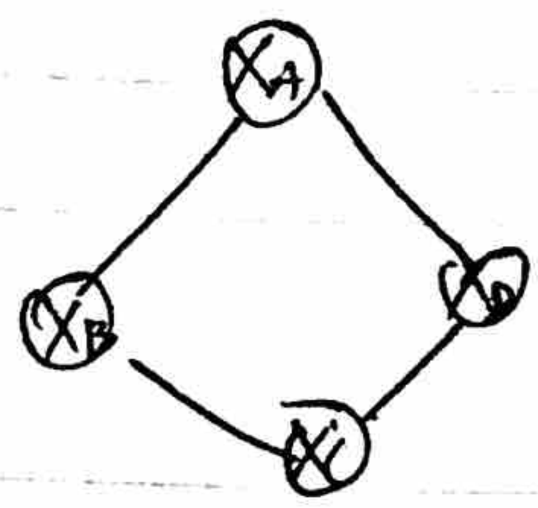
Prob 1.

(1) Let's show MAP assignment with an example. Graph.

even number verticies graph with a cycle.



Assume the joint probability $p(x)$ can factorize in this model.

$$\frac{1}{Z}\prod_{i\in V}\phi_{x_i}\cdot\prod_{(i,j)\in E}\psi_{x_i,x_j}\quad, \text{ here } \phi_{x_i}=e^{w_{x_i}} \text{ and } \psi_{x_i,x_j}=1_{x_i\neq x_j}$$

where $w_a=a \quad \forall a\in\{1,\dots,k\}$.

Converting to Integer programming to show MAP assignment. then we get

$$\max_{q\in Q}\sum_{i\in V}\sum_{x_i}q_i(x_i)\cdot x_i+\underbrace{\sum_{(i,j)}\sum_{x_i x_j}q_{ij}(x_i,x_j)\cdot\log 1_{x_i\neq x_j}}_{②}$$

$\log Z$ goes away since It is a constant value.

Here, term ② has to be 0

since if $x_i\neq x_j$ some value 1 or 0 multiply $\log 1=0$.

1 or $0\times\log 1=0$

or if $x_i=x_j$ then $0\cdot\log 0$ is 0.

Therefore, we have to show by summing up over first term

$$\max_{q\in Q}\sum_{i\in V}\sum_{x_i}q_i(x_i)\cdot x_i$$

and pick mass satisfying constraints and maximizing MAP.

$\forall i,\ \sum_{x_i}q_i(x_i)=1 \Rightarrow q_i(1)+q_i(2)+\cdots+q_i(k)=1$

If we pick one to be 1 rest will be 0.

$\sum_{x_i}q_{ij}(x_i,x_j)=q_i(x_i)$

$$\therefore \max_{q \in Q} [q_A(W_A=1)\cdot 1 + q_A(W_A=2)\cdot 2 + \cdots + q_A(W_A=k)\cdot k +$$
$$q_B(W_B=1)\cdot 1 + q_B(W_B=2)\cdot 2 + \cdots + q_B(W_B=k)\cdot k +$$
$$q_C(W_C=1)\cdot 1 + q_C(W_C=2)\cdot 2 + \cdots + q_C(W_C=k)\cdot k +$$
$$q_D(W_D=1)\cdot 1 + q_D(W_D=2)\cdot 2 + \cdots + q_D(W_D=k)\cdot k \;]$$

To maximize, we pick k for W_A, then adjacent nodes can choose only $k-1$ for corresponding nodes which secondly maximizing ~~probability~~ assignment.

$\therefore q_{X_A}(W_A=k)=1 \quad q_{X_B}(W_B=k-1)=1 \quad q_{X_D}(W_D=k-1)=1 \quad q_{X_C}(W_C=k)=1.$

$\therefore$ If $W_A=k$, $W_C=k$ then $W_B=k-1$, $W_D=k-1$.

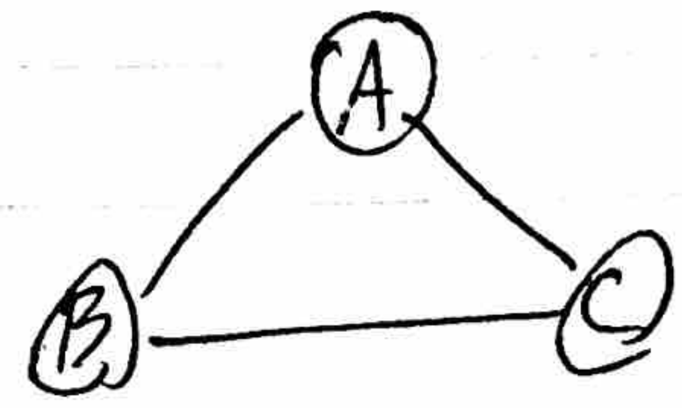
Similarly if $W_B=k, W_D=k$ then $W_A=k-1$, $W_C=k-1$.

Any one of these two is maximizing assignment

If we extend the graph with more vertices, this can be shown same way

We can show using Induction by assuming this example as true.

(2) Suppose a graph



weight $= \{1, 2, 3\}$.

For MAP IP.

$\Rightarrow$ we have $\arg\max_{q \in Q} \Big[q_{X_A}(X_A=1)\cdot 1 + q_{X_A}(X_A=2)\cdot 2 + q_{X_A}(X_A=3)\cdot 3 +$
$q_{X_B}(X_B=1)\cdot 1 + q_{X_B}(X_B=2)\cdot 2 + q_{X_B}(X_B=3)\cdot 3 +$
$q_{X_C}(X_C=1)\cdot 1 + q_{X_C}(X_C=2)\cdot 2 + q_{X_C}(X_C=3)\cdot 3 \Big]$

If we pick $X_A = 3$, maximizing X_B.

$\underbrace{q_{X_AX_B}(3,1)}_{=0} + \underbrace{q_{AB}(3,2)}_{=1} + \underbrace{q_{AB}(3,3)}_{=0} = 1$.

$\underbrace{q_{X_AX_C}(3,1)}_{=1} + \underbrace{q_{AC}(3,2)}_{=0} + \cancel{q_{AC}(3,3)} = 1$

$q_{X_AX_B}(1,2) + \cancel{q_{AB}(2,2)} + \underbrace{q_{AB}(3,2)}_{=1} = \underbrace{q_B(2)}_{=1}$

$\therefore$ we can choose $\arg\max_{q \in Q} \big[q_{X_A}(X_A=3)\cdot 3 + q_{X_B}(X_B=2)\cdot 2 + q_{X_C}(X_C=1)\cdot 1 \big]$

then $3\cdot 1 + 2\cdot 1 + 1\cdot 1 = \underline{6}$ //

For MAP LP

$\frac{2}{3}\cdot 3 + \frac{1}{3}\cdot 2 + \frac{1}{3}\cdot 3 + \frac{2}{3}\cdot 2 + \frac{1}{3}\cdot 3 + \frac{1}{3}\cdot 2 + \frac{1}{3}\cdot 1$

$= \frac{6}{3} + \frac{2}{3} + \frac{3}{3} + \frac{4}{3} + \frac{3}{3} + \frac{2}{3} + \frac{1}{3}$

$= 2 + 1 + 1 + 1 + 2 = 7.$

$\frac{1}{2}\cdot 3 + \frac{1}{2}\cdot 2 + \frac{1}{2}\cdot 3 + \frac{1}{2}\cdot 2 + \frac{1}{2}\cdot 3 + \frac{1}{2}\cdot 2 = \underline{7.5}$ //

MAP optimal solution