

Weak and mild solutions to the MHD equations and the viscoelastic Navier–Stokes equations with damping in Wiener amalgam spaces

Chen-Chih Lai *

Department of Mathematics, Columbia University, New York, NY 10027, USA

Abstract

We study the three-dimensional incompressible magnetohydrodynamic (MHD) equations and the incompressible viscoelastic Navier–Stokes equations with damping. Building on techniques developed by Bradshaw, Lai, and Tsai (Math. Ann. 2024), we prove the existence of mild solutions in Wiener amalgam spaces that satisfy the corresponding spacetime integral bounds. In addition, we construct global-in-time local energy weak solutions in these amalgam spaces using the framework introduced by Bradshaw and Tsai (SIAM J. Math. Anal. 2021). As part of this construction, we also establish several properties of local energy solutions with L^2_{uloc} initial data, including initial and eventual regularity as well as small-large uniqueness, extending analogous results obtained for the Navier–Stokes equations by Bradshaw and Tsai (Comm. Partial Differential Equations 2020).

Contents

1	Introduction	1
1.1	Mild solutions of MHD equations	4
1.2	Mild solutions of viscoelastic Navier–Stokes equations with damping	6
1.3	Weak solutions of MHD equations	7
1.4	Weak solutions of viscoelastic Navier–Stokes equations with damping	9
2	Construction of mild solutions in Wiener amalgam spaces	11
2.1	Mild solutions in subcritical spaces: Proof of Theorem 1.1	11
2.2	Mild solutions in critical spaces with small data: Proof of Theorem 1.2	14
2.3	Mild solutions in critical spaces with enough decay: Proof of Theorem 1.3	21
3	Local energy solutions in Wiener amalgam spaces	27
3.1	Eventual regularity for local energy solutions	33
3.2	A priori bounds and explicit growth rate	34
3.3	Global existence	43

1 Introduction

The incompressible magnetohydrodynamic (MHD) equations describe the interaction of a fluid’s velocity field and a magnetic field within a conducting medium, coupling the incompressible Navier–Stokes equations with Maxwell’s equations of electromagnetism. These fundamental equations are

*Email address: cclai.math@gmail.com

given by

$$\left. \begin{aligned} \partial_t v - \Delta v + v \cdot \nabla v - b \cdot \nabla b + \nabla \pi &= 0 \\ \partial_t b - \Delta b + v \cdot \nabla b - b \cdot \nabla v &= 0 \\ \nabla \cdot v = \nabla \cdot b &= 0 \end{aligned} \right\} \text{ in } \mathbb{R}^3 \times (0, \infty), \quad (\text{MHD})$$

where v is the velocity, b the magnetic field, and π the pressure. The study of MHD equations has attracted considerable attention. The foundational result of Duvaut and Lions [11] established the global existence of weak solutions with finite energy. Building upon this, Sermange and Temam [38] investigated regularity criteria for weak solutions. Subsequent effort refined these regularity conditions under various assumptions. For instance, Wu [41] and Zhou [44] established Serrin-type criteria and scaling-invariant regularity conditions, while He and Xin [15] and Kang and Lee [19] developed partial regularity results for suitable weak solutions. Further improvements were made via harmonic analysis methods, as in [9], and directionally-restricted criteria, such as those by Cao and Wu [7]. More recently, local regularity theory for MHD has been advanced in parabolic Morrey spaces [8], and Fernández-Dalgo and Jarrín [13] provided weak-strong uniqueness results in weighted L^2 spaces, alongside constructions of weak suitable solutions in local Morrey spaces.

On the existence side, Miao, Yuan, and Zhang [35] proved global mild solutions for small data in BMO^{-1} , and He and Xin [16] constructed self-similar solutions under small homogeneous initial data. Moreover, the existence of forward discretely self-similar and self-similar local Leray solutions is established in the critical space $L^{3,\infty}$ [25] and in the weighted L^2 spaces [12]. The criticality of the $L^{3,\infty}$ class was also highlighted in [34], where global regularity of weak solutions was established under this condition. Additional contributions include the construction of global smooth solutions under spectral constraints [30], the use of Morrey spaces to ensure global well-posedness for small data [31], and the construction of forward self-similar solutions via topological and blow-up methods [42].

Complementing the MHD system, the incompressible viscoelastic Navier–Stokes equations with damping (vNSEd) model non-Newtonian fluids with both viscous and elastic characteristics. In the simplified setting where both relaxation and retardation times are infinite, the vNSEd system reads

$$\left. \begin{aligned} \partial_t v - \Delta v + v \cdot \nabla v - \nabla \cdot (\mathbf{F}\mathbf{F}^\top) + \nabla \pi &= 0 \\ \partial_t \mathbf{F} + v \cdot \nabla \mathbf{F} - (\nabla v)\mathbf{F} &= 0 \\ \nabla \cdot v &= 0 \end{aligned} \right\} \text{ in } \mathbb{R}^3 \times (0, \infty), \quad (1.1)$$

with initial data

$$v|_{t=0} = v_0 \quad \text{and} \quad \mathbf{F}|_{t=0} = \mathbf{F}_0 \quad \text{in } \mathbb{R}^3,$$

where v is the velocity field, \mathbf{F} is the local deformation tensor of the fluid, and π is the pressure. This model arises from Oldroyd-type theories for viscoelastic fluids and captures the interplay between fluid motion and elastic stresses. The addition of a damping term in the equation for \mathbf{F} (following Lin–Liu–Zhang [29]) is critical for obtaining global solutions, particularly in the absence of intrinsic dissipative mechanisms. To be more precise, they introduced the following viscoelastic Navier–Stokes equations with damping as a way to approximate solutions of (1.1):

$$\left. \begin{aligned} \partial_t v - \Delta v + v \cdot \nabla v - \nabla \cdot (\mathbf{F}\mathbf{F}^\top) + \nabla \pi &= 0 \\ \partial_t \mathbf{F} - \mu \Delta \mathbf{F} + v \cdot \nabla \mathbf{F} - (\nabla v)\mathbf{F} &= 0 \\ \nabla \cdot v &= 0 \end{aligned} \right\} \text{ in } \mathbb{R}^3 \times (0, \infty), \quad (1.2)$$

for a damping parameter $\mu > 0$. Existence results for smooth solutions under smallness conditions or specific symmetries have been established by various authors [10, 26, 29]. Note that if $\nabla \cdot \mathbf{F} = 0$

at some instance of time, then $\nabla \cdot \mathbf{F} = 0$ at all later times. In fact, by taking divergence of (1.2)₂ and using (1.2)₃, one have the following equation for $\nabla \cdot \mathbf{F}$:

$$\partial_t(\nabla \cdot \mathbf{F}) + v \cdot \nabla(\nabla \cdot \mathbf{F}) = \mu \Delta(\nabla \cdot \mathbf{F}).$$

Hence it is natural to assume

$$\nabla \cdot \mathbf{F} = 0.$$

The authors [29] noted that using standard weak convergence methods to pass the limit of solutions to (1.2) as $\mu \rightarrow 0^+$ does not yield weak solutions of (1.1). Despite this, system (1.2) remains an interesting subject of study. For instance, Lai, Lin, and Wang [24] established the existence of forward self-similar classical solution to (1.2) for locally Hölder continuous, (-1) -homogeneous initial data. Additionally, the existence of forward discretely self-similar and self-similar local Leray solutions in the critical space $L^{3,\infty}$ is established in [25], following the analysis in [2].

Regularity issues for weak solutions of the viscoelastic Navier–Stokes equations with damping have been investigated from several perspectives. Hynd [17] proved a version of the Caffarelli–Kohn–Nirenberg partial regularity theorem adapted to the viscoelastic system with damping, while Kim [22] established Serrin-type regularity criteria in weak- L^p spaces. These results have been further extended in [39], which proved global existence of mild solutions in scaling-invariant spaces for small data and derived various regularity criteria in Lorentz, multiplier, BMO, and Besov spaces. Additional contributions include the construction of global classical solutions with symmetry assumptions in periodic domains by Liu and Lin [32], and refined local energy bounds leading to improved ϵ -regularity conditions in the sense of Caffarelli–Kohn–Nirenberg in [43].

Since the damping parameter μ does not affect our analysis, we set $\mu = 1$ throughout this paper. Then, columnwisely, (1.2) can be rewritten as

$$\left. \begin{aligned} \partial_t v - \Delta v + v \cdot \nabla v - \sum_{n=1}^3 f_n \cdot \nabla f_n + \nabla \pi &= 0 \\ \partial_t f_m - \Delta f_m + v \cdot \nabla f_m - f_m \cdot \nabla v &= 0 \\ \nabla \cdot f_m = \nabla \cdot v &= 0 \end{aligned} \right\} \text{ in } \mathbb{R}^3 \times (0, \infty), \quad m = 1, 2, 3, \quad (\text{vNSEd})$$

where f_m is the m -th column vector of \mathbf{F} .

A central theme in the analysis of both the (MHD) and (vNSEd) systems is the interplay between the nonlinear couplings, scaling symmetries, and the functional framework chosen for solutions. While much progress has been made in classical Lebesgue, Sobolev, and Besov spaces, recent advances have highlighted the utility of Wiener amalgam spaces in studying fluid systems. In this paper, the Wiener amalgam spaces are denoted E_q^p and defined by the norm

$$\|f\|_{E_q^p} := \left\| \|f\|_{L^p(B_1(k))} \right\|_{\ell^q(k \in \mathbb{Z}^3)} < \infty.$$

These spaces, which blend local integrability and global decay properties, provide a flexible setting that accommodates non-decaying or large initial data while retaining control over both local and global behaviors. We identify E_∞^p with L_{uloc}^p with the norm $\|f\|_{L_{\text{uloc}}^p} := \sup_{x_0 \in \mathbb{R}^3} \|f\|_{L^p(B_1(x_0))}$. The closure of $C_c^\infty(\mathbb{R}^3)$ under the L_{uloc}^p norm is denoted by E^p . Note that for $p, p_1, p_2, q, q_1, q_2 \in [1, \infty]$ we have the Hölder inequality:

$$\|fg\|_{E_q^p} \leq \|f\|_{E_{q_1}^{p_1}} \|g\|_{E_{q_2}^{p_2}}, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} \leq \frac{1}{q_1} + \frac{1}{q_2}. \quad (1.3)$$

We will consider two kinds of spacetime integrals: For $0 < T \leq \infty$, $x \in \mathbb{R}^3$, and $1 \leq s, p, q \leq \infty$, define the norms $L_T^s E_q^p$ and $E_{T,q}^{s,p}$ as follows:

$$\|f\|_{L_T^s E_q^p} := \|f\|_{L^s(0,T;E_q^p(\mathbb{R}^3))}, \quad (1.4)$$

and

$$\|f\|_{E_{T,q}^{s,p}} := \left\| \|f\|_{L_T^s L^p(B_1(k))} \right\|_{\ell^q(k \in \mathbb{Z}^3)}. \quad (1.5)$$

These norms are different from each other when $s \neq q$. By Minkowski's integral inequality,

$$\|f\|_{L_T^s E_q^p} \leq \|f\|_{E_{T,q}^{s,p}}, \quad \text{if } q \leq s, \quad (1.6)$$

and

$$\|f\|_{E_{T,q}^{s,p}} \leq \|f\|_{L_T^s E_q^p}, \quad \text{if } q \geq s. \quad (1.7)$$

Previous works [4, 1] developed a detailed theory for the incompressible Navier–Stokes equations in Wiener amalgam spaces, establishing mild and weak solutions, spacetime integral bounds, and eventual regularity results for different ranges of the Lebesgue exponent q . In this paper, we extend these techniques to the (MHD) and (vNSEd) systems. Specifically, we prove the existence of mild solutions for small data in critical and subcritical Wiener amalgam spaces, as well as global weak solutions under appropriate integrability and decay conditions. Our analysis demonstrates the robustness of the Wiener amalgam framework in addressing the intricate coupling structures and nonlinearities in these models.

In the following subsections, we introduce the definitions of mild and local energy solutions for the systems (MHD) and (vNSEd), and present our main results.

1.1 Mild solutions of MHD equations

A pair of vector fields (v, b) is called a *mild solution* to (MHD) if it satisfies

$$(v, b)(x, t) = (e^{t\Delta} v_0, e^{t\Delta} b_0) - B((v, b), (v, b))(t), \quad (1.8)$$

where B is a bilinear operator defined by $B = (B_1, B_2)$,

$$\begin{aligned} B_1((v, b), (u, a))(t) &= \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (v \otimes u - b \otimes a) ds, \\ B_2((v, b), (u, a))(t) &= \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (v \otimes a - b \otimes u) ds, \end{aligned} \quad (1.9)$$

in which \mathbb{P} is the Helmholtz projection operator. More precisely, the vector components of the bilinear operators B_1 and B_2 can be expressed by

$$\begin{aligned} B_1((v, b), (u, a))_i(x, t) &= \int_0^t \int_{\mathbb{R}^d} \partial_l S_{ij}(x - y, t - s) (v_l u_j - b_l a_j)(y, s) dy ds, \quad i = 1, 2, 3, \\ B_2((v, b), (u, a))_i(x, t) &= \int_0^t \int_{\mathbb{R}^d} \partial_l S_{ij}(x - y, t - s) (v_l a_j - b_l u_j)(y, s) dy ds, \quad i = 1, 2, 3, \end{aligned}$$

where S_{ij} is the Oseen tensor derived by Oseen in [37]. We refer the readers to [18, Section 2.2] for a brief introduction of the Oseen tensor.

We first consider the case of data $(v_0, b_0) \in E_q^r \times E_q^r$ with $r > 3$, which we refer to as *subcritical*, and state the existence of mild solutions in the amalgam spaces in the following theorem.

Theorem 1.1 (Subcritical data (MHD)). *Let $r \in (3, \infty]$ and $q \in [1, \infty]$. If $v_0, b_0 \in E_q^r$ are divergence free, then, for any positive time $T = T(\|(v_0, b_0)\|_{E_q^r \times E_q^r})$ chosen so that*

$$T^{1/2-3/(2r)} + T^{1/2} \lesssim \|(v_0, b_0)\|_{E_q^r \times E_q^r}^{-1}, \quad (1.10)$$

there exists a unique mild solution $(v, b) \in L^\infty(0, T; E_q^r \times E_q^r) \cap C((0, T); E_q^r \times E_q^r)$ to (MHD). Moreover, (v, b) satisfies

$$\sup_{0 \leq t \leq T} \|(v, b)(t)\|_{E_q^r \times E_q^r} \leq C \|(v_0, b_0)\|_{E_q^r \times E_q^r}. \quad (1.11)$$

If $q, r < \infty$, then $(v, b) \in C([0, T]; E_q^r \times E_q^r)$. If $q = \infty$ or $r = \infty$, then we still have $\|(e^{t\Delta} v_0 - v(t), e^{t\Delta} b_0 - b(t))\|_{E_q^r \times E_q^r} \rightarrow 0$ as $t \rightarrow 0^+$.

Furthermore, if $r < \infty$, then for any $s \in [r, \infty]$ and $p \in [r, 3r]$ with $\frac{2}{s} + \frac{3}{p} = \frac{3}{r}$,

$$\|(v, b)\|_{E_{T,m}^{s,p} \times E_{T,m}^{s,p}} \leq C \|(v_0, b_0)\|_{E_q^r \times E_q^r}, \quad m \geq q,$$

provided $(1 + T^{\frac{1}{s}+\epsilon})(T^{\frac{1}{2}-\frac{3}{2r}} + T^{1-\frac{1}{s}}) \lesssim \|(v_0, b_0)\|_{E_q^r \times E_q^r}^{-1}$ for all $\epsilon > 0$.

Theorem 1.1 is proved in Section 2.1.

We now turn to the *critical* case, i.e., the case when data $(v_0, b_0) \in E_q^3 \times E_q^3$. When the data is sufficiently small, we have the following existence theorem of mild solutions.

Theorem 1.2 (Critical data I (MHD)). *Let $q \in [1, \infty]$. Fix $T > 0$. There exists $\varepsilon = \varepsilon(T) > 0$ such that for all divergence-free $v_0, b_0 \in E_q^3$ with $\|(v_0, b_0)\|_{E_q^3 \times E_q^3} \leq \varepsilon$, there exists a mild solution (v, b) to (MHD) with*

$$(v, b) \in L^\infty(0, T; E_q^3 \times E_q^3) \quad \text{and} \quad t^{\frac{1}{2}}(v, b) \in L^\infty(0, T; E_q^\infty \times E_q^\infty).$$

The solution is unique in the class

$$\sup_{0 < t < T} t^{\frac{1}{4}} \|(v, b)\|_{E_q^6 \times E_q^6} \leq 2 \sup_{0 < t < T} t^{\frac{1}{4}} \|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{E_q^6 \times E_q^6}. \quad (1.12)$$

Furthermore, $\|(v, b)\|_{L_T^\infty(E_q^3 \times E_q^3)} + \|t^{1/2}(v, b)\|_{L_T^\infty(E_q^\infty \times E_q^\infty)} \lesssim \|(v_0, b_0)\|_{E_q^3 \times E_q^3}$. We have $(v, b) \in C((0, T); E_q^3 \times E_q^3)$ for $q = \infty$ and $(v, b) \in C([0, T]; E_q^3 \times E_q^3)$ for $q < \infty$. If $q = \infty$, then we have for any ball B and $\delta \in (0, 2]$ that

$$\lim_{t \rightarrow 0^+} \|(v, b)(t) - (v_0, b_0)\|_{L^{3-\delta}(B) \times L^{3-\delta}(B)} = 0. \quad (1.13)$$

For any $s \in [3, \infty]$ and $p \in [3, 9]$ given by $\frac{2}{s} + \frac{3}{p} = 1$, by taking $\varepsilon \leq \varepsilon_0(T, s)$ sufficiently small, this solution further satisfies

$$\|(v, b)\|_{E_{T,m}^{s,p} \times E_{T,m}^{s,p}} + \mathbb{1}_{q \leq s} \|(v, b)\|_{L_T^s(E_m^p \times E_m^p)} \leq C \|(v_0, b_0)\|_{E_q^3 \times E_q^3}, \quad \forall m \in [q, \infty]. \quad (1.14)$$

The following theorem concerns the critical case with enough decay, $1 \leq q \leq 3$.

Theorem 1.3 (Critical data II (MHD)). *Let $1 \leq q \leq 3$. For all divergence-free $v_0, b_0 \in E_q^3$, there exist $T = T(v_0, b_0) > 0$ and a unique mild solution (v, b) to (MHD) satisfying*

$$(v, b) \in BC([0, T]; E_q^3 \times E_q^3) \quad \text{and} \quad t^{\frac{1}{2}}(v, b) \in L^\infty(0, T; E_{q_2}^\infty \times E_{q_2}^\infty),$$

with $1/q_2 = 1/q - 1/3$, $q_2 \in [\frac{3}{2}, \infty]$. For any $s \in [3, \infty)$, $\frac{2}{s} + \frac{3}{p} = 1$, and $m \in [q, \infty]$, there is $T_1 \in (0, T]$ such that

$$(v, b) \in E_{T_1, m}^{s, p} \times E_{T_1, m}^{s, p}. \quad (1.15)$$

Furthermore, there is $\varepsilon(q) > 0$ such that $T = \infty$ if $\|(v_0, b_0)\|_{E_q^3 \times E_q^3} \leq \varepsilon(q)$. If we assume further

$$m > p' = \frac{p}{p-1}, \quad \text{and} \quad m \geq m_1, \quad \frac{2}{s} + \frac{3}{m_1} = \frac{3}{q}, \quad (1.16)$$

with $m > m_1(s, q)$ when $q = 1$, then there exists $\varepsilon_1(s, q, m) > 0$ such that $T_1 = \infty$ if $\|(v_0, b_0)\|_{E_q^3 \times E_q^3} \leq \varepsilon_1(s, q, m)$. Instead of (1.25), if we assume

$$m \geq \max(p', m_1), \quad \text{and} \quad \begin{cases} m > m_1 & \text{if } q = 1, \\ m \geq p & \text{if } 3s < 5q, \end{cases} \quad (1.17)$$

then there exists $\varepsilon_2(s, q, m) > 0$ such that $v, b \in L_{T=\infty}^s E_m^p$ if $\|(v_0, b_0)\|_{E_q^3 \times E_q^3} \leq \varepsilon_2(s, q, m)$.

We prove Theorem 1.3 in Section 2.3.

1.2 Mild solutions of viscoelastic Navier–Stokes equations with damping

A pair (v, \mathbf{F}) , $\mathbf{F} = [f_1, f_2, f_3] \in \mathbb{R}^{3 \times 3}$, is called a *mild solution* to (vNSEd) if it satisfies

$$\begin{aligned} v(x, t) &= e^{t\Delta} v_0 - \mathcal{B}_0((v, \mathbf{F}), (v, \mathbf{F}))(t), \\ f_m(x, t) &= e^{t\Delta} (f_m)_0 - \mathcal{B}_m((v, \mathbf{F}), (v, \mathbf{F}))(t), \quad m = 1, 2, 3, \end{aligned} \quad (1.18)$$

where

$$\begin{aligned} \mathcal{B}_0((v, \mathbf{F}), (u, \mathbf{G}))(t) &= \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot \left(v \otimes u - \sum_{n=1}^3 f_n \otimes g_n \right) ds, \quad \mathbf{G} = [g_1, g_2, g_3], \\ \mathcal{B}_m((v, \mathbf{F}), (u, \mathbf{G}))(t) &= \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (v \otimes g_m - f_m \otimes u) ds. \end{aligned}$$

The main results of the mild solutions for the viscoelastic Navier–Stokes equations with damping are stated as follows:

Theorem 1.4 (Subcritical data (vNSEd)). *Let $r \in (3, \infty]$ and $q \in [1, \infty]$. If $(v_0, \mathbf{F}_0) \in E_q^r \times \mathbf{E}_q^r$, $\mathbf{F}_0 = [(f_1)_0, (f_2)_0, (f_3)_0]$, where v_0 and $(f_m)_0$, $m = 1, 2, 3$, are divergence free, then, for any positive time $T = T(\|(v_0, \mathbf{F}_0)\|_{E_q^r \times \mathbf{E}_q^r})$ chosen so that*

$$T^{1/2-3/(2r)} + T^{1/2} \lesssim \|(v_0, \mathbf{F}_0)\|_{E_q^r \times \mathbf{E}_q^r}^{-1}, \quad (1.19)$$

there exists a unique mild solution $(v, \mathbf{F}) \in L^\infty(0, T; E_q^r \times \mathbf{E}_q^r) \cap C((0, T); E_q^r \times \mathbf{E}_q^r)$ to (vNSEd). Moreover, (v, \mathbf{F}) satisfies

$$\sup_{0 \leq t \leq T} \|(v, \mathbf{F})(t)\|_{E_q^r \times \mathbf{E}_q^r} \leq C \|(v_0, \mathbf{F}_0)\|_{E_q^r \times \mathbf{E}_q^r}. \quad (1.20)$$

If $q, r < \infty$, then $(v, \mathbf{F}) \in C([0, T]; E_q^r \times \mathbf{E}_q^r)$. If $q = \infty$ or $r = \infty$, then we still have $\|(e^{t\Delta} v_0 - v(t), e^{t\Delta} \mathbf{F}_0 - \mathbf{F}(t))\|_{E_q^r \times \mathbf{E}_q^r} \rightarrow 0$ as $t \rightarrow 0^+$.

Furthermore, if $r < \infty$, then for any $s \in [r, \infty]$ and $p \in [r, 3r]$ with $\frac{2}{s} + \frac{3}{p} = \frac{3}{r}$,

$$\|(v, \mathbf{F})\|_{E_{T, m}^{s, p} \times \mathbf{E}_{T, m}^{s, p}} \leq C \|(v_0, \mathbf{F}_0)\|_{E_q^r \times \mathbf{E}_q^r}, \quad m \geq q,$$

provided $(1 + T^{\frac{1}{s} + \epsilon})(T^{\frac{1}{2} - \frac{3}{2r}} + T^{1 - \frac{1}{s}}) \lesssim \|(v_0, \mathbf{F}_0)\|_{E_q^r \times \mathbf{E}_q^r}^{-1}$ for all $\epsilon > 0$.

Theorem 1.5 (Critical data I (vNSEd)). *Let $q \in [1, \infty]$. Fix $T > 0$. There exists $\varepsilon = \varepsilon(T) > 0$ such that for all divergence-free $v_0, (f_m)_0 \in E_q^3$, $m = 1, 2, 3$, with $\|(v_0, \mathbf{F}_0)\|_{E_q^3 \times \mathbf{E}_q^3} \leq \varepsilon$, $\mathbf{F}_0 = [(f_1)_0, (f_2)_0, (f_3)_0]$, there exists a mild solution (v, \mathbf{F}) to (vNSEd) with*

$$(v, \mathbf{F}) \in L^\infty(0, T; E_q^3 \times \mathbf{E}_q^3) \quad \text{and} \quad t^{\frac{1}{2}}(v, \mathbf{F}) \in L^\infty(0, T; E_q^\infty \times \mathbf{E}_q^\infty).$$

The solution is unique in the class

$$\sup_{0 < t < T} t^{\frac{1}{4}} \|(v, \mathbf{F})\|_{E_q^6 \times \mathbf{E}_q^6} \leq 2 \sup_{0 < t < T} t^{\frac{1}{4}} \|(e^{t\Delta} v_0, e^{t\Delta} \mathbf{F}_0)\|_{E_q^6 \times \mathbf{E}_q^6}. \quad (1.21)$$

Furthermore, $\|(v, \mathbf{F})\|_{L_T^\infty(E_q^3 \times \mathbf{E}_q^3)} + \|t^{1/2}(v, \mathbf{F})\|_{L_T^\infty(E_q^\infty \times \mathbf{E}_q^\infty)} \lesssim \|(v_0, \mathbf{F}_0)\|_{E_q^3 \times \mathbf{E}_q^3}$. We have $(v, \mathbf{F}) \in C((0, T); E_q^3 \times \mathbf{E}_q^3)$ for $q = \infty$ and $(v, \mathbf{F}) \in C([0, T]; E_q^3 \times \mathbf{E}_q^3)$ for $q < \infty$. If $q = \infty$, then we have for any ball B and $\delta \in (0, 2]$ that

$$\lim_{t \rightarrow 0^+} \|(v, \mathbf{F})(t) - (v_0, \mathbf{F}_0)\|_{L^{3-\delta}(B) \times \mathbf{L}^{3-\delta}(B)} = 0. \quad (1.22)$$

For any $s \in [3, \infty]$ and $p \in [3, 9]$ given by $\frac{2}{s} + \frac{3}{p} = 1$, by taking $\varepsilon \leq \varepsilon_0(T, s)$ sufficiently small, this solution further satisfies

$$\|(v, \mathbf{F})\|_{E_{T,m}^{s,p} \times \mathbf{E}_{T,m}^{s,p}} + \mathbb{1}_{q \leq s} \|(v, \mathbf{F})\|_{L_T^s(E_m^p \times \mathbf{E}_m^p)} \leq C \|(v_0, \mathbf{F}_0)\|_{E_q^3 \times \mathbf{E}_q^3}, \quad \forall m \in [q, \infty]. \quad (1.23)$$

Theorem 1.6 (Critical data II (vNSEd)). *Let $1 \leq q \leq 3$. For all divergence-free $v_0, (f_1)_0, (f_2)_0, (f_3)_0 \in E_q^3$, there exist $T = T(v_0, \mathbf{F}_0) > 0$, $\mathbf{F}_0 = [(f_1)_0, (f_2)_0, (f_3)_0]$, and a unique mild solution (v, \mathbf{F}) to (vNSEd) satisfying*

$$(v, \mathbf{F}) \in BC([0, T]; E_q^3 \times \mathbf{E}_q^3) \quad \text{and} \quad t^{\frac{1}{2}}(v, \mathbf{F}) \in L^\infty(0, T; E_{q_2}^\infty \times \mathbf{E}_{q_2}^\infty),$$

with $1/q_2 = 1/q - 1/3$, $q_2 \in [\frac{3}{2}, \infty]$. For any $s \in [3, \infty)$, $\frac{2}{s} + \frac{3}{p} = 1$, and $m \in [q, \infty]$, there is $T_1 \in (0, T]$ such that

$$(v, \mathbf{F}) \in E_{T_1,m}^{s,p} \times \mathbf{E}_{T_1,m}^{s,p}. \quad (1.24)$$

Furthermore, there is $\varepsilon(q) > 0$ such that $T = \infty$ if $\|(v_0, \mathbf{F}_0)\|_{E_q^3 \times \mathbf{E}_q^3} \leq \varepsilon(q)$. If we assume further

$$m > p' = \frac{p}{p-1}, \quad \text{and} \quad m \geq m_1, \quad \frac{2}{s} + \frac{3}{m_1} = \frac{3}{q}, \quad (1.25)$$

with $m > m_1(s, q)$ when $q = 1$, then there exists $\varepsilon_1(s, q, m) > 0$ such that $T_1 = \infty$ if $\|(v_0, \mathbf{F}_0)\|_{E_q^3 \times \mathbf{E}_q^3} \leq \varepsilon_1(s, q, m)$. Instead of (1.25), if we assume

$$m \geq \max(p', m_1), \quad \text{and} \quad \begin{cases} m > m_1 & \text{if } q = 1, \\ m \geq p & \text{if } 3s < 5q, \end{cases} \quad (1.26)$$

then there exists $\varepsilon_2(s, q, m) > 0$ such that $(v, \mathbf{F}) \in L_{T=\infty}^s(E_m^p \times \mathbf{E}_m^p)$ if $\|(v_0, \mathbf{F}_0)\|_{E_q^3 \times \mathbf{E}_q^3} \leq \varepsilon_2(s, q, m)$.

1.3 Weak solutions of MHD equations

We first introduce the notion of local energy solutions for the MHD equations, which is consistent with the concept introduced in [4] for the Navier–Stokes equations.

Definition 1.7 (local energy solution (MHD)). *Let $0 < T \leq \infty$. A pair of vector fields (v, b) , $v, b \in L^2_{\text{loc}}(\mathbb{R}^3 \times [0, T))$, is a local energy solution to (MHD) with divergence-free initial data $v_0, b_0 \in L^2_{\text{uloc}}(\mathbb{R}^3)$, denoted as $(v, b) \in \mathcal{N}_{\text{MHD}}(v_0, b_0)$, if the following hold:*

1. *There exists $\pi \in L^{3/2}_{\text{loc}}(\mathbb{R}^3 \times [0, T))$ such that (v, b, π) is a distributional solution to (MHD).*
2. *For any $R > 0$, (v, b) satisfies*

$$\begin{aligned} & \text{ess sup}_{0 \leq t < R^2 \wedge T} \sup_{x_0 \in \mathbb{R}^3} \int_{B_R(x_0)} \frac{1}{2} (|v(x, t)|^2 + |b(x, t)|^2) dx \\ & + \sup_{x_0 \in \mathbb{R}^3} \int_0^{R^2 \wedge T} \int_{B_R(x_0)} (|\nabla v(x, t)|^2 + |\nabla b(x, t)|^2) dx dt < \infty. \end{aligned}$$

3. *For any $R > 0$, $x_0 \in \mathbb{R}^3$, and $0 < T' < T$, there exists a function of time $c_{x_0, R}(t) \in L^{3/2}(0, T')$ so that, for every $0 < t < T'$ and $x \in B_{2R}(x_0)$,*

$$\begin{aligned} \pi(x, t) = & -\Delta^{-1} \text{div div} [(v \otimes v - b \otimes b) \chi_{4R}(x - x_0)] \\ & - \int_{\mathbb{R}^3} (K(x - y) - K(x_0 - y)) (v \otimes v - b \otimes b)(y, t) (1 - \chi_{4R}(y - x_0)) dy + c_{x_0, R}(t) \end{aligned} \quad (1.27)$$

in $L^{3/2}(B_{2R}(x_0) \times (0, T'))$ where $K(x)$ is the kernel of $\Delta^{-1} \text{div div}$, $K_{ij}(x) = \partial_i \partial_j \frac{-1}{4\pi|x|}$, and $\chi_{4R}(x)$ is the characteristic function of B_{4R} .

4. *For all compact subsets K of \mathbb{R}^3 we have $v(t) \rightarrow v_0$ and $b(t) \rightarrow b_0$ in $L^2(K)$ as $t \rightarrow 0^+$.*
5. *For all cylinders Q compactly supported in $\mathbb{R}^3 \times (0, T)$ and all nonnegative $\phi \in C_c^\infty(Q)$, we have the local energy inequality*

$$\begin{aligned} 2 \int \int (|\nabla v|^2 + |\nabla b|^2) \phi dx dt & \leq \int \int (|v|^2 + |b|^2) (\partial_t \phi + \Delta \phi) dx dt \\ & + \int \int (|v|^2 + |b|^2 + 2\pi) (v \cdot \nabla \phi) dx dt \\ & - 2 \int \int (v \cdot b) (b \cdot \nabla \phi) dx dt. \end{aligned} \quad (1.28)$$

6. The functions

$$t \mapsto \int_{\mathbb{R}^3} v(x, t) \cdot w(x) dx \quad t \mapsto \int_{\mathbb{R}^3} b(x, t) \cdot w(x) dx$$

are continuous in $t \in [0, T)$ for any compactly supported $w \in L^2(\mathbb{R}^3)$.

For given divergence-free $v_0, b_0 \in L^2_{\text{uloc}}$, let $\mathcal{N}_{\text{MHD}}(v_0, b_0)$ denote the set of all local energy solutions to (MHD) with initial data (v_0, b_0) .

Theorem 1.8 (Eventual regularity in E_q^2 (MHD)). *Assume $v_0, b_0 \in E_q^2$, where $1 \leq q \leq 3$, are divergence free, and $(v, b) \in \mathcal{N}_{\text{MHD}}(v_0, b_0)$. Then (v, b) has eventual regularity, i.e., there is $t_1 < \infty$ such that v and b are regular at (x, t) whenever $t \geq t_1$, and*

$$\|(v, b)(\cdot, t)\|_{L^\infty \times L^\infty} \lesssim t^{1/2},$$

for sufficiently large t .

The proof of Theorem 1.8 is given in Section 3.1

Define the ℓ^q local energy

$$\|(v, b)\|_{\mathbf{LE}_q(0, T)} := \|(v, b)\|_{E_{T, q}^{\infty, 2} \times E_{T, q}^{\infty, 2}} + \|(\nabla v, \nabla b)\|_{E_{T, q}^{2, 2} \times E_{T, q}^{2, 2}}. \quad (1.29)$$

Theorem 1.9 (Explicit growth rate in E_q^2 (MHD)). Assume $v_0, b_0 \in E_q^2$, where $1 \leq q < \infty$, are divergence free, and $(v, b) \in \mathcal{N}_{\text{MHD}}(v_0, b_0)$ satisfies, for some $T_2 > 0$,

$$\|(v, b)\|_{\mathbf{LE}_q(0, T_1)} < \infty, \quad \forall T_1 \in (0, T_2).$$

Then, for any $R \geq 1$, with $T = \min\left(\lambda_1(1 + \|(v_0, b_0)\|_{E_q^2 \times E_q^2})^{-4} R^2, T_2\right)$, we have

$$\left\| \operatorname{ess\,sup}_{0 \leq t \leq T} \int_{B_R(Rk)} (|v|^2 + |b|^2) dx + \int_0^T \int_{B_R(Rk)} (|\nabla v|^2 + |\nabla b|^2) dx dt \right\|_{\ell^{q/2}(k \in \mathbb{Z}^3)} \leq C \|(v_0, b_0)\|_{E_q^2 \times E_q^2}^2, \quad (1.30)$$

for positive constants λ_1 and C independent of (v_0, b_0) and R . In particular, if $T_2 = \infty$ then $T \rightarrow \infty$ as $R \rightarrow \infty$.

The proof of Theorem 1.9 is given in Section 3.2

Theorem 1.10 (Existence in E_q^2 (MHD)). Assume $v_0, b_0 \in E_q^2$, where $1 \leq q < \infty$, are divergence free. Then, there exists a time-global local energy solution (v, b) and associated pressure π to (MHD) in \mathbb{R}^3 with initial data v_0, b_0 so that, for any $0 < T < \infty$,

$$\|(v, b)\|_{\mathbf{LE}_q(0, T)} < \infty. \quad (1.31)$$

In particular, $(v, b) \in L^\infty(0, T; E_q^2 \times E_q^2)$.

The proof of Theorem 1.10 is divided into two cases: $q \geq 2$ and $1 \leq q < 2$. The case $q \geq 2$ is addressed in Section 3.3.1, and the case $1 \leq q < 2$ is handled separately in Section 3.3.2.

1.4 Weak solutions of viscoelastic Navier–Stokes equations with damping

We now define analogous local energy solutions to the viscoelastic Navier–Stokes equations with damping as follows.

Definition 1.11 (local energy solution (vNSEd)). Let $0 < T \leq \infty$. A pair of a vector field and a tensor (v, \mathbf{F}) , $\mathbf{F} = [f_1, f_2, f_3]$, $v, f_m \in L_{\text{loc}}^2(\mathbb{R}^3 \times [0, T])$, $m = 1, 2, 3$, is a local energy solution to (vNSEd) with initial data $v_0, \mathbf{F}_0 \in L_{\text{loc}}^2(\mathbb{R}^3)$, $\mathbf{F}_0 = [(f_1)_0, (f_2)_0, (f_3)_0]$, where $v_0, (f_m)_0$ are divergence free, denoted as $(v, \mathbf{F}) \in \mathcal{N}_{\text{vNSEd}}(v_0, \mathbf{F}_0)$, if the following hold:

1. There exists $\pi \in L_{\text{loc}}^{3/2}(\mathbb{R}^3 \times [0, T])$ such that (v, \mathbf{F}, π) is a distributional solution to (vNSEd).
2. For any $R > 0$, (v, \mathbf{F}) satisfies

$$\begin{aligned} & \operatorname{ess\,sup}_{0 \leq t < R^2 \wedge T} \sup_{x_0 \in \mathbb{R}^3} \int_{B_R(x_0)} \frac{1}{2} (|v(x, t)|^2 + |\mathbf{F}(x, t)|^2) dx \\ & + \sup_{x_0 \in \mathbb{R}^3} \int_0^{R^2 \wedge T} \int_{B_R(x_0)} (|\nabla v(x, t)|^2 + |\nabla \mathbf{F}(x, t)|^2) dx dt < \infty. \end{aligned}$$

3. For any $R > 0$, $x_0 \in \mathbb{R}^3$, and $0 < T' < T$, there exists a function of time $c_{x_0, R}(t) \in L^{3/2}(0, T')$ so that, for every $0 < t < T'$ and $x \in B_{2R}(x_0)$,

$$\begin{aligned} \pi(x, t) = & -\Delta^{-1} \operatorname{div} \operatorname{div} \left[\left(v \otimes v - \sum_{n=1}^3 f_n \otimes f_n \right) \chi_{4R}(x - x_0) \right] \\ & - \int_{\mathbb{R}^3} (K(x - y) - K(x_0 - y)) \left(v \otimes v - \sum_{n=1}^3 f_n \otimes f_n \right) (y, t) (1 - \chi_{4R}(y - x_0)) dy + c_{x_0, R}(t) \end{aligned} \quad (1.32)$$

in $L^{3/2}(B_{2R}(x_0) \times (0, T'))$ where $K(x)$ is the kernel of $\Delta^{-1} \operatorname{div} \operatorname{div}$, $K_{ij}(x) = \partial_i \partial_j \frac{-1}{4\pi|x|}$, and $\chi_{4R}(x)$ is the characteristic function of B_{4R} .

4. For all compact subsets K of \mathbb{R}^3 we have $v(t) \rightarrow v_0$ and $f_m(t) \rightarrow (f_m)_0$, $m = 1, 2, 3$, in $L^2(K)$ as $t \rightarrow 0^+$.

5. For all cylinders Q compactly supported in $\mathbb{R}^3 \times (0, T)$ and all nonnegative $\phi \in C_c^\infty(Q)$, we have the local energy inequality

$$\begin{aligned} 2 \int \int (|\nabla v|^2 + |\nabla \mathbf{F}|^2) \phi \, dx dt &\leq \int \int (|v|^2 + |\mathbf{F}|^2) (\partial_t \phi + \Delta \phi) \, dx dt \\ &+ \int \int (|v|^2 + |\mathbf{F}|^2 + 2\pi) (v \cdot \nabla \phi) \, dx dt \\ &- 2 \sum_{n=1}^3 \int \int (v \cdot f_n)(f_n \cdot \nabla \phi) \, dx dt. \end{aligned} \quad (1.33)$$

6. The functions

$$t \mapsto \int_{\mathbb{R}^3} v(x, t) \cdot w(x) \, dx \quad t \mapsto \int_{\mathbb{R}^3} \mathbf{F}(x, t) \cdot w(x) \, dx$$

are continuous in $t \in [0, T)$ for any compactly supported $w \in L^2(\mathbb{R}^3)$.

For given divergence-free $v_0, (f_m)_0 \in L^2_{\text{loc}}$, $m = 1, 2, 3$, let $\mathcal{N}_{\text{MHD}}(v_0, \mathbf{F}_0)$, $\mathbf{F}_0 = [(f_1)_0, (f_2)_0, (f_3)_0]$, denote the set of all local energy solutions to (vNSEd) with initial data (v_0, \mathbf{F}_0) .

The main results concerning weak solutions of the viscoelastic Navier–Stokes equations with damping are stated as follows:

Theorem 1.12 (Eventual regularity in E_q^2 (vNSEd)). Assume $v_0, (f_1)_0, (f_2)_0, (f_3)_0 \in E_q^2$, where $1 \leq q \leq 3$, are divergence free, and $(v, \mathbf{F}) \in \mathcal{N}_{\text{vNSEd}}(v_0, \mathbf{F}_0)$, $\mathbf{F}_0 = [(f_1)_0, (f_2)_0, (f_3)_0]$. Then (v, \mathbf{F}) has eventual regularity, i.e., there is $t_1 < \infty$ such that v and \mathbf{F} are regular at (x, t) whenever $t \geq t_1$, and

$$\|(v, \mathbf{F})(\cdot, t)\|_{L^\infty \times \mathbf{L}^\infty} \lesssim t^{1/2},$$

for sufficiently large t .

Define the ℓ^q local energy

$$\|(v, \mathbf{F})\|_{\mathbf{LE}_q(0, T)} := \|(v, \mathbf{F})\|_{E_{T, q}^{\infty, 2} \times \mathbf{E}_{T, q}^{\infty, 2}} + \|(\nabla v, \nabla \mathbf{F})\|_{E_{T, q}^{2, 2} \times \mathbf{E}_{T, q}^{2, 2}}.$$

Theorem 1.13 (Explicit growth rate in E_q^2 (vNSEd)). Assume $v_0, (f_1)_0, (f_2)_0, (f_3)_0 \in E_q^2$, where $1 \leq q < \infty$, are divergence free, and $(v, \mathbf{F}) \in \mathcal{N}_{\text{vNSEd}}(v_0, \mathbf{F}_0)$, $\mathbf{F}_0 = [(f_1)_0, (f_2)_0, (f_3)_0]$, satisfies, for some $T_2 > 0$,

$$\|(v, \mathbf{F})\|_{\mathbf{LE}_q(0, T_1)} < \infty, \quad \forall T_1 \in (0, T_2).$$

Then, for any $R \geq 1$, with $T = \min\left(\lambda_1(1 + \|(v_0, \mathbf{F}_0)\|_{E_q^2 \times \mathbf{E}_q^2})^{-4} R^2, T_2\right)$, we have

$$\left\| \operatorname{ess\,sup}_{0 \leq t \leq T} \int_{B_R(Rk)} (|v|^2 + |\mathbf{F}|^2) \, dx + \int_0^T \int_{B_R(Rk)} (|\nabla v|^2 + |\nabla \mathbf{F}|^2) \, dx \, dt \right\|_{\ell^{q/2}(k \in \mathbb{Z}^3)} \leq C \|(v_0, \mathbf{F}_0)\|_{E_q^2 \times \mathbf{E}_q^2}^2,$$

for positive constants λ_1 and C independent of (v_0, \mathbf{F}_0) and R . In particular, if $T_2 = \infty$ then $T \rightarrow \infty$ as $R \rightarrow \infty$.

Theorem 1.14 (Existence in E_q^2 (vNSEd)). *Assume $v_0, (f_1)_0, (f_2)_0, (f_3)_0 \in E_q^2$, where $1 \leq q < \infty$, are divergence free. Then, there exists a time-global local energy solution (v, \mathbf{F}) and associated pressure π to (vNSEd) in \mathbb{R}^3 with initial data v_0, \mathbf{F}_0 , $\mathbf{F}_0 = [(f_1)_0, (f_2)_0, (f_3)_0]$, so that, for any $0 < T < \infty$,*

$$\|(v, \mathbf{F})\|_{\mathbf{LE}_q(0,T)} < \infty.$$

In particular, $(v, \mathbf{F}) \in L^\infty(0, T; E_q^2 \times \mathbf{E}_q^2)$.

The remainder of the paper is organized as follows: In Section 2, we construct mild solutions in critical and subcritical spaces and prove Theorems 1.1, 1.2, 1.3, 1.4, 1.5, and 1.6. In Section 3, we examine properties of local energy solutions, including uniqueness and regularity, establish a priori bounds and explicit growth rate, and prove the global existence results: Theorems 1.8, 1.9, 1.10, 1.12, 1.13, and 1.14.

2 Construction of mild solutions in Wiener amalgam spaces

This section is dedicated to proving Theorems 1.1, 1.2, 1.3, 1.4, 1.5, and 1.6. The proof techniques closely follow those outlined in [1, Section 3]. Specifically, we apply the Picard iteration scheme as in [1, Section 3] to the problems (1.8) and (1.18) to construct mild solutions for both the MHD equations (MHD) and the viscoelastic Navier–Stokes equations with damping (vNSEd), respectively. Since the structure of (vNSEd) is analogous to that of (MHD), we prove Theorems 1.1, 1.2, and 1.3 for (MHD). The proofs of Theorems 1.4, 1.5, and 1.6 for (vNSEd) are omitted for brevity.

2.1 Mild solutions in subcritical spaces: Proof of Theorem 1.1

The proof of Theorem 1.1 is an adaption of the proof of [1, Theorem 1.1] for the Navier–Stokes equations to the MHD equations.

Define

$$\|(v, b)\|_{\mathcal{E}_T} = \sup_{0 \leq t \leq T} \|(v, b)(t)\|_{E_q^r \times E_q^r}.$$

By [1, Lemma 2.1], we have the linear estimate

$$\|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{E_q^r \times E_q^r} \lesssim \|(v_0, b_0)\|_{E_q^r \times E_q^r}.$$

So,

$$\|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{\mathcal{E}_T} \leq C_1 \|(v_0, b_0)\|_{E_q^r \times E_q^r}. \quad (2.1)$$

For bilinear estimate, again by [1, Lemma 2.1], we estimate:

$$\begin{aligned} & \|B_1((v, b), (u, a))(t)\|_{E_q^r} \\ & \lesssim \int_0^t \left(\frac{1}{(t-s)^{\frac{1}{2} + \frac{3}{2r}}} + \frac{1}{(t-s)^{\frac{1}{2}}} \right) \left(\|(v \otimes u)(s)\|_{E_q^{r/2}} + \|(b \otimes a)(s)\|_{E_q^{r/2}} \right) ds \\ & \lesssim \left(t^{\frac{1}{2} - \frac{3}{2r}} + t^{\frac{1}{2}} \right) \left(\sup_{0 < \tau < t} \|v(\tau)\|_{E_q^r} \sup_{0 < \tau < t} \|u(\tau)\|_{E_\infty^r} + \sup_{0 < \tau < t} \|b(\tau)\|_{E_q^r} \sup_{0 < \tau < t} \|a(\tau)\|_{E_\infty^r} \right) \\ & \lesssim \left(t^{\frac{1}{2} - \frac{3}{2r}} + t^{\frac{1}{2}} \right) \left(\sup_{0 < \tau < t} \|v(\tau)\|_{E_q^r} \sup_{0 < \tau < t} \|u(\tau)\|_{E_q^r} + \sup_{0 < \tau < t} \|b(\tau)\|_{E_q^r} \sup_{0 < \tau < t} \|a(\tau)\|_{E_q^r} \right) \\ & \lesssim \left(t^{\frac{1}{2} - \frac{3}{2r}} + t^{\frac{1}{2}} \right) \|(v, b)\|_{\mathcal{E}_T} \|(u, a)\|_{\mathcal{E}_T}, \end{aligned}$$

where we used the embedding $E_q^r \subset E_\infty^r$. Similarly,

$$\begin{aligned}
& \|B_2((v, b), (u, a))(t)\|_{E_q^r} \\
& \lesssim \int_0^t \left(\frac{1}{(t-s)^{\frac{1}{2}+\frac{3}{2r}}} + \frac{1}{(t-s)^{\frac{1}{2}}} \right) \left(\|(v \otimes a)(s)\|_{E_q^{r/2}} + \|(b \otimes u)(s)\|_{E_q^{r/2}} \right) ds \\
& \lesssim \left(t^{\frac{1}{2}-\frac{3}{2r}} + t^{\frac{1}{2}} \right) \left(\sup_{0<\tau<t} \|v(\tau)\|_{E_q^r} \sup_{0<\tau<t} \|a(\tau)\|_{E_\infty^r} + \sup_{0<\tau<t} \|b(\tau)\|_{E_q^r} \sup_{0<\tau<t} \|u(\tau)\|_{E_\infty^r} \right) \\
& \lesssim \left(t^{\frac{1}{2}-\frac{3}{2r}} + t^{\frac{1}{2}} \right) \left(\sup_{0<\tau<t} \|v(\tau)\|_{E_q^r} \sup_{0<\tau<t} \|a(\tau)\|_{E_q^r} + \sup_{0<\tau<t} \|b(\tau)\|_{E_q^r} \sup_{0<\tau<t} \|u(\tau)\|_{E_q^r} \right) \\
& \lesssim \left(t^{\frac{1}{2}-\frac{3}{2r}} + t^{\frac{1}{2}} \right) \|(v, b)\|_{\mathcal{E}_T} \|(u, a)\|_{\mathcal{E}_T}.
\end{aligned}$$

Thus, the full bilinear estimate becomes

$$\|B((v, b), (u, a))\|_{\mathcal{E}_T} \leq C_2 \left(T^{\frac{1}{2}-\frac{3}{2r}} + T^{\frac{1}{2}} \right) \|(v, b)\|_{\mathcal{E}_T} \|(u, a)\|_{\mathcal{E}_T}. \quad (2.2)$$

We look for a solution of the form

$$(v, b) = (e^{t\Delta}v_0, e^{t\Delta}b_0) - B((v, b), (v, b)).$$

Suppose T is small enough so that $\|(v_0, b_0)\|_{E_q^r \times E_q^r} < (8C_1C_2(T^{1/2-3/(2r)} + T^{1/2}))^{-1}$. Then by the Picard contraction principle, there exists a unique strong mild solution satisfying

$$\|(v, b)\|_{\mathcal{E}_T} \leq 2C_1 \|(v_0, b_0)\|_{E_q^r \times E_q^r}. \quad (2.3)$$

To prove continuity at time zero, assume $r, q < \infty$. Then

$$\begin{aligned}
\|v(t) - v_0\|_{E_q^r} & \leq \|B_1(u, u)(t)\|_{E_q^r} + \|e^{t\Delta}v_0 - v_0\|_{E_q^r} \\
& \lesssim \left(t^{\frac{1}{2}-\frac{3}{2r}} + t^{\frac{1}{2}} \right) \left[\sup_{0<\tau<t} \|v(\tau)\|_{E_q^r} \sup_{0<\tau<t} \|v(\tau)\|_{E_\infty^r} + \sup_{0<\tau<t} \|b(\tau)\|_{E_q^r} \sup_{0<\tau<t} \|b(\tau)\|_{E_\infty^r} \right] \\
& \quad + \|e^{t\Delta}v_0 - v_0\|_{E_q^r} \\
& \leq \left(t^{\frac{1}{2}-\frac{3}{2r}} + t^{\frac{1}{2}} \right) \left[\sup_{0<\tau<t} \|v(\tau)\|_{E_q^r} \sup_{0<\tau<t} \|v(\tau)\|_{E_q^r} + \sup_{0<\tau<t} \|b(\tau)\|_{E_q^r} \sup_{0<\tau<t} \|b(\tau)\|_{E_q^r} \right] \\
& \quad + \|e^{t\Delta}v_0 - v_0\|_{E_q^r}.
\end{aligned}$$

Both terms tend to zero as $t \rightarrow 0^+$, the latter by [1, Lemma 2.3]. Hence, $\|v(t) - v_0\|_{E_q^r} \rightarrow 0$ as $t \rightarrow 0^+$. Similarly,

$$\begin{aligned}
\|b(t) - b_0\|_{E_q^r} & \lesssim \left(t^{\frac{1}{2}-\frac{3}{2r}} + t^{\frac{1}{2}} \right) \left[\sup_{0<\tau<t} \|v(\tau)\|_{E_q^r} \sup_{0<\tau<t} \|b(\tau)\|_{E_q^r} + \sup_{0<\tau<t} \|v(\tau)\|_{E_q^r} \sup_{0<\tau<t} \|b(\tau)\|_{E_q^r} \right] \\
& \quad + \|e^{t\Delta}b_0 - b_0\|_{E_q^r} \rightarrow 0 \text{ as } t \rightarrow 0^+.
\end{aligned}$$

The continuity at $t \in (0, T)$ can be shown as usual, see e.g., [40, lines 3-8, page 86], including $r = \infty$ or $q = \infty$.

If either $r = \infty$ or $q = \infty$, the semigroup terms no longer vanish, but the bilinear terms still tend to zero:

$$\left(t^{\frac{1}{2}-\frac{3}{2r}} + t^{\frac{1}{2}} \right) \sup_{0<\tau<t} \|v(\tau)\|_{E_q^r} \sup_{0<\tau<t} \|v(\tau)\|_{E_q^r} \rightarrow 0, \quad \left(t^{\frac{1}{2}-\frac{3}{2r}} + t^{\frac{1}{2}} \right) \sup_{0<\tau<t} \|b(\tau)\|_{E_q^r} \sup_{0<\tau<t} \|b(\tau)\|_{E_q^r} \rightarrow 0,$$

and

$$\left(t^{\frac{1}{2}-\frac{3}{2r}} + t^{\frac{1}{2}}\right) \sup_{0 < \tau < t} \|v(\tau)\|_{E_q^r} \sup_{0 < \tau < t} \|b(\tau)\|_{E_q^r} \rightarrow 0.$$

Hence,

$$\|v(t) - e^{t\Delta}v_0\|_{E_q^r} + \|b(t) - e^{t\Delta}b_0\|_{E_q^r} \rightarrow 0 \text{ as } t \rightarrow 0^+,$$

as asserted in the theorem.

For uniqueness, let $(v, b), (v', b') \in L^\infty(0, T; E_q^r \times E_q^r) \cap C((0, T); E_q^r \times E_q^r)$ be two mild solutions with initial data $(v_0, b_0) \in E_q^r \times E_q^r$. Then, for $0 < t < T' \leq T$,

$$\begin{aligned} \|(v - v')(t)\|_{E_q^r} &\leq \|B_1((v - v', b - b'), (v, b))(t)\|_{E_q^r} + \|B_1((v', b'), (v - v', b - b'))(t)\|_{E_q^r} \\ &\lesssim \left(t^{\frac{1}{2}-\frac{3}{2r}} + t^{\frac{1}{2}}\right) \left(\|(v, b)\|_{\mathcal{E}_T} + \|(v', b')\|_{\mathcal{E}_T}\right) \left(\sup_{0 < t < T'} \|(v - v')(t)\|_{E_q^r} + \sup_{0 < t < T'} \|(b - b')(t)\|_{E_q^r}\right), \end{aligned}$$

so that we have

$$\begin{aligned} \sup_{0 < t < T'} \|(v - v')(t)\|_{E_q^r} &\lesssim \left(T'^{\frac{1}{2}-\frac{3}{2r}} + T'^{\frac{1}{2}}\right) \left(\|(v, b)\|_{\mathcal{E}_T} + \|(v', b')\|_{\mathcal{E}_T}\right) \\ &\quad \cdot \sup_{0 < t < T'} \left(\sup_{0 < t < T'} \|(v - v')(t)\|_{E_q^r} + \sup_{0 < t < T'} \|(b - b')(t)\|_{E_q^r}\right). \end{aligned}$$

Similarly,

$$\begin{aligned} \sup_{0 < t < T'} \|(b - b')(t)\|_{E_q^r} &\lesssim \left(T'^{\frac{1}{2}-\frac{3}{2r}} + T'^{\frac{1}{2}}\right) \left(\|(v, b)\|_{\mathcal{E}_T} + \|(v', b')\|_{\mathcal{E}_T}\right) \\ &\quad \cdot \sup_{0 < t < T'} \left(\sup_{0 < t < T'} \|(v - v')(t)\|_{E_q^r} + \sup_{0 < t < T'} \|(b - b')(t)\|_{E_q^r}\right). \end{aligned}$$

Thus, for small enough $T' > 0$, this implies $(v, b) = (v', b')$ on $(0, T')$, and repeating the argument yields uniqueness on $(0, T)$.

To obtain the spacetime integral bound, assume $3 < r \leq s \leq \infty$, $r \leq p < \infty$, $\frac{2}{s} + \frac{3}{p} = \frac{3}{r}$, and $1 \leq q = m \leq \infty$. Consider the Banach space

$$X_T = \mathcal{E}_T \cap (E_{T,q}^{s,p} \times E_{T,q}^{s,p}).$$

We can assume $m = q$ since $\|f\|_{E_{T,m}^{s,p}} \leq \|f\|_{E_{T,q}^{s,p}}$ for $m \geq q$. From $\frac{3}{p} = \frac{3}{r} - \frac{2}{s} \geq \frac{3}{r} - \frac{2}{r}$ we get $p \leq 3r < \infty$. For the linear term, by (2.1) and [1, Lemma 2.4] (which needs $r < \infty$ and $r \leq s$), we have for a fixed $\epsilon > 0$ that

$$\begin{aligned} \|(e^{t\Delta}v_0, e^{t\Delta}b_0)\|_{X_T} &= \|(e^{t\Delta}v_0, e^{t\Delta}b_0)\|_{\mathcal{E}_T} + \|(e^{t\Delta}v_0, e^{t\Delta}b_0)\|_{E_{T,q}^{s,p} \times E_{T,q}^{s,p}} \\ &\leq C_3(1 + T^{1/s+\epsilon}) \|(v_0, b_0)\|_{E_q^r \times E_q^r}. \end{aligned}$$

For the bilinear term, by (2.2) and [1, Lemma 2.7] with $\tilde{p} = p/2$ and $\tilde{s} = s/2$ so that $\sigma = \frac{1}{2} - \frac{3}{2r} > 0$ due to $r > 3$, (allowing $s = \infty$),

$$\begin{aligned} \|B((v, b), (u, a))\|_{X_T} &= \|B((v, b), (u, a))\|_{\mathcal{E}_T} + \|B_1((v, b), (u, a))\|_{E_{T,q}^{s,p}} + \|B_2((v, b), (u, a))\|_{E_{T,q}^{s,p}} \\ &\leq C_4 \left[\left(T^{\frac{1}{2}-\frac{3}{2r}} + T^{\frac{1}{2}}\right) \|(v, b)\|_{\mathcal{E}_T} \|(u, a)\|_{\mathcal{E}_T} \right. \\ &\quad \left. + \left(T^{\frac{1}{2}-\frac{3}{2r}} + T^{1-\frac{1}{s}}\right) \left(\|v \otimes u\|_{E_{T,q}^{\frac{s}{2}, \frac{p}{2}}} + \|b \otimes a\|_{E_{T,q}^{\frac{s}{2}, \frac{p}{2}}} + \|v \otimes a\|_{E_{T,q}^{\frac{s}{2}, \frac{p}{2}}} + \|b \otimes u\|_{E_{T,q}^{\frac{s}{2}, \frac{p}{2}}} \right) \right]. \end{aligned}$$

Since

$$\|f \otimes g\|_{E_{T,q}^{\frac{s}{2}, \frac{p}{2}}} \leq \|f\|_{E_{T,q}^{s,p}} \|g\|_{E_{T,\infty}^{s,p}} \leq \|f\|_{E_{T,q}^{s,p}} \|g\|_{E_{T,q}^{s,p}},$$

we derive

$$\|B((v, b), (u, a))\|_{X_T} \leq 2C_4 \left(T^{\frac{1}{2} - \frac{3}{2r}} + T^{1 - \frac{1}{s}} \right) \|(v, b)\|_{X_T} \|(u, a)\|_{X_T}.$$

Choose $T > 0$ small enough such that $\|(v_0, b_0)\|_{E_q^r \times E_q^r} < [8C_3C_4(1 + T^{\frac{1}{s} + \epsilon})(T^{\frac{1}{2} - \frac{3}{2r}} + T^{1 - \frac{1}{s}})]^{-1}$, the Picard iteration yields a unique strong mild solution $(\tilde{v}, \tilde{b}) \in X_T$. Since $X_T \subset L_T^\infty(E_q^r \times E_q^r)$, uniqueness implies (\tilde{v}, \tilde{b}) , and the solution satisfies the desired spacetime integral bound. This completes the proof of Theorem 1.1. \square

2.2 Mild solutions in critical spaces with small data: Proof of Theorem 1.2

The proof of Theorem 1.2 is an adaption of the proof of [1, Theorem 1.2] for the Navier–Stokes equations to the MHD equations.

Let

$$\|(v, b)\|_{\tilde{\mathcal{E}}_T} := \sup_{0 < t < T} \|(v, b)(\cdot, t)\|_{E_q^3 \times E_q^3} + \sup_{0 < t < T} t^{\frac{1}{2}} \|(v, b)(\cdot, t)\|_{E_q^\infty \times E_q^\infty}$$

and

$$\|(v, b)\|_{\tilde{\mathcal{F}}_T} := \sup_{0 < t < T} t^{\frac{1}{4}} \|(v, b)(\cdot, t)\|_{E_q^6 \times E_q^6}.$$

The inclusion $\tilde{\mathcal{E}}_T \subset \tilde{\mathcal{F}}_T$ is obvious. We define the spaces

$$\mathcal{E}_T := \tilde{\mathcal{E}}_T \cap E_{T,m}^{s,p} \quad \text{and} \quad \mathcal{F}_T := \tilde{\mathcal{F}}_T \cap E_{T,m}^{s,p}, \quad (2.4)$$

with corresponding norms

$$\|(v, b)\|_{\mathcal{E}_T} := \|(v, b)\|_{\tilde{\mathcal{E}}_T} + \|(v, b)\|_{E_{T,m}^{s,p}} \quad \text{and} \quad \|(v, b)\|_{\mathcal{F}_T} := \|(v, b)\|_{\tilde{\mathcal{F}}_T} + \|(v, b)\|_{E_{T,m}^{s,p}}.$$

The inclusion $\mathcal{E}_T \subset \mathcal{F}_T$ follows immediately.

From [1, (2.16)], and choosing $\epsilon > \frac{3}{2m} - \frac{3}{2q}$ from the definition of β in [1, Lemma 2.4], we have

$$\|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{E_{T,m}^{s,p} \times E_{T,m}^{s,p}} \lesssim (1 + T^{\frac{1}{s} + \epsilon}) \|(v_0, b_0)\|_{E_q^3 \times E_q^3}. \quad (2.5)$$

Also, by [1, Lemma 2.1],

$$t^{\frac{1}{4}} \|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{E_q^6 \times E_q^6} \lesssim (1 + T^{\frac{1}{4}}) \|(v_0, b_0)\|_{E_q^3 \times E_q^3}. \quad (2.6)$$

Combining, we obtain

$$\|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{\mathcal{F}_T} \leq C_1 (1 + T^{\frac{1}{s} + \epsilon} + T^{\frac{1}{4}}) \|(v_0, b_0)\|_{E_q^3 \times E_q^3}. \quad (2.7)$$

Using [1, Lemma 2.7] with $\tilde{s} = s/2$, $\tilde{p} = p/2$, and $\tilde{m} = m$ (so that $\sigma = 0$), and applying Hölder's inequality, we estimate the bilinear terms:

$$\begin{aligned} \|B_1((v, b), (u, a))\|_{E_{T,m}^{s,p}} &\leq C(1 + T^{1 - \frac{1}{s}}) \left(\|v \otimes u\|_{E_{T,m}^{\frac{s}{2}, \frac{p}{2}}} + \|b \otimes a\|_{E_{T,m}^{\frac{s}{2}, \frac{p}{2}}} \right) \\ &\leq C(1 + T^{1 - \frac{1}{s}}) \left(\|v\|_{E_{T,m}^{s,p}} \|u\|_{E_{T,\infty}^{s,p}} + \|b\|_{E_{T,m}^{s,p}} \|a\|_{E_{T,\infty}^{s,p}} \right) \\ &\leq C(1 + T^{1 - \frac{1}{s}}) \left(\|v\|_{E_{T,m}^{s,p}} \|u\|_{E_{T,m}^{s,p}} + \|b\|_{E_{T,m}^{s,p}} \|a\|_{E_{T,m}^{s,p}} \right), \end{aligned} \quad (2.8)$$

where we used the inclusion $E_{T,m}^{s,p} \subset E_{T,\infty}^{s,p}$ in the last inequality. Similarly, we have

$$\begin{aligned} \|B_2((v, b), (u, a))\|_{E_{T,m}^{s,p}} &\leq C(1 + T^{1-\frac{1}{s}}) \left(\|v \otimes a\|_{E_{T,m}^{\frac{s}{2}, \frac{p}{2}}} + \|b \otimes u\|_{E_{T,m}^{\frac{s}{2}, \frac{p}{2}}} \right) \\ &\leq C(1 + T^{1-\frac{1}{s}}) \left(\|v\|_{E_{T,m}^{s,p}} \|a\|_{E_{T,\infty}^{s,p}} + \|b\|_{E_{T,m}^{s,p}} \|u\|_{E_{T,\infty}^{s,p}} \right) \\ &\leq C(1 + T^{1-\frac{1}{s}}) \left(\|v\|_{E_{T,m}^{s,p}} \|a\|_{E_{T,m}^{s,p}} + \|b\|_{E_{T,m}^{s,p}} \|u\|_{E_{T,m}^{s,p}} \right). \end{aligned} \quad (2.9)$$

Hence,

$$\|B((v, b), (u, a))\|_{E_{T,m}^{s,p} \times E_{T,m}^{s,p}} \leq C(1 + T^{1-\frac{1}{s}}) \|(v, b)\|_{\mathcal{F}_T} \|(u, a)\|_{\mathcal{F}_T}. \quad (2.10)$$

Additionally, applying [1, Lemma 2.1] and Hölder inequality (1.3), we obtain

$$\begin{aligned} \|B_1((v, b), (u, a))\|_{E_q^6}(t) &\lesssim \int_0^t \left(\frac{1}{(t-\tau)^{\frac{1}{2}}} + \frac{1}{(t-\tau)^{\frac{3}{4}}} \right) \left(\|(v \otimes u)(\tau)\|_{E_q^3} + \|(b \otimes a)(\tau)\|_{E_q^3} \right) d\tau \\ &\leq \int_0^t \left(\frac{1}{(t-\tau)^{\frac{1}{2}}} + \frac{1}{(t-\tau)^{\frac{3}{4}}} \right) \left(\|v(\tau)\|_{E_q^6} \|u(\tau)\|_{E_q^6} + \|b(\tau)\|_{E_q^6} \|a(\tau)\|_{E_q^6} \right) d\tau \\ &\lesssim (1 + t^{-1/4}) \|(v, b)\|_{\tilde{\mathcal{F}}_T} \|(u, a)\|_{\tilde{\mathcal{F}}_T}. \end{aligned}$$

Similarly,

$$\begin{aligned} \|B_2((v, b), (u, a))\|_{E_q^6}(t) &\lesssim \int_0^t \left(\frac{1}{(t-\tau)^{\frac{1}{2}}} + \frac{1}{(t-\tau)^{\frac{3}{4}}} \right) \left(\|(v \otimes a)(\tau)\|_{E_q^3} + \|(b \otimes u)(\tau)\|_{E_q^3} \right) d\tau \\ &\leq \int_0^t \left(\frac{1}{(t-\tau)^{\frac{1}{2}}} + \frac{1}{(t-\tau)^{\frac{3}{4}}} \right) \left(\|v(\tau)\|_{E_q^6} \|a(\tau)\|_{E_q^6} + \|b(\tau)\|_{E_q^6} \|u(\tau)\|_{E_q^6} \right) d\tau \\ &\lesssim (1 + t^{-1/4}) \|(v, b)\|_{\tilde{\mathcal{F}}_T} \|(u, a)\|_{\tilde{\mathcal{F}}_T}. \end{aligned}$$

Hence

$$\|B((v, b), (u, a))\|_{\mathcal{F}_T} \leq C_2(1 + T^{\frac{1}{4}} + T^{1-\frac{1}{s}}) \|(v, b)\|_{\mathcal{F}_T} \|(u, a)\|_{\mathcal{F}_T}.$$

Taking $\|(v_0, b_0)\|_{E_q^3 \times E_q^3}$ small enough, by (2.7), it is possible to make

$$\|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{\mathcal{F}_T} \leq [4C_2(1 + T^{1/4} + T^{1-\frac{1}{s}})]^{-1}. \quad (2.11)$$

The Picard iteration yields a mild solution $(v, b) \in \mathcal{F}_T$ to (MHD) so that

$$\|(v, b)\|_{\mathcal{F}_T} \leq 2\|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{\mathcal{F}_T}.$$

This solution is unique among all mild solutions (u, a) with data (v_0, b_0) satisfying $\|(u, a)\|_{\mathcal{F}_T} \leq 2\|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{\mathcal{F}_T}$. In fact, since we can also apply the Picard contraction to $\tilde{\mathcal{F}}_T$, the solution is also unique in the class $\|(u, a)\|_{\tilde{\mathcal{F}}_T} \leq 2\|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{\tilde{\mathcal{F}}_T}$.

Next, we show that a solution $(v, b) \in \mathcal{F}_T$ with small enough initial data $(v_0, b_0) \in E_q^3 \times E_q^3$ also belongs to \mathcal{E}_T . Let $\{(v^{(n)}, b^{(n)})\}_{n \geq 1}$ be the Picard iteration sequence in \mathcal{F}_T . By construction,

$$\|(v^{(n)}, b^{(n)})\|_{\mathcal{F}_T} \leq 2C_1(1 + T^{\frac{1}{s}+\epsilon} + T^{\frac{1}{4}}) \|(v_0, b_0)\|_{E_q^3 \times E_q^3} < 1.$$

Note that

$$\left\| (v^{(n)}, b^{(n)}) \right\|_{\tilde{\mathcal{E}}_T} \leq \left\| (e^{t\Delta} v_0, e^{t\Delta} b_0) \right\|_{\tilde{\mathcal{E}}_T} + \left\| B((v^{(n-1)}, b^{(n-1)}), (v^{(n-1)}, b^{(n-1)})) \right\|_{\tilde{\mathcal{E}}_T}.$$

By [1, Lemma 2.1],

$$\left\| (e^{t\Delta} v_0, e^{t\Delta} b_0) \right\|_{\tilde{\mathcal{E}}_T} \leq C(1 + T^{1/2}) \left\| (v_0, b_0) \right\|_{E_q^3 \times E_q^3}.$$

As is usual in arguments like this, we now seek estimates for $B((v, b), (u, a))$ in $\tilde{\mathcal{E}}_T$ in terms of measurements of (v, b) and (u, a) in $\tilde{\mathcal{F}}_T$ and $\tilde{\mathcal{E}}_T$. We have by [1, Lemma 2.1] and Hölder inequality (1.3),

$$\begin{aligned} \|B_1((v, b), (u, a))\|_{E_q^3} &\lesssim \int_0^t \left(\frac{1}{(t-\tau)^{\frac{1}{2}}} + \frac{1}{(t-\tau)^{\frac{3}{4}}} \right) \left(\|(v \otimes u)(\tau)\|_{E_q^2} + \|(b \otimes a)(\tau)\|_{E_q^2} \right) d\tau \\ &\leq \int_0^t \left(\frac{1}{(t-\tau)^{\frac{1}{2}}} + \frac{1}{(t-\tau)^{\frac{3}{4}}} \right) \left(\|v(\tau)\|_{E_q^3} \|u(\tau)\|_{E_q^6} + \|b(\tau)\|_{E_q^3} \|a(\tau)\|_{E_q^6} \right) d\tau \\ &\lesssim (1 + T^{1/4}) \|(v, b)\|_{\tilde{\mathcal{E}}_T} \|(u, a)\|_{\tilde{\mathcal{F}}_T}, \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} \|B_2((v, b), (u, a))\|_{E_q^3} &\lesssim \int_0^t \left(\frac{1}{(t-\tau)^{\frac{1}{2}}} + \frac{1}{(t-\tau)^{\frac{3}{4}}} \right) \left(\|(v \otimes a)(\tau)\|_{E_q^2} + \|(b \otimes u)(\tau)\|_{E_q^2} \right) d\tau \\ &\leq \int_0^t \left(\frac{1}{(t-\tau)^{\frac{1}{2}}} + \frac{1}{(t-\tau)^{\frac{3}{4}}} \right) \left(\|v(\tau)\|_{E_q^3} \|a(\tau)\|_{E_q^6} + \|b(\tau)\|_{E_q^3} \|u(\tau)\|_{E_q^6} \right) d\tau \\ &\lesssim (1 + T^{1/4}) \|(v, b)\|_{\tilde{\mathcal{E}}_T} \|(u, a)\|_{\tilde{\mathcal{F}}_T}, \end{aligned} \tag{2.13}$$

which imply

$$\|B((v, b), (u, a))\|_{E_q^3 \times E_q^3} \lesssim (1 + T^{1/4}) \|(v, b)\|_{\tilde{\mathcal{E}}_T} \|(u, a)\|_{\tilde{\mathcal{F}}_T}. \tag{2.14}$$

Moreover, we have

$$\begin{aligned} t^{\frac{1}{2}} \|B_1((v, b), (u, a))\|_{E_q^\infty} &\lesssim t^{\frac{1}{2}} \int_0^t \left(\frac{1}{(t-\tau)^{\frac{1}{2}}} + \frac{1}{(t-\tau)^{\frac{3}{4}}} \right) \left(\|(v \otimes u)(\tau)\|_{E_q^6} + \|(b \otimes a)(\tau)\|_{E_q^6} \right) d\tau \\ &\leq t^{\frac{1}{2}} \int_0^t \left(\frac{1}{(t-\tau)^{\frac{1}{2}}} + \frac{1}{(t-\tau)^{\frac{3}{4}}} \right) \left(\|v(\tau)\|_{E_q^\infty} \|u(\tau)\|_{E_q^6} + \|b(\tau)\|_{E_q^\infty} \|a(\tau)\|_{E_q^6} \right) d\tau \\ &\lesssim (1 + T^{1/4}) \|(v, b)\|_{\tilde{\mathcal{E}}_T} \|(u, a)\|_{\tilde{\mathcal{F}}_T}, \end{aligned}$$

and

$$\begin{aligned} t^{\frac{1}{2}} \|B_2((v, b), (u, a))\|_{E_q^\infty} &\lesssim t^{\frac{1}{2}} \int_0^t \left(\frac{1}{(t-\tau)^{\frac{1}{2}}} + \frac{1}{(t-\tau)^{\frac{3}{4}}} \right) \left(\|(v \otimes a)(\tau)\|_{E_q^6} + \|(b \otimes u)(\tau)\|_{E_q^6} \right) d\tau \\ &\leq t^{\frac{1}{2}} \int_0^t \left(\frac{1}{(t-\tau)^{\frac{1}{2}}} + \frac{1}{(t-\tau)^{\frac{3}{4}}} \right) \left(\|v(\tau)\|_{E_q^\infty} \|a(\tau)\|_{E_q^6} + \|b(\tau)\|_{E_q^\infty} \|u(\tau)\|_{E_q^6} \right) d\tau \\ &\lesssim (1 + T^{1/4}) \|(v, b)\|_{\tilde{\mathcal{E}}_T} \|(u, a)\|_{\tilde{\mathcal{F}}_T}, \end{aligned}$$

which imply

$$t^{\frac{1}{2}} \|B_2((v, b), (u, a))\|_{E_q^\infty \times E_q^\infty} \lesssim (1 + T^{1/4}) \|(v, b)\|_{\tilde{\mathcal{E}}_T} \|(u, a)\|_{\tilde{\mathcal{F}}_T}.$$

By switching $(v, b), (u, a)$ in the estimates,

$$\|B((v, b), (u, a))\|_{\tilde{\mathcal{E}}_T} \lesssim (1 + T^{1/4}) \min \left(\|(v, b)\|_{\tilde{\mathcal{E}}_T} \|(u, a)\|_{\tilde{\mathcal{F}}_T}, \|(u, a)\|_{\tilde{\mathcal{E}}_T} \|(v, b)\|_{\tilde{\mathcal{F}}_T} \right).$$

We now return to our main objective: proving that the Picard iterates $\{v^{(n)}, b^{(n)}\}$ are uniformly bounded in $\tilde{\mathcal{E}}_T$. From the recursive relation, we have

$$\begin{aligned} \|(v^{(n)}, b^{(n)})\|_{\tilde{\mathcal{E}}_T} &\leq \|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{\tilde{\mathcal{E}}_T} + \|B((v^{(n-1)}, b^{(n-1)}), (v^{(n-1)}, b^{(n-1)}))\|_{\tilde{\mathcal{E}}_T} \\ &\leq C_T \|(v_0, b_0)\|_{E_q^3 \times E_q^3} + C'_T \|(v^{(n-1)}, b^{(n-1)})\|_{\tilde{\mathcal{F}}_T} \|(v^{(n-1)}, b^{(n-1)})\|_{\tilde{\mathcal{E}}_T} \\ &\leq C_T \|(v_0, b_0)\|_{E_q^3 \times E_q^3} + C'_T C''_T \|(v_0, b_0)\|_{E_q^3 \times E_q^3} \|(v^{(n-1)}, b^{(n-1)})\|_{\tilde{\mathcal{E}}_T}. \end{aligned}$$

Thus, if $\|(v_0 \times b_0)\|_{E_q^3 \times E_q^3} \leq (2C'_T C''_T)^{-1}$, then $\|(v^{(n)}, b^{(n)})\|_{\tilde{\mathcal{E}}_T}$ is uniformly bounded by $2C_T \|(v_0, b_0)\|_{E_q^3 \times E_q^3}$. To show that the limit $(v, b) \in \tilde{\mathcal{E}}_T$, consider the difference of successive iterates:

$$\begin{aligned} &\|(v^{(n+1)}, b^{(n+1)}) - (v^{(n)}, b^{(n)})\|_{\tilde{\mathcal{E}}_T} \\ &= \|B((v^{(n)}, b^{(n)}), (v^{(n)}, b^{(n)})) - B((v^{(n-1)}, b^{(n-1)}), (v^{(n-1)}, b^{(n-1)}))\|_{\tilde{\mathcal{E}}_T} \\ &\leq \|B((v^{(n)}, b^{(n)}) - (v^{(n-1)}, b^{(n-1)}), (v^{(n)}, b^{(n)}))\|_{\tilde{\mathcal{E}}_T} + \|B((v^{(n-1)}, b^{(n-1)}), (v^{(n)}, b^{(n)}) - (v^{(n-1)}, b^{(n-1)}))\|_{\tilde{\mathcal{E}}_T} \\ &\lesssim \|(v^{(n)}, b^{(n)}) - (v^{(n-1)}, b^{(n-1)})\|_{\tilde{\mathcal{F}}_T} \left(\|(v^{(n)}, b^{(n)})\|_{\tilde{\mathcal{E}}_T} + \|(v^{(n-1)}, b^{(n-1)})\|_{\tilde{\mathcal{E}}_T} \right) \\ &\lesssim \|(v^{(n)}, b^{(n)}) - (v^{(n-1)}, b^{(n-1)})\|_{\tilde{\mathcal{E}}_T} \left(\|(v^{(n)}, b^{(n)})\|_{\tilde{\mathcal{E}}_T} + \|(v^{(n-1)}, b^{(n-1)})\|_{\tilde{\mathcal{E}}_T} \right). \end{aligned} \tag{2.15}$$

This shows the sequence $\{(v^{(n+1)}, b^{(n+1)}) - (v^{(n)}, b^{(n)})\}$ is Cauchy in $\tilde{\mathcal{E}}_T$, and the limit of $\{(v^{(n)}, b^{(n)})\}$ lies in \mathcal{E}_T , since convergence in \mathcal{F}_T implies convergence in the full \mathcal{E}_T -norm.

If $q \leq s$, we use the embeddings

$$\|f\|_{L_T^s E_m^p} \leq \|f\|_{L_T^s E_q^p} \leq \|f\|_{E_{T,q}^{s,p}},$$

from (1.6) to justify the $L_T^s E_m^p$ -estimate in (1.23).

Next, we prove convergence to the initial data when $q < \infty$. By [1, Lemma 2.3] we have

$$\lim_{T' \rightarrow 0^+} \sup_{0 < t < T'} t^{\frac{1}{4}} \|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{E_q^6 \times E_q^6} = \lim_{T' \rightarrow 0} \|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{\tilde{\mathcal{F}}_{T'}} = 0, \tag{2.16}$$

whenever $(v_0, b_0) \in E_q^3 \times E_q^3$. We extend this to the Picard sequence by induction. The base case $(v^{(0)}, b^{(0)}) = (e^{t\Delta} v_0, e^{t\Delta} b_0)$ satisfies (2.16). Suppose the inductive hypothesis holds:

$$\lim_{T' \rightarrow 0} \|(v^{(n-1)}, b^{(n-1)})\|_{\tilde{\mathcal{F}}_{T'}} = 0. \tag{2.17}$$

Then, from the iteration estimate in the class $\tilde{\mathcal{F}}_{T'}$ where we are taking $T' \leq T$, we have

$$\begin{aligned} \|(v^{(n)}, b^{(n)})\|_{\tilde{\mathcal{F}}_{T'}} &\leq \|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{\tilde{\mathcal{F}}_{T'}} + \|B((v^{(n-1)}, b^{(n-1)}), (v^{(n-1)}, b^{(n-1)}))\|_{\tilde{\mathcal{F}}_{T'}} \\ &\lesssim \|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{\tilde{\mathcal{F}}_{T'}} + \|(v^{(n-1)}, b^{(n-1)})\|_{\tilde{\mathcal{F}}_{T'}}^2, \end{aligned}$$

which implies

$$\lim_{T' \rightarrow 0} \|(v^{(n)}, b^{(n)})\|_{\tilde{\mathcal{F}}_{T'}} = 0 \quad (2.18)$$

by (2.16) and (2.17). This completes the induction.

Since the Picard sequence converges in $\tilde{\mathcal{F}}_T$, the limit (v, b) also satisfies

$$\lim_{T' \rightarrow 0^+} \|(v, b)\|_{\tilde{\mathcal{F}}_{T'}} = 0. \quad (2.19)$$

for $T' > 0$ sufficiently small. Using (2.14) and (2.19), we find

$$\lim_{T' \rightarrow 0^+} \sup_{0 < t < T'} \|B((v, b), (v, b))\|_{E_q^3 \times E_q^3}(t) = 0.$$

Combining this with [1, Lemma 2.3], we conclude

$$\lim_{t \rightarrow 0} \|(v, b) - (v_0, b_0)\|_{E_q^3 \times E_q^3} = 0.$$

If $q = \infty$ then we have a weaker mode of convergence. Fix a ball B . Take $R > 0$ large so that $B \subset B_R(0)$. We re-write the bilinear form as

$$B((v, b), (v, b)) = B((v, b), (v\chi_{B_R(0)}, b\chi_{B_R(0)})) + B((v, b), (v(1 - \chi_{B_R(0)}), b(1 - \chi_{B_R(0)}))).$$

If $1 < \omega < 3$ then we have

$$\begin{aligned} &\|B_1((v, b), (v\chi_{B_R(0)}, b\chi_{B_R(0)}))\|_{L^\omega} \\ &\lesssim \int_0^t \frac{1}{(t - \tau)^{\frac{1}{2} + \frac{3}{2}(\frac{2}{3} - \frac{1}{\omega})}} (||v|^2 \chi_{B_R(0)}\|_{L^{3/2}} + ||b|^2 \chi_{B_R(0)}\|_{L^{3/2}})(\tau) d\tau \\ &\lesssim_R t^{\frac{1}{2} - \frac{3}{2}(\frac{2}{3} - \frac{1}{\omega})} \|(v, b)\|_{L^\infty(L_{\text{uloc}}^3 \times L_{\text{uloc}}^3)}^2, \\ &\|B_2((v, b), (v\chi_{B_R(0)}, b\chi_{B_R(0)}))\|_{L^\omega} \\ &\lesssim \int_0^t \frac{1}{(t - \tau)^{\frac{1}{2} + \frac{3}{2}(\frac{2}{3} - \frac{1}{\omega})}} (||v \otimes b\chi_{B_R(0)}\|_{L^{3/2}} + ||b \otimes v\chi_{B_R(0)}\|_{L^{3/2}})(\tau) d\tau \\ &\lesssim \int_0^t \frac{1}{(t - \tau)^{\frac{1}{2} + \frac{3}{2}(\frac{2}{3} - \frac{1}{\omega})}} (||v|^2 \chi_{B_R(0)}\|_{L^{3/2}} + ||b|^2 \chi_{B_R(0)}\|_{L^{3/2}})(\tau) d\tau \\ &\lesssim_R t^{\frac{1}{2} - \frac{3}{2}(\frac{2}{3} - \frac{1}{\omega})} \|(v, b)\|_{L^\infty(L_{\text{uloc}}^3 \times L_{\text{uloc}}^3)}^2, \end{aligned}$$

by Young's inequality, so that

$$\|B((v, b), (v\chi_{B_R(0)}, b\chi_{B_R(0)}))\|_{L^\omega \times L^\omega} \lesssim_R t^{\frac{1}{2} - \frac{3}{2}(\frac{2}{3} - \frac{1}{\omega})} \|(v, b)\|_{L^\infty(L_{\text{uloc}}^3 \times L_{\text{uloc}}^3)}^2.$$

For any $R > 0$, the above vanishes as $t \rightarrow 0$ provided $\omega < 3$.

By taking $R = 2 \max_{x \in B} |x|$, we can ensure that for all $x \in B$ and $|y| \geq R$ we have $\frac{1}{2}|y| \leq |x - y| \leq \frac{3}{2}|y|$. Hence, for $x \in B$,

$$\begin{aligned}
& |B((v, b), (v\chi_{B_R(0)^c}, b\chi_{B_R(0)^c}))(x, t)| \\
& \lesssim \int_0^t \int_{y \in B_R(0)^c} \frac{1}{(|x - y| + \sqrt{t - \tau})^4} (|v|^2 + |b|^2)(y, \tau) dy d\tau \\
& \lesssim t \sup_{0 < \tau < t} \sum_{k=1}^{\infty} \frac{1}{2^{4k-4} R^4} \int_{R2^{k-1} \leq |y| < R2^k} (|v|^2 + |b|^2)(y, \tau) dy \\
& \lesssim t \sup_{0 < \tau < t} \sum_{k=1}^{\infty} \frac{R2^k}{R^4 2^{4k-4}} \left(\int_{|y| < R2^k} (|v|^3 + |b|^3)(y, \tau) dy \right)^{\frac{2}{3}} \\
& \lesssim t \sum_{k=1}^{\infty} \frac{R^3 2^{3k}}{R^4 2^{4k-4}} \sup_{0 < \tau < t} \|(v, b)(\tau)\|_{L^3_{\text{uloc}} \times L^3_{\text{uloc}}}^2 \\
& \lesssim \frac{t}{R} \sup_{0 < \tau < t} \|(v, b)(\tau)\|_{L^3_{\text{uloc}} \times L^3_{\text{uloc}}}^2.
\end{aligned}$$

Now,

$$\begin{aligned}
& \|B((v, b), (v, b))(t)\|_{L^\omega(B) \times L^\omega(B)} \\
& \lesssim_R \|B((v, b), (v\chi_{B_R(0)^c}, b\chi_{B_R(0)^c}))(t)\|_{L^\omega} + \|B((v, b), (v\chi_{B_R(0)^c}, b\chi_{B_R(0)^c}))(t)\|_{L^\infty(B)} \\
& \lesssim_R t^{\frac{1}{2} - \frac{3}{2}(\frac{2}{3} - \frac{1}{\omega})} \|(v, b)\|_{L^\infty(L^3_{\text{uloc}} \times L^3_{\text{uloc}})}^2 + t \sup_{0 < \tau < t} \|(v, b)(\tau)\|_{L^3_{\text{uloc}} \times L^3_{\text{uloc}}}^2.
\end{aligned}$$

Hence

$$\lim_{t \rightarrow 0^+} \|B((v, b), (v, b))(t)\|_{L^\omega(B) \times L^\omega(B)} = 0.$$

Referring to [33, p. 394], we have

$$\lim_{t \rightarrow 0} \|(e^{t\Delta} v_0, e^{t\Delta} b_0) - (v_0, b_0)\|_{L^\omega(B) \times L^\omega(B)} = 0.$$

It follows that

$$\lim_{t \rightarrow 0} \|(v, b)(\cdot, t) - (v_0, b_0)(\cdot)\|_{L^\omega(B) \times L^\omega(B)} = 0.$$

We now prove continuity at positive times. Let $t_1 > 0$ be fixed. We will send $t \rightarrow t_1$. Note that by [1, Lemma 2.3] we have $(e^{t\Delta} v_0, e^{t\Delta} b_0) - (e^{t_1\Delta} v_0, e^{t_1\Delta} b_0) \rightarrow 0$ in $E_q^3 \times E_q^3$ as $t \rightarrow t_1$. We therefore only need to show $B((v, b), (v, b))(t) \rightarrow B((v, b), (v, b))(t_1)$. Following [40, p. 86], we take ρ slightly less than 1 so that $\rho t_1 < t$ and write

$$\begin{aligned}
B((v, b), (v, b))(t) - B((v, b), (v, b))(t_1) &= \int_{\rho t_1}^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot F d\tau - \int_{\rho t_1}^{t_1} e^{(t_1-\tau)\Delta} \mathbb{P} \nabla \cdot F d\tau \\
&\quad + \int_0^{\rho t_1} (e^{(t-\rho t_1)\Delta} - e^{(t_1-\rho t_1)\Delta}) e^{(\rho t_1-\tau)\Delta} \mathbb{P} \nabla \cdot F d\tau
\end{aligned}$$

where $F = (v \otimes v)(\tau)$, $(b \otimes b)(\tau)$, $(v \otimes b)(\tau)$, or $(b \otimes v)(\tau)$. For the first and second terms, by [1, (2.3)] with $p = \tilde{p} = 3$, $\tilde{q} = q$ and using the embedding $E_q^\infty \subset E_\infty^\infty$, we have

$$\int_{\rho t_1}^t \|e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot F\|_{E_q^3} d\tau \lesssim \int_{\rho t_1}^t \frac{1}{(t-\tau)^{\frac{1}{2}} \tau^{\frac{1}{2}}} \|v, b\|_{E_q^3}(\tau) \tau^{\frac{1}{2}} \|v, b\|_{E_q^\infty} d\tau \lesssim \frac{(t - \rho t_1)^{\frac{1}{2}}}{(\rho t_1)^{\frac{1}{2}}} \|(v, b)\|_{\mathcal{E}_T}^2,$$

and

$$\int_{\rho t_1}^{t_1} \|e^{(t_1-\tau)\Delta} \mathbb{P} \nabla \cdot F\|_{E_q^3} d\tau \lesssim \int_{\rho t_1}^{t_1} \frac{1}{(t_1-\tau)^{\frac{1}{2}} \tau^{\frac{1}{2}}} \|v, b\|_{E_q^3}(\tau) \tau^{\frac{1}{2}} \|v, b\|_{E_q^\infty} d\tau \lesssim \frac{(t_1 - \rho t_1)^{\frac{1}{2}}}{(\rho t_1)^{\frac{1}{2}}} \|(v, b)\|_{\mathcal{E}_T}^2,$$

both of which can be made arbitrarily small by taking ρt_1 close to t_1 and t close to t_1 .

For the third term we note that by [1, Lemma 2.3], for each $0 < \tau < \rho t_1$, we have

$$\|(e^{(t-\rho t_1)\Delta} - e^{(t_1-\rho t_1)\Delta})e^{(\rho t_1-\tau)\Delta} \mathbb{P} \nabla \cdot F(\tau)\|_{E_q^3} \rightarrow 0 \text{ as } t \rightarrow t_1,$$

which follows (even if $q = \infty$) the fact that $e^{(\rho t_1-\tau)\Delta} \mathbb{P} \nabla \cdot F(\tau) \in E_q^3$, which is a consequence of [1, Lemma 2.1]. Additionally,

$$\begin{aligned} & \| (e^{(t-\rho t_1)\Delta} - e^{(t_1-\rho t_1)\Delta}) e^{(\rho t_1-\tau)\Delta} \mathbb{P} \nabla \cdot F(\tau) \|_{E_q^3} \\ & \lesssim \left(\frac{1}{(t-\tau)^{\frac{1}{2}} \tau^{\frac{1}{2}}} + \frac{1}{(t_1-\tau)^{\frac{1}{2}} \tau^{\frac{1}{2}}} \right) \|(v, b)\|_{\mathcal{E}_T}^2 \in L^1(0, \rho t_1), \end{aligned}$$

where integration in $L^1(0, \rho t_1)$ is with respect to τ . So, by Lebesgue's dominated convergence theorem,

$$\int_0^{\rho t_1} \| (e^{(t-\rho t_1)\Delta} - e^{(t_1-\rho t_1)\Delta}) e^{(\rho t_1-\tau)\Delta} \mathbb{P} \nabla \cdot F(\tau) \|_{E_q^3} d\tau \rightarrow 0 \text{ as } t \rightarrow t_1.$$

The above show the continuity of $(v, b)(t)$ at positive times.

To prove the spacetime integral bound (1.23) for $p = 3$, $s = \infty$, we work in the spaces with the norms

$$\|(v, b)\|_{\mathcal{E}_T^*} = \|(v, b)\|_{E_{T,q}^{\infty,3} \times E_{T,q}^{\infty,3}} + \left\| (t^{\frac{1}{2}} v, t^{\frac{1}{2}} b) \right\|_{E_{T,q}^{\infty,\infty} \times E_{T,q}^{\infty,\infty}}, \quad \|(v, b)\|_{\mathcal{F}_T^*} = \left\| (t^{\frac{1}{4}} v, t^{\frac{1}{4}} b) \right\|_{E_{T,q}^{\infty,6} \times E_{T,q}^{\infty,6}}.$$

Note that $\mathcal{E}_T^* \subset \mathcal{F}_T^*$.

For the linear estimate in \mathcal{F}_T^* , taking $p = 6$ so that $a = 1/4$ in [1, Lemma 3.1], we have

$$\|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{\mathcal{F}_T^*} = \left\| (t^{\frac{1}{4}} e^{t\Delta} v_0, t^{\frac{1}{4}} e^{t\Delta} b_0) \right\|_{E_{T,q}^{\infty,6} \times E_{T,q}^{\infty,6}} \leq C_1^* (1 + T^{\frac{1}{4}}) \|(v_0, b_0)\|_{E_q^3 \times E_q^3}. \quad (2.20)$$

For bilinear estimate, taking $a = b = 1/4$, $p = \tilde{p} = 6$ in [1, Lemma 3.2],

$$\begin{aligned} \|B((v, b), (u, a))\|_{\mathcal{F}_T^*} &= \left\| t^{\frac{1}{4}} B((v, b), (u, a)) \right\|_{E_{T,q}^{\infty,6} \times E_{T,q}^{\infty,6}} \\ &\leq C_2^* (1 + T^{\frac{3}{4}}) \left\| (t^{\frac{1}{4}} v, t^{\frac{1}{4}} b) \right\|_{E_{T,q}^{\infty,6} \times E_{T,q}^{\infty,6}} \left\| (t^{\frac{1}{4}} u, t^{\frac{1}{4}} a) \right\|_{E_{T,q}^{\infty,6} \times E_{T,q}^{\infty,6}} \\ &= C_2^* (1 + T^{\frac{3}{4}}) \|(v, b)\|_{\mathcal{F}_T^*} \|(u, a)\|_{\mathcal{F}_T^*}. \end{aligned}$$

By choosing $\|(v_0, b_0)\|_{E_q^3 \times E_q^3}$ small enough so that $\|(v_0, b_0)\|_{E_q^3 \times E_q^3} < \frac{1}{4C_1^* C_2^* (1+T^{\frac{1}{4}})(1+T^{\frac{3}{4}})}$, Picard iteration yields a unique mild solution satisfying

$$\|(v, b)\|_{\mathcal{F}_T^*} \leq 2C_1^* (1 + T^{\frac{1}{4}}) \|(v_0, b_0)\|_{E_q^3 \times E_q^3}.$$

Now, we claim that any solution $(v, b) \in \mathcal{F}_T^*$ with sufficiently small $(v_0, b_0) \in E_q^3 \times E_q^3$ also belongs to \mathcal{E}_T^* . By [1, (3.21)] of [1, Lemma 3.1], we have

$$\|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{E_{T,q}^{\infty,3} \times E_{T,q}^{\infty,3}} \lesssim \ln(2 + T) \|(v_0, b_0)\|_{E_q^3 \times E_q^3}.$$

Taking $p = \infty$ so that $a = 1/2$ in [1, Lemma 3.1], we also obtain

$$\left\| (t^{\frac{1}{2}} v_0, t^{\frac{1}{2}} b_0) \right\|_{E_{T,q}^{\infty,\infty}} \lesssim (1 + T^{\frac{1}{2}}) \|(v_0, b_0)\|_{E_q^3 \times E_q^3}.$$

Combining both estimates, we conclude:

$$\left\| (e^{t\Delta} v_0, e^{t\Delta} b_0) \right\|_{\mathcal{E}_T^*} \lesssim (1 + T^{\frac{1}{2}}) \|(v_0, b_0)\|_{E_q^3 \times E_q^3}. \quad (2.21)$$

Next, we establish the bilinear estimate:

$$\|B((v, b), (u, a))\|_{\mathcal{E}_T^*} \lesssim_T \min \left(\|(v, b)\|_{\mathcal{E}_T^*} \|(u, a)\|_{\mathcal{F}_T^*}, \|(u, a)\|_{\mathcal{E}_T^*} \|(v, b)\|_{\mathcal{F}_T^*} \right). \quad (2.22)$$

Indeed, by applying [1, Lemma 3.2] with $(a, b, p, \tilde{p}) = (0, 1/4, 3, 6)$ and $(a, b, p, \tilde{p}) = (1/2, 1/4, \infty, 6)$, we obtain

$$\begin{aligned} & \|B((v, b), (u, a))\|_{E_{T,q}^{\infty,3} \times E_{T,q}^{\infty,3}} \\ & \lesssim (1 + T^{\frac{3}{4}}) \min \left(\|(v, b)\|_{E_{T,q}^{\infty,3} \times E_{T,q}^{\infty,3}} \left\| (t^{\frac{1}{4}} u, t^{\frac{1}{4}} a) \right\|_{E_{T,q}^{\infty,6} \times E_{T,q}^{\infty,6}}, \right. \\ & \quad \left. \|(u, a)\|_{E_{T,q}^{\infty,3} \times E_{T,q}^{\infty,3}} \left\| (t^{\frac{1}{4}} v, t^{\frac{1}{4}} b) \right\|_{E_{T,q}^{\infty,6} \times E_{T,q}^{\infty,6}} \right), \end{aligned}$$

and

$$\begin{aligned} & \left\| t^{\frac{1}{2}} B((v, b), (u, a)) \right\|_{E_{T,q}^{\infty,\infty} \times E_{T,q}^{\infty,\infty}} \\ & \lesssim (1 + T^{\frac{3}{4}}) \min \left(\left\| (t^{\frac{1}{2}} v, t^{\frac{1}{2}} b) \right\|_{E_{T,q}^{\infty,\infty} \times E_{T,q}^{\infty,\infty}} \left\| (t^{\frac{1}{4}} u, t^{\frac{1}{4}} a) \right\|_{E_{T,q}^{\infty,6} \times E_{T,q}^{\infty,6}}, \right. \\ & \quad \left. \left\| (t^{\frac{1}{2}} u, t^{\frac{1}{2}} a) \right\|_{E_{T,q}^{\infty,\infty} \times E_{T,q}^{\infty,\infty}} \left\| (t^{\frac{1}{4}} v, t^{\frac{1}{4}} b) \right\|_{E_{T,q}^{\infty,6} \times E_{T,q}^{\infty,6}} \right). \end{aligned}$$

Using the same argument as before, we conclude that $(v, b) \in \mathcal{E}_T^*$, possibly after taking a smaller $T > 0$. In particular, $(v, b) \in E_{T,q}^{\infty,3} \times E_{T,q}^{\infty,3}$, with the norm controlled by $\|(v_0, b_0)\|_{E_q^3 \times E_q^3}$. The case $s = \infty$ in Theorem 1.2 then follows from the embeddings $\|(v, b)\|_{L^\infty E_q^3 \times L^\infty E_q^3} \leq \|(v, b)\|_{E_{T,q}^{\infty,3} \times E_{T,q}^{\infty,3}}$ and $\left\| (t^{\frac{1}{2}} v, t^{\frac{1}{2}} b) \right\|_{L_T^\infty(E_q^\infty \times E_q^\infty)} \leq \left\| (t^{\frac{1}{2}} v, t^{\frac{1}{2}} b) \right\|_{E_{T,q}^{\infty,\infty} \times E_{T,q}^{\infty,\infty}}$. \square

2.3 Mild solutions in critical spaces with enough decay: Proof of Theorem 1.3

The proof of Theorem 1.3 is an adaption of the proof of [1, Theorem 1.3] for the Navier–Stokes equations to the MHD equations.

By [1, Lemma 2.1], we have for $1 \leq q \leq 3$,

$$\begin{aligned} & \sup_{0 < t < \infty} \left(\|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{E_q^3 \times E_q^3} + t^{\frac{1}{2}} \|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{E_{q_2}^\infty \times E_{q_2}^\infty} + t^{\frac{1}{4}} \|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{E_{q_1}^6 \times E_{q_1}^6} \right) \\ & \leq C \|(v_0, b_0)\|_{E_q^3 \times E_q^3}, \end{aligned} \quad (2.23)$$

where

$$\frac{1}{q_1} = \frac{1}{q} - \frac{1}{6}, \quad \frac{1}{q_2} = \frac{1}{q} - \frac{1}{3}, \quad q < q_1 < q_2 \leq \infty. \quad (2.24)$$

For $0 < T \leq \infty$ and $1 \leq q \leq 3$, let $\mathcal{E}_T, \mathcal{F}_T$ be Banach spaces defined as

$$\mathcal{E}_T := \left\{ (v, b) \in L^\infty(0, T; E_q^3 \times E_q^3) : t^{\frac{1}{2}}(v, b)(\cdot, t) \in L^\infty(0, T; E_{q_2}^\infty \times E_{q_2}^\infty) \right\}, \quad (2.25)$$

and

$$\mathcal{F}_T := \left\{ (v, b) : t^{\frac{1}{4}}(v, b)(\cdot, t) \in L^\infty(0, T; E_{q_1}^6 \times E_{q_1}^6) \right\}, \quad (2.26)$$

with norms

$$\|(v, b)\|_{\mathcal{E}_T} := \sup_{0 < t < T} \|(v, b)(\cdot, t)\|_{E_q^3 \times E_q^3} + \sup_{0 < t < T} t^{\frac{1}{2}} \|(v, b)(\cdot, t)\|_{E_{q_2}^\infty \times E_{q_2}^\infty},$$

and

$$\|(v, b)\|_{\mathcal{F}_T} := \sup_{0 < t < T} t^{\frac{1}{4}} \|(v, b)(\cdot, t)\|_{E_{q_1}^6 \times E_{q_1}^6},$$

respectively. Note that $\mathcal{E}_T \subset \mathcal{F}_T$.

By [1, Lemma 2.1] again and Hölder inequality (1.3) using $2q \geq q_1$ due to $q \leq 3$,

$$\begin{aligned} \|B_1((v, b), (u, a))\|_{E_{q_1}^6}(t) &\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{3}{4}}} \left(\|v \otimes u(\tau)\|_{E_q^3} + \|b \otimes a(\tau)\|_{E_q^3} \right) d\tau \\ &\leq \int_0^t \frac{1}{(t-\tau)^{\frac{3}{4}}} \left(\|v(\tau)\|_{E_{q_1}^6} \|u(\tau)\|_{E_{q_1}^6} + \|b(\tau)\|_{E_{q_1}^6} \|a(\tau)\|_{E_{q_1}^6} \right) d\tau \\ &\leq \int_0^t \frac{1}{(t-\tau)^{\frac{3}{4}}} \tau^{-1/4} \|(v, b)\|_{\mathcal{F}_T} \tau^{-1/4} \|(u, a)\|_{\mathcal{F}_T} d\tau \\ &\lesssim t^{-1/4} \|(v, b)\|_{\mathcal{F}_T} \|(u, a)\|_{\mathcal{F}_T}, \end{aligned} \quad (2.27)$$

$$\begin{aligned} \|B_2((v, b), (u, a))\|_{E_{q_1}^6}(t) &\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{3}{4}}} \left(\|v \otimes a(\tau)\|_{E_q^3} + \|b \otimes u(\tau)\|_{E_q^3} \right) d\tau \\ &\leq \int_0^t \frac{1}{(t-\tau)^{\frac{3}{4}}} \left(\|v(\tau)\|_{E_{q_1}^6} \|a(\tau)\|_{E_{q_1}^6} + \|b(\tau)\|_{E_{q_1}^6} \|u(\tau)\|_{E_{q_1}^6} \right) d\tau \\ &\leq \int_0^t \frac{1}{(t-\tau)^{\frac{3}{4}}} \tau^{-1/4} \|(v, b)\|_{\mathcal{F}_T} \tau^{-1/4} \|(u, a)\|_{\mathcal{F}_T} d\tau \\ &\lesssim t^{-1/4} \|(v, b)\|_{\mathcal{F}_T} \|(u, a)\|_{\mathcal{F}_T}, \end{aligned} \quad (2.28)$$

so that

$$\|B((v, b), (u, a))\|_{E_{q_1}^6 \times E_{q_1}^6}(t) \lesssim t^{-1/4} \|(v, b)\|_{\mathcal{F}_T} \|(u, a)\|_{\mathcal{F}_T}. \quad (2.29)$$

Hence,

$$\|B((v, b), (u, a))\|_{\mathcal{F}_T} \leq c_* \|(v, b)\|_{\mathcal{F}_T} \|(u, a)\|_{\mathcal{F}_T},$$

where c_* is a universal constant.

Concerning the caloric extension of (v_0, b_0) , we have for $\|(v_0, b_0)\|_{E_q^3 \times E_q^3}$ of any size that

$$\lim_{T \rightarrow 0} \|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{\mathcal{F}_T} = 0,$$

by [1, (2.13)] of [1, Lemma 2.3]. Hence, there exists $T = T(u_0)$ so that

$$\|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{\mathcal{F}_T} \lesssim c_*^{-1}. \quad (2.30)$$

If, on the other hand, $\|(v_0, b_0)\|_{E_q^3 \times E_q^3} \lesssim c_*^{-1}$, then by (2.23), we have (2.30) for $T = \infty$. The Picard contraction theorem then guarantees the existence of a mild solution (v, b) to (MHD) so that

$$\|(v, b)\|_{\mathcal{F}_T} \leq 2\|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{\mathcal{F}_T}.$$

This solution is unique among all mild solutions (u, a) with data (v_0, b_0) satisfying $\|(u, a)\|_{\mathcal{F}_T} \leq 2\|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{\mathcal{F}_T}$.

Next, we show that a solution $(v, b) \in \mathcal{F}_T$ with initial data $(v_0, b_0) \in E_q^3 \times E_q^3$ also belongs to \mathcal{E}_T . Let $\{(v^{(n)}, b^{(n)})\}_{n \geq 1}$ be the Picard iteration sequence in \mathcal{F}_T . By construction,

$$\|(v^{(n)}, b^{(n)})\|_{\mathcal{F}_T} \leq 2\|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{\mathcal{F}_T}. \quad (2.31)$$

Note that

$$\|(v^{(n)}, b^{(n)})\|_{\mathcal{E}_T} \leq \|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{\mathcal{E}_T} + \|B((v^{(n-1)}, b^{(n-1)}), (v^{(n-1)}, b^{(n-1)}))\|_{\mathcal{E}_T}.$$

We now bound $B((v, b), (u, a))$ in \mathcal{E}_T in terms of (v, b) and (u, a) in \mathcal{F}_T and \mathcal{E}_T . We have by [1, Lemma 2.1] and Hölder inequality (1.3) using $q_1 \geq 6$,

$$\begin{aligned} \|B_1((v, b), (u, a))\|_{E_q^3}(t) &\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}} \left(\|v \otimes u(\tau)\|_{E_q^3} + \|b \otimes a(\tau)\|_{E_q^3} \right) d\tau \\ &\leq \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}} \left(\|v(\tau)\|_{E_{q_1}^6} \|u(\tau)\|_{E_{q_1}^6} + \|b(\tau)\|_{E_{q_1}^6} \|a(\tau)\|_{E_{q_1}^6} \right) d\tau \\ &\leq \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}} \tau^{-1/4} \|(v, b)\|_{\mathcal{F}_T} \tau^{-1/4} \|(u, a)\|_{\mathcal{F}_T} d\tau \\ &\lesssim \|(v, b)\|_{\mathcal{F}_T} \|(u, a)\|_{\mathcal{F}_T}, \end{aligned} \quad (2.32)$$

$$\begin{aligned} \|B_2((v, b), (u, a))\|_{E_q^3}(t) &\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}} \left(\|v \otimes a(\tau)\|_{E_q^3} + \|b \otimes u(\tau)\|_{E_q^3} \right) d\tau \\ &\leq \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}} \left(\|v(\tau)\|_{E_{q_1}^6} \|a(\tau)\|_{E_{q_1}^6} + \|b(\tau)\|_{E_{q_1}^6} \|u(\tau)\|_{E_{q_1}^6} \right) d\tau \\ &\leq \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}} \tau^{-1/4} \|(v, b)\|_{\mathcal{F}_T} \tau^{-1/4} \|(u, a)\|_{\mathcal{F}_T} d\tau \\ &\lesssim \|(v, b)\|_{\mathcal{F}_T} \|(u, a)\|_{\mathcal{F}_T}, \end{aligned} \quad (2.33)$$

so that

$$\|B((v, b), (u, a))\|_{E_q^3 \times E_q^3}(t) \lesssim \|(v, b)\|_{\mathcal{F}_T} \|(u, a)\|_{\mathcal{F}_T}. \quad (2.34)$$

By $\mathcal{E}_T \subset \mathcal{F}_T$, we have

$$\|B((v, b), (u, a))\|_{E_q^3 \times E_q^3}(t) \lesssim \|(v, b)\|_{\mathcal{E}_T} \|(u, a)\|_{\mathcal{F}_T} \wedge \|(u, a)\|_{\mathcal{E}_T} \|(v, b)\|_{\mathcal{F}_T}.$$

Also by [1, Lemma 2.1] and Hölder inequality (1.3),

$$\begin{aligned} \|B_1((v, b), (u, a))\|_{E_{q_2}^\infty}(t) &\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{3}{4}}} \left(\|v \otimes u\|_{E_{q_1}^6}(\tau) + \|b \otimes a\|_{E_{q_1}^6}(\tau) \right) d\tau \\ &\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{3}{4}} \tau^{3/4}} \left(\tau^{1/2} \|v\|_{E_{q_2}^\infty} \tau^{1/4} \|u\|_{E_{q_1}^6} \wedge \tau^{1/2} \|v\|_{E_{q_2}^\infty} \tau^{1/4} \|u\|_{E_{q_1}^6} \right. \\ &\quad \left. + \tau^{1/2} \|b\|_{E_{q_2}^\infty} \tau^{1/4} \|a\|_{E_{q_1}^6} \wedge \tau^{1/2} \|b\|_{E_{q_2}^\infty} \tau^{1/4} \|a\|_{E_{q_1}^6} \right) d\tau \\ &\lesssim t^{-\frac{1}{2}} (\|(v, b)\|_{\mathcal{E}_T} \|(u, a)\|_{\mathcal{F}_T} \wedge \|(u, a)\|_{\mathcal{E}_T} \|(v, b)\|_{\mathcal{F}_T}), \end{aligned} \quad (2.35)$$

$$\begin{aligned}
\|B_2((v, b), (u, a))\|_{E_{q_2}^\infty}(t) &\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{3}{4}}} \left(\|v \otimes a\|_{E_{q_1}^6}(\tau) + \|b \otimes u\|_{E_{q_1}^6}(\tau) \right) d\tau \\
&\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{3}{4}} \tau^{3/4}} (\tau^{1/2} \|v\|_{E_{q_2}^\infty} \tau^{1/4} \|a\|_{E_{q_1}^6} \wedge \tau^{1/2} \|v\|_{E_{q_2}^\infty} \tau^{1/4} \|a\|_{E_{q_1}^6} \\
&\quad + \tau^{1/2} \|b\|_{E_{q_2}^\infty} \tau^{1/4} \|u\|_{E_{q_1}^6} \wedge \tau^{1/2} \|b\|_{E_{q_2}^\infty} \tau^{1/4} \|u\|_{E_{q_1}^6}) d\tau \\
&\lesssim t^{-\frac{1}{2}} (\|(v, b)\|_{\mathcal{E}_T} \|(u, a)\|_{\mathcal{F}_T} \wedge \|(u, a)\|_{\mathcal{E}_T} \|(v, b)\|_{\mathcal{F}_T}), \tag{2.36}
\end{aligned}$$

so that

$$\|B((v, b), (u, a))\|_{E_{q_2}^\infty \times E_{q_2}^\infty}(t) \lesssim t^{-\frac{1}{2}} (\|(v, b)\|_{\mathcal{E}_T} \|(u, a)\|_{\mathcal{F}_T} \wedge \|(u, a)\|_{\mathcal{E}_T} \|(v, b)\|_{\mathcal{F}_T}).$$

Based on the above estimates we conclude

$$\|B((v, b), (u, a))\|_{\mathcal{E}_T} \lesssim \|(v, b)\|_{\mathcal{E}_T} \|(u, a)\|_{\mathcal{F}_T} \wedge \|(u, a)\|_{\mathcal{E}_T} \|(v, b)\|_{\mathcal{F}_T}. \tag{2.37}$$

We can now conclude that $\{(v^{(n)}, b^{(n)})\}$ is Cauchy in \mathcal{E}_T by the calculation preceding and including (2.15). However, the smallness of the constant is now provided by (2.30)-(2.31), not by $\|(v_0, b_0)\|_{E_q^3 \times E_q^3}$.

We now show continuity. For small data, we can try to inherit continuity from Theorem 1.2. But we will provide a proof valid for general data. We first address convergence to the initial data. By [1, Lemma 2.3] we have

$$\lim_{T' \rightarrow 0^+} \sup_{0 < t < T'} t^{\frac{1}{4}} \|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{E_{q_1}^6 \times E_{q_1}^6} = \lim_{T' \rightarrow 0^+} \|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{\mathcal{F}_{T'}} = 0, \tag{2.38}$$

whenever $(v_0, b_0) \in E_q^3 \times E_q^3$. By our estimates in the class $\mathcal{F}_{T'}$ where we are taking $T' \leq T$, we have

$$\begin{aligned}
\|(v^{(n)}, b^{(n)})\|_{\mathcal{F}_{T'}} &\leq \|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{\mathcal{F}_{T'}} + \|B((v^{(n-1)}, b^{(n-1)}), (v^{(n-1)}, b^{(n-1)}))\|_{\mathcal{F}_{T'}} \\
&\lesssim \|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{\mathcal{F}_{T'}} + \|(v^{(n-1)}, b^{(n-1)})\|_{\mathcal{F}_{T'}}^2.
\end{aligned}$$

From this and by induction, for any n we have

$$\lim_{T' \rightarrow 0^+} \|(v^{(n)}, b^{(n)})\|_{\mathcal{F}_{T'}} = 0.$$

The limit (2.38), convergence of the Picard iterates in \mathcal{F}_T and the above inequality imply that, by taking T' small, we can make $\sup_{0 < t < T'} t^{\frac{1}{4}} \|(v, b)(t)\|_{E_{q_1}^6 \times E_{q_1}^6}$ small. To elaborate, we have

$$\|(v, b)\|_{\mathcal{F}_{T'}} \leq \|(v, b) - (v^{(n)}, b^{(n)})\|_{\mathcal{F}_{T'}} + \|(v^{(n)}, b^{(n)})\|_{\mathcal{F}_{T'}}. \tag{2.39}$$

We may choose n large so that the first term is small and then make the second term small by taking T' small. Hence,

$$\lim_{T' \rightarrow 0^+} \|(v, b)\|_{\mathcal{F}_{T'}} = 0. \tag{2.40}$$

Using (2.34), this implies

$$\lim_{T' \rightarrow 0^+} \sup_{0 < t < T'} \|B((v, b), (v, b))\|_{E_q^3 \times v}(t) = 0.$$

This and [1, Lemma 2.3] imply

$$\lim_{t \rightarrow 0} \|(v, b) - (v_0, b_0)\|_{E_q^3 \times E_q^3} = 0.$$

The proof of continuity for positive times follows from the same argument used in the proof of Theorem 1.2 and is therefore omitted for brevity.

We now prove the spacetime integral bound (1.24) for $p \in (3, 9]$ and $\frac{2}{s} + \frac{3}{p} = 1$. Note that we exclude $p = 3$, i.e., $s = \infty$. By imbedding $E_q^p \subset E_m^p$ for $m > q$, we may assume $m < \infty$. (We do not take $m = q$ since we need $q < m$ for global existence). Denote the Banach space

$$X_T = \mathcal{E}_T \cap (E_{T,m}^{s,p} \times E_{T,m}^{s,p}).$$

For the linear term, by [1, Lemma 2.1] and [1, Lemma 2.4],

$$\begin{aligned} \|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{X_T} &= \sup_{0 < t < T} \|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{E_q^3 \times E_q^3} \\ &\quad + \sup_{0 < t < T} t^{\frac{1}{2}} \|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{E_{q_2}^\infty \times E_{q_2}^\infty} + \|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{E_{T,m}^{s,p} \times E_{T,m}^{s,p}} \\ &\leq C_3(1 + T^{\beta_0}) \|(v_0, b_0)\|_{E_q^3 \times E_q^3}, \end{aligned} \quad (2.41)$$

for any $\beta_0 \in [0, \infty)$ and $\beta_0 > \alpha_0 = \frac{3}{2m} - \frac{3}{2q} + \frac{1}{s}$. Note that

$$\lim_{T \rightarrow 0_+} \|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{E_{T,m}^{s,p} \times E_{T,m}^{s,p}} = 0. \quad (2.42)$$

For the bilinear term, by [1, Lemma 2.7] with $\tilde{s} = s/2$, $\tilde{p} = p/2$, and $\tilde{m} = \max(1, m/2)$, so that

$$\sigma = 0, \quad \alpha = \begin{cases} \frac{1}{2} - \frac{3}{2m} - \frac{1}{s}, & \text{if } 2 \leq m \leq \infty, \\ -1 + \frac{3}{2m} - \frac{1}{s}, & \text{if } 1 < m < 2, \end{cases} \quad \alpha < 1 - \frac{1}{s}, \quad (2.43)$$

we have

$$\|B((v, b), (u, a))\|_{E_{T,m}^{s,p} \times E_{T,m}^{s,p}} \leq C_4(1 + T^\beta) \left(\|v \otimes u\|_{E_{T,\tilde{m}}^{\frac{s}{2}, \frac{p}{2}}} + \|b \otimes a\|_{E_{T,\tilde{m}}^{\frac{s}{2}, \frac{p}{2}}} + \|v \otimes a\|_{E_{T,\tilde{m}}^{\frac{s}{2}, \frac{p}{2}}} + \|b \otimes u\|_{E_{T,\tilde{m}}^{\frac{s}{2}, \frac{p}{2}}} \right)$$

for any $\beta \in [0, 1 - \frac{1}{s}]$ and $\beta > \alpha$. Note

$$\|f \otimes g\|_{E_{T,\tilde{m}}^{\frac{s}{2}, \frac{p}{2}}} \leq \|f\|_{E_{T,2\tilde{m}}^{s,p}} \|g\|_{E_{T,2\tilde{m}}^{s,p}} \leq \|f\|_{E_{T,m}^{s,p}} \|g\|_{E_{T,m}^{s,p}}$$

no matter $m \geq 2$ or $1 < m < 2$. We conclude, also using (2.37),

$$\begin{aligned} \|B((v, b), (u, a))\|_{E_{T,m}^{s,p} \times E_{T,m}^{s,p}} &\leq C_4(1 + T^\beta) \|(v, b)\|_{E_{T,m}^{s,p} \times E_{T,m}^{s,p}} \|(u, a)\|_{E_{T,m}^{s,p} \times E_{T,m}^{s,p}}, \\ \|B((v, b), (u, a))\|_{X_T} &\leq 2C_4(1 + T^\beta) \|(v, b)\|_{X_T} \|(u, a)\|_{X_T}. \end{aligned} \quad (2.44)$$

By (2.42), we can find $T_1 \in (0, T]$ so that

$$\|(e^{t\Delta} v_0, e^{t\Delta} b_0)\|_{E_{T_1,m}^{s,p} \times E_{T_1,m}^{s,p}} \leq \delta = [4C_4(1 + T^\beta)]^{-1}.$$

Then the Picard sequence $(v^{(k)}, b^{(k)})$ satisfies $\|(v^{(k)}, b^{(k)})\|_{E_{T_1,m}^{s,p} \times E_{T_1,m}^{s,p}} \leq 2\delta$ for all $k \in \mathbb{N}$, and we get $\|(v, b)\|_{E_{T_1,m}^{s,p} \times E_{T_1,m}^{s,p}} \leq 2\delta$. Thus, (v, b) satisfies the spacetime integral bound (1.24).

We now establish the global $E_{T,m}^{s,p}$ -estimates when (v_0, b_0) is sufficiently small in $E_q^3 \times E_q^3$. To this end, we aim to eliminate the dependence of constants on T , i.e., we choose $\beta_0 = 0$ in (2.41) and $\beta = 0$ in (2.44).

Let us analyze the conditions under which $\beta = 0$ in (2.44) is permissible. When $m \geq 2$, we additionally assume that $\frac{2}{s} + \frac{3}{m} \geq 1$ so that the exponent $\alpha \leq 0$. In particular, we can take $\beta = 0$ when $\alpha = 0$ since the condition $1 < \tilde{m} < m < \infty$ required in [1, Lemma 2.7] is satisfied. Note that when $m = 2$, we have $\tilde{m} = 1$, but then $\alpha < 0$.

For $1 < m < 2$, we impose the additional condition $\frac{3}{m} < \frac{2}{s} + 2 = 3 - \frac{3}{p}$, which ensures $\alpha < 0$ and hence again allows us to take $\beta = 0$.

With $\beta_0 = \beta = 0$ in (2.41) and (2.44), we obtain a global-in-time estimate for (v, b) in $E_{T=\infty,m}^{s,p} \times E_{T=\infty,m}^{s,p}$, provided that the initial data satisfies $\|(v_0, b_0)\|_{E_q^p \times E_q^p} < [32C_3C_4]^{-1}$.

Observe that when $m \geq 2$, all required conditions—including the upper bound $\frac{2}{s} + \frac{3}{m} \geq 1$ —are satisfied if we take $m = p$. Once we have established $(v, b) \in E_{T=\infty,m}^{s,p} \times E_{T=\infty,m}^{s,p}$ for some m , then the inclusion property implies $(v, b) \in E_{T=\infty,\tilde{m}}^{s,p} \times E_{T=\infty,\tilde{m}}^{s,p}$ for all $\tilde{m} \in [m, \infty]$. Thus, the condition $\frac{2}{s} + \frac{3}{m} \geq 1$ can be removed entirely.

We now consider the $L_T^s E_m^p$ -estimates of (v, b) , restricting to $T = \infty$ for simplicity. Fix $s \in [3, \infty)$ and $q \in [1, 3]$, and define p by the relation $\frac{3}{p} + \frac{2}{s} = 1$. Using [1, Lemma 2.8] with $\tilde{p} = p/2$, $\tilde{s} = s/2$ and $\tilde{m} \geq 1$ such that $\frac{1}{\tilde{m}} - \frac{1}{m} = \frac{1}{p} - \frac{1}{p} = \frac{1}{p}$, and by applying Hölder inequality, we obtain

$$\begin{aligned} \|B((v, b), (u, a))\|_{L_T^s(E_m^p \times E_m^p)} &\lesssim \|v \otimes u\|_{L_T^{\frac{s}{2}} E_{\tilde{m}}^{\frac{p}{2}}} + \|b \otimes a\|_{L_T^{\frac{s}{2}} E_{\tilde{m}}^{\frac{p}{2}}} + \|v \otimes a\|_{L_T^{\frac{s}{2}} E_{\tilde{m}}^{\frac{p}{2}}} + \|b \otimes u\|_{L_T^{\frac{s}{2}} E_{\tilde{m}}^{\frac{p}{2}}} \\ &\lesssim \|(v, b)\|_{L_T^s(E_m^p \times E_m^p)} \|(u, a)\|_{L_T^s(E_p^p \times E_p^p)} \\ &\lesssim \|(v, b)\|_{L_T^s(E_m^p \times E_m^p)} \|(u, a)\|_{L_T^s(E_p^p \times E_p^p)}, \quad \text{if } p \geq m. \end{aligned}$$

The condition $\tilde{m} \geq 1$ is equivalent to $m \geq p' = \frac{p}{p-1}$. Hence,

$$\|B((v, b), (u, a))\|_{L_T^s(E_m^p \times E_m^p)} \lesssim \|(v, b)\|_{L_T^s(E_m^p \times E_m^p)} \|(u, a)\|_{L_T^s(E_m^p \times E_m^p)}, \quad \text{if } p' \leq m \leq p. \quad (2.45)$$

Let $\mathcal{M}(s, q)$ denote the set of all m for which we can establish $(v, b) \in L^s(E_m^p \times E_m^p)$. Since $L^s E_m^p \subset L^s E_{m_2}^p$ for $m < m_2$, the set $\mathcal{M}(s, q)$, if nonempty, must be an interval of the form

$$\underline{m} < m \leq \infty, \quad \text{or} \quad \underline{m} \leq m \leq \infty, \quad (2.46)$$

for some $\underline{m} = \underline{m}(s, q) \in [1, \infty]$.

Define m_1 by

$$\frac{1}{m_1} = \frac{1}{q} - \frac{2}{3s}.$$

Given $s < \infty$ and $q \leq 3$, and recalling that $\frac{1}{p} = \frac{1}{3} - \frac{2}{3s}$, we deduce that

$$q < m_1 \leq p. \quad (2.47)$$

From [1, Lemma 2.4] with $d = r = 3$, we have

$$\|e^{t\Delta} f\|_{L_{T=\infty}^s E_m^p} \lesssim \|f\|_{E_q^3}, \quad (2.48)$$

with constants independent of R , provided one of the following holds:

A. $m_1 \leq m \leq s$, (and $m_1 < m \leq s$ if $q = 1$),

E₁. $s < p = m$,

E₂. $s < m$ and $\frac{1}{q} \geq \frac{5}{3s}$.

If the linear estimate (2.48) holds, and if $p' \leq m \leq p$ so that the bilinear estimate (2.45) holds, then the Picard iteration $(v^{(n)}, b^{(n)})$ converges in $L^s(E_m^p \times E_m^p)$ for sufficiently small initial data.

Case A: This case applies as soon as $m_1 \leq s$, i.e., $\frac{1}{q} \geq \frac{5}{3s}$. Strict inequality $m_1 < s$ holds when $q = 1$. By (2.47) and since $p' < 2 < s$, the value

$$m^*(s, q) = \max(p', m_1)$$

belongs to both $[p', p]$ and $[m_1, s]$. Thus, $(v^{(n)}, b^{(n)})$ converges in $L^s(E_m^p \times E_m^p)$ for $m = m^*$, or for m slightly larger than m^* when $q = 1$ and $3 \leq s \leq 4$.

Case E₁: This case applies when $3 \leq s < 5$, allowing us to take $m = p$. It thus covers the parameter range $1 \leq q \leq 3 \leq s < 5$, and $\frac{1}{q} < \frac{5}{3s}$.

Case E₂: While this case also requires $\frac{1}{q} \geq \frac{5}{3s}$, it does not yield smaller admissible m than Case A.

This completes the proof of $L^s E_m^p$ -estimates, and concludes the proof of Theorem 1.3. \square

3 Local energy solutions in Wiener amalgam spaces

In this section, we address weak solutions and establish Theorems 1.8, 1.9, 1.10 for the MHD equations (MHD), along with Theorems 1.12, 1.13, 1.14 for the viscoelastic Navier–Stokes equations with damping (vNSEd). Given the similarity between the structures of (vNSEd) and (MHD), we focus on presenting the proofs of Theorems 1.8, 1.9, 1.10 for (MHD). The details of verification of Theorems 1.12, 1.13, 1.14 for (vNSEd) are left to the readers.

Define

$$N_R^0(v_0, b_0) = \sup_{x_0 \in \mathbb{R}^3} \frac{1}{R} \int_{B_R(x_0)} (|v_0|^2 + |b_0|^2) dx,$$

and

$$N_{q,R}^0(v_0, b_0) = \frac{1}{R} \left[\sum_{k \in \mathbb{Z}^3} \left(\int_{B_R(kR)} (|v_0|^2 + |b_0|^2) dx \right)^{q/2} \right]^{2/q},$$

$$N_{\infty,R}^0(v_0, b_0) = N_R^0(v_0, b_0) := \sup_{x_0 \in \mathbb{R}^3} \frac{1}{R} \int_{B_R(x_0)} (|v_0|^2 + |b_0|^2) dx.$$

Lemma 3.1. *Let $v_0, b_0 \in L_{\text{uloc}}^2$ be divergence free, and assume $(v, b) \in \mathcal{N}_{\text{MHD}}(v_0, b_0)$. For all $r > 0$ we have*

$$\text{ess sup}_{0 \leq t \leq \sigma r^2} \sup_{x_0 \in \mathbb{R}^3} \int_{B_r(x_0)} \frac{|v|^2 + |b|^2}{2} dx dt + \sup_{x_0 \in \mathbb{R}^3} \int_0^{\sigma r^2} \int_{B_r(x_0)} (|\nabla v|^2 + |\nabla b|^2) dx dt < C A_0(r), \quad (3.1)$$

$$\sup_{x_0 \in \mathbb{R}^3} \int_0^{\sigma r^2} \int_{B_r(x_0)} \left(|v|^3 + |b|^3 + |p - c_{x_0,r}(t)|^{3/2} \right) dx dt < C r^{\frac{1}{2}} (A_0(r))^{\frac{3}{2}}, \quad (3.2)$$

where

$$A_0(r) = r N_r^0(v_0, b_0) = \sup_{x_0 \in \mathbb{R}^3} \int_{B_r(x_0)} (|v_0|^2 + |b_0|^2) dx,$$

and

$$\sigma = \sigma(r) = c_0 \min \{ (N_r^0(v_0, b_0))^{-2}, 1 \}, \quad (3.3)$$

for a small universal constant $c_0 > 0$.

Proof. The proof of the lemma is an adaption of the proof of [3, Lemma 2.1] for the Navier–Stokes equations to the MHD equations.

By Hölder and Young inequalities, for any $\delta > 0$, we have

$$\|v\|_{L^3(0,T;L^3)}^3 \lesssim \|v\|_{L^6(0,T;L^2)}^{3/2} \|v\|_{L^2(0,T;L^6)}^{3/2} \lesssim (\delta R)^{-3} \|v\|_{L^6(0,T;L^2)}^6 + \delta R \|v\|_{L^2(0,T;L^6)}^2,$$

and

$$\|b\|_{L^3(0,T;L^3)}^3 \lesssim \|b\|_{L^6(0,T;L^2)}^{3/2} \|b\|_{L^2(0,T;L^6)}^{3/2} \lesssim (\delta R)^{-3} \|b\|_{L^6(0,T;L^2)}^6 + \delta R \|b\|_{L^2(0,T;L^6)}^2.$$

In addition, applying Sobolev inequality yields

$$\begin{aligned} \frac{1}{R} \int_0^{\sigma R^2} \int_{B_{2R}(x_0)} |v|^3 dx dt &\leq \frac{C}{\delta^3 R^4} \int_0^{\sigma R^2} \left(\int_{B_{2R}(x_0)} |v|^2 dx \right)^3 dt + \frac{C\delta}{R^2} \int_0^{\sigma R^2} \int_{B_{2R}(x_0)} |v|^2 dx dt \\ &\quad + C\delta \sup_{x_0 \in \mathbb{R}^3} \int_0^{\sigma R^2} \int_{B_{2R}(x_0)} |\nabla v|^2 dx dt, \end{aligned}$$

and

$$\begin{aligned} \frac{1}{R} \int_0^{\sigma R^2} \int_{B_{2R}(x_0)} |b|^3 dx dt &\leq \frac{C}{\delta^3 R^4} \int_0^{\sigma R^2} \left(\int_{B_{2R}(x_0)} |b|^2 dx \right)^3 dt + \frac{C\delta}{R^2} \int_0^{\sigma R^2} \int_{B_{2R}(x_0)} |b|^2 dx dt \\ &\quad + C\delta \sup_{x_0 \in \mathbb{R}^3} \int_0^{\sigma R^2} \int_{B_{2R}(x_0)} |\nabla b|^2 dx dt, \end{aligned}$$

where C is a positive constant independent of σ . For the pressure term, using the local pressure expansion (1.27), we obtain

$$\frac{1}{R} \int_0^{\sigma R^2} \int_{B_{2R}(x_0)} |\pi - c_{x_0,R}(t)|^{3/2} \leq \frac{C}{R} \int_0^{\sigma R^2} \int_{B_{4R}(x_0)} (|v|^3 + |b|^3) dx dt + \frac{C}{R^4} \int_0^{\sigma R^2} \left[\bar{A} \left(\frac{t}{R^2} \right) \right]^{3/2} dt, \quad (3.4)$$

where

$$\bar{A}(\sigma) = \operatorname{ess\,sup}_{0 \leq t \leq \sigma R^2} \sup_{x_0 \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|v|^2 + |b|^2}{2} \phi(x - x_0) dx.$$

Applying the local energy inequality, we deduce

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{|v|^2 + |b|^2}{2} \phi(x - x_0) dx + \int_0^t \int_{\mathbb{R}^3} (|\nabla v|^2 + |\nabla b|^2) \phi(x - x_0) dx ds \\ \leq \frac{A_0(R)}{2} + \frac{C}{R^2} \int_0^{\sigma R^2} \bar{A} \left(\frac{s}{R^2} \right) ds + \frac{C}{R^4} \int_0^{\sigma R^2} \left(\bar{A} \left(\frac{s}{R^2} \right) \right)^3 ds, \end{aligned} \quad (3.5)$$

for sufficiently small δ . Therefore, we conclude that

$$\frac{\bar{A}(\sigma)}{R} \leq \frac{A_0(R)}{2R} + \frac{C}{R^2} \int_0^{\sigma R^2} \frac{\bar{A} \left(\frac{s}{R^2} \right)}{R} ds + \frac{C}{R^2} \int_0^{\sigma R^2} \left(\frac{\bar{A} \left(\frac{s}{R^2} \right)}{R} \right)^2 ds. \quad (3.6)$$

Making the change of variables $\tau = s/R^2$ and applying Grönwall's inequality [3, Lemma 2.2], we derive

$$\overline{A}(\sigma) \leq A_0(R),$$

for $t \in [0, T_R]$ where $T_R = \sigma R^2$, and

$$\sigma = c_0 \min \{ (N_R^0(v_0, b_0))^{-2}, 1 \}$$

for some small constant c_0 independent of R and (v_0, b_0) . Note that $\overline{A}(\sigma)$ is nondecreasing in σ , and its continuity in σ follows directly from the local energy inequality. With the above estimate for $\overline{A}(\sigma)$, the lemma follows by the standard *continuation in σ* argument, provided that c_0 is chosen sufficiently small. \square

We recall a local regularity criterion for *suitable weak solutions* that is a replacement for the case of the Navier–Stokes equations proposed in [6] (see also [28, 36]). See [15] for the definition of suitable weak solutions.

Lemma 3.2 (ϵ -regularity criterion [34, Theorem 3.1]). *There exists a universal constant $\epsilon_* > 0$ such that, if (v, b, π) is a suitable weak solution of (MHD) in $Q_r = Q_r(x_0, t_0) = B_r(x_0) \times (t_0 - r^2, t_0)$, $B_r(x_0) \subset \mathbb{R}^3$, and*

$$\epsilon^3 = \frac{1}{r^2} \int_{Q_r} (|v|^3 + |v|^3 + |\pi|^{3/2}) dx dt < \epsilon_*,$$

then v and b are Hölder continuous on $\overline{Q_{r/2}}$.

The corresponding ϵ -regularity criterion for weak solutions of the viscoelastic Navier–Stokes equations with damping, (vNSEd), is established in [17, Proposition 3.2].

Theorem 3.3 (Initial and eventual regularity). *There is a small positive constant ϵ_1 such that the following holds. Assume that $v_0, b_0 \in L^2_{\text{uloc}}(\mathbb{R}^3)$ are divergence free and that $(v, b) \in \mathcal{N}_{\text{MHD}}(v_0, b_0)$. Let*

$$N_R^0 := \sup_{x_0 \in \mathbb{R}^3} \frac{1}{R} \int_{B_R(x_0)} (|v_0|^2 + |b_0|^2) dx.$$

1. *If there exists $R_0 > 0$ so that*

$$\sup_{R \geq R_0} N_R^0 < \epsilon_1,$$

then (v, b) has eventual regularity. Moreover, if $R_0^2 \lesssim t$, then

$$t^{1/2} \|(v, b)\|_{L^\infty \times L^\infty} \lesssim \left(\sup_{R \geq R_0} N_R^0 \right)^{1/2} < \infty.$$

2. *If there exists $R_0 > 0$ so that*

$$\sup_{R \leq R_0} N_R^0 < \epsilon_1,$$

then (v, b) has initial regularity. Moreover, if $t \leq c_0 R_0^2$, then

$$t^{1/2} \|(v, b)(\cdot, t)\|_{L^\infty \times L^\infty} \lesssim \left(\sup_{R \leq R_0} N_R^0 \right)^{1/2} < \infty.$$

3. *If (v_0, b_0) satisfies*

$$\sup_{R > 0} N_R^0 < \epsilon_1,$$

then the set of singular times of (v, b) in $\mathbb{R}^3 \times (0, \infty)$ is empty. Moreover, for all $t > 0$,

$$t^{1/2} \|(v, b)(\cdot, t)\|_{L^\infty \times L^\infty} \lesssim \left(\sup_{R>0} N_R^0 \right)^{1/2} < \infty.$$

Proof. The proof of the theorem is an adaption of the proof of [3, Theorem 1.2] for the Navier–Stokes equations to the MHD equations.

Assume there exists $R_0 > 0$ such that for all $R \geq R_0$, we have $N_R^0(v_0, b_0) < \epsilon_1$, where $\epsilon_1 \in (0, 1)$ is a small constant to be determined.

Fix $x_0 \in \mathbb{R}^3$ and $R > R_0$. Define $\tilde{p}(x, t) = p(x, t) - c_{x_0, R}(t)$ where $c_{x_0, R}(t)$ is the function of t from the local pressure expansion (1.27). Then (v, b) is a suitable weak solution to (MHD) with associated pressure \tilde{p} . By the estimate (3.2), we have

$$\int_0^{\sigma(R)R^2} \int_{B_R(x_0)} \left(|v|^3 + |b|^3 + |\tilde{p}|^{3/2} \right) dx dt \leq C(N_R^0)^{\frac{3}{2}} R^2.$$

Thus, if $R \geq R_0$ and $\epsilon_1 \leq (c_0 C^{-1} \epsilon_*)^{2/3}$, then the right side is bounded by ϵ_* , and we may apply Lemma 3.2. It follows that

$$v, b \in L^\infty(Q), \quad Q = B_{\frac{c_0^{1/2} R}{2}}(x_0) \times \left[\frac{3c_0 R^2}{4}, c_0 R^2 \right],$$

and for $(x, t) \in Q$,

$$|v(x, t)| + |b(x, t)| \leq C_0 \left(\frac{C}{c_0} (N_R^0)^{3/2} \right)^{\frac{1}{3}} \left(\frac{c_0^{1/2} R}{2} \right)^{-1} \leq C(N_R^0)^{\frac{1}{2}} t^{-\frac{1}{2}}. \quad (3.7)$$

Hence, (v, b) is regular in $\mathbb{R}^3 \times \left(\frac{3c_0 R^2}{4}, c_0 R^2 \right]$. Since $R \geq R_0$ is arbitrary, we conclude that (v, b) is regular at every point $(x, t) \in \mathbb{R}^3 \times \left(\frac{3c_0 R_0^2}{4}, \infty \right)$, with the bound given by (3.7). Note that this threshold $\frac{3c_0 R_0^2}{4}$ depends only on (v_0, b_0) and is uniform for all $(v, b) \in \mathcal{N}_{\text{MHD}}(v_0, b_0)$.

The argument proceeds analogously in the case where $\sup_{R \leq R_0} N_R^0 < \epsilon_1$; we omit the details for brevity.

Finally, if $N_R^0 < \epsilon_1$ for all $R > 0$, then $\sigma(R) = c_0$ for all $R > 0$. In this case, (v, b) is regular with the same bound (3.7) throughout the entire space-time domain $\cup_{0 < R < \infty} \mathbb{R}^3 \times \left(\frac{3c_0 R^2}{4}, c_0 R^2 \right] = \mathbb{R}^3 \times (0, \infty)$. \square

As a consequence of Theorem 3.3, the uniqueness result below can be established by adapting the argument used in the proof of [3, Theorem 1.7].

Theorem 3.4 (Uniqueness for data that is small at high frequencies). *Assume that $v_0, b_0 \in E^2$ are divergence free. Let $(v, b), (u, a) \in \mathcal{N}_{\text{MHD}}(v_0, b_0)$. Then there exist universal constants $0 < \epsilon_3, \tau_0 \leq 1$ so that if*

$$\sup_{0 < r \leq R} N_r^0 \leq \epsilon_3$$

for some $R > 0$, then $(u, a) = (v, b)$ as distributions on $\mathbb{R}^3 \times (0, T)$, $T = \tau_0 R^2$.

Proof. Assume $\lim_{R \rightarrow 0} N_R^0 < \epsilon$ for some $\epsilon > 0$, and either (v_0, b_0) satisfies

$$\lim_{R \rightarrow \infty} \sup_{x_0 \in \mathbb{R}^3} \frac{1}{R^2} \int_{B_R(x_0)} (|v_0(x)|^2 + |b_0(x)|^2) dx = 0, \quad (3.8)$$

or $(v_0, b_0) \in E^2 \times E^2$. Let R_0 satisfy $\sup_{R < R_0} N_R^0 < \epsilon$. By Theorem 3.3 we have for $T = c_0 R_0^2$,

$$t^{1/2} \|(v, b)(\cdot, t)\|_{L^\infty \times L^\infty} \leq C(\epsilon), \quad 0 < t \leq T.$$

Using (3.1) we have the estimate

$$\sup_{r > \sqrt{t/c_0}, x \in \mathbb{R}^3} \frac{1}{r} \int_{B_r(x)} (|v(x, t)|^2 + |b(x, t)|^2) dx < C\epsilon^2, \quad (3.9)$$

where c_0 is defined in (3.3). Moreover, by item 3 of Theorem 3.3, we have

$$\|(v, b)(t)\|_{L^\infty \times L^\infty} < C\epsilon^2 t^{-1/2}, \quad 0 < t < \infty. \quad (3.10)$$

Combining (3.9) and (3.10), we deduce that, for $r < \sqrt{t/c_0}$,

$$\begin{aligned} & \frac{1}{r} \int_{B_r(x)} (|v(x, t)|^2 + |b(x, t)|^2) dx \\ &= \frac{1}{r} \left(\int_{B_r(x)} (|v(x, t)|^2 + |b(x, t)|^2) dx \right)^{1/3} \left(\int_{B_r(x)} (|v(x, t)|^2 + |b(x, t)|^2) dx \right)^{2/3} \\ &\lesssim \|(v, b)(t)\|_{L^\infty \times L^\infty}^{2/3} \left(\int_{B_{\sqrt{t/c_0}}(x)} (|v(x, t)|^2 + |b(x, t)|^2) dx \right)^{2/3} \\ &\lesssim C(\epsilon) t^{-1/3} (t/c_0)^{1/3} = C(\epsilon). \end{aligned} \quad (3.11)$$

Using the estimates (3.9) for $r = \sqrt{T}$ and (3.11) for $r < \sqrt{T}$, we have for all $t \in (0, T)$ and $r \in (0, T^{1/2})$ that

$$\frac{1}{r} \int_{B_r(x)} (|v(x, t)|^2 + |b(x, t)|^2) dx \leq C(\epsilon),$$

which implies that

$$\sup_{0 < t < \infty} \|(v, b)(t)\|_{L^2_{\text{uloc}, r} \times L^2_{\text{uloc}, r}} < C(\epsilon), \quad (3.12)$$

where $\|f\|_{L^2_{\text{uloc}, r}} := \sup_{x \in \mathbb{R}^3} \|f\|_{L^2(x)}$.

We now check that (v, b) satisfies the integral formula (1.8), i.e., it is a mild solution, on $\mathbb{R}^3 \times (0, T)$. If $(v_0, b_0) \in E^2 \times E^2$, then this follows from a direct adaption of [20, §8] for the Navier–Stokes equations to the MHD equations. On the other hand, assume (v_0, b_0) satisfies (3.8). By (3.1), we have

$$\text{ess sup}_{0 \leq t \leq \sigma(r)r^2} \|(v, b)\|_{L^2_{\text{uloc}, r} \times L^2_{\text{uloc}, r}}^2 < C A_0(r), \quad A_0(r) = \|(v_0, b_0)\|_{L^2_{\text{uloc}, r} \times L^2_{\text{uloc}, r}}^2.$$

Since (v_0, b_0) satisfies (3.8), we have $\sigma(r)r^2 \rightarrow \infty$ as $r \rightarrow \infty$. So, there exists \bar{R} so that, for all $R > \bar{R}$, $\sigma(R)R^2 > T$. We conclude that for any $r > 0$

$$\text{ess sup}_{0 \leq t \leq T} \|(v, b)\|_{L^2_{\text{uloc}, r} \times L^2_{\text{uloc}, r}}^2 \leq f(r), \quad (3.13)$$

where $f(r) = CA_0(r) + CA_0(\bar{R})$.

Let (\tilde{v}, \tilde{b}) be defined as

$$(\tilde{v}, \tilde{b})(x, t) = (e^{t\Delta}v_0, e^{t\Delta}b_0) - B((v, b), (v, b))(t), \quad (3.14)$$

where B is a bilinear operator defined by $B = (B_1, B_2)$, where B_1 and B_2 are given in (1.9). Using [33, (1.8), (1.10)], we have

$$\|(e^{t\Delta}v_0, e^{t\Delta}b_0)\|_{L^2_{\text{uloc},r} \times L^2_{\text{uloc},r}} \leq \|(v_0, b_0)\|_{L^2_{\text{uloc},r} \times L^2_{\text{uloc},r}} = (A_0(r))^{1/2},$$

and

$$\begin{aligned} & \left\| \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (v \otimes v - b \otimes b)(s) ds \right\|_{L^2_{\text{uloc},r}} + \left\| \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (v \otimes b - b \otimes v)(s) ds \right\|_{L^2_{\text{uloc},r}} \\ & \leq \int_0^t \frac{C}{(t-s)^{1/2}} \|(v, b)(s)\|_{L^\infty \times L^\infty} \|(v, b)(s)\|_{L^2_{\text{uloc},r} \times L^2_{\text{uloc},r}} ds \\ & \leq \int_0^t \frac{C(\epsilon)}{(t-s)^{1/2} s^{1/2}} ds = C(\epsilon). \end{aligned}$$

Thus

$$\sup_{0 \leq t \leq T} \|(\tilde{v}, \tilde{b})\|_{L^2_{\text{uloc},r} \times L^2_{\text{uloc},r}}^2 \leq CA_0(r) + C(\epsilon) \sup_{0 < s < T} \|(v, b)\|_{L^2_{\text{uloc},r} \times L^2_{\text{uloc},r}}^2 \leq Cf(r). \quad (3.15)$$

Then $(V, B) = (v, b) - (\tilde{v}, \tilde{b})$ satisfies

$$\text{ess sup}_{0 \leq t \leq T} \|(V, B)\|_{L^2_{\text{uloc},r} \times L^2_{\text{uloc},r}}^2 \leq Cf(r), \quad \forall r > 0.$$

Following the same logic in [20, §8] and adapting it to MHD equations, we have that the mollified V_ϵ and B_ϵ are harmonic in x and for fixed $t \in (0, T)$

$$|V_\epsilon(x, t)| + |B_\epsilon(x, t)| \leq C \left(\frac{1}{r^3} \int_{B_r(x)} (|V_\epsilon(y, t)|^2 + |B_\epsilon(y, t)|^2) dy \right)^{1/2} \leq C \left(\frac{f(r)}{r^3} \right)^{1/2} \rightarrow 0 \text{ as } r \rightarrow \infty,$$

This shows $(V_\epsilon, B_\epsilon) = (0, 0)$ for $t < T$, for all $\epsilon > 0$. Hence $(V, B) = (0, 0)$ and $(v, b) = (\tilde{v}, \tilde{b})$. This shows that any local energy solution with data satisfying the assumptions of Theorem 3.4 is a mild solution.

Assume $(u, a) \in \mathcal{N}_{\text{MHD}}(v_0, b_0)$ also and satisfy the assumptions of Theorem 3.4. Then, (u, a) is also a mild solution. Let $(w, d) = (v, b) - (u, a)$. Then

$$w(\cdot, t) = - \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (w \otimes w + u \otimes w + w \otimes u - d \otimes d - a \otimes d - d \otimes a)(\cdot, s) ds,$$

and

$$d(\cdot, t) = - \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (w \otimes d + u \otimes d + w \otimes a - d \otimes w - a \otimes w - d \otimes u)(\cdot, s) ds.$$

By [33, Corollary 3.1] we have for $t > 0$ that

$$\begin{aligned} & \|(w, d)(t)\|_{L^2_{\text{uloc}} \times L^2_{\text{uloc}}} \\ & \leq C \sup_{0 < s < t} \sqrt{s} (\|(v, b)(s)\|_{L^\infty \times L^\infty} + \|(u, a)(s)\|_{L^\infty \times L^\infty}) \|(w, d)\|_{L^\infty(L^2_{\text{uloc}} \times L^2_{\text{uloc}})}. \end{aligned}$$

Since $\sqrt{s}(\|(v, b)(s)\|_{L^\infty \times L^\infty} + \|(u, a)(s)\|_{L^\infty \times L^\infty})$ is small we have

$$\|(w, d)(t)\|_{L^2_{\text{uloc}} \times L^2_{\text{uloc}}} \leq \frac{1}{2} \|(w, d)\|_{L^\infty(L^2_{\text{uloc}} \times L^2_{\text{uloc}})}.$$

Taking the essential supremum on the left hand side leads to uniqueness. \square

As a corollary of Theorem 3.4, the following local uniqueness result in E^3 can be derived by adapting the proof of [3, Corollary 1.8].

Corollary 3.5 (Local uniqueness in E^3). *Assume $v_0, b_0 \in E^3$ are divergence free. Let (v, b) and (u, a) be elements of $\mathcal{N}_{\text{MHD}}(v_0, b_0)$. Then, there exists $T = T(v_0, b_0) > 0$ so that $(v, b) = (u, a)$ as distributions on $\mathbb{R}^3 \times (0, T)$.*

Proof. Assume $(v_0, b_0) \in E^3 \times E^3$. Then, in particular, $(v_0, b_0) \in E^2 \times E^2$. Let $\epsilon > 0$ be given. Since $(v_0, b_0) \in E^3 \times E^3$, there exists R_0 such that

$$\sup_{|x_0| \geq R_0} \int_{B_1(x_0)} (|v_0(x)|^3 + |b_0(x)|^3) dx < \epsilon.$$

On the other hand, since $v_0, b_0 \in E^3$, their local L^3 -norms are uniformly small on sufficiently small balls. That is, there exists $\gamma \in (0, 1]$ such that

$$\sup_{|x_0| \leq R_0, 0 < r \leq \gamma} \int_{B_r(x_0)} (|v_0(x)|^3 + |b_0(x)|^3) dx < \epsilon.$$

Applying Hölder's inequality, we obtain:

$$\sup_{|x_0| \leq R_0, 0 < r \leq \gamma} \frac{1}{r} \int_{B_r(x_0)} (|v_0(x)|^2 + |b_0(x)|^2) dx < |B_1|^{1/3} \epsilon^{2/3}.$$

Therefore, by Theorem 3.4, any local energy solution with initial data (v_0, b_0) is unique in the local energy class, at least for a short time. \square

3.1 Eventual regularity for local energy solutions

Lemma 3.6. *Assume $v_0, b_0 \in E_q^2$, $2 \leq q < \infty$. Then*

$$\lim_{R \rightarrow \infty} R^{\frac{6}{q}-2} N_{q,R}^0(v_0, b_0) = 0.$$

Consequently, if $v_0, b_0 \in E_q^2$, then

$$\lim_{R \rightarrow \infty} N_R^0(v_0, b_0) = 0 \quad \text{if } 2 \leq q \leq 3 \quad \text{and} \quad \lim_{R \rightarrow \infty} R^{-1} N_{q,R}^0(v_0, b_0) = 0 \quad \text{if } 2 \leq q \leq 6.$$

Proof. The proof of the lemma is an adaption of the proof of [4, Lemma 2.2] for the Navier–Stokes equations to the MHD equations.

Let $\epsilon > 0$ be given. Suppose that $v_0, b_0 \in E_q^2$ for some $2 \leq q < \infty$. Then the norms can be expressed as $\|v_0\|_{E_q^2} = \|u\|_{\ell^q(\mathbb{Z}^3)}$ and $\|b_0\|_{E_q^2} = \|a\|_{\ell^q(\mathbb{Z}^3)}$, where

$$u = (u_k)_{k \in \mathbb{Z}^3} \in \ell^q(\mathbb{Z}^3), \quad u_k = \|v_0\|_{L^2(B_1(k))},$$

$$a = (a_k)_{k \in \mathbb{Z}^3} \in \ell^q(\mathbb{Z}^3), \quad a_k = \|b_0\|_{L^2(B_1(k))}.$$

For any $R \geq 1$, we have the estimate

$$N_{q,R}^0(v_0, b_0) \leq \frac{C}{R} \left[\sum_{k \in \mathbb{Z}^3} \left(\sum_{|i-kR| < R} (u_i^2 + a_i^2) \right)^{\frac{q}{2}} \right]^{\frac{2}{q}} \quad (3.16)$$

Now fix any $\delta > 0$. Since $u, a \in \ell^q(\mathbb{Z}^3)$, we can choose $M > 1$ large enough such that $\|u^{>M}\|_{\ell^q} \leq \delta$ and $\|a^{>M}\|_{\ell^q} \leq \delta$, where

$$u_k^{>M} = \begin{cases} 0 & \text{if } |k| \leq M, \\ u_k & \text{if } |k| > M, \end{cases} \quad \text{and} \quad a_k^{>M} = \begin{cases} 0 & \text{if } |k| \leq M, \\ a_k & \text{if } |k| > M. \end{cases}$$

Set $u^{\leq M} = u - u^{>M}$ and $a^{\leq M} = a - a^{>M}$. Following the argument leading to [4, (2.4)] by Hölder's inequality we obtain, for all $R > M$,

$$\left[R^{\frac{6}{q}-2} N_{q,R}^0(v_0, b_0) \right]^{\frac{q}{2}} \leq C \left(\|u^{>M}\|_{\ell^q}^q + \|a^{>M}\|_{\ell^q}^q \right) + CR^{3-\frac{3q}{2}} M^{\frac{3(q-2)}{2}} \left(\|u^{\leq M}\|_{\ell^q}^q + \|a^{\leq M}\|_{\ell^q}^q \right).$$

To conclude, we first choose $\delta > 0$ small enough so that $C(\|u^{>M}\|_{\ell^q}^q + \|a^{>M}\|_{\ell^q}^q) < \epsilon/2$. Then, for this fixed $M = M(\delta)$, we choose R sufficiently large to ensure that $CR^{3-\frac{3q}{2}} M^{\frac{3(q-2)}{2}} (\|u^{\leq M}\|_{\ell^q}^q + \|a^{\leq M}\|_{\ell^q}^q) < \epsilon/2$, which is possible provided $q > 2$.

To prove the final assertions, we begin by noting that $N_R^0(v_0, b_0) \leq N_{q,R}^0(v_0, b_0)$. Moreover, if $v_0, b_0 \in E_q^2$ for $q \leq 3$, then in particular $v_0, b_0 \in E_3^2$. Therefore,

$$\lim_{R \rightarrow \infty} N_R^0(v_0, b_0) \leq \lim_{R \rightarrow \infty} N_{3,R}^0(v_0, b_0) = 0.$$

For the final part, observe that when $q \leq 6$ and $R \geq 1$, we have $R^{-1} \leq R^{\frac{6}{q}-2}$. Thus,

$$\lim_{R \rightarrow \infty} R^{-1} N_{q,R}^0(v_0, b_0) = 0.$$

□

Proof of Theorem 1.8. For $2 \leq q \leq 3$, Lemma 3.6 implies

$$\lim_{R \rightarrow \infty} N_R^0(v_0, b_0) = 0.$$

Applying Theorem 3.3 then yields the desired result.

For $1 \leq q < 2$, the same conclusion follows from the fact that $v_0, b_0 \in E_q^2 \subset L^2$ since E_q^2 embeds into L^2 when $q < 2$. This completes the proof of Theorem 1.8. □

3.2 A priori bounds and explicit growth rate

In this section we prove new a priori bounds for data $(v_0, b_0) \in E_q^2 \times E_q^2$ and use it to prove Theorem 1.9.

Lemma 3.7. *Assume $v_0, b_0 \in E_q^2$ for some $q \geq 1$ are divergence free and that $(v, b) \in \mathcal{N}_{\text{MHD}}(v_0, b_0)$ satisfies, for some $T_2 > 0$,*

$$\left\| \text{ess sup}_{0 \leq T \leq T_1} \int_{B_1(x_0)} (|v|^2 + |b|^2) dx + \int_0^{T_1} \int_{B_1(x_0)} (|\nabla v|^2 + |\nabla b|^2) dx dt \right\|_{\ell^{\frac{q}{2}}(x_0 \in \mathbb{Z}^3)} < \infty \quad \text{for all } T_1 \in (0, T_2). \quad (3.17)$$

Then there are positive constants C_1 and $\lambda_0 < 1$, both independent of q and R such that, for all $R > 0$ with $\lambda_R R^2 \leq T_2$,

$$\left\| \operatorname{ess\,sup}_{0 \leq t \leq \lambda_R R^2} \int_{B_R(x_0 R)} \frac{|v|^2 + |b|^2}{2} dx + \int_0^{\lambda_R R^2} \int_{B_R(x_0 R)} (|\nabla v|^2 + |\nabla b|^2) dx dt \right\|_{\ell^{\frac{q}{2}}(x_0 \in \mathbb{Z}^3)} \leq C_1 A_{0,q}(R), \quad (3.18)$$

where

$$A_{0,q}(R) = R N_{q,R}^0 = \left\| \int_{B_R(x_0 R)} (|v_0|^2 + |b_0|^2) dx \right\|_{\ell^{\frac{q}{2}}(x_0 \in \mathbb{Z}^3)}, \quad \lambda_R = \min \left(\lambda_0, \frac{\lambda_0 R^2}{(A_{0,q}(R))^2} \right).$$

Furthermore, for all $R > 0$,

$$\left\| \int_0^{\lambda_R R^2} \int_{B_R(x_0 R)} |v|^{\frac{10}{3}} + |b|^{\frac{10}{3}} + |\pi - c_{R x_0, R}(t)|^{\frac{5}{3}} dx dt \right\|_{\ell^{\frac{3q}{10}}(x_0 \in \mathbb{Z}^3)} \leq C (A_{0,q}(R))^{\frac{5}{3}}, \quad q \geq 2, \quad (3.19)$$

and

$$\left\| \int_0^{\lambda_R R^2} \int_{B_R(x_0 R)} |v|^{\frac{10}{3}} + |b|^{\frac{10}{3}} + |\pi - c_{R x_0, R}(t)|^{\frac{5}{3}} dx dt \right\|_{\ell^\sigma(x_0 \in \mathbb{Z}^3)} \leq C (A_{0,q}(R))^{\frac{5}{3}}, \quad 1 \leq q < 2. \quad (3.20)$$

for all $\sigma \geq \frac{3}{5}$.

Proof. The proof of the lemma is an adaption of the proof of [4, Lemma 3.1] for the Navier–Stokes equations to the MHD equations.

Let $\phi_0 \in C_c^\infty(\mathbb{R}^3)$ be radial, non-increasing cutoff function such that $\phi_0 \equiv 1$ on $B_1(0)$, $\operatorname{supp} \phi_0 \subset B_2(0)$, and $|\nabla \phi_0(x)| \lesssim 1$, $|\nabla \phi_0^{1/2}(x)| \lesssim 1$. Let $R > 0$ be as in the statement of the lemma, and define the scaled cutoff $\phi(x) := \phi_0(x/R)$. Fix $0 < \lambda \leq 1$.

For each $\kappa \in R\mathbb{Z}^3$, define the localized energy quantity

$$e_{R,\lambda}(\kappa) := \operatorname{ess\,sup}_{0 \leq t \leq \lambda R^2} \int (|v(t)|^2 + |b(t)|^2) \phi(x - \kappa) dx + \int_0^{\lambda R^2} \int (|\nabla v|^2 + |\nabla b|^2) \phi(x - \kappa) dx dt.$$

We begin by deriving bounds on $e_{R,\lambda}(\kappa)$, which will then be used to control the quantity

$$E_{R,q,\lambda} := \left\| \operatorname{ess\,sup}_{0 \leq t \leq \lambda R^2} \int (|v(t)|^2 + |b(t)|^2) \phi(x - Rk) dx + \int_0^{\lambda R^2} \int (|\nabla v|^2 + |\nabla b|^2) \phi(x - Rk) dx dt \right\|_{\ell^{\frac{q}{2}}(k \in \mathbb{Z}^3)}^{\frac{q}{2}}$$

in terms of $A_{0,q}(R)$ for sufficiently small λ . By assumption, $E_{R,q,\lambda} < \infty$. To estimate $e_{R,\lambda}(\kappa)$, we apply the local energy inequality (1.33):

$$\begin{aligned} & \int (|v(t)|^2 + |b(t)|^2) \phi(x - \kappa) dx + 2 \int_0^t \int (|\nabla v|^2 + |\nabla b|^2) \phi(x - \kappa) dx ds \\ & \leq \int (|v_0|^2 + |b_0|^2) \phi(x - \kappa) dx + \int_0^t \int (|v|^2 + |b|^2) \Delta \phi(x - \kappa) dx ds \\ & \quad + \int_0^t \int (|v|^2 + |b|^2) (v \cdot \nabla \phi(x - \kappa)) dx ds + \int_0^t \int 2\pi (v \cdot \nabla \phi(x - \kappa)) dx ds \\ & \quad - 2 \int_0^t \int (b \cdot v) (b \cdot \nabla \phi(x - \kappa)) dx ds. \end{aligned}$$

We now estimate each term on the right-hand side, beginning with the second term. Using the properties of ϕ , we have:

$$\begin{aligned}
\int_0^{\lambda R^2} \int (|v|^2 + |b|^2) |\Delta \phi(x - \kappa)| dx ds &\leq \frac{C}{R^2} \int_0^{\lambda R^2} \int_{B_{2R}(\kappa)} (|v|^2 + |b|^2) dx ds \\
&\leq C\lambda \sum_{\kappa' \in R\mathbb{Z}^3; |\kappa' - \kappa| \leq 2R} \operatorname{ess\,sup}_{0 \leq t \leq \lambda R^2} \int (|v|^2 + |b|^2) \phi(x - \kappa') dx \\
&\leq C\lambda \sum_{\kappa' \in R\mathbb{Z}^3; |\kappa' - \kappa| \leq 2R} e_{R,\lambda}(\kappa').
\end{aligned}$$

For the cubic terms, we apply the Gagliardo–Nirenberg inequality:

$$\begin{aligned}
\int_{B_{2R}} |v|^3 dx &\lesssim \left(\int_{B_{2R}} |\nabla v|^2 \right)^{\frac{3}{4}} \left(\int_{B_{2R}} |v|^2 \right)^{\frac{3}{4}} + R^{-\frac{3}{2}} \left(\int_{B_{2R}} |v|^2 \right)^{\frac{3}{2}}, \\
\int_{B_{2R}} |b|^3 dx &\lesssim \left(\int_{B_{2R}} |\nabla b|^2 \right)^{\frac{3}{4}} \left(\int_{B_{2R}} |b|^2 \right)^{\frac{3}{4}} + R^{-\frac{3}{2}} \left(\int_{B_{2R}} |b|^2 \right)^{\frac{3}{2}}.
\end{aligned}$$

Let $N = \sup_{0 \leq t \leq \lambda R^2} \int_{B_{2R}} (|v(t)|^2 + |b(t)|^2) dx + 2 \int_0^{\lambda R^2} \int_{B_{2R}} (|\nabla v|^2 + |\nabla b|^2) dx dt$. Then, integrating the above estimates over time yields

$$\begin{aligned}
\int_0^{\lambda R^2} \int_{B_{2R}} (|v|^3 + |b|^3) dx ds &\lesssim N^{\frac{3}{4}} \int_0^{\lambda R^2} \left[\left(\int_{B_{2R}} |\nabla v|^2 \right)^{\frac{3}{4}} + \left(\int_{B_{2R}} |\nabla b|^2 \right)^{\frac{3}{4}} \right] ds + R^{-\frac{3}{2}} N^{\frac{3}{2}} \lambda R^2 \\
&\lesssim N^{\frac{3}{2}} (\lambda R^2)^{\frac{1}{4}} + N^{\frac{3}{2}} \lambda R^{\frac{1}{2}} \\
&\lesssim N^{\frac{3}{2}} \lambda^{\frac{1}{4}} R^{\frac{1}{2}},
\end{aligned} \tag{3.21}$$

where we've used $\lambda \leq 1$ in the final step. As a consequence, we have

$$\begin{aligned}
\int_0^{\lambda R^2} \int (|v|^2 + |b|^2) (v \cdot \nabla \phi(x - \kappa)) dx ds &\leq \frac{C}{R} \int_0^{\lambda R^2} \int_{B_{2R}(\kappa)} (|v|^3 + |b|^3) dx ds \\
&\leq CR^{-\frac{1}{2}} \lambda^{\frac{1}{4}} \sum_{\kappa' \in R\mathbb{Z}^3; |\kappa' - \kappa| \leq 4R} (e_{R,\lambda}(\kappa'))^{\frac{3}{2}},
\end{aligned}$$

and

$$\begin{aligned}
-2 \int_0^{\lambda R^2} \int (b \cdot v) (b \cdot \nabla \phi(x - \kappa)) dx ds &\leq \frac{C}{R} \int_0^{\lambda R^2} \int_{B_{2R}(\kappa)} |v| |b|^2 dx ds \\
&\leq \frac{C}{R} \int_0^{\lambda R^2} \int_{B_{2R}(\kappa)} (|v|^3 + |b|^3) dx ds \\
&\leq CR^{-\frac{1}{2}} \lambda^{\frac{1}{4}} \sum_{\kappa' \in R\mathbb{Z}^3; |\kappa' - \kappa| \leq 4R} (e_{R,\lambda}(\kappa'))^{\frac{3}{2}}.
\end{aligned}$$

Now, the only remaining term is the pressure term. To estimate it, we use the local pressure expansion (1.27) to write $\pi(x, t)$ for $x \in B_{2R}(\kappa)$ as

$$\begin{aligned}
\pi(x, t) &= -\Delta^{-1} \operatorname{div} \operatorname{div} [(v \otimes v - b \otimes b) \chi_{4R}(x - \kappa)] \\
&\quad - \int_{\mathbb{R}^3} (K(x - y) - K(\kappa - y)) (v \otimes v - b \otimes b)(y, t) (1 - \chi_{4R}(y - \kappa)) dy + c_{\kappa, R}(t) \\
&=: \pi_1(x, t) + \pi_2(x, t) + c_{\kappa, R}(t).
\end{aligned}$$

Note that

$$|K(x-y) - K(\kappa-y)| \leq \frac{CR}{|\kappa-y|^4} \quad (3.22)$$

for $|x-\kappa| \leq 2R$ and $|\kappa-y| \geq 4R$. This ensures that π_2 is well-defined even if v and b lack decay at infinity.

For the localized term π_1 , standard Calderon-Zygmund theory gives

$$\begin{aligned} \|\pi_1\|_{L^{\frac{3}{2}}(B_{2R}(\kappa))} &\leq \left\| v\chi_{4R}^{1/2}(\cdot - \kappa) \right\|_{L^3}^2 + \left\| b\chi_{4R}^{1/2}(\cdot - \kappa) \right\|_{L^3}^2 \\ &\leq C \sum_{\kappa' \in R\mathbb{Z}^3; |\kappa' - \kappa| \leq 9R} \left(\left\| v\phi^{1/2}(\cdot - \kappa) \right\|_{L^3}^2 + \left\| b\phi^{1/2}(\cdot - \kappa) \right\|_{L^3}^2 \right), \end{aligned}$$

where we used the support and scaling properties of the cutoff functions. Then, applying Hölder's inequality and the inequality $\left(\sum_{j=1}^n a_j^2\right) \left(\sum_{j=1}^n |a_j|\right) \leq n \sum_{j=1}^n |a_j|^3$ along with the bound from (3.21), we obtain

$$\begin{aligned} &\int_0^{\lambda R^2} \int 2\pi_1(v \cdot \nabla \phi(x - \kappa)) dx ds \\ &\leq \frac{C}{R} \int_0^{\lambda R^2} \sum_{\kappa' \in R\mathbb{Z}^3; |\kappa' - \kappa| \leq 9R} \left(\left\| v\phi^{1/2}(\cdot - \kappa') \right\|_{L^3}^3 + \left\| b\phi^{1/2}(\cdot - \kappa') \right\|_{L^3}^3 \right) ds \\ &\leq CR^{-\frac{1}{2}} \lambda^{\frac{1}{4}} \sum_{\kappa' \in R\mathbb{Z}^3; |\kappa' - \kappa| \leq 10R} (e_{R,\lambda}(\kappa'))^{\frac{3}{2}}. \end{aligned}$$

To estimate π_2 , we begin with the following pointwise bound for $x \in B_{2R}(\kappa)$:

$$\begin{aligned} |\pi_2(x, t)| &\leq C \int \frac{R}{|\kappa - y|^4} (|v|^2 + |b|^2)(y, t) (1 - \chi_{4R}(y - \kappa)) dy \\ &\leq C \sum_{\kappa' \in R\mathbb{Z}^3; |\kappa' - \kappa| > 4R} \int_{B_{2R}(\kappa')} \frac{R}{|\kappa - y|^4} (|v(y, t)|^2 + |b(y, t)|^2) \phi(y - \kappa') dy \\ &\leq \frac{C}{R^3} \sum_{\kappa' \in R\mathbb{Z}^3; |\kappa' - \kappa| > 4R} \frac{1}{|\kappa/R - \kappa'/R|^4} \int_{B_{2R}(\kappa')} (|v(y, t)|^2 + |b(y, t)|^2) \phi(y - \kappa') dy \\ &\leq \frac{C}{R^3} (\overline{K} * e_{R,\lambda})(\kappa), \end{aligned}$$

where we used the estimate (3.22), and the convolution $\overline{K} * e_{R,\lambda}$ is taken over the lattice $R\mathbb{Z}^3$, with

$$\overline{K}(x) = \begin{cases} \frac{1}{|x/R|^4} & \text{if } |x| > 4R, \\ \overline{K}(x) = 0 & \text{otherwise,} \end{cases} \quad \text{for } x \in R\mathbb{Z}^3.$$

Assume now that $q \geq 2$. Using $\lambda \leq 1$, we estimate

$$\begin{aligned} &\int_0^{\lambda R^2} \int 2\pi_2(x, s) (v(x, s) \cdot \nabla \phi(x - \kappa)) dx ds \\ &\leq \frac{C}{R} \int_0^{\lambda R^2} \int_{B_{2R}(\kappa)} |\pi_2|^{\frac{3}{2}} dx ds + \frac{C}{R} \int_0^{\lambda R^2} \int_{B_{2R}(\kappa)} |v|^3 dx ds \\ &\leq C\lambda^{\frac{1}{4}} R^{-\frac{1}{2}} ((\overline{K} * e_{R,\lambda})(\kappa))^{\frac{3}{2}} + C\lambda^{\frac{1}{4}} R^{-\frac{1}{2}} \sum_{|\kappa' - \kappa| \leq 4R} (e_{R,\lambda}(\kappa'))^{\frac{3}{2}}. \end{aligned} \quad (3.23)$$

Note that $\int_0^{\lambda R^2} \int 2c_{x_0,R}(s) v \cdot \nabla \phi(x - \kappa) dx ds = 0$.

Combining all estimates, we obtain the key bound:

$$\begin{aligned} e_{R,\lambda}(\kappa) &\leq \int (|v_0|^2 + |b_0|^2) \phi(x - \kappa) dx + C\lambda \sum_{\kappa' \in R\mathbb{Z}^3; |\kappa' - \kappa| \leq 2R} e_{R,\lambda}(\kappa') \\ &\quad + C \frac{\lambda^{\frac{1}{4}}}{R^{\frac{1}{2}}} \sum_{\kappa' \in R\mathbb{Z}^3; |\kappa' - \kappa| \leq 2R} (e_{R,\lambda}(\kappa'))^{\frac{3}{2}} + C \frac{\lambda^{\frac{1}{4}}}{R^{\frac{1}{2}}} ((\overline{K} * e_{R,\lambda})(\kappa))^{\frac{3}{2}}, \end{aligned} \quad (3.24)$$

provided $\lambda \leq 1$. Note that all constants here are independent of q . We now raise both sides of (3.24) to the power $q/2$ and sum over $\kappa \in R\mathbb{Z}^3$. The left-hand side yields $E_{R,q,\lambda}$. For the right-hand side:

- The initial data term is controlled by

$$\sum_{\kappa \in R\mathbb{Z}^3} \left[\int (|v_0|^2 + |b_0|^2) \phi(x - \kappa) dx \right]^{\frac{q}{2}} \leq C^q (A_{0,q}(R))^{\frac{q}{2}},$$

- The linear term gives

$$\sum_{\kappa \in R\mathbb{Z}^3} \left[C\lambda \sum_{\kappa' \in R\mathbb{Z}^3; |\kappa' - \kappa| \leq 2R} e_{R,\lambda}(\kappa') \right]^{\frac{q}{2}} \leq C^q \lambda^{\frac{q}{2}} E_{R,q,\lambda},$$

- For the nonlinear cubic term, using $\left(\sum_{j=1}^n a_j\right)^p \leq n^p \sum_{j=1}^n a_j^p$ for $p \geq 1$ and $a_j \geq 0$, we have

$$\sum_{\kappa \in R\mathbb{Z}^3} \left[C \frac{\lambda^{\frac{1}{4}}}{R^{\frac{1}{2}}} \sum_{\kappa' \in R\mathbb{Z}^3; |\kappa' - \kappa| \leq 2R} (e_{R,\lambda}(\kappa'))^{\frac{3}{2}} \right]^{\frac{q}{2}} \leq C^q \left(\frac{\lambda^{\frac{1}{4}}}{R^{\frac{1}{2}}} \right)^{\frac{q}{2}} \sum_{\kappa \in R\mathbb{Z}^3} (e_{R,\lambda}(\kappa))^{\frac{3q}{4}}.$$

- For the convolution term we use Young's convolution inequality to find

$$\begin{aligned} \sum_{\kappa \in R\mathbb{Z}^3} \left(C \frac{\lambda^{\frac{1}{4}}}{R^{\frac{1}{2}}} ((\overline{K} * e_{R,\lambda})(\kappa))^{\frac{3}{2}} \right)^{\frac{q}{2}} &\leq C^q \left(\frac{\lambda^{\frac{1}{4}}}{R^{\frac{1}{2}}} \right)^{\frac{q}{2}} \sum_{\kappa \in R\mathbb{Z}^3} ((\overline{K} * e_{R,\lambda})(\kappa))^{\frac{3q}{4}} \\ &\leq C^q \left(\frac{\lambda^{\frac{1}{4}}}{R^{\frac{1}{2}}} \right)^{\frac{q}{2}} \|\overline{K}\|_{\ell^1(R\mathbb{Z}^3)}^{\frac{3q}{4}} \|e_{R,\lambda}\|_{\ell^{\frac{3q}{4}}}^{\frac{3q}{4}}, \end{aligned}$$

where $\|\overline{K}\|_{\ell^1(R\mathbb{Z}^3)}$ is bounded independently of R .

Now, since $\|e_{R,\lambda}\|_{\ell^{\frac{3q}{4}}} \leq \|e_{R,\lambda}\|_{\ell^{\frac{q}{2}}}$, we conclude for $E = E_{R,q,\lambda}$ and some constant $C_2 \geq 1$ independent of q, R that

$$E \leq C_2^q (A_{0,q}(R))^{\frac{q}{2}} + C_2^q \lambda^{\frac{q}{2}} E + C_2^q \left(\frac{\lambda^{\frac{1}{4}}}{R^{\frac{1}{2}}} \right)^{\frac{q}{2}} E^{\frac{3}{2}}. \quad (3.25)$$

The right side is finite for $\lambda < R^{-2}T_2$ by assumption (3.17). It follows from the same argument as in [4, p. 2005], $E_{R,q,\lambda}$ is continuous in λ . So, from (3.25) we conclude that

$$E \leq 2E_0, \quad E_0 = C_2^q (A_{0,q}(R))^{\frac{q}{2}},$$

provided $C_2^q \lambda^{\frac{q}{2}} \leq 1/4$ and $C_2^q \left(\frac{\lambda^{\frac{1}{4}}}{R^{\frac{1}{2}}} \right)^{\frac{q}{2}} (2E_0)^{\frac{1}{2}} \leq 1/4$, which is achieved if (using $q \geq 2$)

$$\lambda \leq \lambda_R := \min \left(\lambda_0, \frac{\lambda_0 R^2}{(A_{0,q}(R))^2} \right),$$

where $\lambda_0 = \min((2C_2)^{-2}, (2C_2)^{-12})$. This shows the first estimate (3.18) of Lemma 3.7, for $q \geq 2$, with $C_1 = CC_2^2$. Note that the constants C_2 , λ_0 , and C_1 do not depend on q and R .

For $1 \leq q < 2$, we replace (3.23) by

$$\begin{aligned} \int_0^{\lambda R^2} \int 2\pi_2 v \cdot \nabla \phi(x - \kappa) dx ds &\lesssim \frac{1}{R^4} \lambda R^{\frac{7}{2}} \|v\|_{L^\infty(0, \lambda R^2; L^2(B_{2R}(\kappa)))} |\overline{K} * e_{R,\lambda}(\kappa)| \\ &\lesssim \frac{\lambda}{R^{1/2}} \left(\|v\|_{L^\infty(0, \lambda R^2; L^2(B_{2R}(\kappa)))}^2 + |\overline{K} * e_{R,\lambda}(\kappa)|^2 \right) \\ &\lesssim \frac{\lambda}{R^{1/2}} \left(\sum_{\kappa' \in R\mathbb{Z}^3; |\kappa' - \kappa| \leq 2R} e_{R,\lambda}(\kappa') + |\overline{K} * e_{R,\lambda}(\kappa)|^2 \right). \end{aligned}$$

Raising both sides of the above inequality to the power $q/2$ and sum over $\kappa \in R\mathbb{Z}^3$, we get

$$\sum_{\kappa \in R\mathbb{Z}^3} \left(C \left(\lambda + \frac{\lambda}{R^{1/2}} \right) \sum_{\kappa' \in R\mathbb{Z}^3; |\kappa' - \kappa| \leq 2R} e_{R,\lambda}(\kappa') \right)^{q/2} \leq C^q \left(\lambda + \frac{\lambda}{R^{1/2}} \right)^{q/2} E_{R,q,\lambda},$$

Above we have used $\left(\sum_{j=1}^n a_j \right)^p \leq \sum_{j=1}^n a_j^p$ for $0 < p < 1$ and $a_j \geq 0$. For the convolution term we use Young's convolution inequality to obtain

$$\begin{aligned} \sum_{\kappa \in R\mathbb{Z}^3} \left(\frac{C\lambda}{R^{1/2}} |\overline{K} * e_{R,\lambda}(\kappa)|^2 \right)^{q/2} &= \left(\frac{C\lambda}{R^{1/2}} \right)^{\frac{q}{2}} \sum_{\kappa \in R\mathbb{Z}^3} |\overline{K} * e_{R,\lambda}(\kappa)|^q \\ &\leq \left(\frac{C\lambda}{R^{1/2}} \right)^{\frac{q}{2}} \|\overline{K}\|_{\ell^1}^q \|e_{R,\lambda}\|_{\ell^q}^q \leq \left(\frac{C\lambda}{R^{1/2}} \right)^{\frac{q}{2}} \|e_{R,\lambda}\|_{\ell^{q/2}}^q \leq \left(\frac{C\lambda}{R^{1/2}} \right)^{\frac{q}{2}} E_{R,q,\lambda}^2, \end{aligned}$$

where we used the fact that $\|\overline{K}\|_{\ell^1(R\mathbb{Z}^3)}$ is bounded independently of R . We conclude that, for some constant $C_2 \geq 1$ independent of q, R , we have

$$\begin{aligned} E_{R,q,\lambda} &\leq C_2^q A_{0,q}(R)^{q/2} + C_2^q \left(\lambda + \frac{\lambda}{R^{1/2}} \right)^{q/2} E_{R,q,\lambda} + C_2^q \left(\frac{\lambda}{R^2} \right)^{\frac{q}{8}} E_{R,q,\lambda}^{3/2} \\ &\quad + C_2^q \left(\frac{\lambda}{R^{1/2}} \right)^{\frac{q}{2}} E_{R,q,\lambda}^2 \end{aligned} \tag{3.26}$$

The same argument as in [4, p. 2005] shows that $E_{R,q,\lambda}$ is continuous in λ . Therefore, we conclude from (3.26) and a continuity argument that the estimate (3.18) also holds for $1 < q < 2$.

We now show (3.19) and (3.20). By the Gagliardo–Nirenberg inequality,

$$\begin{aligned} \int_{B_R} |v|^{\frac{10}{3}} dx &\lesssim \left(\int_{B_R} |\nabla v|^2 \right) \left(\int_{B_R} |v|^2 \right)^{\frac{2}{3}} + R^{-2} \left(\int_{B_R} |v|^2 \right)^{\frac{5}{3}}, \\ \int_{B_R} |b|^{\frac{10}{3}} dx &\lesssim \left(\int_{B_R} |\nabla b|^2 \right) \left(\int_{B_R} |b|^2 \right)^{\frac{2}{3}} + R^{-2} \left(\int_{B_R} |b|^2 \right)^{\frac{5}{3}}. \end{aligned}$$

Denoting $N = \sup_{0 \leq t \leq \lambda R^2} \int_{B_R} (|v(t)|^2 + |b(t)|^2) dx + 2 \int_0^{\lambda R^2} \int_{B_R} (|\nabla v|^2 + |\nabla b|^2) dx ds$ with $\lambda = \lambda_R$, we have

$$\begin{aligned} \int_0^{\lambda R^2} \int_{B_R} \left(|v|^{\frac{10}{3}} + |b|^{\frac{10}{3}} \right) dx ds &\lesssim N^{\frac{2}{3}} \int_0^{\lambda R^2} \left(\int_{B_R} |\nabla v|^2 + |\nabla b|^2 \right) ds + R^{-2} N^{\frac{5}{3}} \lambda R^2 \\ &\lesssim N^{\frac{5}{3}} + \lambda N^{\frac{5}{3}} \lesssim N^{\frac{5}{3}}, \end{aligned} \quad (3.27)$$

using $\lambda \leq 1$. For $k \in R\mathbb{Z}^3$ and $Q(k) = B_R(k) \times (0, \lambda R^2)$, by (3.27) with B_R replaced by $B_R(k)$, we have $N \leq e_{R,\lambda}(k)$ and hence

$$\sum_{k \in R\mathbb{Z}^3} \left[\int_{Q(k)} \left(|v|^{\frac{10}{3}} + |b|^{\frac{10}{3}} \right) dx dt \right]^{\frac{3q}{10}} \leq C \sum_{k \in R\mathbb{Z}^3} \left((e_{R,\lambda}(k))^{\frac{5}{3}} \right)^{\frac{3q}{10}} \leq C E_0, \quad q \geq 2,$$

and, for $\sigma > 0$ satisfying $\frac{5\sigma}{3} \geq \frac{q}{2}$ (which is implied if $\sigma \geq \frac{3}{5}$), we have

$$\begin{aligned} \sum_{k \in R\mathbb{Z}^3} \left[\int_{Q(k)} \left(|v|^{\frac{10}{3}} + |b|^{\frac{10}{3}} \right) dx dt \right]^\sigma \\ \leq C \sum_{k \in R\mathbb{Z}^3} (e_{R,\lambda}(k))^{\frac{5\sigma}{3}} \leq C \left[\sum_{k \in R\mathbb{Z}^3} (e_{R,\lambda}(k))^{\frac{q}{2}} \right]^{\frac{2}{q} \cdot \frac{5\sigma}{3}} \leq C E_0^{\frac{2}{q} \cdot \frac{5\sigma}{3}}, \quad 1 \leq q < 2. \end{aligned}$$

By Calderon–Zygmund estimates,

$$\begin{aligned} \sum_{k \in R\mathbb{Z}^3} \left(\int_{Q(k)} |\pi_1|^{\frac{5}{3}} dx dt \right)^{\frac{3q}{10}} \\ \leq C \sum_{k \in R\mathbb{Z}^3} \sum_{k' \in R\mathbb{Z}^3; |k-k'| < 10R} \left[\left(\int_{Q(k')} |v|^{\frac{10}{3}} dx dt \right)^{\frac{3q}{10}} + \left(\int_{Q(k')} |b|^{\frac{10}{3}} dx dt \right)^{\frac{3q}{10}} \right] \\ \leq C E_0, \quad q \geq 2, \end{aligned}$$

and

$$\begin{aligned} \sum_{k \in R\mathbb{Z}^3} \left(\int_{Q(k)} |\pi_1|^{\frac{5}{3}} dx dt \right)^\sigma \\ \leq C \sum_{k \in R\mathbb{Z}^3} \sum_{k' \in R\mathbb{Z}^3; |k-k'| < 10R} \left[\left(\int_{Q(k')} |v|^{\frac{10}{3}} dx dt \right)^\sigma + \left(\int_{Q(k')} |b|^{\frac{10}{3}} dx dt \right)^\sigma \right] \\ \leq C E_0^{\frac{2}{q} \cdot \frac{5\sigma}{3}}, \quad 1 \leq q < 2. \end{aligned}$$

For π_2 , recall π_2 in $B_R(k)$ is bounded by $R^{-3} \overline{K} * e_{R,\lambda}(k)$ and hence

$$\int_{Q(k)} |\pi_2|^{\frac{5}{3}} dx dt \leq C \lambda ((\overline{K} * e_{R,\lambda})(k))^{\frac{5}{3}}.$$

Thus,

$$\begin{aligned} \sum_{k \in R\mathbb{Z}^3} \left(\int_{Q(k)} |\pi_2|^{\frac{5}{3}} dx dt \right)^{\frac{3q}{10}} &\leq C \lambda^{\frac{3q}{10}} \sum_{k \in R\mathbb{Z}^3} ((\bar{K} * e_{R,\lambda})(k))^{\frac{q}{2}} \\ &\leq C \lambda^{\frac{3q}{10}} \|\bar{K}\|_{\ell^1}^{\frac{q}{2}} \sum_{k \in R\mathbb{Z}^3} (e_{R,\lambda}(k))^{\frac{q}{2}} \leq C E_0, \quad q \geq 2, \end{aligned}$$

and, if $\frac{5\sigma}{3} \geq 1$, which is implied by our assumptions, we have

$$\begin{aligned} \sum_{k \in R\mathbb{Z}^3} \left(\int_{Q(k)} |\pi_2|^{\frac{5}{3}} dx dt \right)^{\sigma} &\leq C \lambda^{\sigma} \sum_{k \in R\mathbb{Z}^3} ((\bar{K} * e_{R,\lambda})(k))^{\frac{5\sigma}{3}} \\ &\leq C \lambda^{\sigma} \|\bar{K}\|_{\ell^1}^{\frac{5\sigma}{3}} \sum_{k \in R\mathbb{Z}^3} (e_{R,\lambda}(k))^{\frac{5\sigma}{3}} \leq C E_0^{\frac{2}{3} \cdot \frac{5\sigma}{3}}, \quad 1 \leq q < 2. \end{aligned}$$

We conclude that

$$\left\| \int_{Q(k)} |v|^{\frac{10}{3}} + |b|^{\frac{10}{3}} + |\pi_1 + \pi_2|^{\frac{5}{3}} dx dt \right\|_{\ell^{\frac{3q}{10}}(k \in R\mathbb{Z}^3)} \leq C E_0^{\frac{10}{3q}} = C (A_{0,q}(R))^{\frac{5}{3}}, \quad q \geq 2,$$

and, for $\frac{5\sigma}{3} \geq 1$,

$$\left\| \int_{Q(k)} |v|^{\frac{10}{3}} + |b|^{\frac{10}{3}} + |\pi_1 + \pi_2|^{\frac{5}{3}} dx dt \right\|_{\ell^{\sigma}(k \in R\mathbb{Z}^3)} \leq C E_0^{\frac{10}{3q}} = C (A_{0,q}(R))^{\frac{5}{3}}, \quad 1 \leq q < 2.$$

This shows (3.19) and (3.20) and completes the proof. \square

We now prove Theorem 1.9, which follows directly as a simple consequence of Lemma 3.7.

Proof of Theorem 1.9. The proof of Theorem 1.9 is an adaption of the proof of [4, Theorem 1.4] for the Navier–Stokes equations to the MHD equations.

Observe that, by the assumption $R \geq 1$, the estimate (3.16), and Hölder's inequality, we have

$$\begin{aligned} N_{q,R}^0(v_0, b_0) &\leq \frac{C}{R} \left[\sum_{k \in \mathbb{Z}^3} \left(\sum_{|i-kR| < R} u_i^2 + a_i^2 \right)^{\frac{q}{2}} \right]^{\frac{2}{q}} \\ &\leq \frac{C}{R} \left[\left(\sum_{k \in \mathbb{Z}^3} \sum_{|i-kR| < R} u_i^q R^{(3-\frac{6}{q})\frac{q}{2}} \right)^{\frac{2}{q}} + \left(\sum_{k \in \mathbb{Z}^3} \sum_{|i-kR| < R} a_i^q R^{(3-\frac{6}{q})\frac{q}{2}} \right)^{\frac{2}{q}} \right] \\ &\leq C R^{2-\frac{6}{q}} \left(\|v_0\|_{E_q^2}^2 + \|b_0\|_{E_q^2}^2 \right), \end{aligned} \tag{3.28}$$

where $u_i = \|v_0\|_{L^2(B_1(i))}$ and $a_i = \|b_0\|_{L^2(B_1(i))}$ for $i \in \mathbb{Z}^3$. Now, applying Lemma 3.7, we obtain the bound

$$\left\| \operatorname{ess\,sup}_{0 \leq t \leq \lambda_R R^2} \int_{B_R(x_0 R)} \frac{|v|^2 + |b|^2}{2} dx + \int_0^{\lambda_R R^2} \int_{B_R(x_0 R)} (|\nabla v|^2 + |\nabla b|^2) dx dt \right\|_{\ell^{\frac{q}{2}}(x_0 \in \mathbb{Z}^3)} \leq C_1 A_{0,q}(R).$$

Next, using the definition of λ_R from Lemma 3.7 and the estimate (3.28), we find

$$\begin{aligned}\lambda_R R^2 &= \min \left(\lambda_0 R^2, \frac{\lambda_0 R^2}{\left(N_{q,R}^0(v_0, b_0) \right)^2} \right) \geq \min \left(\lambda_0 R^2, \frac{\lambda_0 R^{\frac{12}{q}-2}}{C^2 \left(\|v_0\|_{E_q^2}^4 + \|b_0\|_{E_q^2}^4 \right)} \right) \\ &\geq \frac{\lambda_1 R^{\min(2, \frac{12}{q}-2)}}{\left(1 + \|(v_0, b_0)\|_{E_q^2 \times E_q^2} \right)^4},\end{aligned}$$

where $\lambda_1 := \lambda_0(1+C)^{-2}$. Furthermore, from (3.28), we also have $A_{0,q}(R) = RN_{q,R}^0(v_0, b_0) \leq CR^{3-\frac{6}{q}} \left(\|(v_0, b_0)\|_{E_q^2 \times E_q^2}^2 \right)$. This yields the desired upper bound in the statement of Theorem 1.9. \square

Lemma 3.8 (Far-field regularity of local energy solutions with data in E_q^2). *Assume $v_0, b_0 \in E_q^2$ for some $q \geq 1$ are divergence free. If (v, b) is a local energy solution on $\mathbb{R}^3 \times (0, T_0)$ evolving from (v_0, b_0) with $(v, b) \in \mathbf{LE}_q(0, T_0)$. Then*

$$(v, b)(t) \in E_q^4 \times E_q^4 \text{ for a.e. } t \in (0, T_0].$$

Proof. By Lemma 3.7 with $R = 1$, we have

$$\|e_k\|_{\ell^{q/2}(k \in \mathbb{Z}^3)} \leq CA_{0,q}, \quad \|f_k\|_{\ell^{q/3}(k \in \mathbb{Z}^3)} \leq CA_{0,q}^{3/2}, \quad (3.29)$$

where

$$\begin{aligned}e_k &= \operatorname{ess\,sup}_{0 \leq t \leq T_0} \int_{B_1(k)} \frac{|v|^2 + |b|^2}{2} dx + \int_0^{T_0} \int_{B_1(k)} |\nabla v|^2 + |\nabla b|^2 dx dt, \\ f_k &= \int_0^{T_0} \int_{B_1(k)} |v|^3 + |b|^3 + |\pi - c_{k,1}(t)|^{3/2} dx dt,\end{aligned}$$

and

$$A_{0,q} = \left\| \int_{B_1(k)} |v_0(x)|^2 + |b_0(x)|^2 dx \right\|_{\ell^{q/2}(k \in \mathbb{Z}^3)} = \|(v_0, b_0)\|_{E_q^2 \times E_q^2}^2.$$

Since

$$\lim_{R \rightarrow \infty} \|f_k\|_{\ell^{q/3}(k \in \mathbb{Z}^3; |k| > R)} = 0,$$

by Lemma 3.2, there exists $R_0 > 0$ such that

$$v, b \in L^\infty \cap C_{\text{loc}}(B_{R_0}^c \times [T_0/2, T_0])$$

and

$$\lim_{R \rightarrow \infty} \|v\|_{L^\infty(B_{R_0}^c \times [T_0/2, T_0])} + \|b\|_{L^\infty(B_{R_0}^c \times [T_0/2, T_0])} = 0.$$

In fact, for $|k| > R_0$,

$$\|v\|_{L^\infty(B_1(k) \times [T_0/2, T_0])}^3 + \|b\|_{L^\infty(B_1(k) \times [T_0/2, T_0])}^3 \leq C \sum_{|k'-k| \leq 2} f_{k'}.$$

Thus

$$\left\| \|v\|_{L^\infty(B_1(k) \times [T_0/2, T_0])} + \|b\|_{L^\infty(B_1(k) \times [T_0/2, T_0])} \right\|_{\ell^q(k \in \mathbb{Z}^3; |k| > R_0)} \leq C \|f_k\|_{\ell^{q/3}(k \in \mathbb{Z}^3; |k| > R_0-2)}^{1/3} \ll 1.$$

By (3.29) and Sobolev imbedding,

$$v, b \in L^{8/3}(0, T_0; L^4(B_{R_0})).$$

Thus

$$(v, b)(t) \in E_q^4 \times E_q^4 \text{ for a.e. } t \in (0, T_0],$$

completing the proof of the lemma. \square

3.3 Global existence

In this section, we prove Theorem 1.10.

For $q \geq 2$, we achieve this by considering the perturbed MHD equations

$$\begin{aligned} \partial_t v - \Delta v + v \cdot \nabla v + u \cdot \nabla v + v \cdot \nabla u - b \cdot \nabla b - a \cdot \nabla b - b \cdot \nabla a + \nabla \pi &= 0, \\ \partial_t b - \Delta b + v \cdot \nabla b + u \cdot \nabla b + v \cdot \nabla a - b \cdot \nabla v - a \cdot \nabla v - b \cdot \nabla u &= 0, \\ \nabla \cdot v = \nabla \cdot b &= 0. \end{aligned} \quad (3.30)$$

where u and a are given divergence-free vector fields. A local energy solution to the perturbed MHD equations, (3.30), is a weak solution (v, b) satisfying Definition 1.7 with the obvious modifications, namely, (v, b) and π satisfy the perturbed system as distributions and also satisfy the perturbed local energy inequality.

For $1 \leq q < 2$, we achieve this via the localized and regularized MHD equations:

$$\begin{aligned} \partial_t v^\epsilon - \Delta v^\epsilon + (\mathcal{J}_\epsilon(v^\epsilon) \cdot \nabla)(v^\epsilon \Phi_\epsilon) - (\mathcal{J}_\epsilon(b^\epsilon) \cdot \nabla)(b^\epsilon \Phi_\epsilon) + \nabla \pi^\epsilon &= 0, \\ \partial_t b^\epsilon - \Delta b^\epsilon + (\mathcal{J}_\epsilon(v^\epsilon) \cdot \nabla)(b^\epsilon \Phi_\epsilon) - (\mathcal{J}_\epsilon(b^\epsilon) \cdot \nabla)(v^\epsilon \Phi_\epsilon) &= 0, \\ \nabla \cdot v^\epsilon = \nabla \cdot b^\epsilon &= 0, \end{aligned} \quad (3.31)$$

where $\mathcal{J}_\epsilon(f) = \eta_\epsilon * f$ for a spatial mollifier $\eta_\epsilon(x) = \epsilon^{-3} \eta(x/\epsilon)$ and $\Phi_\epsilon(x) = \Phi(\epsilon x)$ for a fixed radially decreasing cutoff function Φ satisfying $\Phi = 1$ on $B_1(0)$ and $\text{supp}(\Phi) \subset B_{3/2}(0)$.

3.3.1 The case $q \geq 2$

Lemma 3.9. *Let $\epsilon \in (0, 1]$ and $\delta > 0$ be given, let $T_0 > 0$, and let η be a spatial mollifier in \mathbb{R}^3 . Assume that $v_0, b_0 \in L^2$ are divergence free and that $u, a : \mathbb{R}^3 \times [0, T_0] \rightarrow \mathbb{R}^3$ satisfies $\nabla \cdot u = \nabla \cdot a = 0$ and*

$$\text{ess sup}_{0 < t \leq T_0} \|(u, a)(t)\|_{L_{\text{uloc}}^3 \times L_{\text{uloc}}^3} < \delta.$$

Then there exist $T_{\epsilon, \delta} = \min(T_0, C(\epsilon)(\|(v_0, b_0)\|_{E_q^2 \times E_q^2} + \delta)^{-2})$ and a mild solution (v_ϵ, b_ϵ) of the integral equation

$$\begin{aligned} v_\epsilon(t) &= e^{t\Delta} v_0 + \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot ((\eta_\epsilon * v_\epsilon) \otimes v_\epsilon - (\eta_\epsilon * b_\epsilon) \otimes b_\epsilon) ds + L_t^{(1)}(v_\epsilon, b_\epsilon), \\ b_\epsilon(t) &= e^{t\Delta} b_0 + \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot ((\eta_\epsilon * v_\epsilon) \otimes b_\epsilon - (\eta_\epsilon * b_\epsilon) \otimes v_\epsilon) ds + L_t^{(2)}(v_\epsilon, b_\epsilon), \end{aligned} \quad (3.32)$$

for $0 < t < T_{\epsilon, \delta}$, where

$$L_t^{(1)}(v_\epsilon, b_\epsilon) = \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot ((\eta_\epsilon * u) \otimes v_\epsilon + v_\epsilon \otimes (\eta_\epsilon * u) - (\eta_\epsilon * a) \otimes b_\epsilon - b_\epsilon \otimes (\eta_\epsilon * a)) ds,$$

$$L_t^{(2)}(v_\epsilon, b_\epsilon) = \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot ((\eta_\epsilon * u) \otimes b_\epsilon + b_\epsilon \otimes (\eta_\epsilon * u) - (\eta_\epsilon * a) \otimes v_\epsilon - b_\epsilon \otimes (\eta_\epsilon * u)) ds,$$

with $(v_\epsilon, b_\epsilon) \in \mathbf{LE}_q(0, T_{\epsilon, \delta}) \cap C([0, T_{\epsilon, \delta}]; L^2 \times L^2)$, and (v_ϵ, b_ϵ) satisfies

$$\left\| \left(\operatorname{ess\,sup}_{0 < t < T_{\epsilon, \delta}} \int_{B_1(k)} (|v_\epsilon(x, t)|^2 + |b_\epsilon(x, t)|^2) dx \right)^{\frac{1}{2}} \right\|_{\ell^q} \leq 2C \|(v_0, b_0)\|_{E_q^2 \times E_q^2} \quad \text{and} \quad (3.33)$$

$$\sup_{0 < t < T_{\epsilon, \delta}} \|(v_\epsilon, b_\epsilon)(t)\|_{L^2 \times L^2} \leq 2C \|(v_0, b_0)\|_{L^2 \times L^2}$$

for a universal constant $C > 0$. This is the unique mild solution of (3.32) in the class (3.33). There exists a pressure π_ϵ so that (v_ϵ, b_ϵ) and π_ϵ solve

$$\begin{aligned} \partial_t v_\epsilon - \Delta v_\epsilon + (\eta_\epsilon * v_\epsilon) \cdot \nabla v_\epsilon + (\eta_\epsilon * u) \cdot \nabla v_\epsilon + v_\epsilon \cdot \nabla (\eta_\epsilon * u) \\ - (\eta_\epsilon * b_\epsilon) \cdot \nabla b_\epsilon - (\eta_\epsilon * a) \cdot \nabla b_\epsilon - b_\epsilon \cdot \nabla (\eta_\epsilon * a) + \nabla \pi_\epsilon &= 0, \\ \partial_t b_\epsilon - \Delta b_\epsilon + (\eta_\epsilon * v_\epsilon) \cdot \nabla b_\epsilon + (\eta_\epsilon * u) \cdot \nabla b_\epsilon + v_\epsilon \cdot \nabla (\eta_\epsilon * a) \\ - (\eta_\epsilon * b_\epsilon) \cdot \nabla v_\epsilon - (\eta_\epsilon * a) \cdot \nabla v_\epsilon - b_\epsilon \cdot \nabla (\eta_\epsilon * u) &= 0, \\ \nabla \cdot v_\epsilon = \nabla \cdot b_\epsilon &= 0, \end{aligned} \quad (3.34)$$

in the weak sense on $\mathbb{R}^3 \times (0, T_{\epsilon, \delta})$. Finally, (v_ϵ, b_ϵ) and π_ϵ are smooth by the interior regularity of the Stokes equations with smooth coefficients.

Proof. The proof of the lemma is an adaption of the proof of [4, Lemma 4.1] for the Navier–Stokes equations to the MHD equations.

Note that $v_0, b_0 \in E_q^2$ since $L^2 \subset E_q^2$. We begin by establishing estimates for the iterates of the Picard scheme. Define the initial iterates as

$$(v_\epsilon^1, b_\epsilon^1) = (e^{t\Delta} v_0, e^{t\Delta} b_0),$$

and for $n > 1$, set

$$\begin{aligned} v_\epsilon^n(t) &= e^{t\Delta} v_0 + \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot ((\eta_\epsilon * v_\epsilon^{n-1}) \otimes v_\epsilon^{n-1} - (\eta_\epsilon * b_\epsilon^{n-1}) \otimes b_\epsilon^{n-1}) ds + L_t^{(1)}(v_\epsilon^{n-1}, b_\epsilon^{n-1}), \\ b_\epsilon^n(t) &= e^{t\Delta} b_0 + \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot ((\eta_\epsilon * v_\epsilon^{n-1}) \otimes b_\epsilon^{n-1} - (\eta_\epsilon * b_\epsilon^{n-1}) \otimes v_\epsilon^{n-1}) ds + L_t^{(2)}(v_\epsilon^{n-1}, b_\epsilon^{n-1}). \end{aligned}$$

For the initial iterates, it follows from the same approach obtaining [4, (4.7)] that

$$\left\| \left(\sup_{0 < t < 1} \int_{B_1(k)} |v_\epsilon^1(x, t)|^2 dx \right)^{\frac{1}{2}} \right\|_{\ell^q} \leq C \|v_0\|_{E_q^2} \quad \text{and} \quad \left\| \left(\sup_{0 < t < 1} \int_{B_1(k)} |b_\epsilon^1(x, t)|^2 dx \right)^{\frac{1}{2}} \right\|_{\ell^q} \leq C \|b_0\|_{E_q^2}. \quad (3.35)$$

For the n th iterates, we use the assumption that

$$\left\| \left(\sup_{0 < t < T_\epsilon} \int_{B_1(k)} (|v_\epsilon^{n-1}(x, t)|^2 + |b_\epsilon^{n-1}(x, t)|^2) dx \right)^{\frac{1}{2}} \right\|_{\ell^q} < 2C \|(v_0, b_0)\|_{E_q^2 \times E_q^2}.$$

We have

$$\begin{aligned} \int_{B_1(k)} |v_\epsilon^n(x, t)|^2 dx &\leq \int_{B_1(k)} |e^{t\Delta} v_0(x)|^2 dx + I^{(1)}(k) + J^{(1)}(k), \\ \int_{B_1(k)} |b_\epsilon^n(x, t)|^2 dx &\leq \int_{B_1(k)} |e^{t\Delta} b_0(x)|^2 dx + I^{(2)}(k) + J^{(2)}(k), \end{aligned} \quad (3.36)$$

where

$$\begin{aligned} I^{(1)}(k) &= \int_{B_1(k)} \left| \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot ((\eta_\epsilon * v_\epsilon^{n-1}) \otimes v_\epsilon^{n-1} - (\eta_\epsilon * b_\epsilon^{n-1}) \otimes b_\epsilon^{n-1}) ds \right|^2 dx, \\ J^{(1)}(k) &= \int_{B_1(k)} \left| L_t^{(1)}(v_\epsilon^{n-1}, b_\epsilon^{n-1}) \right|^2 dx, \\ I^{(2)}(k) &= \int_{B_1(k)} \left| \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot ((\eta_\epsilon * v_\epsilon^{n-1}) \otimes b_\epsilon^{n-1} - (\eta_\epsilon * b_\epsilon^{n-1}) \otimes v_\epsilon^{n-1}) ds \right|^2 dx, \\ J^{(2)}(k) &= \int_{B_1(k)} \left| L_t^{(2)}(v_\epsilon^{n-1}, b_\epsilon^{n-1}) \right|^2 dx. \end{aligned}$$

The first two terms on the right-hand side of (3.36) have already been estimated in (3.35). For $I^{(1)}(k)$ and $I^{(2)}(k)$, using the same technique deriving [4, (4.11), (4.12)], we get

$$\begin{aligned} &I^{(1)}(k) + I^{(2)}(k) \\ &\leq C(\epsilon)t^2 \left[\left(\sup_{0 < s < t} \|v_\epsilon^{n-1}\|_{E_q^2}^2 \right) \left((\tilde{K} * u^{n-1})(k) \right)^2 + \left(\sup_{0 < s < t} \|b_\epsilon^{n-1}\|_{E_q^2}^2 \right) \left((\tilde{K} * a^{n-1})(k) \right)^2 \right] \\ &\quad + C(\epsilon)t \left[\sup_{0 < s < t} \|v_\epsilon^{n-1}\|_{E_q^2}^2 \sup_{0 < s < t} \sum_{|k-k'| < 8} \|v_\epsilon^{n-1}\|_{L^2(B_1(k'))}^2 \right. \\ &\quad \left. + \sup_{0 < s < t} \|b_\epsilon^{n-1}\|_{E_q^2}^2 \sup_{0 < s < t} \sum_{|k-k'| < 8} \|b_\epsilon^{n-1}\|_{L^2(B_1(k'))}^2 \right], \end{aligned} \quad (3.37)$$

where

$$u_k^{n-1} = \left(\sup_{0 < s < t} \int_{B_1(k)} |v_\epsilon^{n-1}(y)|^2 dy \right)^{1/2}, \quad a_k^{n-1} = \left(\sup_{0 < s < t} \int_{B_1(k)} |b_\epsilon^{n-1}(y)|^2 dy \right)^{1/2},$$

and

$$\tilde{K}(k) = |k|^{-4} \text{ if } |k| \geq 4, \quad \tilde{K}(k) = 0 \text{ otherwise.} \quad (3.38)$$

We next estimate the terms $J^{(1)}(k)$ and $J^{(2)}(k)$. Using the same argument for [4, (4.13), (4.14)] yields

$$\begin{aligned} J^{(1)}(k) + J^{(2)}(k) &\leq C\delta^2 t^2 \left((\tilde{K} * u^{n-1})(k) + (\tilde{K} * a^{n-1})(k) \right)^2 \\ &\quad + C(\epsilon)t\delta^2 \left(\|v_\epsilon^{n-1}\|_{L^2(B_4(k))}^2 + \|b_\epsilon^{n-1}\|_{L^2(B_4(k))}^2 \right). \end{aligned} \quad (3.39)$$

Combining (3.35), (3.37), and (3.39), we have

$$\begin{aligned}
& \int_{B_1(k)} \left(|v_\epsilon^n(x, t)|^2 + |b_\epsilon^n(x, t)|^2 \right) dx \\
& \leq C \sum_{|k-k'| \leq 4} \int_{B_1(k')} (|v_0(x)|^2 + |b_0(x)|^2) dx + C \left[\left(\tilde{K} * (u + a) \right) (k) \right]^2 \\
& \quad + C(\epsilon) t^2 \left[\left(\sup_{0 < s < t} \|v_\epsilon^{n-1}\|_{E_q^2}^2 \right) \left((\tilde{K} * u^{n-1})(k) \right)^2 + \left(\sup_{0 < s < t} \|b_\epsilon^{n-1}\|_{E_q^2}^2 \right) \left((\tilde{K} * a^{n-1})(k) \right)^2 \right] \\
& \quad + C(\epsilon) t \left[\sup_{0 < s < t} \|v_\epsilon^{n-1}\|_{E_q^2}^2 \sup_{0 < s < t} \sum_{|k-k'| < 8} \|v_\epsilon^{n-1}\|_{L^2(B_1(k'))}^2 \right. \\
& \quad \quad \left. + \sup_{0 < s < t} \|b_\epsilon^{n-1}\|_{E_q^2}^2 \sup_{0 < s < t} \sum_{|k-k'| < 8} \|b_\epsilon^{n-1}\|_{L^2(B_1(k'))}^2 \right] \\
& \quad + C\delta^2 t^2 \left((\tilde{K} * u^{n-1})(k) + (\tilde{K} * a^{n-1})(k) \right)^2 + C(\epsilon) t \delta^2 \left(\|v_\epsilon^{n-1}\|_{L^2(B_4(k))}^2 + \|b_\epsilon^{n-1}\|_{L^2(B_4(k))}^2 \right).
\end{aligned} \tag{3.40}$$

Taking the supremum in time of the left-hand side of (3.40), applying the $\ell^{\frac{q}{2}}$ norm, using Young's convolution inequality, and raising everything to the $1/2$ power yields

$$\begin{aligned}
\|(v_\epsilon^n, b_\epsilon^n)\|_{\mathbf{LE}_q^b(0, t)} & \leq C \|(v_0, b_0)\|_{E_q^2 \times E_q^2} + C(\epsilon) t^{\frac{1}{2}} \|(v_\epsilon^{n-1}, b_\epsilon^{n-1})\|_{\mathbf{LE}_q^b(0, t)}^2 \\
& \quad + C(\epsilon) \delta t^{\frac{1}{2}} \|(v_\epsilon^{n-1}, b_\epsilon^{n-1})\|_{\mathbf{LE}_q^b(0, t)},
\end{aligned} \tag{3.41}$$

where $\|\cdot\|_{\mathbf{LE}_q^b(I)}$ is the first part of the norm $\|\cdot\|_{\mathbf{LE}_q(I)}$ defined by $\|(v, b)\|_{\mathbf{LE}_q^b(I)} = \|(v, b)\|_{E_{T, q}^{\infty, 2} \times E_{T, q}^{\infty, 2}}$. So, if t is small as determined by $C(\epsilon)$, ϵ , and $\|(v_0, b_0)\|_{E_q^2 \times E_q^2}$ (but independently of n), $t \leq C(\epsilon) / \left(\|(v_0, b_0)\|_{E_q^2 \times E_q^2}^2 + \delta^2 \right)$, then the right-hand side of (3.41) is controlled by $2C \|(v_0, b_0)\|_{E_q^2 \times E_q^2}$. This establishes a uniform-in- n bound for $(v_\epsilon^n, b_\epsilon^n)$.

These uniform bounds and the estimation methods above allow us to show the difference estimate

$$\|(v_\epsilon^{n+1}, b_\epsilon^{n+1}) - (v_\epsilon^n, b_\epsilon^n)\|_{\mathbf{LE}_q^b(0, t)} \leq C(\epsilon) \sqrt{t} \left(\|(v_0, b_0)\|_{E_q^2 \times E_q^2} + \delta \right) \|(v_\epsilon^n, b_\epsilon^n) - (v_\epsilon^{n-1}, b_\epsilon^{n-1})\|_{\mathbf{LE}_q^b(0, t)}.$$

Thus, if t is sufficiently small, then $(v_\epsilon^n, b_\epsilon^n)$ is a Cauchy sequence in $\mathbf{LE}_q^b(0, t)$ norm and converges to a limit $(v_\epsilon^n, b_\epsilon^n)$ in the sense that

$$\|(v_\epsilon^n, b_\epsilon^n) - (v_\epsilon, b_\epsilon)\|_{\mathbf{LE}_q^b(0, t)} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

This convergence implies (v_ϵ, b_ϵ) satisfies (3.32) and (3.33).

Uniqueness in the class (3.33) follows from the same difference estimates as before. Suppose (v_1, b_1) and (v_2, b_2) are two mild solutions of (3.32) satisfying (3.33). Then

$$\|(v_1, b_1) - (v_2, b_2)\|_{\mathbf{LE}_q^b(0, t)} \leq C(\epsilon) \sqrt{t} \left(\|(v_0, b_0)\|_{E_q^2 \times E_q^2} + \delta \right) \|(v_1, b_1) - (v_2, b_2)\|_{\mathbf{LE}_q^b(0, t)}.$$

Hence, if $t > 0$ is sufficiently small, we conclude that $(v_1, b_1) = (v_2, b_2)$.

We now recover a pressure associated to (v_ϵ, b_ϵ) . It is known that $(v_\epsilon, b_\epsilon) \in L^\infty(0, T_{\epsilon, \delta}; L^2 \times L^2)$ for some $T_{\epsilon, \delta} > 0$, with $\|(v_\epsilon, b_\epsilon)\|_{L^\infty(0, T_{\epsilon, \delta}; L^2 \times L^2)} \leq 2C \|(v_0, b_0)\|_{L^2 \times L^2}$. Therefore, the following nonlinear terms belongs to $L^\infty(0, T_{\epsilon, \delta}; L^2)$:

$$\eta_\epsilon * v_\epsilon \otimes v_\epsilon + \eta_\epsilon * u \otimes v_\epsilon + v_\epsilon \otimes \eta_\epsilon * u - \eta_\epsilon * b_\epsilon \otimes b - \eta_\epsilon * a \otimes b_\epsilon - b_\epsilon \otimes \eta_\epsilon * a.$$

Consequently $\pi_\epsilon = (-\Delta)^{-1} \partial_i \partial_j (\eta_\epsilon * v_\epsilon \otimes v_\epsilon + \eta_\epsilon * u \otimes v_\epsilon + v_\epsilon \otimes \eta_\epsilon * u - \eta_\epsilon * b_\epsilon \otimes b - \eta_\epsilon * a \otimes b_\epsilon - b_\epsilon \otimes \eta_\epsilon * a)$ is well-defined. It follows that $v_\epsilon - e^{t\Delta} v_0$ solves the Stokes system with pressure π_ϵ and forcing term equal to $\nabla \cdot (\eta_\epsilon * v_\epsilon \otimes v_\epsilon + \eta_\epsilon * u \otimes v_\epsilon + v_\epsilon \otimes \eta_\epsilon * u - \eta_\epsilon * b_\epsilon \otimes b - \eta_\epsilon * a \otimes b_\epsilon - b_\epsilon \otimes \eta_\epsilon * a)$. Adding back the linear term $e^{t\Delta} v_0$, we see that $(v_\epsilon, \pi_\epsilon)$ solve the perturbed, regularized Navier–Stokes equations. The local pressure expansion (1.27) follows from the definition of π_ϵ .

We now establish the estimate

$$\left\| \int_0^t \int_{B_1(k)} (|\nabla v_\epsilon|^2 + |\nabla b_\epsilon|^2) dx ds \right\|_{\ell^{q/2}(k)} < \infty. \quad (3.42)$$

This follow from the local energy equality satisfied by (v_ϵ, b_ϵ) and the associated pressure π_ϵ , valid for $t \leq T_{\epsilon, \delta}$ due to the regularity of the solutions due to smoothness and convergence to the data in L^2_{loc} :

$$\begin{aligned} & \int_{B_1(k)} (|v_\epsilon|^2 + |b_\epsilon|^2) (x, t) \phi(x - k) dx + 2 \int_0^t \int (|\nabla v_\epsilon|^2 + |\nabla b_\epsilon|^2) \phi(x - k) dx ds \\ &= \int (|v_0|^2 + |b_0|^2) \phi(x - k) dx + \int_0^t \int (|v_\epsilon|^2 + |b_\epsilon|^2) \Delta \phi(x - k) dx ds \\ &+ \int_0^t \int (|v_\epsilon|^2 + |b_\epsilon|^2) (\eta_\epsilon * v_\epsilon + \eta_\epsilon * u) \cdot \nabla \phi(x - k) dx ds \\ &- 2 \int_0^t \int (v_\epsilon \cdot \nabla (\eta_\epsilon * u)) \cdot v_\epsilon \phi(x - k) dx ds + 2 \int_0^t \int (b_\epsilon \cdot \nabla (\eta_\epsilon * a)) \cdot v_\epsilon \phi(x - k) dx ds \\ &- 2 \int_0^t \int (v_\epsilon \cdot \nabla (\eta_\epsilon * a)) \cdot b_\epsilon \phi(x - k) dx ds + 2 \int_0^t \int (b_\epsilon \cdot \nabla (\eta_\epsilon * u)) \cdot b_\epsilon \phi(x - k) dx ds \\ &+ 2 \int_0^t \int \pi_\epsilon (v_\epsilon \cdot \nabla \phi(x - k)) dx ds - 2 \int_0^t \int (v_\epsilon \cdot b_\epsilon) (\eta_\epsilon * b_\epsilon + \eta_\epsilon * a) \cdot \nabla \phi(x - k) dx ds. \end{aligned} \quad (3.43)$$

To estimate the nonlinear terms, we use the bound $\|\eta_\epsilon * f\|_{L^\infty(B_2(k))} \leq C(\epsilon) \|f\|_{L^2_{\text{uloc}}}$ for $\epsilon \leq 1$. Hence, we have

$$\begin{aligned} & \int_0^t \int (|v_\epsilon|^2 + |b_\epsilon|^2) (\eta_\epsilon * v_\epsilon) \cdot \nabla \phi(x - k) dx ds \\ & \leq C(\epsilon) \|v_\epsilon\|_{L^2 \infty L^2_{\text{uloc}}} \text{ess sup}_{0 < s < t} \sum_{k' \sim k} \int_{B_1(k')} (|v_\epsilon|^2 + |b_\epsilon|^2) dx, \end{aligned} \quad (3.44)$$

and

$$\begin{aligned} & \int_0^t \int (v_\epsilon \cdot b_\epsilon) (\eta_\epsilon * b_\epsilon) \cdot \nabla \phi(x - k) dx ds \\ & \lesssim \int_0^t \int (|v_\epsilon|^2 + |b_\epsilon|^2) (\eta_\epsilon * b_\epsilon) \cdot \nabla \phi(x - k) dx ds \\ & \leq C(\epsilon) \|b_\epsilon\|_{L^2 \infty L^2_{\text{uloc}}} \text{ess sup}_{0 < s < t} \sum_{k' \sim k} \int_{B_1(k')} (|v_\epsilon|^2 + |b_\epsilon|^2) dx. \end{aligned} \quad (3.45)$$

From the assumptions on u and a , we also have

$$\|\eta_\epsilon * u\|_{L^\infty} + \|\nabla(\eta_\epsilon * u)\|_{L^\infty} \leq C(\epsilon) \|u\|_{L^\infty(0, T_0; L^3_{\text{uloc}})} \leq C(\epsilon)\delta,$$

and

$$\|\eta_\epsilon * a\|_{L^\infty} + \|\nabla(\eta_\epsilon * a)\|_{L^\infty} \leq C(\epsilon) \|a\|_{L^\infty(0, T_0; L^3_{\text{uloc}})} \leq C(\epsilon)\delta.$$

These imply the following estimates:

$$\begin{aligned} & \int_0^t \int (|v_\epsilon|^2 + |b_\epsilon|^2) (\eta_\epsilon * u) \cdot \nabla \phi(x - k) dx ds \\ & \leq C(\epsilon) \delta \operatorname{ess\,sup}_{0 < s < t} \sum_{k' \sim k} \int_{B_1(k')} (|v_\epsilon(x, s)|^2 + |b_\epsilon(x, s)|^2) dx, \end{aligned} \quad (3.46)$$

$$\begin{aligned} & \int_0^t \int (v_\epsilon \cdot b_\epsilon) (\eta_\epsilon * a) \cdot \nabla \phi(x - k) dx ds \\ & \lesssim \int_0^t \int (|v_\epsilon|^2 + |b_\epsilon|^2) (\eta_\epsilon * a) \cdot \nabla \phi(x - k) dx ds \\ & \leq C(\epsilon) \delta \operatorname{ess\,sup}_{0 < s < t} \sum_{k' \sim k} \int_{B_1(k')} (|v_\epsilon(x, s)|^2 + |b_\epsilon(x, s)|^2) dx, \end{aligned} \quad (3.47)$$

$$\int_0^t \int (v_\epsilon \cdot \nabla(\eta_\epsilon * u)) \cdot v_\epsilon \phi(x - k) dx ds \leq C(\epsilon) \delta \operatorname{ess\,sup}_{0 < s < t} \sum_{k' \sim k} \int_{B_1(k')} |v_\epsilon|^2 dx, \quad (3.48)$$

$$\begin{aligned} \int_0^t \int (b_\epsilon \cdot \nabla(\eta_\epsilon * a)) \cdot v_\epsilon \phi(x - k) dx ds & \leq C(\epsilon) \delta \operatorname{ess\,sup}_{0 < s < t} \sum_{k' \sim k} \int_{B_1(k')} |b_\epsilon| |v_\epsilon| dx \\ & \leq C(\epsilon) \delta \operatorname{ess\,sup}_{0 < s < t} \sum_{k' \sim k} \int_{B_1(k')} (|b_\epsilon|^2 + |v_\epsilon|^2) dx, \end{aligned} \quad (3.49)$$

$$\begin{aligned} \int_0^t \int (v_\epsilon \cdot \nabla(\eta_\epsilon * a)) \cdot b_\epsilon \phi(x - k) dx ds & \leq C(\epsilon) \delta \operatorname{ess\,sup}_{0 < s < t} \sum_{k' \sim k} \int_{B_1(k')} |v_\epsilon| |b_\epsilon| dx \\ & \leq C(\epsilon) \delta \operatorname{ess\,sup}_{0 < s < t} \sum_{k' \sim k} \int_{B_1(k')} (|v_\epsilon|^2 + |b_\epsilon|^2) dx, \end{aligned} \quad (3.50)$$

and

$$\int_0^t \int (b_\epsilon \cdot \nabla(\eta_\epsilon * u)) \cdot b_\epsilon \phi(x - k) dx ds \leq C(\epsilon) \delta \operatorname{ess\,sup}_{0 < s < t} \sum_{k' \sim k} \int_{B_1(k')} |b_\epsilon|^2 dx. \quad (3.51)$$

The pressure satisfies the local pressure expansion (1.27), which allows us—after incorporating an additive constant—to express it as a sum of two components: $\pi_\epsilon(x, t) + c = \pi_{\epsilon, \text{near}} + \pi_{\epsilon, \text{far}}$, where $\pi_{\epsilon, \text{near}}$ is a Calderon–Zygmund operator applied to a localized term, and $\pi_{\epsilon, \text{far}}$ is a nonsingular integral operator acting on data supported away from the ball $B_2(k)$. Due to the structure of the pressure term in the local energy inequality, the additive constant c plays no role and may be disregarded. By applying the Calderon–Zygmund inequality, the contribution from $\pi_{\epsilon, \text{near}}$ to the local energy inequality can be estimated in the same way as the nonlinear and perturbative terms

discussed earlier. Specifically, it is controlled by the right-hand sides of estimates (3.44) through (3.51). We are thus left to estimate only the far-filed component $\pi_{\epsilon, \text{far}}$. In $B_2(k) \times (0, T_0)$,

$$\begin{aligned} |\pi_{\epsilon, \text{far}}| &\leq C \sum_{k' \in \mathbb{Z}^3; |k' - k| > 4} \frac{1}{|k - k'|^4} \int_{B_2(k')} (|v_\epsilon| |\eta_\epsilon * v_\epsilon| + |\eta_\epsilon * u| |v_\epsilon| + |b_\epsilon| |\eta_\epsilon * b_\epsilon| + |\eta_\epsilon * a| |b_\epsilon|) dy \\ &\leq C(\epsilon) \sum_{k' \in \mathbb{Z}^3; |k' - k| > 4} \frac{1}{|k - k'|^4} \left(\|v_\epsilon\|_{L^2(B_3(k'))}^2 + \|v_\epsilon\|_{L^2(B_3(k'))} \|u\|_{L^\infty(0, T_0; L^3_{\text{uloc}})} \right. \\ &\quad \left. + \|b_\epsilon\|_{L^2(B_3(k'))}^2 + \|b_\epsilon\|_{L^2(B_3(k'))} \|a\|_{L^\infty(0, T_0; L^3_{\text{uloc}})} \right). \end{aligned}$$

Therefore, the contribution of $\pi_{\epsilon, \text{far}}$ to the local energy equation satisfies

$$\begin{aligned} &\int_0^t \int_{B_2(k)} \pi_{\epsilon, \text{far}}(v_\epsilon \cdot \nabla \phi(x - k)) dx ds \\ &\leq C(\epsilon) T_0 \|v_\epsilon\|_{L^\infty L^2_{\text{uloc}}} \text{ess sup}_{0 < s < t} \sum_{k' \in \mathbb{Z}^3; |k' - k| > 4} \frac{1}{|k - k'|^4} \int_{B_3(k')} (|v_\epsilon|^2 + |b_\epsilon|^2) dy \\ &\quad + C(\epsilon) \delta T_0 \|v_\epsilon\|_{L^\infty L^2(B_2(k))} \text{ess sup}_{0 < s < t} \sum_{k' \in \mathbb{Z}^3; |k' - k| > 4} \frac{1}{|k - k'|^4} \left(\|v_\epsilon\|_{L^2(B_3(k'))} + \|b_\epsilon\|_{L^2(B_3(k'))} \right). \end{aligned} \quad (3.52)$$

Taking the essential supremum in t , raising both sides of (3.52) to the power $q/2$, and summing over $k \in \mathbb{Z}^3$, we apply Hölder's and Young's inequalities to control the far-field pressure term. This establishes the estimate (3.42), completing the proof of Lemma 3.9. \square

Lemma 3.10. *Assume that $v_0, b_0 \in E_q^2$, for some $2 \leq q < \infty$, are divergence free. There exists a small universal constant c_0 so that for all $\delta \in (0, c_0]$ and for all divergence free vector fields $u, a : \mathbb{R}^3 \times [0, T_0] \rightarrow \mathbb{R}^3$ satisfying*

$$\text{ess sup}_{0 < t \leq T_0} \|(u, a)(t)\|_{L^3_{\text{uloc}} \times L^3_{\text{uloc}}} < \delta$$

for some $T_0 > 0$ and if, additionally, a given local energy solution (v, b) to the perturbed MHD equations, (3.30), satisfies

$$\left\| \text{ess sup}_{0 \leq t \leq T_0} \int_{B_1(x_0)} (|v|^2 + |b|^2) dx + \int_0^{T_0} \int_{B_1(x_0)} (|\nabla v|^2 + |\nabla b|^2) dx dt \right\|_{\ell^{\frac{q}{2}}(x_0 \in \mathbb{Z}^3)} < \infty,$$

then there are positive universal constants C_1 and $\lambda_0 < 1$ such that

$$\left\| \text{ess sup}_{0 \leq t \leq \lambda} \int_{B_1(x_0)} \frac{|v|^2 + |b|^2}{2} dx + \int_0^\lambda \int_{B_1(x_0)} (|\nabla v|^2 + |\nabla b|^2) dx dt \right\|_{\ell^{\frac{q}{2}}(x_0 \in \mathbb{Z}^3)} < C_1 A_{0,q},$$

where

$$A_{0,q} = \left\| \int_{B_1(x_0)} (|v_0|^2 + |b_0|^2) dx \right\|_{\ell^{\frac{q}{2}}(x_0 \in \mathbb{Z}^3)}, \quad \lambda = \min \left(T_0, \lambda_0, \frac{\lambda_0}{A_{0,q}^2} \right).$$

Consequently,

$$\left\| \int_0^\lambda \int_{B_1(x_0)} |v|^{\frac{10}{3}} + |b|^{\frac{10}{3}} + |\pi - c_{x_0}(t)|^{\frac{5}{3}} dx dt \right\|_{\ell^{\frac{3q}{10}}(x_0 \in \mathbb{Z}^3)} \leq C A_{0,q}^{\frac{5}{3}}.$$

Proof. The proof of the lemma is an adaption of the proof of [4, Lemma 4.2] for the Navier–Stokes equations to the MHD equations.

Once the perturbation terms in the local energy inequality for (v, b) are estimated, the proof proceeds identically to that of Lemma 3.7 with $R = 1$ and $\lambda_R = \lambda$.

The linear terms in the perturbed local energy inequality can be estimated as follows:

$$\begin{aligned}
& \int_0^\lambda \int (u \cdot \nabla v + v \cdot \nabla u - a \cdot \nabla b - b \cdot \nabla a) \cdot (\phi(x - \kappa)v) \, dx dt \\
&= \int_0^\lambda \int [u \cdot \nabla v \cdot (\phi(x - \kappa)v) - v \otimes u : \nabla(\phi(x - \kappa)v) \\
&\quad - a \cdot \nabla b \cdot (\phi(x - \kappa)v) + b \otimes a : \nabla(\phi(x - \kappa)v)] \, dx dt \\
&\leq C \int_0^\lambda \int_{B_2(\kappa)} [|u|(|v|^2 + |v||\nabla b|) + |a|(|v||b| + |v||\nabla b| + |b||\nabla v|)] \, dx dt \\
&\leq C \|u\|_{L^\infty L^3_{\text{uloc}}} \int_0^\lambda \|v\|_{L^6(B_2(\kappa))} \left(\|v\|_{L^2(B_2(\kappa))} + \|\nabla v\|_{L^2(B_2(\kappa))} \right) dt \\
&\quad + C \|a\|_{L^\infty L^3_{\text{uloc}}} \int_0^\lambda \|v\|_{L^6(B_2(\kappa))} \left(\|b\|_{L^2(B_2(\kappa))} + \|\nabla b\|_{L^2(B_2(\kappa))} \right) dt \\
&\quad + C \|a\|_{L^\infty L^3_{\text{uloc}}} \int_0^\lambda \|b\|_{L^6(B_2(\kappa))} \|\nabla v\|_{L^2(B_2(\kappa))} dt \\
&\leq C \lambda \delta \operatorname{ess\,sup}_{0 < t < \lambda} \sum_{\kappa' \sim \kappa} \int_{B_1(\kappa')} (|v(x, t)|^2 + |b(x, t)|^2) \, dx \\
&\quad + C \delta \operatorname{ess\,sup}_{0 < t < \lambda} \sum_{\kappa' \sim \kappa} \int_0^\lambda \int_{B_1(\kappa')} (|\nabla v(x, t)|^2 + |\nabla b(x, t)|^2) \, dx dt,
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^\lambda \int (u \cdot \nabla b + v \cdot \nabla a - a \cdot \nabla v - b \cdot \nabla u) \cdot (\phi(x - \kappa)b) \, dx dt \\
&= \int_0^\lambda \int [u \cdot \nabla b \cdot (\phi(x - \kappa)b) - v \otimes a : \nabla(\phi(x - \kappa)b) \\
&\quad - a \cdot \nabla v \cdot (\phi(x - \kappa)b) + b \otimes u : \nabla(\phi(x - \kappa)b)] \, dx dt \\
&\leq C \int_0^\lambda \int_{B_2(\kappa)} [|u|(|b|^2 + |b||\nabla v|) + |a|(|v||b| + |v||\nabla b| + |b||\nabla v|)] \, dx dt \\
&\leq C \|u\|_{L^\infty L^3_{\text{uloc}}} \int_0^\lambda \|b\|_{L^6(B_2(\kappa))} \left(\|b\|_{L^2(B_2(\kappa))} + \|\nabla b\|_{L^2(B_2(\kappa))} \right) dt \\
&\quad + C \|a\|_{L^\infty L^3_{\text{uloc}}} \int_0^\lambda \|v\|_{L^6(B_2(\kappa))} \left(\|b\|_{L^2(B_2(\kappa))} + \|\nabla b\|_{L^2(B_2(\kappa))} \right) dt \\
&\quad + C \|a\|_{L^\infty L^3_{\text{uloc}}} \int_0^\lambda \|b\|_{L^6(B_2(\kappa))} \|\nabla v\|_{L^2(B_2(\kappa))} dt \\
&\leq C \lambda \delta \operatorname{ess\,sup}_{0 < t < \lambda} \sum_{\kappa' \sim \kappa} \int_{B_1(\kappa')} (|v(x, t)|^2 + |b(x, t)|^2) \, dx \\
&\quad + C \delta \operatorname{ess\,sup}_{0 < t < \lambda} \sum_{\kappa' \sim \kappa} \int_0^\lambda \int_{B_1(\kappa')} (|\nabla v(x, t)|^2 + |\nabla b(x, t)|^2) \, dx dt.
\end{aligned}$$

The pressure can be decomposed into local and far-field contributions. The local part contains new terms that are handled exactly as in the previous estimates, once the Calderon–Zygmund inequality is applied. The far-field pressure splits as $\pi_{\text{far}} = \pi_{\text{far},(v,b)} + \pi_{\text{far},(u,a)}$, where $\pi_{\text{far},(v,b)}$ matches the far-field term treated in the proof of Lemma 3.7, and $\pi_{\text{far},(u,a)}$ is the remaining contribution. Since the estimate for $\pi_{\text{far},(v,b)}$ is already established in the proof of Lemma 3.7, we focus on bounding $\pi_{\text{far},(u,a)}$ in $B_2(\kappa) \times (0, T)$. Specifically, we have

$$\begin{aligned} |\pi_{\text{far},(u,a)}(x, t)| &\leq C \int \frac{1}{|\kappa - y|^4} (|v(y, t)| |u(y, t)| + |b(y, t)| |a(y, t)|) (1 - \chi_4(y - \kappa)) dy \\ &\leq C \delta \tilde{K} * e_\lambda^{1/2}(\kappa), \end{aligned}$$

where \tilde{K} is defined in (3.38), and

$$e_\lambda(\kappa) = \text{ess sup}_{0 \leq t \leq \lambda} \int_{B_1(\kappa)} (|v(x, t)|^2 + |b(x, t)|^2) dx + \int_0^\lambda \int_{B_1(\kappa)} (|\nabla v(x, t)|^2 + |\nabla b(x, t)|^2) dx dt.$$

This yields the estimate:

$$\begin{aligned} &\int_0^\lambda \int \pi_{\text{far},(u,a)}(x, s) v(x, s) \cdot \nabla \phi(x - \kappa) dx ds \\ &\leq C \int_0^\lambda \int_{B_2(\kappa)} \delta^{1/2} (\tilde{K} * e_\lambda^{1/2}) \delta^{1/2} |v| dx ds \\ &\leq C \delta \int_0^\lambda \int_{B_2(\kappa)} (\tilde{K} * e_\lambda^{1/2})^2 dx ds + \delta \int_0^\lambda \int_{B_2(\kappa)} |v|^2 dx ds \\ &\leq C \delta \lambda \left((\tilde{K} * e_\lambda^{1/2})(\kappa) \right)^2 + \delta \lambda \text{ess sup}_{0 < t < \lambda} \sum_{|\kappa - \kappa'| < 4} \int_{B_1(\kappa')} |v|^2 dx. \end{aligned}$$

Combining the above estimates with the argument in the proof of Lemma 3.7 (see (3.24)), we obtain

$$\begin{aligned} e_\lambda(\kappa) &\leq \int (|v_0|^2 + |b_0|^2) \phi(x - \kappa) dx + C \lambda \sum_{\kappa' \in \mathbb{Z}^3; |\kappa' - \kappa| \leq 2} e_\lambda(\kappa') \\ &\quad + C \lambda^{1/4} \sum_{\kappa' \in \mathbb{Z}^3; |\kappa' - \kappa| \leq 10} (e_\lambda(\kappa'))^{3/2} + C \lambda^{1/4} \left((\tilde{K} * e_\lambda)(\kappa) \right)^{3/2} \\ &\quad + C \delta \sum_{|\kappa - \kappa'| < 10} e_\lambda(\kappa') + C \delta \lambda^{1/4} \left((\tilde{K} * e_\lambda^{1/2})(\kappa) \right)^2, \end{aligned} \tag{3.53}$$

where we are using $\lambda \leq \lambda_0 \leq 1$. The first two lines above coincide exactly with the estimates in the proof of Lemma 3.7, so we focus on the two additional terms in the last line. To control the final term, we apply the $\ell^{q/2}$ norm:

$$\left\| \left((\tilde{K} * e_\lambda^{1/2})(\kappa) \right)^2 \right\|_{\ell^{q/q}}^2 = \left\| (\tilde{K} * e_\lambda^{1/2})(\kappa) \right\|_{\ell^{q/q}}^2 \leq C \left\| \tilde{K} \right\|_{\ell^1} \|e_\lambda\|_{\ell^{q/2}}.$$

We now choose c_0 (which bounds δ) sufficiently small so that, after taking the $\ell^{q/2}$ norm of both sides of the inequality, the δ -weighted terms on the right can be absorbed into the left-hand side. With this absorption, the remaining terms are exactly as in the proof of Lemma 3.7, and the conclusion follows by the same argument. \square

Lemma 3.11. *Let $\epsilon, \delta > 0$ be given and assume $\delta \leq c_0$, where c_0 is given in Lemma 3.11. Assume that $v_0, b_0 \in L^2$ are divergence free and that $u, a : \mathbb{R}^3 \times [0, T_0] \rightarrow \mathbb{R}^3$ are divergence free that satisfy*

$$\operatorname{ess\,sup}_{0 < t \leq T_0} \left(\|u(t)\|_{L^3_{\text{uloc}}} + \|b(t)\|_{L^3_{\text{uloc}}} \right) < \delta.$$

Then there exist $T \in (0, T_0]$ and a weak solution (v_ϵ, b_ϵ) and pressure π_ϵ to (3.34) on $\mathbb{R}^3 \times [0, T]$. Furthermore, we have that $(v_\epsilon, b_\epsilon) \in L^\infty(0, T; E_q^2 \times E_q^2)$ and satisfies

$$\left\| \operatorname{ess\,sup}_{0 \leq t \leq T_0} \int_{B_1(x_0)} \frac{|v_\epsilon|^2 + |b_\epsilon|^2}{2} dx + \int_0^{T_0} \int_{B_1(x_0)} (|\nabla v_\epsilon|^2 + |\nabla b_\epsilon|^2) dx dt \right\|_{\ell^{\frac{4}{3}}(x_0 \in \mathbb{Z}^3)}^2 \leq 2C \|(v_0, b_0)\|_{E_q^2 \times E_q^2}$$

for some positive constant C independent of $\epsilon, \delta, (u, a)$ and (v_0, b_0) . Here, $T = \min(T_0, \lambda_0, \lambda_0 A_{0,q}^{-2})$ depends on $\|(v_0, b_0)\|_{E_q^2 \times E_q^2}$ but not on $\|(v_0, b_0)\|_{L^2 \times L^2}$, (v_ϵ, b_ϵ) , ϵ, δ , or (u, a) .

Proof. The proof of the lemma is an adaption of the proof of [4, Lemma 4.3] for the Navier–Stokes equations to the MHD equations.

Let $(v_\epsilon, b_\epsilon, \pi_\epsilon)$ be a smooth solution of (3.34) on $\mathbb{R}^3 \times [0, T_0]$ with initial data $(v_0, b_0) \in L^2$. The energy equality for the regularized perturbed problem reads:

$$\begin{aligned} & \|v_\epsilon(t)\|_{L^2}^2 + \|b_\epsilon(t)\|_{L^2}^2 + 2 \int_0^t \int_0^s (|\nabla v_\epsilon|^2 + |\nabla b_\epsilon|^2) dx ds \\ &= \|v_0\|_{L^2}^2 + \|b_0\|_{L^2}^2 + 2 \int_0^t \int [(v_\epsilon \cdot \nabla v_\epsilon - b_\epsilon \cdot \nabla b_\epsilon) \cdot (\eta_\epsilon * u) + (v_\epsilon \cdot \nabla b_\epsilon - b_\epsilon \cdot \nabla v_\epsilon) \cdot (\eta_\epsilon * a)] dx ds \end{aligned}$$

By the estimate from [27, p. 217], the right-hand side is uniformly bounded in ϵ , implying that $\|(v_\epsilon, b_\epsilon)(t)\|_{L^2 \times L^2} \leq M_1$ for all $t \in [0, T_0]$ for some constant M_1 independent of ϵ . Furthermore, if $(v_\epsilon, b_\epsilon) \in \mathbf{LE}_q(0, T_0)$, then by Lemma 3.10, the local energy estimates extend up to time $T = \min(T_0, \lambda_0, \lambda_0 A_{0,q}^{-2})$, yielding $\|(v_\epsilon, b_\epsilon)\|_{\mathbf{LE}_q(0, T_0)} < M_2$ for some constant M_2 independent of ϵ .

Now, let $T_{\epsilon, \delta}$ be the time-scale provided in Lemma 3.9 corresponding to initial data of size M_1 in L^2 . Then, Lemma 3.9 ensures that the solution (v_ϵ, b_ϵ) exists on $\mathbb{R}^3 \times [0, T_{\epsilon, \delta}]$ and belongs to $\mathbf{LE}_q(0, T_{\epsilon, \delta})$. Since Lemma 3.10 also applies to the regularized system, we further conclude that $\|(v_\epsilon, b_\epsilon)\|_{\mathbf{LE}_q(0, T_{\epsilon, \delta})} < M_2$ and hence $\operatorname{ess\,sup}_{0 < t \leq T_{\epsilon, \delta}} \|(v, b)(t)\|_{E_q^2 \times E_q^2} \leq M_2$. In addition, the energy estimate gives $\operatorname{ess\,sup}_{0 < t \leq T_{\epsilon, \delta}} \|(v, b)(t)\|_{L^2 \times L^2} \leq M_1$. This allows us to restart the solution at any time $t_* \in [T_{\epsilon, \delta}/2, 3T_{\epsilon, \delta}/4]$, and apply Lemma 3.10 again with the same bounds. By uniqueness, the extended solution coincides with the original one, and hence we obtain a solution on $[0, 3T_{\epsilon, \delta}/2]$ that remains in $\mathbf{LE}_q(0, 3T_{\epsilon, \delta}/2)$. Repeating this argument and iterating the solution step-by-step, we reach the full time interval $[0, T]$. Throughout the iteration, the \mathbf{LE}_q and L^2 norms remain bounded uniformly by M_2 and M_1 , respectively. Therefore, for each $\epsilon > 0$, the solution (v_ϵ, b_ϵ) to the regularized system exists on $[0, T]$ and satisfies $(v_\epsilon, b_\epsilon) \in \mathbf{LE}_q(0, T)$ with bounds independent of ϵ . \square

Lemma 3.12. *Let c_0 and λ_0 be the constants in Lemma 3.10. Assume that $v_0, b_0 \in E_q^2$ are divergence free and that $u, a : \mathbb{R}^3 \times [0, T_0] \rightarrow \mathbb{R}^3$ are divergence free and satisfy*

$$\operatorname{ess\,sup}_{0 < t \leq T_0} \left(\|u(t)\|_{L^3_{\text{uloc}}} + \|a(t)\|_{L^3_{\text{uloc}}} \right) < \delta \leq c_0 \quad \text{and} \quad \operatorname{ess\,sup}_{0 < t \leq T_0} \left(\|u(t)\|_{L^4_{\text{uloc}}} + \|a(t)\|_{L^4_{\text{uloc}}} \right) < \infty.$$

Let $T = \min(T_0, \lambda_0, \lambda_0 A_{0,q}^{-2})$. Then there exist a local energy solution (v, b) and π to the perturbed MHD equations, (3.30), satisfying

$$\|(v, b)\|_{\mathbf{LE}_q(0,T)} \leq C \|(v_0, b_0)\|_{E_q^2 \times E_q^2}$$

for some constant $C > 0$.

Proof. The proof of the lemma is an adaption of the proof of [4, Lemma 4.4] for the Navier–Stokes equations to the MHD equations.

Fix $(v_0, b_0) \in E_q^2 \times E_q^2$. For each $\epsilon > 0$, approximate the data by divergence-free vector fields $(v_0^{(\epsilon)}, b_0^{(\epsilon)}) \in L^2 \times L^2$ satisfying $\|v_0 - v_0^{(\epsilon)}\|_{E_q^2} + \|b_0 - b_0^{(\epsilon)}\|_{E_q^2} < \epsilon$. Such approximations can be constructed using the Bogovskii map (see [40]). Let (v_ϵ, b_ϵ) denote the solutions constructed in Lemma 3.12 corresponding to the initial data $(v_0^{(\epsilon)}, b_0^{(\epsilon)})$. By the uniform estimates from Lemma 3.12, we obtain bounds on $\partial_t v_\epsilon$ and $\partial_t b_\epsilon$ in the dual of $L^3(0, T; W_0^{1,3}(B_M(0)))$, which allow us to extract a subsequence $(v_n, b_n) := (v_{\epsilon_n}, b_{\epsilon_n})$ of (v_ϵ, b_ϵ) and $\pi_n := \pi_{\epsilon_n}$ of π_ϵ , such that, as $n \rightarrow \infty$,

$$\begin{aligned} (v_n, b_n) &\overset{*}{\rightharpoonup} (v, b) \quad \text{in } L^\infty(0, T; L_{\text{loc}}^2 \times L_{\text{loc}}^2), \\ (v_n, b_n) &\rightharpoonup (v, b) \quad \text{in } L^2(0, T; H_{\text{loc}}^1 \times H_{\text{loc}}^1), \\ (v_n, b_n), (\eta_{\epsilon_n} * v_n, \eta_{\epsilon_n} * b_n) &\rightarrow (v, b) \quad \text{in } L^3(0, T; L_{\text{loc}}^3 \times L_{\text{loc}}^3), \\ (\eta_{\epsilon_n} * u, \eta_{\epsilon_n} * a) &\rightarrow (u, a) \quad \text{in } L^3(0, T; L_{\text{loc}}^3 \times L_{\text{loc}}^3), \\ \pi_n^{(k)} &\rightharpoonup \pi^{(k)} \quad \text{in } L^{3/2}(0, T; L^{3/2}(B_k(0))), \end{aligned}$$

where $\pi^{(k)}(x, t) = \pi(x, t) - c_k(t)$ for $x \in B_k(0)$ and $t \in (0, T_0]$ for some $c_k \in L^{3/2}(0, T_0)$, and $\pi_n^{(k)}$ is the local pressure expansion for π_n on ball $B_k(0)$. The limit (v, b, π) is a local energy solution to the perturbed MHD equations with initial data (v_0, b_0) . We claim that (v, b, π) satisfies the *perturbed* local energy inequality: for all nonnegative $\phi \in C_c^\infty(\mathbb{R}^3 \times [0, T))$,

$$\begin{aligned} &2 \iint (|\nabla v|^2 + |\nabla b|^2) \phi \, dx dt \\ &\leq \int (|v_0|^2 + |b_0|^2) \phi \, dx + \iint (|v|^2 + |b|^2) (\partial_t \phi + \Delta \phi) \, dx dt + \iint (|v|^2 + |b|^2 + 2\pi) (v \cdot \nabla \phi) \, dx dt \\ &\quad + \iint (|v|^2 + |b|^2) (u \cdot \nabla \phi) \, dx dt + 2 \iint (v \cdot \nabla v - b \cdot \nabla b) \cdot (u \phi) \, dx dt \\ &\quad + 2 \iint (v \cdot u + b \cdot a) (v \cdot \nabla \phi) \, dx dt + 2 \iint (v \cdot \nabla b - b \cdot \nabla v) \cdot (a \phi) \, dx dt \\ &\quad - 2 \iint (v \cdot a + b \cdot u) (b \cdot \nabla \phi) \, dx dt - 2 \iint (v \cdot b) ((b + a) \cdot \nabla \phi) \, dx dt. \end{aligned} \tag{3.54}$$

The first two lines are inherited via standard compactness arguments. We now focus on the convergence of the remaining terms, especially those not involving $\nabla \phi$, which are of higher order. We

have

$$\begin{aligned}
& \left| \iint (v_n \cdot \nabla v_n) \cdot (\eta_{\epsilon_n} * u) \phi - (v \cdot \nabla v) \cdot u \phi \, dx dt \right| \\
& \leq \left| \iint ((v_n - v) \cdot \nabla v_n) \cdot (u \phi) \, dx dt \right| + \left| \iint (v \cdot \nabla (v_n - v)) \cdot (u \phi) \, dx dt \right| \\
& \quad + \left| \iint (v_n \cdot \nabla v_n) \cdot (\eta_{\epsilon_n} * u - u) \phi \, dx dt \right| \\
& =: I_{1,n} + I_{2,n} + I_{3,n},
\end{aligned} \tag{3.55}$$

$$\begin{aligned}
& \left| \iint (b_n \cdot \nabla b_n) \cdot (\eta_{\epsilon_n} * u) \phi - (b \cdot \nabla b) \cdot u \phi \, dx dt \right| \\
& \leq \left| \iint ((b_n - b) \cdot \nabla b_n) \cdot (u \phi) \, dx dt \right| + \left| \iint (b \cdot \nabla (b_n - b)) \cdot (u \phi) \, dx dt \right| \\
& \quad + \left| \iint (b_n \cdot \nabla b_n) \cdot (\eta_{\epsilon_n} * u - u) \phi \, dx dt \right| \\
& =: I_{4,n} + I_{5,n} + I_{6,n},
\end{aligned} \tag{3.56}$$

$$\begin{aligned}
& \left| \iint (v_n \cdot \nabla b_n) \cdot (\eta_{\epsilon_n} * a) \phi - (v \cdot \nabla b) \cdot a \phi \, dx dt \right| \\
& \leq \left| \iint ((v_n - v) \cdot \nabla b_n) \cdot (a \phi) \, dx dt \right| + \left| \iint (v \cdot \nabla (b_n - b)) \cdot (a \phi) \, dx dt \right| \\
& \quad + \left| \iint (v_n \cdot \nabla b_n) \cdot (\eta_{\epsilon_n} * a - a) \phi \, dx dt \right| \\
& =: I_{7,n} + I_{8,n} + I_{9,n},
\end{aligned} \tag{3.57}$$

and

$$\begin{aligned}
& \left| \iint (b_n \cdot \nabla v_n) \cdot (\eta_{\epsilon_n} * a) \phi - (b \cdot \nabla v) \cdot a \phi \, dx dt \right| \\
& \leq \left| \iint ((b_n - b) \cdot \nabla v_n) \cdot (a \phi) \, dx dt \right| + \left| \iint (b \cdot \nabla (v_n - v)) \cdot (a \phi) \, dx dt \right| \\
& \quad + \left| \iint (b_n \cdot \nabla v_n) \cdot (\eta_{\epsilon_n} * a - a) \phi \, dx dt \right| \\
& =: I_{10,n} + I_{11,n} + I_{12,n}.
\end{aligned} \tag{3.58}$$

Our goal is to show that the above twelve quantities vanish as $n \rightarrow \infty$. Let B be a ball containing $\text{supp } \phi$. Then, using Hölder's inequality and log-convexity of L^p norms, we have

$$I_{1,n} \lesssim \|u\|_{L^\infty L^4_{\text{uloc}}} \|v_n - v\|_{L^2(0,T;L^2(B))}^{1/4} \|v_n\|_{L^2(0,T;H^1(B))}^{7/4} \rightarrow 0,$$

$$I_{4,n} \lesssim \|u\|_{L^\infty L^4_{\text{uloc}}} \|b_n - b\|_{L^2(0,T;L^2(B))}^{1/4} \|b_n\|_{L^2(0,T;H^1(B))}^{7/4} \rightarrow 0,$$

$$I_{7,n} \lesssim \|a\|_{L^\infty L^4_{\text{uloc}}} \|v_n - v\|_{L^2(0,T;L^2(B))}^{1/4} \|b_n\|_{L^2(0,T;H^1(B))}^{7/4} \rightarrow 0,$$

and

$$I_{10,n} \lesssim \|a\|_{L^\infty L^4_{\text{uloc}}} \|b_n - b\|_{L^2(0,T;L^2(B))}^{1/4} \|v_n\|_{L^2(0,T;H^1(B))}^{7/4} \rightarrow 0,$$

as $n \rightarrow \infty$ by strong convergence of (v_n, b_n) to (v, b) in $L^2(0, T; L^2(B) \times L^2(B))$. Next, weak convergence of (v_n, b_n) to (v, b) in $L^2(0, T; H^1(B) \times H^1(B))$ ensures that $I_{2,n}, I_{5,n}, I_{8,n}, I_{11,n} \rightarrow 0$ as $n \rightarrow \infty$, since the products $v_i u_j \phi, b_i u_j \phi, v_i a_j \phi$, and $b_i a_j \phi$ all belong to $L^2(B \times (0, T))$. Finally, for the mollifier terms, we use strong convergence of the mollified quantities in $L^\infty(0, T; L^3(B))$ and uniform bounds on v_n, b_n in $L^2(0, T; H^1(B))$, to deduce $I_{3,n}, I_{6,n}, I_{9,n}$, and $I_{12,n} \rightarrow 0$ as $n \rightarrow \infty$. Hence, all error terms vanish in the limit, and (3.54) holds. Moreover, following the argument in [23, (3.28)-(3.29)], we derive the time-slice version of the perturbed local energy inequality: for any nonnegative $\psi \in C_c^\infty(\mathbb{R}^3)$ and any $t \in (0, T)$,

$$\begin{aligned}
& \int (|v(t)|^2 + |b(t)|^2) \psi \, dx + 2 \int_0^t \int (|\nabla v|^2 + |\nabla b|^2) \psi \, dx dt \\
& \leq \int (|v_0|^2 + |b_0|^2) \psi \, dx + \int_0^t \int (|v|^2 + |b|^2) \Delta \psi \, dx dt + \iint (|v|^2 + |b|^2 + 2\pi) (v \cdot \nabla \psi) \, dx dt \\
& \quad + \int_0^t \int (|v|^2 + |b|^2) (u \cdot \nabla \psi) \, dx dt + 2 \int_0^t \int (v \cdot \nabla v - b \cdot \nabla b) \cdot (u \psi) \, dx dt \\
& \quad + 2 \iint (v \cdot u + b \cdot a) (v \cdot \nabla \psi) \, dx dt + 2 \int_0^t \int (v \cdot \nabla b - b \cdot \nabla v) \cdot (a \psi) \, dx dt \\
& \quad - 2 \int_0^t \int (v \cdot a + b \cdot u) (b \cdot \nabla \psi) \, dx dt - 2 \int_0^t \int (v \cdot b) ((b + a) \cdot \nabla \psi) \, dx dt.
\end{aligned} \tag{3.59}$$

We now establish the bound for $\|(v, b)\|_{\mathbf{LE}_q(0, T)}$. Let ϕ the cutoff function used in the proof of Lemma 3.7, and define $\psi(x) := \phi(x - k)$ for each $k \in \mathbb{Z}^3$. Fix a large integer $K > 0$, and restrict to $|k| \leq K$. Applying (3.59) with this choice of ψ , the right-hand side can be approximated by the corresponding terms for the sequence $\{(v_n, b_n)\}$, since all such quantities converge in the limit. In particular, for all $|k| \leq K$, we can ensure that the difference between the terms involving (v, b) and those for (v_n, b_n) is less than $2^{-K} K^{-3}$ uniformly, provided $n \geq N_K$ for some sufficiently large N_K . Taking these approximate terms, applying standard estimates (which can be derived similar to the proof of (3.19))

$$\left\| \int_0^{\lambda_R R^2} \int_{B_R(x_0 R)} |v|^3 + |b|^3 + |\pi - c_{R x_0, R}(t)|^{3/2} \, dx dt \right\|_{\ell^{\frac{q}{3}}(x_0 \in \mathbb{Z}^3)} \leq C \lambda_R^{\frac{1}{10}} R^{\frac{1}{2}} (A_{0,q}(R))^{\frac{3}{2}}, \quad R > 0. \tag{3.60}$$

Taking the essential supremum in time followed by the $\ell^{q/2}$ -sum over $|k| \leq K$, we obtain a uniform bound

$$\begin{aligned}
& \left[\sum_{|k| \leq K} \left(\operatorname{ess\,sup}_{0 < t < T} \int_{B_1(k)} (|v(x, t)|^2 + |b(x, t)|^2) \, dx + \int_0^T \int_{B_1(k)} (|\nabla v|^2 + |\nabla b|^2) \, dx dt \right)^{q/2} \right]^{1/q} \\
& \leq C \left(\|(v_0, b_0)\|_{E_q^2 \times E_q^2} + \frac{C}{2K} \right).
\end{aligned}$$

Since this bound holds for all $K \in \mathbb{N}$, it follows that $(v, b) \in \mathbf{LE}_q(0, T)$, with the norm estimate $\|(v, b)\|_{\mathbf{LE}_q(0, T)} \leq C \|(v_0, b_0)\|_{E_q^2 \times E_q^2}$. \square

Lemma 3.13. *Let $2 \leq q < \infty$. Assume $v_0, b_0 \in L^2$ are divergence free, and assume $u, a : \mathbb{R}^3 \times [0, T_0] \rightarrow \mathbb{R}^3$ are divergence free and satisfy*

$$\operatorname{ess\,sup}_{0 < t \leq T_0} \left(\|u(t)\|_{L^3_{\text{uloc}}} + \|a(t)\|_{L^3_{\text{uloc}}} \right) < \delta \leq c_0 \quad \text{and} \quad \operatorname{ess\,sup}_{0 < t \leq T_0} \left(\|u(t)\|_{L^4_{\text{uloc}}} + \|a(t)\|_{L^4_{\text{uloc}}} \right) < \infty.$$

where c_0 is from Lemma 3.10. For any $T \in (0, T_0]$, if $\delta \leq \delta_0(T) \leq c_0$ is sufficiently small, then there exists a local energy solution (v, b) to the perturbed MHD equations, (3.30), so that $(v, b) \in \mathbf{LE}_q(0, T)$. In particular, this is true when $(u, a) \equiv (0, 0)$.

Proof. The proof of the lemma is an adaption of the proof of [4, Lemma 5.1] for the Navier–Stokes equations to the MHD equations.

We begin with the special case $(u, a) = (0, 0)$ to highlight the key ideas. Suppose the initial data $(v_0, b_0) \in L^2 \times L^2$, and define $a_k = \int_{B_1(k)} (|v_0|^2 + |b_0|^2) dx$ for $k \in \mathbb{Z}^3$. Then,

$$\sum_k a_k^{q/2} \leq (\max_k a_k)^{q/2-1} \sum_k a_k \leq \left(\sum_k a_k \right)^{q/2},$$

which shows that $(v_0, b_0) \in E_q^2 \times E_q^2$ with $\|(v_0, b_0)\|_{E_q^2 \times E_q^2} \leq M_2 := C \|(v_0, b_0)\|_{L^2 \times L^2}$. Let (v, b) be the solution of the perturbed MHD equations with $(u, a) = (0, 0)$ (so that it is a solution of (MHD)) constructed via Lemma 3.12 with initial data (v_0, b_0) . Then $(v, b) \in \mathbf{LE}_q(0, T_0)$, where $T_0 = T_0(M_2)$ is the existence time depending on the size of the initial data in $E_q^2 \times E_q^2$. For almost every $t \in (0, T)$, we have that $(v, b)(t) \in E^3 \times E^3$ and that $\|(v, b)(t)\|_{L^2 \times L^2} \leq \|(v_0, b_0)\|_{L^2 \times L^2}$. The inclusion $(v, b)(t) \in E^3$ follows from Lemma 3.8 and the embedding $E_q^4 \subset E^3$. Hence, $\|(v, b)(t)\|_{E_q^2 \times E_q^2} \leq M_2$ for almost every $t \in (0, T_0)$. In particular, these bounds hold at some time $t_0 \in (T_0/2, T_0)$. We now restart the MHD equations at time t_0 , treating $(v, b)(t_0)$ as new initial data in $E^3 \times E^3$. By Lemma 3.12, there exists a local energy solution (v_1, b_1) in $\mathbf{LE}_q(t_0, t_0 + T_0)$. By uniqueness of local energy solution with $E^3 \times E^3$ (Corollary 3.5), we have $(v_1, b_1) = (v, b)$ on some short interval $[t_0, t_0 + \Delta t_1]$. This allows us to glue (v_1, b_1) to (v, b) , yielding a local energy solution (still denoted (v, b)) on $[0, 3T_0/2]$ that lies in $\mathbf{LE}_q(t_0, t_0 + T_0) \cap \mathbf{LE}_q(0, T_0)$. Hence, $(v, b) \in \mathbf{LE}_q(0, 3T_0/2)$. Repeating this argument, we obtain a solution $(v, b) \in \mathbf{LE}_q(0, T)$ for any $T > 0$, using the uniform-in-time control of $\|(v, b)\|_{E_q^2 \times E_q^2}$.

Now consider the general case $(u, a) \neq (0, 0)$. Let $(v_0, b_0) \in L^2 \times L^2 \subset E_q^2 \times E_q^2$, and let (v, b) be the local energy solution to the perturbed MHD equations, (3.30), given by Lemma 3.12. Assuming $\delta := \|(u, a)\|_{L^\infty(L^3_{\text{uloc}} \times L^3_{\text{uloc}})} \ll c_0$, we have $(v, b) \in \mathbf{LE}_q(0, T_0)$ for some $T_0 = T_0(\|(v_0, b_0)\|_{E_q^2 \times E_q^2})$. We now derive an energy estimate. Using the following bounds:

$$\begin{aligned} \int_{\mathbb{R}^3} |u||v|(|\nabla v| + |v|) &\leq \sum_k \int_{B_1(k)} |u||v|(|\nabla v| + |v|) \lesssim \|u\|_{L^3_{\text{uloc}}} \sum_k \int_{B_1(k)} (|\nabla v|^2 + |v|^2), \\ \int_{\mathbb{R}^3} |u||b|(|\nabla b| + |b|) &\leq \sum_k \int_{B_1(k)} |u||b|(|\nabla b| + |b|) \lesssim \|u\|_{L^3_{\text{uloc}}} \sum_k \int_{B_1(k)} (|\nabla b|^2 + |b|^2), \\ \int_{\mathbb{R}^3} |a||v|(|\nabla b| + |b|) &\leq \sum_k \int_{B_1(k)} |a||v|(|\nabla b| + |b|) \lesssim \|a\|_{L^3_{\text{uloc}}} \sum_k \int_{B_1(k)} (|\nabla b|^2 + |b|^2 + |v|^2), \end{aligned}$$

and

$$\int_{\mathbb{R}^3} |a||b|(|\nabla v| + |v|) \leq \sum_k \int_{B_1(k)} |a||b|(|\nabla v| + |v|) \lesssim \|a\|_{L^3_{\text{uloc}}} \sum_k \int_{B_1(k)} (|\nabla v|^2 + |v|^2 + |b|^2),$$

we obtain the energy inequality:

$$\begin{aligned}
& \| (v, b)(t) \|_{L^2 \times L^2}^2 + 2 \| (\nabla v, \nabla b) \|_{L^2(0, t; L^2 \times L^2)}^2 \\
& \leq \| (v_0, b_0) \|_{L^2 \times L^2}^2 \\
& + C \| (u, a) \|_{L^\infty(L_{\text{uloc}}^3 \times L_{\text{uloc}}^3)} \left(\| (\nabla v, \nabla b) \|_{L^2(0, t; L^2 \times L^2)}^2 + t \sup_{0 < s < t} \| (v, b)(s) \|_{L^2 \times L^2}^2 \right),
\end{aligned} \tag{3.61}$$

for $0 < t < T$. If $T \leq T_0$, the result follows. Otherwise, we choose δ sufficiently small (depending on T) to absorb the right-hand side, yielding:

$$\sup_{0 < t < T} \| (v, b)(t) \|_{L^2 \times L^2}^2 + 2 \| (\nabla v, \nabla b) \|_{L^2(0, T; L^2 \times L^2)}^2 \leq 2 \| (v_0, b_0) \|_{L^2 \times L^2}^2.$$

This in turn implies

$$\sup_{0 < t < T} \| (v, b)(t) \|_{E_q^2 \times E_q^2} \leq C \| (v_0, b_0) \|_{L^2 \times L^2}.$$

Thus, we obtain uniform-in-time control of $\| (v, b)(t) \|_{E_q^2 \times E_q^2}$ and the argument from the $(u, a) = (0, 0)$ case applies to yield the desired result. \square

Lemma 3.14. *Suppose that $v_0, b_0 \in E_\infty^4$ are divergence free. Assume also that $\delta := \| (v_0, b_0) \|_{E_\infty^4 \times E_\infty^4} < \epsilon_*$ for a universal constant ϵ_* . Then there exists a second universal constant $\tau_0 > 0$ and (v, b) and π comprising a local energy solution to (MHD) in $\mathbb{R}^3 \times (0, \tau_0)$ with initial data (v_0, b_0) so that (v, b) and π are smooth in space and time, $(v, b) \in C([0, \tau_0]; E_\infty^4 \times E_\infty^4)$, and*

$$\sup_{0 \leq t \leq \tau_0} \| (v, b)(t) \|_{L_{\text{uloc}}^4 \times L_{\text{uloc}}^4} < C\delta.$$

Furthermore, if $(u, a) \in \mathcal{N}_{\text{MHD}}(v_0, b_0)$, then $(u, a) = (v, b)$ on $\mathbb{R}^3 \times [0, \tau_0]$.

Proof. The proof of the lemma is an adaption of the proof of [4, Lemma A.1] for the Navier–Stokes equations to the MHD equations.

Since $(v_0, b_0) \in E_\infty^4 \times E_\infty^4$, it follows that $(v_0, b_0) \in E^2 \times E^2$ as $E_\infty^4 \subset E^2$. Viewing the MHD equations as a coupled system of inhomogeneous Stokes systems, we apply the linear theory from [21, §5] and follow the argument of Theorem 1.5 therein to construct a global-in-time local energy solution (v, b) evolving from (v_0, b_0) . We may assume $\| (v_0, b_0) \|_{L_{\text{uloc}}^3 \times L_{\text{uloc}}^3} < \epsilon_3$, where ϵ_3 is given in Theorem 3.4. Then, for all $x_0 \in \mathbb{R}^3$ and $r \leq 1$,

$$\frac{1}{r} \int_{B_r(x_0)} (|v_0|^2 + |b_0|^2) dx \leq C \left(\int_{B_r(x_0)} (|v_0|^3 + |b_0|^3) dx \right) \leq C\epsilon_3.$$

Thus, Theorem 3.4 ensures uniqueness of the local energy solution $(v, b) \in \mathcal{N}_{\text{MHD}}(v_0, b_0)$ up to some time τ_0 .

Now consider the mild solution constructed in Theorem 1.1 with $r = 4, q = \infty$. Since $E^4 \subset \mathcal{L}_{\text{uloc}}^4$, the closure of $BUC(\mathbb{R}^3)$ in the L_{uloc}^4 norm, there exists a time $T > 0$ and a unique mild solution $(u, a) \in C([0, T]; L_{\text{uloc}}^4)$. In the construction of this mild solution (see Section 2.1), we may redefine the space \mathcal{E}_T with the norm $\| (v, b) \|_{\mathcal{E}_T} = \sup_{0 < t < T} \| (v, b)(t) \|_{E_q^r} + \sup_{0 < t < T} t^{3/(2r)} \| (v, b)(t) \|_{L^\infty}$ to ensure that the mild solution $(u(t), a(t)) \in L^\infty \times L^\infty$ for all $t > 0$. Then, by adapting the regularity argument from [14, §4], we conclude that (u, a) is smooth in both space and time for all $t > 0$. By choosing ϵ_* sufficiently small, the existence time T for the mild solution in Theorem 1.1 exceeds τ_0 .

We now verify that (u, a) defines a local energy solution. By embeddings, the same convergence properties at $t = 0$ hold with L^4 and L^4_{uloc} replaced by L^2 and L^2_{uloc} , respectively. This implies that if $w \in L^2(\mathbb{R}^3)$ is compactly supported, then

$$\lim_{t \rightarrow 0} \int (u(x, t) - u_0(x))w(x) dx = \lim_{t \rightarrow 0} \int (a(x, t) - a_0(x))w(x) dx = 0.$$

Moreover, the maps

$$t \mapsto \int u(x, t) \cdot w(x) dx \quad \text{and} \quad t \mapsto \int a(x, t) \cdot w(x) dx$$

are continuous for $t > 0$ due to the smoothness of (u, a) .

To complete the verification, we adapt the pressure construction from [5, Theorem 1.4], as carried out in [5, §6] for the Navier–Stokes equations. This yields a pressure π such that (u, a, π) satisfies the MHD equations in the distributional sense. The local expansion of π also guarantees that $\pi \in L^{3/2}_{\text{loc}}(\mathbb{R}^3 \times (0, T))$. The local energy inequality follows from the space-time smoothness of (u, a) , and item 2 in the definition of local energy solution is satisfied since $(u, a) \in L^\infty L^3_{\text{uloc}}$. Hence, $(u, a) \in \mathcal{N}_{\text{MHD}}(v_0, b_0)$, and uniqueness implies $(v, b) = (u, a)$ on $\mathbb{R}^3 \times (0, \tau_0)$. Therefore,

$$\|(v, b)(t)\|_{L^3_{\text{uloc}} \times L^3_{\text{uloc}}} \leq C \|(v, b)(t)\|_{L^4_{\text{uloc}} \times L^4_{\text{uloc}}} < C\delta$$

for all $t \in (0, \tau_0)$.

The mild solution constructed in Theorem 1.1 satisfies $(v, b) \in C([0, \tau_0]; L^4_{\text{uloc}} \times L^4_{\text{uloc}})$. Since $L^4_{\text{uloc}} = E^4_\infty \subset E^2_\infty$, we may apply Lemma 3.8 with $q = \infty$ and data $(v_0, b_0) \in E^4_\infty \times E^4_\infty \subset E^2_\infty \times E^2_\infty$ to conclude that for almost every $t > 0$, $(v, b)(t) \in E^4_\infty \times E^4_\infty$. This further implies $(v, b) \in C([0, \tau_0]; E^4_\infty \times E^4_\infty)$. \square

Proof of Theorem 1.10 for $q \geq 2$. The proof of Theorem 1.10 for $q \geq 2$ is an adaption of the proof of [4, Theorem 1.5] for the Navier–Stokes equations to the MHD equations.

Assume that $v_0, b_0 \in E^2_q$ are divergence free. By Lemma 3.12 with $(u, a) = (0, 0)$, there exists a local energy solution (v, b) to the MHD equations on $\mathbb{R}^3 \times (0, T_0)$ such that $(v, b) \in \mathbf{LE}_q(0, T_0)$. Moreover, by Lemma 3.8, we have

$$(v, b)(t) \in E^4_q \times E^4_q \quad \text{for a.e. } t \in [T_0/2, T_0].$$

Choose a time $t_0 > T_0/2$ so that $(v, b)(t_0) \in E^4_q \times E^4_q$. We aim to construct a local energy solution in $\mathbf{LE}_q(t_0, t_0 + \tau_0)$ with initial data $(v, b)(t_0) \in E^4_q \times E^4_q$, where τ_0 is the fixed time-scale in Lemma 3.14. Using the Bogovskii map (see [40] for the details), for any $\delta > 0$, we can decompose the initial data as

$$(v, b)(t_0) = (u_0, a_0) + (w_0, d_0), \quad \nabla \cdot w_0 = \nabla \cdot d_0 = 0,$$

where

$$\|(u_0, a_0)\|_{E^4_q \times E^4_q} < \delta, \quad w_0, d_0 \in L^2(\mathbb{R}^3).$$

By Lemma 3.14, choosing δ sufficiently small ensures the existence of a local energy solution (u, a) with pressure π , defined on $\mathbb{R}^3 \times (t_0, t_0 + \tau_0)$, evolving from the initial data (u_0, a_0) , which is smooth in both space and time and satisfies

$$\sup_{t_0 \leq t \leq t_0 + \tau_0} \|(u, a)(t)\|_{L^4_{\text{uloc}} \times L^4_{\text{uloc}}} < C\delta.$$

Furthermore, by the uniqueness result in Lemma 3.14, this solution (u, a) coincides with the one given by Lemma 3.12, and hence $(u, a) \in \mathbf{LE}_q(t_0, t_0 + \tau_0)$.

Next, we apply Lemma 3.13 with the perturbation factor (u, a) , again choosing δ sufficiently small to ensure that the time-scale it yields is at least τ_0 . This gives a local energy solution $(w, d) \in \mathbf{LE}_q(t_0, t_0 + \tau_0)$ to the perturbed MHD equation with initial data (w_0, d_0) and associated pressure π . Define $(v_1, b_1) := (u, a) + (w, d)$. This gives a local energy solution on $\mathbb{R}^3 \times (t_0, t_0 + \tau_0)$. To verify the local energy inequality for (v_1, b_1) , we use approximations $(w^{(n)}, d^{(n)}) \rightarrow (w, d)$, as in the proof of Lemma 3.12, and apply the inequality to $(u, a) + (w^{(n)}, d^{(n)})$.

Since (v, b) and (v_1, b_1) coincide at t_0 , and since $(v, b)(t_0) \in E^3 \times E^3$ since $E_\infty^4 \subset E^3$, Corollary 3.5 implies that $(v, b)(x, t) = (v_1, b_1)(x, t)$ on $\mathbb{R}^3 \times (t_0, t_0 + \gamma)$ for some $\gamma > 0$. Thus, we may glue (v_1, b_1) to (v, b) to extend the solution to $\mathbf{LE}_q(0, t_0 + \tau_0)$. Repeating this procedure n times yields a solution $(v, b) \in \mathbf{LE}_q(0, t_0 + n\tau_0)$. Taking the limit $n \rightarrow \infty$, we obtain the global-in-time local energy solution asserted in Theorem 1.10 for $q \geq 2$. \square

3.3.2 The case $1 \leq q < 2$

Now, we consider the case $1 \leq q < 2$ and look at the localized and regularized MHD equations, (3.31). The following lemma corresponds to [1, Lemma 4.6] for the Navier–Stokes equations.

Lemma 3.15. *Let $q \geq 1$. For each $0 < \epsilon < 1$ and divergence free v_0, b_0 with $\|v_0\|_{E_q^2} \leq B$, $\|b_0\|_{E_q^2} \leq B$, if $0 < T < \min(1, c\epsilon^3 B^{-2})$, we can find a unique solution $(v, b) = (v^\epsilon, b^\epsilon)$ to the integral form of (3.31)*

$$\begin{aligned} v(t) &= e^{t\Delta} v_0 - \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot (\mathcal{J}_\epsilon(v) \otimes v \Phi_\epsilon - \mathcal{J}_\epsilon(b) \otimes b \Phi_\epsilon)(\tau) d\tau, \\ b(t) &= e^{t\Delta} b_0 - \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot (\mathcal{J}_\epsilon(v) \otimes b \Phi_\epsilon - \mathcal{J}_\epsilon(b) \otimes v \Phi_\epsilon)(\tau) d\tau, \end{aligned} \quad (3.62)$$

satisfying

$$\|(v, b)\|_{\mathbf{LE}_q(0, T)} \leq 2C_0 B,$$

where $c > 0$ and $C_0 > 1$ are absolute constants.

Proof. The proof of the lemma is an adaption of the proof of [1, Lemma 4.6] for the Navier–Stokes equations to the MHD equations.

Let $\Psi(v, b)$ denote the mapping defined by the right-hand side of (3.62) for $(v, b) \in \mathbf{LE}_q(0, T)$. By [1, Lemma 2.9] and the assumption $T \leq 1$, we obtain the estimate

$$\begin{aligned} \|\Psi(v, b)\|_{\mathbf{LE}_q(0, T)} &\lesssim \|v_0\|_{E_q^2} + \|\mathcal{J}_\epsilon(v) \otimes v \Phi_\epsilon\|_{E_{T,q}^{2,2}} + \|\mathcal{J}_\epsilon(b) \otimes b \Phi_\epsilon\|_{E_{T,q}^{2,2}} \\ &\quad + \|b_0\|_{E_q^2} + \|\mathcal{J}_\epsilon(v) \otimes b \Phi_\epsilon\|_{E_{T,q}^{2,2}} + \|\mathcal{J}_\epsilon(b) \otimes v \Phi_\epsilon\|_{E_{T,q}^{2,2}} \\ &\lesssim \|v_0\|_{E_q^2} + \|\mathcal{J}_\epsilon(v)\|_{L^\infty(0, T; L^\infty(\mathbb{R}^3))} \|v\|_{E_{T,q}^{2,2}} + \|\mathcal{J}_\epsilon(b)\|_{L^\infty(0, T; L^\infty(\mathbb{R}^3))} \|b\|_{E_{T,q}^{2,2}} \\ &\quad + \|b_0\|_{E_q^2} + \|\mathcal{J}_\epsilon(v)\|_{L^\infty(0, T; L^\infty(\mathbb{R}^3))} \|b\|_{E_{T,q}^{2,2}} + \|\mathcal{J}_\epsilon(b)\|_{L^\infty(0, T; L^\infty(\mathbb{R}^3))} \|v\|_{E_{T,q}^{2,2}} \\ &\lesssim \|v_0\|_{E_q^2} + \|b_0\|_{E_q^2} + \epsilon^{-\frac{3}{2}} \sqrt{T} \left(\|v\|_{E_{T,q}^{\infty,2}}^2 + \|b\|_{E_{T,q}^{\infty,2}}^2 \right). \end{aligned}$$

Therefore, for some constants $C_0, C_1 > 0$,

$$\|\Psi(v, b)\|_{\mathbf{LE}_q(0, T)} \leq C_0 \|(v_0, b_0)\|_{E_q^2 \times E_q^2} + C_1 \epsilon^{-\frac{3}{2}} \sqrt{T} \|(v, b)\|_{\mathbf{LE}_q(0, T)}^2,$$

To estimate the difference, let $(v, b), (u, a) \in \mathbf{LE}_q(0, T)$. Then

$$\begin{aligned} & \|\Psi(v, b) - \Psi(u, a)\|_{\mathbf{LE}_q(0, T)} \\ & \leq C_1 \epsilon^{-\frac{3}{2}} \sqrt{T} \left(\|(v, b)\|_{\mathbf{LE}_q(0, T)} + \|(u, a)\|_{\mathbf{LE}_q(0, T)} \right) \|(v, b) - (u, a)\|_{\mathbf{LE}_q(0, T)}. \end{aligned}$$

Applying the Picard contraction principle, we see that if the time T satisfies $T < \frac{\epsilon^3}{64(C_0 C_1 B)^2} = c\epsilon^3 B^{-2}$, then Ψ has a unique fixed point $(v, b) \in \mathbf{LE}_q(0, T)$ with $\|(v, b)\|_{\mathbf{LE}_q(0, T)} \leq 2C_0 B$, solving the integral system (3.62). \square

Lemma 3.16. *Let $v_0, b_0 \in E_q^2$, $q \geq 1$, be divergence free. For each $\epsilon \in (0, 1)$, we can find (v^ϵ, b^ϵ) in $\mathbf{LE}_q(0, T)$ and π^ϵ in $L^\infty(0, T; L^2(\mathbb{R}^3))$ for some positive $T = T(\epsilon, \|(v_0, b_0)\|_{E_q^2 \times E_q^2})$ which solve the localized and regularized MHD equations, (3.31), in the sense of distributions and $(v^\epsilon, b^\epsilon)(t) \rightarrow (v_0, b_0)$ in $L^2(E) \times L^2(E)$ as $t \rightarrow 0^+$ for any compact subset E of \mathbb{R}^3 .*

Proof. The proof of the lemma is an adaption of the proof of [1, Lemma 4.7] for the Navier–Stokes equations to the MHD equations. We provide the corresponding details by following the same logic used in the proof of [23, Lemma 3.4] for the Navier–Stokes equations, adapting it from the L_{uloc}^2 framework to the E_q^2 setting, and from the Navier–Stokes equations to the MHD equations.

By Lemma 3.15, there is a mild solution $(v^\epsilon, b^\epsilon) \in \mathbf{LE}_q(0, T)$ to (3.62) for $T = T(\epsilon, \|(v_0, b_0)\|_{E_q^2 \times E_q^2})$. Apparently,

$$\begin{aligned} \|v^\epsilon - e^{t\Delta} v_0\|_{E_{t',q}^{\infty,2}} &= \left\| \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot (\mathcal{J}_\epsilon(v) \otimes v \Phi_\epsilon - \mathcal{J}_\epsilon(b) \otimes b \Phi_\epsilon)(\tau) d\tau \right\|_{E_{t',q}^{\infty,2}} \\ &\lesssim \|\mathcal{J}_\epsilon(v) \otimes v \Phi_\epsilon\|_{E_{t',q}^{2,2}} + \|\mathcal{J}_\epsilon(b) \otimes b \Phi_\epsilon\|_{E_{t',q}^{2,2}} \\ &\lesssim \epsilon^{-3/2} \sqrt{t'} \left(\|v\|_{E_{t',q}^{\infty,2}}^2 + \|b\|_{E_{t',q}^{\infty,2}}^2 \right), \end{aligned}$$

and

$$\begin{aligned} \|b^\epsilon - e^{t\Delta} b_0\|_{E_{t',q}^{\infty,2}} &= \left\| \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot (\mathcal{J}_\epsilon(v) \otimes b \Phi_\epsilon - \mathcal{J}_\epsilon(b) \otimes v \Phi_\epsilon)(\tau) d\tau \right\|_{E_{t',q}^{\infty,2}} \\ &\lesssim \|\mathcal{J}_\epsilon(v) \otimes b \Phi_\epsilon\|_{E_{t',q}^{2,2}} + \|\mathcal{J}_\epsilon(b) \otimes v \Phi_\epsilon\|_{E_{t',q}^{2,2}} \\ &\lesssim \epsilon^{-3/2} \sqrt{t'} \|vb\|_{E_{t',q}^{\infty,2}} \lesssim \epsilon^{-3/2} \sqrt{t'} \left(\|v\|_{E_{t',q}^{\infty,2}}^2 + \|b\|_{E_{t',q}^{\infty,2}}^2 \right), \end{aligned}$$

where we have used [1, Lemma 2.9] and assumed $t' \leq T \leq 1$. Also, for any compact subset E of \mathbb{R}^3 , we have $\|e^{t\Delta} v_0 - v_0\|_{L^2(E)}, \|e^{t\Delta} b_0 - b_0\|_{L^2(E)} \rightarrow 0$ as t goes to 0 by Legesgue's convergence theorem. Then, it follows that $\lim_{t \rightarrow 0^+} \|v^\epsilon(t) - v_0\|_{L^2(E)} = 0$ and $\lim_{t \rightarrow 0^+} \|b^\epsilon(t) - b_0\|_{L^2(E)} = 0$ for any compact subset E of \mathbb{R}^3 .

Note that both $e^{t\Delta} v_0$ and $e^{t\Delta} b_0$, with $v_0, b_0 \in E_q^2$, solve the homogeneous heat equation in the distribution sense. Also, using $\nabla \cdot v_0 = \nabla \cdot b_0 = 0$, we can easily see that $\nabla \cdot e^{t\Delta} v_0 = \nabla \cdot e^{t\Delta} b_0 = 0$.

On the other hand, $\mathcal{J}_\epsilon(v^\epsilon), \mathcal{J}_\epsilon(b^\epsilon) \in L^\infty(\mathbb{R}^3 \times [0, T])$ and $(v^\epsilon, b^\epsilon) \in \mathbf{LE}_q(0, T)$ imply

$$\mathcal{J}_\epsilon(v^\epsilon) \otimes v^\epsilon \Phi_\epsilon, \quad \mathcal{J}_\epsilon(b^\epsilon) \otimes b^\epsilon \Phi_\epsilon, \quad \mathcal{J}_\epsilon(v^\epsilon) \otimes b^\epsilon \Phi_\epsilon, \quad \mathcal{J}_\epsilon(b^\epsilon) \otimes v^\epsilon \Phi_\epsilon \in L^\infty(0, T; L^2(\mathbb{R}^3)).$$

Hence by the classical theory, $u^\epsilon = v^\epsilon - e^{t\Delta} v_0$ and π^ϵ defined by

$$\pi^\epsilon = (-\Delta)^{-1} \partial_i \partial_j \left[(\mathcal{J}_\epsilon(v_i^\epsilon) v_j^\epsilon - \mathcal{J}_\epsilon(b_i^\epsilon) b_j^\epsilon) \Phi_\epsilon \right] \in L^\infty(0, T; L^2(\mathbb{R}^3)),$$

solves the Stokes system with the source term $\nabla \cdot [(\mathcal{J}_\epsilon(v^\epsilon) \otimes v^\epsilon - \mathcal{J}_\epsilon(b^\epsilon) \otimes b^\epsilon) \Phi_\epsilon]$ in the distribution sense. Moreover, $a^\epsilon = b^\epsilon - e^{t\Delta} b_0$ solves the forced heat equation with the forcing $\nabla \cdot [(\mathcal{J}_\epsilon(v^\epsilon) \otimes b^\epsilon - \mathcal{J}_\epsilon(b^\epsilon) \otimes v^\epsilon) \Phi_\epsilon]$ in the distribution sense. By adding the homogeneous heat equation for $e^{t\Delta} v_0$ with $\nabla \cdot e^{t\Delta} v_0 = 0$ and the Stokes system for u^ϵ and π^ϵ , $v^\epsilon = e^{t\Delta} v_0 + u^\epsilon$ satisfies

$$\partial_t v^\epsilon - \Delta v^\epsilon + (\mathcal{J}_\epsilon(v^\epsilon) \cdot \nabla)(v^\epsilon \Phi_\epsilon) - (\mathcal{J}_\epsilon(b^\epsilon) \cdot \nabla)(b^\epsilon \Phi_\epsilon) + \nabla \pi^\epsilon = 0$$

in the sense of distribution. Moreover, by adding the homogeneous heat equation for $e^{t\Delta} b_0$ with $\nabla \cdot e^{t\Delta} b_0 = 0$ and the forced heat equation for a^ϵ , $b^\epsilon = e^{t\Delta} b_0 + a^\epsilon$ satisfies

$$\partial_t b^\epsilon - \Delta b^\epsilon + (\mathcal{J}_\epsilon(v^\epsilon) \cdot \nabla)(b^\epsilon \Phi_\epsilon) - (\mathcal{J}_\epsilon(b^\epsilon) \cdot \nabla)(v^\epsilon \Phi_\epsilon) = 0$$

in the sense of distribution. □

We next show global existence for the localized and regularized MHD equations, (3.31).

Lemma 3.17. *Assume $v_0, b_0 \in E_q^2$, $1 \leq q < 2$, are divergence free, and fix $\epsilon \in (0, 1)$. Then, there exists a global solution $(v^\epsilon, b^\epsilon, \pi^\epsilon)$ to the localized and regularized MHD equations, (3.31), such that $(v^\epsilon, b^\epsilon) \in \mathbf{LE}_q(0, T)$ for any $T < \infty$, and $(v^\epsilon, b^\epsilon, \pi^\epsilon)$ satisfies the a priori bounds (3.18) and (3.20) for all $R = n \in \mathbb{N}$ up to time $T_n = \lambda_0 n^2 \min \left(1, n^2 (c_3 \|(v_0, b_0)\|_{E_q^2 \times E_q^2}^2)^{-2} \right)$ for some $c_3 > 0$, where λ_0 is given in Lemma 3.7.*

Proof. The proof of the lemma is an adaption of the proof of [1, Lemma 4.8] for the Navier–Stokes equations to the MHD equations.

We set the radius $R = n \in \mathbb{N}$. By (3.28), we have the bound $A_{0,q}(n) \leq c_3(\|v_0\|_{E_q^2}^2 + \|b_0\|_{E_q^2}^2)$ for all $n \in \mathbb{N}$. Define

$$T_n = \lambda_0 n^2 \min \left\{ 1, n^2 \left(c_3(\|v_0\|_{E_q^2}^2 + \|b_0\|_{E_q^2}^2) \right)^{-2} \right\} \leq \lambda_n n^2,$$

where the constants λ_0 and λ_n are as in Lemma 3.7. Note that T_n is increasing and $T_n \rightarrow \infty$ as $n \rightarrow \infty$. Now, by the same argument used in the proof of Lemma 3.7, if a solution $(v^\epsilon, b^\epsilon, \pi^\epsilon)$ of (3.31) satisfies $(v^\epsilon, b^\epsilon) \in \mathbf{LE}_q(0, T)$, then it satisfies the a priori bounds (3.18) and (3.20) on the interval $[0, \min(T, T_n)]$ with radius $R = n$.

Since the system (3.31) is a coupled system of inhomogeneous Stokes systems with localized and regularized forcing, standard theory guarantees the existence of unique global solution $(v^\epsilon, b^\epsilon, \pi^\epsilon)$. By uniqueness, this solution agrees with the \mathbf{LE}_q -solution constructed in Lemma 3.16, and thus $(v^\epsilon, b^\epsilon) \in \mathbf{LE}_q(0, \tau)$ for some $\tau = \tau(\epsilon, \|(v_0, b_0)\|_{E_q^2 \times E_q^2}) > 0$. Fix $n \in \mathbb{N}$. Applying (3.18) with $R = n$, we obtain $\|(v^\epsilon, b^\epsilon)(\tau)\|_{E_q^2 \times E_q^2} \leq C(n) \|(v_0, b_0)\|_{E_q^2 \times E_q^2}$. Then, by Lemma 3.16, there exists an \mathbf{LE}_q -solution on $(\tau, \tau + \tau_1)$ for some $\tau_1 = \tau_1(\epsilon, C(n) \|(v_0, b_0)\|_{E_q^2 \times E_q^2}) > 0$. By uniqueness, this solution coincides with (v^ϵ, b^ϵ) , and we conclude that $(v^\epsilon, b^\epsilon) \in \mathbf{LE}_q(0, \tau + \tau_1)$, with the a priori bound (3.18) valid up to time $\tau + \tau_1$. This argument can be iterated: by repeatedly extending the solution, we obtain $(v^\epsilon, b^\epsilon) \in \mathbf{LE}_q(0, \tau + k\tau_1)$ for $k \in \mathbb{N}$, until $\tau + k\tau_1 \geq T_n$. Thus, for each $n \in \mathbb{N}$, we obtain $(v^\epsilon, b^\epsilon) \in \mathbf{LE}_q(0, T_n)$, with the a priori bound (3.18) holding for $R = n$ up to time T_n . Since $T_n \rightarrow \infty$, the lemma follows. □

Proof of Theorem 1.10 for $1 \leq q < 2$. The proof of Theorem 1.10 for $1 \leq q < 2$ is an adaption of the proof of [1, Theorem 1.4] for the Navier–Stokes equations to the MHD equations. We provide the necessary details by following the same strategy as in the proof of [3, Theorem 1.5] for the

Navier–Stokes equations, adapting the argument from the L^2_{uloc} framework to the E_q^2 setting, and from the Navier–Stokes to the MHD equations.

For $k \in \mathbb{N}$, let (v^k, b^k, π^k) be the solution of the localized, regularized MHD equations, (3.31), with $\epsilon = 1/k$, given in Lemma 3.17. They share the same a priori bound (3.18) for $R = n$ up to time T_n , thus

$$\sup_{k \in \mathbb{N}} \left\| (v^k, b^k) \right\|_{\mathbf{LE}_q(0, T_n)} < \infty, \quad \forall n \in \mathbb{N}.$$

Using this a priori bound, we now construct the desired global solutions as the limit of v^k defined in $(0, T_k)$, $T_k \rightarrow \infty$ by induction. Lemma 3.17 implies that (v^ϵ, b^ϵ) are uniformly bounded in the class from inequalities

$$\sup_{0 < t < T_1} \int_{B_1} (|v^k|^2 + |b^k|^2) dx + \int_0^{T_1} \int_{B_1} (|\nabla v^k|^2 + |\nabla b^k|^2) dx dt \leq cA \quad (3.63)$$

$$\int_0^{T_1} \int_{B_1} (|v^k|^{10/3} + |b^k|^{10/3}) dx dt \leq cA^{5/3}, \quad (3.64)$$

$$\int_0^{T_1} \int_{B_1} |\pi^k(x, t) - c_{0,2}^k(t)|^{3/2} dx dt \leq C(T_1, A), \quad (3.65)$$

where $c_{0,2}^k(t)$ is the function of t in (3.19) with $x_0 = 0$ and $R = 2$, and

$$\left\| \partial_t v^k \right\|_{\chi_1} + \left\| \partial_t b^k \right\|_{\chi_1} \leq C(T_1, A), \quad (3.66)$$

where χ_1 is the space dual to $L^3(0, T_1; W_0^{1,3}(B_1))$. Hence there exists a sequence $(v^{1,k}, b^{1,k})$ (where the corresponding ϵ are denoted by $\epsilon_{1,k}$) that converges to a solution (v_1, b_1) of (MHD) on $B_1 \times (0, T_1)$ in the following sense:

$$\begin{aligned} (v^{1,k}, b^{1,k}) &\xrightarrow{*} (v_1, b_1) \text{ in } L^\infty(0, T_1; (L^2 \times L^2)(B_1)), \\ (v^{1,k}, b^{1,k}) &\rightharpoonup (v_1, b_1) \text{ in } L^2(0, T_1; (H^1 \times H^1)(B_1)), \\ (v^{1,k}, b^{1,k}) &\rightarrow (v_1, b_1) \text{ in } L^3(0, T_1; (L^3 \times L^3)(B_1)), \\ (\mathcal{J}_{\epsilon_{1,k}} v^{1,k}, \mathcal{J}_{\epsilon_{1,k}} b^{1,k}) &\rightarrow (v_1, b_1) \text{ in } L^3(0, T_1; (L^3 \times L^3)(B_1)). \end{aligned}$$

By Lemma 3.17 all $v^{1,k}$ are also uniformly bounded on $B_n \times [0, T_n]$ for $n \in \mathbb{N}$, $n \geq 2$ and, recursively, we can extract subsequences $\{(v^{n,k}, b^{n,k})\}_{k \in \mathbb{N}}$ from $\{(v^{n-1,k}, b^{n-1,k})\}_{k \in \mathbb{N}}$ which converge to a solution (v_n, b_n) of (MHD) on $B_n \times (0, T_n)$ as $k \rightarrow \infty$ in the following sense:

$$\begin{aligned} (v^{n,k}, b^{n,k}) &\xrightarrow{*} (v_n, b_n) \text{ in } L^\infty(0, T_n; (L^2 \times L^2)(B_n)), \\ (v^{n,k}, b^{n,k}) &\rightharpoonup (v_n, b_n) \text{ in } L^2(0, T_n; (H^1 \times H^1)(B_n)), \\ (v^{n,k}, b^{n,k}) &\rightarrow (v_n, b_n) \text{ in } L^3(0, T_n; (L^3 \times L^3)(B_n)), \\ (\mathcal{J}_{\epsilon_{n,k}} v^{n,k}, \mathcal{J}_{\epsilon_{n,k}} b^{n,k}) &\rightarrow (v_n, b_n) \text{ in } L^3(0, T_n; (L^3 \times L^3)(B_n)). \end{aligned}$$

Let $(\tilde{v}_n, \tilde{b}_n)$ be the extension by 0 of (v_n, b_n) to $\mathbb{R}^3 \times (0, \infty)$. Note that, at each step, $(\tilde{v}_n, \tilde{b}_n)$ agrees with $(\tilde{v}_{n-1}, \tilde{b}_{n-1})$ on $B_{n-1} \times (0, T_{n-1})$. Let $(v, b) = \lim_{n \rightarrow \infty} (\tilde{v}_n, \tilde{b}_n)$. Then $(v, b) = (v_n, b_n)$ on $B_n \times (0, T_n)$ for every $n \in \mathbb{N}$.

Let $(v^k, b^k) = (v^{k,k}, b^{k,k})$ on $B_k \times (0, T_k)$ and equal 0 elsewhere. Let ϵ_k denote the corresponding regularization parameter. Then, for every fixed n and as $k \rightarrow \infty$,

$$\begin{aligned} (v^k, b^k) &\xrightarrow{*} (v, b) \quad \text{in } L^\infty(0, T_n; (L^2 \times L^2)(B_n)), \\ (v^k, b^k) &\rightharpoonup (v, b) \quad \text{in } L^2(0, T_n; (H^1 \times H^1)(B_n)), \\ (v^k, b^k) &\rightarrow (v, b) \quad \text{in } L^3(0, T_n; (L^3 \times L^3)(B_n)), \\ (\mathcal{J}_{\epsilon_k} v^k, \mathcal{J}_{\epsilon_k} b^k) &\rightarrow (v, b) \quad \text{in } L^3(0, T_n; (L^3 \times L^3)(B_n)). \end{aligned} \tag{3.67}$$

Based on the uniform bounds for the approximates, we have that (v, b) satisfies (1.30).

To resolve the pressure, let

$$\begin{aligned} \pi^k(x, t) &= -\frac{1}{3} \left[\mathcal{J}_{\epsilon_k}(v^k) \cdot v^k(x, t) \Phi_{\epsilon_k}(x) - \mathcal{J}_{\epsilon_k}(b^k) \cdot b^k(x, t) \Phi_{\epsilon_k}(x) \right] \\ &\quad + \text{p.v.} \int_{B_2} K_{ij}(x - y) \left[\mathcal{J}_{\epsilon_k}(v_i^k) v_j^k(y, t) - \mathcal{J}_{\epsilon_k}(b_i^k) b_j^k(y, t) \right] \Phi_{\epsilon_k}(y) dy \\ &\quad + \text{p.v.} \int_{B_2^c} (K_{ij}(x - y) - K_{ij}(-y)) \left[\mathcal{J}_{\epsilon_k}(v_i^k) v_j^k(y, t) - \mathcal{J}_{\epsilon_k}(b_i^k) b_j^k(y, t) \right] \Phi_{\epsilon_k}(y) dy, \end{aligned}$$

which, together with $(v^k, b^k) = (v^{\epsilon_k}, b^{\epsilon_k})$, solves (3.31) with $\epsilon = \epsilon_k$ in the distributional sense.

From the convergence properties of (v^k, b^k) , it follows that $\pi^k \rightarrow \pi$ in $L^{3/2}(0, T_n; L^{3/2}(B_n))$ for all n where $\pi(x, t) = \lim_{n \rightarrow \infty} \bar{\pi}^n(x, t)$ in which $\bar{\pi}^n(x, t)$ is defined for $|x| < 2^n$ by

$$\bar{\pi}^n(x, t) = -\frac{1}{3} (|v(x, t)|^2 - |b(x, t)|^2) + \text{p.v.} \int_{B_2} K_{ij}(x - y) (v_i v_j - b_i b_j)(y, t) dy + \bar{\pi}_3^n + \bar{\pi}_4^n,$$

with

$$\begin{aligned} \bar{\pi}_3^n(x, t) &= \text{p.v.} \int_{B_{2^{n+1}} \setminus B_2} (K_{ij}(x - y) - K_{ij}(-y)) (v_i v_j - b_i b_j)(y, t) dy, \\ \bar{\pi}_4^n(x, t) &= \int_{B_{2^{n+1}}^c} (K_{ij}(x - y) - K_{ij}(-y)) (v_i v_j - b_i b_j)(y, t) dy. \end{aligned}$$

We have $\bar{\pi}_3^n, \bar{\pi}_4^n \in L^{3/2}((0, T) \times B_{2^n})$ and

$$\bar{\pi}_3^n + \bar{\pi}_4^n = \bar{\pi}_3^{n+1} + \bar{\pi}_4^{n+1} \quad \text{in } L^{3/2}((0, T) \times B_{2^n}).$$

Thus $\bar{\pi}^n$ is independent of n for $n > \log_2 |x|$.

We now establish the above local pressure expression for all scales. Note that the formula is valid for π^k at all scales, that is, for any $T > 0$, fixed $R > 0$ and $x_0 \in \mathbb{R}^3$, we have the following equality in $L^{3/2}(B_{2R}(x_0) \times (0, T))$,

$$\begin{aligned} \hat{\pi}_{x_0, R}^k(x, t) &:= \pi^k(x, t) - c_{x_0, R}^k(t) \\ &= -\Delta^{-1} \operatorname{div} \operatorname{div} \left[\left((\mathcal{J}_k v^k \otimes v^k - \mathcal{J}_k b^k \otimes b^k) \Phi_k \right) \chi_{4R}(x - x_0) \right] \\ &\quad - \int_{\mathbb{R}^3} (K(x - y) - K(x_0 - y)) \left((\mathcal{J}_k v^k \otimes v^k - \mathcal{J}_k b^k \otimes b^k) \Phi_k \right) (y, t) (1 - \chi_{4R}(y - x_0)) dy, \end{aligned}$$

where $\mathcal{J}_k = \mathcal{J}_{\epsilon_k}$ and $\Phi_k = \Phi_{\epsilon_k}$. Similarly, let

$$\begin{aligned} \hat{\pi}_{x_0, R}(x, t) &= -\Delta^{-1} \operatorname{div} \operatorname{div} [(v \otimes v - b \otimes b) \chi_{4R}(x - x_0)] \\ &\quad - \int_{\mathbb{R}^3} (K(x - y) - K(x_0 - y)) (v \otimes v - b \otimes b)(y, t) (1 - \chi_{4R}(y - x_0)) dy. \end{aligned}$$

Fix $T > 0$, $x_0 \in \mathbb{R}^3$ and $R > 0$. Choose n large enough that $B_{8R}(x_0) \times (0, T) \subset Q_n = B_n \times (0, T_n)$. We claim that $\hat{\pi}_{x_0, R}^k(x, t)$ converges to $\hat{p}_{x_0, R}(x, t)$ in $L^{3/2}(B_{2R}(x_0) \times (0, T))$. If this is the case, by taking the limit of the weak form of (3.31), we can show that $(v, b, \hat{\pi}_{x_0, R})$ also satisfies (MHD) in $B_{2R}(x_0) \times (0, T)$. Hence $\nabla \pi - \nabla \hat{\pi}_{x_0, R} = 0$, and we may define

$$c_{x_0, R}(t) = \pi(x, t) - \hat{\pi}_{x_0, R}(x, t)$$

which is hence a function of t in $L^{3/2}(0, T)$ that is independent of x . This gives the desired local pressure expansion in $B_{2R}(x_0) \times (0, T)$.

To verify the claim we work term by term. Note that the estimate in [23, (3.26)] shows that

$$\left\| v_i v_j - (\mathcal{J}_k v_i^k) v_j^k \Phi_k \right\|_{L^{3/2}(B_M \times [0, T_n])}, \quad \left\| b_i b_j - (\mathcal{J}_k b_i^k) b_j^k \Phi_k \right\|_{L^{3/2}(B_M \times [0, T_n])} \rightarrow 0,$$

as $k \rightarrow \infty$ for every $M > 0$ and $n \in \mathbb{N}$. This implies

$$\begin{aligned} & -\Delta^{-1} \operatorname{div} \operatorname{div} \left[\left((\mathcal{J}_k v^k \otimes v^k - \mathcal{J}_k b^k \otimes b^k) \Phi_k \right) \chi_{4R}(x - x_0) \right] \\ & \rightarrow -\Delta^{-1} \operatorname{div} \operatorname{div} [(v \otimes v - b \otimes b) \chi_{4R}(x - x_0)] \end{aligned}$$

in $L^{3/2}(B_{2R}(x_0) \times (0, T_n))$, and

$$\begin{aligned} & - \int_{|x| < M} (K(x - y) - K(x_0 - y)) \left((\mathcal{J}_k v^k \otimes v^k - \mathcal{J}_k b^k \otimes b^k) \Phi_k \right) (y, t) (1 - \chi_{4R}(y - x_0)) dy \\ & \rightarrow - \int_{|x| < M} (K(x - y) - K(x_0 - y)) (v \otimes v - b \otimes b)(y, t) (1 - \chi_{4R}(y - x_0)) dy \end{aligned}$$

in $L^{3/2}(B_{2R}(x_0) \times (0, T_n))$ for every $M > 8R$. For the far-field part, still assuming $M > 8R$, we have

$$\begin{aligned} & \left\| \int_{|x| \geq M} (K(x - y) - K(x_0 - y)) \left((\mathcal{J}_k v^k \otimes v^k - \mathcal{J}_k b^k \otimes b^k) \Phi_k - (v \otimes v - b \otimes b) \right) (y, t) dy \right\|_{L^{3/2}(B_{2R}(x_0) \times (0, T_n))} \\ & \leq C(R, n, \|(v_0, b_0)\|_{L_{\text{uloc}}^2 \times L_{\text{uloc}}^2}) M^{-1} \leq C(R, n, \|(v_0, b_0)\|_{E_q^2 \times E_q^2}) M^{-1}, \end{aligned}$$

where we've used the embedding $E_q^2 \subset L_{\text{uloc}}^2$. This can be made arbitrarily small by taking M large and noting R and n are fixed. Consequently, and since the other parts of the pressure converge, we conclude that

$$\hat{\pi}_{x_0, R}^k(x, t) \rightarrow \hat{\pi}_{x_0, R}(x, t) \text{ in } L^{3/2}(B_{2R}(x_0) \times (0, T_n)), \quad (3.68)$$

which leads to the desired local pressure expansion. Since n was arbitrary, this gives the pressure formula for arbitrarily large times.

At this point we have established items 1.-3. from the definition of local energy solutions. We now check remaining items.

Fix T_0 and choose n so that $T_n \geq T_0$. Then (3.67) holds for all n with T_n replaced by T_0 . Furthermore, the estimates (3.63)–(3.66) and (3.68) are valid in $B_n \times [0, T_0]$ up to a re-definition of A . Moreover, we have

$$\|\partial_t v\|_{\chi_n} + \|\partial_t b\|_{\chi_n} \leq C(n, T_0, A), \quad (3.69)$$

and

$$\left\| \operatorname{ess\,sup}_{0 \leq t \leq T_0} \int_{B_1(k)} (|v|^2 + |b|^2) dx + \int_0^{T_0} \int_{B_1(k)} (|\nabla v|^2 + |\nabla b|^2) dx dt \right\|_{\ell^{q/2}(k \in \mathbb{Z}^3)} \leq 2A. \quad (3.70)$$

It follows from (3.69) and (3.70) that for every n ,

$$t \mapsto \int_{B_n} v(x, t) \cdot w(x) dx \quad t \mapsto \int_{B_n} b(x, t) \cdot w(x) dx \quad (3.71)$$

are continuous in $t \in [0, T_0]$ for every $w \in L^2(B_2)$. Since T_0 was arbitrary, we can extend this to all times. The local energy inequality follows from the local energy equality for (v^k, b^k) and π^k , and (3.67), (3.68) in $B_n \times [0, T_0]$, (3.69), and $\hat{\pi}_n(x, t) = \pi(x, t) - c_n(t)$ for some $c_n \in L^{3/2}(0, T_0)$. Convergence to initial data in L^2_{loc} follows from (3.71) and the local energy inequality. This confirms that items 4.-6. from the definition of local energy solutions are satisfied and finishes the proof of Theorem 1.10 for $1 \leq q < 2$. \square

Acknowledgments

I warmly thank Zachary Bradshaw and Tai-Peng Tsai for helpful comments. The research was partially support by the AMS-Simons Travel Grant and the Simons Foundation Math + X Investigator Award #376319 (Michael I. Weinstein). The author gratefully acknowledges the unwavering financial and emotional support of his wife, Anyi Bao.

References

- [1] Z. Bradshaw, C.-C. Lai, and T.-P. Tsai. Mild solutions and spacetime integral bounds for Stokes and Navier-Stokes flows in Wiener amalgam spaces. *Math. Ann.*, 388(3):3053–3126, 2024.
- [2] Z. Bradshaw and T.-P. Tsai. Forward discretely self-similar solutions of the Navier-Stokes equations II. *Ann. Henri Poincaré*, 18(3):1095–1119, 2017.
- [3] Z. Bradshaw and T.-P. Tsai. Global existence, regularity, and uniqueness of infinite energy solutions to the Navier-Stokes equations. *Comm. Partial Differential Equations*, 45(9):1168–1201, 2020.
- [4] Z. Bradshaw and T.-P. Tsai. Local energy solutions to the Navier-Stokes equations in Wiener amalgam spaces. *SIAM J. Math. Anal.*, 53(2):1993–2026, 2021.
- [5] Z. Bradshaw and T.-P. Tsai. On the local pressure expansion for the Navier-Stokes equations. *J. Math. Fluid Mech.*, 24:1–32, 2022.
- [6] L. Caffarelli, R. Kohn, and L. Nirenberg. Partial regularity of suitable weak solutions of the Navier-Stokes equations. *Comm. Pure Appl. Math.*, 35(6):771–831, 1982.
- [7] C. Cao and J. Wu. Two regularity criteria for the 3D MHD equations. *J. Differential Equations*, 248(9):2263–2274, 2010.
- [8] D. Chamorro, F. Cortez, J. He, and O. Jarrín. On the local regularity theory for the magneto-hydrodynamic equations. *Doc. Math.*, 26:125–148, 2021.
- [9] Q. Chen, C. Miao, and Z. Zhang. On the regularity criterion of weak solution for the 3D viscous magneto-hydrodynamics equations. *Comm. Math. Phys.*, 284:919–930, 2008.

- [10] Y. Chen and P. Zhang. The global existence of small solutions to the incompressible viscoelastic fluid system in 2 and 3 space dimensions. *Comm. Partial Differential Equations*, 31(10-12):1793–1810, 2006.
- [11] G. Duvaut and J.-L. Lions. Inéquations en thermoélasticité et magnétohydrodynamique. *Arch. Rational Mech. Anal.*, 46:241–279, 1972.
- [12] P. G. Fernández-Dalgo and O. Jarrín. Discretely self-similar solutions for 3D MHD equations and global weak solutions in weighted L^2 spaces. *J. Math. Fluid Mech.*, 23(1):Paper No. 22, 30, 2021.
- [13] P. G. Fernández-Dalgo and O. Jarrín. Weak-strong uniqueness in weighted L^2 spaces and weak suitable solutions in local Morrey spaces for the MHD equations. *J. Differential Equations*, 271:864–915, 2021.
- [14] Y. Giga, K. Inui, and S. Matsui. On the cauchy problem for the Navier-Stokes equations with nondecaying initial data. *Quad. Mat.*, 4:28–68, 1999.
- [15] C. He and Z. Xin. Partial regularity of suitable weak solutions to the incompressible magnetohydrodynamic equations. *J. Funct. Anal.*, 227(1):113–152, 2005.
- [16] C. He and Z. Xin. On the self-similar solutions of the magneto-hydro-dynamic equations. *Acta Math. Sci. Ser. B (Engl. Ed.)*, 29(3):583–598, 2009.
- [17] R. Hynd. Partial regularity of weak solutions of the viscoelastic Navier-Stokes equations with damping. *SIAM J. Math. Anal.*, 45(2):495–517, 2013.
- [18] K. Kang, B. Lai, C.-C. Lai, and T.-P. Tsai. The Green tensor of the nonstationary Stokes system in the half space. *Comm. Math. Phys.*, 399(2):1291–1372, 2023.
- [19] K. Kang and J. Lee. Interior regularity criteria for suitable weak solutions of the magnetohydrodynamic equations. *J. Differential Equations*, 247(8):2310–2330, 2009.
- [20] K. Kang, H. Miura, and T.-P. Tsai. Short time regularity of Navier–Stokes flows with locally L^3 initial data and applications. *Int. Math. Res. Not. IMRN*, 2021(11):8763–8805, 2021.
- [21] N. Kikuchi and G. Seregin. Weak solutions to the Cauchy problem for the Navier-Stokes equations satisfying the local energy inequality. In *Nonlinear equations and spectral theory*, volume 220 of *Amer. Math. Soc. Transl. Ser. 2*, pages 141–164. Amer. Math. Soc., Providence, RI, 2007.
- [22] J.-M. Kim. On regularity criteria of weak solutions to the 3D viscoelastic Navier-Stokes equations with damping. *Appl. Math. Lett.*, 69:153–160, 2017.
- [23] H. Kwon and T.-P. Tsai. Global Navier–Stokes flows for non-decaying initial data with slowly decaying oscillation. *Comm. Math. Phys.*, 375(3):1665–1715, 2020.
- [24] B. Lai, J. Lin, and C. Wang. Forward self-similar solutions to the viscoelastic Navier-Stokes equation with damping. *SIAM J. Math. Anal.*, 49(1):501–529, 2017.
- [25] C.-C. Lai. Forward discretely self-similar solutions of the MHD equations and the viscoelastic Navier-Stokes equations with damping. *J. Math. Fluid Mech.*, 21(3):Paper No. 38, 28, 2019.

- [26] Z. Lei, C. Liu, and Y. Zhou. Global solutions for incompressible viscoelastic fluids. *Arch. Ration. Mech. Anal.*, 188(3):371–398, 2008.
- [27] P. G. Lemarié-Rieusset. *Recent developments in the Navier-Stokes problem*, volume 431 of *Chapman & Hall/CRC Research Notes in Mathematics*. Chapman & Hall/CRC, Boca Raton, FL, 2002.
- [28] F. Lin. A new proof of the Caffarelli-Kohn-Nirenberg theorem. *Comm. Pure Appl. Math.*, 51(3):241–257, 1998.
- [29] F.-H. Lin, C. Liu, and P. Zhang. On hydrodynamics of viscoelastic fluids. *Comm. Pure Appl. Math.*, 58(11):1437–1471, 2005.
- [30] Y. Lin, H. Zhang, and Y. Zhou. Global smooth solutions of MHD equations with large data. *J. Differential Equations*, 261(1):102–112, 2016.
- [31] F. Liu, S. Xi, Z. Zeng, and S. Zhu. Global mild solutions to three-dimensional magnetohydrodynamic equations in Morrey spaces. *J. Differential Equations*, 314:752–807, 2022.
- [32] M. Liu and X. Lin. Global classical solutions to the elastodynamic equations with damping. *J. Inequal. Appl.*, 2021(1):88, 2021.
- [33] Y. Maekawa and Y. Terasawa. The Navier-Stokes equations with initial data in uniformly local L^p spaces. *Differential Integral Equations*, 19(4):369–400, 2006.
- [34] A. Mahalov, B. Nicolaenko, and T. Shilkin. $L_{3,\infty}$ -solutions to the MHD equations. *J. Math. Sci.*, 143:2911–2923, 2007.
- [35] C. Miao, B. Yuan, and B. Zhang. Well-posedness for the incompressible magneto-hydrodynamic system. *Math. Methods Appl. Sci.*, 30(8):961–976, 2007.
- [36] J. Nečas, M. Růžička, and V. Šverák. On Leray’s self-similar solutions of the Navier-Stokes equations. *Acta Math.*, 176(2):283–294, 1996.
- [37] C. W. Oseen. *Neuere methoden und ergebnisse in der hydrodynamik*, volume 1. Akademische Verlagsgesellschaft, 1927.
- [38] M. Sermange and R. Temam. Some mathematical questions related to the MHD equations. *Comm. Pure Appl. Math.*, 36(5):635–664, 1983.
- [39] Z. Tan, W. Wu, and J. Zhou. Existence of mild solutions and regularity criteria of weak solutions to the viscoelastic Navier–Stokes equation with damping. *Commun. Math. Sci.*, 18(1):205–226, 2020.
- [40] T.-P. Tsai. *Lectures on Navier-Stokes equations*, volume 192 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2018.
- [41] J. Wu. Regularity results for weak solutions of the 3D MHD equations. *Discrete Contin. Dyn. Syst.*, 10(1/2):543–556, 2004.
- [42] Y. Yang. Forward self-similar solutions to the mhd equations in the whole space. *arXiv preprint arXiv:2404.02601*, 2024.

- [43] G. Yue, Z. Pang, and Y. Wu. On the interior regularity criteria for the viscoelastic fluid system with damping. *Adv. Calc. Var.*, 18(2):323–338, 2025.
- [44] Y. Zhou. Remarks on regularities for the 3D MHD equations. *Discrete Contin. Dyn. Syst.*, 12(5):881–886, 2005.