Examples

Suppose F only depends on y', $F = F(y')$.

Then $F_y = F_{xy'} = F_{yy'}y' = 0$, and so we solve

$F_{y'y'}y'' = 0$. Thus $F_{y'y'} = 0$ or $y'' = 0$

- If $y'' = 0$, $y(x) = C_1 x + C_2$ is a two-parameter fam of straight lines

- If $F_{y'y'}(y') = 0$ has at least one real root $y' = k_i$, then again $y = k_i x + C$, a one-par. family of straight lines contained in the two-parameter family $y(x) = C_1 x + C_2$.

$\therefore$ for $F = F(y')$, ~~the~~ extremals are arbitrary straight lines.

To use this, consider arclength $l(y(x)) = \int_{x_0}^{x_1} \sqrt{1 + y'^2}\, dx$, where $F(y') = \sqrt{1+(y')^2}$. Hence the extremals are straight lines of form $C_1 x + C_2$.

Suppose F depends on y and y' : $F(y, y')$

Then $F_{xy'} = 0$, so

$$F_y - F_{yy'}y' - F_{y'y'}y'' = 0$$

Multiplying both sides by y', and note LHS is the exact derivative $\frac{d}{dx}(F - y'F_{y'})$,

as $$\frac{d}{dx}(F - y'F_{y'}) = F_y \frac{dy}{dx} + F_{y'}\frac{dy'}{dx} - y''F_{y'} - y'\left(\frac{d}{dx}F_{y'}\right)$$

$$= F_y y' + \cancel{F_{y'}y'' - y''F_{y'}} - \left(y'F_{yy'}\frac{dy}{dx} + y'F_{y'y'}\frac{dy'}{dx}\right)$$

$$= y'\left(F_y - F_{yy'}y' - F_{y'y'}y''\right)$$

Consequently, the euler equation has a first integral

$$F - y'F_{y'} = C_1.$$

Since this nicer equation does not depend on x explicitly, it can be solved analytically easier s.a. solving for y' and separation of variables, or via parameter.

To utilize, consider the classical Brachistochrone Problem.

- Find a curve joining two given points A and B so that a particle moving along this curve starting at A reaches B in the shortest amount of time, friction and rolling neglected.

For coordinate convenience, suppose origin $(0,0)$ is A, let x-axis be horizontal and y-axis vertically downwards.

Kinetic Energy $K = \frac{1}{2}mv^2 \Rightarrow v = \frac{ds}{dt} = \sqrt{2gy} \Rightarrow \frac{ds}{\sqrt{2gy}} = dt$

Potential Energy $Q = mgy$

Additionally, as $ds^2 = dx^2 + dy^2$; $\int_A^B ds = \int \sqrt{dx^2 + dy^2} = \int_{x_1}^{x_2} \sqrt{1+(y')^2}\,dx$

Hence time to reach B from A is

$$t(y(x)) = \frac{1}{\sqrt{2g}} \int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{\sqrt{y}}\,dx, \quad y(0)=0, \; y(x_1)=y_1$$

as $F(y,y') = \frac{\sqrt{1+y'^2}}{\sqrt{y}}$, we can use form

$$\frac{\sqrt{1+y'^2}}{\sqrt{y}} - y'\left(\frac{\cancel{2}y'}{\cancel{2}\sqrt{1+y'^2}\sqrt{y}}\right) = C$$

Simplified, $\frac{\sqrt{1+y'^2}}{\sqrt{y}}\frac{\sqrt{1+y'^2}}{\sqrt{1+y'^2}} - \frac{y'^2}{\sqrt{y}\sqrt{1+y'^2}} = C$

$$\frac{1+y'^2-y'^2}{\sqrt{y}\sqrt{1+y'^2}} = \frac{1}{\sqrt{y(1+y'^2)}} = C,$$

equivalently $y(1+y'^2) = C_1$

Which looks much nicer.

use parameter t, $y' = \cot(t)$, $y = \frac{C_1}{1+\cot^2(t)} = C_1 \sin^2(t)$
$= \frac{1}{2}C_1(1-\cos(2t))$

Moreover, $dx = \frac{dy}{y'} = \frac{2C_1 \sin(t)\cos(t)\,dt}{\cot(t)} = 2C_1 \sin^2(t)\,dt$
$= C_1(1-\cos(2t))dt$

$$\Rightarrow \quad x = C_1\left(t - \frac{\sin(2t)}{2}\right) + C_2 = \frac{C_1}{2}\left(2t - \sin(2t)\right) + C_2$$

Consequently, the parameterized solution becomes

$$x = \frac{C_1}{2}(2t - \sin(2t)) + C_2 \qquad y = \frac{C_1}{2}(1 - \cos(2t))$$

Moreover, $C_2 = 0$ as $x=0$ when $y=0$, at using adjusted parameter $t_1 = t$, we get a family of cycloids

$x = \frac{C_1}{2}(t_1 - \sin(t_1))$, $y = \frac{C_1}{2}(1-\cos(t_1))$* w/ radius of rolling ball $C_1/2$, passing through $B = (x_1, y_1)$.