A quasianalytic class with weakly smooth germs

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1 Introduction

Recall that an expansion $\langle M; <, \ldots \rangle$ of a dense linear order is called **o-minimal** when the definable subsets of M are exactly the finite unions of points and intervals. By the Tarski-Seidenberg Theorem, the structure $\mathbb{R}_{alg} = \langle \mathbb{R}; <, +, -, \cdot, 0, 1 \rangle$ has quantifier elimination. It is easy to deduce that it is also model-complete and o-minimal.

Given a power series $f \in \mathbb{R}[X_1, \dots, X_n]$ that converges in a neighborhood of I^n , where I = [-1, 1], we define

$$\widetilde{f}(x) = \begin{cases} f(x), & x \in I^n \\ 0, & x \notin I^n \end{cases}$$

and call the function \widetilde{f} a **restricted analytic function**. Consider \mathbb{R}_{an} , the structure \mathbb{R}_{alg} augmented by a function symbol for each restricted analytic function. In [4], the authors give a quantifier elimination result for the structure $\langle \mathbb{R}_{an}; \exp, \log \rangle$ and they deduce that the structure $\mathbb{R}_{an,\exp} = \langle \mathbb{R}_{an}; \exp \rangle$ is ominimal.

In particular, the structure \mathbb{R}_{an} is o-minimal. Actually, this is already observed in [1], where this result is obtained as a consequence of Gabrielov's Theorem of the complement proven in [5]. This method, which does not rely on quantifier elimination, is adapted in [3] in the following way. A **generalized** series $f \in \mathbb{R}[X^*]$, where $X = (X_1, \ldots, X_m)$, is a formal sum

$$f = \sum_{\alpha} c_{\alpha} X^{\alpha},$$

where α runs over the set $[0,\infty)^m$ and such that there are well ordered sets $S_1,\ldots,S_m\subseteq[0,\infty)$ with $\{\alpha:c_\alpha\neq 0\}\subseteq S_1\times\cdots\times S_m$. The set $\mathrm{supp}(f)\coloneqq\{\alpha:c_\alpha\neq 0\}$ is called the **support** of f. The condition on the support guarantees that these series can be added and multiplied in the usual way. If $f\in\mathbb{R}[X^*]$, we say that it converges when the series

$$\sum_{\alpha \in \operatorname{supp}(f)} c_{\alpha} x^{\alpha}$$

converges absolutely on I^m . In this case, we define

$$\widetilde{f}(x) = \begin{cases} f(x), & x \in I^m \\ 0, & x \notin I^m \end{cases}$$

as in the analytic case. By adapting Gabrielov's Theorem of the complement and by giving a suitable monomialization procedure, the authors of [3] show that the expansion of \mathbb{R}_{alg} by function symbols for each \widetilde{f} is o-minimal.

Later on, in [9], the authors generalize this method to series that do not necessarily converge. For each $r = (r_1, \ldots, r_n) \in (0, \infty)^n$, define the **polydisk** $I_{n,r}$ as

$$I_{n,r} = (-r_1, r_1) \times \cdots \times (-r_n, r_n).$$

Then, for each $n \ge 0$ and each $r \in (0, \infty)^n$ consider an algebra $\mathcal{A}_{n,r}$ of functions $f \colon I_{n,r} \to \mathbb{R}$ and let \mathcal{A}_n be the algebra of germs at 0 of functions in $\mathcal{A}_{n,r}$. We assume that each germ in \mathcal{A}_n is a C^{∞} germ, meaning that it has a C^{∞} representative. Then, there is a morphism $\mathbf{T} \colon \mathcal{A}_n \to \mathbb{R}[\![X]\!]$ that takes a germ $f \in \mathcal{A}_n$ and that returns its Taylor series at 0. We say that the algebras \mathcal{A}_n are **quasianalytic** when this morphism is injective. In practice, if $f \in \mathcal{A}_n$, we can view f as a sum of the series $\mathbf{T}(f)$, even when this series does not converge in any neighborhood of 0. Under this assumption, and other technical hypotheses about the algebras $\mathcal{A}_{n,r}$, it is then shown in [9] that the structure $\langle \mathbb{R}_{\text{alg}}; (\tilde{f})_{f \in \mathcal{A}} \rangle$ is o-minimal and model-complete, where

$$\widetilde{f}(x) = \begin{cases} f(x), & x \in I_{n,r} \\ 0, & x \notin I_{n,r} \end{cases}$$

for $f \in \mathcal{A}_{n,r}$.

This same method can also be applied with generalized series, although more care is necessary in this latter case. An example of this method is given in [8] while Rolin and Servi provide a proof of o-minimality and model-completeness under general assumptions in [7]. The authors of the latter paper also remark that most known o-minimal structures satisfy the hypotheses of their theorem. Furthermore, in [7, Section 4], the authors also use o-minimality in the form of the o-minimal Preparation Theorem, which is proven in [2], to derive a quantifier elimination result. However, due to the hypotheses made, all the structures to which these results apply necessarily have C^{∞} cell-decomposition.

In [6], Le Gal and Rolin show that there exists an o-minimal expansion of the real field which does not admit C^{∞} cell-decomposition. Recall that a germ f of function $\mathbb{R}^n \to \mathbb{R}$ at 0 is called **weakly** C^{∞} if it is C^p for all $p \geqslant 0$. This means that it has C^p representatives for every $p \geqslant 0$ but these can be defined on smaller and smaller neighborhoods as $p \to \infty$. Thus, in particular, this does not imply that f has a C^{∞} representative. However, it still makes sense to talk about the Taylor series of f since f has partial derivatives of every order. Therefore, the concept of quasianalyticity still makes sense in this context and the authors use the same ideas as the ones in [7] in order to prove o-minimality. The main difference is that they cannot prove a global version of the Fiber Cutting Lemma so that they have to use a local version of the result instead.

Firstly, let us recall some notations from [6]. We let \mathcal{W}_n be the algebra of weakly C^{∞} germs at 0 of functions $\mathbb{R}^n \to \mathbb{R}$.

Definition 1.1. Let $H: \mathbb{R} \to \mathbb{R}$ be a function whose germ at 0 is weakly C^{∞} . We define $\mathcal{A}(H) = (\mathcal{A}_n(H))$ to be the smallest family of subalgebras $\mathcal{A}_n(H) \subseteq \mathcal{W}_n$ such that

- The germ of H belongs to $\mathcal{A}_1(H)$ and polynomial germs in n variables are in $\mathcal{A}_n(H)$.
- If the germ $f \in \mathcal{A}_n(H)$ vanishes on the hyperplane $x_i = 0$ then the germ which continuously extends $\frac{f}{x_i}$ is in $\mathcal{A}_n(H)$.

- If $f \in \mathcal{A}_m(H)$ and $g_1, \ldots, g_m \in \mathcal{A}_n(H)$ are such that $g_i(0) = 0$ for every $1 \leq i \leq m$, then $f(g_1, \ldots, g_m) \in \mathcal{A}_n(H)$.
- Let $f \in \mathcal{A}_n(H)$ be such that f(0) = 0 and $\frac{\partial f}{\partial x_n}(0) \neq 0$. Then, the germ $\varphi \in \mathcal{W}_{n-1}$ such that $f(x', \varphi(x')) = 0$ is in $\mathcal{A}_{n-1}(H)$.

In [6, Section 2], Le Gal and Rolin prove the following theorem.

Theorem 1.2. There exists a function $H: \mathbb{R} \to \mathbb{R}$ which satisfies the following conditions.

- The germ of H at 0 is weakly C^{∞} but not C^{∞} .
- The restriction of H to the complement of any neighborhood of 0 is piecewise given by finitely many polynomials.
- The algebras $A_n(H)$ are quasianalytic, meaning that the morphism $\mathbf{T} \colon A_n(H) \to \mathbb{R}[\![X]\!]$ which sends a germ to its Taylor series at 0 is injective.

Then, [6, Section 3] is dedicated to proving that the structure $\mathbb{R}_H = \langle \mathbb{R}_{\text{alg}}; H, H', H'', \ldots \rangle$, which is the expansion of the real ordered field with function symbols for the derivatives of any order order of H, is o-minimal and model-complete. This structure clearly does not have C^{∞} cell-decomposition so that the result of [7] does not apply in this case. Despite that, the proofs of o-minimality and model-completeness given in [7, Section 3] and in [6, Section 3] are rather similar and rely heavily on a Fiber Cutting Lemma. The main difference between the two is that some global results in [7] have to be stated locally in [6]. However, the local methods used in [6] involve restricting to small neighborhoods of 0 often and, as shown in an example given in the Appendix to this text, their proof of Lemma 3.7 sometimes constructs sets with empty germ meaning that such sets cannot always be restricted. The goal of the present document is to provide a framework that lets us restrict to smaller neighborhoods of 0 at will, and hence produce an alternative proof of the o-minimality and model-completeness of \mathbb{R}_H .

In a forthcoming paper, we weaken the hypotheses of [7] to prove a very general o-minimality and quantifier elimination result for structures generated by generalized quasianalytic classes of weakly C^{∞} germs. In particular, given H as in Theorem 1.2, there is an expansion of \mathbb{R}_H with existentially definable functions that satisfies these weaker hypotheses, so that we obtain a quantifier elimination theorem for an o-minimal structure without smooth cell-decomposition.

We now describe the plan of the present document. We fix $H: \mathbb{R} \to \mathbb{R}$ a function as in Theorem 1.2. Roughtly, a Λ -set $A \subseteq \mathbb{R}^n$ is a bounded and quantifier-free definable subset (see Definition 2.1). A sub- Λ -set is defined as a projection of a Λ -set. Thus, in particular, sub- Λ -sets are existentially definable. Sub- Λ -sets are easily seen to be stable under finite unions and intersections as well as projections. Provided we show that sub- Λ -sets are stable under settheoretic difference, we can establish a bijective correspondence between \mathbb{R}_H -definable sets and sub- Λ -sets. Model-completeness is then immediate while

o-minimality follows if we can prove that Λ -sets have finitely many connected components since projections preserve this property.

In Section 2, we introduce the notion of simple sub- Λ -sets which is at the heart of this new approach to the proof of o-minimality. In Theorem 2.4, we show that germs of quantifier-free definable sets at 0 (namely germs of H-sets, see Definition 2.1) have representatives that are also simple sub- Λ -sets. This is done by showing inductively on the structure of germs $f \in \mathcal{A}_n(H)$ (see Definition 1.1) that the graph of f has a representative which is also a simple sub- Λ -set.

In Section 3, we prove a monomialization result (Theorem 3.11) from which we derive several parametrization results by manifolds for H-sets. In particular, we show in Corollary 3.16 that Λ -sets have finitely many connected components which will later yield o-minimality. Monomialization is a process by which we can replace a germ $f \in \mathcal{A}_n(H)$ with a family of normal germs, namely germs of the form $x^{\alpha}u(x)$ where $u \in \mathcal{A}_n(H)$ is such that $u(0) \neq 0$. The main advantage of such germs is that their sign depends only on the signs of the various coordinates.

In Section 4, we show that every sub- Λ -set is simple (see Theorem 4.16). Combining this with Lemma 4.1, we see that the hypotheses of Gabrielov's Theorem of the Complement [3, Theorem 2.7] are satisfied. Finally, in Theorem 4.18, we establish a bijection between \mathbb{R}_H -definable sets and sub- Λ -sets, allowing us to deduce model-completeness and o-minimality at once. The main novelty of this approach has to do with the Fiber Cutting Lemma, of which there are now two versions, a local one (Proposition 4.10) and a global one (Corollary 4.13). The classical proof relies on \mathcal{A} -analyticity [7, Definition 1.10], which is a global assumption that does not hold in the present case. Finally, the importance of simple sub- Λ -sets is stressed throughout Paragraph 4.4, as they can be replaced seamlessly by Λ -sets.

2 A local property of H-basic sets

Throughout the text, we fix a function H such as in Theorem 1.2 and we let \mathbb{R}_H be the structure $\langle \mathbb{R}_{\text{alg}}; H, H', H'', \dots \rangle$. Notice that \mathbb{R}_H and $\langle \mathbb{R}_{\text{alg}}; H \rangle$ have the same definable sets so that one is o-minimal if and only if the other is as well. However, \mathbb{R}_H is model-complete while $\langle \mathbb{R}_{\text{alg}}; H \rangle$ is not. We start by recalling a few definitions.

Definition 2.1. Firstly, if $r = (r_1, \ldots, r_n) \in \mathbb{R}^n$, then we define the **polydisk** $I_{n,r} = (-r_1, r_1) \times \cdots \times (-r_n, r_n)$. Given a continuous function $f: I_{n,r} \to \mathbb{R}$, we write $f \in \mathcal{A}_n(H)$ to mean that the germ at 0 of f is in $\mathcal{A}_n(H)$. A subset $A \subseteq I_{n,r}$ is said to be H-basic if there are $g_0, g_1, \ldots, g_q \in \mathcal{A}_n(H)$ such that

$$A = \{x \in I_{n,r} : g_0(x) = 0, g_1(x) > 0, \dots, g_q(x) > 0\}.$$

An H-set is a finite union of H-basic sets. A set $A \subseteq \mathbb{R}^n$ is said to be H-semianalytic when, for any $a \in \mathbb{R}^n$, there is a polydisk $I_{n,r}$ such that $(A-a) \cap I_{n,r}$ is an H-set. If $A \subseteq \mathbb{R}^n$ is both H-semianalytic and bounded we say that A

is a Λ -set. Given integers $n, k \ge 0$, we let $\Pi_n : \mathbb{R}^{n+k} \to \mathbb{R}^n$ be the projection on the first n variables. Then, if $A \subseteq \mathbb{R}^n$ is such that there are an integer $k \ge 0$ and a Λ -set $A' \subset \mathbb{R}^{n+k}$ with $\Pi_n(A') = A$, we say that A is a sub- Λ -set.

In the defininition of H-basic sets, the condition $g_0, \ldots, q_q \in \mathcal{A}_n(H)$ is really only a condition on the germs of g_0, \ldots, g_q at 0. Thus, H-basic sets can be quite arbitrary away from 0, meaning that the definition of H-basic sets is local around 0. On the contrary, if $A \subseteq \mathbb{R}^n$ is H-semianalytic then we have information on the germ of A at every point $a \in \mathbb{R}^n$. Thus, this latter definition is global. As a first step towards proving o-minimality, we would like to show that there is a link between the two notions. In [7], such a link is obtained by forcing the functions g_0, g_1, \ldots, g_q that appear in the definition of H-basic sets to be H-analytic, meaning that their germ at every point $a \in I_{n,r}$ is the translation at a of a germ in $A_n(H)$ [7, Definition 1.10]. However, this approach is valid only as long as each germ in $A_n(H)$ has an H-analytic representative. But, such a representative would have partial derivatives of every order showing that it is C^{∞} . Since H has no C^{∞} representative, this method is unavailable in our case. Instead, we introduce the notion of simple sub- Λ -sets, which are a special instance of sub- Λ -sets, and we show that, for every H-set $A \subseteq \mathbb{R}^n$, there exists a polydisk $I_{n,r}$ such that $A \cap I_{n,r}$ is a simple sub- Λ -set.

2.1 Simple sub- Λ -sets

Definition 2.2. A sub- Λ -set $A \subseteq \mathbb{R}^n$ is called **simple** when it is existentially definable in \mathbb{R}_H and when there are an integer $k \geqslant 0$ and a Λ -set $A' \subseteq \mathbb{R}^{n+k}$ such that $\Pi_n(A') = A$ and the fibers of $\Pi_n \upharpoonright A'$ are finite, namely, for every $a \in A$, the set

$$A'_{a} := \{ b \in \mathbb{R}^{k} : (a, b) \in A' \}$$

is finite.

In the notation of the definition, we will show in Proposition 4.5 that $\dim(A) = \dim(A')$ for a reasonable definition of dimension. This, along with the following set theoretical stability properties of simple sub- Λ -sets, explain the usefulness of the notion.

Proposition 2.3. Let $A, B \subseteq \mathbb{R}^n$ and $C \subseteq \mathbb{R}^m$ be simple sub- Λ -sets. Then, the sets $A \cap B$, $A \cup B$ and $A \times C$ are also simple sub- Λ -sets. If $k \leqslant n$ and $x \in \mathbb{R}^k$ then the fiber

$$A_x = \{ y \in \mathbb{R}^{n-k} : (x,y) \in A \}$$

is a simple sub- Λ -set. Furthermore, if $\Pi_k \upharpoonright A$ has finite fibers then $\Pi_k(A)$ is also a simple sub- Λ -set. Finally, if $s(1), \ldots, s(n) \in \{1, \ldots, k\}$ then

$$\{(x_1,\ldots,x_k)\in\mathbb{R}^k:(x_{s(1)},\ldots,x_{s(n)})\in A\}$$

is a simple sub- Λ -set.

Every part of the proposition above is immediate so we omit the proof.

2.2 The main result

As announced in the introduction of this section, the main theorem we are looking to prove is the following.

Theorem 2.4. If $A \subseteq \mathbb{R}^n$ is an H-set then there is a polydisk $I_{n,r}$ such that $A \cap I_{n,r}$ is a simple sub- Λ -set.

The theorem is a consequence of the following lemma.

Lemma 2.5. Given $f: I_{n,r} \to \mathbb{R}$ such that $f \in \mathcal{A}_n(H)$, there is a polydisk $I_{n,r'} \subseteq I_{n,r}$ such that $\Gamma(f \upharpoonright I_{n,r'})$ is a simple sub- Λ -set.

We start by proving the theorem using the lemma.

Proof of Theorem 2.4. Let $g_0, g_1, \ldots, g_q \in \mathcal{A}_n(H)$ be functions defined on a polydisk $I_{n,r}$ such that

$$A = \{x \in I_{n,r} : g_0(x) = 0, \ g_1(x) > 0, \ \dots, \ g_q(x) > 0\}.$$

By the lemma, up to shrinking $I_{n,r}$, we may assume that $\Gamma(g_i)$ is a simple sub- Λ -set for each $0 \leq i \leq q$. In particular, the functions g_0, \ldots, g_q are bounded. Thus, there exists a polyradius $s = (s_0, \ldots, s_q)$ such that $|g_i(x)| < s_i$ for each $0 \leq i \leq q$ and each $x \in I_{n,r}$. Then,

$$A' = \{(x, y) \in I_{n,r} \times I_{q+1,s} : y_i = g_i(x) \text{ for } 0 \le i \le q\}$$

 $\cap \{(x, y) \in I_{n,r} \times I_{q+1,s} : y_0 = 0, y_1 > 0, \dots, y_q > 0\}$

is a simple sub- Λ -set. Furthermore, $A = \Pi_n(A')$ and the fibers of $\Pi_n \upharpoonright A'$ are finite.

The aim of the rest of this section is to prove the lemma. Given the definition of $\mathcal{A}_n(H)$ (see Definition 1.1), the most natural way to prove the lemma would be as follows. First define \mathcal{B}_n to be the set of germs $f \in \mathcal{W}_n$ having a representative $f \colon I_{n,r} \to \mathbb{R}$ such that $\Gamma(f)$ is a simple sub- Λ -set. We have to show that $\mathcal{A}_n(H) \subseteq \mathcal{B}_n$. Since \mathcal{B}_n is a collection of algebras stable under composition and implicit functions, it suffices to prove that it is also stable under monomial division. However, there does not seem to be an easy way of showing this. Thus, we have to inductively prove a stronger property.

Definition 2.6. Consider an open set $U \subseteq \mathbb{R}^n$, a function $f: U \to \mathbb{R}$ and an integer $p \geqslant 0$. We say that f has property $(*_p)$ if f is C^p and, for any multi-index α such that $|\alpha| \leqslant p$, the graph of $\frac{\partial^{|\alpha|} f}{\partial x^{\alpha}}$ is a simple sub- Λ -set.

Remark 2.7. If $f: \mathbb{R}^n \to \mathbb{R}$ is a polynomial and $I_{n,r}$ is a polydisk, then the graph of $f \upharpoonright I_{n,r}$ is a quantifier-free definable Λ -set. Thus, $f \upharpoonright I_{n,r}$ has $(*_p)$ for every $p \geq 0$. Also, since the germ of H at 0 is weakly C^{∞} , for any integer $p \geq 0$, there exists an interval $I_{1,r}$ such that $H \upharpoonright I_{1,r}$ is C^p . Furthermore, since H is piecewise a polynomial in the complement of any neighborhood of 0, the graph of the derivative $H^{(k)} \upharpoonright I_{1,r}$ is a Λ -set for every $k \leq p$. In particular, for any $p \geq 0$, there is some polyradius r such that $H \upharpoonright I_{1,r}$ has $(*_p)$.

Assuming the results proven in the next two paragraphs, we can now prove the lemma as follows.

Proof of Lemma 2.5. We define \mathcal{B}_n to be the set of germs $f \in \mathcal{W}_n$ such that, for every $p \geq 0$, f has a reprensentative $f: I_{n,r} \to \mathbb{R}$ having $(*_p)$. It is clear that \mathcal{B}_n is a subalgebra of \mathcal{W}_n . By the remark above, $H \in \mathcal{B}_1$ and the germs of polynomials in n variables belong to \mathcal{B}_n . Furthermore, by Propositions 2.8, 2.9 and 2.10, this collection of algebras is stable under composition, implicit functions and monomial division respectively. Thus, $\mathcal{A}_n(H) \subseteq \mathcal{B}_n$ by Definition 1.1 whence the result.

2.3 Composition and implicit functions

The goal of this paragraph is to show that the algebras \mathcal{B}_n in the proof of Lemma 2.5 are stable under composition of functions and taking implicit functions. A rather natural approach to these results involves proving stability properties for functions having the property $(*_p)$ for a fixed integer p, which is more than we need. In this paragraph and the next, if $a = (a_1, \ldots, a_n)$, it will be convenient to write $\hat{a} = (a_1, \ldots, a_{n-1})$. Since both of the propositions below are proven in essentially the same way, we omit the proof of Proposition 2.8 about composition.

Proposition 2.8. Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open sets and consider functions $f_1, \ldots, f_m \colon U \to \mathbb{R}$ and $g \colon V \to \mathbb{R}$ that all have $(*_p)$. Assume that $f(U) \subseteq V$ where $f = (f_1, \ldots, f_m)$. Then, the function $g \circ f$ has $(*_p)$.

Proposition 2.9. Let $U \subseteq \mathbb{R}^n$ be an open set and consider a C^1 function $f: U \to \mathbb{R}$ that has $(*_p)$. Consider also $a \in U$ such that f(a) = 0 and $\frac{\partial f}{\partial x_n}(a) \neq 0$. Then, there exists an open neighborhood $V \subseteq \mathbb{R}^{n-1}$ of \widehat{a} and a function $\varphi: V \to \mathbb{R}$ having $(*_p)$ such that $\varphi(\widehat{a}) = a_n$ and $f(\widehat{x}, \varphi(\widehat{x})) = 0$ for all $\widehat{x} \in V$.

Proof. We prove the result by induction on p. Firstly, assume that p=0. Since f is C^1 , the Implicit Functions Theorem yields a polydisk $I_{n,r}$ such that $a+I_{n,r}\subseteq U$ and, for each $\widehat{x}\in\widehat{a}+I_{n-1,\widehat{r}}$, there is a unique x_n such that $x\in a+I_{n,r}$ and f(x)=0. The set

$$A = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : x \in (a + I_{n,r}), y = f(x) \text{ and } y = 0\}$$

is a simple sub- Λ -set. The projection of A on the x-coordinates is the simple sub- Λ -set

$$B = \{x \in a + I_{n,r} : f(x) = 0\}.$$

The results above show that the set B is the graph of a function $\varphi \colon V \to \mathbb{R}$ where $V = \widehat{a} + I_{n-1,\widehat{r}}$. Since the fibers of this projection are finite, the graph of φ is a simple sub- Λ -set. Finally, we have $f(\widehat{x}, \varphi(\widehat{x})) = 0$ for all $\widehat{x} \in V$ by definition.

Before we consider the case p > 0, we need an intermediate result. Consider $0 < \alpha < \beta$ and let $U = \{x \in \mathbb{R} : \alpha < |x| < \beta\}$. Define $g: U \to \mathbb{R}$ as $g(x) = \frac{1}{x}$.

We are going to show that g has $(*_p)$ for every $p \ge 0$. Firstly, if $r = (\beta, \frac{1}{\alpha})$, then

$$\Gamma(g) = \{(x, y) \in I_{n,r} : xy = 1\}$$

is a Λ -set. Also, given $p \ge 0$, the derivative $g^{(p)}$ has the form $g^{(p)} = P \circ g$ where P is a polynomial. By composition, we deduce that the graph of $g^{(p)}$ is a simple sub- Λ -set which concludes.

Going back to the proof of the proposition, assume that p > 0. Then, we already know that there is an open neighborhood $V \subseteq \mathbb{R}^{n-1}$ of \widehat{a} and a function $\varphi \colon V \to \mathbb{R}$ having $(*_{p-1})$ such that $\varphi(\widehat{a}) = a_n$ and $f(\widehat{x}, \varphi(\widehat{x})) = 0$ for all $\widehat{x} \in V$. Up to shrinking V, we may also assume that there are $0 < \alpha < \beta$ such that $\alpha < \left| \frac{\partial f}{\partial x_n}(\widehat{x}, \phi(\widehat{x})) \right| < \beta$ for all $\widehat{x} \in V$. Then, given $1 \leqslant i \leqslant n-1$ and $\widehat{x} \in V$, we have

$$\frac{\partial \varphi}{\partial x_i}(\widehat{x}) = -\frac{\frac{\partial f}{\partial x_i}(\widehat{x}, \varphi(\widehat{x}))}{\frac{\partial f}{\partial x_n}(\widehat{x}, \varphi(\widehat{x}))}.$$

By assumption, we know that the functions $\frac{\partial f}{\partial x_i}$ and $\frac{\partial f}{\partial x_n}$ have $(*_{p-1})$. Furthermore, by the inductive hypothesis, φ also has $(*_{p-1})$. Thus, using Proposition 2.8 about composition, the fact that the polynomial $(x,y)\mapsto xy$ has $(*_{p-1})$ and the above result about the function g, we find that $\frac{\partial \varphi}{\partial x_i}$ has $(*_{p-1})$. Since this holds for all $1 \leq i \leq n-1$, the function φ must have $(*_p)$.

2.4 Monomial division

This paragraph is devoted to showing that the algebras \mathcal{B}_n introduced in the proof of Lemma 2.5 are stable under monomial division. The proof of this result is more delicate than those of the corresponding results for composition and implicit functions. This is because there does not seem to be an easy way to argue that functions having property $(*_p)$ are closed under monomial division.

Proposition 2.10. Let $I_{n,r}$ be a polydisk and consider a function $f: I_{n,r} \to \mathbb{R}$. Assume that, for every $p \geqslant 0$, there is some polydisk $I_{n,r'} \subseteq I_{n,r}$ such that $f \upharpoonright I_{n,r'}$ has $(*_p)$. Consider also $q \geqslant 0$ such that, for each $0 \leqslant k \leqslant q$, the derivative $\frac{\partial^k f}{\partial x_n^k}$ exists and verifies that $\frac{\partial^k f}{\partial x_n^k}(x) = 0$ for all $x \in I_{n,r}$ such that $x_n = 0$. Then, for every $p \geqslant 0$, there is some polydisk $I_{n,r'} \subseteq I_{n,r}$ such that there is a continuous function $g: I_{n,r'} \to \mathbb{R}$ having $(*_p)$ with $x_n^{q+1}g(x) = f(x)$ for every $x \in I_{n,r'}$.

Notice in particular that the assumptions of the lemma imply that the germ of f at 0 is weakly C^{∞} .

Proof. We prove by induction on p that the result holds for all integers q and all functions f with the required properties. Firstly, assume that p=0. Up to shrinking $I_{n,r}$, we may assume that f has $(*_{q+1})$. In particular, since f is C^{q+1} , we can extend $x \mapsto \frac{f(x)}{x_q^{q+1}}$ to a continuous function $g: I_{n,r} \to \mathbb{R}$ such that

$$x_n^{q+1}g(x) = f(x)$$

for every $x \in I_{n,r}$. Since f has $(*_0)$, the set

$$\{(x,y,z)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}:x_n\neq 0,\ y=f(x),\ zx_n^{q+1}=y\}$$

is a simple sub- Λ -set. Also, the projection of this set on the (x, z)-coordinates is the graph of $g \upharpoonright \{x \in I_{n,r} : x_n \neq 0\}$. Furthermore, if $x \in I_{n,r}$ is such that $x_n = 0$, then we have

$$g(x) = \frac{1}{(q+1)!} \frac{\partial^{q+1} f}{\partial x_n^{q+1}}(x).$$

Also, since f has $(*)_{q+1}$, it follows that

$$\left\{ (x,y) \in \mathbb{R}^n \times \mathbb{R} : x_n = 0, \ y = \frac{1}{(q+1)!} \frac{\partial^{q+1} f}{\partial x_n^{q+1}} (x) \right\}$$

is a simple sub- Λ -set. Since this is the graph of $g \upharpoonright \{x \in I_{n,r} : x_n = 0\}$, we see that the graph of g is a simple sub- Λ -set by taking a union.

Now, assume that p > 0. Up to shrinking $I_{n,r}$, we can assume that there is a function $g: I_{n,r} \to \mathbb{R}$ that has $(*_0)$ and such that

$$x_n^{q+1}g(x) = f(x)$$

for every $x \in I_{n,r}$ such that $x_n \neq 0$. Up to shrinking $I_{n,r}$ some more, we can assume that f is C^{q+2} . Next, given an integer $1 \leqslant i \leqslant n-1$, $\frac{\partial g}{\partial x_i}$ is the continuous function such that

$$x_n^{q+1} \frac{\partial g}{\partial x_i}(x) = \frac{\partial f}{\partial x_i}(x)$$

for every $x \in I_{n,r}$. By the inductive hypothesis, up to shrinking $I_{n,r}$ finitely many times, we are reduced to the case when $\frac{\partial g}{\partial x_i}$ has $(*_{p-1})$ for each $1 \leq i \leq n-1$. Finally, $\frac{\partial g}{\partial x_n}$ is the continuous function such that

$$x_n^{q+2} \frac{\partial g}{\partial x_n}(x) = x_n \frac{\partial f}{\partial x_n}(x) - (q+1)f(x)$$

for every $x \in I_{n,r}$. Thus, by the inductive hypothesis and Proposition 2.8, we can assume that $\frac{\partial g}{\partial x_n}$ has $(*_{p-1})$ up to shrinking $I_{n,r}$. Finally, g has $(*_0)$ and, for every $1 \leq i \leq n$, $\frac{\partial g}{\partial x_i}$ has $(*_{p-1})$. Thus, g has $(*_p)$ whence the result.

3 Parametrization results

In this section we prove a local parametrization result for H-basic sets around 0. In Paragraph 3.1, we start by recalling the process of monomialization. This process allows us to replace germs in $\mathcal{A}_n(H)$ with normal germs, namely, germs $f \in \mathcal{A}_n(H)$ such that there are $\alpha \in \mathbb{N}^n$ and $u \in \mathcal{A}_n(H)$ with $u(0) \neq 0$ and $f = x^{\alpha}u$. The advantage of such germs is that, in a neighborhood of 0, the sign of f only depends on the signs of the variables x_1, \ldots, x_n . In particular, H-basic

sets defined by equations and inequations involving only normal germs have a very simple structure. In Paragraph 3.2, we take advantage of this to obtain a first parametrization result, namely Proposition 3.14. We then introduce the notion of H-manifolds which are both manifolds and H-basic sets at the same time. We rephrase Proposition 3.14 in terms of H-manifolds in Corollary 3.19. The two remaining paragraphs are dedicated to proving a refinement of this latter result by showing that the manifolds involved in the parametrization may verify two, rather technical, regularity conditions.

3.1 Monomialization

This paragraph is a summary that aims at proving a slightly stronger monomialization result than [7, Theorem 2.11]. We recall the notations but we refer to [7, Section 2] for the proofs of the results that do not require any modification.

Definition 3.1. An elementary transformation is a map $\nu: I_{n,r'} \to I_{n,r}$ of one of the following forms.

• A blow-up chart: Given $1 \le i, j \le n$ with $i \ne j$ and $\lambda \in \mathbb{R}$, define

$$\pi_{i,j}^{\lambda}(x') = x \text{ where } \begin{cases} x_k = x'_k & k \neq i \\ x_i = x'_j(\lambda + x'_i). \end{cases}$$

Also, define $\pi_{i,j}^{\infty} = \pi_{j,i}^{0}$.

• A Tschirnhausen translation: Given $h \in \mathcal{A}_{n-1}(H)$ such that h(0) = 0, define

$$\tau_h(x') = x \text{ where } \begin{cases} x_k = x'_k & k \neq n \\ x_n = x'_n + h(x'_1, \dots, x'_{n-1}). \end{cases}$$

• A shear transformation: Given $1 \le i \le n$ and c_1, \ldots, c_{i-1} , define

$$L_{i,c}(x') = x \text{ where } \begin{cases} x_k = x'_k & i \leq k \leq n \\ x_k = x'_k + c_k x'_i & 1 \leq k < i. \end{cases}$$

• A ramification: Given $1 \le i \le n$ and $d \in \mathbb{N}$, define

$$r_i^{d,\pm}(x') = x$$
 where
$$\begin{cases} x_k = x'_k & k \neq i \\ x_i = \pm x'_i^d \end{cases}$$

An admissible transformation is a composition of elementary transforma-

Remark 3.2. Let ρ be an admissible transformation. If we write $\rho = (\rho_1, \ldots, \rho_n)$ then $\rho_1, \ldots, \rho_n \in \mathcal{A}_n(H)$. In particular, the germs of ρ_1, \ldots, ρ_n at 0 are weakly C^{∞} . Furthermore, $\rho(0) = 0$ so that, for any series $F \in \mathbb{R}[X]$, we can define

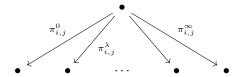
$$F \circ \rho = F(\mathbf{T}(\rho_1), \dots, \mathbf{T}(\rho_n))$$

where $\mathbf{T} \colon \mathcal{A}_n(H) \to \mathbb{R}[\![X]\!]$ is the map which takes a germ to its Taylor series. It is easy to see that the map $\mathbb{R}[\![X]\!] \to \mathbb{R}[\![X]\!], F \mapsto F \circ \rho$ is an algebra homomorphism. Also, by composition of Taylor series, if $f \in \mathcal{A}_n(H)$ then $\mathbf{T}(f \circ \rho) = \mathbf{T}(f) \circ \rho$.

Proposition 3.3 ([7, Lemma 2.5]). If ρ is an admissible transformation then the algebra homomorphism $F \mapsto F \circ \rho$ defined in the remark above is injective.

Definition 3.4. An **elementary tree** is a tree of any of the following forms.

• Given $1 \leq i, j \leq n$ such that $i \neq j$, consider the tree



with one branch for each transformation in the family $(\pi_{i,j}^{\lambda})_{\lambda \in \mathbb{R} \cup \{\infty\}}$.

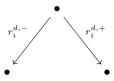
• Given $h \in \mathcal{A}_{n-1}(H)$ with h(0) = 0, consider the tree



• Given $1 \leq i \leq n$ and $c = (c_1, \ldots, c_{i-1})$, consider the tree



• Given $1 \leq i \leq n$ and $d \in \mathbb{N}$, consider the tree



Admissible trees of height at most h are defined inductively on the ordinal h. An admissible tree of height 0 is simply a vertex \bullet . Given an ordinal $h \ge 1$, an admissible tree of height at most h is an elementary tree with admissible trees of height < h attached to each of its leaves.

Remark 3.5. Notice that the height of an admissible tree T is an ordinal so that admissible trees may be infinite. However, each branch of T is finite so that it induces an admissible transformation ρ .

Definition 3.6. Let $X = (X_1, ..., X_n)$ be a tuple of variables and $F \in \mathbb{R}[\![X]\!]$ be non-zero. We say that F is **normal** when there are $\alpha \in \mathbb{N}^n$ and $U \in \mathbb{R}[\![X]\!]$ an invertible series, namely an invertible element of the ring $\mathbb{R}[\![X]\!]$, such that $F = X^{\alpha}U$. If $f \in \mathcal{A}_n(H)$ is non-zero, we say that it is **normal** when $\mathbf{T}(f)$ is normal.

Remark 3.7. A series $F \in \mathbb{R}[\![X]\!]$ is invertible if and only if $F(0) \neq 0$. Also, consider $f \in \mathcal{A}_n(H)$ a normal germ. Then, there are $\alpha \in \mathbb{N}^n$ and $U \in \mathbb{R}[\![X]\!]$ an invertible series such that $\mathbf{T}(f) = X^{\alpha}U$. But then, since germs in $\mathcal{A}_n(H)$ are stable under monomial division, it follows that there is a germ $u \in \mathcal{A}_n(H)$ such that $f(x) = x^{\alpha}u(x)$. It is now clear that $\mathbf{T}(u) = U$ whence, in particular, $u(0) \neq 0$.

Lemma 3.8 ([7, Lemma 2.9]). If $F_1, \ldots, F_p \in \mathbb{R}[\![X]\!]$ are non-zero series then they are all normal if and only if their product $\prod_{i=1}^p F_i$ is normal. Also, if $F, G \in \mathbb{R}[\![X]\!]$ are two non-zero series such that F, G and F - G are all normal then either $F \mid G$ or $G \mid F$.

Definition 3.9. Given $F_1, \ldots, F_p \in \operatorname{im} \mathbf{T}$ and an admissible tree T, we say that T monomializes F_1, \ldots, F_p when $F_1 \circ \rho, \ldots, F_p \circ \rho$ are normal for each admissible transformation ρ induced by a branch of T.

Given a blow-up chart $\nu = \pi_{i,j}^{\lambda}$ with $\lambda \neq \infty$, we have

$$\pi_{i,j}^{\lambda}(X') = X \text{ if and only if } \begin{cases} X'_k = X_k & k \neq i \\ X'_i = \frac{X_i}{X_i} - \lambda \end{cases}$$

Thus, we need to divide by X_j in order to write the inverse of ν . We call X_j the **critical variable** of ν . Since $\pi_{i,j}^{\infty} = \pi_{j,i}^{0}$, this definition applies to all blow-up charts. We say that T *-monomializes F_1, \ldots, F_p whenever it monomializes F_1, \ldots, F_p and, for any blow-up chart ν in T, the sub-tree T' of T below ν monomializes W, where W is the critical variable of ν .

Remark 3.10. Given an elementary transformation ν and $F \in \operatorname{im} \mathbf{T}$, we let $\nu^*(F) = W \cdot (F \circ \nu)$ when ν is a blow-up chart with critical variable W and $\nu^*(F) = F \circ \nu$ otherwise. In view of Lemma 3.8, an admissible tree T *-monomializes F if and only if, for each branch (ν_1, \ldots, ν_h) of T, the germ $\nu_h^* \circ \cdots \circ \nu_1^*(F)$ is normal.

Theorem 3.11. Let $F_1, \ldots, F_p \in \text{im } \mathbf{T}$ be non-zero series. There is an admissible tree T that *-monomializes F_1, \ldots, F_p .

Similar, but slightly different statements can be found in [7, Theorem 2.11] or [9, Section 2] for instance. Nonetheless, we include the proof for clarity, and also because we want to insist on the need to keep the variables X_1, \ldots, X_n monomialized after every step of the algorithm, as explained in the remark below.

Remark 3.12. Suppose that we have series $F, G \in \mathbb{R}[\![X]\!]$ such that $F = X_nG$. It is not necessarily true that any tree T which monomializes G will also monomialize F. For instance, suppose n=2 and write the variables X,Y instead of X_1, X_2 . If G = Y - X then $G \circ \tau_X(X', Y') = Y'$ is normal but $X \circ \tau_X(X', Y') = Y' + X'$ is not.

However, assume that $F, G \in \mathbb{R}[\![X]\!]$ and $H \in \mathbb{R}[\![\widehat{X}]\!]$ are such that F = HG. Assume also that H is normal and that T is an admissible tree satisfying the following conditions:

- There are no shear transformations in T;
- The only Tschirnhausen translations in T act on the variable X_n ;
- The only blow-up charts of the form $\pi_{n,i}^{\infty}$ with $1 \leq i \leq n-1$ in T are at the leaves and there are no blow-ups of the form $\pi_{i,n}$ with $1 \leq i \leq n-1$ in the tree.

It is easy to see that $H \circ \rho$ is still normal, whenever ρ is an admissible transformation induced by a branch of T. In particular, if T monomializes G, then it also monomializes F.

In particular, it is very important that we are done with no further steps whenever we apply a blow-up chart of the form $\pi_{n,i}^{\infty}$ as in case 1 of the proof below. This is also the reason why we pay attention throughout the proof at keeping the variables X_1, \ldots, X_{n-1} monomialized at every step. The additional benefit this provides is that it guarantees that the tree T automatically *-monomializes the series F_1, \ldots, F_p since the only blow-up charts that do not appear at the leaves of the tree have one of X_1, \ldots, X_{n-1} as their critical variable.

Proof. We are going to prove the result by induction on n, the case n=1 being obvious. To begin with, we may assume that p=1 by Lemma 3.8 and we write $F=F_1=\sum_{\alpha}a_{\alpha}X^{\alpha}$. Take $c_1,\ldots,c_{n-1}\in\mathbb{R}$ and let $G=F\circ L_c$. We then have

$$G(0, \dots, 0, X'_n) = F(c_1 X'_n, \dots, c_{n-1} X'_n, X'_n) = \sum_{\alpha} a_{\alpha} c^{\widehat{\alpha}} X'_n^{|\alpha|} = \sum_{k \in \mathbb{N}} Q_k(c) X'_n^{|\alpha|}$$

where $\widehat{\alpha} = (\alpha_1, \dots, \alpha_{n-1})$, and

$$Q_k(Y_1, \dots, Y_{n-1}) = \sum_{|\alpha|=k} a_{\alpha} Y^{\widehat{\alpha}}.$$

We also have

$$F(X_1'X_n',\ldots,X_{n-1}'X_n',X_n') = \sum_{k \in \mathbb{N}} Q_k(\widehat{X'})X_n'^{|\alpha|}$$

and $F(X_1'X_n',\ldots,X_{n-1}'X_n',X_n')$ is obtained by composing F with blow-up charts. Since composition with blow-up charts is injective and $F\neq 0$, we deduce that there must be some $k\in\mathbb{N}$ with $Q_k\neq 0$. But then, there are $c_1,\ldots,c_{n-1}\in$

 \mathbb{R} such that $Q_k(c) \neq 0$. For this tuple $c = (c_1, \ldots, c_{n-1})$, we deduce that $G(0, \ldots, 0, X'_n) \neq 0$, from which it follows that the series G is regular of some order d in X'_n , namely, there are series $G_1, \ldots, G_d \in \mathbb{R}[\widehat{X'}]$ and a unit $U \in \mathbb{R}[X']$ such that

$$G(X') = U(X')X_n'^d + G_1(\widehat{X'})X_n'^{d-1} + \dots + G_d(\widehat{X'})$$

where $\widehat{X'} = (X'_1, \dots, X'_{n-1})$. It suffices to show that we can *-monomialize G so that it suffices to prove the result when F = G. In this latter case, we write

$$F(X) = U(X)X_n^d + F_1(\hat{X})X_n^{d-1} + \dots + F_d(\hat{X}).$$

We now show by induction on d that we can *-monomialize F, X_1, \ldots, X_{n-1} , the result being obvious when d = 0. Assume that d > 0 and let $f \in \mathcal{A}_n(H)$ be such that $F = \mathbf{T}(f)$. We have

$$\mathbf{T}\left(\frac{\partial^d f}{\partial x_n^d}\right) = \frac{\partial^d F}{\partial X_n^d} = d!U$$

so that $\frac{\partial^d f}{\partial x_n^d}(0) \neq 0$. Furthermore,

$$\mathbf{T}\left(\frac{\partial^{d-1}f}{\partial x_n^{d-1}}\right) = \frac{\partial^{d-1}F}{\partial X_n^{d-1}} = (d-1)![F_1(\widehat{X}) + U(X)X_n].$$

If $F_1(0) \neq 0$ then F is regular in X_n of order d-1 and the result follows by induction. Otherwise, $\frac{\partial^{d-1} f}{\partial x_n^{d-1}}(0) = 0$. Thus, by Definition 1.1, there is $b \in \mathcal{A}_{n-1}(H)$ such that $\frac{\partial^{d-1} f}{\partial x_n^{d-1}}(\widehat{x}, b(\widehat{x})) = 0$ and b(0) = 0. Now, if $g = f \circ \tau_b$, we have

$$\frac{\partial^{d-1}g}{\partial x_n'^{d-1}}(\widehat{x'},0) = \frac{\partial^{d-1}f}{\partial x_n^{d-1}}(\widehat{x'},b(\widehat{x'})) = 0.$$

It is also easy to see that $\mathbf{T}(g)$ is regular of order d in X_n so that, up to replacing F with $F \circ \tau_b$, we can assume that $\frac{\partial^{d-1}F}{\partial X_n^{d-1}}(\widehat{x},0) = 0$ which implies that $F_1 = 0$. In particular, notice that F is then normal whenever d = 1.

By the inductive hypothesis on n, there is an admissible tree T that *-monomializes the series $F_2, \ldots, F_d, X_1, \ldots, X_{n-1}$. Let (ν_1, \ldots, ν_k) be a branch of T. We must now show that we can *-monomialize the series

- $F \circ \nu_1 \circ \cdots \circ \nu_k$;
- $X_i \circ \nu_1 \circ \cdots \circ \nu_k$ for every $1 \leqslant i \leqslant n-1$;
- $W \circ \nu_{j+1} \circ \cdots \circ \nu_k$ whenever ν_j is a blow-up chart with critical variable W.

Since the series in the last two points are already normal, it suffices to show that we can *-monomialize $F \circ \nu_1 \circ \cdots \circ \nu_k$ and X_1, \ldots, X_{n-1} . Thus, up to replacing

F with $F \circ \nu_1 \circ \cdots \circ \nu_k$, we can assume that F_2, \ldots, F_d are all normal so that we can write $F_i = \widehat{X}^{\alpha_i} U_i(\widehat{X})$ where U_i is a unit.

Let $\sigma = (\sigma_1, \dots, \sigma_{n-1}) \in \{+, -\}^{n-1}$ and define

$$r^{\sigma} = r_1^{d!,\sigma_1} \circ \cdots \circ r_{n-1}^{d!,\sigma_{n-1}}.$$

Up to replacing F with $F \circ r^{\sigma}$, it suffices to prove the result when α_i is divisible by d! for every $2 \leqslant i \leqslant d$. Now, by the inductive hypothesis on n there is an admissible tree T that *-monomializes X_1, \ldots, X_{n-1} and the series $\widehat{X}^{\frac{\alpha_i}{i}} - \widehat{X}^{\frac{\alpha_j}{j}}$ for every $2 \leqslant i, j \leqslant d$. By the same argument as above, it suffices to show that, for every branch (ν_1, \ldots, ν_k) of T, we can *-monomialize the series $F \circ \nu_1 \circ \cdots \circ \nu_k$ and X_1, \ldots, X_{n-1} . Thus, by Lemma 3.8, up to replacing F with $F \circ \nu_1 \circ \cdots \circ \nu_k$, we may assume that the tuples $\frac{\alpha_i}{i}$ are linearly ordered.

Let $2 \leqslant l \leqslant d$ be minimal such that $\frac{\alpha_l}{l} \leqslant \frac{\alpha_i}{i}$ for every $2 \leqslant i \leqslant d$. Since the partial order on tuples $\beta \in \mathbb{N}^{n-1}$ is well-founded, we may prove the result by induction on $\frac{\alpha_l}{l}$. When $\alpha_l = 0$, F is regular in X_n of order d-l whence the result follows by induction. Otherwise, up to exchanging the order of the variables X_1, \ldots, X_{n-1} , we can assume that $\alpha_{l,1} \neq 0$. We can now do a blow-up between the variables X_1 and X_n . There are three cases to treat. For simplicity of notation, we still write X for the variables after the various blow-up charts.

Case 1: $\lambda = \infty$.

$$\begin{split} F \circ \pi_{n,1}^{\infty}(X) &= \widetilde{U}(X)X_n^d + \sum_{i=2}^d \widetilde{U_i}(X)\widehat{X}^{\alpha_i}X_n^{d-i+\alpha_{i,1}} \\ &= X_n^d \left[\widetilde{U}(X) + \sum_{i=2}^d \widetilde{U_i}(X)\widehat{X}^{\alpha_i}X_n^{\alpha_{i,1}-i}\right] \end{split}$$

where $\widetilde{U} = U \circ \pi_{n,1}^{\infty}$ and $\widetilde{U}_i = U_i \circ \pi_{n,1}^{\infty}$ for every $2 \leqslant i \leqslant d$. The series $F \circ \pi_{n,1}^{\infty}$ is normal since \widehat{X}^{α_i} is never a constant for $2 \leqslant i \leqslant d$. Thus, we are done in this case.

Case 2: $\lambda \notin \{\infty, 0\}$.

$$F \circ \pi_{n,1}^{\lambda}(X) = \widetilde{U}(X)X_1^d(\lambda + X_n)^d + \sum_{i=2}^d \widetilde{U_i}(\widehat{X})\widehat{X}^{\alpha_i}X_1^{d-i}(\lambda + X_n)^{d-i}$$

$$= X_1^d \left[\widetilde{U}(X)X_n^d + d\lambda \widetilde{U}(X)X_n^{d-1} + \sum_{i=2}^d \binom{d}{i}\lambda^i \widetilde{U}(X)X_n^{d-i} + \sum_{i=2}^d \widetilde{F}_i(\widehat{X})X_n^{d-i} \right]$$

where $\widetilde{U} = U \circ \pi_{n,1}^{\lambda}$, $\widetilde{U}_i = U_i \circ \pi_{n,1}^{\lambda}$ for every $2 \leqslant i \leqslant d$ and $\widetilde{F}_i \in \mathbb{R}[\widehat{X}]$. Let $G \in \mathbb{R}[X]$ be the series such that $F(X) = X_1^d G(X)$. Since $\widetilde{U}(X) = X_1^d G(X)$ $U(X_1, \ldots, X_{n-1}, X_1 X_n)$, the series \widetilde{U} cannot be regular in X_n of any order other than 0. Thus, the series G is regular in X_n of order d-1. Since it suffices to show that we can *-monomialize the series G, X_1, \ldots, X_{n-1} by Lemma 3.8, the result now follows by induction on d.

Case 3: $\lambda = 0$.

$$F \circ \pi_{n,1}^0 = \widetilde{U}(X) X_1^d X_n^d + \sum_{i=2}^d \widetilde{U_i}(\widehat{X}) \widehat{X}^{\alpha_i} X_1^{d-i} X_n^{d-i}$$
$$= X_1^d \left[\widetilde{U}(X) X_n^d + \sum_{i=2}^d \widetilde{U_i}(\widehat{X}) \widehat{X}^{\beta_i} X_n^{d-i} \right]$$

where $\widetilde{U} = U \circ \pi^0_{n,1}$, $\widetilde{U_i} = U_i \circ \pi^0_{n,1}$ for every $2 \leqslant i \leqslant d$ and $\beta_{i,j} = \alpha_{i,j}$ for $2 \leqslant i \leqslant d$ and $2 \leqslant j \leqslant n-1$ while $\beta_{i,1} = \alpha_{i,1} - i$. Now, let

$$G(X) = \widetilde{U}(X)X_n^d + \sum_{i=2}^d \widetilde{U_i}(\widehat{X})\widehat{X}^{\beta_i}X_n^{d-i}.$$

Since $F \circ \pi_{n,1}^0 = X_1^d G$, it suffices to show that we can *-monomialize the series G, X_1, \ldots, X_{n-1} by Lemma 3.8. But $\beta_l < \alpha_l$ so that the result follows by induction on $\frac{\alpha_l}{I}$

3.2 A first parametrization

Let $g_0, \ldots, g_n \in \mathcal{A}_n(H)$ be functions defined on $I_{n,r}$ and consider the *H*-basic set

$$A = \{x \in I_{n,r} : g_0(x) = 0, g_1(x) > 0, \dots, g_q(x) > 0\}.$$

If $\rho: I_{n,r'} \to I_{n,r}$ is an admissible transformation then

$$B := \rho^{-1}(A) = \{ x' \in I_{n,r'} : g_0 \circ \rho(x') = 0, g_1 \circ \rho(x') > 0, \dots, g_q \circ \rho(x') > 0 \}.$$

Furthermore, if the germs at 0 of the functions $g_0 \circ \rho, \ldots, g_q \circ \rho$ are all normal, then their sign in a neighborhood of 0 only depends on the signs of the variables x_1, \ldots, x_n . Thus, in this latter case, there is a neighborhood U of 0 in B that is a finite union of sub-quadrants as defined below.

Definition 3.13. A sub-quadrant is a set of the form $Q = D_1 \times \cdots \times D_n$ where each set D_i is either $\{0\}$ or $(0, r_i)$ or $(-r_i, 0)$ for some $r_i > 0$.

The discussion above combined with Theorem 3.11 let us adapt [7, Proposition 3.4] as follows.

Proposition 3.14. Consider an H-basic set A. There is a family $(Q_j, \rho_j)_{j \in J}$ such that

• $Q_j \subseteq \mathbb{R}^n$ is a sub-quadrant and ρ_j is an admissible transformation defined on Q_j such that $\rho_j \upharpoonright Q_j$ is a diffeomorphism onto its image $\rho_j(Q_j) \subseteq A$.

• If $U_j \subseteq \mathbb{R}^n$ is a neighborhood of 0 for each $j \in J$ then there is a finite subset $J_0 \subseteq J$ such that

$$\bigcup_{j\in J_0} \rho_j(Q_j\cap U_j)$$

is a neighborhood of 0 in A, namely, there is a neighborhood $V \subseteq \mathbb{R}^n$ of 0 such that $\bigcup_{j \in J_0} \rho_j(Q_j \cap U_j) = V \cap A$.

Proof. Consider functions $g_0, \ldots, g_q \in \mathcal{A}_n(H)$ defined on some polydisk $I_{n,r}$ such that

$$A = \{x \in I_{n,r} : g_0(x) = 0, g_1(x) > 0, \dots, g_q(x) > 0\}.$$

By Theorem 3.11, there is an admissible tree T that *-monomializes the functions g_0,\ldots,g_q . The idea of the proof is as follows. Consider ρ an admissible transformation induced by a branch of T. The germs of $g_0 \circ \rho,\ldots,g_q \circ \rho$ at 0 are all normal so that, in particular, they do not vanish at 0. Thus, by continuity of $g_0 \circ \rho,\ldots,g_q \circ \rho$ up to shrinking Q we may assume that each function $g_0 \circ \rho,\ldots,g_q \circ \rho$ has constant sign on Q. This implies that either $\rho(Q) \subseteq A$ or $\rho(Q) \cap A = \emptyset$. Now, the elementary transformations that are not blow-up charts have inverses for composition. For the blow-up charts, as remarked before, we need to divide by the critical variable in order to write the inverse. Since T *-monomializes g_0,\ldots,g_q , we can easily track the sign of this critical variable. Finally, the set where the critical variable vanishes is "small" so that we can apply the result inductively to it. We are now going to give the details.

We prove the result by induction on the pairs (n, h) ordered lexicographically, where h is the height of T. If $n \in \{0, 1\}$ or if h = 0, the germs g_0, \ldots, g_q are already normal and the result follows at once. We can even take each ρ_i to be the identity.

Otherwise, h>0. Assume first that the elementary transformations attached to the root of T are not blow-up charts and let $\nu\colon I_{n,r'}\to I_{n,r}$ be one of these elementary transformations. The sub-tree T' of T below ν *-monomializes the functions $g_0\circ\nu,\ldots,g_q\circ\nu$. Thus, the result holds for the H-basic set

$$B_{\nu} = \{ x \in I_{n,r'} : g_0 \circ \nu(x) = 0, g_1 \circ \nu(x) > 0, \dots, g_q \circ \nu(x) > 0 \}.$$

Also, $\nu(B_{\nu}) \subseteq A$ and, since ν can be extended to a diffeomorphism $\mathbb{R}^n \to \mathbb{R}^n$, it follows that $\nu \upharpoonright B_{\nu}$ is a diffeomorphism onto its image. Furthermore, if $U_{\nu} \subseteq \mathbb{R}^n$ is a neighborhood of 0 for each elementary transformation ν attached to the root of T, then $\bigcup_{\nu} \nu(B_{\nu} \cap U_{\nu})$ is a neighborhood of 0 in A. Since there are only finitely many admissible transformations attached to the root of T, the result follows at once.

Assume next that the elementary transformations attached to the root of T are blow-up charts. Then, let w,w' be the two variables involved in this blow-up and, for each ν attached to the root of T, let w_{ν} be its critical variable. Now, consider $\nu \colon I_{n,r'} \to I_{n,r}$ an elementary transformation attached to the root of T.

The sub-tree T' of T below ν *-monomializes the functions $g_0 \circ \nu, \ldots, g_q \circ \nu, w_{\nu}$. Thus, the result holds inductively for the H-set

$$B_{\nu} = \{x \in I_{n,r'} : g_0 \circ \nu(x) = 0, g_1 \circ \nu(x) > 0, \dots, g_q \circ \nu(x) > 0, w_{\nu} \neq 0\}.$$

Also, $\nu(B_{\nu}) \subseteq A$ and $\nu \upharpoonright B_{\nu}$ is a diffeomorphism onto its image. Assume given $U_{\nu} \subseteq \mathbb{R}^n$ a neighborhood of 0 for each ν attached to the root of T. Then, by the compactness of $\mathbb{R} \cup \{\infty\}$ with its usual topology [3, p. 4406], there are ν_1, \ldots, ν_p some elementary transformations attached to the root of T such that

$$\bigcup_{i=1}^{p} \nu_i(B_i \cap U_i)$$

is a neighborhood of 0 in $A \cap \{w \neq 0 \text{ or } w' \neq 0\}$. Furthermore, $A \cap \{w = w' = 0\}$ can be identified with an H-basic set $A' \subseteq \mathbb{R}^{n-2}$ so that the result now follows by induction.

Remark 3.15. In [7, Proposition 3.4], the same compactness as above is used in order to obtain a finite family of charts. However, we cannot do the same in our case. This is because we will use local methods that will force us to restrict the charts to smaller neighborhoods of 0. If we only kept finitely many charts, there would be no way to guarantee that we still cover a neighborhood of 0 in A after such restrictions. Thus, we will first restrict all the charts in the parametrization as much as we need to and then we will use compactness to obtain a finite family.

Nonetheless, using compactness at this stage allows us to prove that every Λ -set has finitely many connected components as follows.

Corollary 3.16. Let $A \subseteq \mathbb{R}^n$ be a Λ -set. There are charts $(Q_1, \rho_1), \ldots, (Q_p, \rho_p)$ such that

- For each $1 \leq i \leq p$, $Q_i \subseteq \mathbb{R}^n$ is a sub-quadrant and ρ_i is an admissible transformation defined on Q_i .
- We have

$$A = \rho_1(Q_1) \cup \cdots \cup \rho_p(Q_p).$$

In particular, every Λ -set has finitely many connected components.

Proof. Since A is locally an H-set around every point of \overline{A} , the result follows at once from the proposition and the compactness of \overline{A} .

The parametrization part of the corollary will not be reused later, it serves only to show that Λ -sets have finitely many connected components.

Definition 3.17. A set $M \subseteq \mathbb{R}^n$ is called an H-manifold when there are a polydisk $I_{n,r}$ and functions $f_1, \ldots, f_{n-d}, g_1, \ldots, g_q \in \mathcal{A}_n(H)$ defined and C^1 on $I_{n,r}$ such that $\nabla f_1(x), \ldots, \nabla f_{n-d}(x)$ are linearly independent for every $x \in I_{n,r}$ and

$$M = \{x \in I_{n,r} : f_1(x) = \dots = f_{n-d}(x) = 0, \ g_1(x) > 0, \dots, \ g_q(x) > 0\}.$$

Remark 3.18. Assume that ρ is an admissible transformation defined on a sub-quadrant Q. Then, the set

$$M = \{ (y, x) \in \mathbb{R}^n \times \mathbb{R}^n : y_1 - \rho_1(x) = \dots = y_n - \rho_n(x) = 0, x \in Q \}$$

= $\{ (\rho(x), x) : x \in Q \} \subseteq \mathbb{R}^{2n}$

is an H-manifold. Furthermore, if $\rho \upharpoonright Q$ is a diffeomorphism onto its image, then $\Pi_n \upharpoonright M$ is also a diffeomorphism onto its image.

This remark provides us at once with the following restatement of Proposition 3.14

Corollary 3.19. Let $A \subseteq \mathbb{R}^n$ be an H-basic set. There is a family $(M_j)_{j \in J}$ of H-manifolds such that

- For every $j \in J$, there is $k_j \ge 0$ such that $M_j \subseteq \mathbb{R}^{n+k_j}$. Furthermore, the projection $\Pi_n \upharpoonright M_j$ is a diffeomorphism onto its image.
- If $U_j \subseteq \mathbb{R}^{n+k_j}$ is a neighborhood of 0 for every $j \in J$, then there is a finite subset $J_0 \subseteq J$ such that $\bigcup_{j \in J_0} \Pi_n(M_j \cap U_j)$ is a neighborhood of 0 in A.

Proof. Let $(Q_j, \rho_j)_{j \in J}$ be the family provided by Proposition 3.14. For each $j \in J$, we define $M_j = \{(\rho_j(x), x) : x \in Q_j\}$. It is clear that the family $(M_j)_{j \in J}$ satisfies the conditions of the corollary.

Remark 3.20. The set J in the statement of the corollary might be infinite.

The next step in the proof is to refine the statement of Corollary 3.19 by improving on the regularity of the manifolds M_j . The result we have in mind is Proposition 3.26 and we will build up to it over the course of the next two paragraphs.

3.3 A parametrization such that Π_n has constant rank

Consider $A \subseteq \mathbb{R}^{n+k}$ an H-basic set. By applying Corollary 3.19 to A, we obtain a family $(M_j)_{j\in J}$ of H-manifolds. Given $j\in J$, we already know that $\Pi_{n+k}\upharpoonright M_j$ is a diffeomorphism onto its image. Our goal in this section will be to show that we can always choose the family (M_j) so that it is also true that the projections $\Pi_n\upharpoonright M_j$ have constant rank. To be more precise, this paragraph is devoted to proving the following proposition.

Proposition 3.21. Let $A \subseteq \mathbb{R}^{n+k}$ be an H-basic set. Given $0 \le l \le n$, there is a family $(M_j)_{j \in J}$ of H-manifolds such that

- For every $j \in J$, there is $h_j \ge 0$ such that $M_j \subseteq \mathbb{R}^{n+k+h_j}$. Furthermore, the projection $\Pi_{n+k} \upharpoonright M_j$ is a diffeomorphism onto its image.
- If $U_j \subseteq \mathbb{R}^{n+k+h_j}$ is a neighborhood of 0 for every $j \in J$, then there is a finite subset $J_0 \subseteq J$ such that $\bigcup_{j \in J_0} \Pi_{n+k}(M_j \cap U_j)$ is a neighborhood of 0 in A.

• For every $j \in J$, the projection $\Pi_n \upharpoonright M_j$ has constant rank.

In order to do so, we start with the parametrization given in Corollary 3.19 and we further refine it by monomializing various determinants involving a basis for $T_x(M_j)$ for each $j \in J$. Thus, the first step is to give a basis of the tangent plane of an H-manifold in a neighborhood of 0 consisting of functions in $\mathcal{A}_n(H)$.

Proposition 3.22. Consider $M \subseteq \mathbb{R}^n$ an H-manifold of dimension d. There are a polydisk $I_{n,r}$ and functions $b_1, \ldots, b_d \colon I_{n,r} \to \mathbb{R}^n$, all of whose components are in $A_n(H)$, and such that $b_1(x), \ldots, b_d(x)$ is a basis of $T_x(M)$ for every $x \in M \cap I_{n,r}$.

Proof. Consider functions $f_1, \ldots, f_{n-d}, g_1, \ldots, g_q \in \mathcal{A}_n(H)$ defined and C^1 on a polydisk $I_{n,r}$ such that $\nabla f_1(x), \ldots, \nabla f_{n-d}(x)$ are independent for every $x \in I_{n,r}$ and

$$M = \{x \in I_{n,r} : f_1(x) = \dots = f_{n-d}(x) = 0, \ g_1(x) > 0, \ \dots, \ g_q(x) > 0\}.$$

Now, let $e_1, \ldots, e_d \in \mathbb{R}^n$ be vectors such that $\nabla f_1(0), \ldots, \nabla f_{n-d}(0), e_1, \ldots, e_d$ is a basis of \mathbb{R}^n . By continuity of $\det(\nabla f_1(x), \ldots, \nabla f_{n-d}(x), e_1, \ldots, e_d)$, we can assume that $\nabla f_1(x), \ldots, \nabla f_{n-d}(x), e_1, \ldots, e_d$ is a basis of \mathbb{R}^n for every $x \in I_{n,r}$ up to shrinking $I_{n,r}$.

For each $x \in I_{n,r}$, we let $a_1(x), \ldots, a_{n-d}(x), b_1(x), \ldots, b_d(x)$ be the result of applying the Gram-Schmidt orthonormalization process to the basis $\nabla f_1(x), \ldots, \nabla f_{n-d}(x), e_1, \ldots, e_d$. Since orthonormalization only involves taking sums, products and dividing by non-zero functions, it follows that all the components of the functions $a_1, \ldots, a_{n-d}, b_1, \ldots, b_d$ are in $\mathcal{A}_n(H)$. Given $x \in M \cap I_{n,r}$, the space $T_x(M)$ is the orthogonal of the space V spanned by the vectors $\nabla f_1(x), \ldots, \nabla f_{n-d}(x)$. Now, let $1 \leq j \leq d$ be an integer. Since V is also spanned by $a_1(x), \ldots, a_{n-d}(x)$ and since $b_j(x)$ is orthogonal to each of these vectors, it follows that $b_j(x) \in T_x(M)$. Thus, the vectors $b_1(x), \ldots, b_d(x)$ make up a linearly independent family in $T_x(M)$. Since $\dim(T_x(M)) = d$, this family must be a basis.

The rank of the projection Π_n on an H-manifold $M \subseteq \mathbb{R}^{n+k}$ can be computed by looking at the sign of various determinants. By monomializing the determinants in question, we can cut up M into smaller pieces on which each determinant has constant sign. In particular, we can guarantee that Π_n has constant rank on the pieces with "large" rank.

Lemma 3.23. Let $M \subseteq \mathbb{R}^{n+k}$ be an H-manifold with $\operatorname{rk}_x(\Pi_n \upharpoonright M) \leqslant l$ for every $x \in M$. Then, there are H-manifolds M_1, \ldots, M_p and an H-basic set M' such that $\Pi_n \upharpoonright M_i$ has constant rank l for every $1 \leqslant i \leqslant p$, $\operatorname{rk}_x(\Pi_n \upharpoonright M) < l$ for every $x \in M'$ and $M_1 \cup \cdots \cup M_p \cup M'$ is a neighborhood of 0 in M.

Proof. By Proposition 3.22, there are a polydisk $I_{n+k,r}$ as well as functions $b_1, \ldots, b_d \colon I_{n+k,r} \to \mathbb{R}^{n+k}$, all of whose components are in $\mathcal{A}_{n+k}(H)$, and such that $b_1(x), \ldots, b_d(x)$ is a basis of $T_x(M)$ for every $x \in M \cap I_{n+k,r}$. Without

loss of generality, we can assume that $M \subseteq I_{n+k,r}$. Given two strictly increasing functions $\iota: \{1, \ldots, l\} \to \{1, \ldots, n\}$ and $\kappa: \{1, \ldots, l\} \to \{1, \ldots, d\}$, we can consider the H-basic set

$$M_{\iota,\kappa} = \{ x \in M : \det(\Pi_{\iota}(b_{\kappa(1)}(x)), \dots, \Pi_{\iota}(b_{\kappa(l)}(x))) \neq 0 \}$$

Notice that $M_{\iota,\kappa}$ is the union of two H-manifolds and that $\Pi_n \upharpoonright M_{\iota,\kappa}$ has constant rank l. Furthermore, the set

$$M' = M \setminus \bigcup_{\iota,\kappa} M_{\iota,\kappa}$$

is H-basic. Finally, for $x \in M'$, we have $\operatorname{rk}_x(\Pi_n \upharpoonright M) < l$.

In Lemma 3.24 below, we refine the statement of the lemma above by using Corollary 3.19 in order to parametrize the set M'. We can thus replace M' with a family of H-manifolds such that the rank of Π_n on each of these manifolds is "small".

Lemma 3.24. Consider $M \subseteq \mathbb{R}^{n+k}$ an H-manifold such that $\operatorname{rk}_x(\Pi_n \upharpoonright M) \leqslant l$ for every $x \in M$. Then, there is a family $(M_j)_{j \in J}$ of H-manifolds such that

- For every $j \in J$, there is $h_j \ge 0$ such that $M_j \subseteq \mathbb{R}^{n+k+h_j}$. Furthermore, the projection $\Pi_{n+k} \upharpoonright M_j$ is a diffeomorphism onto its image.
- If $U_j \subseteq \mathbb{R}^{n+k_j}$ is a neighborhood of 0 for every $j \in J$, then there is a finite subset $J_0 \subseteq J$ such that $\bigcup_{j \in J_0} \Pi_{n+k}(M_j \cap U_j)$ is a neighborhood of 0 in A
- For every $j \in J$, either $\Pi_n \upharpoonright M_j$ has constant rank l or $\operatorname{rk}_x(\Pi_n \upharpoonright M_j) < l$ for every $x \in M_j$.

Proof. By Lemma 3.23, there are $M_1,\ldots,M_p,M'\subseteq\mathbb{R}^{n+k}$ such that the sets M_1,\ldots,M_p are H-manifolds, M' is H-basic, $M_1\cup\cdots\cup M_p\cup M'$ is a neighborhood of 0 in M, $\Pi_n\upharpoonright M_j$ has constant rank l for $1\leqslant j\leqslant p$ and $\mathrm{rk}_x(\Pi_n\upharpoonright M')< l$ for every $x\in M'$. Thus, it suffices to prove the result for M'. Now, consider a family $(M'_j)_{j\in J}$ of H-manifolds obtained by applying Corollary 3.19 to the set M'. For each $j\in J$ and $x\in M'_j$, we must have $\mathrm{rk}_x(\Pi_n\upharpoonright M'_j)\leqslant \mathrm{rk}_y(\Pi_n\upharpoonright M')< l$ where $y=\Pi_{n+k}(x)$, whence the result.

The lemma above allows us to prove Proposition 3.21. The idea is that we can continue applying the lemma to all of the pieces on which Π_n does not have constant rank until the rank of Π_n on these pieces goes to 0. The following lemma makes this idea precise. Notice also that Proposition 3.21 is a restatement of the case l=0 in the lemma below.

Lemma 3.25. Let $A \subseteq \mathbb{R}^{n+k}$ be an H-basic set. Given $0 \leqslant l \leqslant n$, there is a family $(M_j)_{j \in J}$ of H-manifolds such that

• For every $j \in J$, there is $h_j \ge 0$ such that $M_j \subseteq \mathbb{R}^{n+k+h_j}$. Furthermore, the projection $\Pi_{n+k} \upharpoonright M_j$ is a diffeomorphism onto its image.

- If $U_j \subseteq \mathbb{R}^{n+k+h_j}$ is a neighborhood of 0 for every $j \in J$, then there is a finite subset $J_0 \subseteq J$ such that $\bigcup_{j \in J_0} \Pi_{n+k}(M_j \cap U_j)$ is a neighborhood of 0 in A.
- For every $j \in J$, either the projection $\Pi_n \upharpoonright M_j$ has constant rank or $\operatorname{rk}_x(\Pi_n \upharpoonright M_j) \leqslant l$ for all $x \in M_j$.

Proof. The result follows from Lemma 3.24 by a decreasing induction on l.

3.4 Local charts

Consider an H-manifold $M \subseteq \mathbb{R}^{n+k}$ of dimension d such that $\Pi_n \upharpoonright M$ has constant rank $l \leqslant d$. Assume also that there is a strictly increasing sequence $\iota \colon \{1,\ldots,d\} \to \{1,\ldots,n+k\}$ such that $\Pi_\iota \upharpoonright M$ is an immersion. Then, Π_ι is also a local diffeomorphism so that we have a convenient way of identifying M with \mathbb{R}^d locally around every point. Furthermore, assume that $\iota(l) \leqslant n$ and define $\iota' \colon \{1,\ldots,l\} \to \{1,\ldots,n\}$ by $\iota'(i) = \iota(i)$ for every $1 \leqslant i \leqslant l$. Then, the following diagram commutes

$$M \xrightarrow{\Pi_{\iota}} \mathbb{R}^{d}$$

$$\Pi_{n} \downarrow \qquad \qquad \downarrow \Pi_{l}$$

$$\mathbb{R}^{n} \xrightarrow{\Pi_{\iota}} \mathbb{R}^{l},$$

showing that the pair $(\Pi_l, \Pi_{l'})$ provides us with a local identification of $\Pi_n \colon M \to \mathbb{R}^n$ with $\Pi_l \colon \mathbb{R}^d \to \mathbb{R}^l$. We may refine Corollary 3.21 as follows to assume that such sequences exist for each M_i .

Proposition 3.26. Let $A \subseteq \mathbb{R}^{n+k}$ be an H-basic set. There is a family $(M_j)_{j \in J}$ of H-manifolds such that

- For every $j \in J$, there is $h_j \ge 0$ such that $M_j \subseteq \mathbb{R}^{n+k+h_j}$. Furthermore the projection $\Pi_{n+k} \upharpoonright M_j$ is a diffeomorphism onto its image.
- If $U_j \subseteq \mathbb{R}^{n+k+h_j}$ is a neighborhood of 0 for every $j \in J$, then there is a finite subset $J_0 \subseteq J$ such that $\bigcup_{j \in J_0} \Pi_{n+k}(M_j \cap U_j)$ is a neighborhood of 0 in A
- For every $j \in J$, if $d = \dim M_j$ then $\Pi_n \upharpoonright M_j$ has constant rank l and there is some sequence $\iota : \{1, \ldots, d\} \to \{1, \ldots, n+k+h_j\}$ such that ι is strictly increasing, $\Pi_\iota \upharpoonright M_j$ is an immersion and $\iota(l) \leq n$.

The proposition follows at once by applying the lemma below to each of the H-manifolds M_j in the statement of Corollary 3.21. The lemma itself is simply a restatement of the discussion in [9] before Lemma 4.5.

Lemma 3.27. Let $M \subseteq \mathbb{R}^{n+k}$ be an H-manifold of dimension d such that $\Pi_n \upharpoonright M$ has constant rank $l \leqslant d$. Then, there are H-manifolds M_1, \ldots, M_p that are also open submanifolds of M and such that

- $M_1 \cup \cdots \cup M_p$ is a neighborhood of 0 in M.
- For each $1 \le i \le p$, there exists some strictly increasing sequence $\iota : \{1, \ldots, d\} \to \{1, \ldots, n+k\}$ such that $\iota(l) \le n$ and $\Pi_{\iota} \upharpoonright M_{i}$ is an immersion.

4 Cutting fibers

4.1 Dimension

Throughout this document, we use the word manifold to mean C^1 -submanifold of \mathbb{R}^n for some integer $n \ge 1$. Following [3, p. 4379], we say that a set $A \subseteq \mathbb{R}^n$ has dimension when it is a countable union of manifolds. In this case, its dimension is

$$\dim A = \max\{\dim M : M \subseteq A \text{ is a manifold}\}.$$

If A is a manifold then the definition above agrees with the usual notion of dimension. If $A_i \subseteq \mathbb{R}^n$ has dimension for each $i \geq 0$ then $A = \bigcup A_i$ also has dimension and dim $A = \max\{\dim A_i : i \geq 0\}$. Also, assume that $M \subseteq \mathbb{R}^n$ is a manifold and that $f : M \to \mathbb{R}^k$ is a C^1 -map with constant rank r. Then, since M has a countable basis, the rank Theorem gives us that f(M) has dimension and $\dim(f(M)) = r$.

Later on, we will prove Theorem 4.16 by repeatedly decreasing the dimension of the fibers. In order to do so, we need tools to compare the dimensions of various sets. Such results are easy consequences of the following global parametrization for Λ -sets.

Lemma 4.1. Let $A \subseteq \mathbb{R}^{n+k}$ be a Λ -set. There are simple sub- Λ -sets M_1, \ldots, M_p that are also H-manifolds and such that

- For $1 \le i \le p$, there is some $h_i \ge 0$ such that $M_i \subseteq \mathbb{R}^{n+k+h_i}$. Furthermore, the projection $\Pi_{n+k} \upharpoonright M_i$ is a diffeomorphism onto its image.
- $\bullet \ A = \Pi_{n+k}(M_1) \cup \cdots \cup \Pi_{n+k}(M_p).$
- For every $i \in I$, if $d_i = \dim M_i$ then $\Pi_n \upharpoonright M_i$ has constant rank l and there is some sequence $\iota : \{1, \ldots, d_i\} \to \{1, \ldots, n+k+h_i\}$ such that ι is strictly increasing, $\Pi_\iota \upharpoonright M_i$ is an immersion and $\iota(l) \leq n$.

Proof. The result follows easily from Proposition 3.26 and the compactness of \overline{A} .

In particular, using the notation of the lemma above, we have $\Pi_n(A) = \Pi_n(M_1) \cup \cdots \cup \Pi_n(M_p)$. Given $1 \leq i \leq p$, the projection $\Pi_n \upharpoonright M_i$ has constant rank so that $\Pi_n(M_i)$ has dimension. Thus, it follows that $\Pi_n(A)$ also has dimension. All in all, this shows that all sub- Λ -sets have dimension.

Recall that, by Corollary 3.16, Λ -sets have finitely many connected components. The result holds also for sub- Λ -sets since they are continuous images of Λ -sets. Furthermore, a set $A \subseteq \mathbb{R}^n$ has dimension 0 if and only if it is discrete which yields the following lemma.

Lemma 4.2. Let $A \subseteq \mathbb{R}^n$ be a sub- Λ -set. Then, A has dimension 0 if and only if it is finite.

Suppose that $M \subseteq \mathbb{R}^{n+k}$ is a manifold of dimension d such that $\Pi_n \upharpoonright M$ has constant rank l. Then, we know that $\dim M_y = d - l$ for every $y \in \Pi_n(M)$ and that $\dim \Pi_n(M) = l$. In particular, $\dim M = \dim \Pi_n(M) + l$. The parametrization of Lemma 4.1 allows us to replace Λ -sets with manifolds so that we may generalize the results above to Λ -sets to obtain the lemma below, the proof of which we omit.

Lemma 4.3. Consider $A \subseteq \mathbb{R}^{n+k}$ a Λ -set such that there is an integer $\mu \geqslant 0$ with dim $A_y = \mu$ for all $y \in \Pi_n(A)$. Then dim $A = \dim(\Pi_n(A)) + \mu$.

Remark 4.4. In particular, notice that, if $A \subseteq \mathbb{R}^n$ is a sub- Λ -set and if $A' \subseteq \mathbb{R}^{n+k}$ is a Λ -set such that $\Pi_n(A') = A$ and $\Pi_n \upharpoonright A'$ has finite fibers, then $\dim A' = \dim A$.

Actually, we can use the remark above to generalize the lemma to simple $\sinh \Lambda$ -sets.

Proposition 4.5. Consider $A \subseteq \mathbb{R}^{n+k}$ a simple sub- Λ -set and assume that there is an integer $\mu \geqslant 0$ such that $\dim A_y = \mu$ for all $y \in \Pi_n(A)$. Then $\dim A = \dim(\Pi_n(A)) + \mu$.

The proposition gives us a convenient way to argue that some sets have small fibers. Indeed, suppose that $A \subseteq B$ are two simple sub- Λ -sets satisfying the hypotheses of the proposition and such that $\Pi_n(A) = \Pi_n(B)$ and dim $A < \dim B$. Then, the fibers of A must be smaller than those of B.

4.2 A consequence of quasianalyticity

Suppose that $g\colon I_{n,r}\to\mathbb{R}$ is a function such that $g\in\mathcal{A}_n(H)$ and g vanishes on an open set $U\subseteq I_{n,r}$ with $0\in\overline{U}$. Since U is open, all the partial derivatives of g of any order vanish on U so that, by continuity, they all vanish at 0 as well. Thus, by quasianalyticity, it follows that there is an open neighborhood V of 0 such that g vanishes on V. The aim of this paragraph is to show that we have a similar result when we replace $I_{n,r}$ with an H-manifold M. To be more precise, we want to prove the following proposition.

Proposition 4.6. Consider $I_{n,r}$ a polydisk and $M \subseteq I_{n,r}$ an H-manifold with non-empty germ at 0 as well as a function $g \in \mathcal{A}_n(H)$ that is defined on $I_{n,r}$. Assume that the interior of the set $\{x \in M : g(x) = 0\}$ in M has non-empty germ at 0. Then, there is some neighborhood $U \subseteq \mathbb{R}^n$ of 0 such that g vanishes on $M \cap U$.

This result will be useful when trying to decrease dimension. Indeed, consider $M \subseteq \mathbb{R}^n$ an H-manifold of dimension d and let $g \in \mathcal{A}_n(H)$. The zero set of $g \upharpoonright M$ defines a germ of subsets of M which we write $Z_M(g)$. If dim $Z_M(g) = d$ then the interior of $Z_M(g)$ in M has non-empty germ at 0 so that g vanishes in

a neighborhood of 0 in M. Thus, either $Z_M(g)$ and M have the same germ at 0 or dim $Z_M(g) < d$.

Proof. Up to shrinking $I_{n,r}$, there are functions $f_1, \ldots, f_{n-d} \in \mathcal{A}_n(H)$ defined and C^1 on $I_{n,r}$ such that M is an open submanifold of

$$M' = \{x \in I_{n,r} : f_1(x) = \dots = f_{n-d}(x) = 0\}$$

and $\nabla f_1(x), \ldots, \nabla f_{n-d}(x)$ are linearly independent for every $x \in I_{n,r}$. Without loss of generality, we may assume that M = M'.

The map $f = (f_1, \ldots, f_{n-d}) \colon I_{n,r} \to \mathbb{R}^{n-d}$ is a submersion. Thus, by stability of the algebras generated by H under implicit functions, there exist a polydisk $I_{d,s}$, a neighborhood $U \subseteq \mathbb{R}^n$ of 0 and a diffeomorphism $\Phi \colon I_{d,s} \to M \cap U$ all of whose components are in $\mathcal{A}_d(H)$. We may now apply the result to the function $g \circ \Phi$ because it is defined on a polydisk, which allows us to conclude.

4.3 The Local Fiber Cutting Lemma

At the moment, we know from Lemma 4.1 that we can parametrize sub- Λ -sets $A\subseteq \mathbb{R}^n$ as

$$A = \Pi_n(M_1) \cup \cdots \cup \Pi_n(M_p)$$

where each M_i is an H-manifold such that $\Pi_n \upharpoonright M_i$ has constant rank. However, there is no bound on the dimension of the fibers. In order to show that the hypotheses of Gabrielov's Theorem of the Complement are satisfied, we need to have such a parametrization where, for each $1 \leqslant i \leqslant p$, there is a strictly increasing sequence $\iota \colon \{1,\ldots,d_i\} \to \{1,\ldots,n\}$ such that $\Pi_\iota \upharpoonright M_i$ is an immersion, where $d_i = \dim(M_i)$. This implies in particular that $\Pi_n \upharpoonright M_i$ is an immersion so that its fibers must have dimension 0 whence they are finite by Lemma 4.2. We will show over the next two paragraphs that such parametrizations exist. This paragraph is dedicated to proving the Local Fiber Cutting Lemma, which is the main tool we use in the proof, while Paragraph 4.4 is concerned with showing how this lemma can be used to conclude.

In order to prove the Local Fiber Cutting Lemma, we first need two results. If $M \subseteq \mathbb{R}^{n+k}$ is an H-manifold such that $\Pi_n \upharpoonright M$ has constant rank, then each fiber M_x , with $x \in \Pi_n(M)$, is a manifold. As in 3.22, we construct a basis for this manifold. The second result will allow us to derive a contradiction in the proof of the Local Fiber Cutting Lemma.

Lemma 4.7. Let $M \subseteq \mathbb{R}^{n+k}$ be an H-manifold of dimension d and assume that $\Pi_n \upharpoonright M$ has constant rank $l \leq d$. Then there are a polydisk $I_{n+k,r}$ and functions $b_1, \ldots, b_{d-l} \colon I_{n+k,r} \to \mathbb{R}^{n+k}$, all of whose components are in $\mathcal{A}_{n+k}(H)$, and such that for all $z \in M \cap I_{n+k,r}$, $b_1(z), \ldots, b_{d-pl}(z)$ is a basis of $T_z(\{x\} \times M_x)$ where $x = \Pi_n(z)$.

Proof. There are a polydisk $I_{n+k,r}$ and functions $a_1, \ldots, a_d : I_{n+k,r} \to \mathbb{R}^{n+k}$, all of whose components are in $\mathcal{A}_{n+k}(H)$, and such that $a_1(z), \ldots, a_d(z)$ is a

basis of $T_z(M)$ for all $z \in M \cap I_{n+k,r}$. Up to reordering these functions, we may assume that $\Pi_n(a_1(0)), \ldots, \Pi_n(a_l(0))$ is a basis of $\Pi_n(T_0(M))$. By the hypothesis on constant rank, up to shrinking $I_{n+k,r}$, we may also assume that $\Pi_n(a_1(z)), \ldots, \Pi_n(a_l(z))$ is a basis of $\Pi_n(T_z(M))$ for all $z \in M \cap I_{n+k,r}$.

Then, define functions $\widetilde{a_1}, \dots, \widetilde{a_l}: I_{n+k,r} \to \mathbb{R}^{n+k}$ inductively as

$$\widetilde{a_1}(z) = \frac{a_1(z)}{\|a_1(z)\|}$$

$$\widetilde{a_i}(z) = \frac{a_i(z) - \sum_{j < i} \langle \Pi_n a_i(z), \Pi_n \widetilde{a_j}(z) \rangle \widetilde{a_j}(z)}{\left\| \Pi_n a_i(z) - \sum_{j < i} \langle \Pi_n a_i(z), \Pi_n \widetilde{a_j}(z) \rangle \Pi_n \widetilde{a_j}(z) \right\|},$$

where $\langle \cdot, \cdot \rangle$ is the scalar product. It is easy to see that the components of $\widetilde{a}_1, \ldots, \widetilde{a}_l$ are in $\mathcal{A}_{n+k}(H)$ and that $\Pi_n \widetilde{a}_1(z), \ldots, \Pi_n \widetilde{a}_l(z)$ is an orthonormal basis of $\Pi_n(T_z(M))$ for every $z \in M \cap I_{n+k,r}$.

Finally, we define functions $b_1, \ldots, b_{d-l} : I_{n+k,r} \to \mathbb{R}^{n+k}$ as

$$b_i(z) = a_{l+i}(z) - \sum_{j=1}^{l} \langle \Pi_n a_{l+i}(z), \Pi_n \widetilde{a_j}(z) \rangle \widetilde{a_j}(z).$$

The components of the functions b_1, \ldots, b_{d-l} are in $\mathcal{A}_{n+k}(H)$ and the family $\widetilde{a}_1(z), \ldots, \widetilde{a}_l(z), b_1(z), \ldots, b_{d-l}(z)$ is a basis of $T_z(M)$ for all $z \in M \cap I_{n+k,r}$. Furthermore, $\Pi_n(b_i(z)) = 0$ for $z \in M$ so that the vectors $b_1(x), \ldots, b_{d-l}(x)$ make up a basis for $T_z(\{x\} \times M_x)$.

Remark 4.8. Notice that the formulas giving the functions $\widetilde{a_1}, \ldots, \widetilde{a_n}$ are reminiscent of the Gram-Schmidt orthonormalization algorithm. However, we have included the formulas because there is a subtlety, namely, we do not necessarily want $\widetilde{a_1}(z), \ldots, \widetilde{a_n}(z)$ to be orthonormal. Rather, we wish to show that their projections on the first n coordinates are orthonormal.

The following lemma will be crucial in order to obtain a contradiction in the proof of the Local Fiber Cutting Lemma below. It is proven in [9], in the paragraph before their Lemma 4.5.

Lemma 4.9. Let $M \subseteq \mathbb{R}^{n+k}$ be an H-manifold of dimension d such that $\Pi_n \upharpoonright M$ has constant rank l < d. Assume also that there is a strictly increasing sequence $\iota : \{1, \ldots, d\} \to \{1, \ldots, n+k\}$ such that $\Pi_\iota \upharpoonright M$ is an immersion and $\iota(l) \leqslant n$. Consider $y \in \Pi_n(M)$ and let C be a connected component of the fiber M_y . Then, fr $C \neq \emptyset$.

Proposition 4.10 (Local Fiber Cutting Lemma). Let $M \subseteq \mathbb{R}^{n+k}$ be an H-manifold of dimension d such that $\Pi_n \upharpoonright M$ has constant rank l < d. Assume also that there is a strictly increasing sequence $\iota \colon \{1, \ldots, d\} \to \{1, \ldots, n+k\}$ with $\iota(l) \leqslant n$ and such that $\Pi_\iota \upharpoonright M$ is an immersion. Then, there are a polydisk $I_{n,r}$ and a simple sub- Λ -set $A \subseteq M$ such that $\dim A < \dim M$ and $\Pi_n(M \cap I_{n+k,r}) = \Pi_n(A)$.

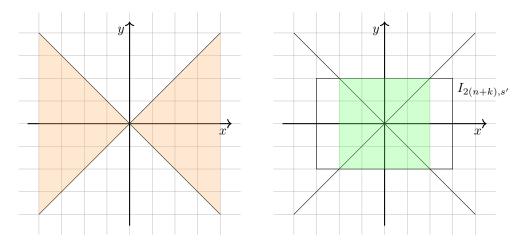


Figure 1

Proof. For $z \in \mathbb{R}^{n+k}$, we will write throughout the proof z = (x,y) where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_k)$. Thus, in particular, $x = \Pi_n(z)$. We assume throughout that M has non-empty germ at 0 since there is nothing to prove otherwise. There are a polydisk $I_{n+k,r} \subseteq \mathbb{R}^{n+k}$, a natural number $q \in \mathbb{N}$ and functions $f_1, \ldots, f_{n+k-d}, g_1, \ldots, g_q \in \mathcal{A}_{n+k}(H)$ that are defined and C^1 on $I_{n+k,r}$, such that $\nabla f_1(x), \ldots, \nabla f_{n+k-d}(x)$ are linearly independent for every $x \in I_{n+k,r}$ and

$$M = \{x \in I_{n+k,r} : f_1(x) = \dots = f_{n+k-d}(x) = 0, g_1(x) > 0, \dots, g_q(x) > 0\}.$$

By Lemma 4.7, up to shrinking $I_{n+k,r}$, we may assume that there are functions $b_1, \ldots, b_{d-l} \colon I_{n+k,r} \to \mathbb{R}^{n+k}$, all of whose components are in $\mathcal{A}_{n+k}(H)$ and such that $b_1(z), \ldots, b_{d-l}(z)$ make up a basis of $T_z(\{x\} \times M_x)$ for each $z \in \mathbb{R}^{n+k}$. We are going to prove the result in two steps. We will begin by proving the result under an additional assumption about M and we will then reduce the general case to the special one.

Special case: For now, we assume that, for each $1 \le i \le k$, there is some $1 \le j \le n$ such that $|y_i| < |x_j|$ whenever $z = (x,y) \in M$. This hypothesis means that M is contained in a "butterfly shape". We have represented the case n = k = 1 in the picture on the left in figure 1, where M would be contained in the orange area.

We define a function $\varphi \in \mathcal{A}_{n+k}(H)$ defined and C^1 on $I_{n+k,r}$ by

$$\varphi(z) = g_1(z) \dots g_q(z) (r_1^2 - z_1^2) \dots (r_{n+k}^2 - z_{n+k}^2).$$

If $z \in M$ then $\varphi(z) > 0$ but $\varphi(z) = 0$ whenever $z \in \text{fr } M = \overline{M} \setminus M$. Now, let $x \in \Pi_n(M)$. Since $\overline{M_x}$ is compact and non-empty, there is $y_0 \in \overline{M_x}$ such that $\varphi(x, y_0)$ is maximal. From $y_0 \in \overline{M_x}$, we deduce that $(x, y_0) \in \overline{M}$. Also, for

every $y \in M_x$, we have $\varphi(x,y) > 0$ so that $\varphi(x,y_0) > 0$ by maximality. Thus, $(x,y_0) \notin \text{fr } M$ so that $(x,y_0) \in M$ whence $y_0 \in M_x$. Furthermore, $\varphi \upharpoonright \{x\} \times M_x$ is critical at z. Thus, if we let

$$A = \{z \in M : \varphi \upharpoonright \{x\} \times M_x \text{ is critical at } z\},\$$

then we obtain $\Pi_n(M) = \Pi_n(A)$. We also have

$$A = \{ z \in M : \langle \nabla \varphi(z), b_1(z) \rangle = \dots = \langle \nabla \varphi(z), b_{d-1}(z) \rangle = 0 \}$$

so that A is a H-basic set.

A polydisk $I_{n+k,r'} \subseteq I_{n+k,r}$, with r' = (s',t'), is said to be **compatible** with M whenever we have $(M \cap I_{n+k,r'})_x = M_x$ for every $x \in \mathbb{R}^n$ such that $(M \cap I_{n+k,r'})_x \neq \emptyset$. Such polydisks form a fundamental system of neighborhoods of 0, as shown in the picture on the right in figure 1, where the green area is a polydisk compatible with M. Furthermore, let $I_{n+k,r'}$ be a polydisk compatible with M and $x \in \Pi_n(M \cap I_{n+k,r'})$. Take y such that $z = (x,y) \in A$. Since $A \subseteq M$, we have $y \in M_x$ so that $z \in I_{n+k,r'}$ by the compatibility assumption. Thus,

$$\Pi_n(M \cap I_{n+k,r'}) = \Pi_n(A \cap I_{n+k,r'}).$$

In view of the equality above, the fact that A is H-basic and Theorem 2.4, it suffices to prove that there exists a polydisk $I_{n+k,r'}$ which is compatible with M and such that $\dim(A \cap I_{n+k,r'}) < \dim(M)$.

Suppose for a contradiction that this is not the case and let B be the interior of A in M. Then, for every polydisk $I_{n+k,r'} \subseteq I_{n+k,r}$ which is compatible with M, we have $\dim(A \cap I_{n+k,r'}) = \dim(M)$ by assumption so that $A \cap I_{n+k,r'}$ has non-empty interior in M. In particular, $B \cap I_{n+k,r'} \neq \emptyset$. Since such polydisks form a fundamental system of neighborhoods of 0, it follows that $0 \in \overline{B}$. By Proposition 4.6, there exists a polydisk $I_{n+k,r'} \subseteq I_{n+k,r}$ which is compatible with M and such that $M \cap I_{n+k,r'} \subseteq A$.

Let r' = (s', t') and consider $x \in \Pi_n(M) \cap I_{n,s'}$. By compatibility of $I_{n+k,r'}$, it follows that $M_x \subseteq I_{n+k,r'}$. Thus, $A_x = M_x$ whence $\varphi_x = \varphi(x, \cdot)$ is constant along any connected component of M_x . Let C be such a connected component. Then, φ_x must also be constant on \overline{C} and, since φ_x is positive on C and vanishes on fr C, it follows that fr $C = \emptyset$ which is a contradiction by Lemma 4.9.

General case: We now return to the general case. We are looking for some polydisk $I_{n+k,r'}$ such that there exists a simple sub- Λ -set $A \subseteq M \cap I_{n+k,r'}$ with $\dim A < \dim M$ and $\Pi_n(M \cap I_{n+k,r'}) = \Pi_n(A)$. We may view r' as some sort of parameter. The idea of the proof is to replace M with a H-manifold in $\mathbb{R}^{2(n+k)}$, where we have turned the parameter r' into a tuple of variables. The precise definition of \widetilde{M} is as follows:

$$\widetilde{M} = \{(r', z) : 0 < r'_i < r_i \text{ for all } 1 \leqslant i \leqslant n + k \text{ and } z \in M \cap I_{n+k,r'}\}.$$

In particular, for every $r' \in \mathbb{R}^{n+k}$ such that $0 < r'_i < r_i$ for each $1 \le i \le n+k$, we have

$$\widetilde{M}_{r'} = M \cap I_{n+k,r'}.$$

Thus, \widetilde{M} can be thought of as a parametrized family of restrictions of M.

Notice also that \widetilde{M} is a H-manifold of dimension n+k+d such that $\Pi_{2n+k} \upharpoonright \widetilde{M}$ has constant rank n+k+l < n+k+d and that \widetilde{M} is contained in a butterfly shape. Indeed, $|y_i| < t_i'$ for every $1 \leqslant i \leqslant k$ and every $(r',z) \in M$, where z = (x,y) and r' = (s',t'). By the special case of the result, there are a simple sub- Λ -set $\widetilde{A} \subseteq \widetilde{M}$ and a polydisk $I_{2(n+k),R}$ such that $\Pi_{2n+k}(\widetilde{M} \cap I_{2(n+k),R}) = \Pi_{2n+k}(\widetilde{A})$ and $\dim(\widetilde{A}) < \dim(\widetilde{M})$.

We now want to use Proposition 4.5 to show that we can find r' arbitrarily close to 0 such that $\dim(\widetilde{A}_{r'}) < \dim(M)$. Thus, suppose for a contradiction that there is r'' such that $\dim(\widetilde{A}_{r'}) = \dim(M)$ whenever $0 < r'_i < r''_i$ for each $1 \le i \le n+k$. It follows that all the fibers of $\widetilde{A} \cap (I_{n+k,r''} \times \mathbb{R}^{n+k})$ have the same dimension. Furthermore,

$$\dim(\Pi_{n+k}(\widetilde{A} \cap (I_{n+k,r''} \times \mathbb{R}^{n+k}))) = \dim(\Pi_{n+k}(\widetilde{A}) \cap I_{n+k,r''})$$
$$= \dim(\Pi_{n+k}(\widetilde{M}) \cap I_{n+k,r''})$$
$$= n+k.$$

Thus, by Proposition 4.5, we obtain $\dim(\widetilde{A} \cap (I_{n+k,r''} \times \mathbb{R}^{n+k})) = \dim(M) + n + k = \dim(\widetilde{M})$ which is a contradiction.

Thus, it is now sufficient to show that there is r'' as above such that $\Pi_n(\widetilde{A}_{r'}) = \Pi_n(M \cap I_{n+k,r'})$ for every $r' \in \mathbb{R}^{n+k}$ such that $0 < r'_i < r''_i$ whenever $1 \le i \le n+k$. Up to shrinking $I_{2(n+k),R}$, we may assume that R has the form (r'',r'') for some polyradius r''. Then, given r' such that $0 < r'_i < r''_i$ for every $1 \le i \le n+k$, we have $\{r'\} \times \widetilde{M}_{r'} \subseteq I_{2(n+k),R}$ whence $\Pi_n(\widetilde{M}_{r'}) = \Pi_n(\widetilde{A}_{r'})$. Furthermore, by definition of \widetilde{M} , we have $\widetilde{M}_{r'} = M \cap I_{n+k,r'}$ which concludes.

Remark 4.11. We reuse the notation of the proof. If we assume \mathcal{A} -analyticity [7, Definition 1.10], then we can show that B is both open and closed. Indeed, consider $a \in \overline{B}$. By \mathcal{A} -analyticity, the translation of A at a is still a set defined by equations so that we may apply Lemma 4.6 to deduce that $a \in B$. Thus, B contains every connected component that it intersects which yields a contradiction whenever $B \neq \emptyset$. Notice that this argument does not depend on M being contained in a butterfly shape and that it proves the stronger result that $\dim A < \dim M$.

However, in the absence of A-analyticity, we can only show that, when $0 \in \overline{B}$, we must also have $0 \in B$. Thus, we only get a contradiction when the germ of B at 0 is non-empty whence we can only prove that the dimension of A is small in a neighborhood of 0. Furthermore, this contradiction depends on the fact that every neighborhood of 0 contains a connected component of a fiber of M. This does not hold in the general case (see Appendix A) but it is true in

the special case that M is contained in a butterfly shape (see the picture on the right in figure 1).

Actually, the proof we give for the special case still applies when the butterfly shape is "distorted", namely when the lines bounding the butterfly are not straight. More precisely, suppose that there are C^1 functions $h_1, \ldots, h_k \in \mathcal{A}_n(H)$ defined on $I_{n,s}$, where r=(s,t), such that $h_1(0)=\cdots=h_k(0)=0$ and $|y_i|< h_i(x)$ for every $z=(x,y)\in M$ and $1\leqslant i\leqslant k$. Then, it is still true that the polydisks compatible with M form a fundamental system of neighborhoods of 0 so that the proof given for the special case above applies in this case as well.

4.4 The Global Fiber Cutting Lemma

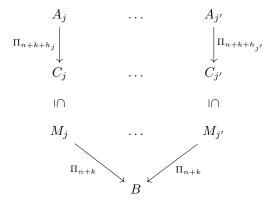
In this paragraph, we start by giving a global version of the Fiber Cutting Lemma. We use it inductively to prove that every sub- Λ -set is simple. Finally, using Gabrielov's Theorem of the Complement [3, Theorem 2.7], we show that the \mathbb{R}_H is o-minimal and model complete.

Proposition 4.12. Consider $B \subseteq \mathbb{R}^{n+k}$ an H-basic set such that $\dim(\Pi_n(B)) < \dim(B)$. Then, there are Λ -sets A_1, \ldots, A_p such that

- For every $1 \leq i \leq p$, there is some integer $h_i \geq 0$ such that $A_i \subseteq \mathbb{R}^{n+k+h_i}$, $\dim(A_i) < \dim(B)$, $\Pi_{n+k}(A_i) \subseteq B$ and $\Pi_{n+k} \upharpoonright A_i$ has finite fibers.
- There is a neighborhood $U \subseteq \mathbb{R}^{n+k}$ of 0 such that

$$\Pi_n(B \cap U) = \Pi_n(A_1) \cup \cdots \cup \Pi_n(A_p).$$

Proof. The proof proceeds by first parametrizing H-basic sets using Proposition 3.26 and then applying the Local Fiber Cutting Lemma to each of these charts. The following diagram illustrates the situation.



Consider $(M_j \subseteq \mathbb{R}^{n+k+h_j})_{j\in J}$ a parametrization of B obtained by Proposition 3.26 and fix $j\in J$. If $\dim(M_j)<\dim(B)$ then we let $C_j=M_j$ and $U_j=\mathbb{R}^{n+k+h_j}$. Notice in particular that $\Pi_n(C_j)=\Pi_n(M_j\cap U_j)$. Now, assume

that $\dim(M_j) = \dim(B)$. Then, $\Pi_n \upharpoonright M_j$ has constant rank $l = \dim(\Pi_n(M_j)) \le \dim(\Pi_n(B)) < \dim(B) = \dim(M_j)$. Thus, we can apply the Local Fiber Cutting Lemma to find $C_j \subseteq M_j$ a simple sub- Λ -set and $U_j \subseteq \mathbb{R}^{n+k+h_j}$ a neighborhood of 0 such that $\dim(C_j) < \dim(M_j)$ and $\Pi_n(C_j) = \Pi_n(M_j \cap U_j)$.

Now, consider $J_0 \subseteq J$ a finite subset such that $V := \bigcup_{j \in J_0} \Pi_{n+k}(M_j \cap U_j)$ is a neighborhood of 0 in B. We then have

$$\Pi_n(V) = \bigcup_{j \in J_0} \Pi_n(M_j \cap U_j) = \bigcup_{j \in J_0} \Pi_n(C_j).$$

For $j \in J_0$, C_j is a simple sub- Λ -set so that there is a Λ -set A_j such that $\Pi_{n+k+h_j}(A_j) = C_j$ and $\dim(A_j) = \dim(C_j)$. In particular, $\dim(A_j) < \dim(B)$ for $j \in J_0$ and

$$\Pi_n(V) = \bigcup_{j \in J_0} \Pi_n(A_j)$$

whence the result.

Corollary 4.13 (Global Fiber Cutting). Let $B \subseteq \mathbb{R}^{n+k}$ be a Λ -set such that $\dim(\Pi_n(B)) < \dim(B)$. Then, there are Λ -sets A_1, \ldots, A_p such that

- For every $1 \leq i \leq p$, there is some integer $h_i \geq 0$ such that $A_i \subseteq \mathbb{R}^{n+k+h_i}$, $\dim(A_i) < \dim(B)$, $\Pi_{n+k}(A_i) \subseteq B$ and $\Pi_{n+k} \upharpoonright A_i$ has finite fibers.
- We have

$$\Pi_n(B) = \Pi_n(A_1) \cup \cdots \cup \Pi_n(A_n).$$

Proof. The result follows from the proposition above and the compactness of \overline{B} .

Remark 4.14. We require that $\Pi_{n+k} \upharpoonright A_i$ has finite fibers and that $\Pi_{n+k}(A_i) \subseteq B$ so that, for every $y \in \Pi_n(A_i)$, we have $\dim((A_i)_y) \leqslant \dim(B_y)$. Indeed, consider $y \in \Pi_n(A_i)$. We have $\Pi_{n+k}((A_i)_y) \subseteq B_y$ and $\Pi_{n+k} \upharpoonright (A_i)_y$ has finite fibers so that, by Proposition 4.5, $\dim((A_i)_y) = \dim(\Pi_{n+k}((A_i)_y)) \leqslant \dim(B_y)$.

We want to work with Λ -sets that respect the hypotheses of Proposition 4.5 in order to have better control over their fibers. This is the purpose of the following lemma, which is an immediate consequence of Lemma 4.1.

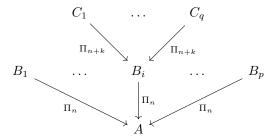
Lemma 4.15. Let $B \subseteq \mathbb{R}^{n+k}$ be a Λ -set. Then, there are Λ -sets A_1, \ldots, A_p such that

- For every $1 \le i \le p$, there is some integer $h_i \ge 0$ such that $A_i \subseteq \mathbb{R}^{n+k+h_i}$ and $\dim((A_i)_y)$ does not depend on $y \in \Pi_n(A_i)$.
- For each $1 \leq i \leq p$, $\Pi_{n+k} \upharpoonright A_i$ has finite fibers and

$$B = \Pi_{n+k}(A_1) \cup \cdots \cup \Pi_{n+k}(A_p).$$

Theorem 4.16. Let $A \subseteq \mathbb{R}^n$ be a sub- Λ -set. Then, A is also a simple sub- Λ -set.

Proof. The following diagram illustrates the proof below.



It suffices to show that there are Λ -sets B_1, \ldots, B_p such that $\Pi_n \upharpoonright B_i$ has finite fibers and $A = \Pi_n(B_1) \cup \cdots \cup \Pi_n(B_p)$. Thus, consider B_1, \ldots, B_p some Λ -sets such that $A = \Pi_n(B_1) \cup \cdots \cup \Pi_n(B_p)$. By applying Lemma 4.15, we may assume that, for every $1 \leqslant i \leqslant p$, the dimension $\mu_i := \dim((B_i)_y)$ does not depend on $y \in \Pi_n(B_i)$. Define also $d_i = \dim(\Pi_n(B_i))$ for each $1 \leqslant i \leqslant p$. Finally, consider $\mu = \max(\mu_1, \ldots, \mu_p)$ and $d = \max\{d_i : \mu_i = \mu\}$ and assume that B_1, \ldots, B_p have been chosen in such a way that the pair (μ, d) is minimal for the lexicographic order. To prove the result, we only need to show that $\mu = 0$ by Lemma 4.2. Thus, assume by contradiction that $\mu > 0$.

Let $1 \le i \le p$ such that $\mu_i = \mu$ and $d_i = d$. For the rest of the proof, we will write $B = B_i$. By Proposition 4.5, we have $\dim(B) = d + \mu > d$ so that, by Corollary 4.13, there are C_1, \ldots, C_q some Λ -sets such that

$$\Pi_n(B) = \Pi_n(C_1) \cup \cdots \cup \Pi_n(C_q)$$

and $\dim(C_j) < \dim(B)$ for $1 \le j \le q$. By Lemma 4.15, we can assume without loss of generality that, for every $1 \le j \le q$, the dimension $\mu_j := \dim((C_j)_y)$ does not depend on $y \in \Pi_n(C_j)$. Define also $d_j = \dim(\Pi_n(C_j))$ for $1 \le j \le q$. Then, fix $1 \le j \le q$. By Remark 4.14, we have $\mu_j \le \mu$. Assume that $\mu_j = \mu$, then $\dim(C_j) = \mu_j + d_j = \mu + d_j$ so that $\mu + d > \mu + d_j$ whence $d_j < d$. Thus, we contradict the minimality of the pair (μ, d) by replacing B with the family C_1, \ldots, C_q and by repeating the same operation for each B_i such that $\mu_i = \mu$ and $d_i = d$.

Remark 4.17. The proof can be seen as a procedure that takes a representation of the form $A = \Pi_n(B_1) \cup \cdots \cup \Pi_n(B_p)$ and that produces a new one such that the pair (μ, d) decreases. Since this procedure can be applied as long as $\mu > 0$, we can apply it repeatedly until $\mu = 0$. Also, the set of pairs (μ, d) is well ordered so that the process must stop eventually.

The theorem along with Lemma 4.1 show that every Λ -set has the Λ -Gabrielov property (see [3, p. 4380]). Thus, if $A, B \subseteq \mathbb{R}^n$ are two sub- Λ -sets, the set $A \setminus B$ is also a sub- Λ -set by Gabrielov's Theorem of the complement [3, Theorem 2.7]. Now, consider $\tau \colon \mathbb{R} \to (-1,1)$ the definable map given by

$$\tau(x) = \frac{x}{\sqrt{1+x^2}}.$$

Theorem 4.18. A set $A \subseteq \mathbb{R}^n$ is \mathbb{R}_H -definable if and only if $\tau^n(A) \subseteq (-1,1)^n$ is a sub- Λ -set. Furthermore, the structure \mathbb{R}_H is o-minimal and model complete.

Proof. Consider \mathcal{D}_n to be the collection of subsets $A \subseteq \mathbb{R}^n$ such that $(\tau^n)(A)$ is a sub- Λ -set. This collection is closed under unions and intersections. Furthermore, if $A \subseteq (-1,1)^n$ is a sub- Λ -set, so is $(-1,1)^n \setminus A$ so that \mathcal{D}_n is also closed under complement. Finally, if $A \in \mathcal{D}_{n+k}$, it is clear that $\Pi_n(A) \in \mathcal{D}_n$. Thus, the collection $\mathcal{D} = (\mathcal{D}_n)_{n \geqslant 0}$ is the collection of definable sets of a certain structure on \mathbb{R} which we write $\mathbb{R}_{\mathcal{D}}$.

We now need to argue that \mathcal{D} is exactly the collection of definable sets of the structure \mathbb{R}_H . Firstly, every sub- Λ -set is clearly definable in \mathbb{R}_H so that every set $A \in \mathcal{D}_n$ must also be definable in \mathbb{R}_H . Since τ is order-preserving, the set $\tau \times \tau(\Gamma(<))$ is the graph of the order on $(-1,1)^n$ which is clearly a sub- Λ -set so that $\Gamma(<) \in \mathcal{D}_2$. Also, given $x, y, z \in (-1,1)$, we have

$$\tau^{-1}(x) + \tau^{-1}(y) = \tau^{-1}(z) \iff \exists u, v, w > 0,$$
$$(u^2, v^2, w^2) = (1 - x^2, 1 - y^2, 1 - z^2)$$
$$\land xvw + yuw = zuv$$

Thus, $\Gamma(+) \in \mathcal{D}_3$. Similarly, it is easy to show that $\Gamma(-), \Gamma(\cdot) \in \mathcal{D}_3$. Therefore, every polynomial is definable in the structure $\mathbb{R}_{\mathcal{D}}$. In particular, the restriction of H to the complement of any neighborhood of 0 is $\mathbb{R}_{\mathcal{D}}$ -definable. Thus, in order to prove that $\mathbb{R}_{\mathcal{D}}$ and \mathbb{R}_H have the same definable sets, it suffices to show that the restriction of H to some neighborhood of 0 is definable in $\mathbb{R}_{\mathcal{D}}$.

Up to considering H - H(0), we may as well assume that H(0) = 0. In this case, the function $\tau^{-1} \circ H \circ \tau$ is well defined in a neighborhood of 0. Furthermore, $\tau \times \tau(\Gamma(\tau^{-1} \circ H \circ \tau))$ is the graph of H restricted to some neighborhood of 0. Since this is a Λ -set, it means that $\tau^{-1} \circ H \circ \tau$ is $\mathbb{R}_{\mathcal{D}}$ -definable. Finally, it is clear that τ is a $\mathbb{R}_{\mathcal{D}}$ -definable function so that τ^{-1} is also $\mathbb{R}_{\mathcal{D}}$ -definable. Putting everything together, we find that H is $\mathbb{R}_{\mathcal{D}}$ -definable whence the result.

Since τ is a homeomorphism and every sub- Λ -set has only finitely many connected components by Corollary 3.16, it follows that every \mathbb{R}_H -definable set also has only finitely many connected components. Thus, \mathbb{R}_H is o-minimal. Furthermore, consider $A \subseteq \mathbb{R}^n$ a definable set. Then, there is $A' \subseteq \mathbb{R}^{n+k}$ a Λ -set such that $\Pi_n(A') = \tau^n(A)$. Then, A' is existentially definable by Proposition 2.4 so that the set

$$B = \{(y,x) \in \mathbb{R}^{n+k} \times \mathbb{R}^{n+k} : \tau^{n+k}(y) = x, x \in A\}$$

is also existentially definable. Furthermore, $\Pi_n(B) = A$ so that \mathbb{R}_H is also model complete.

A A counter example

In this Appendix, we will follow the proof of [6, Lemma 3.7] in a special case. We will see that the set A obtained in this way has empty germ at 0. This is an

issue as the proof of [6, Proposition 3.8] freely restricts such sets A to arbitrary neighborhoods of 0.

Let r = (1, 1) and

$$M = \{(x, y) \in I_{2,r} : y > x\}.$$

Then, the projection Π_1 on the x coordinate has constant rank. As in the proof of Lemma 3.7, we define $\varphi \colon I_r \to \mathbb{R}$ as

$$\varphi(x,y) = (y-x)(1-x^2)(1-y^2)$$

and we let $A = \{(x, y) \in M : \varphi \upharpoonright M_x \text{ is critical at } y\}$. We can then compute

$$A = \left\{ (x, y) \in M : y = \frac{1}{3}x \pm \frac{\sqrt{x^2 + 3}}{3} \right\}.$$

There is $\varepsilon > 0$ such that, for $x \in (-\varepsilon, \varepsilon)$, we have

$$\left| \frac{1}{3}x \pm \frac{\sqrt{x^2 + 3}}{3} \right| > \frac{\sqrt{2}}{3}$$

Thus, $A \cap I_{2,r'} = \emptyset$ where $r' = \left(\varepsilon, \frac{\sqrt{2}}{3}\right)$.

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