# The variation and its properties, along w/ analogous properties of ordinary functions for comparison

*Following Lev. D. Elsgolc

1) The variable $z$ is called a <u>function</u> of a variable $x$, $z = f(x)$, if to each value of $x$ from a certain domain there corresponds a certain value of $z$.

1) The variable $v$ is called a <u>functional</u> depending on a function $y(x)$, $v[y(x)]$, from a certain class of functions, there corresponds a certain value of $v$.

2) The increment $\Delta x$ of the argument $x$ of a function $f(x)$ is the difference of two values of this argument $\Delta x = x - x_1$. If $x$ is the independent variable, the differential coincides with its increment, $dx = \Delta x$

2) The increment, or variation, $\delta y$ of the argument $y(x)$ of the functional $v[y(x)]$ is the difference of two functions $\delta y = y(x) - y_1(x)$. It is assumed $y(x)$ runs through a certain class of functions.

3) A function $f(x)$ is said to be <u>continuous at a point $x_0$</u>, if for each positive number $\varepsilon$, there exists a $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon,$$

whenever

$$|x - x_0| < \delta.$$

It is understood $x$ takes only such values for $f(x)$ to be defined.

3) A functional $v[y(x)]$ is <u>continuous along $y_0(x)$, in the sense of order $k$</u>, if for arbitrary $\varepsilon > 0$, $\exists \delta > 0$ such that

$$|v[y(x)] - v[y_0(x)]| < \varepsilon$$

whenever:

- $|y(x) - y_0(x)| < \delta$
- $|y'(x) - y_0'(x)| < \delta$
- ...
- $|y^{(k)}(x) - y_0^{(k)}(x)| < \delta$

- It is understood $y(x)$ is taken from class of functions $v$ is defined.

4) The function $l(x)$ is called Linear, if the following conditions are satisfied:

- $l(cx) = c\,l(x)$, $c$ arbitrary constant
- $l(x_1 + x_2) = l(x_1) + l(x_2)$.

-Each linear function of one independent variable is of the form

$$l(x) = kx,$$

$k$ constant.

4) The functional $L[y(x)]$ is called Linear if it satisfies the following:

- $L[c \cdot y(x)] = c \cdot L[y(x)]$
- $L[y_1(x) + y_2(x)] = L[y_1(x)] + L_2[y_2(x)]$

5) If the increment $\Delta f = f(x+\Delta x) - f(x)$ is of the form

$$\Delta f = A(x)\Delta x + \beta(x, \Delta x)\Delta x,$$

where $A(x)$ does not depend on $\Delta x$, and $\lim_{\Delta x \to 0} \beta(x, \Delta x) = 0$, then

$f(x)$ is called a differentiable function, and that part of the increment linear in $\Delta x$, ie $A(x)\Delta x$, is the differential of $f(x)$ and designated $df$.

Hence $\lim_{\Delta x \to 0} \frac{df}{\Delta x} = f'(x) = A(x)$, thence:

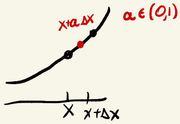
$$df \doteq f'(x)\Delta x$$

5) If the increment $\Delta x = v[y(x) + \delta y] - v[y]$ of a functional is of the form

$$\Delta v = L[y(x), \delta y] + \beta[y(x), \delta y] \max|\delta y|$$

Where $L$ is a linear functional in $\delta y$, $\max|\delta y|$ is maximal value of $|\delta y|$, and $\beta \to 0$ as $\max|\delta y| \to 0$,

Then that part of this increment that is linear in $\delta y$, $L[y(x), \delta y]$, is called variation of the functional, designated $\delta v$.




Note: Other, helpful way, to define differential and variation

- Consider value of $f(x+a\Delta x)$, $x$ and $\Delta x$ fixed and $a$ variable
  if $a=1$, we get increased val. $f(x+\Delta x)$
  if $a=0$, we get original val. $f(x)$

Through chain rule: $\frac{\partial}{\partial a} f(x+a\Delta x)\Big|_{a=0} = f'(x+a\Delta x)\Delta x\Big|_{a=0}$

$$= f'(x)\Delta x = df(x)$$

- Several variables: $z = f(\bar{x}) = f(x_1, x_2, \dots, x_n)$,

$$\frac{\partial}{\partial a} f(x_1 + a\Delta x_1, x_2 + a\Delta x_2, \dots, x_n + a\Delta x_n)\Big|_{a=0}$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \Delta x_i = df$$

Similarly for functionals: we can define variation as derivative of functional $v[y(x) + a\delta y]$ w.r.t. $a$, at $a=0$.

$$\Delta v = v[y(x) + a\delta y] - v[y(x)]$$

$$= L[y, a\delta y] + \beta(y, a\delta y)|a|\max|\delta y|$$, der wrt $a$, @ $a=0$:

$$\lim_{\Delta a \to 0} \frac{\Delta v}{\Delta a} = \lim_{a\to 0} \frac{\Delta v}{a} = \lim_{a \to 0} \frac{L(y, a\delta y) + \beta(y(x), a\delta y)|a|\max|\delta y|}{a}$$

$$= \lim_{a\to 0} \underbrace{\frac{L(y, a\delta y)}{a}}_{= \frac{aL(y,\delta y)}{a}} + \lim_{a\to 0} \frac{\overset{\to 0}{\beta}(y, a\delta y)|a|\max|\delta y|}{a}$$

$$= L(y, \delta y) \qquad \searrow 0$$

* To recap, if variation in sense of main linear increment of a functional exists, then variation in sense of derivative with respect to a parameter at the initial value of this parameter exists too, and ~~the~~ two notions are equivalent

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|---|---|
| 6. The differential of a function $f(x)$ is given by $df = \frac{\partial}{\partial a} f(x + a\Delta x)\Big\|_{a=0}$ | 6. The variation of a functional $v[y(x)]$ is given by $\partial v = \frac{\partial}{\partial a} v[y(x) + a\delta y]\Big\|_{a=0}$ |

<u>Definition</u>: A functional $v$ takes on a <u>maximum value</u> along the curve $y_0(x)$ if for all values of this functional $v[y(x)]$ taken on along arbitrary neighboring to $y = y_0(x)$ curves are not greater than $v[y_0(x)]$, ie,

$$\Delta v = v[y(x)] - v[y_0(x)] \leq 0.$$

If $\Delta v \leq 0$ and $\Delta v = 0$ only when $y(x) = y_0(x)$, then we say the functional $v$ takes on an absolute maximum along $y_0(x)$.

- Similarly we define a curve $y_0(x)$ along which $v$ takes on a minimum value, here $\Delta v \geq 0$ for all curves sufficiently close to $y_0$.

<u>Theorem</u>: If the variation of a functional $v[y(x)]$ exists, and if $v$ takes on a maximum or minimum along $y_0(x)$, then

$$\delta v = 0$$

along $y(x) = y_0(x)$

<u>Proof</u>: Suppose $y_0(x)$ is a maximum or minimum for $v[y(x)]$. When $y_0(x)$ and $\delta y$ are fixed, $v[y_0(x) + a\delta y] = \varphi(a)$ is a function of $a$ which takes up a maximum or minimum for $a=0$.

$$\Rightarrow \varphi'(0) = 0, \text{ or equivalently, } \frac{\partial}{\partial a} v[y_0(x) + a\delta y]\Big|_{a=0} = 0,$$

ie, $\delta v = 0$.

Hence, the variation of a functional vanishes along these curves which makes the functional an extremum.

$\square$

<u>Note</u>: if $y(x) = y_0(x)$ makes $v[y(x)]$ an extremum, then not only:

$$\frac{\partial}{\partial a} v[y_0(x) + a\delta y]\Big|_{a=0} = 0, \text{ but}$$

$$\frac{\partial}{\partial a} v[y(x,a)]\Big|_{a=0} = 0, \text{ where}$$

$y(x,a)$ is an arbitrary class of admitted curves, st

$y(x, a=0) = y_0(x)$, $y(x, a=1) = y_0(x) + \delta y$

$v[y(x,a)]$ is a function of $a$, for any given curve $y(x)$, and hence a value of the functional $v[y(x,a)]$

Since $y_0(x)$ is extremum of $v[y(x)]$, then this function, too, has extremum at $a=0$, and therefore its derivative vanishes at $a=0$.

Consequently $\frac{\partial}{\partial a} v[y(x,a)]\Big|_{a=0} = 0$.