Logic of Computer Science
Lecture 6: Soundness & Completeness of Propositional Logic

Emmanuel Kwesi Tandoh

University of Mines and Technology (UMaT)

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Knowledge | Truth | Excellence

Warm-Up Question

Think-Pair-Share

What does it mean to say a formula is "valid"? Can a valid formula be unprovable?

Learning Objectives

By the end of this lecture, you should be able to:

- State the Soundness and Completeness Theorems.
- Understand their high-level proof strategies.
- Relate these results to computer science applications.
- Discuss trade-offs between expressiveness and decidability.

Key Definitions

- **Proof System** ($\vdash$): syntactic derivations using rules.
- **Semantic Entailment** ($\models$): formula true in all models.
- **Valid:** $\models \varphi$ **Provable:** $\vdash \varphi$

Soundness Theorem

Statement: If $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$.

Soundness Principle

If $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$.
In other words, whenever *you can prove* a formula φ from assumptions Γ, then φ is *true in every interpretation* that makes all of Γ true.

Intuition

Syntactic proofs preserve semantic truth. You can't prove something false.

- Base Case: axioms are tautologies
- Inductive Step: natural deduction rules preserve truth

Visual: Soundness Flow

$$\boxed{\text{Assume } \Gamma} \longrightarrow \boxed{\text{Derive } \varphi \text{ using } \vdash}$$

Soundness: $\Gamma \vdash \varphi \Rightarrow \Gamma \models \varphi$

Completeness Theorem

Statement: If $\models \varphi$, then $\vdash \varphi$.

Completeness Principle

If $\models \varphi$, then $\vdash \varphi$.
In other words, whenever a formula φ is *valid* (true in every interpretation), it is also *provable* within our formal system.

Intuition

If something is true under all interpretations, we can eventually prove it.

- Extend to maximal consistent set
- Build a model from this set
- Use contraposition: if not provable, then not valid

Visual: Completeness Sketch

$$\boxed{\text{Assume } \models \varphi} \longrightarrow \boxed{\text{Show } \varphi \text{ is derivable}}$$

Completeness: $\models \varphi \Rightarrow \vdash \varphi$

Implications for Computer Science

- **Program Verification:** Soundness means verified properties actually hold.
- **Theorem Proving:** Completeness means no truths are unreachable (in theory).
- **SAT Solvers:** Work because propositional logic is decidable.
- **Contrast:** First-order logic is expressive but incomplete and undecidable.

Reflection Discussion

Prompt

Why would you ever use propositional logic instead of full first-order logic in verification?

Reflection Discussion

Prompt

Why would you ever use propositional logic instead of full first-order logic in verification?

- Propositional logic: faster, decidable, better tools
- First-order: more expressive, harder to automate
- **Real-world**: SAT/SMT vs. interactive provers (Coq, Isabelle)

Summary & Takeaways

- **Soundness:** If provable, then valid
- **Completeness:** If valid, then provable
- Together: Trust your proofs & reason about truth
- Enables tools like SAT solvers, type checkers, verifiers

References I

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Thank You!