

# Self-Driving Cars

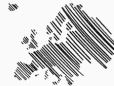
## Lecture 5 – Vehicle Dynamics

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TÜBINGEN



e l l i s  
European Laboratory for Learning and Intelligent Systems

# Agenda

**5.1** Introduction

**5.2** Kinematic Bicycle Model

**5.3** Tire Models

**5.4** Dynamic Bicycle Model

# 5.1

## Introduction

# Credits

1. Kinematic bicycle model


**Kinematics** are about the geometric description of motions in space (e.g. based on different reference frames and coordinate systems).

**Kinetics** (or **dynamics**) are about the laws of the causes of motion (e.g. the effects of forces / moments in Newton's laws).

Inertial, vehicle, and path reference frames

**Inertial frame:** The inertial frame is fixed to the earth. For the inertial frame, the right-handed Cartesian coordinates  $X, Y, Z$  are used. The  $Z$ -axis is vertical (anti-directional to the gravitation) and the  $X$ - $Y$ -axes represent a horizontal plane (perpendicular to gravitation).

**Vehicle frame:** The vehicle frame is attached to the vehicle at a fixed vehicle reference point.



Vehicle Dynamics & Control - 01 Coordinate systems

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Vehicle Dynamics and Control

Prof. Georg Schildbach, University of Luebeck - 1 / 25

Vehicle Dynamics & Control - 01 Coordinate systems

Vehicle Dynamics & Control - 03 Review: Kinematics of a...

Vehicle Dynamics & Control - 04 Ackermann steering...

Vehicle Dynamics & Control - 05 Kinematic bicycle model

Vehicle Dynamics & Control - 02 Vehicle kinematics...

Vehicle Dynamics & Control - 06 Review: Dynamics of a rigl...

Vehicle Dynamics & Control - 07 Tires: Terminology and...

Vehicle Dynamics & Control - 08 Lateral tire models

Vehicle Dynamics & Control - 09 Dynamic bicycle model wit...

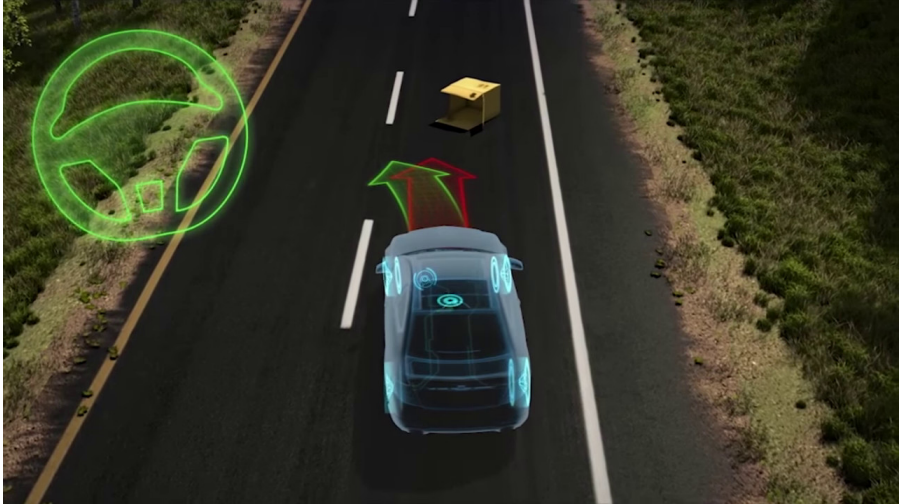
Vehicle Dynamics & Control - 10 Vehicle handling...



► We cover parts of “Vehicle Dynamics & Control” by Prof. Schildbach (Uni Lübeck)

[https://youtu.be/Cg0L\\_HZYxP4](https://youtu.be/Cg0L_HZYxP4)

# Electronic Stability Program



Knowledge of **vehicle dynamics** enables accurate **vehicle control**

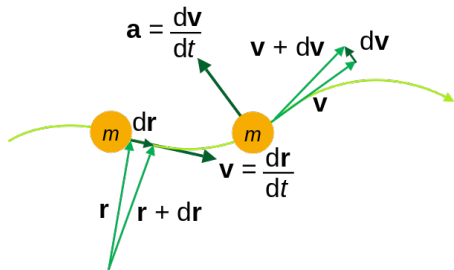
# Kinematics vs. Kinetics

## Kinematics:

- ▶ Greek origin: “motion”, “moving”
- ▶ Describes motion of points and bodies
- ▶ Considers position, velocity, acceleration, ..
- ▶ Examples: Celestial bodies, particle systems, robotic arm, human skeleton

## Kinetics:

- ▶ Describes causes of motion
- ▶ Effects of forces/moments
- ▶ Newton's laws, e.g.,  $F = ma$



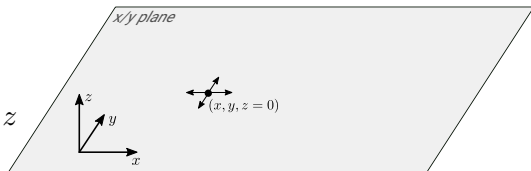
# Holonomic Constraints

**Holonomic constraints** are constraints on the **configuration**:

- ▶ Assume a particle in three dimensions  $(x, y, z) \in \mathbb{R}^3$
- ▶ We can constrain the particle to the x/y plane via:

$$z = 0$$

$$\Leftrightarrow f(x, y, z) = 0 \quad \text{with} \quad f(x, y, z) = z$$



- ▶ Constraints of the form  $f(x, y, z) = 0$  are called holonomic constraints
- ▶ They constrain the configuration space
- ▶ But the system can move freely in that space
- ▶ Controllable degrees of freedom equal total degrees of freedom (2)

# Non-Holonomic Constraints

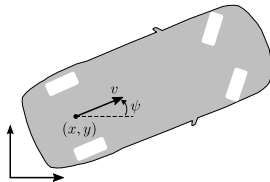
**Non-Holonomic constraints** are constraints on the **velocity**:

- ▶ Assume a vehicle that is parameterized by  $(x, y, \psi) \in \mathbb{R}^2 \times [0, 2\pi]$
- ▶ The 2D vehicle velocity is given by:

$$\dot{x} = v \cos(\psi)$$

$$\dot{y} = v \sin(\psi)$$

$$\Rightarrow \dot{x} \sin(\psi) - \dot{y} \cos(\psi) = 0$$



- ▶ This non-holonomic constraint cannot be expressed in the form  $f(x, y, \psi) = 0$
- ▶ The car cannot freely move in any direction (e.g., sideways)
- ▶ It constrains the velocity space, but not the configuration space
- ▶ Controllable degrees of freedom less than total degrees of freedom (2 vs. 3)



# Holonomic vs. Non-Holonomic Systems

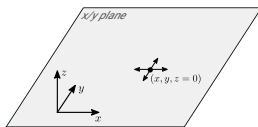
## Holonomic Systems

- ▶ Constrain configuration space
- ▶ Can freely move in any direction
- ▶ Controllable degrees of freedom equal to total degrees of freedom
- ▶ Constraints **can** be described by
$$f(x_1, \dots, x_N) = 0$$

### Example:

3D Particle

$$z = 0$$



## Nonholonomic Systems

- ▶ Constrain velocity space
- ▶ Cannot freely move in any direction
- ▶ Controllable degrees of freedom less than total degrees of freedom
- ▶ Constraints **cannot** be described by
$$f(x_1, \dots, x_N) = 0$$

### Example:

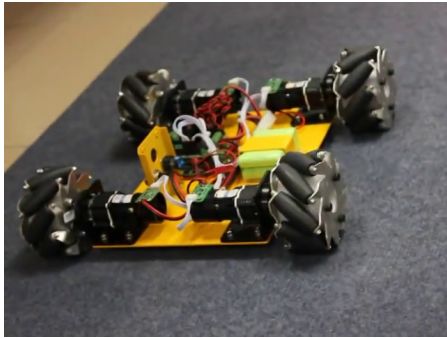
Car

$$\dot{x} \sin(\psi) - \dot{y} \cos(\psi) = 0$$

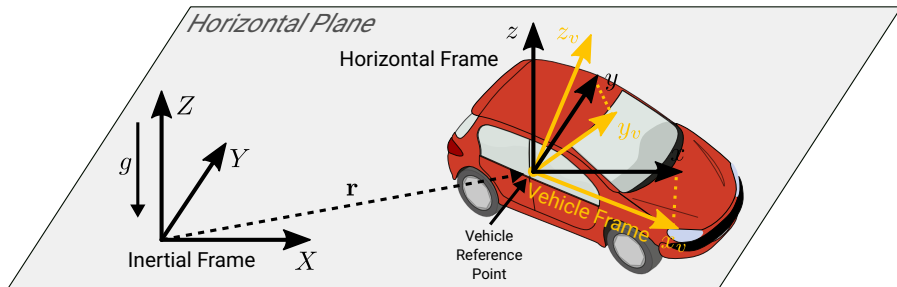


# Holonomic vs. Non-Holonomic Systems

- ▶ A robot can be subject to both holonomic and non-holonomic constraints
- ▶ A car (rigid body in 3D) is kept on the ground by 3 holonomic constraints
- ▶ One additional non-holonomic constraint prevents sideways sliding



# Coordinate Systems



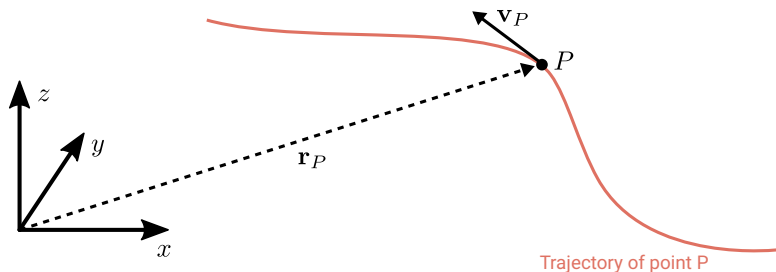
- **Inertial Frame:** Fixed to earth with vertical  $Z$ -axis and  $X/Y$  horizontal plane
- **Vehicle Frame:** Attached to vehicle at fixed reference point;  $x_v$  points towards the front,  $y_v$  to the side and  $z_v$  to the top of the vehicle (ISO 8855)
- **Horizontal Frame:** Origin at vehicle reference point (like vehicle frame) but  $x$ - and  $y$ -axes are projections of  $x_v$ - and  $y_v$ -axes onto the  $X/Y$  horizontal plane

# Kinematics of a Point

The **position**  $\mathbf{r}_P(t) \in \mathbb{R}^3$  of point  $P$  at time  $t \in \mathbb{R}$  is given by 3 coordinates.

**Velocity** and **acceleration** are the first and second derivatives of the position  $\mathbf{r}_P(t)$ .

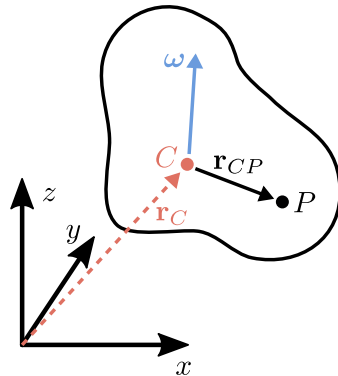
$$\mathbf{r}_P(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \quad \mathbf{v}_P(t) = \dot{\mathbf{r}}_P(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} \quad \mathbf{a}_P(t) = \ddot{\mathbf{r}}_P(t) = \begin{pmatrix} \ddot{x}(t) \\ \ddot{y}(t) \\ \ddot{z}(t) \end{pmatrix}$$



# Kinematics of a Rigid Body

A **rigid body** refers to a collection of infinitely many infinitesimally small mass points which are rigidly connected, i.e., their relative position remains unchanged over time. It's **motion** can be compactly described by the motion of an (arbitrary) reference point  $C$  of the body plus the relative motion of all other points  $P$  with respect to  $C$ .

- ▶  $C$ : Reference point fixed to rigid body
- ▶  $P$ : Arbitrary point on rigid body
- ▶  $\omega$ : Angular velocity of rigid body
- ▶ Position:  $\mathbf{r}_P = \mathbf{r}_C + \mathbf{r}_{CP}$
- ▶ Velocity:  $\mathbf{v}_P = \mathbf{v}_C + \omega \times \mathbf{r}_{CP}$
- ▶ Due to rigidity, points  $P$  can only rotate wrt.  $C$
- ▶ Thus a rigid body has 6 DoF (3 pos., 3 rot.)



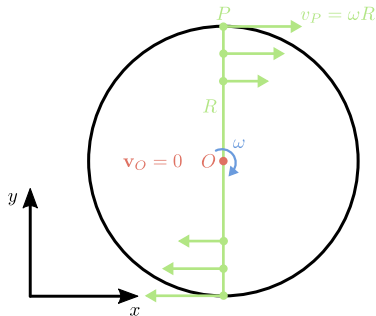
# Instantaneous Center of Rotation

At each time instance  $t \in \mathbb{R}$ , there exists a particular reference point  $O$  (called the **instantaneous center of rotation**) for which  $\mathbf{v}_O(t) = 0$ . Each point  $P$  of the rigid body performs a pure rotation about  $O$ :

$$\mathbf{v}_P = \mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{r}_{OP} = \boldsymbol{\omega} \times \mathbf{r}_{OP}$$

## Example 1: Turning Wheel

- ▶ Wheel is completely lifted off the ground
- ▶ Wheel does not move in  $x$  or  $y$  direction
- ▶ Ang. vel. vector  $\boldsymbol{\omega}$  points into  $x/y$  plane
- ▶ Velocity of point  $P$ :  $v_P = \omega R$  with radius  $R$



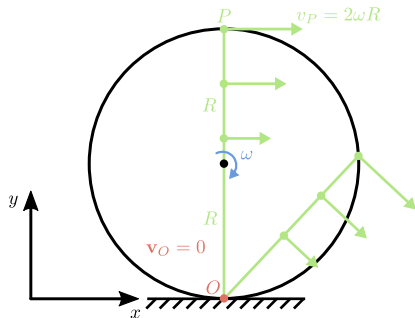
# Instantaneous Center of Rotation

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$$\mathbf{v}_P = \mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{r}_{OP} = \boldsymbol{\omega} \times \mathbf{r}_{OP}$$

## Example 2: Rolling Wheel

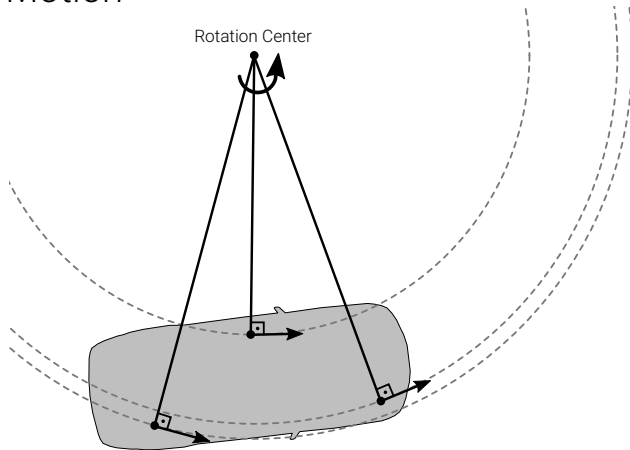
- ▶ Wheel is rolling on the ground without slip
- ▶ Ground is fixed in  $x/y$  plane
- ▶ Ang. vel. vector  $\boldsymbol{\omega}$  points into  $x/y$  plane
- ▶ Velocity of point  $P$ :  $v_P = 2\omega R$  with radius  $R$





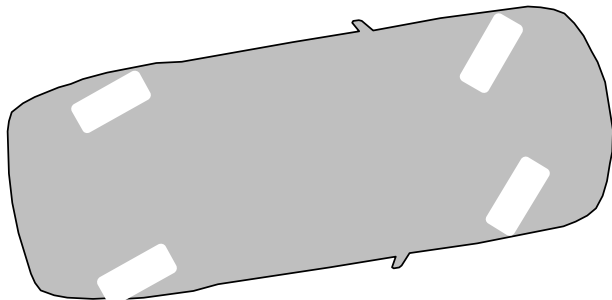


# Rigid Body Motion



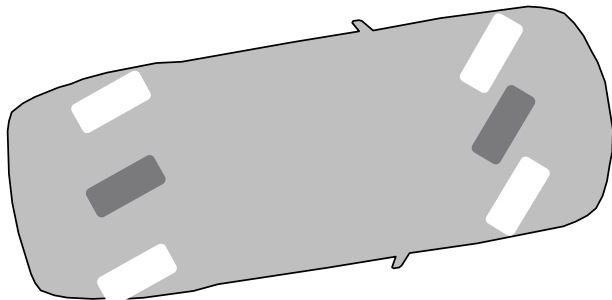
- Different points on the rigid body move along different circular trajectories

# Kinematic Bicycle Model



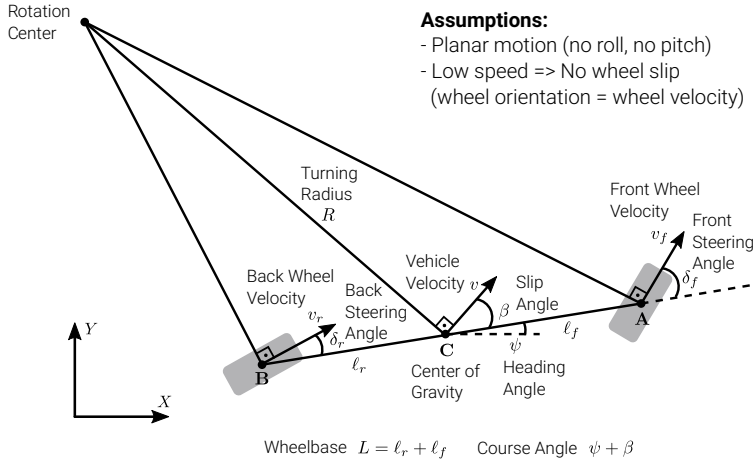
- The **kinematic bicycle model** approximates the 4 wheels with 2 imaginary wheels

# Kinematic Bicycle Model



- The **kinematic bicycle model** approximates the 4 wheels with 2 imaginary wheels

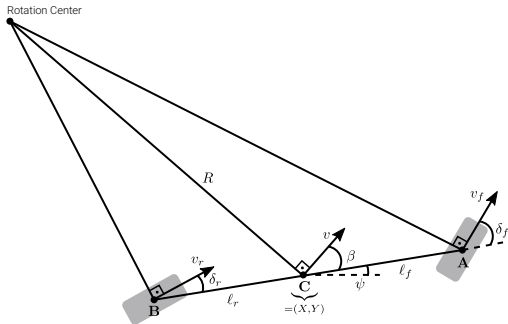
# Kinematic Bicycle Model



► The **kinematic bicycle model** approximates the 4 wheels with 2 imaginary wheels

# Kinematic Bicycle Model

## Model



## Motion Equations

$$\dot{X} = v \cos(\psi + \beta)$$

$$\dot{Y} = v \sin(\psi + \beta)$$

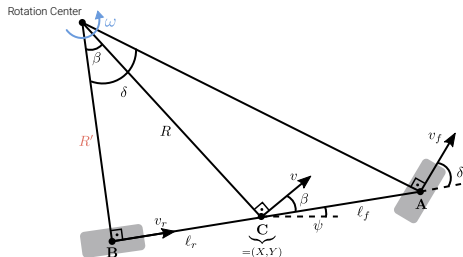
$$\dot{\psi} = \frac{v \cos(\beta)}{\ell_f + \ell_r} (\tan(\delta_f) - \tan(\delta_r))$$

$$\beta = \tan^{-1} \left( \frac{\ell_f \tan(\delta_r) + \ell_r \tan(\delta_f)}{\ell_f + \ell_r} \right)$$

(proof as exercise)

# Kinematic Bicycle Model

## Model



## Motion Equations

$$\dot{X} = v \cos(\psi + \beta)$$

$$\dot{Y} = v \sin(\psi + \beta)$$

$$\dot{\psi} = \frac{v \cos(\beta)}{\ell_f + \ell_r} \tan(\delta)$$

$$\beta = \tan^{-1} \left( \frac{\ell_r \tan(\delta)}{\ell_f + \ell_r} \right)$$

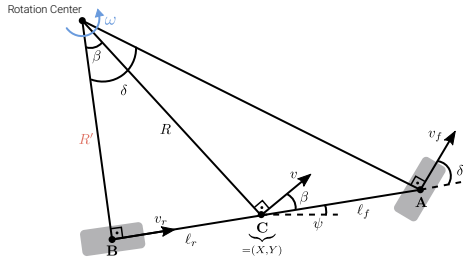
(only front steering)

$$\tan \delta = \frac{\ell_f + \ell_r}{R'} \Rightarrow \frac{1}{R'} = \frac{\tan \delta}{\ell_f + \ell_r} \Rightarrow \tan \beta = \frac{\ell_r}{R'} = \frac{\ell_r \tan \delta}{\ell_f + \ell_r}$$

$$\cos \beta = \frac{R'}{R} \Rightarrow \frac{1}{R} = \frac{\cos \beta}{R'} \Rightarrow \dot{\psi} = \omega = \frac{v}{R} = \frac{v \cos(\beta)}{R'} = \frac{v \cos(\beta)}{\ell_f + \ell_r} \tan(\delta)$$

# Kinematic Bicycle Model

## Model



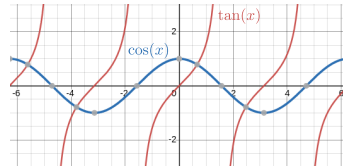
## Motion Equations

$$\dot{X} = v \cos(\psi)$$

$$\dot{Y} = v \sin(\psi)$$

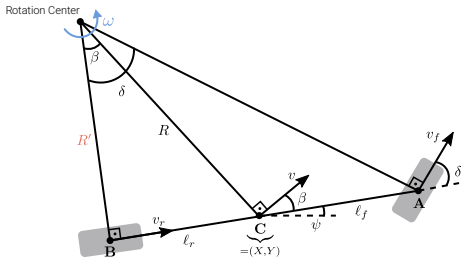
$$\dot{\psi} = \frac{v\delta}{l_f + l_r}$$

(assuming  $\beta$  and  $\delta$  are very small)



# Kinematic Bicycle Model

## Model



## Motion Equations

$$X_{t+1} = X_t + v \cos(\psi_t) \Delta t$$

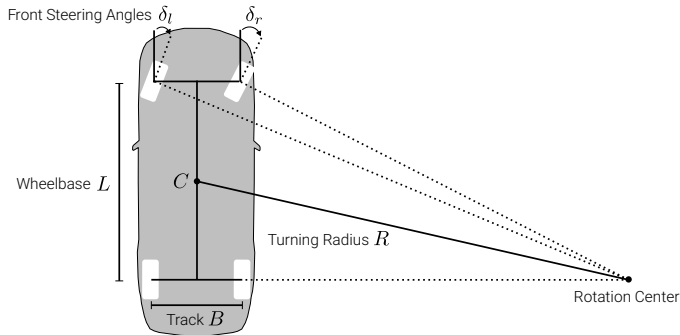
$$Y_{t+1} = Y_t + v \sin(\psi_t) \Delta t$$

$$\psi_{t+1} = \psi_t + \frac{v \delta}{\ell_f + \ell_r} \Delta t$$

(time discretized model)



# Ackermann Steering Geometry

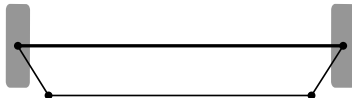


- In practice, the left and right wheel steering angles are not equal if no wheel slip
- Combination of admissible steering angles called Ackerman steering geometry
- If angles are small, the left/right steering wheel angles can be approximated:

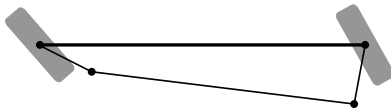
$$\delta_l \approx \tan\left(\frac{L}{R + 0.5B}\right) \approx \frac{L}{R + 0.5B} \qquad \delta_r \approx \tan\left(\frac{L}{R - 0.5B}\right) \approx \frac{L}{R - 0.5B}$$

# Ackermann Steering Geometry

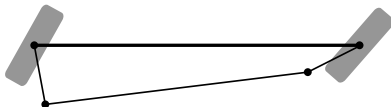
Trapezoidal Geometry



Left Turn



Right Turn



- In practice, this setup can be realized using a trapezoidal tie rod arrangement

## 5.3

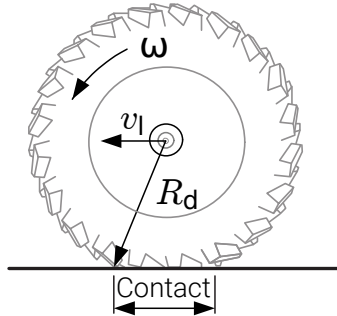
### Tire Models

Kinematics is not enough ..



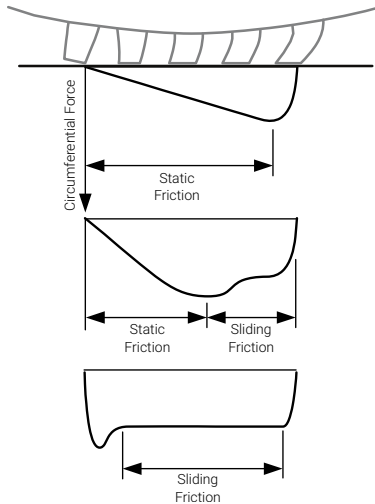
Which assumption of our model is violated in this case?

# Tire Models



- ▶ Tire models describe the lateral and longitudinal forces at the tires
- ▶ There exist many different tire models at various levels of complexity
- ▶ For a simple qualitative description we consider the **tread block model**
- ▶ **Question:** Why do tires “slip”?

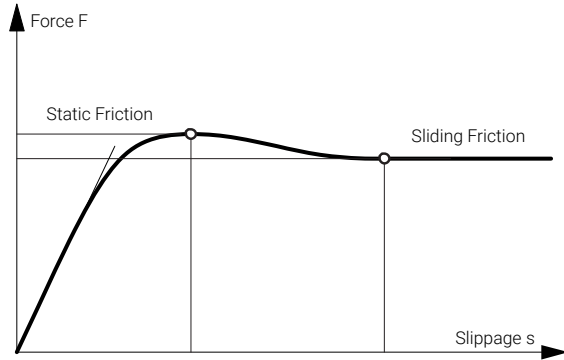
# Tread Block Model



## Longitudinal Force:

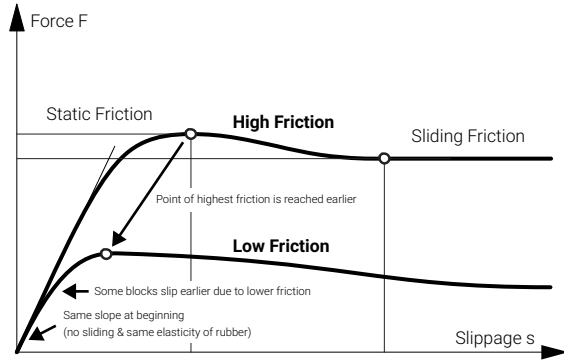
- ▶ As soon as the wheel is driven externally, the **tire tread blocks** start deforming and slipping
- ▶ The tire tread blocks adhere to the ground, **deform** and **slip** when losing contact
- ▶ When the driving force increases and static friction is exceeded the **blocks slip earlier**
- ▶ As **sliding friction** is smaller than **static friction**, this decreases the transmitted driving force
- ▶ If the tire tread blocks start sliding at the beginning, only **sliding friction** can be applied

# Tread Block Model



- ▶ **Slippage:** Difference between surface speed of the wheel and vehicle speed
- ▶ The force  $F$  grows **linearly** with the slippage  $s$  in the beginning (linear deform.)
- ▶ Large slippage  $s$  leads to a **reduction** of  $F$  (sliding friction  $<$  static friction)

# Tread Block Model

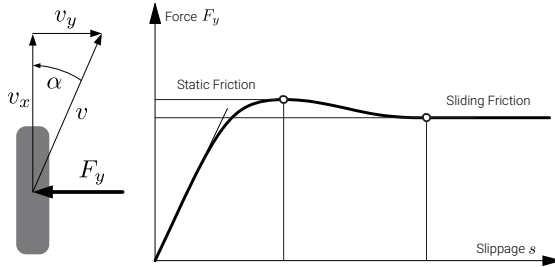


How does the force curve  $F(s)$  change for **slippery terrain** (low friction)?

- ▶ Start of the curve doesn't change as the elasticity of the blocks doesn't change
- ▶ However, the **maximum reduces** due to the decreased static friction, i.e., the tread blocks start sliding earlier due to a decrease in friction



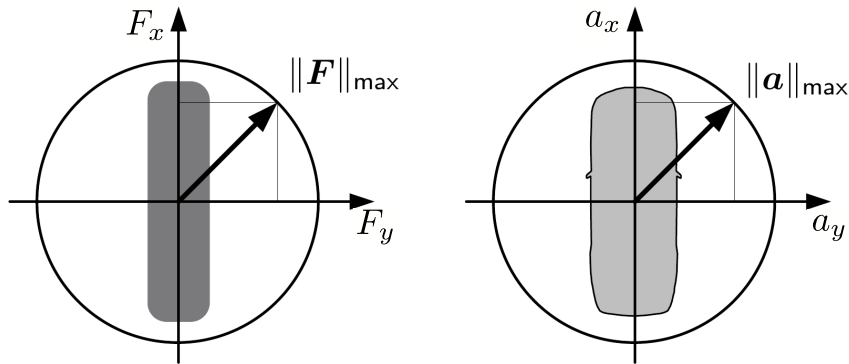
# Tread Block Model



**Lateral Force:** (the figure above shows a top view onto a wheel)

- ▶ Lateral force  $F_y$  analogous to longitudinal force but blocks move laterally now
- ▶ Lateral force for small  $s$  and  $\alpha$  given by:  $F_y = c s = c \tan(\alpha) \approx c \alpha$
- ▶  $v$  = wheel velocity,  $v_x$  = longitudinal vel.,  $v_y$  = lateral vel.,  $c$  = cornering stiffness

# Circle of Forces



## Circle of Forces:

- ▶ Lateral  $F_y$  and longitudinal  $F_x$  force cannot exceed max. friction force  $\|F\|_{\max}$
- ▶ More long. force implies less lat. force; max. acceleration only for straight driving
- ▶ Allows to make statements about maximal possible vehicle accelerations

## 5.4

### Dynamic Bicycle Model

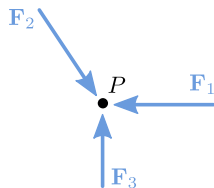
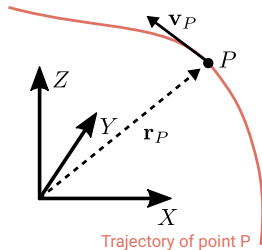
# Dynamics of a Rigid Body

## Translatory Motion of a Point:

- ▶ Consider **point**  $P$  with mass  $m$  in  $\mathbb{R}^3$
- ▶ Let  $\mathbf{r}_P(t) \in \mathbb{R}^3$  be its **position** in an inertial reference frame
- ▶ Let  $\mathbf{v}_P(t)$  denote its **velocity** and  $\mathbf{a}_P(t)$  its **acceleration**
- ▶ The **linear momentum** of  $P$  is defined as  $\mathbf{p}_P(t) = m\mathbf{v}_P(t)$
- ▶ By **Newton's second law** we have

$$\frac{d}{dt}\mathbf{p}_P(t) = m\mathbf{a}_P(t) = \mathbf{F}_{net}(t) = \sum_i \mathbf{F}_i(t)$$

where  $\mathbf{F}_i(t)$  represent all forces acting on the point mass  $P$



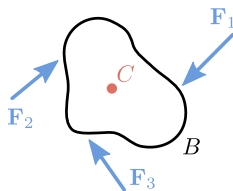
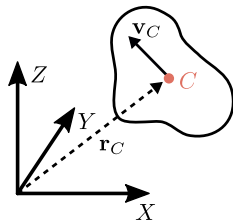
# Dynamics of a Rigid Body

## Translatory Motion of a Rigid Body:

- ▶ Consider a **rigid body**  $B$  with mass  $m$  in  $\mathbb{R}^3$
- ▶ Let  $\mathbf{r}_C(t) \in \mathbb{R}^3$  be the **position** of its **center of gravity C**
- ▶ Let  $\mathbf{v}_C(t)$  denote its **velocity** and  $\mathbf{a}_C(t)$  its **acceleration**
- ▶ The **linear momentum** of  $B$  is defined as  $\mathbf{p}_B(t) = m\mathbf{v}_C(t)$
- ▶ The **center of gravity** of a rigid body **behaves like a point mass** with mass  $m$  and as if all forces act on that point

$$\frac{d}{dt}\mathbf{p}_B(t) = m\mathbf{a}_C(t) = \mathbf{F}_{net}(t) = \sum_i \mathbf{F}_i(t)$$

where  $\mathbf{F}_i(t)$  represent all forces acting on the rigid body  $B$



# Dynamics of a Rigid Body

## Rotatory Motion of a Rigid Body:

- ▶ For the **rotatory motion**, also the geometric shape of  $B$  and the spatial distribution of its mass is important
- ▶ Let  $\rho(x, y, z)$  be the **body's density function**:

$$m = \int_B \rho(x, y, z) dx dy dz = \int_B dm$$

- ▶ The **inertia tensor** of  $B$  is defined as

$$\Theta = \begin{bmatrix} I_x & I_{xy} & I_{xz} \\ I_{yx} & I_y & I_{yz} \\ I_{zx} & I_{zy} & I_z \end{bmatrix}$$

$$I_x = \int_B (y^2 + z^2) dm$$

$$I_y = \int_B (x^2 + z^2) dm$$

$$I_z = \int_B (x^2 + y^2) dm$$

moments of inertia

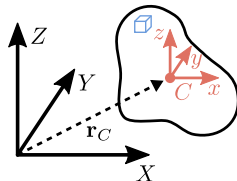
$$I_{xy} = I_{yx} = - \int_B xy dm$$

$$I_{xz} = I_{zx} = - \int_B xz dm$$

$$I_{yz} = I_{zy} = - \int_B yz dm$$

moments of deviation

$$dm = \rho(x, y, z) dx dy dz$$



# Dynamics of a Rigid Body

## Rotatory Motion of a Rigid Body:

- Let  $\omega$  be the vector of **angular velocities**:

$$\omega = (\omega_x \ \omega_y \ \omega_z)^\top$$

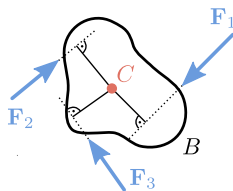
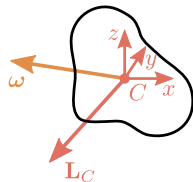
- The **angular momentum**  $\mathbf{L}_C$  of the rigid body  $B$  is given by

$$\mathbf{L}_C = \Theta \omega$$

- By the **angular momentum principle**

$$\frac{d}{dt} \mathbf{L}_C(t) = \Theta \dot{\omega} = \mathbf{M}_{net}(t) = \sum_i \mathbf{M}_i(t)$$

where  $\mathbf{M}_i(t)$  are the moments of all forces acting on  $B$  with respect to the center of gravity  $C$ .



# Dynamics of a Rigid Body

## Rotatory Motion of a Rigid Body with Canonical Coordinates:

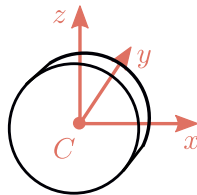
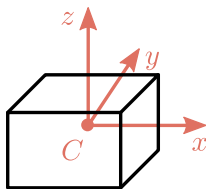
- If the body frame is chosen as a principal axis system for the rigid body (symmetry axes), the inertia tensor is diagonal:

$$\Theta = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}$$

- For the planar motion of a rigid body in the x/y-plane:

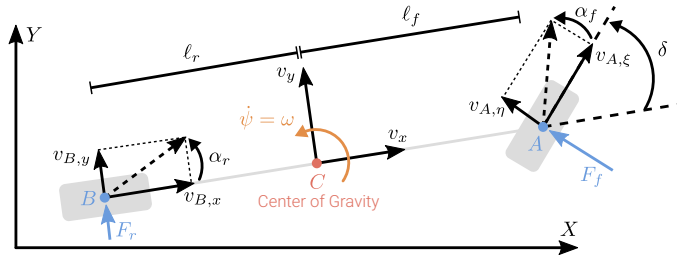
$$\omega_x = \omega_y = 0 \quad \text{and} \quad M_x = M_y = 0$$

- Hence the angular momentum becomes  $L_z = I_z \omega_z(t)$   
and the angular momentum principle yields  $I_z \dot{\omega}_z = \sum_i M_i$





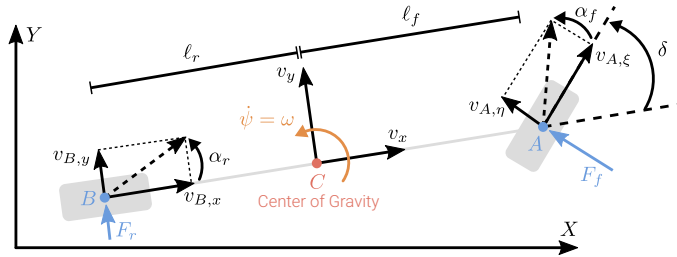
# Dynamic Bicycle Model



## Assumptions:

- ▶ The vehicle's motion is restricted to the X/Y plane
- ▶ The vehicle is considered as a rigid body
- ▶ Only lateral tire forces, generated by a linear tire model
- ▶ Small steering angle  $\delta$ :  $\sin \delta \approx \delta$   $\tan \delta \approx \delta$   $\cos \delta \approx 1$
- ▶ Constant longitudinal velocity  $v_x$

# Dynamic Bicycle Model



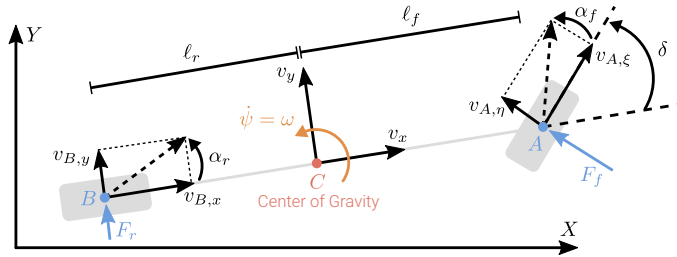
## Lateral Dynamics:

$$ma_y = \sum_i F_{y,i} = F_r + F_f \cos \delta \approx F_r + F_f$$

$$a_y = \dot{v}_y + \omega v_x \quad (\omega v_x = \text{centripetal acc.})$$

$$\Rightarrow m(\dot{v}_y + \omega v_x) = F_r + F_f$$

# Dynamic Bicycle Model

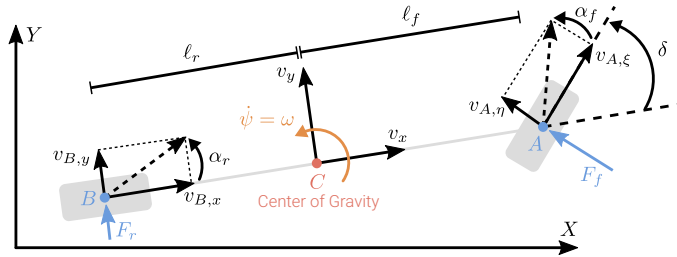


## Yaw Dynamics:

$$I_z \dot{\omega} = \sum_i M_i = -l_r F_r + l_f F_f \underbrace{\cos \delta}_{\approx 1}$$

$$\Rightarrow I_z \dot{\omega} = -l_r F_r + l_f F_f$$

# Dynamic Bicycle Model



## Tire Forces:

$$F_r = -c_r \alpha_r \approx -c_r \tan(\alpha_r) = -c_r \frac{v_{B,y}}{v_{B,x}}$$

$$F_f = -c_f \alpha_f \approx -c_f \tan(\alpha_f) = -c_f \frac{v_{A,\eta}}{v_{A,\xi}}$$

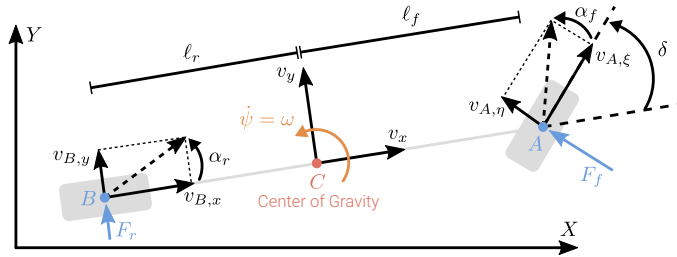
$$v_{B,x} = v_x \quad v_{B,y} = v_y - \omega \ell_r$$

$$v_{A,x} = v_x \quad v_{A,y} = v_y + \omega \ell_f$$

$$v_{A,\xi} = v_{A,x} \underbrace{\cos(\delta)}_{\approx 1} + v_{A,y} \underbrace{\sin(\delta)}_{\approx \delta}$$

$$v_{A,\eta} = -v_{A,x} \underbrace{\sin(\delta)}_{\approx \delta} + v_{A,y} \underbrace{\cos(\delta)}_{\approx 1}$$

# Dynamic Bicycle Model



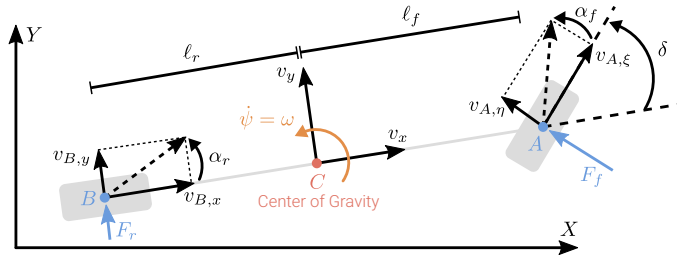
## Tire Forces:

$$F_r = -c_r \frac{v_{B,y}}{v_{B,x}} = -c_r \frac{v_y - \omega l_r}{v_x}$$

$$F_f = -c_f \frac{v_{A,\eta}}{v_{A,\xi}} = -c_f \frac{-v_x \delta + v_y + \omega l_f}{v_x + (v_y + \omega l_f) \delta} \approx c_f \delta - c_f \frac{v_y + \omega l_f}{v_x}$$

Last approximation due to:  $v_x \gg (v_y + \omega l_f) \delta$

# Dynamic Bicycle Model

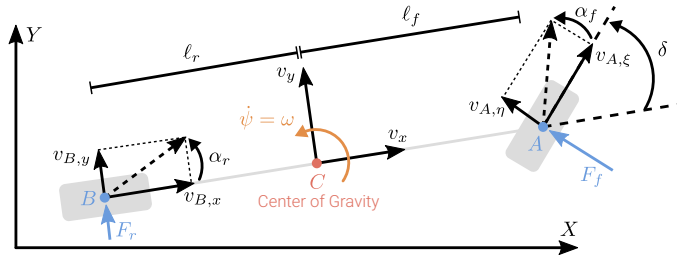


## State Space Representation:

$$m(\dot{v}_y + \omega v_x) = \underbrace{-c_r \frac{v_y - \omega l_r}{v_x}}_{=F_r} + \underbrace{c_f \delta - c_f \frac{v_y + \omega l_f}{v_x}}_{=F_f}$$

$$I_z \dot{\omega} = \underbrace{-l_r \left( -c_r \frac{v_y - \omega l_r}{v_x} \right)}_{F_r} + \underbrace{l_f \left( c_f \delta - c_f \frac{v_y + \omega l_f}{v_x} \right)}_{=F_f}$$

# Dynamic Bicycle Model



## State Space Representation:

$$\begin{bmatrix} \dot{v}_y \\ \dot{\psi} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} -\frac{c_r + c_f}{mv_x} & 0 & \frac{c_r l_r - c_f l_f}{mv_x} - v_x \\ 0 & 0 & 1 \\ \frac{l_r c_r - l_f c_f}{I_z v_x} & 0 & -\frac{l_f^2 c_f + l_r^2 c_r}{I_z v_x} \end{bmatrix} \underbrace{\begin{bmatrix} v_y \\ \psi \\ \omega \end{bmatrix}}_{\text{State}} + \underbrace{\begin{bmatrix} \frac{c_f}{m} \\ 0 \\ \frac{c_f l_f}{I_z} \end{bmatrix} \delta}_{\text{Input}}$$

Can be augmented by the global position to a nonlinear state space model

# Summary

- ▶ A vehicle can be modeled as a rigid body
- ▶ It is subject to holonomic and non-holonomic constraints
- ▶ The bicycle model approximates the vehicle using 2 wheels
- ▶ The kinematic bicycle model assumes no wheel slip (low speeds)
- ▶ However, modeling tires requires to consider slip
- ▶ Sliding friction is smaller than static friction
- ▶ We want to operate in the static friction area of the force curve
- ▶ The circle of forces tells us that lat. and long. forces are dependent
- ▶ The dynamic bicycle model takes into account tire forces and wheel slip