequence $\{y_i\}_{i=1}^{2n+1}$, $\{y_{1,i}\}_{i=1}^{2n+1}$ and $\{y_{2,i}\}_{i=1}^{2n+1}$.

Let $y_1 = 8(\cos\frac{11\pi}{24}, -\sin\frac{11\pi}{24})$ and $y_{1,1} = 8^{\frac{3}{2}}(\cos\frac{\pi}{24}, -\sin\frac{\pi}{24})$. For each $i \in \{1, \cdots, n\}$, we take $y_{2i} = 3\cdot 8^{(\frac{3}{2})^i}(\cos\frac{11\pi}{24}, -\sin\frac{11\pi}{24})$, $y_{1,2i} = (8^{(\frac{3}{2})^{1+i}}\cdot\cos\frac{\pi}{24}, -3\cdot 8^{(\frac{3}{2})^i}\sin\frac{11\pi}{24} - 8^{(\frac{3}{2})^{1+i}}\cdot\sin\frac{\pi}{24})$, $y_{2i+1} = 6\cdot 8^{(\frac{3}{2})^i}(\cos\frac{11\pi}{24}, -\sin\frac{11\pi}{24})$ and $y_{1,2i+1} = (\frac{3\sqrt{3}}{2}\cdot 8^{(\frac{3}{2})^{1+i}}, -6\cdot 8^{(\frac{3}{2})^i}\sin\frac{11\pi}{24} - \frac{3}{2}\cdot 8^{(\frac{3}{2})^{1+i}})$, and let $y_{2,2i+1} = (\frac{3\sqrt{3}}{2}\cdot 8^{(\frac{3}{2})^{1+i}}, -\frac{3\sqrt{3}}{2}\cdot 8^{(\frac{3}{2})^{1+i}}\cdot\tan\frac{\pi}{24})$.

Step 3. The construction of points sequence $\{z_i\}_{i=1}^{2n+1}$, $\{z_{1,i}\}_{i=1}^{2n+1}$ and $\{z_{2,i}\}_{i=1}^{2n+1}$.

Let $z_1 = 8(-\cos\frac{11\pi}{24}, \sin\frac{11\pi}{24})$ and $z_{1,1} = 8^{\frac{3}{2}}(-\cos\frac{\pi}{24}, \sin\frac{\pi}{24})$. For each $i \in \{1, \cdots, n\}$, we take $z_{2i} = 3\cdot 8^{(\frac{3}{2})^i}(-\cos\frac{11\pi}{24}, \sin\frac{11\pi}{24})$, $z_{1,2i} = (-8^{(\frac{3}{2})^{1+i}}\cdot\cos\frac{\pi}{24}, 3\cdot 8^{(\frac{3}{2})^i}\sin\frac{11\pi}{24} + 8^{(\frac{3}{2})^{1+i}}\cdot\sin\frac{\pi}{24})$, $z_{2i+1} = 6\cdot 8^{(\frac{3}{2})^i}(-\cos\frac{11\pi}{24}, \sin\frac{11\pi}{24})$ and $z_{1,2i+1} = (-\frac{3\sqrt{3}}{2}\cdot 8^{(\frac{3}{2})^{1+i}}, 6\cdot 8^{(\frac{3}{2})^i}\sin\frac{11\pi}{24} + \frac{3}{2}\cdot 8^{(\frac{3}{2})^{1+i}})$, and let $z_{2,2i+1} = (-\frac{3\sqrt{3}}{2}\cdot 8^{(\frac{3}{2})^{1+i}}, \frac{3\sqrt{3}}{2}\cdot 8^{(\frac{3}{2})^{1+i}}\cdot\tan\frac{\pi}{24})$.

Step 4. The construction of points sequence $\{w_i\}_{i=1}^{2n+1}$, $\{w_{1,i}\}_{i=1}^{2n+1}$ and $\{w_{2,i}\}_{i=1}^{2n+1}$.

Let $w_1 = 8(-\cos\frac{11\pi}{24}, -\sin\frac{11\pi}{24})$ and $w_{1,1} = 8^{\frac{3}{2}}(-\cos\frac{\pi}{24}, -\sin\frac{\pi}{24})$. For each $i \in \{1, \cdots, n\}$, we take $w_{2i} = 3\cdot 8^{(\frac{3}{2})^i}(-\cos\frac{11\pi}{24}, -\sin\frac{11\pi}{24})$, $w_{1,2i} = (-8^{(\frac{3}{2})^{1+i}}\cdot\cos\frac{\pi}{24}, -3\cdot 8^{(\frac{3}{2})^i}\sin\frac{11\pi}{24} - 8^{(\frac{3}{2})^{1+i}}\cdot\sin\frac{\pi}{24})$, $w_{2i+1} = 6\cdot 8^{(\frac{3}{2})^i}(-\cos\frac{11\pi}{24}, -\sin\frac{11\pi}{24})$ and $w_{1,2i+1} = (-\frac{3\sqrt{3}}{2}\cdot 8^{(\frac{3}{2})^{1+i}}, -6\cdot 8^{(\frac{3}{2})^i}\sin\frac{11\pi}{24} - \frac{3}{2}\cdot 8^{(\frac{3}{2})^{1+i}})$, and let $w_{2,2i+1} = (-\frac{3\sqrt{3}}{2}\cdot 8^{(\frac{3}{2})^{1+i}}, -\frac{3\sqrt{3}}{2}\cdot 8^{(\frac{3}{2})^{1+i}}\cdot\tan\frac{\pi}{24})$.

Construction of domain D (See Figure 3). First we take $x_{2,1} = x_{1,1}$, $z_{2,1} = z_{1,1}$ and $w_{2,1} = w_{1,1}$. For each $i \in \{0, \cdots, n\}$, let $L_{1,i} = [z_{1,1+4i}, z_{1,2+4i}]$, $L_{2,i} = [w_{1,3+4i}, w_{1,4(i+1)}]$, $L_i^{xy} = [x_{2,3+4i}, y_{2,3+4i}]$ and $L_i^{zw} = [z_{2,3+4i}, w_{2,3+4i}]$. For each $i \in \{0, \cdots, n\}$, let $D_{1,i}$ be a domain bounded by $[x_{2+4i}, x_{1,2+4i}]$, $[x_{1,1+4i}, x_{1,2+4i}]$, $[x_{1,1+4i}, x_{2,1+4i}]$, $[x_{2,1+4i}, x_{2,3+4i}]$, $[x_{2,3+4i}, x_{1,3+4i}]$, $[x_{1,3+4i}, x_{1,4(i+1)}]$, $[x_{2+4i}, x_{4(i+1)}]$ and $[x_{4(i+1)}, x_{1,4(i+1)}]$; $D_{2,i}$ be a domain bounded by $[y_{3+4i}, y_{6+4i}]$, $[y_{3+4i}, y_{1,3+4i}]$, $[y_{1,3+4i}, y_{2,3+4i}]$, $[y_{2,3+4i}, y_{2,5+4i}]$, $[y_{6+4i}, y_{1,6+4i}]$, $[y_{1,6+4i}, y_{1,5+4i}]$ and $[y_{1,5+4i}, y_{2,5+4i}]$;

$D_{3,i}$ be a domain bounded by $[z_{4(i+1)}, z_{1,4(i+1)}]$, $[z_{1,4(i+1)}, z_{1,3+4i}]$, $[z_{1,3+4i}, z_{2,3+4i}]$, $[z_{2,3+4i}, z_{2,5+4i}]$, $[z_{4(i+1)}, z_{5+4i}]$, $[z_{5+4i}, z_{1,5+4i}]$ and $[z_{1,5+4i}, z_{2,5+4i}]$, and $D_{4,i}$ be a domain bounded by $[w_{1,1+4i}, w_{1,2+4i}]$, $[w_{1,2+4i}, w_{2+4i}]$, $[w_{1,1+4i}, w_{2,1+4i}]$, $[w_{2+4i}, w_{3+4i}]$, $[w_{3+4i}, w_{1,3+4i}]$, $[w_{1,3+4i}, w_{2,3+4i}]$ and $[w_{2,1+4i}, w_{2,3+4i}]$.

Then we take $\mathcal{D} = \cup_{i=1}^{n} \cup_{j=1}^{4} \{D_{j,i}\}$ and

$$D_n = \mathbb{R}^2 \backslash (\mathcal{D} \cup (\cup_{i=1}^{n}\{L_{1,i}\}) \cup (\cup_{i=1}^{n}\{L_{2,i}\}) \cup (\cup_{i=1}^{n}\{L_i^{xy}\}) \cup (\cup_{i=1}^{n}\{L_i^{zw}\})).$$

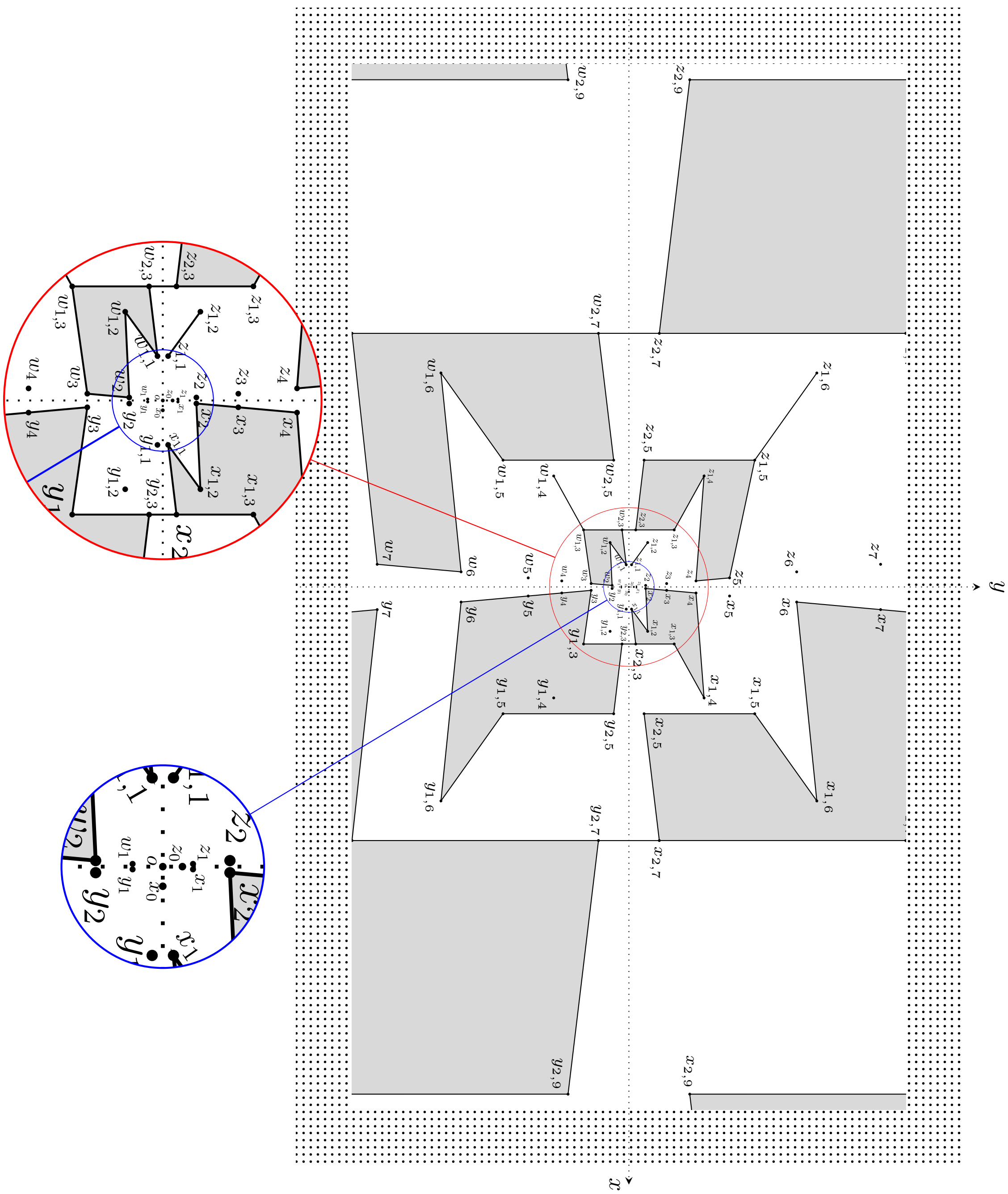

FIGURE 3. Construction of domain D_n

7.2. **Proof of Theorem 1.7.** Set

$$L_1 = \left\{(x,y) : y = \tan\frac{\pi}{24}(x - 8\cos\frac{11\pi}{24}) + 8\sin\frac{11\pi}{24}, \text{where } \ x \geq 8\cos\frac{11\pi}{24}\right\},$$

$$L_2 = \left\{(x,y) : y = \tan\frac{11\pi}{24}(x - 8\cos\frac{11\pi}{24}) + 8\sin\frac{11\pi}{24}, \text{where } \ x \geq 8\cos\frac{11\pi}{24}\right\},$$

$$L_3 = \left\{(x,y) : y = -\tan\frac{11\pi}{24}(x + 8\cos\frac{11\pi}{24}) + 8\sin\frac{11\pi}{24}, \text{where } \ x \leq -8\cos\frac{11\pi}{24}\right\},$$

$$L_4 = \left\{(x,y) : y = -\tan\frac{\pi}{24}(x + 8\cos\frac{11\pi}{24}) + 8\sin\frac{11\pi}{24}, \text{where } \ x \leq -8\cos\frac{11\pi}{24}\right\},$$

$$L_5 = \left\{(x,y) : y = \tan\frac{\pi}{24}(x + 8\cos\frac{11\pi}{24}) - 8\sin\frac{11\pi}{24}, \text{where } \ x \leq -8\cos\frac{11\pi}{24}\right\},$$

$$L_6 = \left\{(x,y) : y = \tan\frac{11\pi}{24}(x + 8\cos\frac{11\pi}{24}) - 8\sin\frac{11\pi}{24}, \text{where } \ x \leq -8\cos\frac{11\pi}{24}\right\},$$

$$L_7 = \left\{(x,y) : y = -\tan\frac{11\pi}{24}(x - 8\cos\frac{11\pi}{24}) - 8\sin\frac{11\pi}{24}, \text{where } \ x \geq 8\cos\frac{11\pi}{24}\right\}$$

and

$$L_8 = \left\{(x,y) : y = -\tan\frac{\pi}{24}(x - 8\cos\frac{11\pi}{24}) - 8\sin\frac{11\pi}{24}, \text{where } \ x \geq 8\cos\frac{11\pi}{24}\right\}.$$

The definitions of L_0 and $\mathcal{C}_i$: Let $\mathcal{C}_1$ be a domain bounded by L_2, L_3 and the segment $[x_1, z_1]$, $\mathcal{C}_2$ be a domain bounded by L_4, L_5 and the segment $[z_1, w_1]$, $\mathcal{C}_4$ be a domain bounded by L_6, L_7 and the segment $[w_1, y_1]$, and $\mathcal{C}_3$ be a domain bounded by L_1, L_8 and the segment $[x_1, y_1]$. Then take $L_0 = \{(x,y) : x = 0 \text{ and } y \geq 8\sin\frac{11\pi}{24}\}$.

Claim 7.1. *Suppose $i \in \{1, \cdots, 4\}$ and $u_1, v_1 \in \mathcal{C}_i$. Then*

(1) *There exists a curve γ joining u_1 and v_1 in $\mathcal{C}_i$ such that for each $u, v \in \gamma$,*

$$\ell(\gamma[u,v]) \leq 6|u - v|,$$

and for each $z \in \gamma$,

$$\min\{\ell(\gamma[u_1, z]), \ell(\gamma[v_1, z])\} < 2(\sin\frac{\pi}{24})^{-1} d_{\mathcal{C}_i}(z) \leq 2(\sin\frac{\pi}{24})^{-1} d_{D_n}(z).$$

(2) *If $u_1, v_1 \in L_0$ and $|v_1| > |u_1|$, then for each $z \in [u_1, v_1]$,*

$$d_{\mathcal{C}_i}(z) = d_{\mathcal{C}_i}(u_1) + \sin\frac{\pi}{24} \cdot |u_1 - z|.$$

Proof of Claim 7.1. (1). We only prove the case $i = 1$, since the proofs of other cases are similar. No loss of generality, we may assume that

$$|v_1| \geq |u_1|.$$

Set $u_0 = (0, 0)$, and let u_2 and v_2 be two points on L_0 with

$$\angle u_1 u_2 u_0 = \angle v_1 v_2 u_0 = \frac{11\pi}{24}. \tag{7.1}$$

Next, we consider two cases.

Case A. $[u_1, v_1] \cap L_0 \neq \emptyset$.

Let $\gamma = [u_1, u_2] \cup [u_2, v_2] \cup [v_1, v_2]$. Then, we obtain from (7.1) that

$$\ell(\gamma) = |u_1 - u_2| + |u_2 - v_2| + |v_1 - v_2| < 3|u_1 - v_1|.$$

Similarly, for each $u, v \in \gamma$,
$$\ell(\gamma[u,v]) \le 3|u-v|$$
and for each $z \in [u_1,u_2] \cup [v_1,v_2]$, it holds
$$\min\{\ell(\gamma[u_1,z]), \ell(\gamma[v_1,z])\} \le d_{\mathcal{C}_1}(z) \le d_{D_n}(z).$$
For each $z \in [u_2,v_2]$, we infer from (7.1) that
$$\max\left\{|u_1-u_2|, \sin\frac{\pi}{24}\cdot|u_2-z|\right\} \le d_{\mathcal{C}_1}(z) \le d_{D_n}(z),$$
and so
$$\ell(\gamma[u_1,z]) = |u_1-u_2| + |u_2-z| < 2(\sin\frac{\pi}{24})^{-1} d_{\mathcal{C}_1}(z) \le 2(\sin\frac{\pi}{24})^{-1} d_{D_n}(z).$$
Hence, Claim 7.1 (1) holds in this case.

Case B. $[u_1,v_1] \cap L_0 = \emptyset$.

Let $u_3 \in [u_1,u_2]$ and $v_3 \in [v_1,v_2]$ be such that
$$|u_1-u_3| = \min\left\{\frac{1}{3}|u_1-v_1|, d_{\mathcal{C}_1}(u_1)\right\} \text{ and } |v_1-v_3| = \min\left\{\frac{1}{3}|u_1-v_1|, d_{\mathcal{C}_1}(v_1)\right\}.$$
Then
$$\begin{aligned}\frac{5}{3}|u_1-v_1| &\ge |u_1-v_1| + |u_1-u_3| + |v_1-v_3| \ge |u_3-v_3|\\ &\ge |u_1-v_1| - |u_1-u_3| - |v_1-v_3| \ge \frac{1}{3}|u_1-v_1|,\end{aligned}$$
which, together with
$$\min\{|u_1-v_3|, |u_3-v_1|\} \ge \frac{2}{3}|u_1-v_1|,$$
implies that
$$\min\{\sin\angle u_1u_3v_3, \sin\angle v_1v_3u_3\} \ge \frac{2}{5}. \tag{7.2}$$
Let $\gamma = [u_1,u_3]\cup[u_3,v_3]\cup[v_1,v_3]$. Then a similar discussion as in Case A, using (7.2), shows that for each $u, v \in \gamma$, it holds
$$\ell(\gamma[u,v]) \le 6|u-v|.$$
Moreover, for each $z \in [u_3,v_3]$, it holds
$$d_{\mathcal{C}_1}(z) > \sin\frac{\pi}{24}\cdot(|u_3-z| + d_{\mathcal{C}_1}(u_3)).$$
Thus, for each $z \in \gamma$, we have
$$\ell(\gamma[u_1,z]) = |u_1-u_3| + |u_3-z| < (\sin\frac{\pi}{24})^{-1} d_{\mathcal{C}_1}(z).$$
Hence, The proof of Claim 7.1 (1) is complete.

(2). This follows from an elementary calculation and thus is omitted here. □

The definitions of $\mathcal{D}_{1,i}$ and $\triangle_{1,i}$: For each $i \in \{1,\cdots,n\}$, let $x_{2,2+4i} \in [x_{2+4i}, x_{1,2+4i}]$ with $\angle x_{1,1+4i}x_{2,2+4i}x_{1,2+4i} = \frac{17\pi}{24}$. We let $\mathcal{D}_{1,i}$ be a domain bounded by $[z_{4i}, z_{2+4i}]$, $[z_{4i}, x_{4i}]$, $[x_{4i}, x_{1,4i}]$, $[x_{1,4i}, x_{1,4i-1}]$, $[x_{1,4i-1}, y_{2,4i-1}]$, $[y_{2,4i-1}, y_{2,1+4i}]$, $[y_{2,1+4i}, x_{1,1+4i}]$, $[x_{1,1+4i}, x_{1,2+4i}]$, $[z_{2+4i}, x_{2+4i}]$, $[x_{2+4i}, x_{1,2+4i}]$ and $[x_{4i}, x_{2+4i}]$, and $\triangle_{1,i}$ be a domain bounded by $[x_{1,1+4i}, x_{2,2+4i}]$, $[x_{2,2+4i}, x_{1,2+4i}]$ and $[x_{1,1+4i}, x_{1,2+4i}]$.

Claim 7.2. *Suppose $i \in \{1,\cdots,n\}$ and $u_1, v_1 \in \mathcal{D}_{1,i}$.*

(1) *If $u_1, v_1 \in \mathcal{D}_{1,i}\backslash\triangle_{1,i}$, then there exists some curve γ joining u_1 and v_1 in $\mathcal{D}_{1,i}\backslash\triangle_{1,i}$ such that for each $u, v \in \gamma$,*

$$\ell(\gamma[u,v]) \leq 6\sigma(u,v),$$

and for each $z \in \gamma$,

$$\min\{\ell(\gamma[u_1,z]), \ell(\gamma[v_1,z])\} < 2(\sin\frac{\pi}{24})^{-1} d_{\mathcal{D}_{1,i}}(z).$$

(2) *If $u_1 \in \triangle_{1,i}$, $v_1 \in \mathcal{D}_{1,i}$ and $\gamma_{u_1v_1} \in \Lambda_{u_1v_1}$, then for each $z \in \gamma_{u_1v_1}$, there exists some curve α, joining u_1 and v_1 in $\mathcal{D}_{1,i}$, such that*

$$\mathbb{B}\left(z, e^{(4(\sin\frac{\pi}{24})^{-1})^8} d_{D_n}(z)\right) \cap \alpha \neq \emptyset.$$

Proof of Claim 7.2. (1). By Claim 7.1, it suffices to consider Claim 7.2 (1) in the following three cases:

(1) $u_1 \in \mathcal{D}_{1,i} \cap \mathcal{C}_1$ and $v_1 \in \mathcal{D}_{1,i}\backslash\{\mathcal{C}_1, \mathcal{C}_4\}$.
(2) $u_1, v_1 \in \mathcal{D}_{1,i}\backslash\{\mathcal{C}_1, \mathcal{C}_4\}$.
(3) $u_1 \in \mathcal{D}_{1,i}\backslash\{\mathcal{C}_1, \mathcal{C}_4\}$ and $v_1 \in \mathcal{D}_{1,i} \cap \mathcal{C}_4$.

As the proof for each cases is similar, we only prove (1) and consider two cases.

Case A. $\sigma(u_1, v_1) \leq \frac{3}{4}|x_{4i} - x_{1,4i}|$.

Select $u_{1,1} \in [x_{4,i}, x_{2+4i}]$ with $\angle u_1u_{1,1}x_{2+4i} = \frac{\pi}{2}$, and $v_{1,1} \in [x_{4i}, x_{1+4i}]$ with $\angle v_1v_{1,1}x_{1,4i} = \frac{\pi}{2}$. Let $L_{x_{2,2+4i}}$ be a line through points x_{2+4i} and $x_{2,2+4i}$, and choose $v_{1,2} \in L_{x_{2,2+4i}}$ with $\angle v_{1,2}v_{1,1}x_{1,4i} = \frac{\pi}{2}$. Then take $v_{1,3} \in [v_{1,1}, v_{1,2}]$ with $|v_{1,1} - v_{1,3}| = 2|x_{4i} - x_{1,4i}|$.

If $|u_1 - x_{4i}| \leq |x_{4i} - x_{1+4i}|$, then select $u_{1,4} \in [v_{1,1}, v_{1,3}]$ with $\angle x_{4i}x_{1+4i}u_{1,4} = \frac{17\pi}{24}$, and $u_{1,2} \in [x_{1+4i}, u_{1,4}]$ with $\angle x_{1+4i}u_{1,1}u_{1,2} = \frac{5\pi}{24}$. At the same time, choose $u_{1,3} \in [x_{1+4i}, x_{2+4i}]$ with $\angle x_{1+4i}u_{1,3}v_{1,3} = \frac{17\pi}{24}$, and $u_{1,4} \in [u_{1,3}, v_{1,3}]$ with $\angle v_{1,4}u_{1,2}v_{1,4} = \frac{5\pi}{12}$. Then set

$$\gamma = [u_1, u_{1,1}] \cup [u_{1,1}, u_{1,2}] \cup [u_{1,2}, u_{1,3}] \cup [u_{1,3}, v_{1,3}] \cup [v_1, v_{1,3}].$$

If $|u_1 - x_{4i}| > |x_{4i} - x_{1+4i}|$, then select $u_{1,2} \in [x_{1+4i}, x_{2+4i}]$ with $|x_{1+4i} - u_{1,2}| = 2|u_1 - v_1|$, and $u_{1,3} \in [v_{1,1}, v_{1,2}]$ with $\angle x_{1+4i}u_{1,2}u_{1,3} = \frac{17\pi}{24}$. Then, set

$$\gamma = [u_1, u_{1,1}] \cup [u_{1,1}, u_{1,2}] \cup [u_{1,2}, u_{1,3}] \cup [u_{1,3}, v_1].$$

In either cases, a similar proof as in Claim 7.1 (1) implies that γ satisfies Claim 7.2 (1).

Case B. $\sigma(u_1, v_1) > \frac{3}{4}|x_{4i} - x_{1,4i}|$.

The proof is similar as the above case and thus is omitted.

(2). If $u_1, v_1 \in \triangle_{1,i}$, then the discussion is similar as in Claim 7.1 (1). Thus we will consider the case: $u_1 \in \triangle_{1,i}$ and $v_1 \in \mathcal{D}_{1,i}\backslash\triangle_{1,i}$.

If $|u_1 - v_1| \leq \frac{1}{2}\max\{d_{\mathcal{D}_{1,i}}(u_1), d_{\mathcal{D}_{1,i}}(v_1)\}$, then Claim 7.2 (2) clearly holds.

If $|u_1 - v_1| > \frac{1}{2}\max\{d_{\mathcal{D}_{1,i}}(u_1), d_{\mathcal{D}_{1,i}}(v_1)\}$, then select

$$u_{1,1} \in [x_{2+4i}, x_{1,2+4i}] \quad \text{with} \quad \angle x_{1,1+4i}u_{1,1}x_{1,2+4i} = \frac{\pi}{2}$$

and $u_{1,2} \in [x_{1,1+4i}, u_{1,1}]$ with $\angle u_{1,1}x_{1,2+4i}u_{1,2} = \angle x_{1,1+4i}x_{1,2+4i}u_{1,2}$. No loss of generality, we may assume that $u_1 \in \triangle_{0,i}$, where $\triangle_{0,i}$ is a domain bounded by $[u_{1,1}, u_{1,2}]$, $[u_{1,1}, x_{1,2+4i}]$ and $[u_{1,2}, x_{1,2+4i}]$.

Take $u_{1,3} \in [u_{1,1}, x_{1,2+4i}]$ and

$$u_{1,4} \in [u_{1,2}, x_{1,2+4i}] \quad \text{with} \quad \angle u_1u_{1,3}x_{1,2+4i} = \angle u_{1,4}u_{1,3}x_{1,2+4i} = \frac{\pi}{2}.$$

Choose $u_{1,5} \in [u_1, u_{1,4}]$ with $|u_1 - u_{1,5}| = \min\{|u_1 - u_{1,1}|, |u_1 - u_{1,4}|\}$. Then, select $u_{1,6} \in [u_{1,1}, u_{1,2}]$ with $\angle u_{1,5}u_{1,6}x_{1,1+4i} = \angle x_{1,2+4i}u_{1,2}x_{1,1+4i}$. By our construction, there is a fold line segment γ_1 in $\mathcal{D}_{1,i}\backslash\triangle_{1,i}$ joining v_1 and $u_{1,6}$ such that γ_1 satisfies Claim 7.2 (1), and so Theorem 2.4 yields that for each $z \in \gamma_1$, it holds

$$\min\{\ell(\gamma_{v_1u_{1,6}}[v_1, z]), \ell(\gamma_{v_1u_{1,6}}[u_{1,6}, z])\} \le e^{(4(\sin\frac{\pi}{24})^{-1})^8} d_{\mathcal{D}_{1,i}}(z) \le e^{(4(\sin\frac{\pi}{24})^{-1})^8} d_{D_n}(z).$$

Set $\gamma = \gamma_1 \cup [u_{1,5}, u_{1,6}] \cup [u_{1,5}, u_1]$. An elementary calculation gives $\ell(\gamma) \le 4\ell(\gamma_{u_1v_1})$ and that $\gamma_{u_1v_1}$ satisfies Claim 7.2 (2). This completes the proof of Claim 7.2. □

The definition of $\mathcal{D}_{2,i}$: For each $i \in \{1, \cdots, n\}$, let $\mathcal{D}_{2,i}$ be a domain bounded by $[w_{2+4i}, y_{2+4i}]$, $[y_{2+4i}, y_{1,2+4i}]$, $[y_{1,2+4i}, y_{1,1+4i}]$, $[y_{1,1+4i}, x_{2,1+4i}]$, $[w_{2+4i}, w_{3+4i}]$, $[w_{3+4i}, y_{3+4i}]$, $[y_{3+4i}, y_{1,3+4i}]$, $[y_{1,3+4i}, x_{2,3+4i}]$ and $[x_{2,1+4i}, x_{2,3+4i}]$.

Claim 7.3. *Suppose $i \in \{1, \cdots, n\}$ and $u_1, v_1 \in \mathcal{D}_{2,i}$. Then there exists some curve γ joining u_1 and v_1 in $\mathcal{D}_{2,i}$ such that for each $u, v \in \gamma$,*

$$\ell(\gamma[u, v]) \le 6\sigma(u, v),$$

and for each $z \in \gamma$,

$$\min\{\ell(\gamma[u_1, z]), \ell(\gamma[v_1, z])\} < 2(\sin\frac{\pi}{24})^{-1} d_{\mathcal{D}_{1,i}}(z).$$

Proof. This is similar to the proof of Claims 7.1 and 7.2 above and thus is omitted. □

The definitions of $\mathcal{D}_{3,i}$, $\mathcal{D}_{4,i}$, $\Delta_{2,i}$ and $\Delta_{3,i}$: For each $i \in \{1, \cdots, n\}$, let $\mathcal{D}_{3,i}$ be a domain bounded by $[w_{2,1+4i}, z_{1,1+4i}]$, $[w_{2,1+4i}, w_{2,3+4i}]$, $[x_{1+4i}, z_{1+4i}]$, $[z_{1+4i}, z_{1,1+4i}]$, $[z_{1,1+4i}, z_{1,2+4i}]$, $[x_{1+4i}, x_{4(i+1)}]$, $[x_{4(i+1)}, z_{4(i+1)}]$, $[z_{4(i+1)}, z_{1,4(i+1)}]$, $[z_{1,4(i+1)}, z_{1,3+4i}]$ and $[z_{1,3+4i}, w_{2,3+4i}]$; and let $\mathcal{D}_{4,i}$ be a domain bounded by $[y_{4i-1}, w_{4i-1}]$, $[w_{4i-1}, w_{1,4i-1}]$, $[w_{1,4i-1}, w_{1,4i}]$, $[w_{1,4i-1}, z_{2,4i-1}]$, $[z_{2,4i-1}, z_{2,1+4i}]$, $[z_{2,1+4i}, w_{1,1+4i}]$, $[w_{1,1+4i}, w_{1,2+4i}]$, $[w_{1,2+4i}, w_{2+4i}]$, $[w_{2+4i}, y_{2+4i}]$ and $[y_{2+4i}, y_{4i-1}]$.

Take

$$z_{2,4(i+1)} \in [z_{4(i+1)}, z_{1,4(i+1)}] \quad \text{with} \quad \angle z_{1,3+4i}z_{2,4(i+1)}z_{1,4(i+1)} = \frac{17\pi}{24}$$

and $w_{2,6+4i} \in [w_{6+4i}, w_{1,6+4i}]$ with $\angle w_{1,5+4i}w_{2,6+4i}w_{1,6+4i} = \frac{17\pi}{24}$. Then, let $\Delta_{2,i}$ be a domain bounded by $[z_{1,3+4i}, z_{1,4(i+1)}]$, $[z_{2,4(i+1)}, z_{1,4(i+1)}]$ and $[z_{2,4(i+1)}, z_{1,3+4i}]$; and $\Delta_{3,i}$ be a domain bounded by $[w_{1,5+4i}, w_{2,6+4i}]$, $[w_{2,6+4i}, w_{1,6+4i}]$ and $[w_{1,5+4i}, w_{1,6+4i}]$.

Claim 7.4. *Suppose $i \in \{1, \cdots, n\}$, $j \in \{3, 4\}$ and $u_1, v_1 \in \mathcal{D}_{j,i}$.*

(1) *If $u_1, v_1 \in \mathcal{D}_{j,i}\backslash\triangle_{j-1,i}$, then there exists some curve γ joining u_1 and v_1 in $\mathcal{D}_{j,i}\backslash\triangle_{j-1,i}$ such that for each $u, v \in \gamma$,*

$$\ell(\gamma[u, v]) \le 4\sigma(u, v),$$

and for each $z \in \gamma$,

$$\min\{\ell(\gamma[u_1, z]), \ell(\gamma[v_1, z])\} < 2(\sin\frac{\pi}{24})^{-1} d_{\mathcal{D}_{j,i}}(z).$$

(2) *For any $t \in \{2, 3\}$, if $u_1 \in \triangle_{t,i}$, $v_1 \in \mathcal{D}_{t+1,i}$ and $\gamma_{u_1v_1} \in \Lambda_{u_1v_1}$, then for each $z \in \gamma_{u_1v_1}$, there exists a curve α joining u_1 and v_1 in $\mathcal{D}_{t+1,i}$ such that*

$$\mathbb{B}\left(z, e^{(4(\sin\frac{\pi}{24})^{-1})^8} d_{D_n}(z)\right) \cap \alpha \ne \emptyset.$$

Proof. This is similar to the proof of Claims 7.1 and 7.2 above and thus is omitted. □

Proof of Theorem 1.7. Let D_0 be a domain bounded by $[x_2, x_{1,2}]$, $[x_{1,2}, x_{1,1}]$, $[x_{1,1}, x_{2,3}]$, $[x_{2,3}, x_{1,3}]$, $[x_{1,4}, x_{1,3}]$, $[x_{1,4}, x_4]$, $[x_4, z_4]$, $[z_4, z_{1,4}]$, $[z_{1,4}, z_{1,3}]$, $[z_{1,3}, w_{2,3}]$, $[w_{1,1}, w_{1,2}]$, $[w_{1,2}, w_2]$, $[w_2, w_3]$, $[w_3, y_3]$, $[y_3, y_{1,3}]$, $[y_{1,3}, x_{2,3}]$ and $[z_{1,1}, z_{1,2}]$. For each $i \in \{1, \cdots, n\}$, we take $D_{0,i} = \cup_{j=1}^{4}\{\mathcal{D}_{j,i}\}$.

For each $x, y \in D_0$ and $\gamma_{xy} \in \Lambda_{xy}$, a similar elementary calculation as in Claim 7.2 (2) shows that γ_{xy} satisfies the C-ball separation condition with $C = e^{(4(\sin\frac{\pi}{24})^{-1})^8}$.

For each $i \in \{1, \cdots, n\}$, $x, y \in D_{0,i}$ and $\gamma_{xy} \in \Lambda_{xy}$, by an elementary calculation shows that $\gamma_{xy} \subset D_{0,i}$, and so it follows from Claims 7.1 $\sim$ 7.4 and Theorem 2.4 that γ_{xy} satisfies the C-ball separation condition with $C = e^{(4(\sin\frac{\pi}{24})^{-1})^8}$.

For each $i \neq j \in \{1, \cdots, n\}$, and for any $x \in D_{0,i}$, $y \in D_{0,j}$ and $\gamma_{xy} \in \Lambda_{xy}$, an elementary calculation shows that $\gamma_{xy} \subset D_{0,i} \cup D_{0,j}$. Then we infer again from Claims 7.1 $\sim$ 7.4 and Theorem 2.4 that γ_{xy} satisfies the C-ball separation condition with $C = e^{(4(\sin\frac{\pi}{24})^{-1})^8}$.

Finally, let $u_i \in D_i$ with $|u_i - x_{4i}| = 1$ and $v_i \in D_i$ with $|v_i - y_{2+4i}| = 1$. By the preceding discussion, we know that $\gamma_{xy} \cap [x_1, z_1] = \emptyset$ for $i > 2$. Clearly, we have

$$\ell(\gamma_{u_i v_i}) > 8^{(\frac{3}{2})^{2+2i}}$$

and

$$\sigma(u_i, v_i) \leq |u_i - x_{4i}| + |v_i - y_{2+4i}| + |x_{4i} - y_{2+4i}| \leq 2 + 6 \cdot 8^{(\frac{3}{2})^{2i+1}} + 3 \cdot 8^{(\frac{3}{2})^{2i}}.$$

Thus

$$\lim_{i\to\infty} \frac{\ell(\gamma_{u_i v_i})}{\sigma(u_i, v_i)} = \infty.$$

This implies that the Gehring-Hayman inequality fails and thus the proof of Theorem 1.7 is complete. □

7.3. **Proof via Matlab software.** Using Matlab software, one obtains that each quasihyperbolic geodesic γ satisfies the C-ball separation condition with a slightly larger constant $C = e^{(4(\sin\frac{\pi}{24})^{-1})^8}$; see Figure 4 $\sim$ 9 below for illustrations.

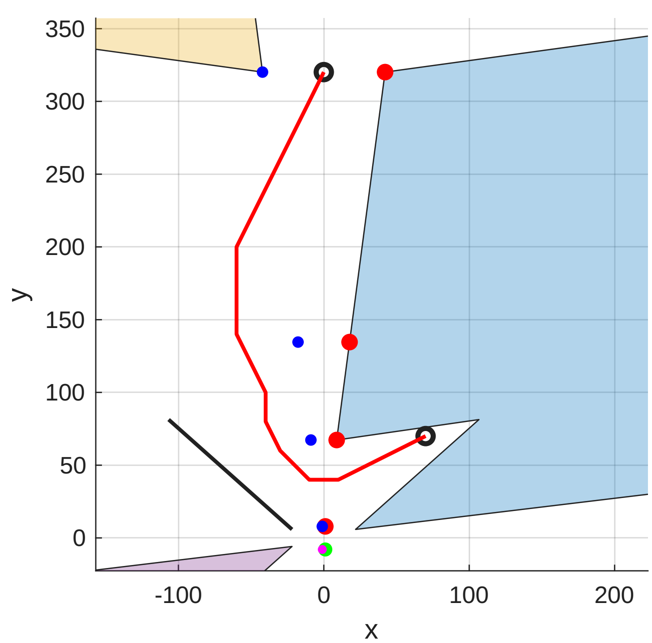

FIGURE 4.

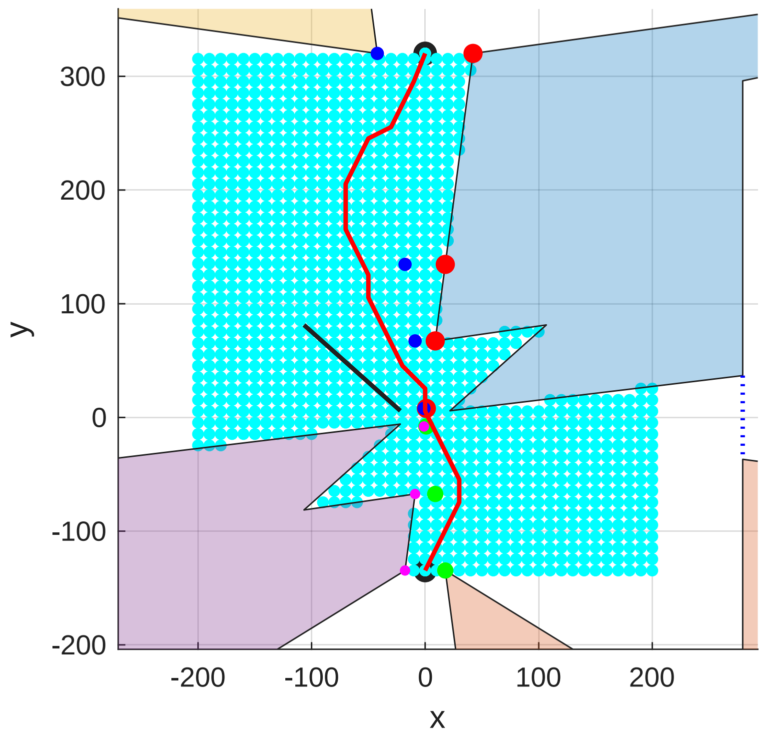

FIGURE 5.

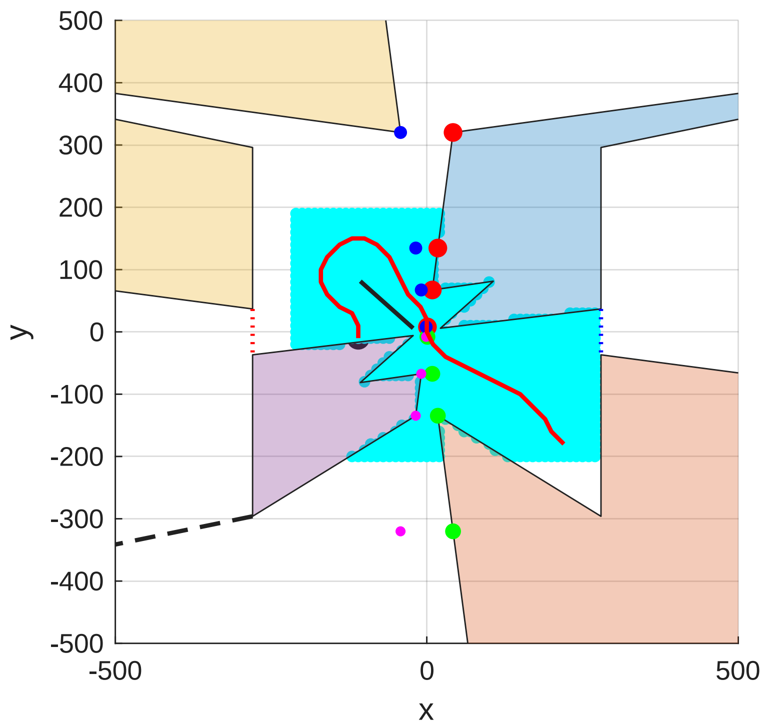

FIGURE 6.

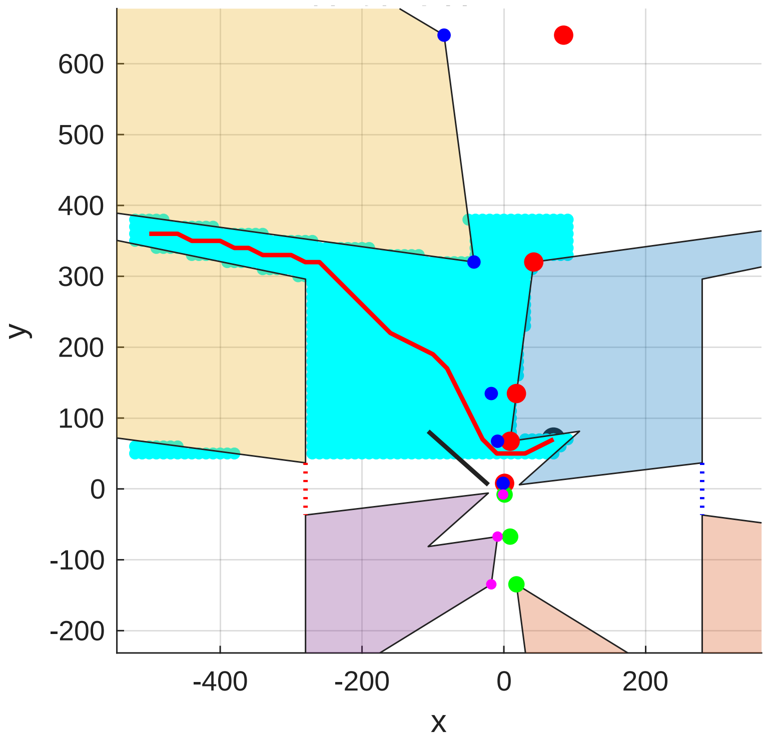

FIGURE 7.

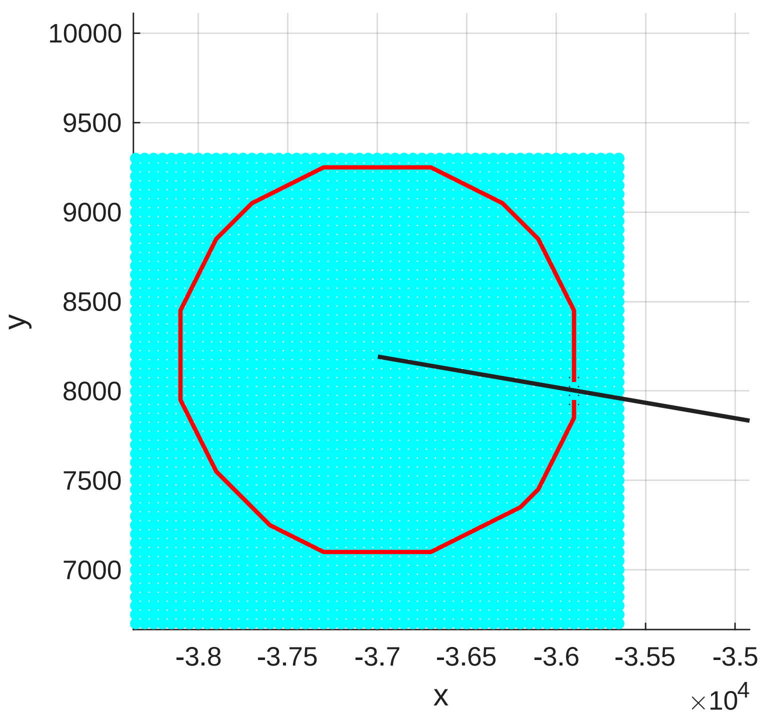

FIGURE 8.

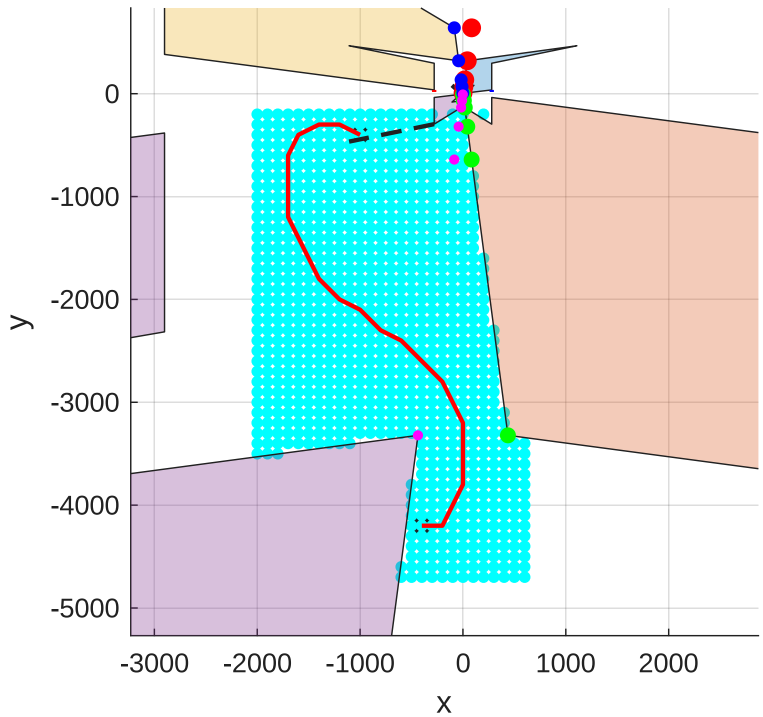

FIGURE 9.

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