Uniqueness and nonuniqueness of p-harmonic Green functions on weighted $\mathbf{R}^n$ and metric spaces

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Abstract. We study uniqueness of p-harmonic Green functions in domains Ω in a complete metric space equipped with a doubling measure supporting a p-Poincaré inequality, with $1 < p < \infty$. For bounded domains in unweighted $\mathbf{R}^n$, the uniqueness was shown for the p-Laplace operator Δ_p and all p by Kichenassamy–Véron (*Math. Ann.* **275** (1986), 599–615), while for $p = 2$ it is an easy consequence of the linearity of the Laplace operator Δ. Beyond that, uniqueness is only known in some particular cases, such as in Ahlfors p-regular spaces, as shown by Bonk–Capogna–Zhou (`arXiv:2211.11974`). When the singularity x_0 has positive p-capacity, the Green function is a particular multiple of the capacitary potential for $\operatorname{cap}_p(\{x_0\}, \Omega)$ and is therefore unique. Here we give a sufficient condition for uniqueness in metric spaces, and provide an example showing that the range of p for which it holds (while x_0 has zero p-capacity) can be a nondegenerate interval. In the opposite direction, we give the first example showing that uniqueness can fail in metric spaces, even for $p = 2$.

Key words and phrases: doubling measure, metric space, nonuniqueness, p-harmonic Green function, Poincaré inequality, weighted Euclidean space, uniqueness.

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1. Introduction

In the Euclidean space $\mathbf{R}^n$, a function u is said to be the (p-harmonic) Green function for a bounded domain $\Omega \subset \mathbf{R}^n$ with singularity at $x_0 \in \Omega$ if it is the weak solution of the equation

$$-\Delta_p u = \delta_{x_0} \quad \text{in } \Omega \tag{1.1}$$

with the Dirac measure δ_{x_0} in the right-hand side and zero boundary values on $\partial\Omega$ in a weak sense. Here

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u), \quad 1 < p < \infty,$$

is the p-Laplace operator. The Green function is p-harmonic outside of its singularity, that is, it satisfies the p-Laplace equation $\Delta_p u = 0$.

In metric spaces, Holopainen–Shanmugalingam [22] introduced a notion of singular functions which extended the notion of Green functions from $\mathbf{R}^n$ and Riemannian manifolds to more general settings. More precise definitions and various characterizations of Green functions in metric spaces were later given in Björn–Björn–Lehrbäck [7] and Björn–Björn [5], see Definitions 4.1 and 4.4 below. For domains in complete metric spaces $X = (X, d)$, equipped with a doubling measure μ supporting a p-Poincaré inequality with $1 < p < \infty$, it was shown in [5] and [7] that Green functions exist in a domain Ω if and only if the p-capacity $C_p(X \setminus \Omega) > 0$ or X is p-hyperbolic.

Roughly speaking, by [7, Theorem 8.5], $u : \Omega \to (0, \infty]$ is a (p-harmonic) *Green function* with singularity at $x_0 \in \Omega$ for a (bounded) domain Ω in a metric space X if it is p-harmonic in $\Omega \setminus \{x_0\}$, has zero boundary values on $\partial\Omega$ in a suitable sense and is properly normalized so that

$$\operatorname{cap}_p(\{x : u(x) \geq b\}, \Omega) = b^{1-p}, \quad \text{when } 0 < b < u(x_0) = \lim_{x \to x_0} u(x).$$

This normalization captures in a quantitative way the right-hand side δ_{x_0} in (1.1).

In this paper we study whether such Green functions in metric spaces are unique. Our first result answers this question in the affirmative in spaces satisfying a critical volume growth condition. Here $B_r = B(x_0, r) = \{y \in X : d(y, x_0) < r\}$.

Theorem 1.1. *Let X be a complete metric space and assume that μ is doubling and supports a p-Poincaré inequality on X, where $1 < p < \infty$. Assume that*

$$\limsup_{r \to 0} \left(\frac{\mu(B_r)}{r^p} \right)^{1/(p-1)} \int_r^1 \left(\frac{\rho^p}{\mu(B_\rho)} \right)^{1/(p-1)} \frac{d\rho}{\rho} = \infty. \tag{1.2}$$

If u and v are (p-harmonic) Green functions in a domain Ω with singularity at $x_0 \in \Omega$, then $u = v$, i.e. the Green function in Ω with singularity at x_0 is unique.

Remark 1.2. When $C_p(\{x_0\}) > 0$, the Green function is a particular multiple of the capacitary potential for $\operatorname{cap}_p(\{x_0\}, \Omega)$ and is therefore unique, see Björn–Björn–Lehrbäck [7, Theorem 1.3] (for bounded Ω) and Björn–Björn [5, Corollary 10.3] (for general Ω).

Theorem 1.1 provides a new sufficient condition (1.2) for the uniqueness of the Green function in the case $C_p(\{x_0\}) = 0$. When the space X is Ahlfors Q-regular and $p = Q$, (1.2) is satisfied and our result recovers the uniqueness of Green functions on a bounded regular domain $\Omega \subset X$ in Bonk–Capogna–Zhou [10, Theorem 1.1(ii) and Remark 1.1].

More generally, condition (1.2) holds if $p > q_0$, where

$$q_0 := \sup\biggl\{ q > 0 : \frac{\mu(B_r)}{\mu(B_R)} \lesssim \Bigl(\frac{r}{R}\Bigr)^q \text{ for all } 0 < r < R \le 1 \biggr\},$$

see Proposition 5.3(a). Example 5.4 shows that the range of new exponents p covered by Theorem 1.1 can be a nondegenerate arbitrarily large interval.

At the same time, in Example 6.1 we prove the following nonuniqueness result with a p-admissible weight in $\mathbf{R}^2$.

Theorem 1.3. *Let $1 < p < \infty$ and either $\Omega = \{x \in \mathbf{R}^2 : |x| < R\}$ or $\Omega = \mathbf{R}^2$.*

Then there is a metric d on $\mathbf{R}^2$, biLipschitz equivalent to the Euclidean metric, and a doubling measure $d\mu = w\,dx$ supporting a p-Poincaré inequality, such that for the metric space $X := (\mathbf{R}^2, d, \mu)$ there are uncountably many (p-harmonic) Green functions in Ω with a singularity at 0, that is, the Green functions in Ω are not unique.

The metric d is independent of p and Ω. If $1 < p < 2$ then we can choose $w \equiv 1$.

For domains in $\mathbf{R}^n$, equipped with the Euclidean metric and the Lebesgue measure, Green functions are known to be unique. For $p = 2$, this is an easy consequence of the linearity of the Laplace operator and the maximum principle for harmonic functions. When $p \ne 2$, the p-Laplace operator is nonlinear and the above argument does not apply. In this case, uniqueness was proved only in 1986 by Kichenassamy–Véron [24, Theorem 2.1].

Beyond that, uniqueness has been shown in the following cases: for regular relatively compact domains in n-dimensional Riemannian manifolds (and $p = n$) by Holopainen [21, Theorem 3.22], and more recently for bounded regular domains in Ahlfors Q-regular metric spaces (and $p = Q$) in Bonk–Capogna–Zhou [10, Theorem 1.1]. The uniqueness also holds when $C_p(\{x_0\}) > 0$, by [5, Corollary 10.3] and [7, Theorem 1.3], since the Green function is then just a suitable multiple of the capacitary potential for $\{x_0\}$ in Ω.

The *p-harmonic functions* in a general metric measure space (X, d, μ) are defined as continuous local minimizers of the p-energy integral

$$\int_\Omega g_u^p \, d\mu, \tag{1.3}$$

where g_u is the minimal p-weak upper gradient of u, see Sections 2 and 3. This is a natural generalization of the fact that the p-Laplace equation $\Delta_p u = 0$ in $\mathbf{R}^n$ is the Euler–Lagrange equation for minimizers of the p-energy $\int_\Omega |\nabla u|^p \, dx$.

If X is equipped with a smooth differentiable structure, such as for Riemannian manifolds, then one can derive an Euler–Lagrange equation which characterizes p-harmonic functions. In particular, if $\mathbf{R}^n$ is equipped with a p-admissible weight w, as in Heinonen–Kilpeläinen–Martio [17], then p-harmonic functions are local minimizers of the weighted p-energy integral

$$\int_\Omega |\nabla u(x)|^p w(x) \, dx$$

and solve the weighted p-Laplace equation

$$\operatorname{div}(w(x)|\nabla u|^{p-2}\nabla u) = 0.$$

In general metric measure spaces, such a description is not always possible – see however Gigli–Mondino [15].

A seminal result of Cheeger [11] states that every complete metric measure space (X, d, μ), with μ doubling and supporting a p-Poincaré inequality, can be equipped with a *measurable differentiable structure*. In such cases, the p-energy can be modified to yield an Euler–Lagrange-type equation. This is obtained via differential calculus, as developed in [11]. Namely, one can define a vector-valued differential Du, as a measurable section of a measurable tangent bundle, whose pointwise Euclidean norm $|Du|$ is comparable to the minimal p-weak upper gradient g_u of u. For such a choice of a fixed measurable differentiable structure, functions minimizing the Cheeger p-energy

$$\int_\Omega |Du|^p \, d\mu$$

satisfy the equation

$$\int_\Omega |Du|^{p-2} Du \cdot D\varphi = 0 \quad \text{for all functions } \varphi \in \operatorname{Lip}_0(\Omega) \tag{1.4}$$

and their continuous representatives are called *Cheeger p-harmonic functions*.

If X is infinitesimally Hilbertian, as in e.g. [15], the inner product in (1.4) can be chosen in such a way that

$$|Du|^2 := Du \cdot Du = g_u^2$$

at almost every point, see also [11, Theorem 6.1 and Corollary 4.41]. In this case, Cheeger p-harmonic and (upper gradient) p-harmonic functions coincide, see [15, Theorem 4.2]. This case occurs for example in all analytically one-dimensional cases, such as the Laakso spaces considered in [27], see also Cheeger–Kleiner [12, Theorem 9.1].

On the other hand, in Theorem 1.3 we do not have a scalar product, and as a result, the (upper gradient) p-harmonic Green functions differ significantly from the Cheeger p-harmonic Green functions. From the metric perspective, the space $(\mathbf{R}^2, d)$ in Theorem 1.3 is biLipschitz equivalent to the Euclidean space, but from a PDE point of view their Laplacians are very different. The metric d is defined through an ℓ^1 norm in the tangent bundle and is similar to a Finsler structure studied in Finsler geometry – see e.g. [1], [13] and [28] for introductions to this vast field.

The notion of p-harmonic functions (based on (1.3)) only depends on the metric and the measure of the underlying space. In contrast, the definition of Cheeger p-harmonic functions also depends on the choice of a measurable differentiable structure. In particular, when the space fails to be infinitesimally Hilbertian, the upper gradient g_u in (1.3) is only comparable to the $|Du|$ norm and there is no Euler–Lagrange equation for the p-harmonic functions (based on (1.3)), which therefore differ from the Cheeger p-harmonic functions.

Note, however, that any "positive" result proved for the energy minimizing p-harmonic functions (such as our uniqueness Theorem 1.1) holds also for the Cheeger p-harmonic functions. Indeed, it suffices to replace g_u by $|Du|$ in all the proofs. In particular, Theorem 1.1 provides new uniqueness results for Green functions in weighted $\mathbf{R}^n$ as in Heinonen–Kilpeläinen–Martio [17], cf. Example 5.4. However, we do not know whether nonuniqueness as in Theorem 1.3 can be obtained for Cheeger p-harmonic Green functions.

The examples in Theorem 1.3 also yield counterexamples to other problems. The strong comparison principle for the p-Laplace operator Δ_p, $1 < p < \infty$, states that if u and v are two p-harmonic functions in a domain Ω satisfying $u \geq v$, then either $u > v$ in Ω or $u \equiv v$ in Ω. When $p = 2$, this is a direct consequence of the linearity of the Laplace operator and the maximum principle for harmonic functions. For the nonlinear p-Laplace operator when $p \neq 2$, this argument does not hold. Using the theory of quasiregular mappings, Manfredi [29, Theorem 2] proved the strong comparison principle for planar domains (in unweighted $\mathbf{R}^2$). In higher dimensional Euclidean spaces $\mathbf{R}^n$, with $n \geq 3$, the strong comparison principle is still an open problem. In metric spaces, we have the following negative result.

Corollary 1.4. *Let $1 < p < \infty$. Then there is a metric d on $\mathbf{R}^2$, biLipschitz equivalent to the Euclidean metric, and a p-admissible measure $d\mu = w\,dx$, such that the following hold for $X = (\mathbf{R}^2, d, \mu)$:*

(a) *The strong comparison principle fails for p-harmonic functions in the punctured unit disc $\{x \in \mathbf{R}^2 : 0 < |x| < 1\}$.*

(b) *The theory for 2-harmonic functions is nonlinear: there are 2-harmonic functions u and v in the punctured unit disc $\{x \in \mathbf{R}^2 : 0 < |x| < 1\}$ such that $u + v$ is not 2-harmonic.*

The paper is organized in the following way. Section 2 contains some definitions and preliminary results about upper gradients and Newtonian functions. In Section 3, p-harmonic functions and some auxiliary results for them are presented. Section 4 deals with the definitions of singular and Green functions and their properties. The uniqueness Theorem 1.1 is proved in Section 5, while Theorem 1.3 and Corollary 1.4 are proved in Section 6.

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2. Upper gradients and Newtonian spaces

We assume throughout the paper that $1 < p < \infty$ and that $X = (X, d, \mu)$ is a metric space equipped with a metric d and a positive complete Borel measure μ such that $0 < \mu(B) < \infty$ for all balls $B \subset X$. We also assume that $\operatorname{diam} X > 0$. *Note that it follows from these assumptions that X is separable. Additional standing assumptions are added at the beginning of Section 3.*

In this section, we introduce the necessary metric space concepts used in this paper. Proofs of most of the results mentioned in this section can be found in the monographs Björn–Björn [4] and Heinonen–Koskela–Shanmugalingam–Tyson [20].

We denote by $B(x,r) := \{y \in X : d(x,y) < r\}$ the open ball of radius $r > 0$ centred at $x \in X$. For $B := B(x,r)$ we use the notation $\lambda B = B(x, \lambda r)$. In some parts, we will also write B_r, when the centre of the ball is fixed. In metric spaces it can happen that balls with different centres or radii denote the same set. We will, however, make the convention that a ball B comes with a predetermined centre and radius. Unless said otherwise, balls are assumed to be open in this paper. By $\overline{B}$ we mean the closure of the open ball $B = B(x,r)$, not the (possibly larger) set $\{y : d(x,y) \leq r\}$. We write $\operatorname{diam} A$ for the diameter of a set $A \subset X$.

The measure μ is *doubling* if there exists a *doubling constant* $C_\mu \geq 1$ such that

$$\mu(2B) \leq C_\mu \mu(B) \quad \text{for every open ball } B.$$

The measure μ is *Ahlfors Q-regular* if there is a constant $C \geq 1$ such that

$$C^{-1} r^Q \leq \mu(B(x,r)) \leq C r^Q \quad \text{for all } x \in X \text{ and } 0 < r < 2 \operatorname{diam} X.$$

A *curve* is a continuous mapping from an interval, and a *rectifiable* curve is a curve with finite length. A rectifiable curve can be parameterized by its arc length ds. A property holds for *p-almost every curve* if the curve family Γ for which it fails has zero p-modulus, i.e. there is $\rho \in L^p(X)$ such that $\int_\gamma \rho\, ds = \infty$ for every $\gamma \in \Gamma$.

Definition 2.1. Let $u : X \to \overline{\mathbf{R}} := [-\infty, \infty]$ be a function. A Borel function $g : X \to [0, \infty]$ is an *upper gradient* of u if for all nonconstant rectifiable curves $\gamma : [0, l_\gamma] \to X$,

$$|u(\gamma(0)) - u(\gamma(l_\gamma))| \le \int_\gamma g\, ds, \tag{2.1}$$

where the left-hand side is considered to be ∞ whenever at least one of the terms therein is infinite. A measurable function $g : X \to [0, \infty]$ is a *p-weak upper gradient* of u if it satisfies (2.1) for p-almost every nonconstant rectifiable curve γ.

The upper gradients were introduced in Heinonen–Koskela [18], [19] and p-weak upper gradients in Koskela–MacManus [26]. It was also shown in [26] that if $g \in L^p_{\mathrm{loc}}(X)$ is a p-weak upper gradient of u, then one can find a sequence $\{g_j\}_{j=1}^\infty$ of upper gradients of u such that $\|g_j - g\|_{L^p(X)} \to 0$. If u has an upper gradient in $L^p_{\mathrm{loc}}(X)$, then it has an a.e. unique *minimal p-weak upper gradient* $g_u \in L^p_{\mathrm{loc}}(X)$ in the sense that $g_u \le g$ a.e. for every p-weak upper gradient $g \in L^p_{\mathrm{loc}}(X)$ of u, see Shanmugalingam [31].

Following Shanmugalingam [30], we define a version of Sobolev spaces on the metric space X.

Definition 2.2. For a measurable function $u : X \to \overline{\mathbf{R}}$, let

$$\|u\|_{N^{1,p}(X)} = \biggl(\int_X |u|^p\, d\mu + \inf_g \int_X g^p\, d\mu \biggr)^{1/p},$$

where the infimum is taken over all upper gradients g of u. The *Newtonian space* on X is

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}(X)} < \infty\}.$$

The quotient space $N^{1,p}(X)/\sim$, where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$, is a Banach space and a lattice, see Shanmugalingam [30].

In this paper it is convenient to assume that functions in $N^{1,p}(X)$ are defined everywhere (with values in $\overline{\mathbf{R}}$), not just up to an equivalence class in the corresponding quotient space. This is important for upper gradients and p-weak upper gradients to make sense, and is also convenient in other places, e.g. when defining the capacities below.

We say that $u \in N^{1,p}_{\mathrm{loc}}(X)$ if for every $x \in X$ there is a ball $B(x, r_x)$ such that $u \in N^{1,p}(B(x, r_x))$. If $u, v \in N^{1,p}_{\mathrm{loc}}(X)$, then $g_u = g_v$ a.e. in $\{x \in X : u(x) = v(x)\}$. In particular $g_{\min\{u,c\}} = g_u \chi_{\{u<c\}}$ a.e. in X for $c \in \mathbf{R}$, where χ denotes the characteristic function. For any nonempty open set $\Omega \subset X$, the spaces $N^{1,p}(\Omega)$ and $N^{1,p}_{\mathrm{loc}}(\Omega)$ are defined by considering $(\Omega, d|_\Omega, \mu|_\Omega)$ as a metric space in its own right.

The *Sobolev capacity* of an arbitrary set $E \subset X$ is

$$C_p(E) = \inf_u \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u \ge 1$ on E. The Sobolev capacity is countably subadditive.

A property of points $x \in X$ holds *quasieverywhere* (q.e.) if it is true outside a set of capacity zero. The Sobolev capacity is the correct gauge for distinguishing between two Newtonian functions. If $u \in N^{1,p}(X)$, then $v \sim u$ if and only if $v = u$ q.e. Moreover, if $u, v \in N^{1,p}(X)$ and $u = v$ a.e., then $u = v$ q.e.

The Newtonian space with zero boundary values is defined by

$$N_0^{1,p}(\Omega) := \{f|_\Omega : f \in N^{1,p}(X) \text{ and } f = 0 \text{ on } X \setminus \Omega\}.$$

Definition 2.3. Let $\Omega \subset X$ be a (possibly unbounded) open set. The *condenser capacity* of a bounded set $E \subset \Omega$ with respect to Ω is

$$\operatorname{cap}_p(E, \Omega) = \inf_u \int_\Omega g_u^p \, d\mu,$$

where the infimum is taken over all $u \in N_0^{1,p}(\Omega)$ such that $u = 1$ on E. If no such function u exists then $\operatorname{cap}_p(E, \Omega) = \infty$. If $E \subset \Omega$ is unbounded, we define (as in Björn–Björn [5])

$$\operatorname{cap}_p(E, \Omega) = \lim_{j \to \infty} \operatorname{cap}_p(E \cap B_j, \Omega).$$

Definition 2.4. Let $q \geq 1$. We say that X or μ supports a *q-Poincaré inequality* if there exist constants $C_{\mathrm{PI}} > 0$ and $\lambda \geq 1$ such that for all balls $B = B(x, r)$, every integrable function u on X, and all upper gradients g of u on X,

$$\fint_B |u - u_B| \, d\mu \leq C_{\mathrm{PI}} r \left(\fint_{\lambda B} g^q \, d\mu \right)^{1/q}, \tag{2.2}$$

where $u_B := \fint_B u \, d\mu := \mu(B)^{-1} \int_B u \, d\mu$.

There are many equivalent formulations of Poincaré inequalities, see e.g. [4, Proposition 4.13] and [20]. In particular, (2.2) can equivalently be required for all p-weak upper gradients. At the same time, in complete spaces with a doubling measure μ, it suffices if (2.2) holds for all compactly supported Lipschitz functions with compactly supported Lipschitz upper gradients, see Keith [23].

A *weight* w on $\mathbf{R}^n$ is a nonnegative locally integrable function. If $d\mu = w \, dx$ is a doubling measure, which supports a p-Poincaré inequality on $\mathbf{R}^n$, then w is called a *p-admissible weight*. See Corollary 20.9 in Heinonen–Kilpeläinen–Martio [17] and Proposition A.17 in [4] for why this is equivalent to other definitions in the literature. Moreover, in this case $g_u = |\nabla u|$ a.e. if $u \in N^{1,p}(\mathbf{R}^n)$, where ∇u is the gradient from [17], and the Sobolev and condenser capacities coincide with those in [17], see Theorem 6.7 and Proposition A.12 in [4] and Proposition 7.2 in Björn–Björn [5]. In particular, $\mathbf{R}^n$ equipped with a p-admissible weight is a special case of the metric spaces considered in this paper and all our results apply in that case. A rich potential theory for such weighted $\mathbf{R}^n$ was developed in [17]. Also many manifolds and other spaces are included in our study.

Throughout the paper, we write $a \lesssim b$ and $b \gtrsim a$ if there is an implicit comparison constant $C > 0$ such that $a \leq Cb$, and $a \simeq b$ if $a \lesssim b \lesssim a$. The implicit comparison constants are allowed to depend on the fixed data.

A *domain* is a nonempty connected open set. As usual, we write $u_+ = \max\{u, 0\}$, and by $E \Subset \Omega$ we mean that $\overline{E}$ is a compact subset of Ω. In this paper, a continuous function is always assumed to be real-valued (as opposed to $\overline{\mathbf{R}}$-valued).

3. p-harmonic and superharmonic functions

In addition to the assumptions from the beginning of Section 2, we assume from now on that X is a complete metric space equipped with a doubling measure μ that supports a p-Poincaré inequality, where $1 < p < \infty$. We also fix a point $x_0 \in X$, write $B_r = B(x_0, r)$, and assume that $\Omega \subset X$ is a nonempty open set.

It follows from these assumptions that X is proper (i.e. every closed bounded set is compact) and connected. Moreover, X is *quasiconvex*, i.e. there is a constant C such that for every pair of points $x, y \in X$ there is a rectifiable curve γ from x to y with length $\ell_\gamma \le Cd(x,y)$. In particular X is locally connected, and components of open sets are open. See e.g. [4, Proposition 3.1 and Theorem 4.32] or [20, Lemma 4.1.14 and Theorem 8.3.2].

A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is a *minimizer* in Ω if

$$\int_{\varphi \ne 0} g_u^p \, d\mu \le \int_{\varphi \ne 0} g_{u+\varphi}^p \, d\mu \quad \text{for all } \varphi \in N^{1,p}_0(\Omega). \tag{3.1}$$

Equivalently, one can require (3.1) only for $\varphi \in \mathrm{Lip}_c(\Omega)$, see Björn [3, Proposition 3.2] (or [4, Proposition 7.9]) for this and other characterizations. A continuous minimizer is called a *p-harmonic function*. (It follows from Kinnunen–Shanmugalingam [25] that every minimizer has a continuous representative.)

Definition 3.1. A function $u : \Omega \to (-\infty, \infty]$ is *superharmonic* in Ω if
- (i) u is lower semicontinuous;
- (ii) u is not identically ∞ in any component of Ω;
- (iii) for every nonempty open set $G \Subset \Omega$ with $C_p(X \setminus G) > 0$ and every function $v \in C(\overline{G})$ which is p-harmonic in G and such that $v \le u$ on ∂G, we have $v \le u$ in G.

This definition of superharmonicity is the same as the one usually used in the Euclidean literature, e.g. in Heinonen–Kilpeläinen–Martio [17, Section 7]. It is equivalent to other definitions of superharmonicity on metric spaces, by Theorem 6.1 in Björn [2] (or [4, Theorem 14.10]). It is not difficult to see that a function u is p-harmonic if and only if both u and $-u$ are superharmonic.

The Harnack inequality for nonnegative p-harmonic function was obtained in Kinnunen–Shanmugalingam [25, Corollary 7.3] using De Giorgi's method (see however [9, Section 10] or [4, Section 8.4] for some necessary modifications). A different proof, using Moser iteration, was given by Björn–Marola [9, Theorem 9.3], and one using a combination of both methods by Björn–Björn [4, Theorem 8.12]. The strong maximum principle is a direct consequence of the Harnack inequality.

Theorem 3.2. (Harnack's inequality) *There exists a constant $A > 0$ depending only on p and the doubling and Poincaré constants C_μ, C_{PI} and λ, such that for any p-harmonic function $u \ge 0$ in Ω, one has*

$$\sup_B u \le A \inf_B u,$$

for every ball $B \subset 50\lambda B \subset \Omega$.

We need the following version of Harnack's inequality, which is obtained by iteration.

Proposition 3.3. *There are constants $C_0, \alpha > 0$, depending only on p and the doubling and Poincaré constants C_μ, C_{PI} and λ, such that if $u \ge 0$ is p-harmonic in a ball B, then for all $0 < \delta \le 1/50\lambda$,*

$$\sup_{\delta B} u \le (1 + C_0 \delta^\alpha) \inf_{\delta B} u.$$

Proof. Let k be the smallest integer such that $\delta > (50\lambda)^{-k-2}$. Note that $k \ge 0$. For $j = 0, 1, \dots, k+1$, let

$$B^j = (50\lambda)^j \delta B, \quad M_j = \sup_{B^j} u \quad \text{and} \quad m_j = \inf_{B^j} u.$$

Then $u - m_{j+1}$ is a nonnegative p-harmonic function in $B^{j+1} = 50\lambda B^j \subset B$. Let A be the constant in Harnack's inequality (Theorem 3.2). Thus

$$M_j - m_{j+1} \le A(m_j - m_{j+1})$$

and hence

$$A(M_j - m_j) = (A-1)M_j + M_j - Am_j \le (A-1)(M_j - m_{j+1}).$$

Dividing by Am_{j+1}, we get

$$\frac{M_j}{m_j} - 1 \le \frac{M_j - m_j}{m_{j+1}} \le \frac{(A-1)(M_j - m_{j+1})}{Am_{j+1}} \le \frac{A-1}{A}\biggl(\frac{M_{j+1}}{m_{j+1}} - 1\biggr).$$

Iterating this inequality and another use of the Harnack inequality yields

$$\frac{M_0}{m_0} - 1 \le \biggl(\frac{A-1}{A}\biggr)^k \biggl(\frac{M_k}{m_k} - 1\biggr) \le \biggl(\frac{A-1}{A}\biggr)^{k+1} A \le C_0 \delta^\alpha,$$

where $\alpha \log 50\lambda = \log(A/(A-1)) > 0$ and $C_0 = A(50\lambda)^\alpha$. □

The following lemma will serve as a substitute for the Loewner-type estimate [10, Lemmas 2.11 and 2.12] used in the proof of Lemma 3.4 in Bonk–Capogna–Zhou [10] for Q-harmonic functions in Ahlfors Q-regular spaces. Here we do not assume Ahlfors regularity and consider p-harmonic functions for any $1 < p < \infty$.

Lemma 3.4. *Let u be a superharmonic function in a domain Ω, which is p-harmonic in $\Omega \setminus \overline{B}$ for some ball $B = B(x_0, r)$ such that $4B \subset \Omega$. Let*

$$M = \sup_{3B\setminus 2B} u \quad \textit{and} \quad m = \inf_{3B\setminus 2B} u$$

and assume that u is normalized so that for all $m \le a < b \le M$,

$$\int_{\{x\in\Omega : a<u(x)<b\}} g_u^p \, d\mu = b - a. \tag{3.2}$$

If $\Omega_M := \{x \in \Omega : u(x) > M\} \Subset \Omega$, then

$$M - m \lesssim \biggl(\frac{\mu(B)}{r^p}\biggr)^{1/(1-p)},$$

where the comparison constant in $\lesssim$ only depends on p and the doubling and Poincaré constants C_μ, C_{PI} and λ.

Proof. By the lower semicontinuity and the strong minimum principle for superharmonic functions [4, Theorem 9.13], we see that

$$m = \min_{\overline{3B}} u$$

is attained at some $x_m \in \partial(3B)$ and that $3B \subset \{x \in \Omega : u(x) \ge m\}$.

Next, let G be a component of Ω_M. Since $u \le M$ in $3B \setminus 2B$, we have either $G \subset 2B$ or $G \subset \Omega \setminus 3B$. However, the latter is impossible because of the strong maximum principle for p-harmonic functions and the assumption that $G \Subset \Omega$. We therefore conclude that $\Omega_M \subset 2B$ and by the continuity of u in $\Omega \setminus \overline{B}$, there is $x_M \in \partial(2B)$ such that $M = u(x_M)$.

Next, note that $u - m$ and $M - u$ are nonnegative p-harmonic functions in

$$B^M := B(x_M, r) \subset 3B \setminus B \quad \text{and} \quad B^m := B(x_m, r) \subset 4B \setminus 2B,$$

respectively. Proposition 3.3 then provides us with $0 < \delta < 1$, depending only on p, C_μ, C_{PI} and λ, so that

$$\sup_{\delta B^M}(u-m) \le \frac{4}{3}\inf_{\delta B^M}(u-m) \quad \text{and} \quad \sup_{\delta B^m}(M-u) \le \frac{4}{3}\inf_{\delta B^m}(M-u).$$

Then

$$M' := \inf_{\delta B^M} u \ge m + \frac{3}{4}(M-m) \quad \text{and} \quad m' := \sup_{\delta B^m} u \le M - \frac{3}{4}(M-m).$$

Subtracting the second inequality from the first, we get that

$$M'-m' \ge \tfrac{1}{2}(M-m). \tag{3.3}$$

Next, let $v = \min\{M', \max\{m', u\}\}$ be the truncation of u at the levels m' and M'. Then, by the doubling property of μ,

$$\fint_{4B} |v - v_{4B}|\,d\mu \ge \frac{1}{2}(M'-m')\frac{\min\{\mu(\delta B^M), \mu(\delta B^m)\}}{\mu(4B)} \gtrsim M'-m'.$$

Inserting this into the p-Poincaré inequality for v in $4B$ implies that

$$M'-m' \lesssim r\biggl(\fint_{4\lambda B} g_v^p\,d\mu\biggr)^{1/p} \lesssim \frac{r}{\mu(B)^{1/p}}\biggl(\int_{\{x\in\Omega: m'<u(x)<M'\}} g_u^p\,d\mu\biggr)^{1/p},$$

where λ is the dilation constant in the p-Poincaré inequality. Note that $m \le m' \le M' \le M$. The normalization (3.2) then yields

$$M'-m' \lesssim \frac{r}{\mu(B)^{1/p}}(M'-m')^{1/p}$$

and hence

$$M'-m' \lesssim \biggl(\frac{r^p}{\mu(B)}\biggr)^{1/(p-1)}.$$

Finally, (3.3) concludes the proof. $\square$

4. Singular and Green functions

Green functions are properly normalized singular functions, see Definition 4.5 and also Theorem 4.6 below for the close relation between singular and Green functions. We begin by defining singular functions. The following definition was formulated in Björn–Björn–Lehrbäck [7] for bounded Ω.

Definition 4.1. ([7, Definition 1.1]) Let $\Omega \subset X$ be a bounded domain. A positive function $u : \Omega \to (0,\infty]$ is a *singular function* in Ω with singularity at $x_0 \in \Omega$ if it satisfies the following properties:

(S1) u is p-harmonic in $\Omega \setminus \{x_0\}$;
(S2) u is superharmonic in Ω;
(S3) $u(x_0) = \sup_\Omega u$;
(S4) $\inf_\Omega u = 0$;
(S5) $\tilde{u} \in N^{1,p}_{\mathrm{loc}}(X \setminus \{x_0\})$, where

$$\tilde{u} = \begin{cases} u & \text{in } \Omega, \\ 0 & \text{on } X \setminus \Omega. \end{cases}$$

The last condition (S5) says that u has "zero boundary values in the Sobolev sense". This makes it possible to treat arbitrary bounded domains Ω and not only regular bounded domains, where the zero boundary data are attained as limits $\lim_{\Omega\ni y\to x} u(y) = 0$ for all $x \in \partial\Omega$.

However, Definition 4.1 is not suitable for unbounded domains Ω, including X itself, since the condition $\tilde{u} \in N^{1,p}_{\mathrm{loc}}(X \setminus \{x_0\})$ does not capture the zero boundary condition at ∞. This is demonstrated by the following example. To simplify the notation, we use $\lim_{x\to\infty}$ as a shorthand for $\lim_{d(x,x_0)\to\infty}$.

Example 4.2. Let $X = \mathbf{R}^n$, $n > 2 = p$ and $\Omega = \mathbf{R}^n \setminus \overline{B(0,1)}$. Let u be the classical 2-harmonic Green function in Ω with singularity at some $x_0 \in \Omega$. Then $\lim_{x\to\infty} u(x) = 0$ and u satisfies Definition 4.1. Let $c > 0$. Then also the function $\bar{u}(x) := u(x) + c(1 - |x|^{2-n})$ satisfies the conditions in Definition 4.1, but $\lim_{x\to\infty} \bar{u}(x) = c > 0$ is not desirable for Green functions, since $\mathbf{R}^n$ is 2-hyperbolic in the sense of Definition 4.7.

Replacing $\tilde{u} \in N^{1,p}_{\mathrm{loc}}(X \setminus \{x_0\})$ in Definition 4.1 by $\tilde{u} \in N^{1,p}(X \setminus \overline{B}_r)$ for every $r > 0$ does not help since the desired Green function (such as $u(x) = c|x|^{(p-n)/(p-1)}$ in $\mathbf{R}^n$) is typically not L^p-integrable at ∞.

Thus, for unbounded domains, the zero boundary value at ∞ needs to be captured in a different way than by Newtonian spaces. One way of doing this simultaneously both for finite boundary points and for ∞ is as in Björn–Björn [5] by means of Perron solutions with respect to the extended boundary

$$\partial^*\Omega := \begin{cases} \partial\Omega \cup \{\infty\}, & \text{if } \Omega \text{ is unbounded,} \\ \partial\Omega, & \text{otherwise,} \end{cases}$$

where ∞ is the point added in the *one-point compactification* $X^* := X \cup \{\infty\}$ of X when X is unbounded.

Definition 4.3. Assume that $\partial^*\Omega \neq \varnothing$. Given $f : \partial^*\Omega \to \overline{\mathbf{R}}$, let $\mathcal{U}_f(\Omega)$ be the collection of all superharmonic functions u in Ω that are bounded from below and such that

$$\liminf_{\Omega\ni y\to x} u(y) \geq f(x) \quad \text{for all } x \in \partial^*\Omega.$$

The *upper Perron solution* of f is defined by

$$\overline{P}_\Omega f(x) = \inf_{u\in\mathcal{U}_f(\Omega)} u(x), \quad x \in \Omega.$$

The *lower Perron solution* is given by $\underline{P}_\Omega f := -\overline{P}_\Omega(-f)$. If $\overline{P}_\Omega f = \underline{P}_\Omega f$, then we denote the common value by $P_\Omega f$.

Perron solutions make it possible to define singular functions also in unbounded domains as follows.

Definition 4.4. ([5, Definition 9.1]) Let $\Omega \subset X$ be a (possibly unbounded) domain. A positive function $u : \Omega \to (0,\infty]$ is a *singular function* in Ω with singularity at $x_0 \in \Omega$ if it satisfies the following properties:

(S1) u is p-harmonic in $\Omega \setminus \{x_0\}$,
(S2) u is superharmonic in Ω,
(S3) $u(x_0) = \sup_\Omega u$,
(S4) $\inf_\Omega u = 0$,
(S5′) $u = P_{\Omega\setminus\overline{B}_r} \tilde{u}$ in $\Omega \setminus \overline{B}_r$ whenever $B_r \Subset \Omega$ and

$$\tilde{u} = \begin{cases} u & \text{in } \Omega, \\ 0 & \text{on } X^* \setminus \Omega. \end{cases}$$

Note that it is only in the last conditions (S5) and (S5′) that Definitions 4.1 and 4.4 differ. For bounded domains Ω, Definitions 4.1 and 4.4 are equivalent by Proposition 9.2 in [5].

Definition 4.5. A *Green function* in a domain Ω is a singular function that is normalized so that

$$\operatorname{cap}_p(\{u \ge b\}, \Omega) = b^{1-p}, \quad \text{when } 0 < b < u(x_0). \tag{4.1}$$

In fact, it follows from [5, Theorem 1.2] and [7, Theorem 9.3] that Green functions u satisfy

$$\operatorname{cap}_p(\{u \ge b\}, \{u > a\}) = (b-a)^{1-p}, \quad \text{when } 0 \le a < b \le u(x_0), \tag{4.2}$$

where we interpret ∞^{1-p} as 0. Here, we use the shorthand notation $\{u \ge b\} = \{x \in \Omega : u(x) \ge b\}$ and similarly for other functions and constants. The normalizations (4.1) and (4.2) reflect the fact that the Green function in $\mathbf{R}^n$ satisfies the p-Laplace equation (1.1) with the Dirac measure δ_{x_0}.

The following theorem summarizes some of the main results in [5] and [7], which will be used in the sequel.

Theorem 4.6. ([5, Theorems 1.1 and 1.2] and [7, Theorem 1.3])

(a) *If v is a singular function in Ω with singularity at $x_0 \in \Omega$ then there is a unique $A > 0$ such that Av is a Green function in Ω.*

(b) *There is a Green function in a domain Ω with singularity at $x_0 \in \Omega$ if and only if either $C_p(X \setminus \Omega) > 0$ or X is p-hyperbolic, according to Definition 4.7 below.*

Definition 4.7. Assume that X is unbounded. Then X is called *p-hyperbolic* if $\operatorname{cap}_p(K, X) > 0$ for some compact set $K \subset X$. Otherwise, X is *p-parabolic.*

If X is p-hyperbolic and u is a Green function in an unbounded domain Ω, then (under our standing assumptions)

$$\lim_{\Omega \ni x \to \infty} u(x) = 0, \tag{4.3}$$

by Lemma 11.1 in [5]. The following example shows that this is not always true in p-parabolic spaces, see also Proposition 4.9 below. This reflects the fact that in p-parabolic spaces, the point at ∞ has zero p-capacity in a generalized sense.

Thus, (4.3) cannot be used to capture the zero condition at infinity provided by (S5′), which similarly as for finite boundary points only requires the boundary value zero in a weaker sense.

Example 4.8. Let $X = \mathbf{R}^2$, $p = 2$, $\Omega = \mathbf{R}^2 \setminus \overline{B(0,1)}$ and $z_0 = 2$ (using complex notation). Then the Green function for Ω with singularity at z_0 can be obtained by a Möbius transformation from the Green function in the unit disc $B(0,1)$ with singularity at 0 and can be calculated to be

$$u(z) = A \log \left| \frac{2z-1}{2-z} \right|$$

for some multiplicative constant $A > 0$. Then $\lim_{z \to \infty} u(z) = A \log 2 > 0$. Further, (S5′) is still true for this function, since ∞ has 2-capacity zero in a generalized sense, and thus has harmonic measure zero. This example is not just a coincidence, but illustrates a more general phenomenon.

The following result shows that for unbounded sets in p-parabolic spaces, we cannot require that Green functions tend to zero at ∞.

Proposition 4.9. *Assume that X is p-parabolic and that K is a compact set with $C_p(K) > 0$, such that $\Omega = X \setminus K$ is a domain. Let u be a Green function in Ω with singularity at $x_0 \in \Omega$, which exists by Theorem 4.6(b). Then*

$$\liminf_{x \to \infty} u(x) > 0.$$

Proof. Let $R > 0$ be so large that $K \Subset B_R$ and let $m = \min_{\partial B_R} u > 0$. Consider the open set $G = X \setminus \overline{B}_R$ and the continuous function f on $\partial^* G$ given by $f(x) = u(x)$ for $x \in \partial B_R$ and $f(\infty) = m$. Note that G, being a subset of the p-parabolic space X, is a p-parabolic set in the sense of Definition 4.1 in Hansevi [16]. Since u is p-harmonic in G and attains the boundary values f on ∂G, Corollary 7.9 in [16] implies that $u = P_G f$ and hence $u \geq m > 0$ in G. □

5. Uniqueness of Green functions

In this section, we prove Theorem 1.1 when $C_p(\{x_0\}) = 0$. The main step in the proof is captured by the following lemma.

Lemma 5.1. *Let u be a Green function in a domain Ω with singularity at x_0. Assume that $C_p(\{x_0\}) = 0$ and that there is a sequence $r_j \searrow 0$ such that*

$$\lim_{j \to \infty} \biggl(\frac{\mu(B_{r_j})}{r_j^p} \biggr)^{1/(p-1)} \int_{r_j}^1 \biggl(\frac{\rho^p}{\mu(B_\rho)} \biggr)^{1/(p-1)} \frac{d\rho}{\rho} = \infty. \tag{5.1}$$

For $r > 0$, let

$$m(r) = \min_{d(x,x_0)=r} u(x) \quad \textit{and} \quad M(r) = \max_{d(x,x_0)=r} u(x).$$

Then for every ball $B_R \Subset \Omega$,

$$\lim_{j \to \infty} \frac{m(r_j)}{\operatorname{cap}_p(B_{r_j}, B_R)^{1/(1-p)}} = \lim_{j \to \infty} \frac{M(r_j)}{\operatorname{cap}_p(B_{r_j}, B_R)^{1/(1-p)}} = 1. \tag{5.2}$$

Proof. Since $C_p(\{x_0\}) = 0$, we have $u(x_0) = \lim_{x \to x_0} u(x) = \infty$, by Lemma 9.5 in Björn–Björn [5]. Fix $R > 0$ and let $r > 0$ be sufficiently small so that $m(r) > M(R)$. By the continuity of u and the maximum principle for p-harmonic functions we see that

$$B_r \subset \{u \geq m(r)\}.$$

Next, Lemma 9.9 in [5] shows that the set $\{x \in \Omega : u(x) > M(r)\}$ is connected and hence (using also that $u(x_0) = \infty$)

$$\{u > M(r)\} \subset B_r.$$

Similar statements hold with r replaced by R and we get

$$\{u > M(r)\} \subset B_r \subset \{u \geq m(r)\} \subset \{u > M(R)\} \subset B_R \subset \{u \geq m(R)\}.$$

Since u is a Green function (and thus normalized), we then have by (4.2) and the monotonicity of the capacity that

$$m(r) - M(R) = \operatorname{cap}_p(\{u \geq m(r)\}, \{u > M(R)\})^{1/(1-p)} \leq \operatorname{cap}_p(B_r, B_R)^{1/(1-p)}$$

and

$$\begin{aligned} M(r) - m(R) &= \lim_{\varepsilon \to 0+} \operatorname{cap}_p(\{u \geq M(r) + \varepsilon\}, \{u > m(R) - \varepsilon\})^{1/(1-p)} \\ &\geq \operatorname{cap}_p(B_r, B_R)^{1/(1-p)}. \end{aligned}$$

Hence

$$\frac{m(r)}{M(r)} - \frac{M(R)}{M(r)} \le \frac{\operatorname{cap}_p(B_r, B_R)^{1/(1-p)}}{M(r)} \le 1 - \frac{m(R)}{M(r)}.$$

Since R is fixed and $M(r) \to \infty$ as $r \to 0$, we have

$$\lim_{r\to 0} \frac{M(R)}{M(r)} = \lim_{r\to 0} \frac{m(R)}{M(r)} = 0.$$

It therefore suffices to show that $m(r_j)/M(r_j) \to 1$, or equivalently that

$$\frac{M(r_j) - m(r_j)}{M(r_j)} \to 0 \quad \text{as } j \to \infty.$$

To this end, Lemma 3.4 together with the estimate for $M(r)$ from Björn–Björn–Lehrbäck [8, Theorem 7.1] and the doubling property of μ give that, for sufficiently small r_j,

$$0 \le \frac{M(r_j) - m(r_j)}{M(r_j)} \lesssim \biggl(\frac{\mu(B_{r_j})}{r_j^p}\biggr)^{1/(1-p)} \biggl(\int_{r_j}^1 \biggl(\frac{\rho^p}{\mu(B_\rho)}\biggr)^{1/(p-1)} \frac{d\rho}{\rho}\biggr)^{-1} \to 0,$$

as $j \to \infty$, by the assumption (5.1). This proves (5.2). □

Corollary 5.2. *Let u_k be a Green function in a domain $\Omega_k \ni x_0$ with singularity at x_0, $k = 1, 2$. Assume that $C_p(\{x_0\}) = 0$ and that (1.2) holds. Then*

$$\lim_{x\to x_0} \frac{u_1(x)}{u_2(x)} = 1.$$

Proof. Choose $r_j \searrow 0$ such that (5.1) holds. Lemma 5.1 implies that for each $\varepsilon > 0$, there is j_0 such that for all $j \ge j_0$,

$$\max_{d(x,x_0)=r_j} u_1 \le (1+\varepsilon) \operatorname{cap}_p(B_{r_j}, B_R)^{1/(1-p)} \le (1+\varepsilon)^2 \min_{d(x,x_0)=r_j} u_2.$$

The maximum principle applied to the annuli $B_{r_j} \setminus \overline{B}_{r_{j+1}}$ then gives that $u_1 \le (1+\varepsilon)^2 u_2$ also in $B_{r_j} \setminus \overline{B}_{r_{j+1}}$for every $j \ge j_0$, and hence in B_{r_j}. Letting $\varepsilon \to 0$ and applying this also with the roles of u_1 and u_2 interchanged completes the proof. □

Proof of Theorem 1.1. When $C_p(\{x_0\}) > 0$, the Green function is a particular multiple of the capacitary potential for $\operatorname{cap}_p(\{x_0\}, \Omega)$ and is therefore unique, see Björn–Björn–Lehrbäck [7, Theorem 1.3] (for bounded Ω) and Björn–Björn [5, Corollary 10.3] (for general Ω). Assume therefore that $C_p(\{x_0\}) = 0$. Let $\varepsilon > 0$. By Corollary 5.2, there is $r < \varepsilon$ such that

$$u \le (1+\varepsilon)v \quad \text{in } B_{2r} \setminus \{x_0\}.$$

Since also $u = v = 0$ on $\partial^*\Omega$, it follows from the definitions of singular functions and Perron solutions that

$$u = P_{\Omega\setminus\overline{B}_r} u \le (1+\varepsilon) P_{\Omega\setminus\overline{B}_r} v = (1+\varepsilon)v \quad \text{in } \Omega \setminus \overline{B}_r.$$

Hence $u \le (1+\varepsilon)v$ in Ω. Letting $\varepsilon \to 0$ and applying this also with the roles of u and v interchanged shows that $u = v$ in $\Omega \setminus \{x_0\}$, while $u(x_0) = v(x_0) = \infty$ follows from $C_p(\{x_0\}) = 0$. □

Condition (1.2) can be determined by the following exponent sets:

$$\underline{Q}_0 := \biggl\{ q>0 : \frac{\mu(B_r)}{\mu(B_R)} \lesssim \Bigl(\frac{r}{R}\Bigr)^q \text{ for all } 0<r<R\le 1 \biggr\}, \tag{5.3}$$

and

$$\overline{Q}_0 := \biggl\{ q>0 : \frac{\mu(B_r)}{\mu(B_R)} \gtrsim \Bigl(\frac{r}{R}\Bigr)^q \text{ for all } 0<r<R\le 1 \biggr\}. \tag{5.4}$$

Note that $\underline{Q}_0$ and $\overline{Q}_0$ are nonempty intervals, under our standing assumptions. The general case when condition (1.2) holds in Remark 1.2 follows from Proposition 5.3(a) below.

Proposition 5.3.
(a) *If $p \notin \underline{Q}_0$ or $p \in \overline{Q}_0$, then (1.2) holds.*
(b) *If $p < \sup \underline{Q}_0$, then (1.2) fails.*

Proof. (a) If $p \notin \underline{Q}_0$, it follows from the definition of $\underline{Q}_0$ that there are $0<r_j<R_j \searrow 0$ such that for each $j=1,2,\dots$,

$$\frac{\mu(B_{r_j})}{\mu(B_{R_j})} \ge j\Bigl(\frac{r_j}{R_j}\Bigr)^p.$$

By passing to a subsequence, we can assume that also $r_j \searrow 0$ and that $R_1 \le \frac{1}{2}$. Now, by the doubling property of μ we have for each j,

$$\int_{r_j}^1 \biggl(\frac{\rho^p}{\mu(B_\rho)}\biggr)^{1/(p-1)} \frac{d\rho}{\rho} \ge \int_{R_j}^{2R_j} \biggl(\frac{\rho^p}{\mu(B_\rho)}\biggr)^{1/(p-1)} \frac{d\rho}{\rho} \simeq \biggl(\frac{R_j^p}{\mu(B_{R_j})}\biggr)^{1/(p-1)}.$$

Hence

$$\biggl(\frac{\mu(B_{r_j})}{r_j^p}\biggr)^{1/(p-1)} \int_{r_j}^1 \biggl(\frac{\rho^p}{\mu(B_\rho)}\biggr)^{1/(p-1)} \frac{d\rho}{\rho} \gtrsim \biggl(\frac{\mu(B_{r_j})}{\mu(B_{R_j})}\frac{R_j^p}{r_j^p}\biggr)^{1/(p-1)} \ge j^{1/(p-1)}.$$

Letting $j\to\infty$ concludes the proof when $p \notin \underline{Q}_0$.

If $p \in \overline{Q}_0$, then for any $0<r<1$ by the definition of $\overline{Q}_0$,

$$\int_r^1 \biggl(\frac{\rho^p}{\mu(B_\rho)}\biggr)^{1/(p-1)} \frac{d\rho}{\rho} \gtrsim \int_r^1 \biggl(\frac{r^p}{\mu(B_r)}\biggr)^{1/(p-1)} \frac{d\rho}{\rho} = \biggl(\frac{r^p}{\mu(B_r)}\biggr)^{1/(p-1)} \log\frac{1}{r}$$

and the rest of the argument is the same as for $p \notin \underline{Q}_0$, upon letting $r\to 0$.

(b) If $p < \sup \underline{Q}_0$, then there is $q \in \underline{Q}_0$ with $q>p$. Let $0<r<\frac{1}{2}$. Then

$$\frac{\mu(B_r)}{r^q} \lesssim \frac{\mu(B_\rho)}{\rho^q} \quad \text{for } r<\rho<1,$$

and thus

$$\begin{aligned}\biggl(\frac{\mu(B_r)}{r^p}\biggr)^{1/(p-1)} &\int_r^1 \biggl(\frac{\rho^p}{\mu(B_\rho)}\biggr)^{1/(p-1)} \frac{d\rho}{\rho} \\ &\lesssim r^{(q-p)/(p-1)} \int_r^1 \biggl(\frac{\mu(B_\rho)}{\rho^q}\frac{\rho^p}{\mu(B_\rho)}\biggr)^{1/(p-1)} \frac{d\rho}{\rho} \simeq 1. \qquad \square\end{aligned}$$

The following example is based on Example 3.3 in Björn–Björn–Lehrbäck [6] and shows that the range of p's for which (5.1) holds and $C_p(\{0\})=0$ can be large.

Example 5.4. Fix $1 < a < b < c < d$. We are now going to construct a weight w on $\mathbf{R}^n$ so that for $a < p \le c$ we have uniqueness in Theorem 1.1 with $x_0 = 0$ and at the same time the associated capacity $C_{p,w}(\{0\}) = 0$, thus showing that this range can be a nondegenerate interval.

Let

$$\lambda = \frac{(c-a)(d-b)}{(b-a)(d-c)}$$

and

$$\alpha_k = 2^{-\lambda^k} \text{ and } \beta_k = \alpha_k^{(d-b)/(d-c)} = \alpha_{k+1}^{(b-a)/(c-a)}, \quad k = 0, 1, 2, \dots .$$

Note that $\lambda > 1$ and thus $\alpha_k \to 0$ as $k \to \infty$. Also, $\alpha_{k+1} \ll \beta_k \ll \alpha_k$. Then the weight $w(x) = w(|x|)$ given by

$$w(\rho) = \begin{cases} \beta_k^{c-a}\rho^{a-n} = \alpha_{k+1}^{b-a}\rho^{a-n}, & \text{if } \alpha_{k+1} \le \rho \le \beta_k,\ k = 0,1,2,\dots, \\ \beta_k^{c-d}\rho^{d-n} = \alpha_k^{b-d}\rho^{d-n}, & \text{if } \beta_k \le \rho \le \alpha_k,\ k = 0,1,2,\dots, \\ \alpha_0, & \text{if } \rho \ge \alpha_0, \end{cases}$$

is continuous and 1-admissible on $\mathbf{R}^n$, by Theorem 10.5 in Björn–Björn–Lehrbäck [6]. Moreover by Example 3.4 in [6], we have

$$\underline{Q}_0 = (0, a] \quad \text{and} \quad \overline{Q}_0 = [d, \infty). \tag{5.5}$$

Let $\Omega \ni 0$ be a domain such that $C_{p,w}(\mathbf{R}^n \setminus \Omega) > 0$ or $p < n$ (which guarantees that the Green function below exists). Theorem 1.1 and Remark 1.2 then show that the Green function with respect to the measure $d\mu = w\,dx$ in Ω with singularity at 0 is unique whenever $p > a$. Next, note that $\alpha_{k+1} \le \frac{1}{2}\beta_k$ for sufficiently large k. Since μ is a doubling measure, a direct calculation shows that

$$\mu(B_{\beta_k}) \simeq \mu(\tfrac{1}{2}B_{\beta_k}) \simeq \mu(B_{\beta_k} \setminus \tfrac{1}{2}B_{\beta_k}) \simeq \beta_k^{c-a}\int_{\frac{1}{2}\beta_k}^{\beta_k} \rho^{a-1}\,d\rho \simeq \beta_k^c.$$

If $p = c$, the general capacity estimate (1.7) in Björn–Björn–Lehrbäck [8] then implies that in the weighted space $(\mathbf{R}^n, w\,dx)$,

$$\begin{aligned} C_{c,w}(\{0\}) &\simeq \left(\int_0^1 \left(\frac{\rho^c}{\mu(B_\rho)}\right)^{1/(p-1)} \frac{d\rho}{\rho}\right)^{1-p} \\ &\le \left(\sum_{k=1}^\infty \left(\frac{\beta_k^c}{\mu(B_{\beta_k})}\right)^{1/(p-1)} \int_{\frac{1}{2}\beta_k}^{\beta_k} \frac{d\rho}{\rho}\right)^{1-p} = 0. \end{aligned}$$

It follows that $C_{p,w}(\{0\}) = 0$ for $p \le c$. In particular, we have uniqueness and at the same time $C_{p,w}(\{0\}) = 0$ (i.e. the Green function u is unbounded and $u(x_0) = \infty$) whenever $a < p \le c$. At the same time, when $p > c$, uniqueness still holds since $C_{p,w}(\{0\}) > 0$ by Proposition 8.2 in [6].

We can make two modifications in this example:

(i) Let $\widetilde{w}(x)$ be a bounded measurable function on $\mathbf{R}^n$ such that $\inf_{\mathbf{R}^n} \widetilde{w} > 0$, and let $w_2(x) = w(x)\widetilde{w}(x)$. Then w_2 is also 1-admissible and has the same exponent sets (5.5) as w, so we get the same conclusions also for this nonradial weight.

(ii) We can moreover equip $X = \mathbf{R}^n$ with a different metric, e.g. an ℓ^q metric with $1 \le q \le \infty$. As long as it is biLipschitz equivalent to the Euclidean metric, the measure will still be globally doubling and supporting a global 1-Poincaré inequality on X, by [4, Proposition 4.16], and the exponent sets (5.5) will remain the same. So the conclusion is the same also in this case.

6. Nonuniqueness of Green functions

In the following example we construct a nonstandard metric on $\mathbf{R}^2$ so that Green functions with singularity at the origin are not unique, both in $\mathbf{R}^2$ and in discs $\{x \in \mathbf{R}^2 : |x| < R\}$. This proves Theorem 1.3. The metric is comparable to the Euclidean metric and possesses a Finsler type structure.

Example 6.1. (*Proof of Theorem* 1.3) Let $X = \mathbf{R}^2$ be equipped with the weighted Lebesgue measure $d\mu = w\,dx$, where $w(x) = |x|^\alpha$ with $\alpha > -1$, and a nonstandard metric d that we will now define. We also let $1 < p < 2 + \alpha$. Note that the weight w is p-admissible, see Heinonen–Kilpeläinen–Martio [17, Corollary 15.35].

For $z \in \mathbf{R}^2 \setminus \{0\}$ we use polar coordinates with complex notation and write $z = re^{i\theta}$ with $r > 0$ and $\theta \in \mathbf{R}$. Thus our functions will be 2π-periodic in θ. Equivalently, we restrict θ to the unit circle $\mathbf{S}^1 = [0, 2\pi)$ (identifying 0 and 2π). In polar coordinates, the standard orthonormal basis for the tangent space $T_z\mathbf{R}^2$ at $z = re^{i\theta} \neq 0$ is given by

$$\partial_r := \frac{\partial}{\partial r} \quad \text{and} \quad \frac{1}{r}\partial_\theta := \frac{1}{r}\frac{\partial}{\partial \theta}.$$

The metric d on $\mathbf{R}^2$ is defined using an ℓ^1-norm on $T_z\mathbf{R}^2$ by

$$d(x,y) = \inf_{\gamma_{x,y}} \int_0^1 \|\dot\gamma(t)\|_{\gamma(t)}\,dt, \quad x,y \in \mathbf{R}^2,$$

where the infimum is taken over all Lipschitz curves $\gamma_{x,y}$ connecting x and y, and

$$\|\dot\gamma(t)\|_z = |a| + |b| \quad \text{when } z \neq 0 \text{ and } \dot\gamma(t) = a\partial_r + \frac{b}{r}\partial_\theta,$$

while $\|\dot\gamma(t)\|_0 = 0$.

The metric d on $\mathbf{R}^2$ is comparable to the Euclidean metric by a factor of $\sqrt{2}$, since the Euclidean norm of $\dot\gamma$ is given by $\sqrt{a^2+b^2}$. It follows that the measure μ is doubling and supports a p-Poincaré inequality also on $(\mathbf{R}^2, d)$, by e.g. [4, Proposition 4.16].

Let $u \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}^2 \setminus \{0\})$. By Rademacher's theorem (see e.g. [14, Theorem 3.2]) for a.e. $z \in \mathbf{R}^2$ there exists a linear mapping (the differential) $du(z) : \mathbf{R}^2 \to \mathbf{R}^2$ such that

$$\lim_{h \to 0} \frac{|u(z+h) - u(z) - du(z)(h)|}{\|h\|_z} = 0.$$

Expressed in polar coordinates with $h = a\partial_r + b\frac{1}{r}\partial_\theta$, the differential can be written as

$$du(z)(h) = a\partial_r u + b\frac{1}{r}\partial_\theta u.$$

Using this and Theorem 6.1 in Cheeger [11] (or Theorem 13.5.1 in Heinonen–Koskela–Shanmugalingam–Tyson [20]), we obtain that for any $u \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}^2 \setminus \{0\})$ and a.e. $z = re^{i\theta} \in \mathbf{R}^2$,

$$\begin{aligned} g_u(z) = \mathrm{Lip}\, u(z) &:= \limsup_{\rho \to 0} \sup_{\|h\|_z \le \rho} \frac{|u(z+h) - u(z)|}{\|h\|_z} \\ &= \lim_{\rho \to 0} \sup_{|a|+|b| \le \rho} \frac{|a\partial_r u(z) + b\frac{1}{r}\partial_\theta u(z)|}{|a|+|b|} = \max\biggl\{|\partial_r u(z)|, \frac{|\partial_\theta u(z)|}{r}\biggr\}. \end{aligned}$$

This calculation expresses the fact that the dual basis of $\{\partial_r, \frac{1}{r}\partial_r\}$ is given by $\{dr, r\,d\theta\}$, and the dual of the ℓ^1-norm on the tangent space is the ℓ^∞-norm on the cotangent space.

Next, we let $f : \mathbf{S}^1 \to \mathbf{R}$ be a 1-Lipschitz function and

$$a_p = \frac{2+\alpha-p}{p-1} > 0. \tag{6.1}$$

Recall that we have assumed that $1 < p < 2+\alpha$ and that $\alpha > -1$. Let $0 < R < \infty$ be fixed and define

$$u(re^{i\theta}) = (r^{-a_p} - R^{-a_p})e^{a_p f(\theta)} \quad \text{for } 0 < r < R \text{ and } \theta \in \mathbf{S}^1.$$

Also let $u(0) = \infty$. We shall show that u is a singular function in $\Omega = \{x \in \mathbf{R}^2 : |x| < R\}$ with singularity at $x_0 = 0$, with respect to $(\mathbf{R}^2, d, \mu)$.

First, we show that u is p-harmonic in $\Omega \setminus \{0\}$. To this end, we have

$$|\partial_r u(re^{i\theta})| = a_p r^{-a_p-1} e^{a_p f(\theta)}$$

and, since $|f'(\theta)| \le 1$ for a.e. $\theta \in \mathbf{S}^1$,

$$\frac{|\partial_\theta u(re^{i\theta})|}{r} = a_p \frac{r^{-a_p} - R^{-a_p}}{r} e^{a_p f(\theta)} |f'(\theta)| \le |\partial_r u(re^{i\theta})| \quad \text{a.e.}$$

Hence

$$g_u(re^{i\theta}) = |\partial_r u(re^{i\theta})| = a_p r^{-a_p-1} e^{a_p f(\theta)} \quad \text{a.e.}$$

Let $0 < a < b < R$ and $\varphi \in \operatorname{Lip}_c(G_{a,b})$, where $G_{a,b} = \{x : a < |x| < b\}$). Fix $\theta \in \mathbf{S}^1$ for the moment. A straightforward calculation shows that the function $r \mapsto u(re^{i\theta})$ is a p-harmonic function on $(0,\infty)$ with respect to the measure $r^{1+\alpha}\,dr$. Indeed, the corresponding Euler–Lagrange equation is $-(|u'|^{p-2}u'r^{1+\alpha})' = 0$, see Heinonen–Kilpeläinen–Martio [17, p. 102]. (They consider $\mathbf{R}^n$ with $n \ge 2$, but the same calculation applies for $n = 1$.)

Thus, comparing the functions $r \mapsto u(re^{i\theta})$ and $r \mapsto (u+\varphi)(re^{i\theta})$ on the interval (a, b), we get

$$\begin{aligned} \int_a^b g_u(re^{i\theta})^p r^{1+\alpha}\,dr &= \int_a^b |\partial_r u(re^{i\theta})|^p r^{1+\alpha}\,dr \\ &\le \int_a^b |\partial_r (u+\varphi)(re^{i\theta})|^p r^{1+\alpha}\,dr \le \int_a^b g_{u+\varphi}(re^{i\theta})^p r^{1+\alpha}\,dr. \end{aligned}$$

Integrating over $\theta \in \mathbf{S}^1$ yields

$$\int_{G_{a,b}} g_u^p\,dx \le \int_{G_{a,b}} g_{u+\varphi}^p\,dx.$$

Hence u is p-harmonic in $G_{a,b}$ for every $0 < a < b < R$ and thus in $\Omega \setminus \{0\}$.

Since $u(0) = \infty$ and $\lim_{G \ni x \to z} u(x) = 0$ whenever $|z| = R$, it follows from Theorem 7.2 in Björn–Björn–Lehrbäck [7] that u is a singular function in Ω. Thus by Theorem 4.6(a), there is $A > 0$ such that Au is a Green function in Ω with singularity at 0. Since f was an arbitrary 1-Lipschitz function on $\mathbf{S}^1$, this shows the nonuniqueness of Green functions in Ω with singularity at 0.

If we instead let

$$u(re^{i\theta}) = r^{-a_p} e^{a_p f(\theta)} \quad \text{for } r > 0 \text{ and } \theta \in \mathbf{S}^1,$$

with $u(0) = \infty$, then the same calculation shows that u is p-harmonic in $\mathbf{R}^2 \setminus \{0\}$. Since $C_p(\{0\}) = 0$, it follows from Lemma 4.3 in [7] that u is superharmonic in $\mathbf{R}^2$, and thus, by Definition 4.4, u is a singular function in $X = (\mathbf{R}^2, d, \mu)$ with singularity at 0. Therefore, by Theorem 4.6(a), there is $A > 0$ such that Au is a Green function in X with singularity at 0. Again, since f was an arbitrary 1-Lipschitz function on $\mathbf{S}^1$, the nonuniqueness follows also in this case.

Finally, since the exponent sets in (5.3) and (5.4) for the measure μ are

$$\underline{Q}_0 = (0, 2+\alpha] \quad \text{and} \quad \overline{Q}_0 = [2+\alpha, \infty),$$

the Green function with singularity at 0 is unique for $p \geq 2+\alpha$ in every domain $\Omega \ni 0$ in $(\mathbf{R}^2, d, \mu)$, by Theorem 1.1 and Remark 1.2.

Note that $(\mathbf{R}^2, d, \mu)$ is p-hyperbolic if and only if $p < 2+\alpha$, while $C_p(\{0\}) = 0$ if and only if $p \leq 2+\alpha$. Note also that in the unweighted case (with $\alpha = 0$) the range when the Green functions are not unique is $1 < p < 2$.

Remark 6.2. One can make similar examples in unweighted and weighted $\mathbf{R}^n$, $n \geq 3$. Since the circle $\mathbf{S}^1$ has to be replaced by the $(n-1)$-dimensional sphere $\mathbf{S}^{n-1}$, the norm on $T_z\mathbf{R}^n$ is for $z = r\omega$ and $h \in \mathbf{R}^n$, with $\omega \in \mathbf{S}^{n-1}$ and $r > 0$, given by

$$\|h\|_z = |h_r| + |h_{\mathbf{S}^{n-1}}|,$$

where $h_r = \frac{h \cdot z}{|z|}$ is the length of the projection onto the radial direction at z and $h_{\mathbf{S}^{n-1}} = h - \frac{h \cdot z}{|z|^2} z$ is the component tangential to the sphere $\{x \in \mathbf{R}^n : |x| = r\}$ at z. The details become somewhat technical. With unweighted $\mathbf{R}^n$ this gives an Ahlfors n-regular example with nonunique Green functions for $1 < p < n$. We leave the details to the interested reader.

Example 6.1 also proves Corollary 1.4:

Proof of Corollary 1.4. Consider (X, d, μ) as in Example 6.1. Let $f_1(\theta) \equiv 0$ and f_2 be a nonnegative 1-Lipschitz function on $\mathbf{S}^1$ such that $f_2(\theta) = 0$ if and only if $\theta = 0$. Next let

$$u_j(re^{i\theta}) = (r^{-a_p} - 1)e^{a_p f_j(\theta)} \quad \text{for } 0 < r < 1 \text{ and } \theta \in \mathbf{S}^1, \ j = 1, 2,$$

where a_p is as in (6.1). By Example 6.1, u_1 and u_2 are p-harmonic in the punctured disc $\Omega := \{x \in \mathbf{R}^2 : 0 < |x| < 1\}$. Moreover, $u_1 \leq u_2$ in Ω and $u_1(x) = u_2(x)$ if and only if $x = re^{i\theta} \in \Omega$ and $\theta = 0$. This shows that the strong comparison principle fails for p-harmonic functions on X.

For the second part consider $p = 2$ above and let $v = u_2 - u_1$. Then $v \geq 0$ in Ω and $v(x) = 0$ if and only if $x = re^{i\theta} \in \Omega$ and $\theta = 0$. It thus follows from the strong maximum principle that v cannot be p-harmonic in Ω. □

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