Propositional Logic

Lecture 2 (Chapter 7)

September 9, 2016

TU/e Technische Universiteit Eindhoven University of Technology

Table of standard equivalences

Equivalences for connectives	
Commutativity: $P \wedge Q \stackrel{val}{=\!=} Q \wedge P$, $P \vee Q \stackrel{val}{=\!=} Q \vee P$, $P \Leftrightarrow Q \stackrel{val}{=\!=} Q \Leftrightarrow P$	**Associativity:** $(P \wedge Q) \wedge R \stackrel{val}{=\!=} P \wedge (Q \wedge R)$, $(P \vee Q) \vee R \stackrel{val}{=\!=} P \vee (Q \vee R)$, $(P \Leftrightarrow Q) \Leftrightarrow R \stackrel{val}{=\!=} P \Leftrightarrow (Q \Leftrightarrow R)$
Idempotence: $P \wedge P \stackrel{val}{=\!=} P$, $P \vee P \stackrel{val}{=\!=} P$	**Double Negation:** $\neg\neg P \stackrel{val}{=\!=} P$
Inversion: $\neg\texttt{True} \stackrel{val}{=\!=} \texttt{False}$, $\neg\texttt{False} \stackrel{val}{=\!=} \texttt{True}$	**True/False-elimination:** $P \wedge \texttt{True} \stackrel{val}{=\!=} P$, $P \wedge \texttt{False} \stackrel{val}{=\!=} \texttt{False}$, $P \vee \texttt{True} \stackrel{val}{=\!=} \texttt{True}$, $P \vee \texttt{False} \stackrel{val}{=\!=} P$
Negation: $\neg P \stackrel{val}{=\!=} P \Rightarrow \texttt{False}$	**Contradiction:** $P \wedge \neg P \stackrel{val}{=\!=} \texttt{False}$ **Excluded Middle:** $P \vee \neg P \stackrel{val}{=\!=} \texttt{True}$
Distributivity: $P \wedge (Q \vee R) \stackrel{val}{=\!=} (P \wedge Q) \vee (P \wedge R)$, $P \vee (Q \wedge R) \stackrel{val}{=\!=} (P \vee Q) \wedge (P \vee R)$	**De Morgan:** $\neg(P \wedge Q) \stackrel{val}{=\!=} \neg P \vee \neg Q$, $\neg(P \vee Q) \stackrel{val}{=\!=} \neg P \wedge \neg Q$
Implication: $P \Rightarrow Q \stackrel{val}{=\!=} \neg P \vee Q$	**Contraposition:** $P \Rightarrow Q \stackrel{val}{=\!=} \neg Q \Rightarrow \neg P$
Bi-implication: $P \Leftrightarrow Q \stackrel{val}{=\!=} (P \Rightarrow Q) \wedge (Q \Rightarrow P)$	**Self-equivalence:** $P \Leftrightarrow P \stackrel{val}{=\!=} \texttt{True}$

For the collection of *all* standard equivalences, see page 372 of the book!

You will have to know them by heart (including their names!).

Start memorising them today!

TU/e Technische Universiteit Eindhoven University of Technology

Calculation

Recall the following *calculation*:

$$
\begin{array}{ll}
 & \neg P \Rightarrow Q \\
\stackrel{val}{=\!=} & \{ \text{ Implication } \} \\
 & \neg\neg P \vee Q \\
\stackrel{val}{=\!=} & \{ \text{ Double Negation } \} \\
 & P \vee Q
\end{array}
$$

Can we conclude

$$\neg P \Rightarrow Q \stackrel{val}{=\!=} P \vee Q \ ?$$

---

1. What about applying two standard equivalences in a row? Does it preserve equivalence?
2. First step: not a *literal* application of Implication. Can we do substitutions?
3. Second step: literal application of Double Negation. Is it safe to apply standard equivalences in a larger context?

TU/e Technische Universiteit Eindhoven University of Technology

$\stackrel{val}{=\!=}$ is a decent *equivalence*

Lemma 6.1.1

1. (Reflexivity:) $P \stackrel{val}{=\!=} P$
2. (Symmetry:) If $P \stackrel{val}{=\!=} Q$, then $Q \stackrel{val}{=\!=} P$
3. (Transitivity:) If $P \stackrel{val}{=\!=} Q$ and $Q \stackrel{val}{=\!=} R$, then $P \stackrel{val}{=\!=} R$

TU/e Technische Universiteit Eindhoven University of Technology

Substitution

Substitution is the replacement of all occurrences of a 'letter' by a formula.

Examples:

1. If we substitute $Q \wedge P$ for P in the valid equivalence

$$P \Rightarrow Q \overset{val}{==} \neg P \vee Q \;,$$

then we get the valid equivalence

$$(Q \wedge P) \Rightarrow Q \overset{val}{==} \neg(Q \wedge P) \vee Q \;.$$

Technische Universiteit Eindhoven University of Technology

Substitution

Substitution is the replacement of all occurrences of a 'letter' by a formula.

Examples:

2. If we substitute $\neg R$ for Q in the valid equivalence

$$(Q \wedge P) \Rightarrow Q \overset{val}{==} \neg(Q \wedge P) \vee Q \;,$$

then we get the valid equivalence

$$(\neg R \wedge P) \Rightarrow \neg R \overset{val}{==} \neg(\neg R \wedge P) \vee \neg R \;.$$

Technische Universiteit Eindhoven University of Technology

Substitution

Substitution is the replacement of all occurrences of a 'letter' by a formula.

Examples:

3. If we (simultaneously) substitute $Q \wedge P$ for P and $\neg R$ for Q in the valid equivalence

$$P \Rightarrow Q \overset{val}{==} \neg P \vee Q \;,$$

then we get the valid equivalence

$$(Q \wedge P) \Rightarrow \neg R \overset{val}{==} \neg(Q \wedge P) \vee \neg R \;.$$

Technische Universiteit Eindhoven University of Technology

Substitution Rule

$\boxed{\text{SUBSTITUTION PRESERVES EQUIVALENCE}}$

Important remarks:

1. Substitution operates on entire equivalences
2. If you substitute for some letter P in an equivalence, then you have to replace all occurrences of P in that equivalence!

Technische Universiteit Eindhoven University of Technology

Leibniz's rule

Leibniz's rule is about the replacement of a subformula by an equivalent subformula.

Example:

From the valid equivalence

$$P \Rightarrow Q \stackrel{val}{==} \neg P \vee Q$$

we can *make new valid equivalences* by replacing $P \Rightarrow Q$ in some complex formula by $\neg P \vee Q$, for instance:

$$(\neg P \wedge (P \Rightarrow Q)) \vee R \stackrel{val}{==} (\neg P \wedge (\neg P \vee Q)) \vee R$$

TU/e Technische Universiteit Eindhoven University of Technology

Leibniz's rule

Leibniz's rule is about the replacement of a subformula by an equivalent subformula.

Schematically:

$$\frac{P \stackrel{val}{==} Q}{\cdots P \cdots \stackrel{val}{==} \cdots Q \cdots}$$

TU/e Technische Universiteit Eindhoven University of Technology

Proving tautologies—method 1

To *prove with a calculation* that P is a tautology:

Give calculation that shows $P \stackrel{val}{==} \texttt{True}$.

TU/e Technische Universiteit Eindhoven University of Technology

Proving tautologies—method 1 (example)

Prove with a calculation that $\neg(P \wedge \neg P)$ is a tautology.

We have the following calculation:

$$\begin{array}{ll} & \neg(P \wedge \neg P) \\ \stackrel{val}{==} & \{ \text{ De Morgan } \} \\ & \neg P \vee \neg\neg P \\ \stackrel{val}{==} & \{ \text{ Double Negation } \} \\ & P \vee \neg P \\ \stackrel{val}{==} & \{ \text{ Excluded Middle } \} \\ & \texttt{True} \end{array}$$

So $\neg(P \wedge \neg P)$ is a tautology.

TU/e Technische Universiteit Eindhoven University of Technology

Proving tautologies—method 2

Lemma 6.1.3
If $P \stackrel{val}{==} Q$, then $P \Leftrightarrow Q$ is a tautology, and vice versa.

To *prove with a calculation* that P is a tautology:

Give a calculation that shows $P \stackrel{val}{==} Q$.

Proving tautologies—method 2 (example)

Prove with a calculation that $\neg(Q \Rightarrow R) \Leftrightarrow (\neg R \wedge Q)$ is a tautology.

First, we establish, with a calculation, that $\neg(Q \Rightarrow R) \stackrel{val}{==} (\neg R \wedge Q)$:

$\neg(Q \Rightarrow R)$
$\stackrel{val}{==}$ { Implication }
$\neg(\neg Q \vee R)$
$\stackrel{val}{==}$ { De Morgan }
$\neg\neg Q \wedge \neg R$
$\stackrel{val}{==}$ { Double negation }
$\neg R \wedge Q$

Explanation:

Substituting Q for P and R for Q in
$P \Rightarrow Q \stackrel{val}{==} \neg P \vee Q$ (Implication)
we get, by the substitution rule:
$Q \Rightarrow R \stackrel{val}{==} \neg Q \vee R$.

(The application of this equivalence in the calculation involves an application of Leibniz.)

Proving tautologies—method 2 (example)

Prove with a calculation that $\neg(Q \Rightarrow R) \Leftrightarrow (\neg R \wedge Q)$ is a tautology.

First, we establish, with a calculation, that $\neg(Q \Rightarrow R) \stackrel{val}{==} (\neg R \wedge Q)$:

$\neg(Q \Rightarrow R)$
$\stackrel{val}{==}$ { Implication }
$\neg(\neg Q \vee R)$
$\stackrel{val}{==}$ { De Morgan }
$\neg\neg Q \wedge \neg R$
$\stackrel{val}{==}$ { Double negation }
$\neg R \wedge Q$

Explanation:

Substituting $\neg Q$ for P and R for Q in
$\neg(P \vee Q) \stackrel{val}{==} \neg P \wedge \neg Q$
(De Morgan)
we get, by the substitution rule:
$\neg(\neg Q \vee R) \stackrel{val}{==} \neg\neg Q \wedge \neg R$.

Proving tautologies—method 2 (example)

Prove with a calculation that $\neg(Q \Rightarrow R) \Leftrightarrow (\neg R \wedge Q)$ is a tautology.

First, we establish, with a calculation, that $\neg(Q \Rightarrow R) \stackrel{val}{==} (\neg R \wedge Q)$:

$\neg(Q \Rightarrow R)$
$\stackrel{val}{==}$ { Implication }
$\neg(\neg Q \vee R)$
$\stackrel{val}{==}$ { De Morgan }
$\neg\neg Q \wedge \neg R$
$\stackrel{val}{==}$ { Double negation }
$\neg R \wedge Q$

Explanation:

Substituting Q for P in
$\neg\neg P \stackrel{val}{==} P$ (Double negation)
we get, by the substitution rule:
$\neg\neg Q \stackrel{val}{==} Q$.

(The application of this equivalence in the calculation involves an application of Leibniz, and is followed by an application of Commutativity.)

Proving tautologies—method 2 (example)

Prove with a calculation that $\neg(Q \Rightarrow R) \Leftrightarrow (\neg R \wedge Q)$ is a tautology.

First, we establish, with a calculation, that $\neg(Q \Rightarrow R) \stackrel{val}{=\!=} (\neg R \wedge Q)$:

$$
\begin{array}{ll}
 & \neg(Q \Rightarrow R) \\
\stackrel{val}{=\!=} & \{ \text{ Implication } \} \\
 & \neg(\neg Q \vee R) \\
\stackrel{val}{=\!=} & \{ \text{ De Morgan } \} \\
 & \neg\neg Q \wedge \neg R \\
\stackrel{val}{=\!=} & \{ \text{ Double negation } \} \\
 & \neg R \wedge Q
\end{array}
$$

From $\neg(Q \Rightarrow R) \stackrel{val}{=\!=} \neg R \wedge Q$ it follows (by Lemma 6.1.3) that $\neg(Q \Rightarrow R) \Leftrightarrow (\neg R \wedge Q)$ is a tautology.

Logical Consequence

Recall:

$$P \stackrel{val}{=\!=} Q \quad \text{means} \quad \begin{cases} \text{(a) whenever } P \text{ is } 1, \text{ then also } Q \text{ is } 1 \\ \text{(b) whenever } Q \text{ is } 1, \text{ then also } P \text{ is } 1 \end{cases}$$

Define:

$$\boxed{P \stackrel{val}{\models} Q \quad \text{means} \quad \{ \text{ (a) whenever } P \text{ is } 1, \text{ then also } Q \text{ is } 1}$$

Pronounce $P \stackrel{val}{\models} Q$ as "P *is stronger than* Q."

P	Q
1	1
0	1/0

$P \stackrel{val}{\models} Q$: 1s are *carried over* from P to Q.

Logical Consequence (example 1)

$$\neg P \left\{ \begin{array}{c} \stackrel{val}{\models} ? \\ \stackrel{val}{\Dashv} ? \end{array} \right\} P \Rightarrow Q$$

P	Q	$\neg P$	$P \Rightarrow Q$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	0	1

extra true

So $\neg P$ is stronger than $P \Rightarrow Q$ (i.e., $\neg P \stackrel{val}{\models} P \Rightarrow Q$).

Logical Consequence (example 2)

$$P \Rightarrow Q \left\{ \begin{array}{c} \stackrel{val}{\models} ? \\ \stackrel{val}{\Dashv} ? \end{array} \right\} P \vee Q$$

P	Q	$P \Rightarrow Q$	$P \vee Q$
0	0	1	0
0	1	1	1
1	0	0	1
1	1	1	1

So $P \Rightarrow Q$ and $P \vee Q$ are incomparable.

Standard weakenings

$\wedge$-$\vee$-weakening:

$P \wedge Q \stackrel{val}{\models} P$

$P \stackrel{val}{\models} P \vee Q$

P	Q	$P \wedge Q$	P	$P \vee Q$
0	0	0	0	0
0	1	0	0	1
1	0	0	1	1
1	1	1	1	1

Also:

$P \wedge Q \stackrel{val}{\models} Q$ and

$Q \stackrel{val}{\models} P \vee Q$.

Extremes:

$\mathtt{False} \stackrel{val}{\models} P$

$P \stackrel{val}{\models} \mathtt{True}$

`False` is strongest of all
`True` is weakest of all

Basic properties of $\stackrel{val}{\models}$, $\stackrel{val}{=\!\!|}$

Lemma 7.3.1

(1a) $P \stackrel{val}{\models} P$.

(2) If $P \stackrel{val}{\models} Q$, then $Q \stackrel{val}{=\!\!|} P$, and vice versa.

(3) If $P \stackrel{val}{\models} Q$ and $Q \stackrel{val}{\models} R$, then $P \stackrel{val}{\models} R$.

Basic properties of $\stackrel{val}{\models}$, $\stackrel{val}{=\!\!|}$

Lemma 7.3.2

$P \stackrel{val}{==} Q$ if, and only if, $P \stackrel{val}{\models} Q$ and $P \stackrel{val}{=\!\!|} Q$.

So, if you need to prove $P \stackrel{val}{\models} Q$ or $P \stackrel{val}{=\!\!|} Q$ by a calculation, then it is enough to prove $P \stackrel{val}{==} Q$. But $P \stackrel{val}{\models} Q$ (or $P \stackrel{val}{=\!\!|} Q$) alone is not enough to conclude $P \stackrel{val}{==} Q$!

Basic properties of $\stackrel{val}{\models}$, $\stackrel{val}{=\!\!|}$

Lemma 7.3.4

$P \stackrel{val}{\models} Q$ if, and only if, $P \Rightarrow Q$ is a tautology.

Substitution Rule for $\stackrel{val}{\models}$, $\stackrel{val}{=\!\!|}$

The Substitution Rule also works for $\stackrel{val}{\models}$ and $\stackrel{val}{=\!\!|}$:

$\boxed{\text{SUBSTITUTION PRESERVES WEAKENING/STRENGHTENING}}$

Example

We have the following valid weakening:

$$P \wedge Q \stackrel{val}{\models} P \vee R$$

and hence, according to the Substitution Rule, if we substitute $(Q \Rightarrow R)$ for P and $(P \vee Q)$ for Q, we get another valid weakening:

$$(Q \Rightarrow R) \wedge (P \vee Q) \stackrel{val}{\models} (Q \Rightarrow R) \vee R \ .$$

TU/e Technische Universiteit Eindhoven University of Technology

Leibniz for $\stackrel{val}{\models}$, $\stackrel{val}{=\!\!|}$?

Recall Leibniz's Rule for making new equivalences:

$$\frac{P \stackrel{val}{=} Q}{\ldots P \ldots \stackrel{val}{=} \ldots Q \ldots}$$

Can we replace $\stackrel{val}{=}$ by $\stackrel{val}{\models}$ in this rule?

Examples

Note that, by $\wedge$-$\vee$-weakening, $P \wedge Q \stackrel{val}{\models} P \vee Q$. Now consider:

1. $\neg(P \wedge Q) \stackrel{val}{\not\models} \neg(P \vee Q)$;
2. $R \Rightarrow (P \wedge Q) \stackrel{val}{\models} R \Rightarrow (P \vee Q)$; and
3. $(P \wedge Q) \Rightarrow R \stackrel{val}{\not\models} (P \vee Q) \Rightarrow R$.

Conclusion: replacing $\stackrel{val}{=}$ by $\stackrel{val}{\models}$ does not yield a valid rule!

TU/e Technische Universiteit Eindhoven University of Technology

Monotonicity

We do have the following weaker variant of Leibniz's Rule:

Monotonicity:
(1) If $P \stackrel{val}{\models} Q$, then $P \wedge R \stackrel{val}{\models} Q \wedge R$
(2) If $P \stackrel{val}{\models} Q$, then $P \vee R \stackrel{val}{\models} Q \vee R$

Example:

Since $P \stackrel{val}{\models} P \vee Q$ by $\wedge$-$\vee$-weakening, we have:

$$\begin{array}{ll} & P \wedge R \\ \stackrel{val}{\models} & \{ \wedge\text{-}\vee\text{-weakening + Monotonicity} \} \\ & (P \vee Q) \wedge R \ . \end{array}$$

TU/e Technische Universiteit Eindhoven University of Technology

Example

Prove with a calculation that $\neg(P \Rightarrow Q) \Rightarrow (\neg Q \wedge (P \vee R))$ is a tautology.

First, we establish that $\neg(P \Rightarrow Q) \stackrel{val}{\models} \neg Q \wedge (P \vee R)$:

$$\begin{array}{ll} & \neg(P \Rightarrow Q) \\ \stackrel{val}{=} & \{ \text{Implication} \} \\ & \neg(\neg P \vee Q) \\ \stackrel{val}{=} & \{ \text{De Morgan} \} \\ & \neg\neg P \wedge \neg Q \\ \stackrel{val}{=} & \{ \text{Double Negation} \} \\ & P \wedge \neg Q \\ \stackrel{val}{\models} & \{ \wedge\text{-}\vee\text{-weakening + Monotonicity} \} \\ & \neg Q \wedge (P \vee R) \end{array}$$

So, by Lemma 7.3.4, the formula

$$\neg(P \Rightarrow Q) \Rightarrow (\neg Q \wedge (P \vee R))$$

is a tautology.

TU/e Technische Universiteit Eindhoven University of Technology

Formal System for Calculation

We now have a precisely defined formal system for calculating with abstract propositions:

- **standard equivalences** and **standard weakenings**;
- **inference rules** (viz. reflexivity, symmetry, transitivity, substitution, Leibniz for equality, Monotonicity for weakening)

It gives a method to prove in a structured manner that

- two abstract propositions are equivalent, or one is stronger/weaker than the other;
- an abstract proposition is a tautology.