Introduction to Local Stability

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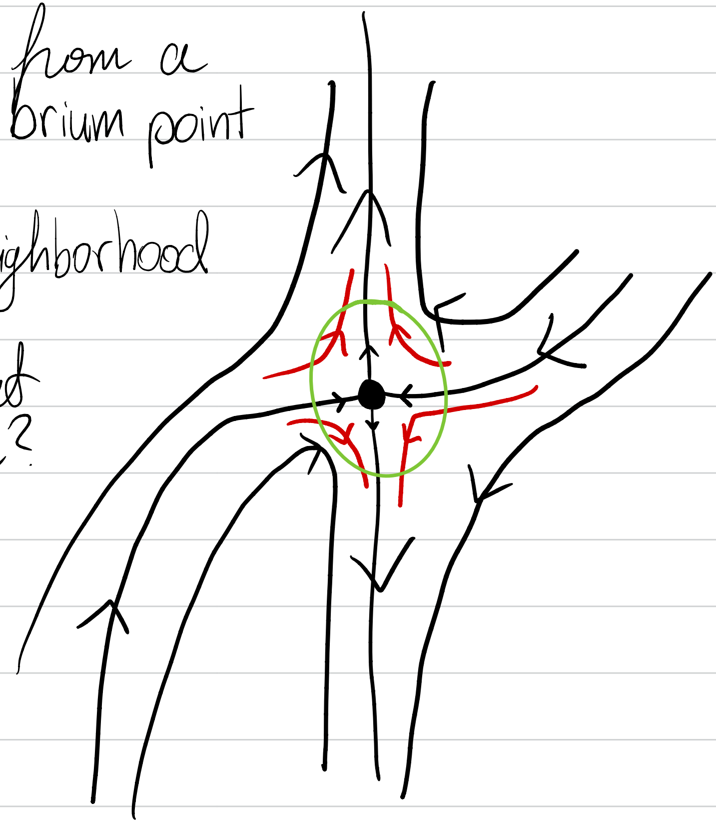
all ~~the~~ preliminary info for the Stable Manifold Theorem

Q: What can we learn from a hyperbolic fixed point/equilibrium point

- Local behavior of neighborhood
- How much information can we get and what can we demonstrate?



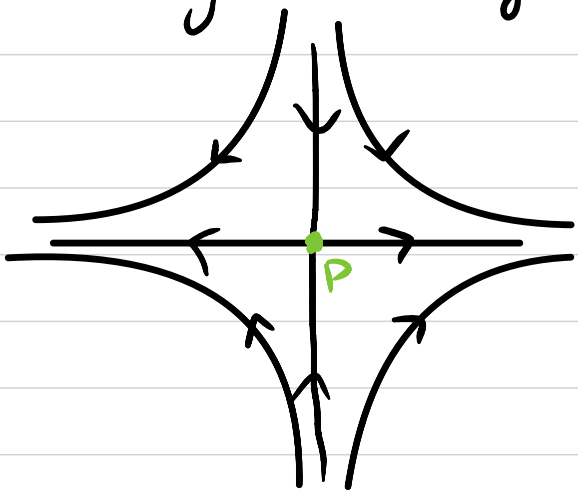
Def

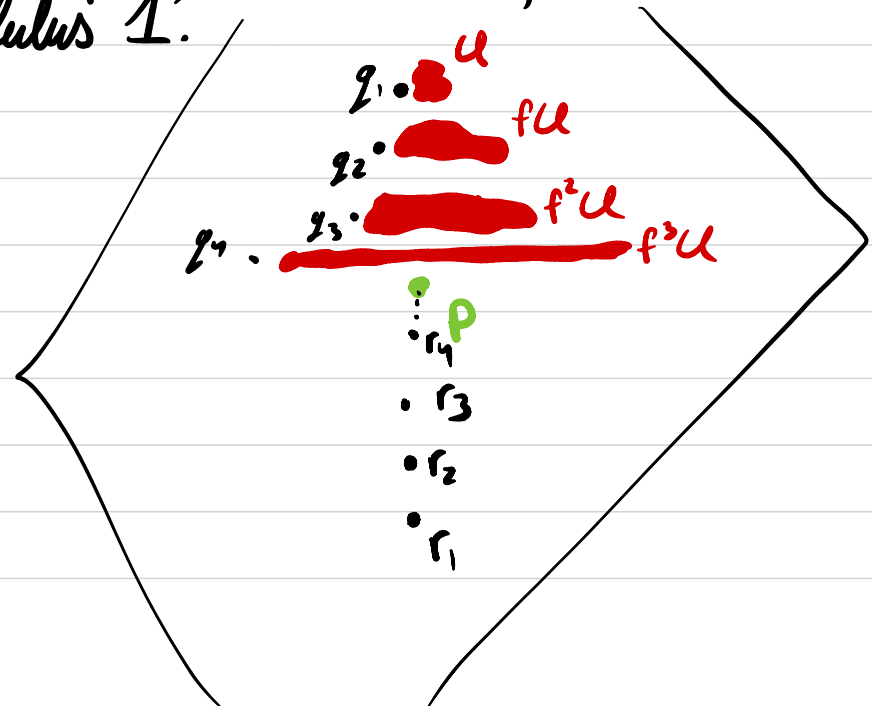
Let $X \in \mathfrak{X}^r(M)$ be a vector field of class C^r over a manifold M and let $p \in M$ be a singularity of M, that is, $X(p)=0$. Then $\varphi_t(p) = p$ $\forall t \in \mathbb{R}$, where $\varphi_t(x)$ is the flow curve passing through $x \in M$

We say p is a <u>hyperbolic singularity</u> of X if $DX_p : TM_p \to TM_p$ is a hyperbolic vector field, that is, DX_p has no eigenvalues on the imaginary axis (all have nonzero real part).

Extending to Maps

Let $p \in M$ be a fixed point of $f \in \mathrm{Diff}^r(M)$. We say that p is a hyperbolic fixed point if $Df_p : TM_p \to TM_p$ is a hyperbolic isomorphism, that is, Df_p has no eigenvalues of modulus 1.





Step 1: What can we compare locally?

Hartman-Grobman (Linearization) Theorem:

The behavior of a dynamical system in a domain near a hyperbolic fixed point is qualitatively the same as the behaviour of its linearization near that point.

Let $f \in \mathrm{Diff}^r(M)$ and $p \in M$ be a hyperbolic fixed point of f. Let $A = Df_p : TM_p \to TM_p$. Then there exist neighborhoods $V(p) \subset M$ and $U(0) \subset TM_p$ and a homeomorphism $h : U \to V$ such that

$$hA = fh.$$

* A diffeomorphism is locally conjugate to its linear part at a hyperbolic fixed point, & analogously a vector field X is locally equivalent to its linear part at a hyperbolic singularity.

Stable & Unstable "Splitting" Decomposition:

- If $L \in \mathcal{L}(\mathbb{R}^n)$ is a hyperbolic vector field then there exists a unique decomposition of $\mathbb{R}^n$ as a direct sum $\mathbb{R}^n = E^s \oplus E^u$, where E^s and E^u are invariant subspaces for L and the flow defined by L such that the eigenvalues of $L^s = L|E^s$ have a negative real part & the eigenvalues of $L^u = L|E^u$ have a positive real part.

$$x \in \mathbb{R}^n \Rightarrow x = x_s \oplus x_u; \quad x_s \in E^s \text{ and } L^s x_s \in E^s$$
$$x_u \in E^u \text{ and } L^u x_u \in E^u$$

$$Lx = \begin{pmatrix} L^s & 0 \\ 0 & L^u \end{pmatrix} \begin{pmatrix} x_s \\ x_u \end{pmatrix} \quad \begin{matrix} Re(\lambda_i) < 0 \\ Re(\lambda_j) > 0 \end{matrix}$$

Def: $W^s(p)$, $W^u(p)$

- The set $W^s(p) \subset M$ is of the points in M that have p as their omega-limit is called the stable manifold of p, ie, if $x \in W^s(p)$ then $f^n(x) \to p$ as $n \to \infty$
- Similarly $W^u(p)$ is set of points with p as α-limit, ie if $y \in W^u(p) \Rightarrow f^{-n}(y) \to p$ as $n \to \infty$.

Important Ex:

Let $GL(\mathbb{R}^n)$ be the group of invertible linear operators

- If $A \in GL(\mathbb{R}^m)$ is hyperbolic, then there is an invariant splitting $\mathbb{R}^m = E^s \oplus E^u$ st for
  - $q \in E^s$, $A^n q \to 0$ as $n \to \infty$
  - $q \in E^u$, $A^{-n} q \to 0$ as $n \to \infty$
  - otherwise, $\|A^n q\| \to +\infty$ as $n \to \infty$

Thus $W^s(0) = E^s$

$W^u(0) = E^u$

Def: B_β, local Stable/Unstable Manifolds

Suppose $M \subset \mathbb{R}^k$; for $\beta > 0$, $B_\beta \subset M$ is the open ball of size β centered at a hyperbolic fixed point p. Then the sets

$$W^s_\beta(p) = \{ q \in B_\beta : f^n(q) \in B_\beta \text{ for all } n \geq 0 \}$$

$$W^u_\beta(p) = \{ q \in B_\beta : f^{-n}(q) \in B_\beta \text{ for all } n \geq 0 \}$$

are called ~~the~~ local stable & unstable manifolds of size β, of the point p.

Manifold terminology

Topological Immersion of $\mathbb{R}^m$ in M is a continuous map $F: \mathbb{R}^m \to M$ such that every point $x \in \mathbb{R}^m$ has a neighborhood V with the following property:
$F|_V$ is a homeomorphism onto its image.
Then we say that $F(\mathbb{R}^m) \subset M$ is an immersed topological submanifold of dimension m

Topological Embedding: A top. embedding of $\mathbb{R}^m$ is an injective topological immersion which is a homeomorphism with its image

Note: If $p \in M$ is a fixed point of f then the stable manifold of p for f coincides w/ unstable manifold for f^{-1}. Then any property of the stable manifold can be translated to apply to unstable.

6.1 Proposition: Given $\beta > 0$ sufficiently small,

(1) $W^s_\beta(p) \subset W^s(p)$ and $W^u_\beta(p) \subset W^u(p)$

↳ The points in a neighborhood of p whose positive orbit remains in the neighborhood of p have p as their ω-limit

(2) $W^s_\beta(p)$ is an embedded topological disc in M whose dimension is that of the stable subspace E^s of $A = Df_p$.

(3) $W^s(p) = \bigcup_{n \geq 0} f^{-n}(W^s_\beta(p))$; and $\exists$ injective top. immersion $\varphi : E^s \to M$ whose image is $W^s(p)$.

Proof (1)+(2):

By Hartman-Grobman, we know there is a neighborhood $\mathcal{U}$ of 0 in TM_p and a homeomorphism $h: B_\beta \to \mathcal{U}$ which conjugates f and A. As A is hyperbolic then for $x \in \mathcal{U}$ st. $A^n(x) \in \mathcal{U}$ $\forall n \geq 0$ so $x \in E^s$, as otherwise x would explode out of the neighborhood $\mathcal{U}$. This means $A^n x \to 0$ as $n \to \infty$.

Furthermore, For any $q \in W^s_\beta \subset B_\beta$, since $hf = Ah$,

- $f(q) = h^{-1}Ah(q)$, and for $y = h(q)$,
- $(f \circ h^{-1})(y) = f(h^{-1}(y)) = (h^{-1}Ah)(h^{-1}(y)) = (h^{-1}A)(y)$

  so $f \circ h^{-1} = h^{-1}A$.
- Therefore $f^2 q = (f(h^{-1}Ah))(q) = (h^{-1}AAh)(q) = (h^{-1}A^2h)(q)$

  so $hf^2 = A^2h$, and hence $hf^n = A^nh$.

Cont.

Therefore since $A^n h(q) \in \mathcal{U}$ for $n \geq 0$, so $A^n h(q) \to 0$ as $n \to \infty$,

we get that $f^n(q) = h^{-1} A^n h(q) \to h^{-1}(0) = p$ as $n \to \infty$, which demonstrates $q \in W^s(p)$, therefore $W^s_\beta \subset W^s(p)$.

- Moreover, $h^{-1}(E^s \cap \mathcal{U}) = W^s_\beta(p)$ since if $x \in h^{-1}(E^s \cap \mathcal{U})$ implies $h(x) \in E^s$ and $h(x) \in \mathcal{U}$ so the orbit of $h(x)$ stays in $\mathcal{U}$, hence ~~the~~ orbit of x stays in B_β and $W^s(p)$, so $x \in W^s_\beta(p)$
- Similarly if $x \in W^s_\beta(p)$, $x \in B_\beta$ and $W^s(p)$, so $h^{-1}(x)$ remains in $h^{-1}(E^s$ and $\mathcal{U})$.

Proof (3):

- As $W^s(p)$ is invariant by f, and $W^s_\beta(p) \subset W^s(p)$, we have that the preimages of $W^s_\beta(p)$ remain in $W^s(p)$, or $\bigcup_{n\geq 0} f^{-n}(W^s_\beta(p)) \subset W^s(p)$.

- Let $q \in W^s(p) \Rightarrow \lim_{n\to\infty} f^n(q) \to p$, so $\exists n_0 \in \mathbb{N}$ st $\|f^n(q) - p\| < \beta$;

$f^n(q) \in B_\beta$ for $n \geq n_0$. Thus $f^{n_0}(q) \in B^s_\beta(p)$ and hence

$q \in f^{-n_0}(W^s_\beta(p))$. Now define a map $\varphi_s : E^s \to M$ whose image is $W^s(p)$. If $x \in E^s$, $\exists n_1 \in \mathbb{N}$ s.t. $A^{n_1} x \in \mathcal{U}$. Define $\varphi_s(x) = (f^{-n_0} h^{-1} A^{n_1})(x)$. Since h^{-1} conjugates A and f, $n_1 = n_0$ exists and φ_s is well defined, or not dependent on choice of n_0.

Cont.

Want to show φ_s is injective topological immersion of E^s in M, ie, every point $x \in E^s$ has a neighborhood $V_s \subset E^s$ s.t. $\varphi_s|_{V_s}$ is an injective homeomorphism onto its image. But we know $\varphi_s|_{V_s} = f^{-n_0} h^{-1} A^{n_0}|_{V_s}$, which is a composition of homeomorphisms and hence itself homeomorphic and thus injective.

- Suppose $x \in \varphi_s(E^s)$. Then $h f^{n_0}(x) \in A^{n_0}(E^s)$, so by the fact E^s is invariant under A,

$h f^{n_0}(x) \in E^s$, so $f^{n_0}(x) \in W^s(p)$, therefore $x \in W^s(p)$.

Similarly we can show $W^s(p) \subset \varphi_s(E^s)$, and therefore

$$W^s(p) = \varphi_s(E^s).$$

Remarks

- Although the local stable manifold is an embedded topological disk, it may not be true globally.
- Limits of Grobman-Hartman Theorem: only provides $W^s(p)$ the structure of a locally injectively immersed submanifold

By the Stable Manifold Theorem, we can show $W^s(p)$ is a differentiable immersed submanifold of the same class as the diffeomorphism.

Def's

Let S and S' be C^r submanifolds of M and let $\varepsilon > 0$. We can compare the closeness of S and S' by saying they are ε-C^r close if there exists a C^r diffeomorphism $h: S \to S'$ such that, for $i: S \to M$ and $i': S' \to M$, $i'h$ is ε-close to i in C^r topology.

$$\begin{array}{ccc} S & \xrightarrow{h} & S' \\ & {\scriptstyle i}\searrow \quad \swarrow{\scriptstyle i'} & \\ & M & \end{array}$$

$$\| i - i'h \| < \varepsilon$$

$$\| i(x) - i'h(x) \| < \varepsilon$$

Recall Implicit Function Theorem

Let $U \subset \mathbb{R}^m \times \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}^n$ a C^r map, $r \geq 1$. Let $z_0 = (x_0, y_0) \in U$ and $f(z_0) = c$. Suppose that the partial derivative w.r.t. the second argument, $D_2 f(z_0) : \mathbb{R}^n \to \mathbb{R}^n$, is an isomorphism. Then there exists open subsets $V \subset \mathbb{R}^m$ containing x_0 and $W \subset U$ containing z_0 such that, for each $x \in V$, $\exists$ unique $\xi(x) \in \mathbb{R}^m$ with $(x, \xi(x)) \in W$ and $f(x, \xi(x)) = c$. The map $\xi : V \to \mathbb{R}^n$ is defined in this way, is of class C^r, and its derivative is given by

$$d\xi(x) = [D_2 f(x, \xi(x))]^{-1} \circ D_1 f(x, \xi(x)).$$

... in Banach Spaces

Let X, Y, Z be Banach spaces, let $f: X \times Y \to Z$ be continuously Fréchet differentiable, ie, for $a \in X \times Y$, there exists a bounded linear operator $A: X \times Y \to Z$ such that

$$\lim_{\|h\| \to 0} \frac{\|f(a+h) - f(a) - Ah\|_Z}{\|h\|_{X \times Y}} = 0$$

If $(x_0, y_0) \in X \times Y$, $f(x_0, y_0) = 0$, and $y \mapsto Df(x_0, y_0)(0, y)$ is an isomorphism from Y onto Z, then there exist neighborhoods U of x_0 and V of y_0 and a fréchet differentiable function $g: U \to V$ such that

$$f(x, g(x)) = 0 \quad \text{and} \quad f(x,y) = 0$$

$$\Longleftrightarrow$$

$$y = g(x) \text{ for all } (x,y) \in U \times V.$$

More prep for SMT

Lemma 6.3: Let $A = Df(0)$, Consider the A-invariant splitting
$\mathbb{R}^m = E^s \oplus E^u$ and norms $\|\cdot\|_s$, $\|\cdot\|_u$ on E^s, E^u
such that $\|A^s\|_s < a < 1$ and $\|(A^u)^{-1}\| < a < 1$.

· On $\mathbb{R}^m$ use ~~the~~ norm $\|x_s \oplus x_u\| = \max\{\|x_s\|_s, \|x_u\|_u\}$. For
$\beta > 0$, let B_β be open ball of radius β centered at 0, and
put $B^s_\beta = B_\beta \cap E^s$, $B^u_\beta = B_\beta \cap E^u$.

Choose $\beta > 0$ so that in B_β, and some $\varepsilon > 0$ s.t. $0 < \varepsilon < \frac{1}{2}(a^{-1} - 1)$,
$f = A + \Phi$, $\Phi(0) = 0$, $\|D\Phi\| < \varepsilon$.

We will write $A = (A^s, A^u)$, $f = (f^s, f^u)$.

Lemma 6.3

If $z = (x_s, x_u)$ and $z' = (x_s, x_u')$ satisfies $f^n(z) \in B_\beta$ and $f^n(z') \in B_\beta$, $\forall n \geq 0$, then $z = z'$.

Proof: Consider the two points $y = (y_s, y_u)$ and $y' = (y_s', y_u')$ in B_β such that $\|y_s - y_s'\| \leq \|y_u - y_u'\|$.

We want to show
- $\|f^u(y) - f^u(y')\| \geq (a^{-1} - \varepsilon)\|y_u - y_u'\|$
- $\|f^s(y) - f^s(y')\| \leq \|f^u(y) - f^u(y')\|$.

$f^u(y) = A^u(y) + \phi^u(y)$, so $f^u(y) - f^u(y') = A^u(y) - A^u(y') + \phi^u(y) - \phi^u(y')$.

Since $\|D\phi\| < \varepsilon < \frac{1}{2}(a^{-1} - \beta)$, then for all $q \in B_\beta$,

$$\|Df(q)\| = \|(A + D\phi)(q)\| \leq a + \varepsilon$$

By Mean Value theorem, $\exists q' \in B_\beta$ s.t.

$f(y) - f(y') = Df(q')(y - y')$ so then ...

Since A is invariant splitting (A^s, A^u), $A^u(y)$ only changes in y_u, so $\|DA^u(y_s+y_u)\| = 0 + \|A^u(y_u)\| > a^{-1}$.

$$\underbrace{\|A^u(y)-A^u(y')\|}_{\substack{*\text{ identity on } y_s, y_s' \\ \text{so only } y_u \,\&\, y_u' \\ \text{change} \\ \leq a^{-1}|y_u - y_u'|}} - \underbrace{\|\phi^u(y)-\phi^u(y')\|}_{\leq \underset{\varepsilon}{\|D\phi\|}\,\|y-y'\|} \leq \|f^u(y)-f^u(y')\| \Rightarrow$$

because

$$\|y-y'\| = \|(y_s - y_s') \oplus (y_u + y_u')\|$$
$$= \max\{\|y_s - y_s'\|, \overset{\text{greater}}{\|y_u - y_u'\|}\}$$
$$= \|y_u - y_u'\|, \text{ then}$$

$$\|f^u(y) - f^u(y')\| \geq (a^{-1} - \varepsilon)\|y_u - y_u'\|.$$

On the other hand, Since $f^s(y)-f^s(y')=A^s(y)-A^s(y')+\Phi^s(y)-\Phi^s(y')$, we similarly get $\|f^s(y)-f^s(y')\|\leq \underbrace{\|A^s(y)-A^s(y')\|}_{\leq a\|y_s-y'_s\|}+\underbrace{\|\Phi^s(y)-\Phi^s(y')\|}_{\leq \varepsilon\|y-y'\|}$.

Since $\|y-y'\|=\|y_u-y'_u\|\geq \|y_s-y'_s\|$,

$\|f^s(y)-f^s(y')\|\leq (a+\varepsilon)\|y_u-y'_u\|$, and since $(a+\varepsilon)\leq(a^{-1}-\varepsilon)$,

[because $\varepsilon<\frac{1}{2}(a^{-1}-1)<\frac{1}{2}(a^{-1}+a)$, so $2\varepsilon<a^{-1}+a \Rightarrow (\varepsilon-a)<(a^{-1}-\varepsilon)$]

then $(a+\varepsilon)\|y_u-y'_u\|\leq (a^{-1}-\varepsilon)\|y_u-y'_u\|\leq \|f^u(y)-f^u(y')\|$.

Taking the inequalities together,

$$\|f(y) - f(y')\| = \|(f^s(y) - f^s(y')), (f^u(y) - f^u(y'))\| = \|f^u(y) - f^u(y')\|$$

$$\geq (a^{-1} - \varepsilon)\|y - y'\|.$$

- Now consider the original points z & z', let $y = f^n(z)$, $y' = f^n(z')$, for $n > 0$. Thus we get

$\|f^n(z) - f^n(z')\| \geq (a^{-1} - \varepsilon)\|z - z'\|$. We know $\varepsilon < \frac{1}{2}(\underbrace{a^{-1}}_{>1} - 1)$,

so $\varepsilon < a^{-1} - 1$, or $1 < a^{-1} - \varepsilon$. Therefore, $z - z' = 0$, as otherwise the distance $\|f^n(z) - f^n(z')\|$ would expand without bound, impossible because of hypothesis $f^n(z)$ and $f^n(z')$ belong to B_β $\forall n \geq 0$.

Coinciding of the stable manifold for a vector field + its maps

- Suppose a vector field $X \in \mathfrak{X}^r(M)$ and let $p \in M$ be a hyperbolic singularity of X.

· Let $W^s(p,X)$ be the stable manifold of p for the vector field X, the set of points of M whose ω-limit is p.

· Let $f = X_1$ be the diffeomorphism induced at time $t=1$, and $W^s(p,f)$ the stable manifold of p for f). p is a hyperbolic fixed point, and $W^s(p,f) = W^s(p,X)$.