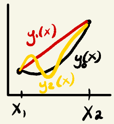


Look for extrema of functionals of form

$$v[y(x)] = \int_{x_0}^{x_1} F(x, y(x), y'(x))\,dx$$

with fixed end-coordinates $(x_0, y(x_0) = y_0), (x_1, y(x_1) = y_1)$.

Recall a necessary condition for an extremum to exist of a functional is that its variation vanishes

First, we show how this applies ~~to the~~ functional in question

Suppose that an extremum occurs along a curve $y = y(x)$ possessing a second-order derivative.

Take any admissable curve $y = y^*(x)$ neighboring to $y(x)$, and construct a one-parameter family of curves

$$y(x, a) = y(x) + a(y^*(x) - y(x))$$

containing the curves $y(x) = y(x, a=0)$, and $y^*(x) = y(x, a=1)$.

Recall variation of the function $\delta y = y^*(x) - y(x)$

As δy is a function of x, it can be differentiated, so

$(\delta y)' = y^{*\prime}(x) - y'(x) = \delta y'$, in other words,
the derivative of the variation is the variation of the variation.

Likewise $(\delta y)'' = y^{*\prime\prime}(x) - y''(x) = \delta y''$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

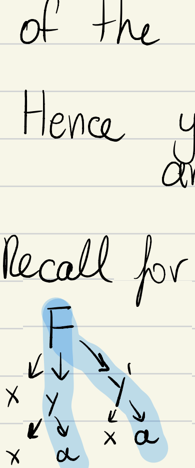
$$(\delta y)^{(k)} = y^{*(k)}(x) - y^{(k)}(x) = \delta y^{(k)}$$

Going back to a family $y = y(x,a)$ or $y = y(x) + a\delta y$,
that for $a=0$ gives the function which is extremum,
and for $a=1$ gives neighboring admissable "comparison" curve,
then the values $v[y(x)] = \int_{x_0}^{x_1} F(x, y(x), y'(x))\,dx$
taken on only along family $y(x,a)$, then it becomes a function
of the variable a: $v[y(x,a)] = \varphi(a)$

Hence $y(x)$ still extrema for local values of family $y(x,a)$,
and so still $\varphi'(0) = 0$

Recall for multivariate chain rule, since $F = F(x, y(x,a), y'(x,a))$,

$$\frac{\partial F}{\partial a} = \frac{\partial F}{\partial y}\frac{\partial y}{\partial a} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial a}$$



Since $\varphi(a) = \int_{x_0}^{x_1} F(x, y(x,a), y'(x,a))\,dx$,

$$\varphi'(a) = \int_{[x_0,x_1]} \left[F_y \frac{\partial}{\partial a} y(x,a) + F_{y'} \frac{\partial}{\partial a} y'(x,a) \right] dx$$

Now because $\frac{\partial}{\partial a} y(x,a) = \frac{\partial}{\partial a}\left(y(x) + a\delta y\right) = \delta y = y^*(x) - y(x)$

and similarly $\partial/\partial a\, y'(x,a) = \frac{\partial}{\partial a}\left(y'(x) + a\delta y'\right) = \delta y' = y^{*\prime}(x) - y'(x)$,

then it follows that:

$$\begin{cases} \varphi'(a) = \int_{x_0}^{x_1} \left(F_y(x, y(x,a), y'(x,a))\delta y + F_{y'}(x, y(x,a), y'(x,a))\delta y' \right) dx \\ \varphi'(0) = \int_{x_0}^{x_1} F_y(x, y(x), y'(x))\delta y + F_{y'}(x, y(x), y'(x))\delta y')\,dx \end{cases}$$

As $\varphi'(0) = \delta v = 0$ for extrema, then the condition is $\Rightarrow$

$$\int_{x_1}^{x_2} (F_y \delta y + F_{y'} \delta y')\,dx = 0.$$

Integrating by parts and recalling $(\delta y)' = \delta y'$,

$$\int_{x_1}^{x_2} F_{y'} \delta y'\,dx = F_{y'} \delta y \Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} \delta y \frac{d}{dx} F_{y'}\,dx$$

Together, this means

$$\delta v = 0 = \int_{x_0}^{x_1} (F_y \delta y + F_{y'} \delta y') dx = F_{y'} \delta y \Big|_{x_0}^{x_1} + \int_{x_0}^{x_1} (F_y \delta y - \frac{d}{dx} F_{y'} \delta y) dx$$

Since the endpoints of $y(x)$ and $y^*(x)$ coincide by the hypothesis,
$y^*(x_1) = y(x_1)$, $y^*(x_0) = y(x_0)$,

and so $F_{y'} \delta y \Big|_{x_0}^{x_1} = F_{y'}(x_1, y(x_1), y'(x_1))(0) - F_{y'}(x_0, y(x_0) y'(x_1))(0) = 0$

Therefore, the necessary condition for an extremum takes the following form:

$$\int_{x_0}^{x_1} \underbrace{\left(F_y - \frac{d}{dx} F_{y'}\right)}_{\substack{\text{taken along extremum} \\ y(x) \text{ is some continuous} \\ \text{function}}} \underbrace{\delta y}_{\substack{\text{Arbitrary function subject} \\ \text{to some general conditions as} \\ \text{comparison } y^*(x) \text{ was chosen arbitrarily.}}} dx = 0$$

Those conditions of δy should be that it vanishes on endpoints, continuity, with first derivative, and $|\delta y|$ small or $|\delta y|$ and $|\delta y'|$ small

To continue, we will introduce a valuable lemma:

<u>Lemma</u>: If a function $\phi(x)$ is continuous on an interval (x_0, x_1), and if

$$\int_{x_1}^{x_2} \phi(x)\eta(x)\,dx = 0$$

for an arbitrary function $\eta(x)$ subject to some conditions of general character only, then $\phi(x) \equiv 0$ throughout $[x_0, x_1]$.

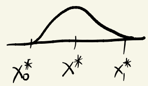
For instance, the conditions that $\eta(x)$ should be a first-order or higher-order differentiable function, ~~that~~ $\eta(x)$ should vanish at ends x_0 and x_1, and $|\eta(x)| < \varepsilon$ or both $|\eta(x)| < \varepsilon$ and $|\eta'(x)| < \varepsilon$ are such general conditions.

Proof: Assume for contradiction that at a point $x = x^*$ of the interval $x_0 \leq x \leq x_1$, $\phi(x^*) \neq 0$.
Since $\phi(x)$ is continuous, and $\phi(x^*) \neq 0$, there exists a neighborhood $x_0^* \leq x \leq x_1^*$ of the point x^* throughout which $\phi(x) > 0$.
Now, choosing $\eta(x)$ to be positive on this neighborhood and vanishes elsewhere, then we have

$$\int_{x_0}^{x_1} \phi(x)\eta(x)\,dx = \int_{x_0^*}^{x_1^*} \phi(x)\eta(x)\,dx \neq 0$$

since $\phi(x)\eta(x) > 0$ on the interval and vanishes outside.

Since this is a contradiction, $\phi(x) \equiv 0$.
As an example of such a function,



positive within interval w/ roots on x_0^* and x_1^*

define $\eta(x) = k \cdot (x - x_0^*)^{2n} (x - x_1^*)^{2n}$

on the interval $x_0^* \leq x \leq x_1^*$,

where $n \in \mathbb{N}$, $k > 0$, and $\eta(x) = 0$ otherwise.

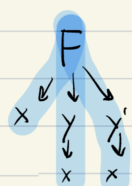
Here, $\eta(x)$ satisfies the condition of the lemma, it has all continuous derivates up to order $2n-1$, and vanishes at x_0 and x_1, and its absolute value can be made arbitrarily small as necessary by choosing coefficient k so that norm is sufficiently small.

Therefore, we can apply the aforementioned lemma and simplify the condition

$$\int_{x_0}^{x_1} \left(F_y - \frac{d}{dx} F_{y'}\right) \delta y \, dx = 0,$$

and so we have that along a curve that makes the functional v an extremum $F_y - \frac{d}{dx} F_{y'} = 0$, ie $y = y(x)$ is a solution of the second-order differential equation

$$F_y - \frac{d}{dx} F_{y'} = 0,$$



explicitly, $F_y - F_{xy'} - F_{yy'} y' - F_{y'y'} y'' = 0$

*Note: to make sure the solutions yield extrema, other analysis must be undertaken.