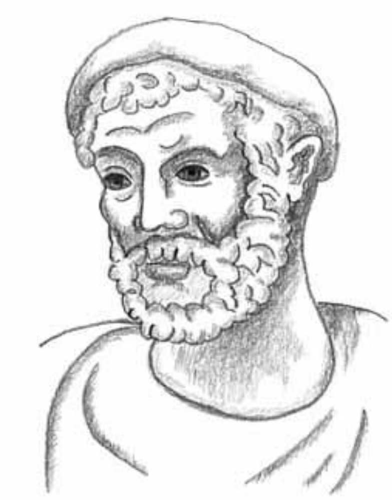
Rigor and Proof

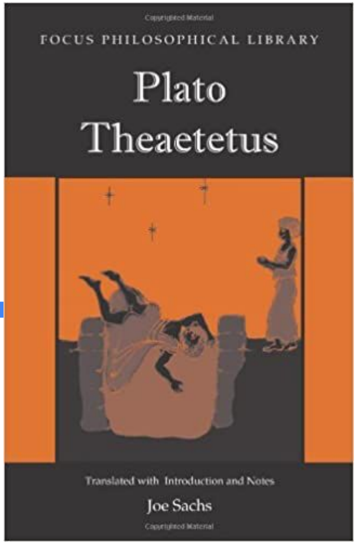
Patryk Kwoczak

Brief Historical Context

Euclid From Raphael's School of Athens



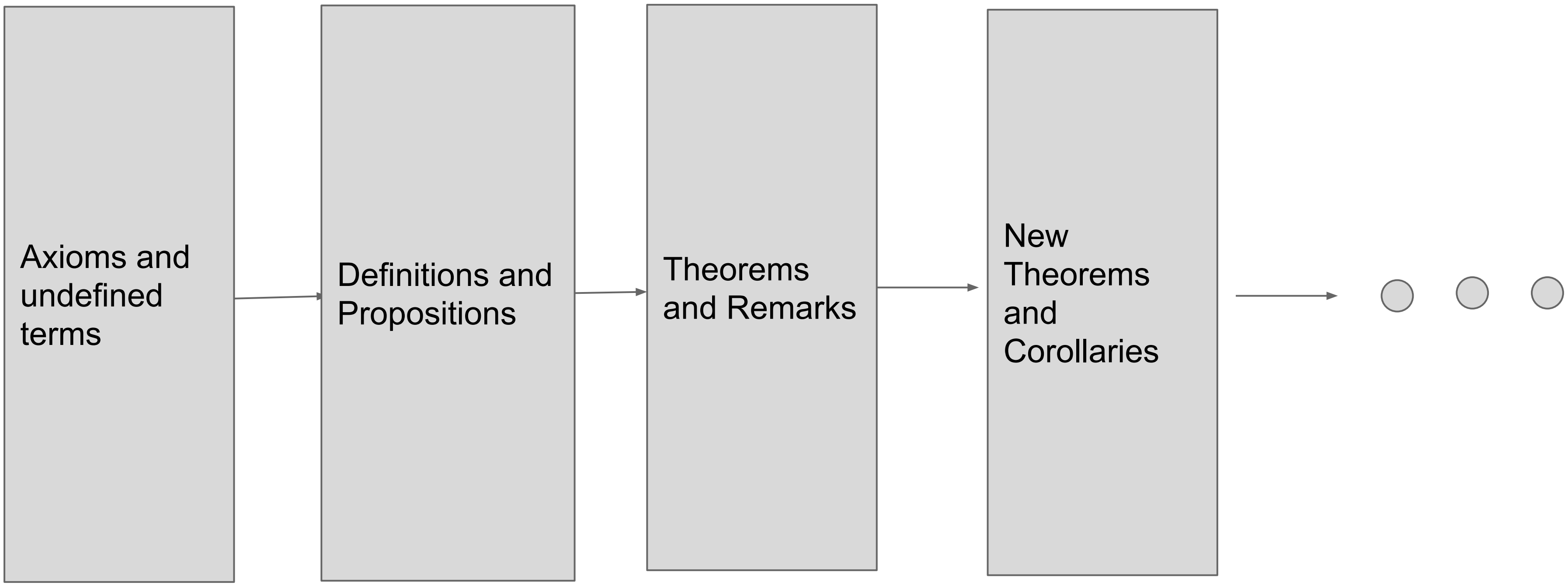
Eudoxus, Drawing by Andreas Strick



Cover from Joe Sachs's translation of Plato's Theaetetus

Euclid: Greatest Systamizer of his era. Carefully selected and organized the material to flow logically one from the other, adding his own contributions but most importantly, he built up, "from a few definitions and axioms, a proud and lofty structure" (Cajori pg 30).

Mathematical Model



A implies B; B implies C; Then A implies C.

Fast Forward ~ 1,800 years

What new tools did we have to our advantage?

The Hindu-Arabic Number-System

The development of Algebra

1. Rhetorical Algebra
   a. Pure use of words
   b. “The thing and one gives two”
2. Syncopated Algebra
   a. Transitional
   b. Alexandria, India, China
3. Symbolic Algebra
   a. Ful symbolic operations
   b. Islamic Mathematicians and Europeans

(Cajori pg 83-112)

Analytic Geometry

- Descartes and Fermat

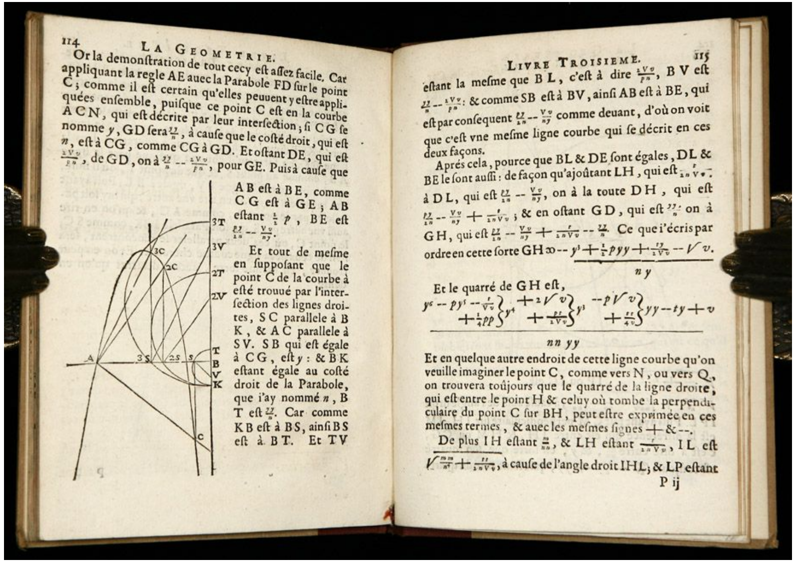
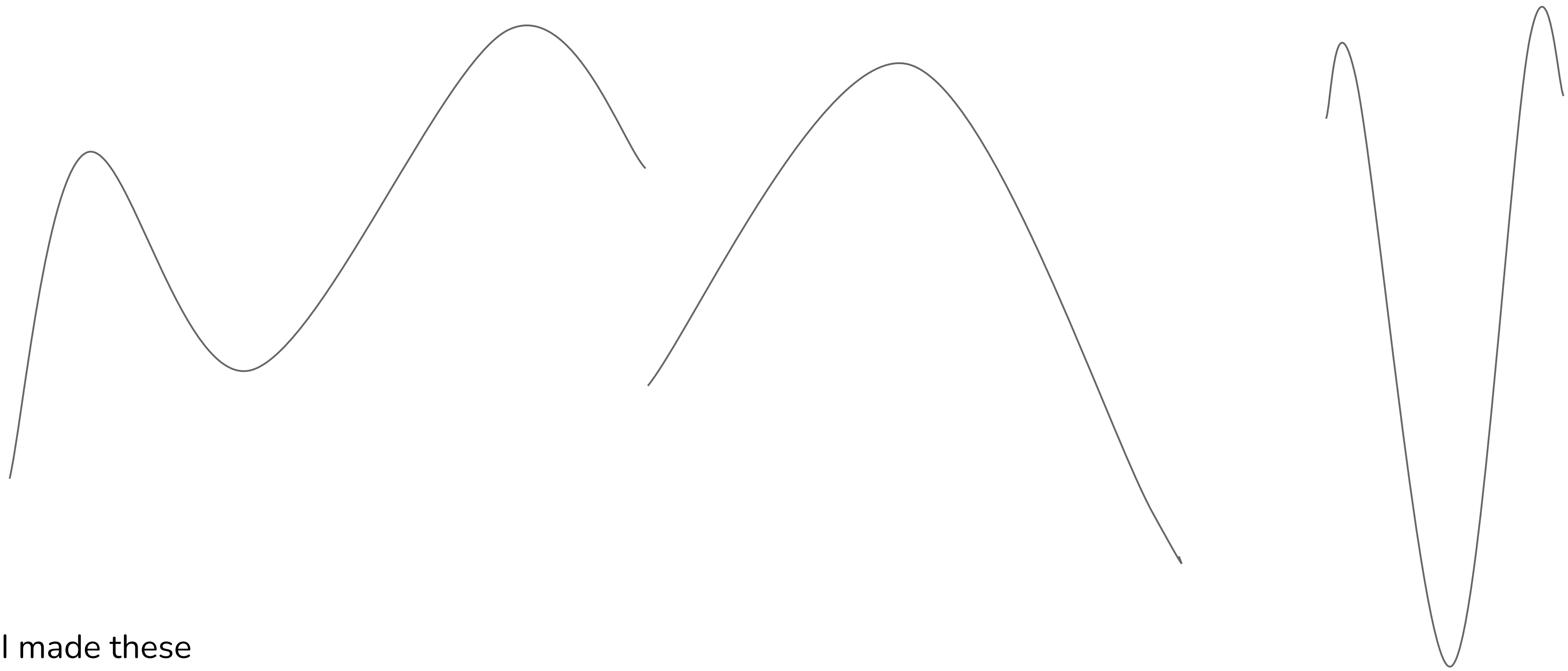


Image from *La Géométrie (1637)* by Descartes

Explain how you would find a maxima or minima geometrically



I made these

Differential Calculations: Children of Analytical Geometric Thought

Fermat (wikimedia)

- Fermat's Rule for maxima and minima:
- take a function f, "adequate" $f(x) \cong f(x+e)$,
- Do algebra and factor out an e.
- discard all remaining e (set equal to 0).
- Find the roots of this equation

Example: $f(x) = 2x-x^2$

From this method he arrived at a way to find tangents to his curves.

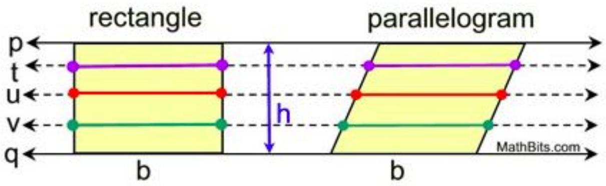
In fact, Newton later wrote that his own early ideas of calculus were inspired by the way Fermat drew his tangents.

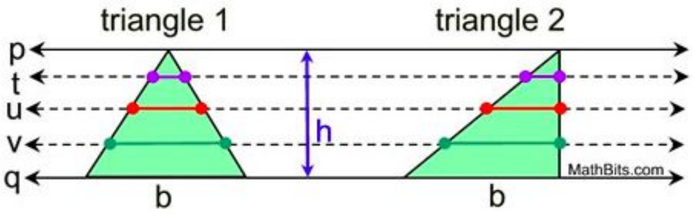
(Cajori pg 164, Simmons (pg. 98)

Indivisible Calculations of Area: Children of Analytical Geometric Thought

Cavalieri's Method of Indivisibles, 'philosophic' idea of math, postulated that a point is the "indivisible" of a line, a line the "indivisible" of a surface, etc. Then, he did applied math to this *line* of thought (haha).

3D Application: The relative proportions and magnitudes of two solids can be found by summations of series of their 'indivisible' areas ("cross-section")





Pyramids and Cubes: Since for a triangle: 1^2 + 2^2 + 3^2 + ... + n^2 = n(n+1)(2n+1)/6, and for a square each length is n and their number n, then n^2 + n^2 + ... + n^2 = n*(n^2) = n^3, then comparing their relative ratio: n(n+1)(2n+1) / (6 n^3), since n is infinitely large, we get 2/6 = ⅓. And thus a pyramid is ⅓ the volume of the prism sharing its base and altitude.

Cajori pg 162

What patterns are beginning to emerge?

This exciting New Math seemed to get the job done, but things are beginning to feel different, lose their precision. Some mechanics aren't being defined, but implemented anyways. Does it get the job done? Can you think of criticisms?

Cavalieri also used his method of Indivisibles to prove that the area under the curve of a polynomial from 0 to 1 for x^2 is ⅓, and generalize the formula to 1/(n+1), proving up to the case of n=9.

Encyclopedia Britannica,Cajori
pg 190

Meaning of Infinitesimals and Differentials

Is the use of an infinitely small quantity justifiable?

Newton developed a theory of fluents and fluxions, extending Fermat's work by describing fluents (quantities that vary, such as a function) with their respective fluxion (velocity of that fluent at every value) through the language of fluxes and moments (the infinitesimal change of fluent, or fluxion * time)

Did arithmetic on these infinitesimals and dropped higher orders of infinitesimal terms.

Small example with x^2.

Newton's Example

Newton's The Method of Fluxions, translated by J. Colson from Newton's Latin, as copied into *A History of Mathematics* by Florian Cajori. Newton described. The fluxions, variables appearing with dots, are not infinitely small but represent the variation of the variable itself, the infinitely small quantity is the moment, or o*x_dot.

so; because $\dot{v}o$, $\dot{x}o$, $\dot{y}o$, and $\dot{z}o$ are to each other as $\dot{v}$, $\dot{x}$, $\dot{y}$, and $\dot{z}$. ... will be represented by $\dot{v}o$, $\dot{y}o$,

"Now since the moments, as $\dot{x}o$ and $\dot{y}o$, are the indefinitely little accessions of the flowing quantities x and y, by which those quantities are increased through the several indefinitely little intervals of time, it follows that those quantities, x and y, after any indefinitely small interval of time, become $x+\dot{x}o$ and $y+\dot{y}o$, and therefore the equation, which at all times indifferently expresses the relation of the flowing quantities, will as well express the relation between $x+\dot{x}o$ and $y+\dot{y}o$, as between x and y; so that $x+\dot{x}o$ and $y+\dot{y}o$ may be substituted in the same equation for those quantities, instead of x and y. Thus let any equation $x^3-ax^2+axy-y^3=o$ be given, and substitute $x+\dot{x}o$ for x, and $y+\dot{y}o$ for y, and there will arise

$$\left.\begin{array}{l} x^3+3x^2\dot{x}o+3x\dot{x}o\dot{x}o+\dot{x}^3o^3 \\ -ax^2-2ax\dot{x}o-a\dot{x}o\dot{x}o \\ +axy+ay\dot{x}o+a\dot{x}o\dot{y}o \\ \qquad +ax\dot{y}o \\ -y^3-3y^2\dot{y}o-3y\dot{y}o\dot{y}o-\dot{y}^3o^3 \end{array}\right\} = o.$$

"Now, by supposition, $x^3-ax^2+axy-y^3=o$, which therefore, being expunged and the remaining terms being divided by o, there will remain

$$3x^2\dot{x}-2ax\dot{x}+ay\dot{x}+ax\dot{y}-3y^2\dot{y}+3x\dot{x}\dot{x}o-a\dot{x}\dot{x}o+a\dot{x}\dot{y}o-3y\dot{y}\dot{y}o +\dot{x}^3oo-\dot{y}^3oo=o.$$

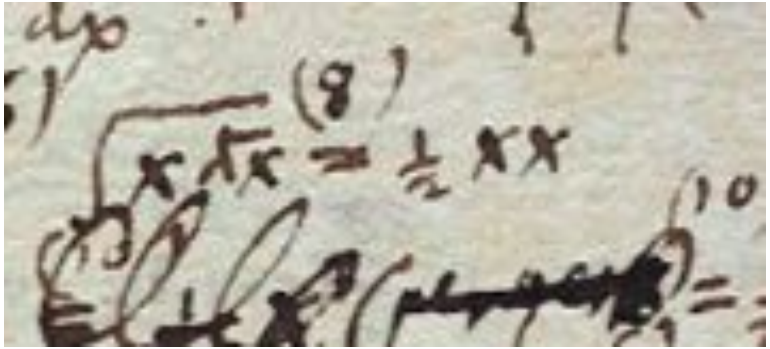
But whereas zero is supposed to be infinitely little, that it may represent the moments of quantities, the terms that are multiplied by it will be nothing in respect of the rest (*termini in eam ducti pro nihilo possunt haberi cum aliis collati*); therefore I reject them, and there remains

$$3x^2\dot{x}-2ax\dot{x}+ay\dot{x}+ax\dot{y}-3y^2\dot{y}=o,$$

as above in Example I." Newton here uses infinitesimals. ... were the difficulties en-

Extensions and Advancements

Both of these methods were developed and expanded in the following century by Newton and Leibniz among others, really pushing the limits of infinitesimal arithmetic.



Leibniz's example of integration. Did he understand dx the same way as you? Images from *Lilly Library, Indiana University, Bloomington, Indiana.*

Quotes by Newton; Yay or Nay?
Think like mathematicians!

Newton clearly wasn't satisfied, because in later years his explanations involved talking about limiting values, "limit of the ratio of vanishing increments."

Does he explain or define "it" well?

"Quantities and the ratios of quantities which in any finite time converge continually to equality, and before the end of that time approach nearer to each other than by any given difference, become ultimately equal" (**115,**Book I, Lemma I, p. 291).

"Rhetorical construction of Limits"

JUDITH V . **GRABINE, MAA, The changing concept of Change: The Derivative from Fermat to Weierstrass**

Why was the math so flawed?

The core of the criticism could be summarized by Bishop George Berkeley's maliciously accurate essay condemning infinitesimal proofs of Calculus,

The Analyst: Or, A Discourse Addressed to an Infidel Mathematician: Wherein It Is Examined Whether the Object, Principles, and Inferences of the Modern Analysis Are More Distinctly Conceived, or More Evidently Deduced, Than Religious Mysteries and Points of Faith.

Detailing the method of flux for arriving at the fluxion n*x^(n-1) for the fluent x^n.

Berkeley: “But it should seem that this reasoning is not fair or conclusive. For when it is said, let the Increments vanish, *i. e.* let the Increments be nothing, or let there be no Increments, the former Supposition that the Increments were something, or that there were Increments, is destroyed, and yet a Consequence of that Supposition, *i. e.* an Expression got by virtue thereof, is retained. Which, by the foregoing Lemma, is a false way of reasoning. Certainly when we suppose the Increments to vanish, we must suppose their Proportions, their Expressions, and every thing else derived from the Supposition of their Existence to vanish with them.” (Chapter 13)

A -> B -> C (Through the proof B, based on a definition of infinitesimals laid out in A)

But A changes definition and suddenly B wouldn’t apply, then we can’t just pretend B -> C and therefore claim C is the result.

Quotes From the Book

XIII. Now the other Method of obtaining a Rule to find the Fluxion of any Power is as follows. Let the Quantity x flow uniformly, and be it propoſed to find the Fluxion of x^n. In the ſame time that x by flowing becomes $x+o$, the Power x^n becomes $\overline{x+o}|^n$, i. e. by the Method of infinite Series $x^n + nox^{n-1} + \frac{nn-n}{2}oox^{n-2} +$ &c. and the Increments o and $nox^{n-1} + \frac{nn-n}{2}oox^{n-2} +$ &c. are one to another as 1 to $nx^{n-1} + \frac{nn-n}{2}ox^{n-2} +$ &c. Let now the Increments vaniſh, and their laſt Proportion will be 1 to nx^{n-1}. But it ſhould ſeem that

that this reaſoning is not fair or concluſive. For when it is ſaid, let the Increments vaniſh, *i. e.* let the Increments be nothing, or let there be no Increments, the former Suppoſition that the Increments were ſomething, or that there were Increments, is deſtroyed, and yet a Conſequence of that Suppoſition, *i. e.* an Expreſſion got by virtue thereof, is retained. Which, by the foregoing Lemma, is a falſe way of reaſoning. Certainly when we ſuppoſe the Increments to vaniſh, we muſt ſuppoſe their Proportions, their Expreſſions, and every thing elſe derived from the Suppoſition of their Exiſtence to vaniſh with them.

Mathematics lost its credibility of being a Science

“I say that in every other Science Men prove their Conclusions by their Principles, and not their Principles by the Conclusions. But if in yours you should allow yourselves this unnatural way of proceeding, the Consequence would be that you must take up with the Induction, and bid adieu to Demonstration. And if you submit to this, your Authority will no longer lead the way in Points of Reason and Science” (Chapter 19)

You can’t just point to its applications, say that “It has given us the right answer, and so it is true!” and still consider yourself scientifically grounded and true.

The truth of the result doesn’t guarantee the validity of the proof.

It is clear what needs to be fixed, but how?

What needed to be done was create a structurally sound method based on clear definitions such that A -> B -> C can be followed without contradicting the rules once they are assigned.

Core: Give rules that explain limiting dy/dx (0/0).

Euler tried to rationalize proportion 0:0 by saying multiplication still follows the same rules: 2:1 = 0:0; n:1 = 0:0

Lagrange invoked an argument through algebra, showing

$$f(x+i) = f(x) + a*i + b*i^2 + \dots$$

And arguing that f’ can be derived from the coefficients of the above without using limits, infinitesimals, and the things he didn’t like (Jeremy Gray)

The Salvation of Cauchy

Cauchy gave definitions for Sequences and Series that led him to examine convergence and divergence.

He described what it meant for a sum to converge (sequence of partial sums), and for it to be useful, he gave several tests for convergence.

He went on to describe continuity, and finally, differentiation.

Rigorous proof of Taylor's Theorem

Did not entirely distinguish between Continuity and uniform Continuity (Weierstrass Resolved)

Real (and Complex) Analysis grew out of this investigation to work with, fix, reinvent, and generalize these definitions.

(Jeremy Gray)

Ode to the rest...

Countless other names and cultures have been emitted or sidelined for this presentation, and I don't want all of them to have been left out entirely.

- China
- Arabic, European, Indian Algebra
  - Diophantus
  - Brahmagupta
  - Muhammed ibn Musa Al-Khwarizmi
  - Al-Kalsadi
  - Vieta and Oughtred
  - (Cajori pg 83-112)
- Indian and Pakistani Region (*Proofs in Indian Mathematics*, Srinivas)
  - Govindasvāmin (c.800)
  - Caturveda Pṛthūdakasvāmin (c.860)
  - Bhäskarächärya II (c.1150)
  - Śaṅkara Vāriyar (c.1535)
  - Jyṣṭesthadeva (c.1530)
- Other than Cauchy:
  - Euler, Gauss, Fourier, Legendre, Lagrange, Abel, Jacobi, Riemann, Weierstrass

Sources

Primary Source: Bishop George Berkeley's The Analyst
Free text: https://en.wikisource.org/wiki/The_Analyst:_a_Discourse_addressed_to_an_Infidel_Mathematician#I
Newton's Example from "The Method of Fluxions" translated by J. Colson, from Florian Cajori's "A History of Mathematics"

Images Of Leibniz's Integral *Lilly Library, Indiana University, Bloomington, Indiana.*
Images from Mathbits.com
Secondary Sources:

- Florian Cajori's "A History of Mathematics"
- The Real and the Complex: A History of Analysis in the 19th Century, Jeremy Gray
- Mathematics by its History, Stillwell
- Corresponding lectures from Stillwell's book by professor N J Wildbergerg of the University of New South Wales on youtube
    - https://www.youtube.com/watch?v=dW8Cy6WrO94&list=PL55C7C83781CF4316
    - (I'm mainly citing this because it was what gave me the ideas of this presentation, and pointed me in a lot of helpful directions, especially the videos titled 'Calculus' and 'Infinite Series'.)
- *Proofs in Indian Mathematics*, Srinivas

https://www.britannica.com/science/mathematics