

REFINED REGULARITY AT CRITICAL POINTS FOR LINEAR ELLIPTIC EQUATIONS

JONGKEUN CHOI, HONGJIE DONG, AND SEICK KIM

ABSTRACT. We investigate the regularity of solutions to linear elliptic equations in both divergence and non-divergence forms, particularly when the principal coefficients have Dini mean oscillation. We show that if a solution u to a divergence-form equation satisfies $Du(x^0) = 0$ at a point, then the second derivative $D^2u(x^0)$ exists and satisfies sharp continuity estimates. As a consequence, we obtain “ $C^{2,\alpha}$ regularity” at critical points when the coefficients of L are C^α . This result refines a theorem of Teixeira (Math. Ann. 358 (2014), no. 1–2, 241–256) in the linear setting, where both linear and nonlinear equations were considered. We also establish an analogous result for equations in non-divergence form.

1. INTRODUCTION AND MAIN RESULTS

We consider the elliptic operator L in divergence form:

$$Lu = D_i(a^{ij}D_ju) + b^iD_iu + cu = \operatorname{div}(\mathbf{A}Du) + \mathbf{b} \cdot Du + cu,$$

and the corresponding operator L in non-divergence form:

$$Lu = a^{ij}D_{ij}u + b^iD_iu + cu = \operatorname{tr}(\mathbf{A}D^2u) + \mathbf{b} \cdot Du + cu$$

defined on a domain $\Omega \subset \mathbb{R}^d$, where $d \geq 2$. Throughout this article, we adopt the standard summation convention over repeated indices.

For a function f defined on $\Omega \subset \mathbb{R}^d$, we define its mean oscillation function $\omega_f(\cdot)$ by

$$\omega_f(r) := \sup_{x \in \Omega} \int_{\Omega \cap B_r(x)} |f - (f)_{\Omega \cap B_r(x)}|, \quad (f)_{\Omega \cap B_r(x)} := \int_{\Omega \cap B_r(x)} f.$$

We say that f has Dini mean oscillation (abbreviated as DMO), and write $f \in \operatorname{DMO}$ if ω_f satisfies the Dini condition:

$$\int_0^1 \frac{\omega_f(r)}{r} dr < \infty.$$

It is clear that if f is Dini continuous (i.e., its modulus of continuity satisfies the Dini condition), then f has Dini mean oscillation. However, the converse is not true: the DMO condition is strictly weaker than Dini continuity; see [2] for examples. Note that the DMO condition implies uniform continuity, with a modulus of continuity controlled by ω_f ; see the appendix of [4].

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This article establishes that if u is a solution to the divergence-form equation $Lu = 0$, and its first derivative vanishes at a point x^0 (i.e., $Du(x^0) = 0$), then the second derivative $D^2u(x^0)$ exists and satisfies quantitative estimates at that point. This conclusion is derived under the assumption that the principal coefficient matrix $\mathbf{A} \in \text{DMO}$, together with suitable conditions on the lower-order coefficients of L . In particular, if $\mathbf{A} \in C^\alpha$, $\mathbf{b} \in L^p$ with $p \geq d/(1 - \alpha)$, and $c \in C^\alpha$ for some $\alpha \in (0, 1)$, then u enjoys “ $C^{2,\alpha}$ regularity” at any critical point x^0 . That is, the second derivative $D^2u(x^0)$ exists, and u satisfies the following estimate for x sufficiently close to x^0 :

$$|Du(x) - D^2u(x^0)(x - x^0)| \lesssim |x - x^0|^{1+\alpha}.$$

This result improves upon a notable theorem of Teixeira [7] in the linear setting, which, under the assumption $\mathbf{A} \in C^\alpha$, $\mathbf{b} = 0$, and $c = 0$, establishes “ $C^{1,1^-}$ regularity” at x^0 . That is, for every $0 < \gamma < 1$, the solution satisfies the estimate

$$|u(x) - u(x^0)| \lesssim |x - x^0|^{1+\gamma}$$

for x sufficiently close to x^0 .

An analogous result holds for the non-divergence form equation $Lu = 0$. Under the assumption that $\mathbf{A} \in \text{DMO}$ and suitable regularity conditions on the lower-order coefficients, if u is a solution of $Lu = 0$ and satisfies $D^2u(x^0) = 0$ at some point x^0 , then the third derivative $D^3u(x^0)$ exists and satisfies appropriate continuity estimates. In particular, if $\mathbf{A} \in C^\alpha$ and $\mathbf{b}, c \in C^{1,\alpha}$ for some $\alpha \in (0, 1)$, then u enjoys “ $C^{3,\alpha}$ regularity” at any point x^0 where its Hessian vanishes.

This article builds upon our recent work in [1], which investigated improved regularity for solutions of the double divergence form equation $L^*u = 0$ (the formal adjoint of the non-divergence form equation $Lu = 0$) on the set where the solution u vanishes. In [1], we specifically established “ $C^{1,\alpha}$ regularity” for solutions of $L^*u = 0$ at points x^0 where $u(x^0) = 0$, under the assumptions that $\mathbf{A} \in C^\alpha$, $\mathbf{b} \in L^p$ for some $p \geq d/(1 - \alpha)$, and $c \in L^p$ for some $p \geq d/(2 - \alpha)$. This result refines an earlier notable theorem by Leitão, Pimentel, and Santos [6].

We now present our main results. In the following theorem, for $\xi \in \mathbb{R}^d$, the notation $D^2u(x^0)\xi$ denotes the product of the symmetric $d \times d$ matrix $D^2u(x^0)$ and the vector ξ , regarded as a $d \times 1$ column vector. We define

$$|D^2u(x^0)| = \sup_{|\xi|=1} |D^2u(x^0)\xi \cdot \xi| = \sup_{|\xi|=1} |D_{ij}u(x^0)\xi^i\xi^j|.$$

Since $D^2u(x^0)$ is symmetric, this definition coincides with the usual operator norm and is equivalent to other matrix norms.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^d$ be a domain. Assume the matrix $\mathbf{A} = (a^{ij})$ satisfies the following ellipticity and boundedness conditions for some constants $\lambda, \Lambda > 0$:*

$$\lambda|\xi|^2 \leq a^{ij}(x)\xi^i\xi^j \quad \text{and} \quad |a^{ij}(x)\xi^i\eta^j| \leq \Lambda|\xi||\eta| \quad \text{for all } x \in \Omega, \xi, \eta \in \mathbb{R}^d. \quad (1.2)$$

Furthermore, assume $\mathbf{A} \in \text{DMO}$, $\mathbf{b} \in L^p$ for some $p > d$, and $c \in \text{DMO}$. Let $u \in C^1_{\text{loc}}(\Omega)$ be a weak solution to the equation:

$$Lu = \text{div}(\mathbf{A}Du) + \mathbf{b} \cdot Du + cu = f \quad \text{in } \Omega, \quad (1.3)$$

where $f \in \text{DMO}$. If $Du(x^0) = 0$ for some $x^0 \in \Omega$, then Du is differentiable at x^0 . More precisely, there exist positive constants r_0 and C , depending only on d, λ, Λ , and the

coefficients of L , such that for $r := \min\{r_0, \frac{1}{2} \text{dist}(x^0, \partial\Omega)\}$, the second derivative D^2u at x^0 satisfies the estimate:

$$|D^2u(x^0)| \leq C \left(\frac{1}{r} \int_{B_{2r}(x^0)} |Du| + \varrho_{\text{lot}}(r) + \varrho_f(r) \right).$$

Here, the terms $\varrho_{\dots}(\cdot)$ represent moduli of continuity, where $\varrho_{\text{coef}}(\cdot)$ is determined by the coefficients of L , $\varrho_{\text{lot}}(\cdot)$ depends on $\|u\|_{L^\infty(B_r(x^0))}$ and c , and $\varrho_f(\cdot)$ is determined by f . Furthermore, for $0 < |x - x^0| < r$, the following estimate holds:

$$\frac{|Du(x) - D^2u(x^0)(x - x^0)|}{|x - x^0|} \leq C \varrho_{\text{coef}}(|x - x^0|) \left\{ \frac{1}{r} \int_{B_{2r}(x^0)} |Du| + \varrho_{\text{lot}}(r) + \varrho_f(r) \right\} + C \varrho_{\text{lot}}(|x - x^0|) + C \varrho_f(|x - x^0|).$$

Here are a few quick remarks concerning Theorem 1.1: First, under the assumptions of the theorem, any weak solution u belongs to $C_{\text{loc}}^1(\Omega)$. In fact, it suffices to assume $c \in L^p$ for some $p > d$, rather than requiring $c \in \text{DMO}$. For further details, refer to Theorem 1.3 in [3] and Theorem 1.5 in [2]. Second, the precise dependence of the moduli of continuity $\varrho_{\dots}(t)$ on the relevant parameters is described in Section 3. Specifically:

- If $\mathbf{A} \in C^\alpha$ for some $\alpha \in (0, 1)$ and $\mathbf{b} \in L^p$ with $p \geq d/(1 - \alpha)$, then $\varrho_{\text{coef}}(t) \lesssim t^\alpha$.
- If $c \in C^\alpha$ for some $\alpha \in (0, 1)$, then $\varrho_{\text{lot}}(t) \lesssim t^\alpha$.
- If $f \in C^\alpha$ for some $\alpha \in (0, 1)$, then $\varrho_f(t) \lesssim t^\alpha$.

Also, we note that $\varrho_{\text{lot}} \equiv 0$ if c is constant, and $\varrho_f \equiv 0$ if f is constant. Finally, while it may seem more natural to consider the equation

$$\text{div}(\mathbf{A}Du + \tilde{\mathbf{b}}u) + \mathbf{b} \cdot Du + cu = \text{div} \tilde{\mathbf{f}} + f$$

instead of (1.3), doing so would require additional regularity assumptions on $\tilde{\mathbf{b}}$ and $\tilde{\mathbf{f}}$ to ensure that the equation reduces properly to (1.3).

In the following theorem, for $\xi \in \mathbb{R}^d$, the notation $\langle D^3u(x^0), \xi \rangle$ denotes a $d \times d$ matrix whose (i, j) -th entry is given by:

$$\langle D^3u(x^0), \xi \rangle_{i,j} = D_{ijk}u(x^0)\xi^k.$$

We also define the norm of the third-order derivative tensor as

$$|D^3u(x^0)| = \sup_{|\xi|=1} |D_{ijk}u(x^0)\xi^i\xi^j\xi^k|.$$

This is equivalent (up to a constant depending only on the dimension) to the supremum of $|\langle D^3u(x^0), e \rangle|$, where the norm is taken in the sense of operator norm for matrices, over all unit vectors $e \in \mathbb{R}^d$.

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^d$ be a domain. Consider a symmetric matrix $\mathbf{A} = (a^{ij})$ satisfying the uniform ellipticity condition:*

$$\lambda|\xi|^2 \leq a^{ij}(x)\xi^i\xi^j \leq \Lambda|\xi|^2 \quad \text{for all } x \in \Omega, \quad \xi \in \mathbb{R}^d, \quad (1.5)$$

where $0 < \lambda \leq \Lambda$ are constants. Additionally, we assume that \mathbf{A} , \mathbf{b} , $D\mathbf{b}$, c and Dc all belong to DMO . Let $u \in C_{\text{loc}}^2(\Omega)$ be a solution of the equation

$$Lu = \text{tr}(\mathbf{A}D^2u) + \mathbf{b} \cdot Du + cu = f \quad \text{in } \Omega, \quad (1.6)$$

where f and Df also belong to DMO . If $D^2u(x^0) = 0$ for some $x^0 \in \Omega$, then D^2u is differentiable at x^0 . More precisely, there exist positive constants r_0 and C , depending only

on d , λ , Λ , and the coefficients of L , such that for $r := \min\{r_0, \frac{1}{2} \text{dist}(x^0, \partial\Omega)\}$, the third derivative D^3u at x^0 satisfies the estimate:

$$|D^3u(x^0)| \leq C \left(\frac{1}{r} \int_{B_{2r}(x^0)} |D^2u| + \varrho_{\text{lot}}(r) + \varrho_{Df}(r) \right).$$

Here, the terms $\varrho_{\dots}(\cdot)$ represent moduli of continuity: $\varrho_{\text{coef}}(\cdot)$ is determined by the coefficients \mathbf{A} , \mathbf{b} , and c ; $\varrho_{\text{lot}}(\cdot)$ depends on $\|u\|_{L^\infty(B_r(x^0))}$, $\|Du\|_{L^\infty(B_r(x^0))}$, $D\mathbf{b}$, and Dc ; and $\varrho_{Df}(\cdot)$ is determined by Df . Furthermore, for $0 < |x - x^0| < r$, the following estimate holds:

$$\frac{|D^2u(x) - \langle D^3u(x^0), x - x^0 \rangle|}{|x - x^0|} \leq C\varrho_{\text{coef}}(|x - x^0|) \left\{ \frac{1}{r} \int_{B_{2r}(x^0)} |D^2u| + \varrho_{\text{lot}}(r) + \varrho_{Df}(r) \right\} + C\varrho_{\text{lot}}(|x - x^0|) + C\varrho_{Df}(|x - x^0|).$$

Here are a few brief remarks concerning Theorem 1.4: First, under the assumptions of the theorem, any strong solution u belongs to $C_{\text{loc}}^2(\Omega)$. In fact, it is not necessary to assume that $D\mathbf{b}$, Dc , and Df belong to DMO; see [3, Theorem 1.5] and [2, Theorem 1.6]. Second, the precise dependence of the moduli of continuity $\varrho_{\dots}(t)$ on the parameters specified in the theorem is discussed in detail in Section 5. Specifically:

- If $\mathbf{A} \in C^\alpha$ for some $\alpha \in (0, 1)$, then $\varrho_{\text{coef}}(t) \lesssim t^\alpha$.
- If $\mathbf{b} \in C^{1,\alpha}$ and $c \in C^{1,\alpha}$ for some $\alpha \in (0, 1)$, then $\varrho_{\text{lot}}(t) \lesssim t^\alpha$.
- If $f \in C^{1,\alpha}$ for some $\alpha \in (0, 1)$, then $\varrho_{Df}(t) \lesssim t^\alpha$.

Additionally, we note that $\varrho_{\text{lot}} \equiv 0$ if $D\mathbf{b}$ and c are constant, and $\varrho_{Df} \equiv 0$ if Df is constant.

The proof of Theorem 1.1 is presented in Sections 2 and 3. Section 2 considers a simplified setting in which the lower-order coefficients \mathbf{b} and c are absent, and the inhomogeneous term f vanishes. This preliminary case is intended to highlight the core ideas without the distraction of technical complications. The full proof of Theorem 1.1 in the general setting is then developed in Section 3. Similarly, the proof of Theorem 1.4 is given in Sections 4 and 5. Section 4 addresses the special case where \mathbf{b} , c , and f are all zero, while Section 5 treats the general case in full detail.

2. PROOF OF THEOREM 1.1: SIMPLE CASE

In this section, we consider an elliptic operator L in the form

$$Lu = D_i(a^{ij}D_ju) = \text{div}(\mathbf{A}Du).$$

We also assume that the inhomogeneous term f is identically zero. These assumptions allow us to present the main idea of the proof more transparently.

We aim to show that if $Du(x^0) = 0$ for some point $x^0 \in \Omega$, then Du is differentiable at x^0 . For simplicity, we assume, without loss of generality, that $x^0 = 0$ and $\text{dist}(x^0, \partial\Omega) \geq 1$. This ensures $B_1(0) \subset \Omega$.

Let $\bar{\mathbf{A}} := (\mathbf{A})_{B_r}$ denote the average of \mathbf{A} over the ball $B_r = B_r(0)$, where $r \in (0, \frac{1}{2}]$. We decompose u as $u = v + w$, where $w \in W_0^{1,p}(B_r)$ (for some $p > 1$) is the weak solution to the problem

$$\text{div}(\bar{\mathbf{A}}Dw) = -\text{div}((\mathbf{A} - \bar{\mathbf{A}})Du) \text{ in } B_r, \quad w = 0 \text{ on } \partial B_r.$$

Lemma 2.1. *Let $B = B_1(0)$. Let \mathbf{A}_0 be a constant matrix satisfying condition (1.2). For $f \in L^p(B)$ and $g \in L^p(B)$ with some $p > 1$, let $u \in W_0^{1,p}(B)$ be the weak solution to the problem:*

$$\begin{cases} \operatorname{div}(\mathbf{A}_0 Du) = \operatorname{div} f + g & \text{in } B, \\ u = 0 & \text{on } \partial B. \end{cases}$$

Then, for any $t > 0$, we have

$$|\{x \in B : |Du(x)| > t\}| \leq \frac{C}{t} \left(\int_B |f| + |g| \right),$$

where $C = C(d, \lambda, \Lambda)$.

Proof. Refer to the proof of [1, Lemma 3.3] and [2, Lemma 2.2]. \blacksquare

By Lemma 2.1, we have $Dw \in L^{\frac{1}{2}}(B_r)$ and

$$\left(\int_{B_r} |Dw|^{\frac{1}{2}} \right)^2 \leq C \omega_{\mathbf{A}}(r) \|Du\|_{L^\infty(B_r)}, \quad (2.2)$$

where $C = C(d, \lambda, \Lambda)$. Although $Dw \in L^p(B_r)$ for any $p \in (0, 1)$, we use $p = \frac{1}{2}$ for simplicity. For more details, refer to the proof of Theorem 1.5 in [2].

Note that $v = u - w$ satisfies the following equation:

$$\operatorname{div}(\bar{\mathbf{A}} Dv) = 0 \quad \text{in } B_r.$$

By interior regularity estimates for solutions of elliptic equations with constant coefficients, we know that $v \in C^\infty(B_r)$. In particular, we have

$$\|D^2 v\|_{L^\infty(B_{r/2})} \leq \frac{C}{r^2} \left(\int_{B_r} |v|^{\frac{1}{2}} \right)^2, \quad (2.3)$$

where C is a constant depending only on d , λ , and Λ . Moreover, this estimate remains valid if v is replaced by $Dv - l$, where l is any affine function of the form:

$$l(x) = x_1 c_1 + \cdots + x_d c_d + p, \quad \text{with } c_1, \dots, c_d, p \in \mathbb{R}^d.$$

We define the set \mathfrak{A} as

$$\mathfrak{A} = \{l(x) = \mathbf{S}x + p : \mathbf{S} \in \mathbb{S}^d, p \in \mathbb{R}^d\},$$

where \mathbb{S}^d denotes the set of all real symmetric $d \times d$ matrices.

Therefore, for any $l \in \mathfrak{A}$, we derive from (2.3) that

$$\|D^3 v\|_{L^\infty(B_{r/2})} \leq \frac{C}{r^2} \left(\int_{B_r} |Dv - l|^{\frac{1}{2}} \right)^2. \quad (2.4)$$

On the other hand, by Taylor's theorem, for any $\rho \in (0, r]$, we have

$$\sup_{x \in B_\rho} |Dv(x) - D^2 v(0)x - Dv(0)| \leq C(d) \|D^3 v\|_{L^\infty(B_\rho)} \rho^2.$$

Consequently, for any $\kappa \in (0, \frac{1}{2})$ and $l \in \mathfrak{A}$, we obtain

$$\left(\int_{B_{\kappa r}} |Dv - D^2 v(0)x - Dv(0)|^{\frac{1}{2}} \right)^2 \leq C \|D^3 v\|_{L^\infty(B_{r/2})} (\kappa r)^2 \leq C \kappa^2 \left(\int_{B_r} |Dv - l|^{\frac{1}{2}} \right)^2, \quad (2.5)$$

where $C = C(d, \lambda, \Lambda) > 0$.

Now, define the function

$$\varphi(r) := \frac{1}{r} \inf_{l \in \mathbb{N}} \left(\int_{B_r} |Du - l|^{\frac{1}{2}} \right)^2. \quad (2.6)$$

Since $u = v + w$, we apply the quasi-triangle inequality, together with estimates (2.2) and (2.5), to obtain

$$\begin{aligned} \kappa r \varphi(\kappa r) &\leq \left(\int_{B_{\kappa r}} |Du - D^2 v(0)x - Dv(0)|^{\frac{1}{2}} \right)^2 \\ &\leq C \left(\int_{B_{\kappa r}} |Dv - D^2 v(0)x - Dv(0)|^{\frac{1}{2}} \right)^2 + C \left(\int_{B_{\kappa r}} |Dw|^{\frac{1}{2}} \right)^2 \\ &\leq C \kappa^2 \left(\int_{B_r} |Dv - l|^{\frac{1}{2}} \right)^2 + C \kappa^{-2d} \left(\int_{B_r} |Dw|^{\frac{1}{2}} \right)^2 \\ &\leq C \kappa^2 \left(\int_{B_r} |Du - l|^{\frac{1}{2}} \right)^2 + C (\kappa^2 + \kappa^{-2d}) \left(\int_{B_r} |Dw|^{\frac{1}{2}} \right)^2 \\ &\leq C \kappa^2 \left(\int_{B_r} |Du - l|^{\frac{1}{2}} \right)^2 + C (\kappa^2 + \kappa^{-2d}) \omega_{\mathbf{A}}(r) \|Du\|_{L^\infty(B_r)}. \end{aligned} \quad (2.7)$$

Let $\beta \in (0, 1)$ be an arbitrary but fixed number. With this β , choose $\kappa = \kappa(d, \lambda, \Lambda, \beta) \in (0, \frac{1}{2})$ such that $C\kappa \leq \kappa^\beta$. Then, inequality (2.7) yields

$$\varphi(\kappa r) \leq \kappa^\beta \varphi(r) + C \omega_{\mathbf{A}}(r) \frac{1}{r} \|Du\|_{L^\infty(B_r)}, \quad (2.8)$$

where $C = C(d, \lambda, \Lambda, \kappa) = C(d, \lambda, \Lambda, \beta)$.

Let $r_0 \in (0, \frac{1}{2}]$ be a number to be chosen later. By iterating (2.8), we obtain, for each $j = 1, 2, \dots$,

$$\varphi(\kappa^j r_0) \leq \kappa^{\beta j} \varphi(r_0) + C \sum_{i=1}^j \kappa^{(i-1)\beta} \omega_{\mathbf{A}}(\kappa^{j-i} r_0) \frac{1}{\kappa^{j-i} r_0} \|Du\|_{L^\infty(B_{\kappa^{j-i} r_0})}.$$

Define

$$M_j(r_0) := \max_{0 \leq i < j} \frac{1}{\kappa^i r_0} \|Du\|_{L^\infty(B_{\kappa^i r_0})} \quad \text{for } j = 1, 2, \dots, \quad (2.9)$$

so that we can estimate

$$\varphi(\kappa^j r_0) \leq \kappa^{\beta j} \varphi(r_0) + C M_j(r_0) \tilde{\omega}_{\mathbf{A}}(\kappa^j r_0), \quad (2.10)$$

where, following [2, (2.15)], we define

$$\tilde{\omega}_{\mathbf{A}}(t) := \sum_{i=1}^{\infty} \kappa^{i\beta} \left\{ \omega_{\mathbf{A}}(\kappa^{-i} t) [\kappa^{-i} t \leq 1] + \omega_{\mathbf{A}}(1) [\kappa^{-i} t > 1] \right\}. \quad (2.11)$$

Here, we adopt Iverson bracket notation: for any statement P , we set $[P] = 1$ if P is true, and $[P] = 0$ otherwise. We emphasize that $\tilde{\omega}_{\mathbf{A}}(t)$ satisfies the Dini condition whenever $\omega_{\mathbf{A}}(t)$ does. In particular, if $\mathbf{A} \in C^\alpha$ for some $\alpha \in (0, 1)$, then choosing $\beta \in (\alpha, 1)$ yields $\tilde{\omega}_{\mathbf{A}}(t) \lesssim t^\alpha$ for all $t \in (0, 1]$. Moreover, we note that $\omega_{\mathbf{A}}(t) \lesssim \tilde{\omega}_{\mathbf{A}}(t)$ for $t \in (0, 1]$. Additionally, we will use the following estimate (see [2, Lemma 2.7]):

$$\sum_{j=0}^{\infty} \tilde{\omega}_{\mathbf{A}}(\kappa^j r) \lesssim \int_0^r \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t} dt, \quad (2.12)$$

which, in particular, implies

$$\tilde{\omega}_A(r) \lesssim \int_0^r \frac{\tilde{\omega}_A(t)}{t} dt. \quad (2.13)$$

For any fixed r , the infimum in (2.6) is achieved by some $l \in \mathfrak{A}$. For $j = 0, 1, 2, \dots$, let $\mathbf{S}_j \in \mathbb{S}^d$ and $\mathbf{p}_j \in \mathbb{R}^d$ be such that:

$$\varphi(\kappa^j r_0) = \frac{1}{\kappa^j r_0} \left(\int_{B_{\kappa^j r_0}} |Du(x) - \mathbf{S}_j x - \mathbf{p}_j|^{\frac{1}{2}} dx \right)^2. \quad (2.14)$$

From (2.6) and Hölder's inequality, we have

$$\varphi(r_0) \leq \frac{1}{r_0} \int_{B_{r_0}} |Du|. \quad (2.15)$$

Combining (2.10) with (2.15), we obtain the estimate

$$\varphi(\kappa^j r_0) \leq \frac{\kappa^{\beta j}}{r_0} \int_{B_{r_0}} |Du| + CM_j(r_0) \tilde{\omega}_A(\kappa^j r_0). \quad (2.16)$$

Next, observe that for each $j = 0, 1, 2, \dots$, we have

$$\int_{B_{\kappa^j r_0}} |\mathbf{S}_j x + \mathbf{p}_j|^{\frac{1}{2}} dx \leq \int_{B_{\kappa^j r_0}} |Du - \mathbf{S}_j x - \mathbf{p}_j|^{\frac{1}{2}} dx + \int_{B_{\kappa^j r_0}} |Du|^{\frac{1}{2}} \leq 2 \int_{B_{\kappa^j r_0}} |Du|^{\frac{1}{2}}. \quad (2.17)$$

Furthermore, we observe that

$$|\mathbf{p}_j|^{\frac{1}{2}} = |\mathbf{S}_j x + \mathbf{p}_j - 2(\mathbf{S}_j(x/2) + \mathbf{p}_j)|^{\frac{1}{2}} \leq |\mathbf{S}_j x + \mathbf{p}_j|^{\frac{1}{2}} + 2^{\frac{1}{2}} |\mathbf{S}_j(x/2) + \mathbf{p}_j|^{\frac{1}{2}}.$$

In addition, we have

$$\int_{B_{\kappa^j r_0}} |\mathbf{S}_j(x/2) + \mathbf{p}_j|^{\frac{1}{2}} dx = \frac{2^d}{|B_{\kappa^j r_0}|} \int_{B_{\kappa^j r_0/2}} |\mathbf{S}_j x + \mathbf{p}_j|^{\frac{1}{2}} dx \leq 2^d \int_{B_{\kappa^j r_0}} |\mathbf{S}_j x + \mathbf{p}_j|^{\frac{1}{2}} dx.$$

Combining these estimates, we deduce

$$|\mathbf{p}_j| \leq C \left(\int_{B_{\kappa^j r_0}} |\mathbf{S}_j x + \mathbf{p}_j|^{\frac{1}{2}} \right)^2 \leq C \left(\int_{B_{\kappa^j r_0}} |Du|^{\frac{1}{2}} \right)^2, \quad j = 0, 1, 2, \dots \quad (2.18)$$

Since $u \in C^1(\bar{B}_{1/2})$ by [2, Theorem 1.5] and $Du(0) = 0$, the estimate (2.18) immediately yields

$$\lim_{j \rightarrow \infty} \mathbf{p}_j = 0. \quad (2.19)$$

Estimate of \mathbf{S}_j . By the quasi-triangle inequality, we have

$$|(\mathbf{S}_j - \mathbf{S}_{j-1})x + (\mathbf{p}_j - \mathbf{p}_{j-1})|^{\frac{1}{2}} \leq |Du - \mathbf{S}_j x - \mathbf{p}_j|^{\frac{1}{2}} + |Du - \mathbf{S}_{j-1} x - \mathbf{p}_{j-1}|^{\frac{1}{2}}.$$

Taking the average over $B_{\kappa^j r_0}$ and using the fact that $|B_{\kappa^{j-1} r_0}|/|B_{\kappa^j r_0}| = \kappa^{-d}$, we obtain

$$\frac{1}{\kappa^j r_0} \left(\int_{B_{\kappa^j r_0}} |(\mathbf{S}_j - \mathbf{S}_{j-1})x + (\mathbf{p}_j - \mathbf{p}_{j-1})|^{\frac{1}{2}} dx \right)^2 \leq C\varphi(\kappa^j r_0) + C\varphi(\kappa^{j-1} r_0) \quad (2.20)$$

for $j = 1, 2, \dots$, where $C = C(d, \lambda, \Lambda, \beta)$.

Next, observe that

$$|\mathbf{p}_j - \mathbf{p}_{j-1}|^{\frac{1}{2}} = |(\mathbf{S}_j - \mathbf{S}_{j-1})x + (\mathbf{p}_j - \mathbf{p}_{j-1}) - 2((\mathbf{S}_j - \mathbf{S}_{j-1})(x/2) + (\mathbf{p}_j - \mathbf{p}_{j-1}))|^{\frac{1}{2}}.$$

Using the quasi-triangle inequality, we obtain

$$|\mathbf{p}_j - \mathbf{p}_{j-1}|^{\frac{1}{2}} \leq |(\mathbf{S}_j - \mathbf{S}_{j-1})x + (\mathbf{p}_j - \mathbf{p}_{j-1})|^{\frac{1}{2}} + 2^{\frac{1}{2}} |(\mathbf{S}_j - \mathbf{S}_{j-1})(x/2) + (\mathbf{p}_j - \mathbf{p}_{j-1})|^{\frac{1}{2}}.$$

Moreover,

$$\int_{B_{\kappa^j r_0}} |(\mathbf{S}_j - \mathbf{S}_{j-1})(x/2) + (\mathbf{p}_j - \mathbf{p}_{j-1})|^{\frac{1}{2}} dx \leq 2^d \int_{B_{\kappa^j r_0}} |(\mathbf{S}_j - \mathbf{S}_{j-1})x + (\mathbf{p}_j - \mathbf{p}_{j-1})|^{\frac{1}{2}} dx.$$

Combining these estimates, we conclude that

$$|\mathbf{p}_j - \mathbf{p}_{j-1}| \leq C(d) \left(\int_{B_{\kappa^j r_0}} |(\mathbf{S}_j - \mathbf{S}_{j-1})x + (\mathbf{p}_j - \mathbf{p}_{j-1})|^{\frac{1}{2}} dx \right)^2, \quad j = 1, 2, \dots$$

Substituting this into (2.20), we derive

$$\frac{1}{\kappa^j r_0} |\mathbf{p}_j - \mathbf{p}_{j-1}| \leq C\varphi(\kappa^j r_0) + C\varphi(\kappa^{j-1} r_0), \quad j = 1, 2, \dots \quad (2.21)$$

On the other hand, for any matrix $\mathbf{S} \in \mathbb{S}^d$, we may write $\mathbf{S} = |\mathbf{S}|\mathbf{U}$, where $\mathbf{U} \in \mathbb{S}^d$ satisfies $|\mathbf{U}| = 1$. Then we have

$$\int_{B_r} |\mathbf{S}x|^{\frac{1}{2}} dx \geq |\mathbf{S}|^{\frac{1}{2}} \inf_{|\mathbf{U}|=1} \int_{B_r} |\mathbf{U}x|^{\frac{1}{2}} dx = |\mathbf{S}|^{\frac{1}{2}} \inf_{|\mathbf{U}|=1} \int_{B_1} r^{\frac{1}{2}} |\mathbf{U}x|^{\frac{1}{2}} dx = C(d)r^{\frac{1}{2}} |\mathbf{S}|^{\frac{1}{2}}. \quad (2.22)$$

Then, by using (2.22), the quasi-triangle inequality, (2.20), (2.21), (2.16), and the observation that $M_{j-1}(r_0) \leq M_j(r_0)$, we obtain

$$\begin{aligned} |\mathbf{S}_j - \mathbf{S}_{j-1}| &\leq \frac{C}{\kappa^j r_0} \left(\int_{B_{\kappa^j r_0}} |(\mathbf{S}_j - \mathbf{S}_{j-1})x|^{\frac{1}{2}} dx \right)^2 \\ &\leq \frac{C}{\kappa^j r_0} \left(\int_{B_{\kappa^j r_0}} |(\mathbf{S}_j - \mathbf{S}_{j-1})x + (\mathbf{p}_j - \mathbf{p}_{j-1})|^{\frac{1}{2}} dx \right)^2 + \frac{C}{\kappa^j r_0} |\mathbf{p}_j - \mathbf{p}_{j-1}| \\ &\leq C\varphi(\kappa^j r_0) + C\varphi(\kappa^{j-1} r_0) \\ &\leq \frac{C\kappa^{\beta j}}{r_0} \int_{B_{r_0}} |Du| + CM_j(r_0) \{ \tilde{\omega}_{\mathbf{A}}(\kappa^j r_0) + \tilde{\omega}_{\mathbf{A}}(\kappa^{j-1} r_0) \}. \end{aligned} \quad (2.23)$$

To estimate $|\mathbf{S}_0|$, we proceed similarly to (2.23) by applying (2.17) and (2.18) with $j = 0$, and using Hölder's inequality to obtain

$$|\mathbf{S}_0| \leq \frac{C}{r_0} \int_{B_{r_0}} |Du|, \quad (2.24)$$

where $C = C(d, \lambda, \Lambda, \beta)$.

For $k > l \geq 0$, we derive from (2.23) and the definition of $M_j(r_0)$ that

$$\begin{aligned} |\mathbf{S}_k - \mathbf{S}_l| &\leq \sum_{j=l}^{k-1} |\mathbf{S}_{j+1} - \mathbf{S}_j| \leq \sum_{j=l}^{k-1} \frac{C\kappa^{\beta(j+1)}}{r_0} \int_{B_{r_0}} |Du| + CM_k(r_0) \sum_{j=l}^k \tilde{\omega}_{\mathbf{A}}(\kappa^j r_0) \\ &\leq \frac{C\kappa^{\beta(l+1)}}{(1-\kappa^\beta)r_0} \int_{B_{r_0}} |Du| + CM_k(r_0) \int_0^{\kappa^l r_0} \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t} dt, \end{aligned} \quad (2.25)$$

where we used (2.12). In particular, by taking $k = j$ and $l = 0$ in (2.25), and using (2.24), we obtain for $j = 1, 2, \dots$ that

$$|\mathbf{S}_j| \leq |\mathbf{S}_j - \mathbf{S}_0| + |\mathbf{S}_0| \leq \frac{C}{r_0} \int_{B_{r_0}} |Du| + CM_j(r_0) \int_0^{r_0} \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t} dt. \quad (2.26)$$

Similarly, we obtain from (2.21) that for $k > l \geq 0$, we have

$$|\mathbf{p}_k - \mathbf{p}_l| \leq C \frac{\kappa^{(\beta+1)(l+1)}}{1-\kappa^{\beta+1}} \int_{B_{r_0}} |Du| + C\kappa^l r_0 M_k(r_0) \int_0^{\kappa^l r_0} \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t} dt. \quad (2.27)$$

Estimate for \mathbf{p}_j . We shall derive improved estimates for $|\mathbf{p}_j|$ using the following lemma, where we set

$$v(x) := u(x) - \frac{1}{2} \mathbf{S}_j x \cdot x - \mathbf{p}_j \cdot x$$

so that

$$Dv(x) = Du(x) - \mathbf{S}_j x - \mathbf{p}_j. \quad (2.28)$$

Lemma 2.29. For $0 < r \leq \frac{1}{2}$, we have

$$\sup_{B_r} |Dv| \leq C \left\{ \left(\int_{B_{2r}} |Dv|^{\frac{1}{2}} \right)^2 + (r|\mathbf{S}_j| + |\mathbf{p}_j|) \int_0^r \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t} dt \right\},$$

where $C = C(d, \lambda, \Lambda, \omega_{\mathbf{A}}, \beta)$.

Proof. Since $\operatorname{div}(\mathbf{A}Du) = 0$ in B_1 , it follows that

$$\operatorname{div}(\mathbf{A}Dv) = -\operatorname{div}(\mathbf{A}(\mathbf{S}_j x + \mathbf{p}_j)) \quad \text{in } B_{2r},$$

for any $0 < r \leq \frac{1}{2}$. Let $x_0 \in B_{3r/2}$ and $0 < t \leq r/4$, and define $\bar{\mathbf{A}} := (\mathbf{A})_{B_t(x_0)}$.

We decompose v as $v = v_1 + v_2$, where $v_1 \in W_0^{1,p}(B_t(x_0))$ (for some $p > 1$) is the weak solution to the problem

$$\operatorname{div}(\bar{\mathbf{A}}Dv_1) = -\operatorname{div}((\mathbf{A} - \bar{\mathbf{A}})(\mathbf{S}_j x + \mathbf{p}_j) + (\mathbf{A} - \bar{\mathbf{A}})Dv) \quad \text{in } B_t(x_0),$$

with the boundary condition $v_1 = 0$ on $\partial B_t(x_0)$

By Lemma 2.1, we obtain (cf. (2.2))

$$\left(\int_{B_t(x_0)} |Dv_1|^{\frac{1}{2}} \right)^2 \leq C \left(\int_{B_t(x_0)} |\mathbf{A} - \bar{\mathbf{A}}| \right) (r|\mathbf{S}_j| + |\mathbf{p}_j|) + C \left(\int_{B_t(x_0)} |\mathbf{A} - \bar{\mathbf{A}}| \right) \|Dv\|_{L^\infty(B_t(x_0))},$$

and thus, we have

$$\left(\int_{B_t(x_0)} |Dv_1|^{\frac{1}{2}} \right)^2 \leq C\omega_{\mathbf{A}}(t) (r|\mathbf{S}_j| + |\mathbf{p}_j|) + C\omega_{\mathbf{A}}(t) \|Dv\|_{L^\infty(B_t(x_0))}.$$

On the other hand, observe that $v_2 = v - v_1$ satisfies

$$L_0 v_2 := \operatorname{div}(\bar{\mathbf{A}}Dv_2) = -\operatorname{div}(\bar{\mathbf{A}}(\mathbf{S}_j x + \mathbf{p}_j)) = \text{constant} \quad \text{in } B_t(x_0).$$

Since L_0 is a constant-coefficients operator, it follows that $L_0(Dv_2) = 0$. The remainder of the proof then proceeds identically to that of [2, Theorem 1.5]. \blacksquare

By Lemma 2.29, and using (2.28), (2.14), and (2.16), we have for $j = 1, 2, \dots$:

$$\begin{aligned} \|Du - \mathbf{S}_j x - \mathbf{p}_j\|_{L^\infty(B_{\frac{1}{2}\kappa^j r_0})} &\leq C\kappa^j r_0 \varphi(\kappa^j r_0) + C(\kappa^j r_0 |\mathbf{S}_j| + |\mathbf{p}_j|) \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t} dt \\ &\leq C\kappa^{(1+\beta)j} \oint_{B_{r_0}} |Du| + C\kappa^j r_0 M_j(r_0) \tilde{\omega}_{\mathbf{A}}(\kappa^j r_0) \\ &\quad + C(\kappa^j r_0 |\mathbf{S}_j| + |\mathbf{p}_j|) \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t} dt, \end{aligned} \quad (2.30)$$

where $C = C(d, \lambda, \Lambda, \omega_{\mathbf{A}}, \beta)$. Since $Du(0) = 0$, we infer from (2.30) that

$$\begin{aligned} |\mathbf{p}_j| &\leq C\kappa^{(1+\beta)j} \oint_{B_{r_0}} |Du| + C\kappa^j r_0 M_j(r_0) \tilde{\omega}_{\mathbf{A}}(\kappa^j r_0) \\ &\quad + C|\mathbf{S}_j| \kappa^j r_0 \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t} dt + C|\mathbf{p}_j| \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t} dt. \end{aligned}$$

Let us fix $r_1 > 0$ such that

$$C \int_0^{r_1} \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t} dt \leq \frac{1}{2}. \quad (2.31)$$

Note that r_1 depends only on $d, \lambda, \Lambda, \omega_{\mathbf{A}}$, and β . We have not yet chosen $r_0 \in (0, \frac{1}{2}]$. We will require $r_0 \leq r_1$, which implies

$$|\mathbf{p}_j| \leq C\kappa^{(1+\beta)j} \oint_{B_{r_0}} |Du| + C\kappa^j r_0 M_j(r_0) \tilde{\omega}_{\mathbf{A}}(\kappa^j r_0) + C|\mathbf{S}_j| \kappa^j r_0 \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t} dt.$$

This estimate, combined with (2.26) and (2.13), yields

$$|\mathbf{p}_j| \leq C\kappa^j r_0 \left\{ \kappa^{\beta j} + \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t} dt \right\} \frac{1}{r_0} \oint_{B_{r_0}} |Du| + C\kappa^j r_0 M_j(r_0) \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t} dt. \quad (2.32)$$

Convergence of \mathbf{S}_j . By (2.30), (2.26), (2.32), and (2.13), we have

$$\begin{aligned} \|Du - \mathbf{S}_j x - \mathbf{p}_j\|_{L^\infty(B_{\frac{1}{2}\kappa^j r_0})} &\leq C\kappa^j r_0 \left\{ \kappa^{\beta j} + \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t} dt \right\} \frac{1}{r_0} \oint_{B_{r_0}} |Du| \\ &\quad + C\kappa^j r_0 M_j(r_0) \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t} dt. \end{aligned} \quad (2.33)$$

Then, from (2.33), (2.26), (2.32), and (2.13), we infer that

$$\frac{1}{\kappa^j r_0} \|Du\|_{L^\infty(B_{\frac{1}{2}\kappa^j r_0})} \leq \frac{C}{r_0} \oint_{B_{r_0}} |Du| + CM_j(r_0) \int_0^{r_0} \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t} dt, \quad (2.34)$$

where $C = C(d, \lambda, \Lambda, \omega_{\mathbf{A}}, \beta)$.

Lemma 2.35. *There exists a constant $r_0 = r_0(d, \lambda, \Lambda, \omega_{\mathbf{A}}, \beta) \in (0, \frac{1}{2})$ such that*

$$\sup_{j \geq 1} M_j(r_0) = \sup_{i \geq 0} \frac{1}{\kappa^i r_0} \|Du\|_{L^\infty(B_{\kappa^i r_0})} \leq \frac{C}{r_0} \oint_{B_{2r_0}} |Du|, \quad (2.36)$$

where $C = C(d, \lambda, \Lambda, \omega_{\mathbf{A}}, \beta)$.

Proof. We define

$$c_i = \frac{1}{\kappa^i r_0} \|Du\|_{L^\infty(B_{\kappa^i r_0})}, \quad i = 0, 1, 2, \dots$$

Then, by (2.9), it is clear that

$$M_1(r_0) = c_0 \quad \text{and} \quad M_{j+1}(r_0) = \max(M_j(r_0), c_j), \quad j = 1, 2, \dots \quad (2.37)$$

Recall that $\kappa = \kappa(d, \lambda, \Lambda, \beta) \in (0, \frac{1}{2})$ has already been chosen. Then, from (2.34), we deduce that there exists a constant $C = C(d, \lambda, \Lambda, \omega_A, \beta) > 0$ such that

$$c_{j+1} \leq C \left\{ \frac{1}{r_0} \int_{B_{r_0}} |Du| + M_j(r_0) \int_0^{r_0} \frac{\tilde{\omega}_A(t)}{t} dt \right\}, \quad j = 1, 2, \dots \quad (2.38)$$

By [2, Theorem 1.5], we obtain

$$c_1 = \frac{1}{\kappa r_0} \|Du\|_{L^\infty(B_{\kappa r_0})} \leq \frac{1}{\kappa r_0} \|Du\|_{L^\infty(B_{r_0})} = \frac{1}{\kappa} c_0 \leq \frac{C}{r_0} \int_{B_{2r_0}} |Du|, \quad (2.39)$$

where $C = C(d, \lambda, \Lambda, \omega_A, \beta)$.

By combining (2.38) and (2.39), we establish the existence of a constant $\gamma = \gamma(d, \lambda, \Lambda, \omega_A, \beta) > 0$ such that the following inequalities hold:

$$\begin{aligned} c_0, c_1 &\leq \frac{\gamma}{r_0} \int_{B_{2r_0}} |Du| \quad \text{and} \\ c_{j+1} &\leq \gamma \left\{ \frac{1}{r_0} \int_{B_{2r_0}} |Du| + M_j(r_0) \int_0^{r_0} \frac{\tilde{\omega}_A(t)}{t} dt \right\}, \quad j = 1, 2, \dots \end{aligned}$$

Now, we fix a number $r_0 \in (0, \frac{1}{2}]$ such that

$$\gamma \int_0^{r_0} \frac{\tilde{\omega}_A(t)}{t} dt \leq \frac{1}{2}$$

and also ensure that $r_0 \leq r_1$, is as defined in (2.31). With this choice, we obtain

$$c_0, c_1 \leq \frac{\gamma}{r_0} \int_{B_{2r_0}} |Du| \quad \text{and} \quad c_{j+1} \leq \frac{\gamma}{r_0} \int_{B_{2r_0}} |Du| + \frac{1}{2} M_j(r_0), \quad j = 1, 2, \dots \quad (2.40)$$

By induction, it follows from (2.37) and (2.40) that

$$c_{2k}, c_{2k+1}, M_{2k+1}(r_0), M_{2k+2}(r_0) \leq \frac{\gamma}{r_0} \int_{B_{2r_0}} |Du| \cdot \sum_{i=0}^k \frac{1}{2^i}, \quad k = 0, 1, 2, \dots$$

This completes the proof of the lemma. ■

Now, Lemma 2.35 and (2.25) imply that the sequence $\{\mathbf{S}_j\}$ is a Cauchy sequence in \mathbb{S}^d , and thus $\mathbf{S}_j \rightarrow \mathbf{S}$ for some $\mathbf{S} \in \mathbb{S}^d$. Moreover, by taking the limit as $k \rightarrow \infty$ in (2.25) and (2.27) (while recalling (2.36) and (2.19)), respectively, and then setting $l = j$, we obtain the following estimates:

$$\begin{aligned} |\mathbf{S}_j - \mathbf{S}| &\leq C \left\{ \kappa^{\beta j} + \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_A(t)}{t} dt \right\} \frac{1}{r_0} \int_{B_{2r_0}} |Du|, \\ |p_j| &\leq C \kappa^j \left\{ \kappa^{\beta j} + \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_A(t)}{t} dt \right\} \int_{B_{2r_0}} |Du|. \end{aligned} \quad (2.41)$$

By the triangle inequality, (2.33), (2.41), and (2.36), we obtain

$$\begin{aligned} \|Du - \mathbf{S}x\|_{L^\infty(B_{\frac{1}{2}\kappa^j r_0})} &\leq \|Du - \mathbf{S}_j x - \mathbf{p}_j\|_{L^\infty(B_{\frac{1}{2}\kappa^j r_0})} + \frac{\kappa^j r_0}{2} |\mathbf{S}_j - \mathbf{S}| + |\mathbf{p}_j| \\ &\leq C\kappa^j r_0 \left\{ \kappa^{\beta j} + \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t} dt \right\} \frac{1}{r_0} \int_{B_{2r_0}} |Du|. \end{aligned} \quad (2.42)$$

Conclusion. It follows from (2.42) that

$$\frac{1}{r} \|Du - \mathbf{S}x\|_{L^\infty(B_r)} \leq \varrho_{\mathbf{A}}(r) \left(\frac{1}{r_0} \int_{B_{2r_0}} |Du| \right), \quad (2.43)$$

where

$$\varrho_{\mathbf{A}}(r) = C \left\{ \left(\frac{2r}{\kappa r_0} \right)^\beta + \int_0^{2r/\kappa} \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t} dt \right\}. \quad (2.44)$$

Note that $\varrho_{\mathbf{A}}$ is a modulus of continuity determined by $d, \lambda, \Lambda, \omega_{\mathbf{A}}$, and $\beta \in (0, 1)$.

In particular, we conclude from (2.43) that Du is differentiable at 0. Moreover, it follows from (2.26) and (2.36) that (noting that $D^2 u(0) = \mathbf{S} = \lim_{j \rightarrow \infty} \mathbf{S}_j$) we have

$$|D^2 u(0)| \leq \frac{C}{r_0} \int_{B_{2r_0}} |Du|,$$

where $C = C(d, \lambda, \Lambda, \omega_{\mathbf{A}}, \beta)$.

From (2.44) and (2.11), we observe that when $\mathbf{A} \in C^\alpha$ for some $\alpha \in (0, 1)$, we have $\varrho_{\mathbf{A}}(r) \lesssim r^\alpha$ by choosing $\beta \in (\alpha, 1)$. This completes the proof in the special case. ■

3. PROOF OF THEOREM 1.1: GENERAL CASE

We now proceed with the proof of Theorem 1.1 in the general setting.

Define $\omega_{\text{coef}}(\cdot)$ by

$$\omega_{\text{coef}}(r) := \omega_{\mathbf{A}}(r) + r^{1-d} \sup_{x \in \Omega} \int_{\Omega \cap B_r(x)} |\mathbf{b}| + r^{1-d} \sup_{x \in \Omega} \int_{\Omega \cap B_r(x)} |c|. \quad (3.1)$$

We observe that $\omega_{\text{coef}}(\cdot)$ satisfies the Dini condition:

$$\int_0^1 \frac{\omega_{\text{coef}}(t)}{t} dt < \infty.$$

Although one might expect a factor of r^{2-d} in the last term of the definition (3.1), we retain the current form for consistency with the notation $\omega_{\text{coef}}(\cdot, x_0)$ introduced in the proof of Lemma 3.9.

Let u be a solution to (1.3), i.e., u satisfies

$$\text{div}(\mathbf{A}Du) + \mathbf{b} \cdot Du + cu = f \quad \text{in } \Omega.$$

From [3, Theorem 1.3] (and also [3, Proposition 2.6]), it follows that $u \in C_{\text{loc}}^1(\Omega)$. Assume that $Du(x^0) = 0$ for some $x^0 \in \Omega$. Without loss of generality, we can set $x^0 = 0$ and assume $B_2(0) \subset \Omega$.

For $0 < r \leq 1/2$, we define $\bar{\mathbf{A}}, \bar{c}, \bar{u}$, and \bar{f} as the averages of \mathbf{A}, c, u , and f over the ball B_r , respectively.

We decompose u as $u = v + w$, where $w \in W_0^{1,p}(B_r)$ (for some $p > 1$) is the solution to the problem:

$$\text{div}(\bar{\mathbf{A}}Dw) = -\text{div}((\mathbf{A} - \bar{\mathbf{A}})Du) - \mathbf{b} \cdot Du - (cu - \bar{c}\bar{u}) + f - \bar{f} \quad \text{in } B_r,$$

with $w = 0$ on ∂B_r .

By Lemma 2.1, we obtain the following estimate via rescaling (cf. (2.2)):

$$\begin{aligned} \left(\int_{B_r} |Dw|^{\frac{1}{2}} \right)^2 &\leq C\omega_{\mathbf{A}}(r)\|Du\|_{L^\infty(B_r)} + Cr \left(\int_{B_r} |\mathbf{b}| \right) \|Du\|_{L^\infty(B_r)} \\ &\quad + Cr \left\{ \omega_c(r)\|u\|_{L^\infty(B_r)} + r \left(\int_{B_r} |c| \right) \|Du\|_{L^\infty(B_r)} \right\} + Cr\omega_f(r), \end{aligned} \quad (3.2)$$

where we utilized the identity

$$cu - \bar{c}\bar{u} = (c - \bar{c})u + \bar{c}(u - \bar{u})$$

and applied the Poincaré inequality to derive:

$$\begin{aligned} \int_{B_r} |cu - \bar{c}\bar{u}| &\leq \|u\|_{L^\infty(B_r)} \int_{B_r} |c - \bar{c}| + C|\bar{c}|r \int_{B_r} |Du| \\ &\leq \omega_c(r)\|u\|_{L^\infty(B_r)} + Cr \left(\int_{B_r} |c| \right) \|Du\|_{L^\infty(B_r)}. \end{aligned} \quad (3.3)$$

Since $v = u - w$ satisfies

$$\operatorname{div}(\bar{\mathbf{A}}Dv) = \bar{f} - \bar{c}\bar{u} = \text{constant} \quad \text{in } B_r,$$

the function $\tilde{v} := D_k v - l$, where $k = 1, \dots, d$ and $l(x)$ is any affine function, satisfies

$$\operatorname{div}(\bar{\mathbf{A}}D\tilde{v}) = 0 \quad \text{in } B_r.$$

Therefore, the same reasoning that led to estimate (2.4) also applies here, yielding the identical estimate.

Let $\varphi(r)$ be as defined in (2.6), and let $\beta \in (0, 1)$ be an arbitrary but fixed constant.

By using (3.2) in place of (2.2) and carrying out computations analogous to those in (2.7), we arrive at the following estimate, which is analogous to (2.8):

$$\varphi(\kappa r) \leq \kappa^\beta \varphi(r) + C\omega_{\text{coef}}(r) \frac{1}{r} \|Du\|_{L^\infty(B_r)} + C\omega_f(r) + C\omega_c(r)\|u\|_{L^\infty(B_r)} \quad (3.4)$$

where $C = C(d, \lambda, \Lambda, \beta)$. Recall that for a given β , the constant $\kappa = \kappa(d, \lambda, \Lambda, \beta)$ is chosen to satisfy $\kappa \in (0, 1/2)$.

Let $M_j(r_0)$ be defined as in (2.9), where $r_0 \in (0, \frac{1}{2}]$ is a parameter to be chosen later. Define $\tilde{\omega}_{\dots}(\cdot)$ as in (2.11), replacing $\omega_{\mathbf{A}}(\cdot)$ with $\omega_{\dots}(\cdot)$.

Then, by replicating the argument from Section 2, we obtain the following estimate, analogous to (2.16):

$$\varphi(\kappa^j r_0) \leq \frac{\kappa^{\beta j}}{r_0} \int_{B_{r_0}} |Du| + CM_j(r_0)\tilde{\omega}_{\text{coef}}(\kappa^j r_0) + C\tilde{\omega}_f(\kappa^j r_0) + C\tilde{\omega}_c(\kappa^j r_0)\|u\|_{L^\infty(B_{r_0})}. \quad (3.5)$$

For $j = 0, 1, 2, \dots$, let $\mathbf{S}_j \in \mathbb{S}^d$ and $\mathbf{p}_j \in \mathbb{R}^d$ be chosen as in (2.14). Then, by using (3.5) in place of (2.16) and replicating the argument from Section 2, we arrive at the following conclusion:

$$\lim_{j \rightarrow \infty} \mathbf{p}_j = 0. \quad (3.6)$$

Additionally, by repeating the same computations that lead to (2.25) and (2.27), we obtain the following estimate for $k > l \geq 0$:

$$|\mathbf{S}_k - \mathbf{S}_l| + \frac{|\mathbf{p}_k - \mathbf{p}_l|}{\kappa^l r_0} \leq \frac{C\kappa^{\beta l}}{r_0} \oint_{B_{r_0}} |Du| + CM_k(r_0) \int_0^{\kappa^l r_0} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt \\ + C \int_0^{\kappa^l r_0} \frac{\tilde{\omega}_f(t)}{t} dt + C\|u\|_{L^\infty(B_{r_0})} \int_0^{\kappa^l r_0} \frac{\tilde{\omega}_c(t)}{t} dt, \quad (3.7)$$

where $C = C(d, \lambda, \Lambda, \beta)$. In particular, similar to (2.26), we obtain the following bound for \mathbf{S}_j :

$$|\mathbf{S}_j| \leq \frac{C}{r_0} \oint_{B_{r_0}} |Du| + CM_j(r_0) \int_0^{r_0} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt \\ + C \int_0^{r_0} \frac{\tilde{\omega}_f(t)}{t} dt + C\|u\|_{L^\infty(B_{r_0})} \int_0^{r_0} \frac{\tilde{\omega}_c(t)}{t} dt, \quad (3.8)$$

where $C = C(d, \lambda, \Lambda, \beta)$.

The following lemma corresponds to Lemma 2.29.

Lemma 3.9. *Let v be defined by*

$$v(x) := u(x) - \frac{1}{2} \mathbf{S}_j x \cdot x - \mathbf{p}_j \cdot x. \quad (3.10)$$

Then, for $0 < r \leq \frac{1}{2}$, we have

$$\sup_{B_r} |Dv| \leq C \left\{ \left(\oint_{B_{2r}} |Dv|^{\frac{1}{2}} \right)^2 + r \int_0^r \frac{\tilde{\omega}_f(t)}{t} dt + r\|u\|_{L^\infty(B_r)} \int_0^r \frac{\tilde{\omega}_c(t)}{t} dt \right. \\ \left. + (r|\mathbf{S}_j| + |\mathbf{p}_j|) \int_0^r \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt \right\},$$

where $C = C(d, \lambda, \Lambda, \omega_{\text{coef}}, \beta)$.

Proof. Note that Dv is given by

$$Dv(x) = Du(x) - \mathbf{S}_j x - \mathbf{p}_j. \quad (3.11)$$

For $x_0 \in B_{3r/2}$ and $0 < t \leq r/4$, let $\bar{\mathbf{A}}$, \bar{c} , \bar{v} , and \bar{f} denote the averages of \mathbf{A} , c , v , and f over the ball $B_t(x_0)$, respectively. Observe that v satisfies

$$\text{div}(\bar{\mathbf{A}} Dv) = f - \text{div}((\mathbf{A} - \bar{\mathbf{A}}) Dv) - \mathbf{b} \cdot Dv - cv \\ - \text{div}(\mathbf{A}(\mathbf{S}_j x + \mathbf{p}_j)) - \mathbf{b} \cdot (\mathbf{S}_j x + \mathbf{p}_j) - c(\frac{1}{2} \mathbf{S}_j x \cdot x + \mathbf{p}_j \cdot x) \quad \text{in } B_t(x_0).$$

We decompose v as $v = v_1 + v_2$, where $v_1 \in W_0^{1,p}(B_t(x_0))$ for some $p > 1$, and v_1 solves

$$\text{div}(\bar{\mathbf{A}} Dv_1) = f - \bar{f} - \text{div}((\mathbf{A} - \bar{\mathbf{A}}) Dv) - \mathbf{b} \cdot Dv - (cv - \bar{c}\bar{v}) \\ - \text{div}((\mathbf{A} - \bar{\mathbf{A}})(\mathbf{S}_j x + \mathbf{p}_j)) - \mathbf{b} \cdot (\mathbf{S}_j x + \mathbf{p}_j) - c(\frac{1}{2} \mathbf{S}_j x \cdot x + \mathbf{p}_j \cdot x) \quad \text{in } B_t(x_0)$$

with boundary condition $v_1 = 0$ on $\partial B_t(x_0)$.

Then, applying Lemma 2.1 via rescaling, we obtain the following estimate:

$$\begin{aligned}
\left(\int_{B_t(x_0)} |Dv_1|^{\frac{1}{2}} \right)^2 &\lesssim t \int_{B_t(x_0)} |f - \bar{f}| + \|Dv\|_{L^\infty(B_t(x_0))} \int_{B_t(x_0)} |\mathbf{A} - \bar{\mathbf{A}}| + t \|Dv\|_{L^\infty(B_t(x_0))} \int_{B_t(x_0)} |\mathbf{b}| \\
&\quad + t \int_{B_t(x_0)} |cv - \bar{c}\bar{v}| + (r|\mathbf{S}_j| + |\mathbf{p}_j|) \int_{B_t(x_0)} |\mathbf{A} - \bar{\mathbf{A}}| \\
&\quad + t(r|\mathbf{S}_j| + |\mathbf{p}_j|) \int_{B_t(x_0)} |\mathbf{b}| + tr(r|\mathbf{S}_j| + |\mathbf{p}_j|) \int_{B_t(x_0)} |c|. \tag{3.12}
\end{aligned}$$

Similar to estimate (3.3), we have:

$$\begin{aligned}
\int_{B_t(x_0)} |cv - \bar{c}\bar{v}| &\leq \|v\|_{L^\infty(B_t(x_0))} \int_{B_t(x_0)} |c - \bar{c}| + Ct \|Dv\|_{L^\infty(B_t(x_0))} \int_{B_t(x_0)} |c| \\
&\leq \left\{ \|u\|_{L^\infty(B_t(x_0))} + 2r^2|\mathbf{S}_j| + 2r|\mathbf{p}_j| \right\} \int_{B_t(x_0)} |c - \bar{c}| + Ct \|Dv\|_{L^\infty(B_t(x_0))} \int_{B_t(x_0)} |c|, \\
&\leq \omega_c(t) \|u\|_{L^\infty(B_t(x_0))} + 4r(r|\mathbf{S}_j| + |\mathbf{p}_j|) \int_{B_t(x_0)} |c| + Ct \|Dv\|_{L^\infty(B_t(x_0))} \int_{B_t(x_0)} |c|, \tag{3.13}
\end{aligned}$$

where we used (3.10) in the second inequality.

Thus, by the definition (3.1), and using (3.12) and (3.13), we obtain

$$\begin{aligned}
\left(\int_{B_t(x_0)} |Dv_1|^{\frac{1}{2}} \right)^2 &\leq Ct \omega_f(t) + C \omega_{\text{coef}}(t) \|Dv\|_{L^\infty(B_t(x_0))} + Ct \omega_c(t) \|u\|_{L^\infty(B_t(x_0))} \\
&\quad + C(r|\mathbf{S}_j| + |\mathbf{p}_j|) \omega_{\text{coef}}(t, x_0),
\end{aligned}$$

where $C = C(d, \lambda, \Lambda)$, and we define (noting that $r \leq 1$).

$$\omega_{\text{coef}}(t, x_0) := \omega_{\mathbf{A}}(t) + t \int_{B_t(x_0)} |\mathbf{b}| + t \int_{B_t(x_0)} |c|.$$

On the other hand, note that $v_2 = v - v_1$ satisfies

$$L_0 v_2 := \text{div}(\bar{\mathbf{A}} Dv_2) = \text{constant} \quad \text{in } B_t(x_0).$$

Therefore, $L_0(Dv_2) = 0$ in $B_t(x_0)$. The remainder of the proof proceeds by arguments analogous to those in [5, Lemma 2.2] and [3, Theorem 1.3]. \blacksquare

By applying Lemma 3.9 together with (3.5), (3.11), and (2.13), we derive the following estimate (cf. (2.30)):

$$\begin{aligned}
\|Du - \mathbf{S}_j x - \mathbf{p}_j\|_{L^\infty(B_{\frac{1}{2}\kappa^j r_0})} &\leq C \kappa^{(1+\beta)j} \int_{B_{r_0}} |Du| + C \kappa^j r_0 M_j(r_0) \tilde{\omega}_{\text{coef}}(\kappa^j r_0) \\
&\quad + C \left(|\mathbf{S}_j| \kappa^j r_0 + |\mathbf{p}_j| \right) \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt + C \kappa^j r_0 \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_f(t)}{t} dt \\
&\quad + C \kappa^j r_0 \|u\|_{L^\infty(B_{r_0})} \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_c(t)}{t} dt. \tag{3.14}
\end{aligned}$$

Since $Du(0) = 0$, we infer from (3.14) that

$$\begin{aligned} |p_j| &\leq C\kappa^{(1+\beta)j} \oint_{B_{r_0}} |Du| + C\kappa^j r_0 M_j(r_0) \tilde{\omega}_{\text{coef}}(\kappa^j r_0) \\ &\quad + C\kappa^j r_0 |\mathbf{S}_j| \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt + C|p_j| \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt \\ &\quad + C\kappa^j r_0 \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_f(t)}{t} dt + C\kappa^j r_0 \|u\|_{L^\infty(B_{r_0})} \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_c(t)}{t} dt. \end{aligned} \quad (3.15)$$

We require $r_0 \in (0, \frac{1}{2}]$ to satisfy $r_0 \leq r_1$, where $r_1 = r_1(d, \lambda, \Lambda, \omega_{\text{coef}}, \beta) > 0$ is chosen so that

$$C \int_0^{r_1} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt \leq \frac{1}{2}.$$

Similarly to (2.32), we derive from (3.15) and (3.8) the following inequality:

$$\begin{aligned} |p_j| &\leq C\kappa^j r_0 \left\{ \kappa^{\beta j} + \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt \right\} \frac{1}{r_0} \oint_{B_{r_0}} |Du| + C\kappa^j r_0 M_j(r_0) \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt \\ &\quad + C\kappa^j r_0 \int_0^{r_0} \frac{\tilde{\omega}_f(t)}{t} dt + C\kappa^j r_0 \|u\|_{L^\infty(B_{r_0})} \int_0^{r_0} \frac{\tilde{\omega}_c(t)}{t} dt, \end{aligned} \quad (3.16)$$

where $C = C(d, \lambda, \Lambda, \omega_{\text{coef}}, \beta)$.

Then, by applying (3.14), (3.8), and (3.16), we obtain the following estimate, which is analogous to (2.33):

$$\begin{aligned} \|Du - \mathbf{S}_j x - p_j\|_{L^\infty(B_{\frac{1}{2}\kappa^j r_0})} &\leq C\kappa^j r_0 \left\{ \kappa^{\beta j} + \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt \right\} \frac{1}{r_0} \oint_{B_{r_0}} |Du| \\ &\quad + C\kappa^j r_0 M_j(r_0) \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt + C\kappa^j r_0 \left(\int_0^{\kappa^j r_0} \frac{\tilde{\omega}_f(t)}{t} dt + \|u\|_{L^\infty(B_{r_0})} \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_c(t)}{t} dt \right) \\ &\quad + C\kappa^j r_0 \left(\int_0^{r_0} \frac{\tilde{\omega}_f(t)}{t} dt + \|u\|_{L^\infty(B_{r_0})} \int_0^{r_0} \frac{\tilde{\omega}_c(t)}{t} dt \right) \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt. \end{aligned} \quad (3.17)$$

Additionally, similar to (2.34), we also obtain

$$\begin{aligned} \frac{2}{\kappa^j r_0} \|Du\|_{L^\infty(B_{\frac{1}{2}\kappa^j r_0})} &\leq \frac{C}{r_0} \oint_{B_{r_0}} |Du| + CM_j(r_0) \int_0^{r_0} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt \\ &\quad + C \int_0^{r_0} \frac{\tilde{\omega}_f(t)}{t} dt + C\|u\|_{L^\infty(B_{r_0})} \int_0^{r_0} \frac{\tilde{\omega}_c(t)}{t} dt, \end{aligned} \quad (3.18)$$

where $C = C(d, \lambda, \Lambda, \omega_{\text{coef}}, \beta)$.

Then, by following the same proof as in Lemma 2.35, we conclude from (3.18) that there exists $r_0 = r_0(d, \lambda, \Lambda, \omega_{\text{coef}}, \beta) \in (0, \frac{1}{2}]$ such that

$$\sup_{j \geq 1} M_j(r_0) \leq \frac{C}{r_0} \oint_{B_{2r_0}} |Du| + C \int_0^{r_0} \frac{\tilde{\omega}_f(t)}{t} dt + C\|u\|_{L^\infty(B_{r_0})} \int_0^{r_0} \frac{\tilde{\omega}_c(t)}{t} dt, \quad (3.19)$$

where $C = C(d, \lambda, \Lambda, \omega_{\text{coef}}, \beta)$.

It follows from (3.7) and (3.19) that the sequence $\{\mathbf{S}_j\}$ converges to some $\mathbf{S} \in \mathbb{S}^d$, as it is a Cauchy sequence in \mathbb{S}^d . Taking the limit as $k \rightarrow \infty$ in (3.7) (with reference to (3.19) and (3.6)), and subsequently setting $l = j$, yields the following estimate:

$$\begin{aligned} |\mathbf{S}_j - \mathbf{S}| + \frac{|p_j|}{\kappa^j r_0} &\leq C \left(\kappa^{\beta j} + \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt \right) \frac{1}{r_0} \int_{B_{2r_0}} |Du| \\ &\quad + C \left(\int_0^{r_0} \frac{\tilde{\omega}_f(t)}{t} dt + \|u\|_{L^\infty(B_{r_0})} \int_0^{r_0} \frac{\tilde{\omega}_c(t)}{t} dt \right) \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt \\ &\quad + C \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_f(t)}{t} dt + C \|u\|_{L^\infty(B_{r_0})} \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_c(t)}{t} dt, \end{aligned} \quad (3.20)$$

where $C = C(d, \lambda, \Lambda, \omega_{\text{coef}}, \beta)$.

Then, similar to (2.42), it follows from (3.17), (3.19), and (3.20) that

$$\begin{aligned} \|Du - \mathbf{S}x\|_{L^\infty(B_{\frac{1}{2}\kappa^j r_0})} &\leq C \kappa^j r_0 \left\{ \kappa^{\beta j} + \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt \right\} \frac{1}{r_0} \int_{B_{2r_0}} |Du| \\ &\quad + C \kappa^j r_0 \left\{ \int_0^{r_0} \frac{\tilde{\omega}_f(t)}{t} dt + \|u\|_{L^\infty(B_{r_0})} \int_0^{r_0} \frac{\tilde{\omega}_c(t)}{t} dt \right\} \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt \\ &\quad + C \kappa^j r_0 \left\{ \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_f(t)}{t} dt + \|u\|_{L^\infty(B_{r_0})} \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_c(t)}{t} dt \right\}. \end{aligned}$$

The previous inequality yields the following estimate:

$$\begin{aligned} \frac{1}{r} \|Du - \mathbf{S}x\|_{L^\infty(B_r)} &\leq \varrho_{\text{coef}}(r) \left\{ \frac{1}{r_0} \int_{B_{2r_0}} |Du| + \varrho_c(r_0) \|u\|_{L^\infty(B_{r_0})} + \varrho_f(r_0) \right\} \\ &\quad + \varrho_c(r) \|u\|_{L^\infty(B_{r_0})} + \varrho_f(r) \quad \text{for all } r \in (0, r_0/2). \end{aligned} \quad (3.21)$$

Here, the moduli of continuity $\varrho_{\text{coef}}(\cdot)$, $\varrho_c(\cdot)$, and $\varrho_f(\cdot)$ are defined as:

$$\begin{aligned} \varrho_{\text{coef}}(r) &:= C \left\{ \left(\frac{2r}{\kappa r_0} \right)^\beta + \int_0^{2r/\kappa} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt \right\}, \\ \varrho_c(r) &:= C \int_0^{2r/\kappa} \frac{\tilde{\omega}_c(t)}{t} dt, \\ \varrho_f(r) &:= C \int_0^{2r/\kappa} \frac{\tilde{\omega}_f(t)}{t} dt, \end{aligned}$$

where $C = C(d, \lambda, \Lambda, \omega_{\text{coef}}, \beta)$.

In particular, it follows from (3.21) that Du is differentiable at 0. Moreover, by combining (3.8) and (3.19), and noting that $D^2u(0) = \mathbf{S} = \lim_{j \rightarrow \infty} \mathbf{S}_j$, we obtain

$$|D^2u(0)| \leq C \left\{ \frac{1}{r_0} \int_{B_{2r_0}} |Du| + \int_0^{r_0} \frac{\tilde{\omega}_f(t)}{t} dt + \|u\|_{L^\infty(B_{r_0})} \int_0^{r_0} \frac{\tilde{\omega}_c(t)}{t} dt \right\},$$

where $C = C(d, \lambda, \Lambda, \omega_{\text{coef}}, \beta)$. Here we define

$$\varrho_{\text{lot}}(r) := \|u\|_{L^\infty(B_{r_0})} \int_0^r \frac{\tilde{\omega}_c(t)}{t} dt.$$

Note that $\varrho_{\text{lot}}(r) \lesssim r^\alpha$ if $c \in C^\alpha$, and $\varrho_{\text{lot}} \equiv 0$ if c is constant. This completes the proof of the theorem. \blacksquare

4. PROOF OF THEOREM 1.4: SIMPLE CASE

In this section, we consider an elliptic operator L of the form

$$Lu = a^{ij}D_{ij}u = \text{tr}(\mathbf{A}D^2u).$$

We also assume that the inhomogeneous term f is zero.

We prove that if $D^2u(x^o) = 0$ for some $x^o \in \Omega$, then D^2u is differentiable at x^o . For simplicity, assume $x^o = 0$ and $B_1(0) \subset \Omega$.

Let $\bar{\mathbf{A}} = (\mathbf{A})_{B_r}$ denote the average of \mathbf{A} over the ball $B_r = B_r(0)$, where $r \in (0, \frac{1}{2}]$. Decompose u as $u = v + w$, where $w \in W^{2,p}(B_r) \cap W_0^{1,p}(B_r)$ (for some $p > 1$) is the strong solution of the problem

$$\text{tr}(\bar{\mathbf{A}}D^2w) = -\text{tr}((\mathbf{A} - \bar{\mathbf{A}})D^2u) \text{ in } B_r, \quad w = 0 \text{ on } \partial B_r.$$

Lemma 4.1. *Let $B = B_1(0)$ and \mathbf{A}_0 be a constant symmetric matrix satisfying the uniform ellipticity condition (1.5). For $f \in L^p(B)$ with $p > 1$, let $u \in W^{2,p}(B) \cap W_0^{1,p}(B)$ be the unique solution to the Dirichlet problem*

$$\begin{cases} \text{tr}(\mathbf{A}_0D^2u) = f & \text{in } B, \\ u = 0 & \text{on } \partial B. \end{cases}$$

Then, for any $t > 0$, the following estimate holds:

$$|\{x \in B : |D^2u(x)| > t\}| \leq \frac{C}{t} \int_B |f|,$$

where $C = C(d, \lambda, \Lambda)$.

Proof. Refer to the proof of [1, Lemma 3.3] and [2, Lemma 2.20]. \blacksquare

Applying Lemma 4.1, we establish that $D^2w \in L^{\frac{1}{2}}(B_r)$, satisfying the estimate:

$$\left(\int_{B_r} |D^2w|^{\frac{1}{2}} \right)^2 \leq C\omega_{\mathbf{A}}(r) \|D^2u\|_{L^\infty(B_r)}, \quad (4.2)$$

where $C = C(d, \lambda, \Lambda)$.

The function $v = u - w$ satisfies

$$\text{tr}(\bar{\mathbf{A}}D^2v) = 0 \text{ in } B_r.$$

By interior regularity for constant-coefficient elliptic equations, $v \in C^\infty(B_r)$, and

$$\|D^2v\|_{L^\infty(B_{r/2})} \leq \frac{C}{r^2} \left(\int_{B_r} |v|^{\frac{1}{2}} \right)^2,$$

where $C = C(d, \lambda, \Lambda)$. This estimate remains valid if v is replaced by $D^2v - \mathbf{L}$, where \mathbf{L} is any affine symmetric matrix-valued function of the form

$$\mathbf{L}(x) = x_1 \mathbf{S}^1 + \cdots + x_d \mathbf{S}^d + \mathbf{P}, \quad \text{with } \mathbf{S}^1, \dots, \mathbf{S}^d, \mathbf{P} \in \mathbb{S}^d.$$

Let $\text{Sym}^3(\mathbb{R}^d)$ denote the space of symmetric $(0,3)$ -tensors – that is, fully symmetric trilinear forms – over \mathbb{R}^d . Given $\mathbf{S} \in \text{Sym}^3(\mathbb{R}^d)$, we define its contraction with a vector $x \in \mathbb{R}^d$ as the matrix

$$\langle \mathbf{S}, x \rangle := x_1 \mathbf{S}^1 + \cdots + x_d \mathbf{S}^d,$$

where each slice $\mathbf{S}^k \in \mathbb{S}^d$ is defined by

$$(\mathbf{S}^k)_{i,j} := \mathcal{S}^{i,j,k}.$$

In index notation, this contraction can be equivalently expressed as

$$(\langle \mathcal{S}, x \rangle)_{i,j} = x_k \mathcal{S}^{i,j,k}.$$

With this notation, we define the affine space

$$\mathfrak{S} = \left\{ \mathbf{L}(x) = \langle \mathcal{S}, x \rangle + \mathbf{P} : \mathcal{S} \in \text{Sym}^3(\mathbb{R}^d), \mathbf{P} \in \mathbb{S}^d \right\}.$$

Then, for any $\mathbf{L} \in \mathfrak{S}$, the following estimate holds:

$$\|D^4 v\|_{L^\infty(B_{r/2})} \leq \frac{C}{r^2} \left(\int_{B_r} |D^2 v - \mathbf{L}|^{\frac{1}{2}} \right)^2. \quad (4.3)$$

By Taylor's theorem, for any $\rho \in (0, r]$,

$$\sup_{x \in B_\rho} |D^2 v(x) - \langle D^3 v(0), x \rangle - D^2 v(0)| \leq C(d) \|D^4 v\|_{L^\infty(B_\rho)} \rho^2,$$

where $D^3 v(0)$ is interpreted as an element of $\text{Sym}^3(\mathbb{R}^d)$.

Consequently, for any $\kappa \in (0, \frac{1}{2})$ and any $\mathbf{L} \in \mathfrak{S}$, we obtain

$$\left(\int_{B_{\kappa r}} |D^2 v - \langle D^3 v(0), x \rangle - D^2 v(0)|^{\frac{1}{2}} \right)^2 \leq C \|D^4 v\|_{L^\infty(B_{r/2})} (\kappa r)^2 \leq C \kappa^2 \left(\int_{B_r} |D^2 v - \mathbf{L}|^{\frac{1}{2}} \right)^2,$$

where $C = C(d, \lambda, \Lambda) > 0$.

We now define the function

$$\Phi(r) := \frac{1}{r} \inf_{\mathbf{L} \in \mathfrak{S}} \left(\int_{B_r} |D^2 u - \mathbf{L}|^{\frac{1}{2}} \right)^2. \quad (4.4)$$

Since $u = v + w$, similar to (2.7), we obtain

$$\kappa r \Phi(\kappa r) \leq C \kappa^2 \left(\int_{B_r} |D^2 u - \mathbf{L}|^{\frac{1}{2}} \right)^2 + C \left(\kappa^2 + \kappa^{-2d} \right) \omega_{\mathbf{A}}(r) \|D^2 u\|_{L^\infty(B_r)}.$$

Let $\beta \in (0, 1)$ be an arbitrary but fixed number. Choose $\kappa = \kappa(d, \lambda, \Lambda, \beta) \in (0, \frac{1}{2})$ such that $C\kappa \leq \kappa^\beta$. Then, we have

$$\Phi(\kappa r) \leq \kappa^\beta \Phi(r) + C \omega_{\mathbf{A}}(r) \frac{1}{r} \|D^2 u\|_{L^\infty(B_r)}, \quad (4.5)$$

where $C = C(d, \lambda, \Lambda, \kappa) = C(d, \lambda, \Lambda, \beta)$.

Let $r_0 \in (0, \frac{1}{2}]$ be a number to be chosen later. By iterating (4.5), we have, for $j = 1, 2, \dots$,

$$\Phi(\kappa^j r_0) \leq \kappa^{\beta j} \Phi(r_0) + C \sum_{i=1}^j \kappa^{(i-1)\beta} \omega_{\mathbf{A}}(\kappa^{j-i} r_0) \frac{1}{\kappa^{j-i} r_0} \|D^2 u\|_{L^\infty(B_{\kappa^{j-i} r_0})}.$$

By defining

$$\mathbf{M}_j(r_0) := \max_{0 \leq i < j} \frac{1}{\kappa^i r_0} \|D^2 u\|_{L^\infty(B_{\kappa^i r_0})} \quad \text{for } j = 1, 2, \dots, \quad (4.6)$$

we obtain

$$\Phi(\kappa^j r_0) \leq \kappa^{\beta j} \Phi(r_0) + C \mathbf{M}_j(r_0) \tilde{\omega}_{\mathbf{A}}(\kappa^j r_0). \quad (4.7)$$

For each fixed r , the infimum in (4.4) is realized by some $\mathbf{L} \in \mathfrak{S}$. For each $j = 0, 1, 2, \dots$, let $\mathcal{S}_j \in \text{Sym}^3(\mathbb{R}^d)$ and $\mathbf{P}_j \in \mathbb{S}^d$ be chosen such that

$$\Phi(\kappa^j r_0) = \frac{1}{\kappa^j r_0} \left(\oint_{B_{\kappa^j r_0}} |D^2 u - \langle \mathcal{S}_j, x \rangle - \mathbf{P}_j|^{\frac{1}{2}} \right)^2. \quad (4.8)$$

From (4.4) and Hölder's inequality, we have

$$\Phi(r_0) \leq \frac{1}{r_0} \oint_{B_{r_0}} |D^2 u|. \quad (4.9)$$

Combining (4.7) and (4.9), we deduce

$$\Phi(\kappa^j r_0) \leq \frac{\kappa^{\beta j}}{r_0} \oint_{B_{r_0}} |D^2 u| + C M_j(r_0) \tilde{\omega}_{\mathbf{A}}(\kappa^j r_0). \quad (4.10)$$

Next, observe that for $j = 0, 1, 2, \dots$, we have

$$\oint_{B_{\kappa^j r_0}} |\langle \mathcal{S}_j, x \rangle + \mathbf{P}_j|^{\frac{1}{2}} \leq \oint_{B_{\kappa^j r_0}} |D^2 u - \langle \mathcal{S}_j, x \rangle - \mathbf{P}_j|^{\frac{1}{2}} + \oint_{B_{\kappa^j r_0}} |D^2 u|^{\frac{1}{2}} \leq 2 \oint_{B_{\kappa^j r_0}} |D^2 u|^{\frac{1}{2}}. \quad (4.11)$$

Furthermore,

$$|\mathbf{P}_j|^{\frac{1}{2}} = |\langle \mathcal{S}_j, x \rangle + \mathbf{P}_j - 2\langle \mathcal{S}_j, x/2 \rangle + \mathbf{P}_j|^{\frac{1}{2}} \leq |\langle \mathcal{S}_j, x \rangle + \mathbf{P}_j|^{\frac{1}{2}} + 2^{\frac{1}{2}} |\langle \mathcal{S}_j, x/2 \rangle + \mathbf{P}_j|^{\frac{1}{2}}$$

and

$$\oint_{B_{\kappa^j r_0}} |\langle \mathcal{S}_j, x/2 \rangle + \mathbf{P}_j|^{\frac{1}{2}} = \frac{2^d}{|B_{\kappa^j r_0}|} \int_{B_{\kappa^j r_0/2}} |\langle \mathcal{S}_j, x \rangle + \mathbf{P}_j|^{\frac{1}{2}} \leq 2^d \oint_{B_{\kappa^j r_0}} |\langle \mathcal{S}_j, x \rangle + \mathbf{P}_j|^{\frac{1}{2}}.$$

Thus,

$$|\mathbf{P}_j| \leq C \left(\oint_{B_{\kappa^j r_0}} |\langle \mathcal{S}_j, x \rangle + \mathbf{P}_j|^{\frac{1}{2}} \right)^2 \leq C \left(\oint_{B_{\kappa^j r_0}} |D^2 u|^{\frac{1}{2}} \right)^2, \quad j = 0, 1, 2, \dots \quad (4.12)$$

Since $u \in C^2(\overline{B}_{1/2})$ by [2, Theorem 1.6] and $D^2 u(0) = 0$, the estimate (4.12) immediately implies

$$\lim_{j \rightarrow \infty} \mathbf{P}_j = 0. \quad (4.13)$$

Estimate of \mathcal{S}_j . By the quasi-triangle inequality, we have

$$|\langle \mathcal{S}_j - \mathcal{S}_{j-1}, x \rangle + \mathbf{P}_j - \mathbf{P}_{j-1}|^{\frac{1}{2}} \leq |D^2 u - \langle \mathcal{S}_j, x \rangle - \mathbf{P}_j|^{\frac{1}{2}} + |D^2 u - \langle \mathcal{S}_{j-1}, x \rangle - \mathbf{P}_{j-1}|^{\frac{1}{2}}.$$

Taking the average over $B_{\kappa^j r_0}$ and using the fact that $|B_{\kappa^{j-1} r_0}|/|B_{\kappa^j r_0}| = \kappa^{-d}$, we obtain

$$\frac{1}{\kappa^j r_0} \left(\oint_{B_{\kappa^j r_0}} |\langle \mathcal{S}_j - \mathcal{S}_{j-1}, x \rangle + \mathbf{P}_j - \mathbf{P}_{j-1}|^{\frac{1}{2}} \right)^2 \leq C \Phi(\kappa^j r_0) + C \Phi(\kappa^{j-1} r_0) \quad (4.14)$$

for $j = 1, 2, \dots$, where $C = C(d, \lambda, \Lambda, \beta)$.

Next, observe that

$$|\mathbf{P}_j - \mathbf{P}_{j-1}|^{\frac{1}{2}} = |\langle \mathcal{S}_j - \mathcal{S}_{j-1}, x \rangle + \mathbf{P}_j - \mathbf{P}_{j-1} - 2\langle \mathcal{S}_j - \mathcal{S}_{j-1}, x/2 \rangle + \mathbf{P}_j - \mathbf{P}_{j-1}|^{\frac{1}{2}}.$$

Using the quasi-triangle inequality, we obtain

$$|\mathbf{P}_j - \mathbf{P}_{j-1}|^{\frac{1}{2}} \leq |\langle \mathcal{S}_j - \mathcal{S}_{j-1}, x \rangle + \mathbf{P}_j - \mathbf{P}_{j-1}|^{\frac{1}{2}} + 2^{\frac{1}{2}} |\langle \mathcal{S}_j - \mathcal{S}_{j-1}, x/2 \rangle + \mathbf{P}_j - \mathbf{P}_{j-1}|^{\frac{1}{2}}.$$

Moreover,

$$\oint_{B_{\kappa^j r_0}} |\langle \mathcal{S}_j - \mathcal{S}_{j-1}, x/2 \rangle + \mathbf{P}_j - \mathbf{P}_{j-1}|^{\frac{1}{2}} \leq 2^d \oint_{B_{\kappa^j r_0}} |\langle \mathcal{S}_j - \mathcal{S}_{j-1}, x \rangle + \mathbf{P}_j - \mathbf{P}_{j-1}|^{\frac{1}{2}}.$$

Combining these estimates, we conclude that

$$|\mathbf{P}_j - \mathbf{P}_{j-1}| \leq C(d) \left(\oint_{B_{\kappa^j r_0}} |\langle \mathcal{S}_j - \mathcal{S}_{j-1}, x \rangle + \mathbf{P}_j - \mathbf{P}_{j-1}|^{\frac{1}{2}} \right)^2, \quad j = 1, 2, \dots$$

Substituting this into (4.14), we derive

$$\frac{1}{\kappa^j r_0} |\mathbf{P}_j - \mathbf{P}_{j-1}| \leq C\Phi(\kappa^j r_0) + C\Phi(\kappa^{j-1} r_0), \quad j = 1, 2, \dots \quad (4.15)$$

On the other hand, for any $\mathcal{S} \in \text{Sym}^3(\mathbb{R}^d)$, we may write $\mathcal{S} = |\mathcal{S}| \mathcal{T}$, where $\mathcal{T} \in \text{Sym}^3(\mathbb{R}^d)$ satisfies $|\mathcal{T}| = 1$. Then we have

$$\oint_{B_r} |\langle \mathcal{S}, x \rangle|^{\frac{1}{2}} \geq |\mathcal{S}|^{\frac{1}{2}} \inf_{|\mathcal{T}|=1} \oint_{B_r} |\langle \mathcal{T}, x \rangle|^{\frac{1}{2}} = |\mathcal{S}|^{\frac{1}{2}} \inf_{|\mathcal{T}|=1} \oint_{B_1} r^{\frac{1}{2}} |\langle \mathcal{T}, x \rangle|^{\frac{1}{2}} = C(d) r^{\frac{1}{2}} |\mathcal{S}|^{\frac{1}{2}}. \quad (4.16)$$

Then, by using (4.16), the quasi-triangle inequality, (4.14), (4.15), (4.10), and the observation that $M_{j-1}(r_0) \leq M_j(r_0)$, we obtain

$$\begin{aligned} |\mathcal{S}_j - \mathcal{S}_{j-1}| &\leq \frac{C}{\kappa^j r_0} \left(\oint_{B_{\kappa^j r_0}} |\langle \mathcal{S}_j - \mathcal{S}_{j-1}, x \rangle|^{\frac{1}{2}} \right)^2 \\ &\leq \frac{C}{\kappa^j r_0} \left(\oint_{B_{\kappa^j r_0}} |\langle \mathcal{S}_j - \mathcal{S}_{j-1}, x \rangle + \mathbf{P}_j - \mathbf{P}_{j-1}|^{\frac{1}{2}} \right)^2 + \frac{C}{\kappa^j r_0} |\mathbf{P}_j - \mathbf{P}_{j-1}| \\ &\leq C\Phi(\kappa^j r_0) + C\Phi(\kappa^{j-1} r_0) \\ &\leq \frac{C\kappa^{\beta j}}{r_0} \oint_{B_{r_0}} |D^2 u| + CM_j(r_0) \left\{ \tilde{\omega}_{\mathbf{A}}(\kappa^j r_0) + \tilde{\omega}_{\mathbf{A}}(\kappa^{j-1} r_0) \right\}. \end{aligned} \quad (4.17)$$

To estimate $|\mathcal{S}_0|$, we proceed similarly to (4.17) by applying (4.11) and (4.12) with $j = 0$, and using Hölder's inequality to obtain

$$|\mathcal{S}_0| \leq \frac{C}{r_0} \oint_{B_{r_0}} |D^2 u|, \quad (4.18)$$

where $C = C(d, \lambda, \Lambda, \beta)$.

For $k > l \geq 0$, we derive from (4.17) and the definition of $M_j(r_0)$ that

$$\begin{aligned} |\mathcal{S}_k - \mathcal{S}_l| &\leq \sum_{j=l}^{k-1} |\mathcal{S}_{j+1} - \mathcal{S}_j| \leq \sum_{j=l}^{k-1} \frac{C\kappa^{\beta(j+1)}}{r_0} \oint_{B_{r_0}} |D^2 u| + CM_k(r_0) \sum_{j=l}^k \tilde{\omega}_{\mathbf{A}}(\kappa^j r_0) \\ &\leq \frac{C\kappa^{\beta(l+1)}}{(1 - \kappa^\beta) r_0} \oint_{B_{r_0}} |D^2 u| + CM_k(r_0) \int_0^{\kappa^l r_0} \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t} dt. \end{aligned} \quad (4.19)$$

In particular, by taking $k = j$ and $l = 0$ in (4.19), and using (4.18), we obtain for $j = 1, 2, \dots$ that

$$|\mathcal{S}_j| \leq |\mathcal{S}_j - \mathcal{S}_0| + |\mathcal{S}_0| \leq \frac{C}{r_0} \oint_{B_{r_0}} |D^2 u| + CM_j(r_0) \int_0^{r_0} \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t} dt. \quad (4.20)$$

Similarly, we obtain from (2.21) that for $k > l \geq 0$, we have

$$|\mathbf{P}_k - \mathbf{P}_l| \leq C \frac{\kappa^{(\beta+1)(l+1)}}{1 - \kappa^{\beta+1}} \int_{B_{r_0}} |D^2 u| + C \kappa^l r_0 M_k(r_0) \int_0^{\kappa^l r_0} \frac{\tilde{\omega}_A(t)}{t} dt. \quad (4.21)$$

Estimate for \mathbf{P}_j . We shall derive improved estimates for $|\mathbf{P}_j|$ using the following lemma, where we set

$$v(x) := u(x) - \frac{1}{6} \langle \mathbf{S}_j, x \rangle x \cdot x - \frac{1}{2} \mathbf{P}_j x \cdot x.$$

Note that

$$D^2 v(x) = D^2 u(x) - \langle \mathbf{S}_j, x \rangle - \mathbf{P}_j. \quad (4.22)$$

Lemma 4.23. *For $0 < r \leq \frac{1}{2}$, we have*

$$\sup_{B_r} |D^2 v| \leq C \left\{ \left(\int_{B_{2r}} |D^2 v|^{\frac{1}{2}} \right)^2 + (r|\mathbf{S}_j| + |\mathbf{P}_j|) \int_0^r \frac{\tilde{\omega}_A(t)}{t} dt \right\},$$

where $C = C(d, \lambda, \Lambda, \omega_A, \beta)$.

Proof. Since $\text{tr}(\mathbf{A} D^2 u) = 0$ in B_1 , it follows that

$$\text{tr}(\mathbf{A} D^2 v) = -\text{tr}(\mathbf{A}(\langle \mathbf{S}_j, x \rangle + \mathbf{P}_j)) \quad \text{in } B_{2r},$$

for any $0 < r \leq \frac{1}{2}$. Let $x_0 \in B_{3r/2}$ and $0 < t \leq r/4$, and denote $\bar{\mathbf{A}} := (\mathbf{A})_{B_t(x_0)}$.

We decompose v as $v = v_1 + v_2$, where $v_1 \in W^{2,p} \cap W_0^{1,p}(B_t(x_0))$ (for some $p > 1$) is the strong solution to the problem

$$\text{tr}(\bar{\mathbf{A}} D^2 v_1) = -\text{tr}((\mathbf{A} - \bar{\mathbf{A}})(\langle \mathbf{S}_j, x \rangle + \mathbf{P}_j) + (\mathbf{A} - \bar{\mathbf{A}}) D^2 v) \quad \text{in } B_t(x_0),$$

with boundary condition $v_1 = 0$ on $\partial B_t(x_0)$.

By Lemma 4.1 and rescaling, we obtain

$$\left(\int_{B_t(x_0)} |D^2 v_1|^{\frac{1}{2}} \right)^2 \leq C \left(\int_{B_t(x_0)} |\mathbf{A} - \bar{\mathbf{A}}| \right) (r|\mathbf{S}_j| + |\mathbf{P}_j|) + C \left(\int_{B_t(x_0)} |\mathbf{A} - \bar{\mathbf{A}}| \right) \|D^2 v\|_{L^\infty(B_t(x_0))},$$

and thus, we have

$$\left(\int_{B_t(x_0)} |D^2 v_1|^{\frac{1}{2}} \right)^2 \leq C \omega_A(t) (r|\mathbf{S}_j| + |\mathbf{P}_j|) + C \omega_A(t) \|D^2 v\|_{L^\infty(B_t(x_0))}.$$

On the other hand, observe that $v_2 = v - v_1$ satisfies

$$L_0 v_2 := \text{tr}(\bar{\mathbf{A}} D^2 v_2) = -\text{tr} \bar{\mathbf{A}}(\langle \mathbf{S}_j, x \rangle + \mathbf{P}_j) = \text{affine function} \quad \text{in } B_t(x_0).$$

Since L_0 is a constant-coefficients operator, it follows that $L_0(D^2 v_2) = 0$. The remainder of the proof then proceeds identically to that of [2, Theorem 1.6]. \blacksquare

By Lemma 4.23, (4.22), (4.8), and (4.10), we have (cf. (2.30))

$$\begin{aligned} \|D^2 u - \langle \mathbf{S}_j, x \rangle - \mathbf{P}_j\|_{L^\infty(B_{\frac{1}{2}\kappa^j r_0})} &\leq C \kappa^{(1+\beta)j} \int_{B_{r_0}} |D^2 u| + C \kappa^j r_0 M_j(r_0) \tilde{\omega}_A(\kappa^j r_0) \\ &\quad + C \left(\kappa^j r_0 |\mathbf{S}_j| + |\mathbf{P}_j| \right) \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_A(t)}{t} dt, \quad j = 1, 2, \dots, \end{aligned} \quad (4.24)$$

where $C = C(d, \lambda, \Lambda, \omega_A, \beta)$. Using $D^2u(0) = 0$, we infer from (4.24) that

$$\begin{aligned} |\mathbf{P}_j| &\leq C\kappa^{(1+\beta)j} \int_{B_{r_0}} |D^2u| + C\kappa^j r_0 M_j(r_0) \tilde{\omega}_A(\kappa^j r_0) \\ &\quad + C|\mathcal{S}_j| \kappa^j r_0 \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_A(t)}{t} dt + C|\mathbf{P}_j| \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_A(t)}{t} dt. \end{aligned}$$

Let us fix $r_1(d, \lambda, \Lambda, \omega_A, \beta) > 0$ such that

$$C \int_0^{r_1} \frac{\tilde{\omega}_A(t)}{t} dt \leq \frac{1}{2}.$$

We will later require $r_0 \leq r_1$. This ensures that

$$|\mathbf{P}_j| \leq C\kappa^{(1+\beta)j} \int_{B_{r_0}} |D^2u| + C\kappa^j r_0 M_j(r_0) \tilde{\omega}_A(\kappa^j r_0) + C|\mathcal{S}_j| \kappa^j r_0 \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_A(t)}{t} dt.$$

This, together with (4.20) and (2.13), yields

$$|\mathbf{P}_j| \leq C\kappa^j r_0 \left\{ \kappa^{\beta j} + \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_A(t)}{t} dt \right\} \frac{1}{r_0} \int_{B_{r_0}} |D^2u| + C\kappa^j r_0 M_j(r_0) \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_A(t)}{t} dt. \quad (4.25)$$

Convergence of \mathcal{S}_j . By (4.24), (4.20), (4.25), and (2.13), we have

$$\begin{aligned} \|D^2u - \langle \mathcal{S}_j, x \rangle - \mathbf{P}_j\|_{L^\infty(B_{\frac{1}{2}\kappa^j r_0})} &\leq C\kappa^j r_0 \left\{ \kappa^{\beta j} + \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_A(t)}{t} dt \right\} \frac{1}{r_0} \int_{B_{r_0}} |D^2u| \\ &\quad + C\kappa^j r_0 M_j(r_0) \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_A(t)}{t} dt. \end{aligned} \quad (4.26)$$

Then, from (4.26), (4.20), and (4.25), we infer that

$$\frac{1}{\kappa^j r_0} \|D^2u\|_{L^\infty(B_{\frac{1}{2}\kappa^j r_0})} \leq \frac{C}{r_0} \int_{B_{r_0}} |D^2u| + CM_j(r_0) \int_0^{r_0} \frac{\tilde{\omega}_A(t)}{t} dt, \quad (4.27)$$

where $C = C(d, \lambda, \Lambda, \omega_A, \beta)$.

Lemma 4.28. *There exists a constant $r_0 = r_0(d, \lambda, \Lambda, \omega_A, \beta) \in (0, \frac{1}{2})$ such that*

$$\sup_{j \geq 1} M_j(r_0) = \sup_{i \geq 0} \frac{1}{\kappa^i r_0} \|D^2u\|_{L^\infty(B_{\kappa^i r_0})} \leq \frac{C}{r_0} \int_{B_{2r_0}} |D^2u|, \quad (4.29)$$

where $C = C(d, \lambda, \Lambda, \omega_A, \beta)$.

Proof. Refer to the proof of Lemma 2.35. ■

Now, Lemma 4.28 and (4.19) imply that the sequence $\{\mathcal{S}_j\}$ is a Cauchy sequence in $\text{Sym}^3(\mathbb{R}^d)$, and thus $\mathcal{S}_j \rightarrow \mathcal{S}$ for some $\mathcal{S} \in \text{Sym}^3(\mathbb{R}^d)$. Moreover, by taking the limit as $k \rightarrow \infty$ in (4.19) and (4.21) (while recalling (4.29) and (4.13)), respectively, and then setting $l = j$, we obtain the following estimates:

$$\begin{aligned} |\mathcal{S}_j - \mathcal{S}| &\leq C \left\{ \kappa^{\beta j} + \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_A(t)}{t} dt \right\} \frac{1}{r_0} \int_{B_{2r_0}} |D^2u|, \\ |\mathbf{P}_j| &\leq C\kappa^j \left\{ \kappa^{\beta j} + \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_A(t)}{t} dt \right\} \int_{B_{2r_0}} |D^2u|. \end{aligned} \quad (4.30)$$

By the triangle inequality, (4.26), (4.30), and (4.29), we obtain

$$\begin{aligned} \|D^2u - \langle S, x \rangle\|_{L^\infty(B_{\frac{1}{2}\kappa^j r_0})} &\leq \|D^2u - \langle S_j, x \rangle - \mathbf{P}_j\|_{L^\infty(B_{\frac{1}{2}\kappa^j r_0})} + \frac{\kappa^j r_0}{2} |S_j - S| + |\mathbf{P}_j| \\ &\leq C\kappa^j r_0 \left\{ \kappa^{\beta j} + \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t} dt \right\} \frac{1}{r_0} \int_{B_{2r_0}} |D^2u|. \end{aligned} \quad (4.31)$$

Conclusion. It follows from (4.31) that

$$\frac{1}{r} \|D^2u - \langle S, x \rangle\|_{L^\infty(B_r)} \leq \varrho_{\mathbf{A}}(r) \left(\frac{1}{r_0} \int_{B_{2r_0}} |D^2u| \right), \quad (4.32)$$

where

$$\varrho_{\mathbf{A}}(r) = C \left\{ \left(\frac{2r}{\kappa r_0} \right)^\beta + \int_0^{2r/\kappa} \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t} dt \right\}. \quad (4.33)$$

Note that $\varrho_{\mathbf{A}}$ is a modulus of continuity determined by $d, \lambda, \Lambda, \omega_{\mathbf{A}}$, and $\beta \in (0, 1)$.

In particular, we conclude from (4.32) that D^2u is differentiable at 0. Moreover, it follows from (4.20) and (4.29) that (noting that $D^3u(0) = S = \lim_{j \rightarrow \infty} S_j$) we have

$$|D^3u(0)| \leq \frac{C}{r_0} \int_{B_{2r_0}} |D^2u|,$$

where $C = C(d, \lambda, \Lambda, \omega_{\mathbf{A}}, \beta)$.

If $\mathbf{A} \in C^\alpha$ for some $\alpha \in (0, 1)$, we have $\varrho_{\mathbf{A}}(r) \lesssim r^\alpha$ by choosing $\beta \in (\alpha, 1)$ in (4.33). This complete the proof in the special case. \blacksquare

5. PROOF OF THEOREM 1.4: GENERAL CASE

We now proceed with the proof of Theorem 1.4 in the general setting.

Define $\omega_{\text{coef}}(\cdot)$ by

$$\begin{aligned} \omega_{\text{coef}}(r) &:= \omega_{\mathbf{A}}(r) + r^{1-d} \sup_{x \in \Omega} \int_{\Omega \cap B_r(x)} |\mathbf{b}| + r^{1-d} \sup_{x \in \Omega} \int_{\Omega \cap B_r(x)} |D\mathbf{b}| \\ &\quad + r^{1-d} \sup_{x \in \Omega} \int_{\Omega \cap B_r(x)} |\mathbf{c}| + r^{1-d} \sup_{x \in \Omega} \int_{\Omega \cap B_r(x)} |D\mathbf{c}|. \end{aligned} \quad (5.1)$$

It is clear that $\omega_{\text{coef}}(\cdot)$ satisfies the Dini condition:

$$\int_0^1 \frac{\omega_{\text{coef}}(t)}{t} dt < \infty.$$

Although one might expect different powers of r to appear in the definition (5.1), we retain the current form to maintain consistency with the notation $\omega_{\text{coef}}(\cdot, x_0)$ introduced in the proof of Lemma 5.11.

Let u be a solution to (1.6), meaning that u satisfies

$$\text{tr}(\mathbf{A}D^2u) + \mathbf{b} \cdot Du + cu = f \quad \text{in } \Omega.$$

By [3, Theorem 1.5] (see also [3, Proposition 2.24]), we know that D^2u is continuous in Ω . Suppose $D^2u(x^0) = 0$ for some $x^0 \in \Omega$. As before, we may assume without loss of generality that $x^0 = 0$ and $B_2(0) \subset \Omega$.

For $0 < r \leq \frac{1}{2}$, denote $\bar{\mathbf{A}} := (\mathbf{A})_{B_r}$, the average of \mathbf{A} over B_r , and define

$$\begin{aligned}\widetilde{\mathbf{b} \cdot Du} &= (\mathbf{b} - (\mathbf{b})_{B_r} - (D\mathbf{b})_{B_r} \cdot x) \cdot Du + (\mathbf{b})_{B_r} \cdot (Du - (Du)_{B_r}) + (Du - (Du)_{B_r}) \cdot (D\mathbf{b})_{B_r} \cdot x, \\ \widetilde{cu} &= (c - (c)_{B_r} - (Dc)_{B_r} \cdot x)u + (c)_{B_r}(u - (u)_{B_r} - (Du)_{B_r} \cdot x) + (u - (u)_{B_r})(Dc)_{B_r} \cdot x, \\ \widetilde{f} &= f - (f)_{B_r} - (Df)_{B_r} \cdot x.\end{aligned}\quad (5.2)$$

We decompose u as $u = v + w$, where $w \in W^{2,p} \cap W_0^{1,p}(B_r)$ (for some $p > 1$) is the solution of the problem

$$\operatorname{tr}(\bar{\mathbf{A}} D^2 w) = -\operatorname{tr}((\mathbf{A} - \bar{\mathbf{A}}) D^2 u) + \widetilde{f} - \widetilde{\mathbf{b} \cdot Du} - \widetilde{cu} \quad \text{in } B_r, \quad w = 0 \quad \text{on } \partial B_r.$$

By Lemma 4.1, we obtain the following estimate via rescaling:

$$\left(\int_{B_r} |D^2 w|^{\frac{1}{2}} dx \right)^2 \leq C \omega_{\mathbf{A}}(r) \|D^2 u\|_{L^\infty(B_r)} + C \int_{B_r} |\widetilde{f}| + \int_{B_r} |\widetilde{\mathbf{b} \cdot Du}| + \int_{B_r} |\widetilde{cu}|. \quad (5.3)$$

Note that the Poincaré inequality implies

$$\int_{B_r} |f - (f)_{B_r} - (Df)_{B_r} \cdot x| \leq Cr \int_{B_r} |Df - (Df)_{B_r}|. \quad (5.4)$$

Therefore, we obtain

$$\int_{B_r} |\widetilde{f}| \leq Cr \omega_{Df}(r).$$

By applying (5.4) to \mathbf{b} , c , and u , we also derive the following estimates:

$$\begin{aligned}\int_{B_r} |\widetilde{\mathbf{b} \cdot Du}| &\leq Cr \omega_{D\mathbf{b}}(r) \|Du\|_{L^\infty(B_r)} + Cr \|D^2 u\|_{L^\infty(B_r)} \int_{B_r} |\mathbf{b}| + Cr^2 \|D^2 u\|_{L^\infty(B_r)} \int_{B_r} |D\mathbf{b}|, \\ \int_{B_r} |\widetilde{cu}| &\leq Cr \omega_{Dc}(r) \|u\|_{L^\infty(B_r)} + Cr^2 \|D^2 u\|_{L^\infty(B_r)} \int_{B_r} |c| + Cr^2 \|Du\|_{L^\infty(B_r)} \int_{B_r} |Dc|.\end{aligned}$$

We define $\omega_{\text{lot}}(r)$ as follows:

$$\omega_{\text{lot}}(r) := \|Du\|_{L^\infty(B_r)} \left(\omega_{D\mathbf{b}}(r) + r^{1-d} \sup_{x \in \Omega} \int_{\Omega \cap B_r(x)} |Dc| \right) + \|u\|_{L^\infty(B_r)} \omega_{Dc}(r). \quad (5.5)$$

It is clear that $\omega_{\text{lot}}(\cdot)$ satisfy the Dini condition:

$$\int_0^1 \frac{\omega_{\text{lot}}(t)}{t} dt < \infty.$$

Considering the definitions (5.1), (5.5), and above estimates, we obtain from (5.3) that:

$$\left(\int_{B_r} |D^2 w|^{\frac{1}{2}} dx \right)^2 \leq C \omega_{\text{cof}}(r) \|D^2 u\|_{L^\infty(B_r)} + Cr \omega_{Df}(r) + Cr \omega_{\text{lot}}(r). \quad (5.6)$$

It is clear that $f - \widetilde{f}$ is an affine function. Similarly, observe that both $cu - \widetilde{cu}$ and $\mathbf{b} \cdot Du - \widetilde{\mathbf{b} \cdot Du}$ are affine functions as well. Therefore, $v = u - w$ satisfies

$$L_0 v := \operatorname{tr}(\bar{\mathbf{A}} D^2 v) = \text{affine function in } B_r.$$

Note that $L_0(D^2 v - \mathbf{L}) = 0$ in B_r for any $\mathbf{L} \in \mathfrak{S}$. Therefore, the same reasoning that led to estimate (4.3) also applies here, yielding the identical estimate.

Let $\Phi(r)$ be defined as in (4.4), and let $\beta \in (0, 1)$ be an arbitrary but fixed constant. By employing (5.6) instead of (4.2), we derive an estimate similar to (4.5) (cf. (3.4)). Specifically, there exists a constant $\kappa = \kappa(d, \lambda, \Lambda, \beta) \in (0, \frac{1}{2})$ such that

$$\Phi(\kappa r) \leq \kappa^\beta \Phi(r) + C\omega_{\text{coef}}(r) \frac{1}{r} \|D^2 u\|_{L^\infty(B_r)} + C\omega_{Df}(r) + C\omega_{\text{lot}}(r),$$

where $C = C(d, \lambda, \Lambda, \beta)$.

Let $M_j(r_0)$ be as defined in (4.6), where $r_0 \in (0, \frac{1}{2}]$ is a number to be chosen later. Then we obtain, similar to (4.10), the following estimate:

$$\Phi(\kappa^j r_0) \leq \frac{\kappa^{\beta j}}{r_0} \int_{B_{r_0}} |D^2 u| + CM_j(r_0) \tilde{\omega}_{\text{coef}}(\kappa^j r_0) + C\tilde{\omega}_{Df}(\kappa^j r_0) + C\tilde{\omega}_{\text{lot}}(\kappa^j r_0). \quad (5.7)$$

Let $S_j \in \text{Sym}^3(\mathbb{R}^d)$ and $\mathbf{P}_j \in \mathbb{S}^d$ for $j = 0, 1, 2, \dots$ be chosen as in (4.8). By utilizing (5.7) instead of (4.10), we conclude (cf. (4.13)) that:

$$\lim_{j \rightarrow \infty} \mathbf{P}_j = 0. \quad (5.8)$$

We similarly obtain the following estimate for $k > l \geq 0$ (cf. (4.19) and (4.21)):

$$\begin{aligned} |S_k - S_l| + \frac{|\mathbf{P}_k - \mathbf{P}_l|}{\kappa^l r_0} &\leq \frac{C\kappa^{\beta l}}{r_0} \int_{B_{r_0}} |D^2 u| + CM_k(r_0) \int_0^{\kappa^l r_0} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt \\ &\quad + C \int_0^{\kappa^l r_0} \frac{\tilde{\omega}_{Df}(t)}{t} dt + C \int_0^{\kappa^l r_0} \frac{\tilde{\omega}_{\text{lot}}(t)}{t} dt, \end{aligned} \quad (5.9)$$

where $C = C(d, \lambda, \Lambda, \beta)$. This also yields the following bound for S_j (cf. (4.20)):

$$\begin{aligned} |S_j| &\leq \frac{C}{r_0} \int_{B_{r_0}} |D^2 u| + CM_j(r_0) \int_0^{r_0} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt \\ &\quad + C \int_0^{r_0} \frac{\tilde{\omega}_{Df}(t)}{t} dt + C \int_0^{r_0} \frac{\tilde{\omega}_{\text{lot}}(t)}{t} dt, \end{aligned} \quad (5.10)$$

where $C = C(d, \lambda, \Lambda, \beta)$.

The following lemma serves as a counterpart to Lemma 3.9.

Lemma 5.11. *Let v be defined by*

$$v(x) := u(x) - \frac{1}{6} \langle S_j, x \rangle x \cdot x - \frac{1}{2} \mathbf{P}_j x \cdot x. \quad (5.12)$$

Then, for any $0 < r \leq \frac{1}{2}$, we have

$$\begin{aligned} \sup_{B_r} |D^2 v| &\leq C \left(\int_{B_{2r}} |D^2 v|^{\frac{1}{2}} \right)^2 + Cr \int_0^r \frac{\tilde{\omega}_{Df}(t)}{t} dt + Cr \int_0^r \frac{\tilde{\omega}_{\text{lot}}(t)}{t} dt \\ &\quad + C(r|S_j| + |\mathbf{P}_j|) \int_0^r \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt, \end{aligned}$$

where $C = C(d, \lambda, \Lambda, \omega_{\text{coef}}, \beta)$.

Proof. For $x_0 \in B_{3r/2}$ and $0 < t \leq r/4$, define $\bar{\mathbf{A}} := (\mathbf{A})_{B_t(x_0)}$ as the average of \mathbf{A} over the ball $B_t(x_0)$. Note that v satisfies

$$\begin{aligned} \operatorname{tr}(\bar{\mathbf{A}}D^2v) &= f - \mathbf{b} \cdot Dv - cv - \operatorname{tr}((\mathbf{A} - \bar{\mathbf{A}})D^2v) - \operatorname{tr}(\mathbf{A}(\langle \mathbf{S}_j, x \rangle + \mathbf{P}_j)) \\ &\quad - \mathbf{b} \cdot \left(\frac{1}{2} \langle \mathbf{S}_j, x \rangle x + \mathbf{P}_j x \right) - c \left(\frac{1}{6} \langle \mathbf{S}_j, x \rangle x \cdot x + \frac{1}{2} \mathbf{P}_j x \cdot x \right). \end{aligned}$$

Define the functions \widetilde{f} , $\widetilde{\mathbf{b} \cdot Dv}$, and \widetilde{cv} as in (5.2). More precisely, we set:

$$\begin{aligned} \widetilde{\mathbf{b} \cdot Dv} &= (\mathbf{b} - (\mathbf{b})_{B_t(x_0)} - (D\mathbf{b})_{B_t(x_0)}(x - x_0)) \cdot Dv + (\mathbf{b})_{B_t(x_0)} \cdot (Dv - (Dv)_{B_t(x_0)}) \\ &\quad + (Dv - (Dv)_{B_t(x_0)}) \cdot (D\mathbf{b})_{B_t(x_0)}(x - x_0), \\ \widetilde{cv} &= (c - (c)_{B_t(x_0)} - (Dc)_{B_t(x_0)} \cdot (x - x_0))v + (v - (v)_{B_t(x_0)}) (Dc)_{B_t(x_0)} \cdot (x - x_0) \\ &\quad + (c)_{B_t(x_0)} (v - (v)_{B_t(x_0)} - (Dv)_{B_t(x_0)} \cdot (x - x_0)), \\ \widetilde{f} &= f - (f)_{B_t(x_0)} - (Df)_{B_t(x_0)} \cdot (x - x_0). \end{aligned} \tag{5.13}$$

Note that the differences $f - \widetilde{f}$, $cv - \widetilde{cv}$ and $\mathbf{b} \cdot Dv - \widetilde{\mathbf{b} \cdot Dv}$ are all affine functions.

We decompose v as $v = v_1 + v_2$, where $v_1 \in W^{2,p} \cap W_0^{1,p}(B_t(x_0))$ for some $p > 1$, and it solves the following Dirichlet problem:

$$\begin{aligned} \operatorname{tr}(\bar{\mathbf{A}}D^2v_1) &= \widetilde{f} - \widetilde{\mathbf{b} \cdot Dv} - \widetilde{cv} - \operatorname{tr}((\mathbf{A} - \bar{\mathbf{A}})D^2v) - \operatorname{tr}\{(\mathbf{A} - \bar{\mathbf{A}})(\langle \mathbf{S}_j, x \rangle + \mathbf{P}_j)\} \\ &\quad - \mathbf{b} \cdot \left\{ \left(\frac{1}{2} \langle \mathbf{S}_j, x \rangle x + \mathbf{P}_j x \right) - \left(\frac{1}{2} \langle \mathbf{S}_j, x_0 \rangle x_0 + \mathbf{P}_j x_0 \right) \right\} \\ &\quad - (\mathbf{b} - (\mathbf{b})_{B_t(x_0)}) \cdot \left(\frac{1}{2} \langle \mathbf{S}_j, x_0 \rangle x_0 + \mathbf{P}_j x_0 \right) \\ &\quad - c \left\{ \left(\frac{1}{6} \langle \mathbf{S}_j, x \rangle x \cdot x + \frac{1}{2} \mathbf{P}_j x \cdot x \right) - \left(\frac{1}{6} \langle \mathbf{S}_j, x_0 \rangle x_0 \cdot x_0 + \frac{1}{2} \mathbf{P}_j x_0 \cdot x_0 \right) \right\} \\ &\quad - (c - (c)_{B_t(x_0)}) \left(\frac{1}{6} \langle \mathbf{S}_j, x_0 \rangle x_0 \cdot x_0 + \frac{1}{2} \mathbf{P}_j x_0 \cdot x_0 \right) \quad \text{in } B_t(x_0), \end{aligned}$$

with $v_1 = 0$ on $\partial B_t(x_0)$.

Then, by applying Lemma 4.1 with a rescaling argument, and using the mean value theorem together with the Poincaré inequality, we obtain the following estimate:

$$\begin{aligned} \left(\int_{B_t(x_0)} |D^2v_1|^{\frac{1}{2}} \right)^2 &\lesssim \int_{B_t(x_0)} (|\widetilde{f}| + |\widetilde{\mathbf{b} \cdot Dv}| + |\widetilde{cv}|) + \omega_{\mathbf{A}}(t) \|D^2v\|_{L^\infty(B_t(x_0))} \\ &\quad + \omega_{\mathbf{A}}(t) (r|\mathbf{S}_j| + |\mathbf{P}_j|) + \left(\int_{B_t(x_0)} |\mathbf{b}| \right) t (r|\mathbf{S}_j| + |\mathbf{P}_j|) \\ &\quad + t \left(\int_{B_t(x_0)} |D\mathbf{b}| \right) r (r|\mathbf{S}_j| + |\mathbf{P}_j|) + t \left(\int_{B_t(x_0)} |c| \right) r (r|\mathbf{S}_j| + |\mathbf{P}_j|) \\ &\quad + t \left(\int_{B_t(x_0)} |Dc| \right) r^2 (r|\mathbf{S}_j| + |\mathbf{P}_j|). \end{aligned} \tag{5.14}$$

Next, combining (5.13) and (5.4) with (5.12), we obtain

$$\begin{aligned}
\int_{B_t(x_0)} |\widetilde{f}| + |\widetilde{\mathbf{b}} \cdot \widetilde{Dv}| + |\widetilde{cv}| &\lesssim t\omega_{Df}(t) + t\omega_{D\mathbf{b}}(t)\|Du\|_{L^\infty(B_t(x_0))} + tr(r|\mathcal{S}_j| + |\mathbf{P}_j|) \int_{B_t(x_0)} |D\mathbf{b}| \\
&+ t\|D^2v\|_{L^\infty(B_t(x_0))} \int_{B_t(x_0)} |\mathbf{b}| + t^2\|D^2v\|_{L^\infty(B_t(x_0))} \int_{B_t(x_0)} |D\mathbf{b}| \\
&+ t\omega_{Dc}(t)\|u\|_{L^\infty(B_t(x_0))} + tr^2(r|\mathcal{S}_j| + |\mathbf{P}_j|) \int_{B_t(x_0)} |Dc| + t^2\|Du\|_{L^\infty(B_t(x_0))} \int_{B_t(x_0)} |Dc| \\
&+ t^2r(r|\mathcal{S}_j| + |\mathbf{P}_j|) \int_{B_t(x_0)} |Dc| + t^2\|D^2v\|_{L^\infty(B_t(x_0))} \int_{B_t(x_0)} |c|. \tag{5.15}
\end{aligned}$$

Define (cf. (5.5))

$$\omega_{\text{lot}}(t, x_0) := \|Du\|_{L^\infty(B_t(x_0))} \left(\omega_{D\mathbf{b}}(t) + t^{1-d} \int_{B_t(x_0)} |Dc| \right) + \|u\|_{L^\infty(B_t(x_0))} \omega_{Dc}(t).$$

Then, using (5.14) and (5.15), we obtain

$$\begin{aligned}
\left(\int_{B_t(x_0)} |Dv_1|^{\frac{1}{2}} \right)^2 &\leq Ct\omega_{Df}(t) + C\omega_{\text{coef}}(t)\|D^2v\|_{L^\infty(B_t(x_0))} + Ct\omega_{\text{lot}}(t, x_0) \\
&+ C(r|\mathcal{S}_j| + |\mathbf{P}_j|)\omega_{\text{coef}}(t, x_0),
\end{aligned}$$

where $C = C(d, \lambda, \Lambda)$, and we define (noting that $r \leq 1$)

$$\omega_{\text{coef}}(t, x_0) := \omega_{\mathbf{A}}(t) + t \int_{B_t(x_0)} |\mathbf{b}| + t \int_{B_t(x_0)} |c| + t \int_{B_t(x_0)} |D\mathbf{b}| + t \int_{B_t(x_0)} |Dc|.$$

On the other hand, recall that $f - \widetilde{f}$, $cv - \widetilde{cv}$ and $\mathbf{b} \cdot Dv - \widetilde{\mathbf{b}} \cdot \widetilde{Dv}$ are all affine functions. Thus, $v_2 = v - v_1$ satisfies

$$L_0 v_2 := \text{tr}(\bar{\mathbf{A}} D^2 v_2) = \text{affine function} \quad \text{in } B_t(x_0).$$

Therefore, $L_0(D^2 v_2) = 0$ in $B_t(x_0)$. The remainder of the proof follows the same arguments as those in the proof of [2, Theorem 1.6] and [5, Lemma 2.2]. \blacksquare

By using Lemma 5.11, (5.7) and (5.12), we obtain (cf. (4.24))

$$\begin{aligned}
\|D^2u - \langle \mathcal{S}_j, x \rangle - \mathbf{P}_j\|_{L^\infty(B_{\frac{1}{2}\kappa^j r_0})} &\leq C\kappa^{(1+\beta)j} \int_{B_{r_0}} |D^2u| + C\kappa^j r_0 M_j(r_0) \tilde{\omega}_{\text{coef}}(\kappa^j r_0) \\
&+ C(|\mathcal{S}_j|\kappa^j r_0 + |\mathbf{P}_j|) \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt + C\kappa^j r_0 \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{Df}(t)}{t} dt \\
&+ C\kappa^j r_0 \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\text{lot}}(t)}{t} dt. \tag{5.16}
\end{aligned}$$

Using $D^2u(0) = 0$, we deduce from (5.16) that

$$\begin{aligned}
|\mathbf{P}_j| &\leq C\kappa^{(1+\beta)j} \int_{B_{r_0}} |D^2u| + C\kappa^j r_0 M_j(r_0) \tilde{\omega}_{\text{coef}}(\kappa^j r_0) + C\kappa^j r_0 |\mathcal{S}_j| \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt \\
&+ C|\mathbf{P}_j| \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt + C\kappa^j r_0 \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{Df}(t)}{t} dt + C\kappa^j r_0 \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\text{lot}}(t)}{t} dt. \tag{5.17}
\end{aligned}$$

We require $r_0 \leq r_1$, where $r_1 = r_1(d, \lambda, \Lambda, \omega_{\text{coef}}, \beta) > 0$ is chosen so that

$$C \int_0^{r_1} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt \leq \frac{1}{2}.$$

We derive from (5.17) and (5.10) the following inequality:

$$\begin{aligned} |\mathbf{P}_j| &\leq C\kappa^j r_0 \left\{ \kappa^{\beta j} + \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt \right\} \frac{1}{r_0} \oint_{B_{r_0}} |D^2 u| + C\kappa^j r_0 M_j(r_0) \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt \\ &\quad + C\kappa^j r_0 \int_0^{r_0} \frac{\tilde{\omega}_{Df}(t)}{t} dt + C\kappa^j r_0 \int_0^{r_0} \frac{\tilde{\omega}_{\text{lot}}(t)}{t} dt, \end{aligned} \quad (5.18)$$

where $C = C(d, \lambda, \Lambda, \omega_{\text{coef}}, \beta)$.

Then, similar to (4.26), we obtain from (5.16), (5.10), and (5.18) the following estimate:

$$\begin{aligned} \|D^2 u - \langle S_j, x \rangle - \mathbf{P}_j\|_{L^\infty(B_{\frac{1}{2}\kappa^j r_0})} &\leq C\kappa^j r_0 \left\{ \kappa^{\beta j} + \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt \right\} \frac{1}{r_0} \oint_{B_{r_0}} |D^2 u| \\ &\quad + C\kappa^j r_0 M_j(r_0) \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt + C\kappa^j r_0 \left\{ \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{Df}(t)}{t} dt + \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\text{lot}}(t)}{t} dt \right\} \\ &\quad + C\kappa^j r_0 \left\{ \int_0^{r_0} \frac{\tilde{\omega}_{Df}(t)}{t} dt + \int_0^{r_0} \frac{\tilde{\omega}_{\text{dat}}(t)}{t} dt \right\} \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt. \end{aligned} \quad (5.19)$$

Additionally, similar to (4.27), we obtain

$$\begin{aligned} \frac{2}{\kappa^j r_0} \|D^2 u\|_{L^\infty(B_{\frac{1}{2}\kappa^j r_0})} &\leq \frac{C}{r_0} \oint_{B_{r_0}} |D^2 u| + CM_j(r_0) \int_0^{r_0} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt \\ &\quad + C \int_0^{r_0} \frac{\tilde{\omega}_{Df}(t)}{t} dt + C \int_0^{r_0} \frac{\tilde{\omega}_{\text{lot}}(t)}{t} dt, \end{aligned} \quad (5.20)$$

where $C = C(d, \lambda, \Lambda, \omega_{\text{coef}}, \beta)$.

Then, by following the same proof as in Lemma 4.28, we conclude from (5.20) that there exists $r_0 = r_0(d, \lambda, \Lambda, \omega_{\text{coef}}, \beta) \in (0, \frac{1}{2}]$ such that

$$\sup_{j \geq 1} M_j(r_0) \leq \frac{C}{r_0} \oint_{B_{2r_0}} |D^2 u| + C \int_0^{r_0} \frac{\tilde{\omega}_{Df}(t)}{t} dt + C \int_0^{r_0} \frac{\tilde{\omega}_{\text{lot}}(t)}{t} dt, \quad (5.21)$$

where $C = C(d, \lambda, \Lambda, \omega_{\text{coef}}, \beta)$.

From (5.9) and (5.21), we can conclude that the sequence $\{S_j\}$ is a Cauchy sequence in $\text{Sym}^3(\mathbb{R}^d)$. Therefore, it converges to some $S \in \text{Sym}^3(\mathbb{R}^d)$. By taking the limit as $k \rightarrow \infty$ in (5.9) (while recalling (5.21) and (5.8)), and then setting $l = j$, we obtain the following estimate:

$$\begin{aligned} |S_j - S| + \frac{|\mathbf{P}_j|}{\kappa^j r_0} &\leq C \left\{ \kappa^{\beta j} + \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt \right\} \frac{1}{r_0} \oint_{B_{2r_0}} |D^2 u| \\ &\quad + C \left\{ \int_0^{r_0} \frac{\tilde{\omega}_{Df}(t)}{t} dt + \int_0^{r_0} \frac{\tilde{\omega}_{\text{lot}}(t)}{t} dt \right\} \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt \\ &\quad + C \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{Df}(t)}{t} dt + C \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\text{lot}}(t)}{t} dt, \end{aligned} \quad (5.22)$$

where $C = C(d, \lambda, \Lambda, \omega_{\text{coef}}, \beta)$.

Then, similar to (4.31), it follows from (5.19), (5.21), and (5.22) that

$$\begin{aligned} \|D^2u - \langle \mathcal{S}, x \rangle\|_{L^\infty(B_{\frac{1}{2}\kappa^j r_0})} &\leq C\kappa^j r_0 \left\{ \kappa^{\beta j} + \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt \right\} \frac{1}{r_0} \int_{B_{2r_0}} |D^2u| \\ &\quad + C\kappa^j r_0 \left\{ \int_0^{r_0} \frac{\tilde{\omega}_{Df}(t)}{t} dt + \int_0^{r_0} \frac{\tilde{\omega}_{\text{lot}}(t)}{t} dt \right\} \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt \\ &\quad + C\kappa^j r_0 \left\{ \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{Df}(t)}{t} dt \int_0^{\kappa^j r_0} \frac{\tilde{\omega}_{\text{lot}}(t)}{t} dt \right\}. \end{aligned}$$

From the previous inequality, we derive the following uniform estimate for all $r \in (0, r_0/2)$:

$$\begin{aligned} \frac{1}{r} \|D^2u - \langle \mathcal{S}, x \rangle\|_{L^\infty(B_r)} &\leq \varrho_{\text{coef}}(r) \left\{ \frac{1}{r_0} \int_{B_{2r_0}} |D^2u| + \varrho_{Df}(r_0) + \varrho_{\text{lot}}(r_0) \right\} \\ &\quad + \varrho_{Df}(r) + \varrho_{\text{lot}}(r), \quad (5.23) \end{aligned}$$

where the moduli of continuity $\varrho_{\dots}(r)$ are defined by

$$\begin{aligned} \varrho_{\text{coef}}(r) &:= C \left\{ \left(\frac{2r}{\kappa r_0} \right)^\beta + \int_0^{2r/\kappa} \frac{\tilde{\omega}_{\text{coef}}(t)}{t} dt \right\}, \\ \varrho_{Df}(r) &:= C \int_0^{2r/\kappa} \frac{\tilde{\omega}_{Df}(t)}{t} dt, \\ \varrho_{\text{lot}}(r) &:= C \int_0^{2r/\kappa} \frac{\tilde{\omega}_{\text{lot}}(t)}{t} dt, \end{aligned}$$

where $C = C(d, \lambda, \Lambda, \omega_{\text{coef}}, \beta)$.

In particular, from (5.23), we conclude that D^2u is differentiable at 0. Moreover, it follows from (5.10) and (5.21) that (noting $D^3u(0) = \mathcal{S} = \lim_{j \rightarrow \infty} \mathcal{S}_j$):

$$|D^3u(0)| \leq C \left\{ \frac{1}{r_0} \int_{B_{2r_0}} |D^2u| + \int_0^{r_0} \frac{\tilde{\omega}_{Df}(t)}{t} dt + \int_0^{r_0} \frac{\tilde{\omega}_{\text{lot}}(t)}{t} dt \right\},$$

where $C = C(d, \lambda, \Lambda, \omega_{\text{coef}}, \beta)$.

To analyze the modulus of continuity ϱ_{lot} , recall the definition:

$$\omega_{\text{lot}}(r) := \|Du\|_{L^\infty(B_r)} \left\{ \omega_{Db}(r) + r^{1-d} \sup_{x \in \Omega} \int_{\Omega \cap B_r(x)} |Dc| \right\} + \|u\|_{L^\infty(B_r)} \omega_{Dc}(r).$$

We isolate a component of this expression by defining:

$$\omega_{Dc,1}(r) := r^{1-d} \sup_{x \in \Omega} \int_{\Omega \cap B_r(x)} |Dc|.$$

It is clear that $\omega_{Dc,1}(r)$ satisfies the Dini condition. Then, for $r \in (0, r_0/2)$, we obtain the following estimate for $\varrho_{\text{lot}}(r)$:

$$\varrho_{\text{lot}}(r) \leq C \|Du\|_{L^\infty(B_{r_0})} \{ \varrho_{Db}(r) + \hat{\varrho}_{Dc}(r) \} + C \|u\|_{L^\infty(B_{r_0})} \varrho_{Dc}(r), \quad (5.24)$$

where the moduli of continuity are defined by

$$\begin{aligned}\varrho_{Db}(r) &:= C \int_0^{2r/\kappa} \frac{\tilde{\omega}_{Db}(t)}{t} dt, \\ \varrho_{Dc}(r) &:= C \int_0^{2r/\kappa} \frac{\tilde{\omega}_{Dc}(t)}{t} dt, \\ \hat{\varrho}_{Dc}(r) &:= C \int_0^{2r/\kappa} \frac{\tilde{\omega}_{Dc,1}(t)}{t} dt.\end{aligned}$$

Note that inequality (5.24) implies that if Db and Dc are C^α functions, then choosing $\beta \in (\alpha, 1)$ yields $\varrho_{\text{lot}}(t) \lesssim t^\alpha$. The theorem is proved. \blacksquare

REFERENCES

- [1] Choi, Jongkeun; Dong, Hongjie; Kim, Dong-ha; Kim, Seick. *Regularity of elliptic equations in double divergence form and applications to Green's function estimates*. arXiv:2401.06621 [math.AP]
- [2] Dong, Hongjie; Kim, Seick. *On C^1 , C^2 , and weak type-(1, 1) estimates for linear elliptic operators*. Comm. Partial Differential Equations **42** (2017), no. 3, 417–435.
- [3] Dong, Hongjie; Escauriaza, Luis; Kim, Seick. *On C^1 , C^2 , and weak type-(1, 1) estimates for linear elliptic operators: part II*. Math. Ann. **370** (2018), no. 1-2, 447–489.
- [4] Hwang, Sukjung; Kim, Seick. *Green's function for second order elliptic equations in non-divergence form*. Potential Anal. **52** (2022), no. 1, 27–39.
- [5] Kim, Seick; Lee, Sungjin. *Estimates for Greens functions of elliptic equations in non-divergence form with continuous coefficients*. Ann. Appl. Math. **37** (2021), no. 2, 111–130.
- [6] Leitão, Raimundo; Pimentel, Edgard A.; Santos, Makson S. *Geometric regularity for elliptic equations in double-divergence form*. Anal. PDE **13** (2020), no. 4, 1129–1144.
- [7] Teixeira, Eduardo V. *Regularity for quasilinear equations on degenerate singular sets*. Math. Ann. **358** (2014), no. 1-2, 241–256.

(J. Choi) DEPARTMENT OF MATHEMATICS EDUCATION, PUSAN NATIONAL UNIVERSITY, BUSAN, 46241, REPUBLIC OF KOREA

Email address: jongkeun.choi@pusan.ac.kr

(H. Dong) DIVISION OF APPLIED MATHEMATICS, BROWN UNIVERSITY, 182 GEORGE STREET, PROVIDENCE, RI 02912, UNITED STATES OF AMERICA

Email address: Hongjie.Dong@brown.edu

(S. Kim) DEPARTMENT OF MATHEMATICS, YONSEI UNIVERSITY, 50 YONSEI-RO, SEODAEMUN-GU, SEOUL 03722, REPUBLIC OF KOREA

Email address: kimseick@yonsei.ac.kr