Weighted minimum α -Green energy problems

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In memory of Bent Fuglede (1925 – 2023)

Abstract. For the α -Green kernel g_D^{α} on a domain $D \subset \mathbb{R}^n$, $n \geq 2$, associated with the α -Riesz kernel $|x-y|^{\alpha-n}$, where $\alpha \in (0,n)$ and $\alpha \leq 2$, and a relatively closed set $F \subset D$, we investigate the problem on minimizing the Gauss functional

$$\int g_D^{\alpha}(x,y) d(\mu \otimes \mu)(x,y) - 2 \int g_D^{\alpha}(x,y) d(\vartheta \otimes \mu)(x,y),$$

 ϑ being a given positive (Radon) measure concentrated on $D \setminus F$, and μ ranging over all probability measures of finite energy, supported in D by F. For suitable ϑ , we find necessary and/or sufficient conditions for the existence of the solution to the problem, give a description of its support, provide various alternative characterizations, and prove convergence theorems when F is approximated by partially ordered families of sets. The analysis performed is substantially based on the perfectness of the α -Green kernel, discovered by Fuglede and Zorii (Ann. Acad. Sci. Fenn. Math., 2018).

1. General conventions and preliminaries

Fix an (open, connected) domain $D \subset \mathbb{R}^n$, $n \geq 2$, a relatively closed set $F \subset D$, $F \neq D$, and a number $\alpha \in (0, n)$, $\alpha \leq 2$. This paper deals with minimum α -Green energy problems in the presence of external fields over positive (Radon) measures μ on D with $\mu(D) = 1$, supported (in D) by F, known as the Gauss variational problem (cf. e.g. Ohtsuka [18], pertaining to general kernels on a locally compact space).

The α -Green kernel $g_D^{\alpha}(x,y)$, $x \in D$, $y \in \mathbb{R}^n$, is related to the α -Riesz kernel $\kappa_{\alpha}(x,y) := |x-y|^{\alpha-n}$ (|x-y| being the Euclidean distance between $x,y \in \mathbb{R}^n$) via the α -Riesz balayage of ε_x , the unit Dirac measure at $x \in D$, onto the set $Y := \mathbb{R}^n \setminus D$. The present work is concerned with analytic aspects of the theory of g_D^{α} -potentials, initiated by Frostman [12] and Riesz [19] and developed further by Fuglede and Zorii [15], Landkof [17], and Zorii [30] (in case $\alpha = 2$, see e.g. the monographes by Armitage and Gardiner [1], Brelot [7, 8], and Doob [10]); some details of this theory can also be found below. For the probabilistic counterpart of g_D^{α} -potential theory we refer to the monographes by Bliedtner and Hansen [2], Bogdan, Byczkowski, Kulczycki, Ryznar, Song, and Vondraček [3], and Doob [10] (see also numerous references therein).

To formulate results related to both α -Riesz and α -Green kernels, it is convenient to begin with some basic concepts of the theory of potentials with respect to general function kernels κ on a locally compact space X.

1.1. Basic concepts of the theory of potentials on a locally compact space. For any locally compact (Hausdorff) space X, we denote by $\mathfrak{M}(X)$ the linear space of all (real-valued Radon) measures μ on X, equipped with the so-called vague (=weak*) topology, i.e. the topology of pointwise convergence of the class $C_0(X)$ of all continuous functions $\varphi: X \to (-\infty, \infty)$ with compact support, and by $\mathfrak{M}^+(X)$ the cone of

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all positive $\mu \in \mathfrak{M}(X)$, where μ is *positive* if and only if $\mu(\varphi) \geq 0$ for every positive $\varphi \in C_0(X)$. A measure $\mu \in \mathfrak{M}^+(X)$ is said to be bounded if $\mu(X) < \infty$.

The reader is expected to be familiar with principal concepts of the theory of measures and integration on a locally compact space. For its exposition we refer to Bourbaki [6] or Edwards [11]; see also Fuglede [13] for a brief survey.

For the purposes of this study, it is enough to assume X to have a *countable* base of open sets. Then it is σ -compact (that is, representable as a countable union of compact sets [4, Section I.9, Definition 5]), see [5, Section IX.2, Corollary to Proposition 16]; and hence negligibility is the same as local negligibility [6, Section IV.5, Corollary 3 to Proposition 5]. Furthermore, then every measure $\mu \in \mathfrak{M}(X)$ has a countable base of vague neighborhoods [26, Lemma 4.4], and therefore any vaguely bounded (hence vaguely relatively compact, cf. [6, Section III.1, Proposition 15]) set $\mathfrak{B} \subset \mathfrak{M}(X)$ has a sequence $(\mu_i) \subset \mathfrak{B}$ that converges vaguely to some $\mu_0 \in \mathfrak{M}(X)$.

A kernel on X is thought to be a symmetric, lower semicontinuous (l.s.c.) function $\kappa: X \times X \to [0, \infty]$. Given $\mu, \nu \in \mathfrak{M}(X)$, the mutual energy and the potential with respect to the kernel κ are defined by means of the equalities

$$I_{\kappa}(\mu,\nu) := \int \kappa(x,y) \, d(\mu \otimes \nu)(x,y),$$
$$U_{\kappa}^{\mu}(x) := \int \kappa(x,y) \, d\mu(y), \quad x \in X,$$

respectively, provided the value on the right is well defined as a finite number or $\pm \infty$. For $\mu = \nu$, the mutual energy $I_{\kappa}(\mu, \nu)$ defines the energy $I_{\kappa}(\mu, \mu) =: I_{\kappa}(\mu)$ of μ . For more details about these definitions, see Fuglede [13, Section 2.1].

A kernel κ is said to be strictly positive definite if $I_{\kappa}(\mu) \geq 0$ for every (signed) $\mu \in \mathfrak{M}(X)$, and moreover $I_{\kappa}(\mu) = 0 \iff \mu = 0$. Then all (signed) $\mu \in \mathfrak{M}(X)$ of finite energy (i.e., with $I_{\kappa}(\mu) < +\infty$) form a pre-Hilbert space $\mathcal{E}_{\kappa} := \mathcal{E}_{\kappa}(X)$ with the inner product $\langle \mu, \nu \rangle_{\kappa} := I_{\kappa}(\mu, \nu)$ and the energy norm $\|\mu\|_{\kappa} := \sqrt{I_{\kappa}(\mu)}$, see [13, Section 3.1]. The topology on \mathcal{E}_{κ} defined by means of this norm is said to be strong.

Furthermore, a strictly positive definite kernel κ is said to satisfy the consistency principle (or to be perfect) if the cone $\mathcal{E}_{\kappa}^{+} := \mathcal{E}_{\kappa}^{+}(X) := \mathcal{E}_{\kappa} \cap \mathfrak{M}^{+}(X)$ is complete in the induced strong topology, and moreover the strong topology on \mathcal{E}_{κ}^{+} is finer than the vague topology on \mathcal{E}_{κ}^{+} , see Fuglede [13, Section 3.3]. Thus, if a kernel κ is perfect, then any strongly Cauchy sequence (net) $(\mu_{j}) \subset \mathcal{E}_{\kappa}^{+}$ converges both strongly and vaguely to one and the same measure $\mu_{0} \in \mathcal{E}_{\kappa}^{+}$, the strong topology on \mathcal{E}_{κ} as well as the vague topology on $\mathfrak{M}(X)$ being Hausdorff.

For any $A \subset X$, we denote by $\mathfrak{M}^+(A;X)$ the set of all $\mu \in \mathfrak{M}^+(X)$ concentrated on A, which means that $A^c := X \setminus A$ is μ -negligible, or equivalently that A is μ -measurable and $\mu = \mu|_A$, $\mu|_A$ being the trace of μ to A, cf. [6, Section V.5.7]. For any $\mu \in \mathfrak{M}^+(A;X)$, then necessarily $\mu(X) = \mu(A)$. Also note that if A is closed, then $\mu \in \mathfrak{M}^+(A;X)$ if and only if $S(\mu;X) \subset A$, where $S(\mu;X)$ is the support of μ in X.

Let $\mathfrak{M}^+(A,q;X)$, q>0, consist of all $\mu\in\mathfrak{M}^+(A;X)$ with $\mu(X)=q$, and let

$$\mathcal{E}_{\kappa}^{+}(A) := \mathcal{E}_{\kappa} \cap \mathfrak{M}^{+}(A;X), \quad \mathcal{E}_{\kappa}^{+}(A,q) := \mathcal{E}_{\kappa} \cap \mathfrak{M}^{+}(A,q;X).$$

All the sets, to appear in this paper, are *Borel*, while all the kernels are *perfect*. By virtue of [29, Theorem 3.5] (cf. [13, Theorem 4.5]), all those sets are therefore *capacitable*, and hence their *inner* and *outer capacities* coincide:

$$\underline{c}_{\kappa}(A) = \overline{c}_{\kappa}(A) =: c_{\kappa}(A).$$

(For the theory of inner and outer capacities of arbitrary sets in a locally compact space endowed with a perfect kernel, we refer to Fuglede [13], cf. also Zorii [25].)

Thus,¹

$$c_{\kappa}(A) := \left[\inf_{\mu \in \mathcal{E}_{\kappa}^{+}(A,1)} \|\mu\|_{\kappa}^{2} \right]^{-1}, \tag{1.1}$$

and therefore (cf. [13, Lemma 2.3.1])

$$c_{\kappa}(A) = 0 \iff \mathcal{E}_{\kappa}^{+}(A) = \{0\}. \tag{1.2}$$

Alternatively (see [13, Theorem 4.1] and [25, Theorem 6.1]),

$$c_{\kappa}(A) = \inf_{\mu \in \Gamma_{\kappa}^{+}(A)} \|\mu\|_{\kappa}^{2}, \tag{1.3}$$

where $\Gamma_{\kappa}^{+}(A)$ consists of all $\mu \in \mathcal{E}_{\kappa}^{+}$ such that $U_{\kappa}^{\mu} \geqslant 1$ q.e. (quasi-everywhere) on A (that is, on all of A except for a subset of capacity zero).²

If $c_{\kappa}(A) < \infty$, then the infimum in (1.3) is attained at the unique $\gamma_{A,\kappa} \in \Gamma_{\kappa}^{+}(A)$, called the κ -capacitary distribution for A (Fuglede [13, Theorem 4.1]).³ If moreover Frostman's maximum principle holds, then $\gamma_{A,\kappa}$ is the only measure in $\Gamma_{\kappa}^{+}(A)$ with

$$U_{\kappa}^{\gamma_{A,\kappa}} = 1$$
 q.e. on A ; (1.4)

in this case, $\gamma_{A,\kappa}$ is also referred to as the κ -equilibrium measure for A. (Frostman's maximum principle means that for any $\mu \in \mathfrak{M}^+(X)$ with $U^{\mu}_{\kappa} \leq c_{\mu}$ μ -a.e., where $c_{\mu} \in (0, \infty)$, the same inequality holds on all of X. See e.g. Ohtsuka [18, p. 143].)

All the kernels, to appear in this paper, satisfy the Frostman and domination maximum principles, where the latter means that for any $\mu, \nu \in \mathfrak{M}^+(X)$ such that both $U^{\mu}_{\kappa} < \infty$ and $U^{\mu}_{\kappa} \leqslant U^{\nu}_{\kappa}$ hold true μ -a.e., we have $U^{\mu}_{\kappa} \leqslant U^{\nu}_{\kappa}$ on all of X.

A measure $\mu \in \mathfrak{M}^+(X)$ is said to be c_{κ} -absolutely continuous if $\mu(K) = 0$ for every compact set $K \subset X$ with $c_{\kappa}(K) = 0$. Each $\mu \in \mathfrak{M}^+(X)$ of finite κ -energy is certainly c_{κ} -absolutely continuous, cf. (1.2), but not the other way around. With regard to the latter, see the example in Landkof [17, pp. 134–135], pertaining to the Newtonian kernel $\kappa_2(x,y) := |x-y|^{2-n}$ on \mathbb{R}^n , $n \geq 3$.

In the rest of this subsection, a set $A \subset X$ is closed. Then the cone $\mathfrak{M}^+(A;X)$ is vaguely closed [6, Section III.2, Proposition 6], which entails, by use of the perfectness of the kernel κ , that the subcone $\mathcal{E}_{\kappa}^+(A)$ of the strongly complete cone \mathcal{E}_{κ}^+ is strongly closed, whence strongly complete. By utilizing Edwards [11, Theorem 1.12.3], this implies, in turn, that for every $\lambda \in \mathcal{E}_{\kappa}^+$, there exists the unique orthogonal projection λ_{κ}^A of λ onto $\mathcal{E}_{\kappa}^+(A)$, minimizing $\|\lambda - \mu\|_{\kappa}$ over all $\mu \in \mathcal{E}_{\kappa}^+(A)$. This λ_{κ}^A actually serves as the balayage of $\lambda \in \mathcal{E}_{\kappa}^+$ onto the set A, for, according to [24, Theorem 4.3, Corollary 4.5], it is uniquely characterized within $\mathcal{E}_{\kappa}^+(A)$ by means of the equality

$$U_{\kappa}^{\lambda_{\kappa}^{A}} = U_{\kappa}^{\lambda}$$
 q.e. on A .

Theorem 1.1. Given a closed set A and a perfect kernel κ , assume $c_{\kappa}(A) < \infty$. Then for any $q \in (0, \infty)$, $\mathcal{E}_{\kappa}^{+}(A, q)$ is complete in the induced strong topology.

Proof. See Zorii [29, Theorem 7.1]. (This certainly no longer holds if $c_{\kappa}(A) = \infty$.)

¹As usual, the infimum over the empty set is interpreted as $+\infty$. We also agree that $1/0 = +\infty$ and $1/(+\infty) = 0$.

²It is used here that for every $\mu \in \mathfrak{M}^+(X)$, U^{μ}_{κ} is l.s.c. on X (see [13, Lemma 2.2.1(c)]), and therefore U^{μ}_{κ} is ν -measurable for every $\nu \in \mathfrak{M}^+(X)$, cf. [6, Section IV.5, Corollary 3 to Theorem 2].

³Unless $\mathcal{E}_{\kappa}^{+}(A)$ is strongly closed (which occurs, in particular, if A is closed or even quasiclosed, see [14, Definition 2.1] and [29, Theorem 3.1] for details), $\gamma_{A,\kappa}$ is *not* necessarily concentrated on A, and hence problem (1.1) is in general unsolvable.

Let ∞_X be the Alexandroff point of a locally compact space X [4, Section I.9.8].

1.2. α -Riesz balayage. In this and the following two subsections, $X := \mathbb{R}^n$ and $\kappa := \kappa_{\alpha}$, where n, α , and κ_{α} are as indicated at the top of Section 1. When speaking of $\nu \in \mathfrak{M}^+(\mathbb{R}^n)$, we always understand that its κ_{α} -potential $U^{\nu}_{\kappa_{\alpha}}$ is not identically infinite on \mathbb{R}^n , which according to Landkof [17, Section I.3.7] occurs if and only if

$$\int_{|y|>1} \frac{d\nu(y)}{|y|^{n-\alpha}} < \infty. \tag{1.5}$$

Then, actually, $U^{\nu}_{\kappa_{\alpha}}$ is finite q.e. on \mathbb{R}^n [17, Section III.1.1]. Note that (1.5) necessarily holds if ν is either bounded, or of finite κ_{α} -energy $I_{\kappa_{\alpha}}(\nu)$; with regard to the latter, see [13, Corollary to Lemma 3.2.3] applied to κ_{α} , the kernel κ_{α} being strictly positive definite according to Riesz [19, Section I.4, Eq. (13)].

Moreover, κ_{α} is perfect and meets the Frostman and domination maximum principles. See Deny [9, p. 121, Corollary (a)], Doob [10, Theorem 1.V.10(b)], Landkof [17, Lemma 1.3, Theorems 1.10, 1.29]; cf. Section 1.1 above for definitions.

Referring to Bliedtner and Hansen [2] (resp. Zorii [22, 23]) for a general theory of outer (resp. inner) κ_{α} -balayage of any $\nu \in \mathfrak{M}^+(\mathbb{R}^n)$ to any $Q \subset \mathbb{R}^n$, in this study we limit ourselves to *closed* Q; this limitation will generally not be repeated henceforth.

Let Q^r denote the set of all α -regular points of Q; by the Wiener type criterion [17, Theorem 5.2],

$$x \in Q^r \iff \sum_{j \in \mathbb{N}} \frac{c_{\kappa_{\alpha}}(Q_j)}{q^{j(n-\alpha)}} = \infty,$$

where $q \in (0,1)$ and $Q_j := Q \cap \{y \in \mathbb{R}^n : q^{j+1} < |x-y| \leq q^j\}$. The set Q^r is Borel measurable [23, Theorem 5.2], while $Q \setminus Q^r$, the set of all α -irregular points of Q, is of κ_{α} -capacity zero (the Kellogg–Evans type theorem, see [22, Theorem 6.6]).

Definition 1.2. Given a (closed) set $Q \subset \mathbb{R}^n$, fix $\xi \in \mathfrak{M}^+(\mathbb{R}^n)$, $\xi \neq 0$, such that $\xi|_{Q^r}$ is c_{κ_α} -absolutely continuous. The κ_α -balayage of ξ onto Q is the unique c_{κ_α} -absolutely continuous measure $\xi^Q_{\kappa_\alpha} \in \mathfrak{M}^+(Q;\mathbb{R}^n)$ having the property

$$U_{\kappa_{\alpha}}^{\xi_{\kappa_{\alpha}}^{Q}} = U_{\kappa_{\alpha}}^{\xi} \quad \text{on } Q^{r} \quad \text{(hence, q.e. on } Q\text{)}.$$
 (1.6)

Regarding the existence and uniqueness of this $\xi_{\kappa_{\alpha}}^{Q}$, see Zorii [23, Corollary 5.2]. (Compare with Fuglede and Zorii [15, Corollary 3.19], pertaining to $\xi \in \mathfrak{M}^{+}(Q^{c}; \mathbb{R}^{n})$. Being based on Theorem 3.17 of the same paper, it however had a gap in its proof; see Remark 1.3 below for details.)

In the rest of this subsection, $\dot{\xi}$ and $\xi_{\kappa_{\alpha}}^{Q}$ are as indicated in Definition 1.2. Since $\xi_{\kappa_{\alpha}}^{Q}$ is $c_{\kappa_{\alpha}}$ -absolutely continuous, (1.6) holds true $\xi_{\kappa_{\alpha}}^{Q}$ -a.e. By the domination principle,

$$U_{\kappa_{\alpha}}^{\xi_{\kappa_{\alpha}}^{Q}} \leqslant U_{\kappa_{\alpha}}^{\xi}$$
 on all of \mathbb{R}^{n} ,

whence, by the principle of positivity of mass for κ_{α} -potentials [15, Theorem 3.11],

$$\xi_{\kappa_{\alpha}}^{Q}(\mathbb{R}^{n}) \leqslant \xi(\mathbb{R}^{n}). \tag{1.7}$$

In the theory of κ_{α} -balayage, the following integral representation is particularly useful (see [23, Theorem 5.1]):

$$\xi_{\kappa_{\alpha}}^{Q} = \int (\varepsilon_{x})_{\kappa_{\alpha}}^{Q} d\xi(x), \qquad (1.8)$$

where

$$(\varepsilon_x)_{\kappa_\alpha}^Q := \varepsilon_x \quad \text{for all } x \in Q^r.$$

Remark 1.3. Compare with [15, Theorem 3.17], dealing with $\xi \in \mathfrak{M}^+(Q^c; \mathbb{R}^n)$. However, this theorem from [15] was based on Lemma 3.16 of the same paper, whose proof was incomplete. Indeed, the functions $y \mapsto \int f_j d\varepsilon_y^A$, appearing in [15, Proof of Lemma 3.16(a)], might be of noncompact support, and hence the suggested use of [6, Section IV.3, Proposition 4] (see the last line on p. 133 in [15]) was unjustified.

Thus, in view of (1.8), for every $\varphi \in C_0(\mathbb{R}^n)$,

$$\int \varphi(z) d\xi_{\kappa_{\alpha}}^{Q}(z) = \int \left(\int \varphi(z) d(\varepsilon_{x})_{\kappa_{\alpha}}^{Q}(z) \right) d\xi(x). \tag{1.9}$$

Moreover, by virtue of [6, Section V.3, Proposition 2], equality (1.9) remains valid when φ is replaced by any positive l.s.c. function on \mathbb{R}^n . For a given $y \in \mathbb{R}^n$, we apply this to the (positive l.s.c.) function $\kappa_{\alpha}(y, z), z \in \mathbb{R}^n$, and thus obtain

$$U_{\kappa_{\alpha}}^{\xi_{\kappa_{\alpha}}^{Q}}(y) = \int \left(\int \kappa_{\alpha}(y, z) \, d(\varepsilon_{x})_{\kappa_{\alpha}}^{Q}(z) \right) d\xi(x) = \int U_{\kappa_{\alpha}}^{(\varepsilon_{x})_{\kappa_{\alpha}}^{Q}}(y) \, d\xi(x). \tag{1.10}$$

Similarly, applying (1.9) to the (positive l.s.c.) constant function 1 on \mathbb{R}^n gives

$$\xi_{\kappa_{\alpha}}^{Q}(\mathbb{R}^{n}) = \int (\varepsilon_{x})_{\kappa_{\alpha}}^{Q}(\mathbb{R}^{n}) d\xi(x). \tag{1.11}$$

1.3. α -thinness of a set at infinity. By Kurokawa and Mizuta [16], a (closed) set $Q \subset \mathbb{R}^n$ is said to be α -thin at infinity if

$$\sum_{j \in \mathbb{N}} \frac{c_{\kappa_{\alpha}}(Q_j)}{q^{j(n-\alpha)}} < \infty,$$

where $q \in (1, \infty)$ and $Q_j := Q \cap \{y \in \mathbb{R}^n : q^j < |y| \leqslant q^{j+1}\}.$

Theorem 1.4. The following (i)–(vi) are equivalent.

- (i) Q is α -thin at infinity.
- (ii) For some (equivalently, every) $x \in \mathbb{R}^n$,

$$x \notin (Q_x^*)^r$$
,

where Q_x^* is the inversion of $Q \cup \{\infty_{\mathbb{R}^n}\}$ with respect to $\{y: |y-x|=1\}$.

(iii) There exists $\nu \in \mathfrak{M}^+(\mathbb{R}^n)$ such that

$$\operatorname{ess inf}_{x \in Q} U^{\nu}_{\kappa_{\alpha}}(x) > 0,$$

the infimum being taken over all of Q except for a subset of κ_{α} -capacity zero.

(iv) There exists the unique measure $\gamma_{Q,\kappa_{\alpha}} \in \mathfrak{M}^+(Q;\mathbb{R}^n)$ having the property

$$U_{\kappa_{\alpha}}^{\gamma_{Q,\kappa_{\alpha}}} = 1$$
 q.e. on Q ;

such $\gamma_{Q,\kappa_{\alpha}}$ is said to be the (generalized) κ_{α} -equilibrium measure on Q^{4}

- (v) There exists $x \in Q^c$ such that $(\varepsilon_x)_{\kappa_\alpha}^Q(\mathbb{R}^n) < 1.5$
- (vi) There exists $\xi \in \mathfrak{M}^+(Q^c; \mathbb{R}^n)$ for which strict inequality in (1.7) holds:

$$\xi_{\kappa_{\alpha}}^{Q}(\mathbb{R}^{n}) < \xi(\mathbb{R}^{n}). \tag{1.12}$$

Proof. See [17, Theorem 5.1], [22, Theorems 5.5, 8.6, 8.7], and [23, Theorems 2.1, 2.2, Corollary 5.3]. We emphasize that in [22, Proofs of Theorems 8.6, 8.7], one has to use [23, Theorem 5.1] in place of [22, Theorem 8.2] (cf. Remark 1.3 above). \Box

⁴Both $I_{\kappa_{\alpha}}(\gamma_{Q,\kappa_{\alpha}})$ and $\gamma_{Q,\kappa_{\alpha}}(Q)$ may be $+\infty$; these are finite if and only if so is $c_{\kappa_{\alpha}}(Q)$. Also note that $\gamma_{Q,\kappa_{\alpha}}$ is $c_{\kappa_{\alpha}}$ -absolutely continuous, for $U_{\kappa_{\alpha}}^{\gamma_{Q,\kappa_{\alpha}}} \leq 1$ on \mathbb{R}^n by Frostman's maximum principle.

⁵For such x, $(\varepsilon_x)_{\kappa_{\alpha}}^Q(\mathbb{R}^n) = U_{\kappa_{\alpha}}^{\gamma_{Q,\kappa_{\alpha}}}(x)$, $\gamma_{Q,\kappa_{\alpha}}$ being introduced by (iv). See [23, Theorem 2.2].

Remark 1.5. Unless $\alpha < 2$, assume that Q^c is connected. Then (v) and (vi) in Theorem 1.4 can be refined as follows (see [22, Theorem 8.7], cf. (1.11)):

- (v') Q is α -thin at infinity if and only if for some (equivalently, for all) $x \in Q^c$, we have $(\varepsilon_x)_{\kappa_\alpha}^Q(\mathbb{R}^n) < 1$.
- (vi') Q is α -thin at infinity if and only if for some (equivalently, for all) nonzero $\xi \in \mathfrak{M}^+(Q^c; \mathbb{R}^n)$, (1.12) holds true.

1.4. α -harmonic measure on $\overline{\mathbb{R}^n} := \mathbb{R}^n \cup \{\infty_{\mathbb{R}^n}\}$. We call a set $e \subset \overline{\mathbb{R}^n}$ Borel if so is $e \cap \mathbb{R}^n$. For any open $\Delta \subset \mathbb{R}^n$ and any Borel $e \subset \overline{\mathbb{R}^n}$, we define the (fractional) α -harmonic measure $\omega_{\alpha}(x, e; \Delta)$, $x \in \Delta$, by means of the formula (see [30, Section 3.2])

$$\omega_{\alpha}(x, e; \Delta) = \begin{cases} (\varepsilon_x)_{\kappa_{\alpha}}^{\Delta^c}(e) & \text{if } e \subset \mathbb{R}^n, \\ (\varepsilon_x)_{\kappa_{\alpha}}^{\Delta^c}(e \cap \mathbb{R}^n) + \omega_{\alpha}(x, \{\infty_{\mathbb{R}^n}\}; \Delta) & \text{otherwise,} \end{cases}$$
(1.13)

where

$$\omega_{\alpha}(x, \{\infty_{\mathbb{R}^n}\}; \Delta) := 1 - (\varepsilon_x)_{\kappa_{\alpha}}^{\Delta^c}(\mathbb{R}^n). \tag{1.14}$$

Thus,

$$\omega_{\alpha}(x, \overline{\mathbb{R}^n}; \Delta) = 1$$
 for all $x \in \Delta$.

Since for $\alpha = 2$, $S((\varepsilon_x)^{\Delta^c}_{\kappa_2}; \mathbb{R}^n) \subset \partial_{\mathbb{R}^n} \Delta$ for every $x \in \Delta$, cf. [22, Theorem 8.5], the concept of α -harmonic measure in $\overline{\mathbb{R}^n}$, introduced by (1.13) and (1.14), generalizes that of 2-harmonic measure, defined in [17, Section IV.3.12] for Borel subsets of $\partial_{\mathbb{R}^n} \Delta$.

It is clear from (1.7) that $(\varepsilon_x)^{\Delta^c}_{\kappa_\alpha}(\mathbb{R}^n) \leq 1$ for every $x \in \Delta$, whence

$$\omega_{\alpha}(x, \{\infty_{\mathbb{R}^n}\}; \Delta) \geqslant 0$$
 for all $x \in \Delta$.

On account of Remark 1.5(v'), we arrive at the following observation.

Corollary 1.6. Unless $\alpha < 2$, assume that Δ is connected. Then Δ^c is not α -thin at infinity if and only if for some (equivalently, for all) $x \in \Delta$,

$$\omega_{\alpha}(x, \{\infty_{\mathbb{R}^n}\}; \Delta) = 0.$$

1.5. α -Green kernel. In the rest of this paper, we fix a domain $D \subset \mathbb{R}^n$. A measure $\mu \in \mathfrak{M}^+(D)$, D being treated as a locally compact space, is said to be *extendible* by zero outside D to all of \mathbb{R}^n if there is the (unique) $\widehat{\mu} \in \mathfrak{M}^+(\mathbb{R}^n)$ such that

$$\int \varphi|_D d\mu = \widehat{\mu}(\varphi) \quad \text{for all } \varphi \in C_0(\mathbb{R}^n);$$

let $\check{\mathfrak{M}}^+(D)$ denote the class of all those μ . If no confusion can arise, this extension of $\mu \in \check{\mathfrak{M}}^+(D)$ will be denoted by the same symbol μ , i.e. $\widehat{\mu} := \mu$.

Note that $\check{\mathfrak{M}}^+(D)$ can equivalently be introduced as the class of all traces $\nu|_D$, ν ranging over all of $\mathfrak{M}^+(\mathbb{R}^n)$. We also remark that, in general, $\check{\mathfrak{M}}^+(D) \neq \mathfrak{M}^+(D)$ (unless, of course, $Y := \mathbb{R}^n \setminus D$ is compact). Another useful observation is that a measure $\mu \in \mathfrak{M}^+(D)$ is obviously extendible if it is bounded. For any $A \subset D$, denote

The α -Green kernel $g_{\alpha} := g_D^{\alpha}$ on D is defined by

$$g_{\alpha}(x,y) := \kappa_{\alpha}(x,y) - U_{\kappa_{\alpha}}^{(\varepsilon_{x})_{\kappa_{\alpha}}^{Y}}(y) = U_{\kappa_{\alpha}}^{\varepsilon_{x}}(y) - U_{\kappa_{\alpha}}^{(\varepsilon_{x})_{\kappa_{\alpha}}^{Y}}(y), \quad x \in D, \ y \in \mathbb{R}^{n}, \quad (1.15)$$

where $(\varepsilon_x)_{\kappa_\alpha}^Y$ is the κ_α -balayage of ε_x , $x \in D$, onto Y, cf. Definition 1.2. (Here we have used the fact that the measure $\varepsilon_x \in \mathfrak{M}^+(D)$, $x \in D$, is extendible, and hence it can be treated as an element of $\mathfrak{M}^+(\mathbb{R}^n)$.) For more details, see e.g. [12, 15, 17].

Lemma 1.7. For any $\mu \in \mathfrak{M}^+(D)$, the g_{α} -potential $U_{g_{\alpha}}^{\mu}$ is well defined and finite q.e. on \mathbb{R}^n and given by

$$U^{\mu}_{g_{\alpha}} = U^{\mu}_{\kappa_{\alpha}} - U^{\mu_{\kappa_{\alpha}}^{Y}}_{\kappa_{\alpha}}.$$
(1.16)

Proof. This follows in the same manner as in [15, Proof of Lemma 4.4], the only difference being in applying (1.10) of the present paper in place of the unjustified Eq. (3.32) in [15] (see Remark 1.3 above for details).

As shown in [15, Theorems 4.9, 4.11], the kernel g_{α} is *perfect*. Furthermore, it satisfies the Frostman and domination maximum principles in the following sense.

Theorem 1.8. Given $\mu, \nu \in \mathfrak{M}^+(D)$, the following (a) and (b) are fulfilled.

- (a) If $U^{\mu}_{g_{\alpha}} \leq 1$ μ -a.e., then the same inequality holds true on all of \mathbb{R}^n .
- (b) Assume μ is $c_{g_{\alpha}}$ -absolutely continuous.⁶ If moreover

$$U_{q_{\alpha}}^{\mu} \leqslant U_{q_{\alpha}}^{\nu} \quad \mu\text{-a.e.},$$

then the same inequality holds true on all of D.

- *Proof.* (a) Applying Lemma 1.7 gives $U^{\mu}_{\kappa_{\alpha}} \leq 1 + U^{\mu^{Y}_{\kappa_{\alpha}}}_{\kappa_{\alpha}}$ μ -a.e., whence the claim in consequence of [10, Theorem 1.V.10(b)] if $\alpha = 2$ or [17, Theorem 1.29] otherwise.
- (b) For $\alpha = 2$, see [10, Theorem 1.V.10(b)]. For $\alpha < 2$, this follows by a slight modification of [15, Proof of Theorem 4.6].
- 1.6. α -Green balayage. In what follows, fix a proper, relatively closed subset F of the domain D with $c_{q_{\alpha}}(F) > 0$, and denote $\Omega := D \setminus F$. Define

$$F^r := (F \cup Y)^r \cap D.$$

Theorem 1.9. For any $\mu \in \check{\mathfrak{M}}^+(D)$ such that $\mu|_{F^r}$ is $c_{\kappa_{\alpha}}$ -absolutely continuous, there exists the unique $c_{\kappa_{\alpha}}$ -absolutely continuous measure $\mu_{g_{\alpha}}^F \in \check{\mathfrak{M}}^+(F;D)$ with

$$U_{g_{\alpha}}^{\mu_{g_{\alpha}}^{F}}(y) = U_{g_{\alpha}}^{\mu}(y) \quad \text{for all } y \in F^{r} \quad \text{(hence, q.e. on } F); \tag{1.17}$$

this $\mu_{g_{\alpha}}^{F}$ is said to be the g_{α} -balayage of μ onto F. Actually,

$$\mu_{g_{\alpha}}^{F} = \mu_{\kappa_{\alpha}}^{F \cup Y} \big|_{F}. \tag{1.18}$$

If moreover $\mu \in \mathcal{E}_{g_{\alpha}}^+$, then the same $\mu_{g_{\alpha}}^F$ can alternatively be found as the only measure in the cone $\mathcal{E}_{g_{\alpha}}^+(F)$ such that

$$\|\mu - \mu_{g_{\alpha}}^F\|_{g_{\alpha}} = \min_{\nu \in \mathcal{E}_{g_{\alpha}}^+(F)} \|\mu - \nu\|_{g_{\alpha}}.$$

Proof. See [30, Theorem 2.1]. For $\mu := \varepsilon_x$, $x \in \Omega$, cf. Frostman [12, Section 5].

It follows from (1.17) by utilizing Theorem 1.8(b) that

$$U_{g_{\alpha}}^{\mu_{g_{\alpha}}^{F}} \leqslant U_{g_{\alpha}}^{\mu}$$
 on all of D ;

whence, by the principle of positivity of mass for g_{α} -potentials [15, Theorem 4.13],

$$\mu_{q_{\alpha}}^{F}(D) \leqslant \mu(D). \tag{1.19}$$

$$c_{g_{\alpha}}(K) = 0 \iff c_{\kappa_{\alpha}}(K) = 0 \text{ for any compact } K \subset D,$$

which follows by noting that $U_{\kappa_{\alpha}}^{(\varepsilon_x)_{\kappa_{\alpha}}^Y}(y)$, $x \in D$, is bounded when y ranges over compact $K \subset D$.

⁶For extendible measures, the concepts of $c_{g_{\alpha}}$ - and $c_{\kappa_{\alpha}}$ -absolute continuity coincide. Indeed,

1.7. When does equality in (1.19) hold? This question is answered by the following slight improvement of [30, Theorem 3.7].

Theorem 1.10. Unless $\alpha < 2$, assume that Ω is connected. Fix $\mu \in \check{\mathfrak{M}}^+(D)$ such that $\mu|_F$ is $c_{\kappa_{\alpha}}$ -absolutely continuous while $\mu|_{\Omega} \neq 0$. Then (i_1) - (ii_1) are equivalent.

- $(i_1) \ \mu_{g_{\alpha}}^F(D) = \mu(D).$
- (ii₁) $\omega_{\alpha}(x, \overline{\mathbb{R}^n} \setminus D; \Omega) = 0$ μ -a.e. on Ω .
- (iii₁) Ω^c is not α -thin at infinity, and

$$\omega_{\alpha}(x,Y;\Omega) = 0$$
 μ -a.e. on Ω .

Proof. This follows in a manner similar to that in [30, Proof of Theorem 3.7]. \Box

2. The Gauss variational problem for g_{α} -potentials

In what follows, we keep all the conventions introduced in the preceding Section 1.

2.1. **Statement of the problem.** In the rest of this paper, we fix a bounded (hence, extendible) measure $\vartheta \in \check{\mathfrak{M}}^+(\Omega; D)$, $\vartheta \neq 0$, such that

$$\varrho := \operatorname{dist}(S(\vartheta; D), F) := \inf_{(x,y) \in S(\vartheta; D) \times F} |x - y| > 0.$$
(2.1)

Treating ϑ as a charge creating the external field

$$f := -U_{q_{\alpha}}^{\vartheta}, \tag{2.2}$$

we are interested in the problem on minimizing the Gauss functional $I_{q_{\alpha},f}(\mu)$,

$$I_{g_{\alpha},f}(\mu) := I_{g_{\alpha}}(\mu) + 2 \int f \, d\mu = \|\mu\|_{g_{\alpha}}^2 - 2 \int U_{g_{\alpha}}^{\vartheta} \, d\mu, \tag{2.3}$$

 μ ranging over the class $\mathcal{E}_{g_{\alpha}}^{+}(F,1)$. That is, does there exist $\lambda_{F,f} \in \mathcal{E}_{g_{\alpha}}^{+}(F,1)$ with

$$I_{g_{\alpha},f}(\lambda_{F,f}) = \inf_{\mu \in \mathcal{E}_{g_{\alpha}}^{+}(F,1)} I_{g_{\alpha},f}(\mu) =: w_{g_{\alpha},f}(F)$$
? (2.4)

Since $\mathcal{E}_{g_{\alpha}}^{+}(F,1) \neq \emptyset$ because of $c_{g_{\alpha}}(F) > 0$ (cf. (1.2)), problem (2.4) makes sense. In view of (1.15) and (2.1),

$$I_{g_{\alpha}}(\vartheta,\mu) \leqslant I_{\kappa_{\alpha}}(\vartheta,\mu) \leqslant \vartheta(D)/\varrho^{n-\alpha} =: M < \infty \text{ for all } \mu \in \mathcal{E}_{g_{\alpha}}^{+}(F,1),$$
 (2.5)

whence, by virtue of (2.3) and the strict positive definiteness of the kernel g_{α} ,

$$-\infty < -2M \leqslant w_{g_{\alpha},f}(F) < \infty. \tag{2.6}$$

This enables us to show, by means of standard arguments based on the convexity of the class $\mathcal{E}_{g_{\alpha}}^{+}(F,1)$, the strict positive definiteness of the kernel g_{α} , and the parallelogram identity in the pre-Hilbert space $\mathcal{E}_{g_{\alpha}}$, that the solution $\lambda_{F,f}$ to problem (2.4) is unique (if it exists). See e.g. [21, Lemma 6].

By (2.1) and (2.2), $f|_F$ is continuous.⁸ Thus, if F =: K is compact, then $\int f d\mu$ is vaguely continuous on $\mathfrak{M}^+(K;D)$, and therefore, by the principle of descent [13, Lemma 2.2.1(e)], the Gauss functional $I_{g_{\alpha},f}(\cdot)$ is vaguely l.s.c. on $\mathfrak{M}^+(K;D)$. Since the class $\mathcal{E}_{g_{\alpha}}^+(K,1)$ is vaguely compact [6, Section III.1, Corollary 3 to Proposition 15],

⁷In constructive function theory, $I_{g_{\alpha},f}(\cdot)$ is also referred to as the *f*-weighted α -Green energy. For the terminology used here, see e.g. [17, 18, 20, 28].

⁸When speaking of a continuous function, we generally understand that the values are *finite* real numbers.

the existence of $\lambda_{K,f}$ immediately follows. But if F is noncompact, then these arguments, based on the vague topology only, fail down, and the problem on the existence of the solution $\lambda_{F,f}$ to problem (2.4) becomes "rather difficult" (Ohtsuka [18, p. 219]).

2.2. Main results. Problem (2.4) will be analyzed below in the framework of the approach suggested in our recent paper [28], which is based on the perfectness of the kernel in question, and hence on the simultaneous use of both the strong and vague topologies on the pre-Hilbert space $\mathcal{E}_{g_{\alpha}}$ (see Sections 1.1, 1.5 above). The theories of α -Green balayage and α -Green equilibrium measures are also particularly helpful.

Necessary and/or sufficient conditions for the existence of the solution $\lambda_{F,f}$ to problem (2.4) are obtained in Theorems 2.2–2.4. We also give alternative characterizations of $\lambda_{F,f}$ (Theorems 2.2–2.4), and we prove assertions on convergence when F is approximated by partially ordered families of sets (Theorems 2.6, 2.7). Furthermore, we establish a description of the support $S(\lambda_{F,f}; D)$ (Theorems 2.5, 2.10), thereby answering the question raised by Ohtsuka in [18, p. 284, Open question 2.1].

To present those results, we first recall the following well-known theorem, providing characteristic properties of the solution $\lambda_{F,f}$. It can be derived from our earlier paper [21] (see Theorems 1, 2 and Proposition 1 therein), dealing with an arbitrary positive definite kernel on a locally compact space.

Theorem 2.1. For $\lambda \in \mathcal{E}_{g_{\alpha}}^+(F,1)$ to serve as the (unique) solution $\lambda_{F,f}$ to problem (2.4), it is necessary and sufficient that either of the two inequalities holds

$$U_{g_{\alpha},f}^{\lambda} \geqslant c_{F,f}$$
 q.e. on F , (2.7)

$$U_{q_{\alpha},f}^{\lambda} \leqslant c_{F,f} \quad \lambda\text{-a.e. on } D,$$
 (2.8)

where

$$U_{g_{\alpha},f}^{\lambda} := U_{g_{\alpha}}^{\lambda} + f$$

is said to be the f-weighted potential of λ , while

$$c_{F,f} := \int U_{g_{\alpha},f}^{\lambda} d\lambda \in (-\infty, \infty)$$
 (2.9)

is referred to as the f-weighted equilibrium constant.

Thus, if the solution $\lambda_{F,f}$ to problem (2.4) exists, then necessarily

$$\lambda_{F,f} \in \Lambda_{F,f}$$
,

where

$$\Lambda_{F,f} := \left\{ \mu \in \mathcal{E}_{g_{\alpha}}^{+}(D) : U_{g_{\alpha},f}^{\mu} \geqslant c_{F,f} \quad \text{q.e. on } F \right\}.$$
 (2.10)

(Note that for any $\nu \in \mathcal{E}_{g_{\alpha}}$, $U_{g_{\alpha}}^{\nu}$ is well defined and finite q.e. on D, cf. [13, Corollary to Lemma 3.2.3], the kernel g_{α} being strictly positive definite; and hence so is $U_{g_{\alpha},f}^{\nu}$.) We are now in a position to formulate the main results of the current paper.

Theorem 2.2. If $\vartheta_{g_{\alpha}}^{F}(F) = 1$, then the solution $\lambda_{F,f}$ to problem (2.4) does exist. Furthermore, then

$$\lambda_{F,f} = \vartheta_{g_{\alpha}}^F, \quad c_{F,f} = 0, \quad w_{g_{\alpha},f}(F) = I_{g_{\alpha},f}(\vartheta_{g_{\alpha}}^F) = -I_{g_{\alpha}}(\vartheta_{g_{\alpha}}^F) \in (-\infty,0), \quad (2.11)$$

while $\lambda_{F,f}$ can alternatively be characterized by any one of the following (i₂)-(iii₂):

(i₂) $\lambda_{F,f}$ is the unique measure in the class $\mathcal{E}_{g_{\alpha}}^{+}(F)$ such that

$$U_{g_{\alpha},f}^{\lambda_{F,f}} = c_{F,f}$$
 q.e. on F .

⁹It is worth noting that in the case $c_{\kappa_{\alpha}}(Y) = 0$, problem (2.4) is reduced to that on minimizing $I_{\kappa_{\alpha},f}(\mu) := \|\mu\|_{\kappa_{\alpha}}^2 - 2 \int U_{\kappa_{\alpha}}^{\vartheta} d\mu$ over the class $\mathcal{E}_{\kappa_{\alpha}}^+(F \cup Y, 1)$, investigated in details in [27, 31].

(ii₂) $\lambda_{F,f}$ is the unique measure in the class $\Lambda_{F,f}$ such that

$$U_{g_{\alpha}}^{\lambda_{F,f}} = \min_{\mu \in \Lambda_{F,f}} U_{g_{\alpha}}^{\mu}$$
 q.e. on D ,

 $\Lambda_{F,f}$ being introduced by means of (2.10).

(iii₂) $\lambda_{F,f}$ is the unique measure in the class $\Lambda_{F,f}$ such that

$$\|\lambda_{F,f}\|_{g_{\alpha}} = \min_{\mu \in \Lambda_{F,f}} \|\mu\|_{g_{\alpha}}.$$

Theorem 2.3. If $c_{g_{\alpha}}(F) < \infty$, then the solution $\lambda_{F,f}$ to problem (2.4) does exist. Assume in addition that $\vartheta_{g_{\alpha}}^{F}(F) \leq 1$. Then this $\lambda_{F,f}$ can be written in the form¹⁰

$$\lambda_{F,f} = \vartheta_{g_{\alpha}}^{F} + c_{F,f} \gamma_{F,g_{\alpha}}, \tag{2.12}$$

where the f-weighted equilibrium constant $c_{F,f}$ admits the representation

$$c_{F,f} = \frac{1 - \vartheta_{g_{\alpha}}^{F}(F)}{c_{g_{\alpha}}(F)} \in [0, \infty), \tag{2.13}$$

and moreover $\lambda_{F,f}$ is uniquely characterized by any one of the above (i_2) – (iii_2) .

Theorem 2.4. Unless $\alpha < 2$, assume that Ω is connected. If moreover

$$\omega_{\alpha}(x, \overline{\mathbb{R}^n} \setminus D; \Omega) = 0 \quad \text{on all of } \Omega,$$
 (2.14)

then

$$\lambda_{F,f} \text{ exists } \iff \vartheta(D) \geqslant 1,$$

or equivalently

$$\lambda_{F,f} \text{ exists } \iff \vartheta_{q_{\alpha}}^{F}(D) \geqslant 1.$$

In the particular case $\vartheta(D) = 1$, we actually have $\lambda_{F,f} = \vartheta_{g_{\alpha}}^{F}$, and so Theorem 2.2 is fully applicable.

Let \check{F} be the reduced kernel of F, defined as the set of all $x \in F$ such that $c_{g_{\alpha}}(F \cap U_x) > 0$ for any open neighborhood U_x of x in D (cf. [17, p. 164]). By virtue of the countable subadditivity of $c_{g_{\alpha}}(\cdot)$ on Borel subsets of D, cf. [13, Lemma 2.3.5], we have $c_{g_{\alpha}}(F \setminus \check{F}) = 0$. It is also easy to see that the reduced kernel of a relatively closed subset of D is likewise relatively closed.

Theorem 2.5. Unless $\alpha < 2$, assume Ω is connected. Also assume that either

$$\vartheta_{q_{\alpha}}^{F}(F) = 1, \tag{2.15}$$

or

$$c_{g_{\alpha}}(F) < \infty \quad and \quad \vartheta_{g_{\alpha}}^{F}(F) \leqslant 1.$$
 (2.16)

Then

$$S(\lambda_{F,f}; D) = \begin{cases} \check{F} & \text{if } \alpha < 2, \\ \partial_D \check{F} & \text{otherwise.} \end{cases}$$
 (2.17)

We denote by \mathfrak{C}_F the upward directed set of all compact subsets K of F, where $K_1 \leq K_2$ if and only if $K_1 \subset K_2$. If a net $(x_K)_{K \in \mathfrak{C}_F} \subset Y$ converges to $x_0 \in Y$, Y being a topological space, then we shall indicate this fact by writing

$$x_K \to x_0$$
 in Y as $K \uparrow F$.

¹⁰Here $\gamma_{F,g_{\alpha}}$ denotes the g_{α} -equilibrium measure on F, normalized by $\gamma_{F,g_{\alpha}}(F) = c_{g_{\alpha}}(F)$, cf. Section 1.1. Also note that $\vartheta_{g_{\alpha}}^{F}(F) \leq 1$ necessarily holds if $\vartheta(D) \leq 1$, cf. (1.19).

¹¹This fails to hold if $\vartheta(D) = 1$ is replaced by $\vartheta(D) > 1$, for then, by Theorem 1.10, $\vartheta_{g_{\alpha}}^{F}(D) > 1$, whence $\lambda_{F,f} \neq \vartheta_{g_{\alpha}}^{F}$.

In particular, it follows from (3.20) and (3.21) (see below) that 12

$$w_{g_{\alpha},f}(K) \downarrow w_{g_{\alpha},f}(F)$$
 in \mathbb{R} as $K \uparrow F$. (2.18)

Theorem 2.6. Assume that $\lambda_{F,f}$ exists.¹³ If $K \uparrow F$, then

$$\lambda_{K,f} \to \lambda_{F,f}$$
 strongly and vaguely in $\mathcal{E}_{g_{\alpha}}^{+}$, (2.19)

whence

$$c_{K,f} \to c_{F,f}, \tag{2.20}$$

and also there is a subsequence $(\lambda_{K_j,f})$ of the net $(\lambda_{K,f})_{K\in\mathfrak{C}_F}$ such that

$$U_{g_{\alpha}}^{\lambda_{K_{j},f}} \to U_{g_{\alpha}}^{\lambda_{F,f}}$$
 pointwise q.e. on D as $j \to \infty$. (2.21)

If moreover $\vartheta_{q_0}^F(D) \leqslant 1$, then the limit relation (2.20) can be refined as follows:

$$c_{K,f} \downarrow c_{F,f}$$
 as $K \uparrow F$. (2.22)

Theorem 2.7. Let $(F_s)_{s\in S}$ be a decreasing net of relatively closed $F_s \subset D$ with the given intersection F, and such that for some $s_1 \in S$, $c_{g_{\alpha}}(F_{s_1}) \in (0, \infty)$ and $\operatorname{dist}(S(\vartheta; D), F_{s_1}) > 0$. Then

$$w_{g_{\alpha},f}(F_s) \uparrow w_{g_{\alpha},f}(F)$$
 as s ranges through S, (2.23)

and moreover (2.19)–(2.21) hold true for $(F_s)_{s\in S}$ in place of $(K)_{K\in\mathfrak{C}_F}$. If, in addition, $\vartheta_{g_\alpha}^{F_{s_1}}(D)\leqslant 1$, then also

$$c_{F_s,f} \uparrow c_{F,f}$$
 as s ranges through S.

Remark 2.8. Theorem 2.6 remains valid if the net $(K)_{K \in \mathfrak{C}_F}$ is replaced by an increasing sequence of relatively closed sets $F_k \subset D$ with the union F and such that the minimizers $\lambda_{F_k,f}$ exist. This can be seen in a manner similar to that in the proof of Theorem 2.6 (Section 8), the only difference being in applying the monotone convergence theorem [6, Section IV.1, Theorem 3] in place of [13, Lemma 1.2.2] (see the proof of the preparatory Lemma 3.6 in Section 3.2).

Example 2.9. Let $c_{\kappa_{\alpha}}(Y) = 0$, and let $F \subset D$ be not α -thin at infinity. Also assume that either $\alpha < 2$, or Ω is connected.

Theorem 2.10. For these particular D and F, the solution $\lambda_{F,f}$ to problem (2.4) exists if and only if $\vartheta(D) \geqslant 1$. Besides, if $\vartheta(D) = 1$, then $S(\lambda_{F,f}; D)$ is given by means of formula (2.17), while otherwise $S(\lambda_{F,f}; D)$ is a compact subset of F.

For the proofs of Theorems 2.2–2.7 and 2.10, see Sections 4–10.

3. Preparatory assertions

To verify the above theorems, we first need to establish some auxiliary results.

Lemma 3.1. Unless $\alpha < 2$, assume Ω is connected. If there exists $x_0 \in \Omega$ with

$$\omega_{\alpha}(x_0, \overline{\mathbb{R}^n} \setminus D; \Omega) = 0, \tag{3.1}$$

then

$$c_{g_{\alpha}}(F) = \infty.$$

¹²Relation (2.18) remains valid if the net $(K)_{K \in \mathfrak{C}_F}$ is replaced by an increasing sequence (F_j) of relatively closed subsets of D with the union F. See also Remark 2.8.

¹³See Theorems 2.2–2.4 for sufficient conditions for this to occur.

Proof. Assuming to the contrary that $c_{g_{\alpha}}(F) < \infty$, we conclude from the perfectness of the kernel g_{α} (Section 1.5) that there exists the (unique) g_{α} -equilibrium measure $\gamma_{F,g_{\alpha}}$ (cf. [15, Theorem 4.12]), and moreover

$$U_{g_{\alpha}}^{\gamma_F} < 1$$
 on all of Ω

(for more details, see [30, Lemma 6.5]). Therefore, by virtue of [30, Lemma 6.6] with $\mu := \varepsilon_x$, where $x \in \Omega$ is arbitrary,

$$(\varepsilon_x)_{g_\alpha}^F(D) = \int U_{g_\alpha}^{\gamma_F} d\varepsilon_x = U_{g_\alpha}^{\gamma_F}(x) < 1,$$

whence

$$(\varepsilon_x)_{q_\alpha}^F(D) < \varepsilon_x(D)$$
 for all $x \in \Omega$. (3.2)

Applying Theorem 1.10 to $\mu := \varepsilon_{x_0}$, where $x_0 \in \Omega$ meets (3.1), we thus arrive at a contradiction with (3.2).

Lemma 3.2. The Gauss functional $I_{g_{\alpha},f}(\cdot)$ is strongly continuous on the strongly complete cone $\mathcal{E}_{g_{\alpha}}^{+}(F)$. Thus, if a net $(\mu_{s}) \subset \mathcal{E}_{g_{\alpha}}^{+}(F)$ is strongly Cauchy, then (μ_{s}) converges strongly (hence, vaguely) to some unique $\mu_{0} \in \mathcal{E}_{g_{\alpha}}^{+}(F)$, and moreover

$$\lim_{s} I_{g_{\alpha},f}(\mu_{s}) = I_{g_{\alpha},f}(\mu_{0}). \tag{3.3}$$

Proof. As the kernel g_{α} is perfect (see Section 1.1 for details), the cone $\mathcal{E}_{g_{\alpha}}^{+}$ is strongly complete, and moreover the strong topology on $\mathcal{E}_{g_{\alpha}}^{+}$ is finer than the vague topology. Being vaguely (hence strongly) closed subcone of the (strongly complete) cone $\mathcal{E}_{g_{\alpha}}^{+}$, cf. [6, Section III.2, Proposition 6], $\mathcal{E}_{g_{\alpha}}^{+}(F)$ is likewise strongly complete, and therefore a strongly Cauchy net $(\mu_{s}) \subset \mathcal{E}_{g_{\alpha}}^{+}(F)$ converges strongly (hence, also vaguely) to some unique $\mu_{0} \in \mathcal{E}_{g_{\alpha}}^{+}(F)$, the strong topology on $\mathcal{E}_{g_{\alpha}}(D)$ as well as the vague topology on $\mathfrak{M}(D)$ being Hausdorff. It thus remains to verify the limit relation (3.3).

In a manner similar to that in (2.5), we obtain, by use of (1.17) and (1.19),

$$I_{g_{\alpha}}(\vartheta_{g_{\alpha}}^{F}) = \int U_{g_{\alpha}}^{\vartheta_{g_{\alpha}}^{F}} d\vartheta_{g_{\alpha}}^{F} = \int U_{g_{\alpha}}^{\vartheta} d\vartheta_{g_{\alpha}}^{F} \leqslant \int U_{\kappa_{\alpha}}^{\vartheta} d\vartheta_{g_{\alpha}}^{F} \leqslant \vartheta(D)^{2}/\varrho^{n-\alpha} < \infty,$$

whence

$$\vartheta_{q_{\alpha}}^{F} \in \mathcal{E}_{q_{\alpha}}^{+}(F). \tag{3.4}$$

Noting from (1.17) that for any $\mu \in \mathcal{E}_{g_{\alpha}}^+(F)$, $U_{g_{\alpha}}^{\vartheta} = U_{g_{\alpha}}^{\vartheta_{g_{\alpha}}^F} \mu$ -a.e., and so

$$\int U_{g_{\alpha}}^{\vartheta} d\mu = \int U_{g_{\alpha}}^{\vartheta_{g_{\alpha}}^{F}} d\mu, \tag{3.5}$$

we thus get

$$I_{g_{\alpha},f}(\mu) = \|\mu\|_{g_{\alpha}}^{2} - 2\langle \vartheta_{g_{\alpha}}^{F}, \mu \rangle_{g_{\alpha}} = \|\mu - \vartheta_{g_{\alpha}}^{F}\|_{g_{\alpha}}^{2} - \|\vartheta_{g_{\alpha}}^{F}\|_{g_{\alpha}}^{2}, \tag{3.6}$$

and the strong continuity of $I_{g_{\alpha},f}(\cdot)$ on $\mathcal{E}_{g_{\alpha}}^{+}(F)$ follows.

Corollary 3.3. We have

$$w_{g_{\alpha},f}(F) \geqslant -I_{g_{\alpha}}(\vartheta_{g_{\alpha}}^F) = I_{g_{\alpha},f}(\vartheta_{g_{\alpha}}^F).$$
 (3.7)

Proof. In view of the strict positive definiteness of g_{α} , this follows from (3.6).

3.1. A dual extremal problem. Parallel with $f = -U_{g_{\alpha}}^{\vartheta}$, we shall consider the dual external field \tilde{f} , given by

$$\tilde{f} := -U_{g_{\alpha}}^{\vartheta_{g_{\alpha}}^F}. \tag{3.8}$$

It is seen from (3.5) that

$$I_{g_{\alpha},f}(\mu) = I_{g_{\alpha},\tilde{f}}(\mu)$$
 for all $\mu \in \mathcal{E}_{g_{\alpha}}^{+}(F)$,

whence

$$w_{g_{\alpha},f}(F) = w_{g_{\alpha},\tilde{f}}(F) := \inf_{\mu \in \mathcal{E}_{g_{\alpha}}^{+}(F,1)} I_{g_{\alpha},\tilde{f}}(\mu). \tag{3.9}$$

Furthermore, the (original) problem (2.4) is solvable if and only if so is the (dual) problem (3.9), and in the affirmative case

$$\lambda_{F,f} = \lambda_{F,\tilde{f}}, \quad c_{F,f} = c_{F,\tilde{f}}. \tag{3.10}$$

An advantage of the dual problem (3.9) if compared with the original problem (2.4) is that $\vartheta_{g_{\alpha}}^{F}$ is of finite g_{α} -energy, cf. (3.4), and so [28] is fully applicable.

3.2. Extremal measures. A net $(\mu_s) \subset \mathcal{E}_{g_\alpha}^+(F,1)$ is said to be minimizing in problem (2.4) if

$$\lim_{s} I_{g_{\alpha},f}(\mu_s) = w_{g_{\alpha},f}(F); \tag{3.11}$$

let $\mathbb{M}_{g_{\alpha},f}(F)$ consist of all those (μ_s) . It is clear from (2.6) that $\mathbb{M}_{g_{\alpha},f}(F) \neq \emptyset$.

Lemma 3.4. There exists a unique $\xi_{F,f} \in \mathcal{E}_{g_{\alpha}}^+(F)$ called the extremal measure in problem (2.4) and such that for every minimizing net $(\mu_s) \in \mathbb{M}_{g_{\alpha},f}(F)$,

$$\mu_s \to \xi_{F,f}$$
 strongly and vaguely in $\mathcal{E}_{g_\alpha}^+(F)$. (3.12)

This yields

$$I_{g_{\alpha},f}(\xi_{F,f}) = w_{g_{\alpha},f}(F). \tag{3.13}$$

Proof. It follows by use of standard arguments, based on the convexity of the cone $\mathcal{E}_{g_{\alpha}}^{+}(F)$, the parallelogram identity in the pre-Hilbert space $\mathcal{E}_{g_{\alpha}}$, and the strict positive definiteness of the kernel g_{α} , that any $(\mu_{s}) \in \mathbb{M}_{g_{\alpha},f}(F)$ is strongly Cauchy (cf. [28, Proof of Lemma 4.1]). This yields (3.12), the cone $\mathcal{E}_{g_{\alpha}}^{+}(F)$ being strongly complete in view of the perfectness of the kernel g_{α} . Since $I_{g_{\alpha},f}(\cdot)$ is strongly continuous on $\mathcal{E}_{g_{\alpha}}^{+}(F)$ (Lemma 3.2), (3.13) is deduced from (3.12) by substituting (3.3) into (3.11).

Noting that the mapping $\mu \mapsto \mu(D)$ is vaguely l.s.c. on $\mathfrak{M}^+(D)$ [6, Section IV.1, Proposition 4], we infer from (3.12) that, in general,

$$\xi_{F,f}(D) \leqslant 1. \tag{3.14}$$

Corollary 3.5. The solution $\lambda_{F,f}$ to problem (2.4) exists if and only if equality prevails in (3.14), i.e.

$$\xi_{F,f}(D) = 1,$$
 (3.15)

and in the affirmative case,

$$\xi_{F,f} = \lambda_{F,f}.\tag{3.16}$$

Proof. If (3.15) holds, then combining it with both $\xi_{F,f} \in \mathcal{E}_{g_{\alpha}}^+(F)$ and (3.13) gives (3.16). For the opposite, assume $\lambda_{F,f}$ exists. Since the trivial sequence $(\lambda_{F,f})$ is obviously minimizing, it must converge strongly to each of $\lambda_{F,f}$ and $\xi_{F,f}$ (Lemma 3.4), which immediately results in (3.16), the strong topology on $\mathcal{E}_{g_{\alpha}}$ being Hausdorff. \square

Lemma 3.6. For the extremal measure $\xi := \xi_{F,f}$, we have

$$U_{q_{\alpha},f}^{\xi} \geqslant C_{\xi}$$
 q.e. on F , (3.17)

where

$$C_{\xi} := \int U_{g_{\alpha},f}^{\xi} d\xi \in (-\infty, \infty). \tag{3.18}$$

Proof. As noted in Section 2.1, problem (2.4) is (uniquely) solvable for every $K \in \mathfrak{C}_F$. Our first aim is to show that those solutions form a minimizing net, i.e.

$$(\lambda_{K,f})_{K \in \mathfrak{C}_F} \in \mathbb{M}_{g_{\alpha},f}(F). \tag{3.19}$$

Since $\mathcal{E}_{g_{\alpha}}^+(K_1,1) \subset \mathcal{E}_{g_{\alpha}}^+(K_2,1) \subset \mathcal{E}_{g_{\alpha}}^+(F,1)$ for any $K_1, K_2 \in \mathfrak{C}_F$ such that $K_1 \leqslant K_2$, the net $(w_{g_{\alpha},f}(K))_{K \in \mathfrak{C}_F}$ decreases, and moreover

$$\lim_{K \in \mathfrak{C}_F} w_{g_{\alpha},f}(K) = \lim_{K \in \mathfrak{C}_F} I_{g_{\alpha},f}(\lambda_{K,f}) \geqslant w_{g_{\alpha},f}(F). \tag{3.20}$$

We are thus reduced to showing that

$$w_{g_{\alpha},f}(F) \geqslant \lim_{K \in \mathfrak{C}_F} w_{g_{\alpha},f}(K).$$
 (3.21)

Applying [13, Lemma 1.2.2] to each of the positive, l.s.c., μ -integrable functions $U_{g_{\alpha}}^{\mu}$ and $U_{g_{\alpha}}^{\vartheta}$, we see that for every $\mu \in \mathcal{E}_{q_{\alpha}}^{+}(F,1)$,

$$I_{g_{\alpha},f}(\mu) \geqslant \lim_{K \uparrow A} I_{g_{\alpha},f}(\mu|_K).$$
 (3.22)

(Regarding the μ -integrability of $U_{g_{\alpha}}^{\vartheta}$, see (3.5).) Noting that $\mu(K) \uparrow \mu(F) = 1$ as $K \uparrow F$, and letting μ range over $\mathcal{E}_{g_{\alpha}}^{+}(F,1)$, we get (3.21) from (3.22), whence (3.19). Thus, according to Lemma 3.4,

$$\lambda_{K,f} \to \xi$$
 strongly and vaguely as $K \uparrow A$. (3.23)

The strong topology on $\mathcal{E}_{g_{\alpha}}$ being first-countable, there exists a subsequence (K_j) of the net $(K)_{K \in \mathfrak{C}_A}$ such that

$$U_{g_{\alpha}}^{\lambda_{K_{j},f}} \to U_{g_{\alpha}}^{\xi}$$
 pointwise q.e. on D as $j \to \infty$, (3.24)

cf. the paragraph following Lemma 4.3.3 in [13]. Now, applying (2.7) and (2.9) to each of those $\lambda_{K_j,f}$, and then passing to the limits in the inequalities thereby obtained, we deduce (3.17) from (3.23) and (3.24), for

$$\lim_{j \to \infty} c_{K_j,f} = \lim_{j \to \infty} \left(\|\lambda_{K_j,f}\|_{g_{\alpha}}^2 - \int U_{g_{\alpha}}^{\vartheta} d\lambda_{K_j,f} \right) = \lim_{j \to \infty} \left(\|\lambda_{K_j,f}\|_{g_{\alpha}}^2 - \langle \vartheta_{g_{\alpha}}^F, \lambda_{K_j,f} \rangle_{g_{\alpha}} \right)$$
$$= \|\xi\|_{g_{\alpha}}^2 - \langle \vartheta_{g_{\alpha}}^F, \xi \rangle_{g_{\alpha}} = \|\xi\|_{g_{\alpha}}^2 - \int U_{g_{\alpha}}^{\vartheta} d\xi = \int U_{g_{\alpha},f}^{\xi} d\xi =: C_{\xi}.$$

(While doing that, we have used the countable subadditivity of the outer g_{α} -capacity, cf. [13, Lemma 2.3.5].)

4. Proof of Theorem 2.2

If $\vartheta_{g_{\alpha}}^{F}(F) = 1$, then $\vartheta_{g_{\alpha}}^{F} \in \mathcal{E}_{g_{\alpha}}^{+}(F,1)$, cf. (3.4), and therefore $I_{g_{\alpha},f}(\vartheta_{g_{\alpha}}^{F}) \geqslant w_{g_{\alpha},f}(F)$. Combined with (3.7), this implies the solvability of problem (2.4) as well as the first and the last relations in (2.11). According to (1.17) with $\mu := \vartheta$,

$$U_{g_{\alpha}}^{\vartheta} = U_{g_{\alpha}}^{\vartheta_{g_{\alpha}}^{F}}$$
 q.e. on F (hence, $\vartheta_{g_{\alpha}}^{F}$ -a.e.),

and substituting this equality into (2.9) with $\lambda := \vartheta_{g_{\alpha}}^F$ gives $c_{F,f} = 0$, whence (2.11).

To verify the characterizations (i_2) – (iii_2) of the solution $\lambda_{F,f}$, we first show that

$$\Lambda_{F,f} = \Lambda_{F,\tilde{f}},\tag{4.1}$$

 $\Lambda_{F,\tilde{f}}$ being defined by (2.10) with f replaced by the dual external field \tilde{f} , cf. (3.8). In fact, $f = \tilde{f}$ q.e. on F, and so for any $\mu \in \mathcal{E}_{g_{\alpha}}^{+}(D)$,

$$U^{\mu}_{g_{\alpha},f} = U^{\mu}_{g_{\alpha},\tilde{f}} \quad \text{q.e. on } F.$$

$$\tag{4.2}$$

As $c_{F,f} = c_{F,\tilde{f}}$ by (3.10), this results in (4.1) by use of the countable subadditivity of the outer g_{α} -capacity, cf. [13, Lemma 2.3.5]. Since, again by virtue of (3.10), $\lambda_{F,f} = \lambda_{F,\tilde{f}}$, utilizing [28, Theorem 1.6, (i) and (ii)] with $\kappa := g_{\alpha}$ and $f := -U_{g_{\alpha}}^{\vartheta_{g_{\alpha}}^{F}}$ proves (ii₂) and (iii₂). (Recall that [28, Theorem 1.6] is applicable here, for $\vartheta_{g_{\alpha}}^{F} \in \mathcal{E}_{g_{\alpha}}^{+}$.)

Finally, according to [28, Theorem 1.6(iii)] applied to $\kappa := g_{\alpha}$ and $f := -U_{g_{\alpha}}^{\vartheta_{g_{\alpha}}^{F}}$, $\lambda_{F,\tilde{f}}$ is the unique measure in $\mathcal{E}_{g_{\alpha}}^{+}(D)$ such that $U_{g_{\alpha},\tilde{f}}^{\lambda_{F,\tilde{f}}} = c_{F,\tilde{f}}$ q.e. on F. Therefore, employing (4.2) with $\mu := \lambda_{F,\tilde{f}} = \lambda_{F,f}$ yields the remaining assertion (i₂).

5. Proof of Theorem 2.3

As seen from Lemma 3.4 and Corollary 3.5, the solution $\lambda_{F,f}$ exists if and only if $\xi_{F,f}$, the extremal measure in problem (2.4), has unit total mass. Since $\xi_{F,f} \in \mathcal{E}_{g_{\alpha}}^+(F)$ is the strong limit of any minimizing net $(\mu_s) \subset \mathcal{E}_{g_{\alpha}}^+(F,1)$ (Lemma 3.4), whereas $\mathcal{E}_{g_{\alpha}}^+(F,1)$ is strongly complete due to the assumption $c_{g_{\alpha}}(F) < \infty$ (see Theorem 1.1, applied to the perfect kernel g_{α}), the required equality $\xi_{F,f}(F) = 1$ follows.

In the rest of this proof, $\vartheta_{g_{\alpha}}^{F}(F) \leqslant 1$. Then

$$\eta_{F,f} := \frac{1 - \vartheta_{g_{\alpha}}^F(F)}{c_{g_{\alpha}}(F)} \in [0, \infty), \tag{5.1}$$

whence

$$\lambda := \vartheta_{g_{\alpha}}^{F} + \eta_{F,f} \gamma_{F,g_{\alpha}} \in \mathcal{E}_{g_{\alpha}}^{+}(F,1). \tag{5.2}$$

Moreover,

$$U_{g_{\alpha},f}^{\lambda} = U_{g_{\alpha}}^{\lambda} + f = \left(U_{g_{\alpha}}^{\vartheta_{g_{\alpha}}^{F}} - U_{g_{\alpha}}^{\vartheta}\right) + \eta_{F,f}U_{g_{\alpha}}^{\gamma_{F,g_{\alpha}}} = \eta_{F,f} \quad \text{q.e. on } F,$$

the last equality being derived from (1.4) and (1.17) by use of the countable subadditivity of the outer capacity. By virtue of Theorem 2.1, we thus have

$$\lambda = \lambda_{F,f}$$
 and $\eta_{F,f} = c_{F,f}$,

which substituted into (5.1) and (5.2) proves (2.13) and (2.12), respectively.

Noting that under the accepted assumptions $c_{g_{\alpha}}(F) < \infty$ and $\vartheta_{g_{\alpha}}^{F}(F) \leq 1$, [28, Theorem 1.6] is fully applicable to $\kappa := g_{\alpha}$ and $f := -U_{g_{\alpha}}^{\vartheta_{g_{\alpha}}^{F}}$, we arrive at (i₂)-(iii₂) in the same manner as in the last two paragraphs of the preceding Section 4.

6. Proof of Theorem 2.4

Assume that (2.14) holds, and that either $\alpha < 2$ or $\Omega := D \setminus F$ is connected. Then, by virtue of Lemma 3.1,

$$c_{g_{\alpha}}(F) = \infty. (6.1)$$

Furthermore, according to Theorem 1.10, $\vartheta(D) = \vartheta_{g_{\alpha}}^{F}(D)$, and hence, in view of Theorem 2.2, we are reduced to the case where $\vartheta(D) \neq 1$, or equivalently

$$\vartheta_{g_{\alpha}}^{F}(D) \neq 1.$$

Suppose first that $q:=\vartheta_{g_{\alpha}}^F(D)<1$. The locally compact space D being σ -compact, there is a sequence of relatively compact, open sets U_j with the union D and such that $\operatorname{Cl}_D U_j \subset U_{j+1}$, see [4, Section I.9, Proposition 15]. Because of (6.1), $c_{g_{\alpha}}(F\setminus U_j)=\infty,\ j\in\mathbb{N},^{14}$ and hence one can choose $(\tau_j)\subset\mathcal{E}_{g_{\alpha}}^+(F,1)$ such that

$$S(\tau_j; D) \subset D \setminus \operatorname{Cl}_D U_j, \quad \|\tau_j\|_{q_\alpha} < 1/j.$$
 (6.2)

Define

$$\mu_j := \vartheta_{q_\alpha}^F + (1 - q)\tau_j. \tag{6.3}$$

Clearly, $\mu_j \in \mathcal{E}_{q_\alpha}^+(F,1)$ for all j, and therefore, on account of Corollary 3.3,

$$-\|\vartheta_{g_{\alpha}}^{F}\|_{g_{\alpha}}^{2} \leqslant w_{g_{\alpha},f}(F) \leqslant \lim_{j \to \infty} I_{g_{\alpha},f}(\mu_{j}) = \lim_{j \to \infty} \left(\|\mu_{j}\|_{g_{\alpha}}^{2} - 2 \int U_{g_{\alpha}}^{\vartheta} d\mu_{j} \right)$$

$$= \lim_{j \to \infty} \left(\|\mu_{j}\|_{g_{\alpha}}^{2} - 2 \langle \vartheta_{g_{\alpha}}^{F}, \mu_{j} \rangle_{g_{\alpha}} \right) = \|\vartheta_{g_{\alpha}}^{F}\|_{g_{\alpha}}^{2} - 2 \|\vartheta_{g_{\alpha}}^{F}\|_{g_{\alpha}}^{2} = -\|\vartheta_{g_{\alpha}}^{F}\|_{g_{\alpha}}^{2},$$

the last but one equality being obtained from (6.3) and $\|\tau_j\|_{g_\alpha} < 1/j$ by making use of the Cauchy–Schwarz (Bunyakovski) inequality. This implies that the sequence (μ_j) is minimizing in problem (2.4), i.e. $(\mu_j) \in \mathbb{M}_{g_\alpha,f}(F)$, and hence it converges strongly and vaguely in $\mathcal{E}_{g_\alpha}^+(F)$ to the (unique) extremal measure $\xi := \xi_{F,f}$ (Lemma 3.4). Since any compact subset of D is contained in U_j for all j large enough, $\tau_j \to 0$ vaguely in consequence of the former relation in (6.2), whence $\xi = \vartheta_{g_\alpha}^F$. Thus $\xi(D) = q < 1$, and applying Corollary 3.5 shows that problem (2.4) is indeed unsolvable.

It thus remains to analyze the case where $\vartheta_{g_{\alpha}}^{F}(D) > 1$. Assume first that $C_{\xi} \ge 0$, C_{ξ} being introduced by means of (3.18). Then, according to (3.17),

$$U_{g_{\alpha}}^{\xi} \geqslant C_{\xi} + U_{g_{\alpha}}^{\vartheta}$$
 q.e. on F ,

whence

$$U_{g_{\alpha}}^{\xi} \geqslant U_{g_{\alpha}}^{\vartheta_{g_{\alpha}}^{F}}$$
 q.e. on F .

Applying to ξ , $\vartheta_{g_{\alpha}}^{F} \in \mathcal{E}_{g_{\alpha}}^{+}(F)$ the refined principle of positivity of mass for g_{α} -potentials as stated in [30, Theorem 5.1], which is possible due to (2.14), we therefore get

$$\xi(D) \geqslant \vartheta_{g_{\alpha}}^{F}(D) > 1,$$

which however contradicts (3.14). Thus necessarily

$$C_{\varepsilon} < 0. \tag{6.4}$$

Integrating (3.17) with respect to ξ , and then substituting (3.18) into the inequality thereby obtained, we have

$$C_{\xi} = \int U_{g_{\alpha},f}^{\xi} d\xi \geqslant C_{\xi} \cdot \xi(D),$$

whence $\xi(D) \ge 1$ in view of (6.4). Combined with (3.14) this implies that, actually, $\xi(D) = 1$, and an application of Corollary 3.5 then shows that problem (2.4) is indeed solvable with ξ serving as the solution $\lambda_{F,f}$. The proof is complete.

¹⁴This follows from the subadditivity of the outer g_{α} -capacity by use of the fact that, due to the strict positive definiteness of the kernel g_{α} , the g_{α} -capacity of any compact set $K \subset D$ is finite.

7. Proof of Theorem 2.5

Unless $\alpha < 2$, assume Ω is connected. Also assume that either (2.15) or (2.16) is fulfilled. Then according to Theorems 2.2 and 2.3, the solution $\lambda_{F,f}$ to problem (2.4) does exist, and moreover

$$\lambda_{F,f} = \begin{cases} \vartheta_{g_{\alpha}}^{F} & \text{if (2.15) holds,} \\ \vartheta_{g_{\alpha}}^{F} + c_{F,f} \gamma_{F,g_{\alpha}} & \text{otherwise,} \end{cases}$$
 (7.1)

where $c_{F,f}$, the f-weighted equilibrium constant for the set F, is ≥ 0 .

On account of (1.18), we conclude from [22, Theorem 7.2], providing a description of the support $S(\mu_{\kappa_{\alpha}}^{Q}; \mathbb{R}^{n})$, where $Q \subset \mathbb{R}^{n}$ is closed and $\mu \in \mathfrak{M}^{+}(\mathbb{R}^{n})$, that

$$S(\vartheta_{g_{\alpha}}^{F}; D) = \begin{cases} \check{F} & \text{if } \alpha < 2, \\ \partial_{D}\check{F} & \text{otherwise.} \end{cases}$$
 (7.2)

In case $c_{g_{\alpha}}(F) < \infty$, the same formula (7.2) holds true for the g_{α} -equilibrium measure $\gamma_{F,g_{\alpha}}$ in place of $\vartheta_{g_{\alpha}}^{F}$, which can be seen with the aid of arguments similar to those in [22, Proof of Theorem 7.2], now applied to the functions $U_{\kappa_{\alpha}}^{\gamma_{F,g_{\alpha}}}$ and $1 + U_{\kappa_{\alpha}}^{(\gamma_{F,g_{\alpha}})_{\kappa_{\alpha}}^{Y}}$, α -superharmonic and α -harmonic on D, respectively. While doing that, we utilize a description of the g_{α} -equilibrium potential $U_{g_{\alpha}}^{\gamma_{F,g_{\alpha}}}$, provided by [30, Lemma 6.5], as well as the representation (1.16) for g_{α} -potentials of extendible measures on D.

In view of (7.1), all this results in (2.17), thereby completing the proof.

8. Proof of Theorem 2.6

As noticed in Section 2.1, problem (2.4) is (uniquely) solvable for every $K \in \mathfrak{C}_F$. Moreover, it is shown in the proof of Lemma 3.6, see (3.19), that those solutions form a minimizing net, i.e. $(\lambda_{K,f})_{K \in \mathfrak{C}_F} \in \mathbb{M}_{g_{\alpha},f}(F)$. Thus, by virtue of Lemma 3.4,

$$\lambda_{K,f} \to \xi$$
 strongly and vaguely in $\mathcal{E}_{g_{\alpha}}^{+}(F)$ as $K \uparrow F$, (8.1)

 $\xi := \xi_{F,f}$ being the extremal measure. On the other hand, problem (2.4) is assumed to be solvable; therefore, according to Corollary 3.5,

$$\xi = \lambda_{F,f},\tag{8.2}$$

which substituted into (8.1) leads to (2.19).

In the same manner as in the proof of (3.24), we deduce from (8.1) that there exists a subsequence (K_j) of the net $(K)_{K \in \mathfrak{C}_A}$ such that

$$U_{g_{\alpha}}^{\lambda_{K_{j},f}} \to U_{g_{\alpha}}^{\xi}$$
 pointwise q.e. on D as $j \to \infty$,

which combined with (8.2) results in (2.21).

It follows from (2.9) that

$$c_{K,f} = \int U_{g_{\alpha},f}^{\lambda_{K,f}} d\lambda_{K,f} = \|\lambda_{K,f}\|_{g_{\alpha}}^2 + \int U_{g_{\alpha}}^{\vartheta} d\lambda_{K,f} = \|\lambda_{K,f}\|_{g_{\alpha}}^2 + \left\langle \vartheta_{g_{\alpha}}^F, \lambda_{K,f} \right\rangle_{g_{\alpha}},$$

and similarly

$$c_{F,f} = \|\lambda_{F,f}\|_{g_{\alpha}}^2 + \langle \vartheta_{g_{\alpha}}^F, \lambda_{F,f} \rangle_{g_{\alpha}}.$$

By the strong convergence of $(\lambda_{K,f})_{K \in \mathfrak{C}_F}$ to $\lambda_{F,f}$, this yields (2.20).

To complete the proof, assume now that $\vartheta_{g_{\alpha}}^{F}(F) \leqslant 1$. For any two relatively closed $F_1, F_2 \subset D$ such that $F_1 \subset F_2$, and any bounded $\mu \in \mathfrak{M}^+(D \setminus F_2; D)$, we have

$$\mu_{g_{\alpha}}^{F_1} = \left(\mu_{g_{\alpha}}^{F_2}\right)_{g_{\alpha}}^{F_1}$$

(balayage "with a rest", see [30, Corollary 2.5]), and therefore, by virtue of (1.19),

$$\mu_{q_{\alpha}}^{F_1}(D) \leqslant \mu_{q_{\alpha}}^{F_2}(D).$$

Thus the net $(\vartheta_{g_{\alpha}}^{K}(D))_{K \in \mathfrak{C}_{F}}$ increases and does not exceed $\vartheta_{g_{\alpha}}^{F}(D)$ ($\leqslant 1$), which implies that Theorem 2.3 is applicable to any $K \in \mathfrak{C}_{F}$. This gives

$$c_{K,f} = \frac{1 - \vartheta_{g_{\alpha}}^{K}(D)}{c_{g_{\alpha}}(K)},$$

and so $(c_{K,f})_{K\in\mathfrak{C}_F}$ decreases. In view of (2.20), we get (2.22), whence the theorem.

9. Proof of Theorem 2.7

We first observe from the monotonicity of the net $(F_s)_{s\in S}$ that $(w_{g_{\alpha},f}(F_s))_{s\in S}$ is bounded and increasing, and moreover

$$\lim_{s \in S} w_{g_{\alpha},f}(F_s) \leqslant w_{g_{\alpha},f}(F). \tag{9.1}$$

Since both $c_{g_{\alpha}}(F_{s_1})$ and $c_{g_{\alpha}}(F)$ are nonzero and finite, so are $c_{g_{\alpha}}(F_s)$ for all $s \geq s_0$, whence the minimizers $\lambda_{F_s,f}$ do exist (Theorem 2.3). Furthermore, for any $t \geq s_0$,

$$(\lambda_{F_s,f})_{s\geqslant t}\subset \mathcal{E}_{q_\alpha}^+(F_t,1).$$

By the convexity of $\mathcal{E}_{g_{\alpha}}^{+}(F_{t},1)$, $(\lambda_{F_{t},f}+\lambda_{F_{s},f})/2\in\mathcal{E}_{g_{\alpha}}^{+}(F_{t},1)$ for all $s\geqslant t$, hence

$$4w_{g_{\alpha},f}(F_t) \leqslant 4I_{g_{\alpha},f}\left(\frac{\lambda_{F_t,f} + \lambda_{F_s,f}}{2}\right) = \|\lambda_{F_t,f} + \lambda_{F_s,f}\|_{g_{\alpha}}^2 + 4\int f \, d(\lambda_{F_t,f} + \lambda_{F_s,f}).$$

Using the parallelogram identity in the pre-Hilbert space $\mathcal{E}_{g_{\alpha}}$ therefore gives

$$0 \leqslant \|\lambda_{F_t,f} - \lambda_{F_s,f}\|_{g_{\alpha}}^2 \leqslant -4w_{g_{\alpha},f}(F_t) + 2I_{g_{\alpha},f}(\lambda_{F_t,f}) + 2I_{g_{\alpha},f}(\lambda_{F_s,f})$$
$$= 2w_{g_{\alpha},f}(F_s) - 2w_{g_{\alpha},f}(F_t),$$

whence the net $(\lambda_{F_s,f})_{s\geqslant t}\subset \mathcal{E}_{g_\alpha}^+(F_t,1)$ is strongly Cauchy. Since $\mathcal{E}_{g_\alpha}^+(F_t,1)$ is complete in the (induced) strong topology (cf. Theorem 1.1 with $\kappa:=g_\alpha$), $(\lambda_{F_s,f})_{s\geqslant t}$ converges strongly (hence vaguely, the kernel g_α being perfect) to some unique $\lambda\in\mathcal{E}_{g_\alpha}^+(F_t,1)$. Noting that this holds for each $t\geqslant s_0$, we thus get $\lambda\in\mathcal{E}_{g_\alpha}^+(F,1)$, and therefore

$$\lim_{s \in S} w_{g_{\alpha},f}(F_s) = \lim_{s \in S} I_{g_{\alpha},f}(\lambda_{F_s,f}) = I_{g_{\alpha},f}(\lambda) \geqslant w_{g_{\alpha},f}(F), \tag{9.2}$$

the latter equality being derived from the strong continuity of $I_{g_{\alpha},f}(\cdot)$ on $\mathcal{E}_{g_{\alpha}}(F_{s_1})$ (Lemma 3.2). Combining (9.2) with (9.1) proves (2.23) as well as $\lambda = \lambda_{F,f}$.

It has thus been shown that $\lambda_{F_s,f} \to \lambda_{F,f}$ strongly and vaguely in $\mathcal{E}_{g_\alpha}^+$ as s ranges through S. The rest of the proof runs in a way similar to that in the proof of Theorem 2.6 (see the last three paragraphs in the preceding section).

10. Proof of Theorem 2.10

To begin with, we first observe that under the requirements of Theorem 2.10, the set F must be unbounded in \mathbb{R}^n , whence the following lemma holds true.

Lemma 10.1. We have

$$\lim_{|x| \to \infty, \ x \in F} U^{\vartheta}_{\kappa_{\alpha}}(x) = 0. \tag{10.1}$$

Proof. This only needs to be verified when $S(\vartheta; D)$ is unbounded in \mathbb{R}^n . Since $\vartheta(D)$ is finite, for any $\varepsilon \in (0, \infty)$, there is an open neighborhood U of $\infty_{\mathbb{R}^n}$ such that

$$\vartheta(U) < \varepsilon \varrho^{n-\alpha},$$

 $\varrho \in (0, \infty)$ being defined by means of (2.1), whence

$$\int \kappa_{\alpha}(x,y) \, d\vartheta|_{U}(y) < \varepsilon \quad \text{for all } x \in F.$$
(10.2)

On the other hand, one can choose a neighborhood U' of $\infty_{\mathbb{R}^n}$ so that $U' \subset U$ and

$$\int \kappa_{\alpha}(x,y) d(\vartheta - \vartheta|_{U})(y) < \varepsilon \quad \text{for all } x \in U',$$

which together with (10.2) proves (10.1).

We further note that, since $c_{\kappa_{\alpha}}(Y) = 0$ while F is not α -thin at infinity,

$$\omega_{\alpha}(x, \overline{\mathbb{R}^n} \setminus D; \Omega) = 0 \quad \text{for all } x \in \Omega,$$
 (10.3)

cf. definitions (1.13), (1.14) and Corollary 1.6. Therefore, according to Theorem 2.4, the minimizer $\lambda_{F,f}$ exists if and only if $\vartheta(D) \geqslant 1$. Furthermore, if $\vartheta(D) = 1$, then, in consequence of (10.3), $\vartheta_{g_{\alpha}}^{F}(D) = 1$ (see Theorem 1.10), and applying Theorem 2.5 shows that a description of $S(\lambda_{F,f}; D)$ is given, indeed, by means of (2.17).

It remains to analyze the case $\vartheta(D) > 1$. Assume, to the contrary, that $S(\lambda_{F,f}; D)$ is unbounded, and so there is a sequence $(x_j) \subset S(\lambda_{F,f}; D) \subset F$ approaching $\infty_{\mathbb{R}^n}$. Noting that $U_{g_\alpha}^{\vartheta}|_F$ is continuous while $U_{g_\alpha}^{\lambda_{F,f}}|_F$ is l.s.c., we deduce from (2.8) that

$$U_{g_{\alpha},f}^{\lambda_{F,f}}(x_j) = U_{g_{\alpha}}^{\lambda_{F,f}}(x_j) - U_{g_{\alpha}}^{\vartheta}(x_j) \leqslant c_{F,f} \quad \text{for all } j,$$
(10.4)

 $c_{F,f}$ being the f-weighted equilibrium constant. According to Corollary 3.5,

$$\lambda_{F,f} = \xi_{F,f} =: \xi,$$

where $\xi_{F,f}$ is the extremal measure, introduced in Lemma 3.4. Comparing (2.9) and (3.18) therefore gives $c_{F,f} = C_{\xi}$, whence, by virtue of (6.4),

$$c_{F,f} < 0.$$

Substituting this into (10.4) and then letting $j \to \infty$, we see from Lemma 10.1 that

$$\liminf_{x \to \infty_{\mathbb{R}^n}, \ x \in F} U_{\kappa_{\alpha}}^{\lambda_{F,f}}(x) = \liminf_{x \to \infty_{\mathbb{R}^n}, \ x \in F} U_{g_{\alpha}}^{\lambda_{F,f}}(x) < 0,$$

the equality being valid since $g_D^{\alpha}(x,y) = \kappa_{\alpha}(x,y)$ for all $(x,y) \in D \times \mathbb{R}^n$ because of the assumption $c_{\kappa_{\alpha}}(Y) = 0$. This, however, contradicts [16, Remark 4.12], the set F not being α -thin at infinity. The proof is complete.

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This manuscript has no associated data.

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