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1. Cicadas

The first task is to predict the evolving population of cicadas by their annual reproduction cycle, which can be modelled with the Hassel equation $N_{n+1} = \frac{R_0 N_n}{(1+aN_n)^b}$, where the parameter R_0 represents the per capita growth rate (1 + births – deaths), and a, b are parameters altering the shape of the population curve; in particular, the stable limit, or population cap, will be discussed below, and if b is 0, then the population experiences exponential unrestricted growth, and if b = 1, then the population graph is a smooth version of the contest competition model; if b > 1, then the model provides overcompensation, in particular, a reduction of population beyond that of the maximum population restriction. Within the example of Cicada populations, we take $R_0 = 5, a = 1e(-4)$, and vary b in the set {1,2,0.5,0 }.

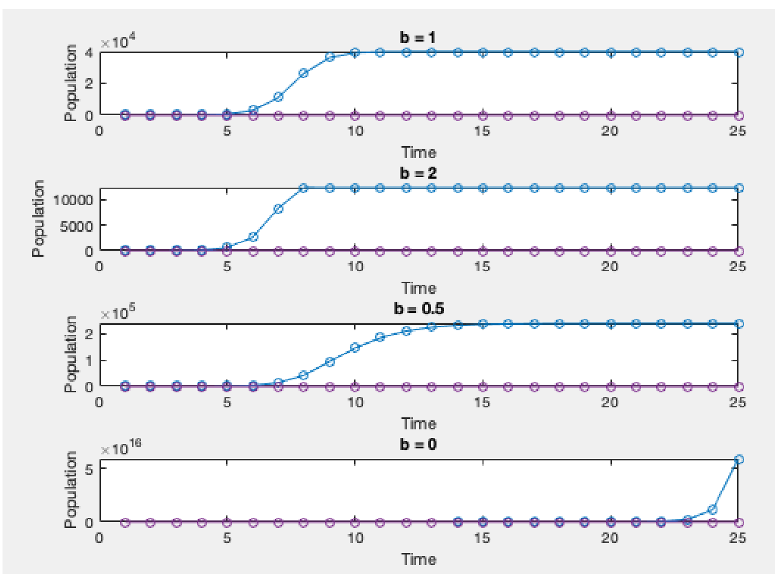
To calculate the stable fixed points, we will solve for when $N_{n+1} = N_n$, ie, if for notational convenience we are solving for $0 < N = x = \frac{R_0 x}{(1+ax)^b}$,

$$1 = \frac{R_0}{(1+ax)^b}$$

$$1 + ax = \sqrt[b]{R_0}$$

$$x = \frac{\left(\sqrt[b]{R_0} - 1\right)}{a}$$

The graphs of the four cases of b are below in figure 1.



(fig 1)

Furthermore, the stable fixed points, respectively, are 40,000; 12,360.67; 240,000; and the final population is unbounded. These results are from what the sequences, for each b value, $\{N_i\}_{i\in\mathbb{N}}$ converge to, if they converge, and the values supported by the analytic solution.

2. House Fly

The following Task is simulating the population growth of house flies, first through an unrestricted period-doubling model, and then by a logistic model. Recall the former model is as follows, $N(t) = N_0 2^{t/\tau}$, where N_0 expresses the initial population, which we fix at 100, and τ represents the doubling time, which we are to take to be 48 hours. It is obviously exponential and unbounded. It is convenient to measure the population in time intervals of one hour. In figure 2 below, we can see that the exponential fly population model grows exponentially.

Next, recall that the logistic differential equation, describing the rate at which the population grows, and representing observed data of populations under restrictions such as finite resources, can be expressed as $\frac{dN}{dt} = R_0 N(t)\left[1 - \frac{N(t)}{K}\right]$, and analytically solved to the family of equations $N(t) = \frac{KN_0 e^{R_0 t}}{K - N_0 + N_0 e^{R_0 t}}$. Note that K assigns the carrying capacity (which we are fixing as 400 flies), the constant R_0 shows the rate of unrestricted growth (corresponding to per capita increase, birth minus deaths), and N_0 is still the initial population of 100 flies. To find R_0, we note that the initial rate of increase should be equal among both models, as we are assuming that initially the population of the flies follows period-doubling. Hence

$$\frac{dN}{dt}\Big|_{t=0} = \frac{d}{dt} N_0 2^{\frac{t}{\tau}}\Big|_{t=0} = \frac{N_0 \ln 2 * 2^{\frac{t}{\tau}}}{\tau}\Big|_{t=0}$$

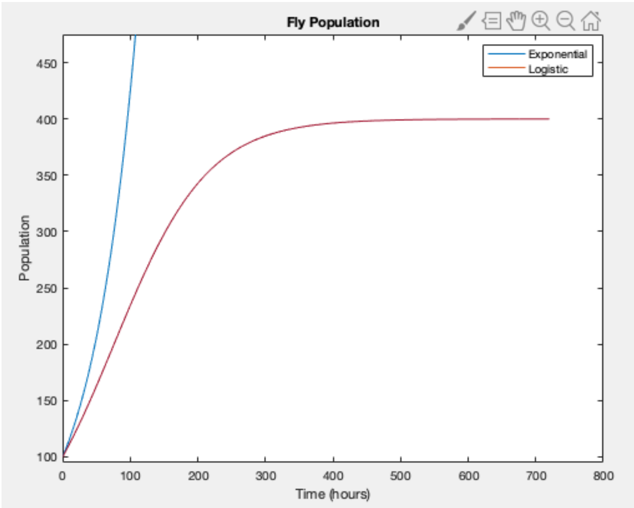
$$R_0 N(0)\left[1 - \frac{N(0)}{K}\right] = \frac{N_0 \ln 2}{\tau}$$

$$R_0\left[1 - \frac{N_0}{K}\right] = \frac{\ln 2}{\tau}$$

$$R_0 \frac{3}{4} = \frac{\ln 2}{48}$$

$$\therefore R_0 = \frac{4}{3}\frac{\ln 2}{48} \approx 0.0193$$

It is clear from figure 2 that the logistic model follows our expectations, approaching and being bounded by K, 400; moreover, the slopes graphically agree at the initial time.



(fig 2)

We want to show that 400 is the fixed point of the equation by solving for a value N^* such that for any N_i following N^*, $N^* = N_i$; simplifying for $i = 1$, and assigning $x = N$ for notational convenience and to emphasize that it is the variable we are iterating through, we can accomplish this numerically by using Newton's Method to find the root of

$$f(x) = \frac{Kxe^{R_0*1}}{K - x + xe^{R_0*1}} - x$$

With the derivative

$$f'(x) = \frac{\left(Ke^{R_0}(K - x + xe^{R_0}) - Kxe^{R_0}(-1 + e^{R_0})\right)}{(K - x + xe^{R_0})^2} - 1$$

For clarification, the Newton root finding algorithm is an iterative process such that

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Halting if a stopping criterion is satisfied.

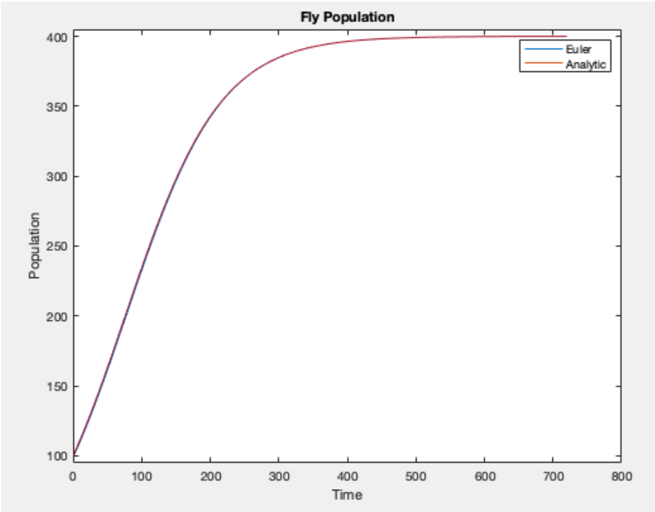
Iterating through Newton's method, starting at $0.9K$, until the error is within 10^{-2}, we get the following table, figure 3.

Figure 3			
i	N_i	Error = $\lvert N_i - N_{i-1} \rvert$	Error = $\lvert f(N_i) \rvert$
1	360	-	39.7008
2	400.0333	40.0333	0.0331
3	400.00000001865	0.0333	1.8524e-08
4	400.	1.8650e-08	0.

Furthermore, we can get an estimate to the analytical logistic curve through implementing Euler's Method, since we have information from the differential equation, and from the initial point N_0, we can approximate the following value by the current value and how much (linear) change there is, as told by the differential value. Specifically,

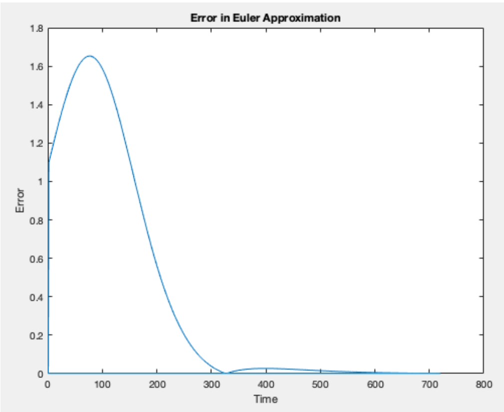
$$N_{i+1} \approx N_i + \Delta t * \frac{dN}{dt}|_{N_i}$$

We will assign the step-size to be 1, and by defining substituting $\frac{dN}{dt}|_{N_i}$ with $R_0 N_i \left[1 - \frac{N_i}{K}\right]$, N_i the previous guess/iteration with Euler's method, and starting at the given N_0, we can iteratively build a numeric solution curve. Figure 4 shows the result in blue.



(fig 4)

Moreover, the pointwise difference between the two solutions is plotted in figure 5, calculated as $E_i = \left|N_i^{\text{Analytic}} - N_i^{\text{Euler}}\right|$,



(fig 5)

An important condition for an accurate approximation of a solution using Euler's Method is an appropriately small step-size, and one can say it is in fact a true solution when the step-size is infinitesimal. However, infinitesimals don't live in numerical algorithms, and one needs to consider the tradeoff between truncation error from the inherent approximation, and the roundoff error of imprecise numerical arithmetic; for example, if the step-size is too small, then the computer won't be able to handle such sensitive precision and the roundoff error can overtake the truncation error minimized. Therefore, one can see what a good fit can be by varying the step-size and using the optimized (minimizing the total error) step-size.

Difference equations are discrete population models defined by the previous iteration's population and a given rule, such as $N_{i+1} = (1 + \beta - \gamma)N_i$, or the Hassel Equation. However, growths defined by a differential equation, such as the Logistic Equation, are more topologically descriptive since they convey behavior of solution curves through the language of growths. Especially when we are dealing with partial differential equations, there is more flexibility in adding additional constraints and group-species interactions.