CE 474 - LOGIC OF COMPUTER SCIENCE

Lecture Note 3

Motivation

Natural deduction mirrors everyday reasoning: "if I know A and $A \to B$, then I may conclude B." Mastering its rules unlocks rigorous proofs, logical clarity, and forms the backbone of automated theorem proving.

Learning Objectives

- Explain the structure and notation of a natural deduction proof.
- Apply introduction and elimination rules for $\to, \wedge, \vee, \neg$.
- Construct simple proofs from given premises to a conclusion.
- Recognize and employ common proof patterns (e.g. proof by contradiction).

1. Proof Structure & Notation

- Each line in a proof has:
  1. A *formula*.
  2. A *rule name* (e.g. "$\to$E").
  3. *Line references* used.
- **Subproofs:** Indented blocks for temporary assumptions, discharged by rules like $\to$I or $\neg$I.

2. Introduction Rules (I)

Connective	Rule	Description
$\wedge$	$\wedge$I	From A and B, infer $A \wedge B$.
$\vee$	$\vee I_1$, $\vee I_2$	From A, infer $A \vee B$; or from B, infer $A \vee B$.
$\to$	$\to$I	Assume A, derive B, then infer $A \to B$.
$\neg$	$\neg$I	Assume A, derive a contradiction ($\perp$), then infer $\neg A$.

3. Elimination Rules (E)

Connective	Rule	Description
$\land$	$\land E_1$, $\land E_2$	From $A \land B$, infer A or infer B.
$\lor$	$\lor E$	From $A \lor B$, and subproofs deriving C from A and from B, infer C.
$\to$	$\to E$	From A and $A \to B$, infer B.
$\neg$	$\neg E$	From A and $\neg A$, infer $\perp$.
$\perp$	$\perp E$	From contradiction infer any formula (explosion).

4. Worked Example: Modus Ponens & $\to$I

Goal: From

$$1.\ P \to (Q \to R), \quad 2.\ P, \quad 3.\ Q$$

derive R.

Proof. 1. $P \to (Q \to R)$ Premise

2. P Premise
3. Q Premise
4. $Q \to R$ $\to$E, 1,2
5. R $\to$E, 4,3

□

5. Proof by Cases ($\lor$E)

> If you have $A \lor B$, and you can derive C from A and also derive C from B, then you may infer C.

Skeleton:

Proof. 1. $A \lor B$ Premise

- 1.1. A Assumption
- 1.2. … derive C

- 1.1. B Assumption
- 1.2. … derive C

2. C $\lor$E, 1,2–3,4–5

□

6. Proof by Contradiction ($\neg$I & $\perp$E)

- **$\neg$I:** Assume A, derive $\perp$, then infer $\neg A$.
- **$\perp$E:** From $\perp$, infer any formula (explosion).

Example: Prove $\neg(P \wedge \neg P)$.

Proof. 1. $P \wedge \neg P$ Assumption

2. P $\wedge E_1$, 1
3. $\neg P$ $\wedge E_2$, 1
4. $\perp$ $\neg$E, 2,3
5. $\neg(P \wedge \neg P)$ $\neg$I, 1–4

□

7. Common Pitfalls & Tips

- Always *discharge* your assumptions (check each $\rightarrow$I or $\neg$I).
- Label rules precisely (e.g. "$\wedge E_1$" not just "$\wedge$E").
- Don't skip justifications—each inference must cite its premises.

8. In-Class Practice

1. From $P \wedge (Q \vee R)$ derive $(P \wedge Q) \vee (P \wedge R)$.
2. Prove $(P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P)$ using $\neg$I and $\rightarrow$I.
3. *Challenge:* From $\neg P \rightarrow \neg Q$ and Q, derive P.

9. Summary & Next Steps

- Reviewed natural deduction proof format and core rules.
- Practiced introduction/elimination for $\rightarrow, \wedge, \vee, \neg$.
- Learned proof by cases and proof by contradiction.
- **Up Next:** Formal proof system properties (soundness, completeness) and extension to predicate logic.