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MAT 336
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History of the birth of Complex Analysis

Origins

Complex numbers have had a controversial relationship with mathematicians well until the exciting age of Euler. On one hand, they couldn't be interpreted by the methods available on the real line; on the other hand, they united a gap left behind by the lack of real solutions to some functions of a real variable, such as many polynomials.

For example, already in the 16th century Gerolamo Cardano, known for spreading the methods of arriving at solutions to cubic and quartic equations, also acknowledged the existence of complex numbers as solutions, but classified them as fictitious (Cajori, page 135), and Rafael Bombelli went further and showed the algebraic properties of imaginary numbers under addition and multiplication; in fact, he also explicitly laid out the rules for multiplying with negative numbers and how the sign of the numbers being multiplied affects the sign of the outcome, an important milestone for European mathematicians that were still sometimes uncomfortable with negative numbers. Bombelli wrote that, here using more modern notation in substitution for his syncopated algebra, $+\sqrt{-n} * +\sqrt{-n} = -n; +\sqrt{-n} * -\sqrt{-n} = +n; -\sqrt{-n} * -\sqrt{-n} = -n$ (O'Connor).

Complex Functions

Indeed, the initial acceptance of complex numbers was very limited, as can generally be seen by how long it took mathematics to become comfortable with even considering them, and their discovered usefulness was small. Leonhard Euler was an important figure, and since he was revered, his gradual acceptance of complex numbers encouraged others too; in fact, Euler's experimentation of extended the logarithm to negative and complex values showed the first glimpses of how powerful it can be to extend the domain of functions that are understood, and a testament to his influence was that he began the custom of writing i for $\sqrt{-1}$. That said, even his contemporaries argued with him over the use of complex numbers. The idea of considering logarithms of negative numbers was debated even in the correspondences of Leibniz and Johann Bernoulli (236, Cajori), in which the former maintained it to be impossible, since a positive logarithm corresponds to a number larger than 1, a negative logarithm to a positive number less than 1, then the logarithm of -1 was not really true; however, even supposing it did exist, then one-half of it would be equivalent to the logarithm of $\sqrt{-1}$, which Leibniz considered absurd (since $\frac{1}{2}\log a = \log \sqrt{a}$), just the very implication that it would naturally bring out imaginary values was a contradiction for Leibniz. Bernoulli disagreed, saying that if one can integrate from the proportion $dx: x$, then surely the equivalent proportion $-dx: -x$ must result in the same answer, with the resulting integral implying that $\log x = \log -x,$ and from there arguing that the logarithmic curve $y = \log{(x)}$ has two branches and symmetric to the y-axis, similar to the hyperbola (236, Cajori). As is known now, this isn't right, but its examination shows how confusing it can be to consider complex numbers and particularly their relationship to calculus.

An impressive feat in 1714 was Roger Cotes' theorem demonstrating a relationship between complex logarithms and the trigonometric functions, published in his work *Philosophical Transactions*, and utilizing modern notation, he shows that

$$i\theta = \log(\cos\theta + i\sin\theta)$$

which had been discovered once more by Euler in exponential form in 1748. More than that, Euler was quickly studying the connections of complex numbers to various areas of mathematics and was even able to demonstrate a counterexample to his teacher, Johann Bernoulli, about his conjectured idea of the logarithms of negative numbers. He also wrote a letter later to Bernoulli stating that both $y = 2\cos x$ and $y = e^{ix} + e^{-ix}$ are integrals to the differential equation $\frac{d^2y}{dx^2} + y = 0$ and equal to each other (236, Cajori). Corresponding with D'Alembert by a letter in 1747, Euler disproved D'Alembert's conclusion that $\log(-1) = 0$, and stated his result that demonstrates Euler's understanding and insight into the topic: he stated that $\log x$ has infinitely many values, all imaginary, except when x is positive, in which case one logarithm of the infinitely many values is real (237, Cajori).

Furthermore, Caspar Wessel innovated the idea of complex numbers as coordinates on a plane, followed also by the more influential Jean-Robert Argand's geometric representation of numbers in the form $x + \sqrt{-1}y$; they helped to establish the relationship of a complex value to a vector and in fact, it is thanks to Argand that we still use the term modulus of a vector or complex number to denote absolute distance from the origin $(265, \text{Cajori})$. Motivated by Euler's use of substituting in complex values when trying to solve real integral problems, Laplace, Poisson, and Legendre all added to the complex canon through their creative approaches at bringing in complex "intermediate steps," permitting it on grounds that the end result is purely real. However, Laplace was unsatisfied that their methods left something to be desired and wished to construct more rigorous ways of understanding the application of complex numbers in calculus. One must note that their contributions were serious work, and even the first appearance of the celebrated Cauchy-Riemann Equations appeared in the work of d'Alembert in his essay on hydrodynamics, *Essai d'une nouvelle théorie de la resistance des fluides*. However, it was only to describe flows in the plane and introduce imaginary quantities that, as mentioned above, always had to vanish by the end. Since the application itself didn't correspond to any foundational material on the subject, it wasn't studied as such (page 60, Gray).

Analysis and Cauchy

One shortcoming in the growth of Complex Analysis is due to Carl Friedrich Gauss's prudence and capabilities of working in a vacuum: he was enough of a powerhouse to generate ground-breaking innovations, but by keeping his accomplishments to himself, usually out of patience that his mathematics hadn't been refined enough to publish, he didn't spread his ideas and they tended to be reinvented, sometimes decades later, sometimes for him to claim propriety at the expense of the thinker that worked hard to get there. Though not to that extreme, such is the case with his discussion of the closed line integral of 1/z around 0 (Crelle Journal, pg 91), the simplest nontrivial case of the residue integral theorem, which he wrote about in a private letter to a friend, Friedrich Bessel, all the way back in 1811, and Gauss gave the correct answer of $2\pi i$. However, he hadn't spread his results publicly nor the underlying foundations that allowed him to construct the answer, and so, however powerful and innovative it might have been, its discussion cannot be written about at length since it is just lacking in material to talk about, and speculation is unnecessary when there are other mathematicians whose work did survive, and their influence significant.

The story we will focus on continues with Cauchy's work, as he is generally considered to jumpstart the rigorous formalization of complex analysis. In fact, in his 1814 *Memoir* writing out how to define a complex derivative he managed to put down on paper equations that could, through substitution and rearrangement, can be simplified to the Cauchy-Riemann Equations, but since he did not yet record their significance, it is generally treated that he 'missed' this relationship. What he did do was outline the apparent validity of the equation

$$\frac{\partial\left[f(y)\frac{\partial y}{\partial x}\right]}{\partial z}=\frac{\partial\left[f(y)\frac{\partial y}{\partial x}\right]}{\partial x}$$

Going on with the substitution $y = M(x,z) + i * N(x,z), f(y) = P' + i * P''$,

$$\frac{\partial S}{\partial z} = \frac{\partial U}{\partial x}, \frac{\partial T}{\partial z} = \frac{\partial V}{\partial x}$$

Where

$$S = P' \frac{\partial M}{\partial x} - P'' \frac{\partial N}{\partial x}, U = P' \frac{\partial M}{\partial z} - P'' \frac{\partial N}{\partial x}$$

$$T = P' \frac{\partial N}{\partial x} + P'' \frac{\partial M}{\partial x}, V = P' \frac{\partial N}{\partial z} + P'' \frac{\partial M}{\partial z}$$

From there Cauchy stated that the equations above hold generally, and went on to demonstrate with several examples. However, as mentioned above, Cauchy missed that by simplifying through setting $M(x,z) = x, N(x,z) = z$, we can obtain the familiar pair of differential equations (Gray 60),

$$\frac{\partial P'}{\partial x} = \frac{\partial P''}{\partial z}, \frac{\partial P'}{\partial z} = -\frac{\partial P''}{\partial x}.$$

However, the story continues with the investigation of the theory of double integrals, for Euler and Laplace stated that the order of the integrands can be switched for free, and it was Cauchy who was able to demonstrate that this is true if the expression that is to be integrated doesn't become indeterminant in the given interval. This motivated Cauchy to study the relationship of integrating two separate paths with a pole in between them over the complex plane; by pole we need to be careful about how we define it as to avoid non-chronically using ideas that weren't exactly the same at that time, but the important fact Cauchy realized is that a pole is indeterminant and breaks continuity. Already in 1825 Cauchy found that when there is no pole or infinity within the closed curve of two closed integrals, then the two integrals arrive at the same value (Gray 112).

Cauchy was able to demonstrate in 1846 that if X, Y are continuous functions of x and y within a closed area, then $\int Xdx + Ydy = \pm \iint \left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right) dxdy$, where the left-hand side extends over the boundary, and the right-hand side over the inner area of the complex plane. By considering the integration first along a closed path surrounding a pole, and later along a closed path that surrounds a line on which the function is discontinuous, he found where things can break down. In 1837 Cauchy stated his momentous theory for series, that a function can always be expanded into a series in x as long as the modulus of x is less than that for which the function stops being finite and continuous. In 1840, Cauchy gave a proof that just rested on the mean value theorem. Later, Cauchy went so far as to demonstrate that more than continuity is demanded, in fact, the first derivative must be continuous (Cajori, 420). It was only as late as 1851 that Cauchy realized that the core of his theory on complex functions was not in the concept of continuity, but analyticity connected to the Cauchy-Riemann equations. By putting this condition on the functions to which the theory corresponded to, he entirely eliminated the uncertainty about the right formulation of the expansion into series: he managed to find the right hypothesis needed to ensure analyticity, the existence of a unique derivative independent from the path. With all this information and investigating the limit of a complex function $f(x + iy)$ he finally arrived at the conditions for the statement that $f(t) = \frac{1}{2\pi i} \int \frac{f(z)dx}{z-t}_{rect}$ $(Gray\ 113)$.

Riemann’s Analysis

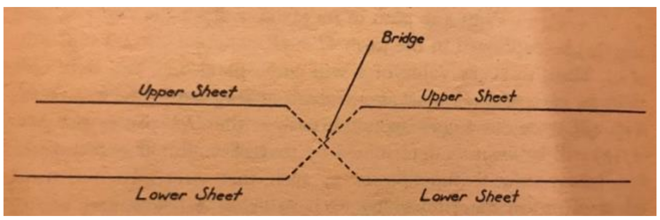
In the dissertation of Bernhard Riemann, Riemann made a similar realization as Cauchy about the connection of differentiation being the core concept of complex analysis in a similar fashion to the role of continuity in real analysis, in the sense that a class of functions can be studied with rigor and without ambiguity. He proved the differential relationship of the Cauchy-Riemann equations and their implications for the differentiability of a complex function w, and gave the shorthand $\frac{\partial w}{\partial y} = i\frac{\partial w}{\partial x}$ (Laugwitz, page 69). On the other hand, Riemann set his eyes on the study of functions that follow the partial differential equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \Delta u = 0$, known as Laplace's Equation, which holds for a given analytic function $w(z) = u(x, y) + iv(x, y)$, for the complex variable $z = x + iy$. He wanted to develop the theory of potential, which had mostly been used in mathematical physics, but apply its ideas to pure mathematics, and they happened to fit into place well within the foundational theory of functions of a complex variable. For example, it had been stated by Peter Gustav Lejeune Dirichlet that for a plane there always is a unique function of two variables, $f(x, y)$, which satisfies $\Delta u = 0$ and which, together with the first and second order differential quotients, is for all values of x, y within a defined area single-valued and continuous, and for all boundary points of this area it has arbitrarily given values. Riemann called this *Dirichlet's Principle*, and based his theory on it, but one should note that the same conclusions were stated by George Green and Sir William Thomas (Gray, 145). The consequence of this principle is that if u is arbitrarily given for all points on a curve, and if v is given for one point within the curve, then the function w is uniquely determined for all points within a closed surface.

However, the use of *Dirichlet's Principle* was suspicious to some of Riemann's contemporaries, with criticism led by Karl Weierstrass. The core issue people expressed with the principle is that it provides a function that minimizes a particular integral, in physical nature, this claim seems reasonable enough: nature tends to follow a path of equilibrium that minimizes a specific quantity; in mathematics, on the other hand, it needs to be proven a minimum exists. In fact, there are three statements that need to be proven: (i) there exist functions that extend the function defined on the surface to the whole of the domain in question, (ii) there are functions of this kind for which the integral is finite, (iii) there are functions of this kind that minimize the integral. Dirichlet asserts that “It is evident that this integral has a minimum because it cannot be negative,” but that doesn’t conclusively prove anything: the function $f(x) = x$ is non-negative on $(0,1)$ but it does not attain a minimum on this domain (Gray, 146). Riemann attempted to prove it and generalize it to cases where prescribed discontinuities were permitted. He did so by constructing an integral over a surface T such that

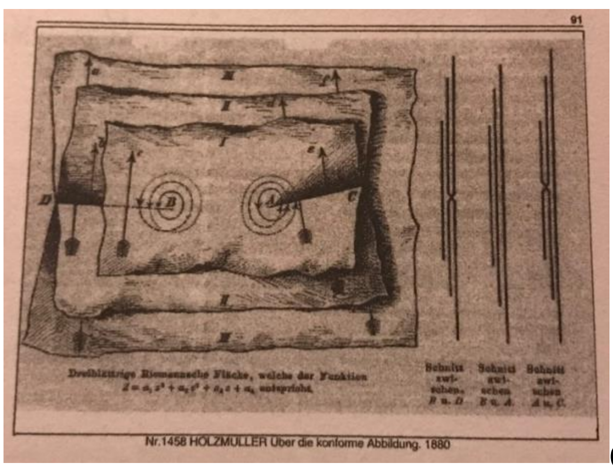
$$L(\alpha, \beta) = \int \left(\frac{\partial \alpha}{\partial x} - \frac{\partial \beta}{\partial y}\right)^2 + \left(\frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x}\right)^2 dT$$

Riemann required the integral to be finite for arbitrary $\alpha(x, y), \beta(x, y)$, and in fact the integral vanishes for such functions satisfying the Cauchy-Riemann equations. He argues that through varying $\alpha(x, y)$ by a continuous function with discontinuities only at single isolated points and being zero everywhere on the boundary of the surface T, the integral can achieve a minimal value. In notation, given such a function λ that vanishes on the boundary and with only isolated discontinuities, and with an integral $L(\lambda) = \int \left(\frac{\partial \lambda}{\partial x}\right)^2 + \left(\frac{\partial \lambda}{\partial y}\right)^2 dT$, then considering $\Omega = L(\alpha + \lambda, \beta)$, Riemann states that “The totality of these functions represents a connected domain closed in itself, in which each function can be transformed continuously into every other, and a function cannot approach indefinitely closely to one which is discontinuous along a curve without $L(\lambda)$ becoming infinite” (Riemann 1851, in Gray 169). For each λ, Ω becomes infinite only with L, and that depending continuously with λ and always nonnegative, then there must be at least one minimum (Gray 169). This was the controversial part, afterwards, it is possible to show deductively that the minimizing function is unique and harmonic. The argument isn’t rigorous enough to satisfy a definitive proof, but it was certainly an attempt, and one cannot say that Riemann followed the *Dirichlet Principle* blindly or on a whim.

Riemann was able to insightfully pierce into how this conclusion applied to complex analysis, and to expand on the technicalities he came up with the famous Riemann surfaces. In his own words, "[These] principles open the way to the study of definite functions of a complex variable independent of an expression of it" (Riemann 1851, 19, in Gray 170). To hold on to the observed continuity of complex functions except at singular points, and make sense of it for cases where the function w has, say, n values for a single value of z, then he devised an n-sheeted (or n-planed) surface glued at its branch points, meaning at some set of points where the values form a connection from one sheet to another, then, the n sheets together form a connected surface, which can be viewed as a stitched-together version of a singly-connected surface which is sown in at the mentioned branch-cuts (Cajori, 422).



(Figure 1, Bell, page 494)



(Figure 2, Holzmuller in Gray, page 173)

In the figures above, illustrations of a Riemann surface are shown to represent the surfaces, since a multivalued function can have different outputs for one input, this is illustrated through the layered sheets; moreover, the surface is still continuous except at the branch point.

Figure 1 is more of an abstract representation to characterize the structure, while Figure 2, drawn by the Mathematician Gustav Holzmuller, is a pictorial interpretation of how a surface with two branch points of order three would look like, drawn back in 1880. Holzmuller was known for providing pictures to illustrate theorems and concepts (Gray, 173).

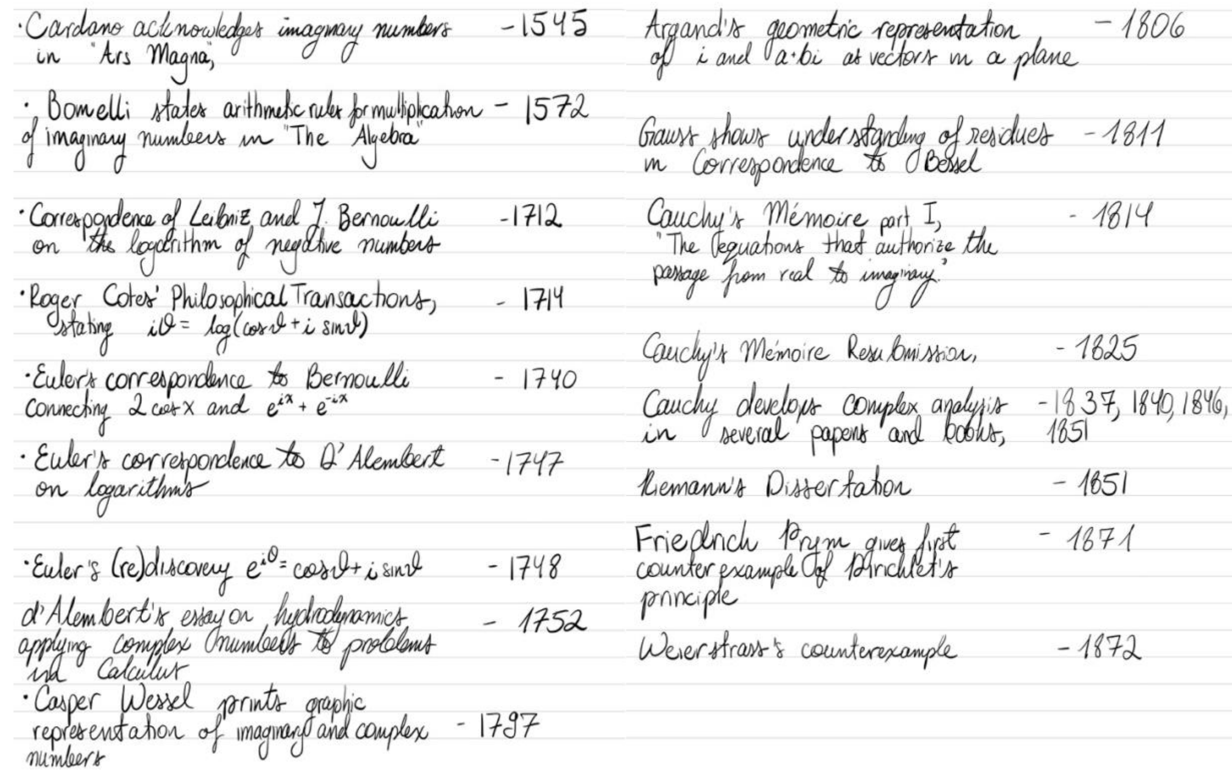
As for the contention of the *Dirichlet Principle*, interestingly enough it was Riemann's own student Friedrich Prym in 1871 (Gray, 186) that first refuted the claim with a counterexample, but most contemporaries referred to Weiertrass's counterexample of an integral bounded below that does not attain its bound, thus showing that the principle's argument is faulty in general. Unfortunately, even though neither counterexample directly disproves the principle, because the form of the sample integral is different, it did successfully challenge the assumption stated in the principle, and because the principle was so widespread in Riemann's work then once it was called into question, the rest of his work was cast in a shadow of doubt. Weierstrass showed that

$$J = \int_{-1}^{1} \left(x \frac{d\phi}{dx} \right)^2 dx \,, \phi(-1) = a \neq b = \phi(1)$$

is always positive and can take any value greater than zero but does not vanish unless $\frac{d\phi}{dx} = 0$ on the entire interval $[-1,1]$, which is not possible by the inclusion of the boundary values of ϕ (Gray, page 187).

Closing Remarks

The understanding of complex analysis had really grown throughout the 19th century, and while there is so much deeper to go, even with the topologic characterization of Riemann surfaces and how his theory survived controversy through the next generation of mathematicians, or how complex functions can instead be characterized with grueling analytic detail, it is no longer in the first chapter of how Complex Analysis arose and its foundational history. Where the theory is left about halfway into the 1800s would be nearly unrecognizable to those who first thought hard about complex numbers, but every person in that extensive timeline held their own contribution into this beautiful and tricky subject. To clarify the jumbled dates, figure 3 below organizes the timeline of the contributions discussed in the paper, with the same source used as when mentioned in the paper. A final, positive remark is that Riemann's use of the Dirichlet principle was justified by Poincaré and Hilbert at the dawn of the 20th century (Friedel, 2018).



- Cardano acknowledges imaginary numbers in "Ars Magna" - 1545
- Bombelli states arithmetic rules for multiplication of imaginary numbers in "The Algebra" - 1572
- Correspondence of Leibniz and J. Bernoulli on the logarithm of negative numbers - 1712
- Roger Cotes' Philosophical Transactions, stating $i\theta = \log(\cos\theta + i\sin\theta)$ - 1714
- Euler's correspondence to Bernoulli connecting $2\cos x$ and $e^{ix} + e^{-ix}$ - 1740
- Euler's correspondence to D'Alembert on logarithms - 1747
- Euler's (re)discovery $e^{i\theta} = \cos\theta + i\sin\theta$ - 1748
- d'Alembert's essay on hydrodynamics applying complex numbers to problems in Calculus - 1752
- Casper Wessel prints graphic representation of imaginary and complex numbers - 1797

Argand's geometric representation of i and $a+bi$ as vectors in a plane - 1806

Gauss shows understanding of residues in Correspondence to Bessel - 1811

Cauchy's Mémoire part I, "The equations that authorize the passage from real to imaginary" - 1814

Cauchy's Mémoire Resubmission, - 1825

Cauchy develops complex analysis in several papers and books, - 1837, 1840, 1846, 1851

Riemann's Dissertation - 1851

Friedrich Prym gives first counterexample of Dirichlet's principle - 1871

Weierstrass's counterexample - 1872

(Figure 3)

Sources

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