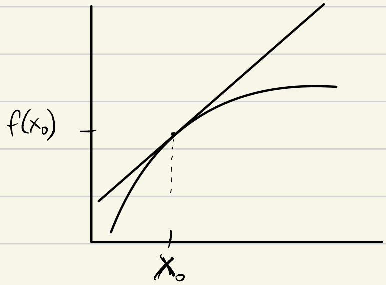
- Following Amol Sasane's text on Functional Analysis

Taking a step back to ordinary calculus for a function $f: \mathbb{R} \to \mathbb{R}$, and let's rewrite definition of the derivative of f at x_0 in a manner that generalizes to maps between normed spaces

- Recall that for $f: \mathbb{R} \to \mathbb{R}$, the derivative at a point x_0 is the approximation of f around x_0 by a straight line.



f is said to be differentiable at x_0 w/ derivative $f'(x_0) \in \mathbb{R}$ if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

In other words, $\forall\, \varepsilon > 0, \exists\, \delta > 0$ st whenever $x \in \mathbb{R}$ satisfies $0 < |x - x_0| < \delta$, there holds that $\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < \varepsilon$

That is, $\dfrac{|f(x) - f(x_0) - f'(x_0)(x - x_0)|}{|x - x_0|} < \varepsilon$.

Now to generalize to maps from a normed space X to normed space Y, we can do most of it easily as

$\|x - x_0\|_X$, and numerator as $\|\cdot\|_Y$, since $f: X \to Y$.

But what object must $f'(x_0)$ turn into?

Since $f(x)$, $f(x_0)$ live in Y, we expect $f'(x_0)(x-x_0)$ to also live in Y

Since $x-x_0 \in X$, we would like $f'(x_0)$ to map it to Y.

Hence $f'(x_0)$ will be a mapping from X to Y.

<u>Definition</u>: Let X, Y be normed spaces, $f: X \to Y$ be a map, $x_0 \in X$.

Then f is said to be differentiable at x_0 if there exists a continuous linear transformation $L: X \to Y$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $x \in X$ satisfies $0 < \|x - x_0\| < \delta$,

we have $$\frac{\|f(x) - f(x_0) - L(x-x_0)\|}{\|x - x_0\|} < \varepsilon$$

* If f is differentiable at x_0, then it can be shown there is at most one such L so that above holds, which we will prove later.

Derivative as a local linear approximation

Tangent as local linear approximation in $f:\mathbb{R}\to\mathbb{R}$ generalizes similarly to f as a map from normed space X to normed space Y, that is differentiable @ point $x_0 \in X$, $f'(x_0)$ can be interpreted as giving a local linear approximation to mapping f near x_0.

Let $\varepsilon > 0$. Then we know for all x close enough & distinct from x_0,
$$\|f(x) - f(x_0) - f'(x_0)(x - x_0)\| < \varepsilon \|x - x_0\|.$$

For x close enough to x_0,
$$\| f(x) - f(x_0) - f'(x_0)(x - x_0)\| \approx 0$$

that is, $f(x) - f(x_0) - f'(x_0)(x - x_0) \approx \bar{0}$, or
$$f(x) - f(x_0) \approx f'(x_0)(x - x_0).$$

* The above saying that near x_0, $f(x) - f(x_0)$ looks like the action of the linear transformation $f'(x_0)$ acting on $(x - x_0)$.

↳ This gives us a way of finding $L = f'(x_0)$ by calculating $f(x) - f(x_0) = L(x - x_0) +$ error, and showing error is suff. mild.

Note: Our demand the map $f'(x_0): X \to Y$ should be a continuous linear transformation is to preserve the property that

$$\text{differentiability at } x_0 \to \text{continuity at } x_0$$

Ex: Let X, Y be a normed space, and let $T: X \to Y$ be a continuous linear transformation.
- Is T differentiable at $x_0 \in X$?
- If so, what is $T'(x_0)$?

Sketch: Want to fill-in-box below w/ cont. linear trans. s.t.:

$$T(x) - T(x_0) \approx \boxed{?}(x - x_0),$$

for x near x_0. But as T linear, $T(x) - T(x_0) = T(x - x_0)$

Good guess: $T'(x_0) = T$

Verify:

Let $\varepsilon > 0$. Choose any δ, say, $\delta = 1$. Then whenever $x \in X$ satisfies $0 < \|x - x_0\| < \delta = 1$, we have

$$\frac{\|T(x) - T(x_0) - T(x - x_0)\|}{\|x - x_0\|} = \frac{\|0\|}{\|x - x_0\|} = 0 < \varepsilon$$

Hence $T'(x_0) = T$

As $x_0 \in X$ arbitrary, we have that $T'(x) = T \ \forall x \in X$!

Ex: Consider $f: C[a,b] \to \mathbb{R}$,

$$f(x) = \int_a^b (x(t))^2 \, dt \quad , \quad x \in C[a,b]$$

Let $x_0 \in C[a,b]$. What is $f'(x_0)$?

Sketch for guess of $f'(x_0)$ and seek C.L. map L so that $x \in C[a,b]$ near x_0, $f(x) - f(x_0) \approx L(x - x_0)$

$$f(x) - f(x_0) = \int_a^b ((x(t))^2 - (x_0(t))^2 \, dt$$

$$= \int_a^b (x(t) + x_0(t))(x(t) - x_0(t)) \, dt$$

$$\approx \int_a^b (x_0(t) + x_0(t))(x(t) - x_0(t)) \, dt$$

$$= \int_a^b 2 x_0(t)(x(t) - x_0(t)) \, dt$$

$$= L(x - x_0),$$

$L: C[a,b] \to \mathbb{R}$ is given by

$$L(h) = \int_a^b 2 x_0(t) h(t) \, dt \quad , \quad h \in C[a,b],$$

which is a continuous linear transformation.

Confirm w/ ε-δ definition: for $x \in C[a,b]$,

$$f(x) - f(x_0) - L(x - x_0)$$

$$= \int_a^b \Big(\underbrace{(x(t))^2 - (x_0(t))^2}_{(x-x_0)(x+x_0)} \Big) dt - \int_a^b 2x_0(t)(x(t) - x_0(t))\,dt$$

$$= \int_a^b (x(t) + x_0(t) - 2x_0(t))(x(t) - x_0(t))\,dt$$

$$= \int_a^b (x(t) - x_0(t))^2\,dt$$

and so $|f(x) - f(x_0) - L(x - x_0)|$

$$= \left| \int_a^b (x(t) - x_0(t))^2 dt \right| \leq \int_a^b |(x - x_0)(t)|^2 dt$$

$$\leq \int_a^b \|x - x_0\|_\infty^2 \, dt$$

$$= (b-a)\|x - x_0\|_\infty^2$$

Thus if $0 < \|x - x_0\|_\infty$,

$$\frac{|f(x) - f(x_0) - L(x - x_0)|}{\|x - x_0\|_\infty} \leq (b-a)\|x - x_0\|_\infty$$

Let $\varepsilon > 0$. assign $\delta := \varepsilon/(b-a)$, then for $0 < \|x - x_0\|_\infty < \delta$,

$$\frac{|f(x) - f(x_0) - L(x - x_0)|}{\|x - x_0\|_\infty} < (b-a)\delta = \varepsilon$$

Hence $f'(x_0) = L$. In other words, $f'(x_0) : C[a,b] \to \mathbb{R}$ given by

$$(f'(x_0))(h) = \int_a^b 2x_0(t)h(t)\,dt \quad \text{for } h \in C[a,b].$$

Moreover, $f'(x_0)$ is a continuous linear transformation.

<u>Theorem</u>: Uniqueness Theorem

Let X, Y be normed spaces. If $f: X \to Y$ is differentiable at $x_0 \in X$, then there exists a <u>unique</u> continuous linear transformation L such that for every $\varepsilon > 0$, $\exists \delta > 0$ s.t. whenever $x \in X$ satisfies $0 < \|x - x_0\| < \delta$, there holds

$$\frac{\|f(x) - f(x_0) - L(x - x_0)\|}{\|x - x_0\|} < \varepsilon.$$

<u>Proof</u>: Suppose that $L_1, L_2 : X \to Y$ are cont. linear transformations st $\forall \varepsilon > 0$, $\exists \delta > 0$ st whenever $x \in X$ satisfies $0 < \|x - x_0\| < \delta$, there holds

$$\frac{\|f(x) - f(x_0) - L_1(x - x_0)\|}{\|x - x_0\|} < \varepsilon \qquad (*)$$

and

$$\frac{\|f(x) - f(x_0) - L_2(x - x_0)\|}{\|x - x_0\|} < \varepsilon.$$

Assume for contradiction $L_1(h_0) \neq L_2(h_0)$ for some $h_0 \in X$.

Clearly $h_0 \neq \bar{0}$, as $L_1(0) = 0 = L_2(0)$.

Choose $\varepsilon = 1/n$ for some $n \in \mathbb{N}$. Then $\exists \delta_n > 0$ st whenever $\|x - x_0\| < \delta_n$, ~~the~~ ⊛ inequalities hold.

With $x := x_0 + \frac{\delta_n}{2\|h_0\|} h_0 \in X$, we have $x \neq x_0$,

$\|x - x_0\| = \left\| \frac{\delta_n}{2\|h_0\|} h_0 \right\| = \frac{\delta_n}{2} \cdot 1 < \delta_n$, so ⊛ hold for x.

Note $$L_1(x - x_0) - L_2(x - x_0) = L_1\left(\frac{\delta_n}{2\|h_0\|} h_0\right) - L_2\left(\frac{\delta_n}{2}\|h_0\| h_0\right)$$
$$= \frac{\delta_n}{2\|h_0\|}\left(L_1 h_0 - L_2 h_0\right)$$

Thus $$\frac{\|L_1(x - x_0) - L_2(x - x_0)\|}{\|x - x_0\|} = \frac{\cancel{\delta_n/2\|h_0\|}\,\|L_1(h_0) - L_2(h_0)\|}{\cancel{\delta_n/2\|h_0\|}\,\|h_0\|}$$

By using triangle inequality, $$\frac{\|f(\cancel{x}) - f(\cancel{x_0}) - L_2(x - x_0) - [f(\cancel{x}) - f(\cancel{x_0}) - L_1(x - x_0)]\|}{\|x - x_0\|}$$

$$= \frac{\|L_1(x - x_0) - L_2(x - x_0)\|}{\|x - x_0\|} < 2\varepsilon = 2/n.$$

$$\Rightarrow \|L_1(h_0) - L_2(h_0)\| < \frac{2}{n}\|h_0\|.$$

As choice of n arbitrary, it follows that $\|L_1(h_0) - L_2(h_0)\| = 0$, $\Rightarrow L_1(h_0) = L_2(h_0)$, a contradiction.

$\square$