Prob 1

1-1. $\max_{x_j} b^*_{ij}(x_i, x_j)$ & $b_i^*(x_i)$ for max-product algorithm

$$\max_{x_j} b_{ij}(x_i,x_j) = \max_{x_j} \psi_{ij}(x_i,x_j)\cdot\phi_i(x_i)\phi_j(x_j)\cdot\prod_{k\in N(i)\setminus j} m_{k\to i}(x_i)\cdot\prod_{k\in N(j)\setminus i} m_{k\to j}(x_j)$$

$$= \phi_i(x_i)\cdot\prod_{k\in N(i)\setminus j} m_{k\to i}(x_i)\cdot\underbrace{\max_{x_j}\psi_{ij}(x_i,x_j)\cdot\phi_j(x_j)\cdot\prod_{k\in N(j)\setminus i} m_{k\to j}(x_j)}$$

This term is $m_{j\to i}$ by definition of message in max product.

$$= \phi_i(x_i)\cdot\prod_{k\in N(i)\setminus j} m_{k\to i}(x_i)\cdot m_{j\to i} = \phi_i(x_i)\cdot\prod_{k\in N(i)} m_{k\to i}(x_i) = b_i(x_i).$$

∴ proved.

1-2. $\sum_{x_j} b_{ij}(x_i,x_j)$ & $b_i^*(x_i)$ for sum-product.

$$\sum_{x_j} b_{ij}(x_i,x_j) = \sum_{x_j}\psi_{ij}(x_i,x_j)\cdot\phi_i(x_i)\cdot\phi_j(x_j)\cdot\prod_{k\in N(i)\setminus j} m_{k\to i}(x_i)\cdot\prod_{k\in N(j)\setminus i} m_{k\to j}(x_j)$$

$$= \phi_i(x_i)\cdot\prod_{k\in N(i)\setminus j} m_{k\to i}(x_i)\cdot\underbrace{\sum_{x_j}\psi_{ij}(x_i,x_j)\cdot\phi_j(x_j)\cdot\prod_{k\in N(j)\setminus i} m_{k\to j}(x_j)}$$

Like above This term is $m_{j\to i}$ in sum-product algorithm for same reason.

$$= \phi_i(x_i)\cdot\prod_{k\in N(i)\setminus j} m_{k\to i}(x_i)\cdot m_{j\to i} = \phi_i(x_i)\cdot\prod_{k\in N(i)} m_{k\to i}(x_i) = b_i(x_i).$$

∴ Proved.

Prob 1-3.

MAX-product convex belief propagation algorithm update equations of the sum-product algorithm at temperature T, as the T approaches zero by standard form of zero limit equation. Formally, If a set of messages form a fixed point of sum-product at temperature T. As $T \to 0$, then these same messages raised to the $1/T$ power, will form a fixed-point of the max-product algorithm, and Under some assumption. ex) $p(x)$ is non-negative, Bethe free energy is strictly convex, and will give global minima,

∴ Zero-temperature limit of convexified sum-product algorithms are guaranteed to solve MAP LP relaxation as $T \to 0$, free energy objective tends to the LP objective

∴ belief can be reformulated to converge to a the set of beliefs there are MAX-product calibrated and these are fixed points of the convex max-product BP algorithm. Thus, we can hope to use Max-product BP to find limiting belief (using zero-limit property)

Prob 3.

$$p(x) = \frac{1}{Z} \prod_{i \in V} \exp^{h_i x_i} \cdot \prod_{(i,j) \in E} \exp^{J_{ij} x_i, x_j}$$

Here, $h_i = 0$ and $J_{ij} = J$ for every edges $\therefore$ we can reformulate as.

$$p(x) = \frac{1}{Z} \prod_{(i,j) \in E} e^{J x_i, x_j}$$

$$\log p(x) = \sum_{(i,j) \in E} J \cdot x_i \cdot x_j - \log Z .$$

Since we are given samples from unknown Ising model over

$(-1,-1,1) \cdots ①$
$(1,-1,-1) \cdots ②$
$(1,1,1) \cdots ③$
$(-1,-1,-1) \cdots ④$
$(1,-1,-1) \cdots ⑤$

we have to consider
① 15 difference combination of x_i, x_j
~~② 5 different Z~~. ② 5 Zs

number ①~⑤ are model numbers.

$$\therefore \log p(x) = \cancel{J \cdot (-1) \cdot (-1) + J(-1)(1) + J(-1) \cdot (1) - \log Z^{①}}$$
$$+ J(1)(-1) + J(1)(-1) + J(-1) \cdot (-1) - \log Z^{②}$$
$$+ J(1)(1) + J(1)(1) + J(1)(1) - \log Z^{③}$$
$$+ J(-1)(-1) + J(-1)(-1) + J(-1)(-1) - \log Z^{④}$$
$$+ J(1)(-1) + J(1)(-1) + J(-1)(-1) - \log Z^{⑤}$$

$$- \sum_{i=①}^{⑤} \log Z^{i} = 5 \log Z .$$

$$= 9J - 6J \cancel{- \log(Z^{①} + \cdots + Z^{⑤})} \underset{-5\log Z}{} = 3J \cancel{- \log(Z^{①} + \cdots + Z^{⑤})} - 5 \log Z .$$

$$\log Z = \log \left(\sum_{|V|} \prod_{(i,j) \in E} e^{J x_i x_j} \right)$$
$$= \log 3 (2 \cdot e^{3J} + 3 \cdot e^{-J}) = \log (6 \cdot e^{3J} + 9 \cdot e^{-J})$$

$$\therefore \log p(x) = 3J - 5 \cdot \log (6 \cdot e^{3J} + 9 \cdot e^{-J})$$

$$\frac{d}{dJ} \log p(x) = 3 - 5 \cdot \frac{1}{6 \cdot e^{3J} + 9 e^{-J}} (18 \cdot e^{3J} - 9 e^{-J}) = 0 .$$

$$\Rightarrow 3 = \frac{5}{6 \cdot e^{3J} + 9 \cdot e^{-J}} (18 \cdot e^{3J} - 9 e^{-J})$$

$$\Rightarrow 1 = \frac{5}{6 \cdot e^{3J} + 9 \cdot e^{-J}} (6 \cdot e^{3J} - 3 e^{-J})$$

$$\Rightarrow 1 = \frac{5}{2 \cdot e^{3J} + 3 \cdot e^{-J}} (2 - e^{3J} - e^{-J})$$

$$2 \cdot e^{3J} + 3 \cdot e^{-J} = 10 e^{3J} - 5 e^{-J}$$
$$\Rightarrow 8 e^{3J} = 8 e^{-J}$$
$$\Rightarrow e^{4J} = 1$$
$$\Rightarrow 4J \log e = \log 1$$
$$\Rightarrow 4J = 0 .$$

$$\therefore J = 0$$