Phugoid Flight Model

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Introduction

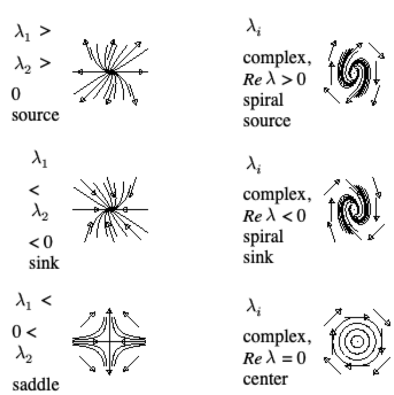
The Phugoid model allows for analysis of the flight of a plane under certain conditions by taking into account gravity, drag, and force due to a propeller. Through investigating the differential equation one can derive the path of flight as well as understand its behavior under varying initial speed, angle, and magnitude of propulsion. The system of Differential Equations for the model is presently shown:

$$\left\{\frac{d\theta}{dt} = \frac{v^2 - \cos(\theta)}{v}, \frac{dv}{dt} = -\sin(\theta) - R * v^2 + k\right\}$$

We are setting R, the drag on the plane, to 0.4, and varying the parameter k, constant acceleration brought upon by the propulsion. Note that because it is in fact constant, the Jacobian Matrix will be the same regardless of the value for k, as any partial derivative will make the parameter vanish. The Jacobian is:

$$\begin{bmatrix} \frac{\sin(\theta)}{v} & 2v + \frac{\cos(\theta)}{v^2} \\ -\cos(\theta) & -0.8v \end{bmatrix}$$

The Jacobian is a powerful tool for analyzing local behavior at a point, as it gives a linear approximation for how the system will change around that point. By calculating the Jacobian at a fixed point, the nature is revealed, illustrated in this handy graphic found here by Professor Sutherland of Stony Brook University: if we consider λ_1, λ_2 as the eigenvalues of the Jacobian evaluated at a fixed point, then



However, when the system is nonlinear more investigation must be done to determine if a point is a center (though such will not be encountered in this paper), this is due to a center being topologically unstable, so small influences from higher-order terms not considered in the linearization can push it into being either a spiral source or sink.

The system is undefined at $v = 0$, however, to understand the geometry of the phase space, one can multiply the differential system by a scalar $|v|$, so as to preserve solution curves and still retain the proper direction. In fact, because only positive velocities are considered, it is only necessary to multiply by v.This produces well-defined saddle points at $(\theta = \frac{\pi}{2} \pm \pi, v = 0)$, allowing the model to survive, as well as analysis to be performed, at an otherwise singularity.

To locate any fixed point ρ_0, one must solve for when $\left\{\frac{d\theta}{dt} = F_1(\theta, v) = 0, \frac{dv}{dt} = F_2(\theta, v) = 0\right\}$, arriving at $\rho_0 = \{\theta_0, v_0\}$, so $F_1(\theta_0 \pm 2\pi j, v_0) = 0 \cap F_2(\theta_0 \pm 2\pi j, v_0) = 0$ for $j \in \mathbb{Z}^+$. Through the labor of Maple, this can be completed by the getFixedPoints procedure, which takes in an initial k value, and outputs $\{\theta_i, v_i\}$, of which only the real solutions are valuable, and as previously mentioned only positive values of v are considered.

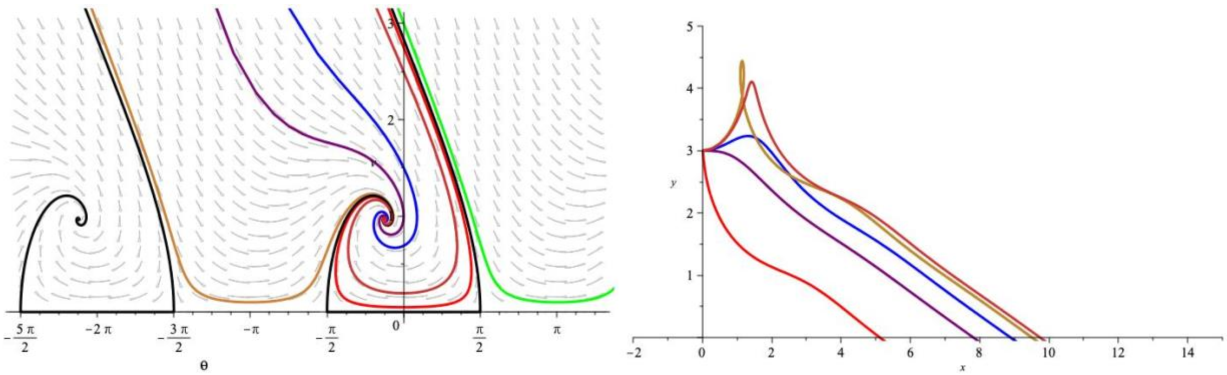
From there, the eigenvalues of the evaluated Jacobian will reveal the local nature of the phase space.

Various procedures are created to illustrate the phugoid phase space at varying ranges for θ and v. All of them use the DEplot command packaged in DEtools internal to Maple, which takes some chosen initial conditions and runs them through the Differential Equation producing solution curves in advancing time, as well as in negative time if asked. Some procedures also output the Jacobian of the System or evaluate the eigenvalues and eigenvectors through the use of Maple.

***k* = 0**

The base case at k = 0 involves no constant acceleration to support the glider. It is trivial that all planes will eventually collapse, as gravity will take its toll. This is supported by the fixed point at $(\theta = -0.3805, v = 0.9636)$, which testifies that eventually all valid conditions will limit to a negative angle. An issue appears when the velocity is approximately 2.825 when the angle is 0; this curve will limit at the singularity and therefore be undefined. What happens physically to the plane is that the speed forces it to move up through the air, but at the exact moment the plane is parallel to the positive vertical axis, it runs out of velocity leading to a discontinuity breaking the model. Since the phase is periodic at 2π, and the separatrix runs through the $\theta = 0$ axis, only those two basins need to be considered.

The fixed point is a spiral sink, with eigenvalues $\lambda_{1,2} - 0.5781 \pm 1.349i.$ All the solution curves originating at the positive v-axis will either converge to this fixed point, if they are below the point where the separatrix intersects with the axis, or to $\theta = -0.3895 + \pi, v = 0.9744$ if they are above. The physical meaning of this is that either the initial velocity is slow enough so it will not do a whole loop, or it goes fast enough to complete one, such as the green solution curve. This solution curve corresponds to the identical flight path of the gold, originating from the same velocity, and an angle 2π less. Though they are attracted to different fixed points, the line integral is the same.

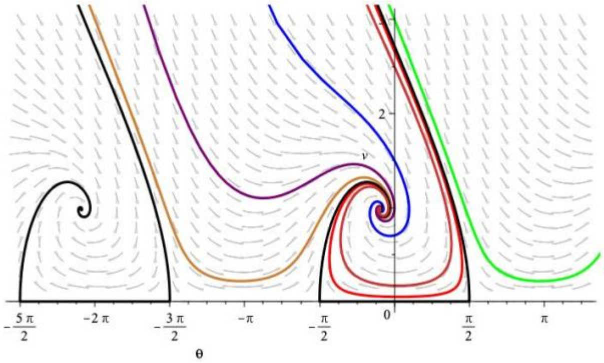


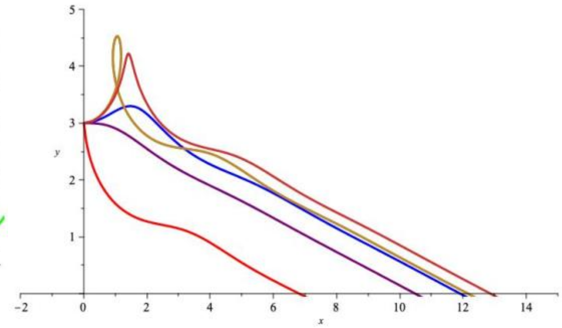
k = 0.1

Topologically very little is different about the nature of the phase space, however important changes must be noted about how it impacts the flight paths. The extra push due the propeller influences the point of intersection of where the separatrix curve connects to the velocity axis: it is lower than when *k* was 0. This signifies that some solution curves originally in that basin are now going to be in the other, such as the curves starting with velocities 1.5 and 2.05. The fixed point also shifted somewhat, locating itself at

$$\rho_0 = (\theta = -0.2875, v \; = \; 0.9793)$$

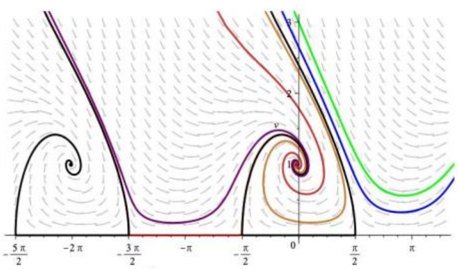
All the planes also flew further, which isn't surprising. Observing the real part of the eigenvalues, it is slightly greater than before at $Re(\lambda_{1,2}) = -0.5365$, as opposed to -0.5781, thus decreasing the rate at which angles change, and so increasing the length of time a plane is airborne. As can also be noted, the planes reach higher as a result.

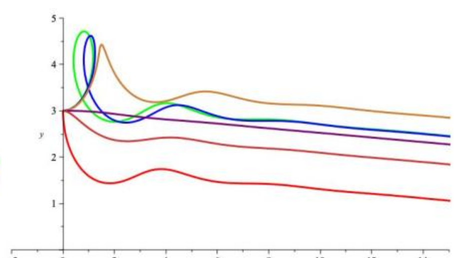




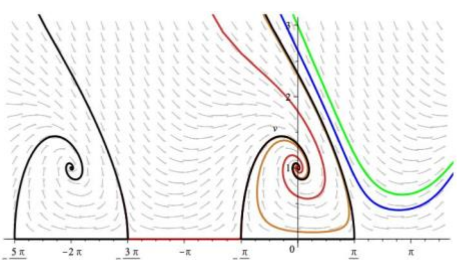
k **= 0.35, 0.4, 0.45**

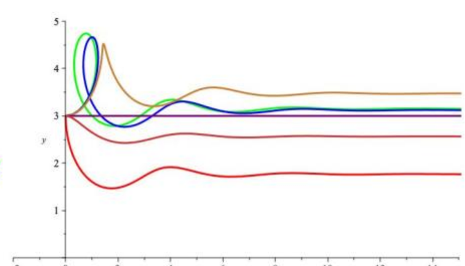
Though the geometric nature of the vector field is similar, these values represent a very important transition in the manor of flight of the planes: when *k* is less than 4, planes will still eventually collapse to the ground, when it is higher, they will always end up flying higher, and when it is exactly 0.4, planes will converge on a constant altitude. What this translates to is the value of θ when at the fixed point: if it is less than 0, plane goes down; if it is higher, plane goes up; 0 signifies that is remains at the same height. I will line them up in ascending order of *k*, and then discuss the meaning in the following paragraph.



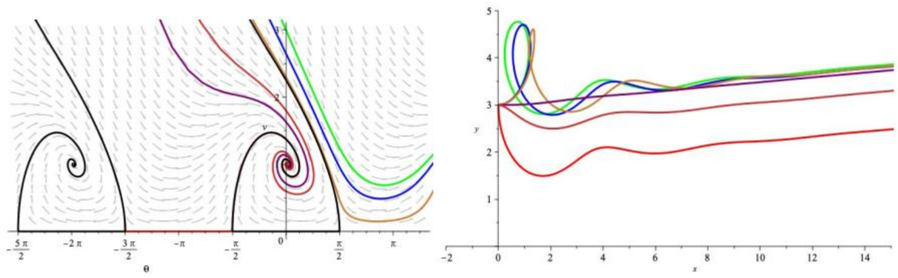


k=0.35



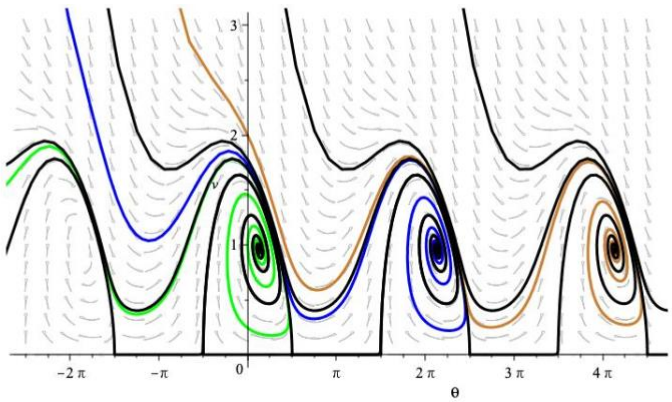


k=0.4



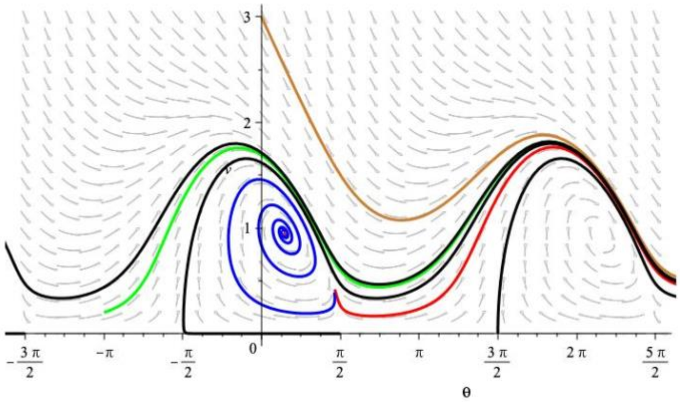
k=0.45

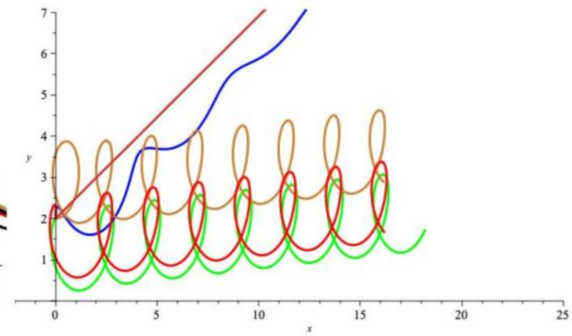
Things are evidently changing, and it is no coincidence that when the constant acceleration surpasses the drag, the plane will surpass the point where it will eventually crash. Interestingly, the real part of the eigenvalue also goes through the value of the drag: at $k = 0.35$, it is about -0.425; at $k = 0.4$: -0.4; finally, at $k = 0.45$: -0.375. But this is not the only difference: more intuitively, the eigenvector corresponding to $\dot{\theta}$ transitions from negative to positive: $\binom{-0.5}{-0.999} \rightarrow \binom{0}{-1} \rightarrow \binom{0.05}{-0.999}$. It represents a shift that the location of the equilibrium point mirrors, as the value of the angle also transitions from negative to positive. The same topological behavior lasts until approximately $k = 0.765$, although with a steadily distorting basin boundary. For example, at $k = 0.76$, the separatrices are still unique, but the angle at which these basins increase in velocity (go up) is much more dramatic. The image is to showcase this example. Initial Velocities are much less stable. However, nothing is different in the content by the fixed point: it is still a spiral sink.



***k* = 0.8**

Now things get more interesting. Although the System generates only 1 fixed point, which still remains a spiral sink at $(\theta = 0.457, v = 0.947)$, it has now formed into varying trajectories: 2 closed and one open. Of those, all solution curves above the top separatrix converge to it, joining a periodic flow. However, only some solution curves below it converge to that same periodic solution. Instead, certain solutions within the isolated basins converge to the sink, while others join to the periodic solution. This isolated basin extends down to the singularity. To tell which go where, the stable manifold to the saddle point defines that territory. Visually, the red solution is just to the right of that manifold, while the blue is just within it, thus converging to the sink, shown in orange. The green curve illustrates another path converging to the periodic flow from below the periodic separatrix.

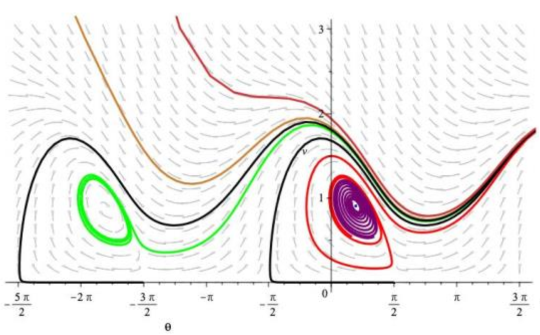


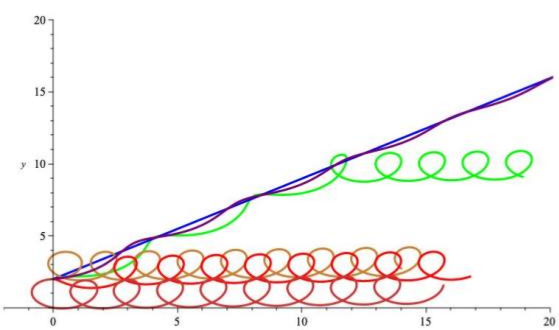


While the gold curve originally starts with a greater velocity, resulting in a loop with a greater arc length, it soon becomes stabilized in the periodic solution.

k = 0.9

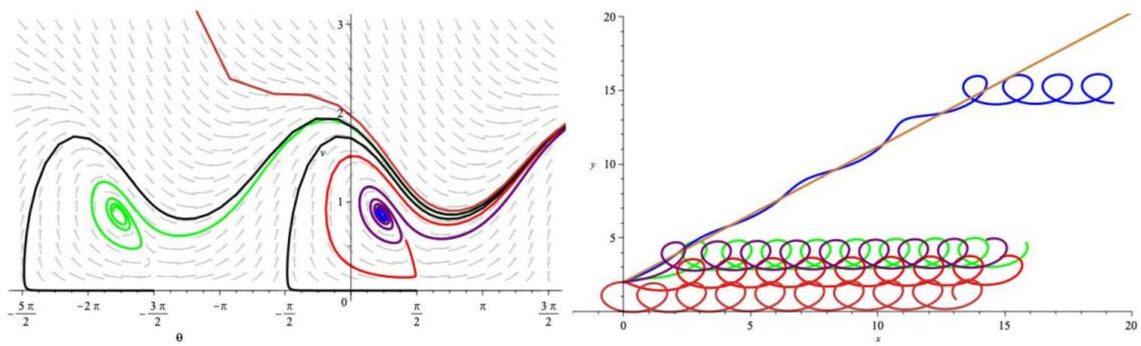
Between approximately $0.81 \leq k \leq 0.9373$, the isolated sink abandons its ground at the θ axis and becomes a fully-fledged island, from which a wonderful sea of well-defined solution curves flow. Essentially, from all outer sides of the basin solution curves converge to the periodic solution. In negative time, the manifolds which converge to the periodic solution are themselves unstable manifolds to the island limit cycle, illustrated in the green curve. The blue dot is the fixed point within the limit cycle, itself a local spiral sink, and within its basin curves will converge to that point. Outside, they will converge to the periodic solution shown in the red, and the two curves above the periodic solution. The green curve, due to the fact it is very close, although outside the island basin, will appear to navigate around the flight path of the fixed-point solution; however, eventually it still converged onto the periodic solution. More detailed analysis can be found in [this Maple Worksheet (alternatively found as a link on this page)], by Professor Sutherland of SUNY Stony Brook.





k = 0.97

Between $0.9373 < k < 1.0$, the fixed point is finally a spiral source! Now all curves are converging to the periodic solution. This fixed point is at $\{\theta = 0.741, v = 0.859\}$, with eigenvalues of the evaluated Jacobian $0.05 \pm 0.97i$. Thus, it is a source, and all flights will eventually be doing loops except for the fixed-point initial condition itself (shown in gold). As would also make sense, all solution curves under the periodic solution would converge to the fixed point in negative time.



$$\mathbf{1.0 \leq k < 1.075}$$

Things get even more interesting.

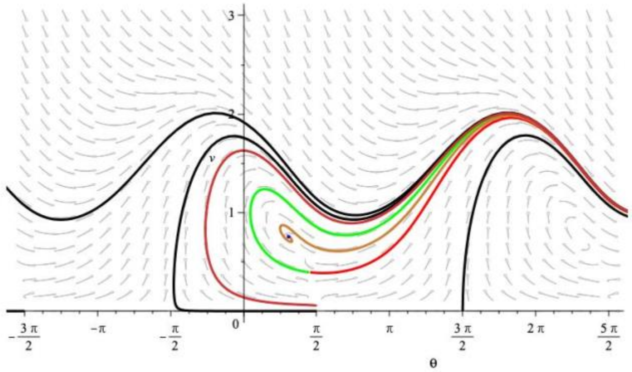
At k = 1, the equilibrium points are undefined due to the singularity. Solving for the fixed points,

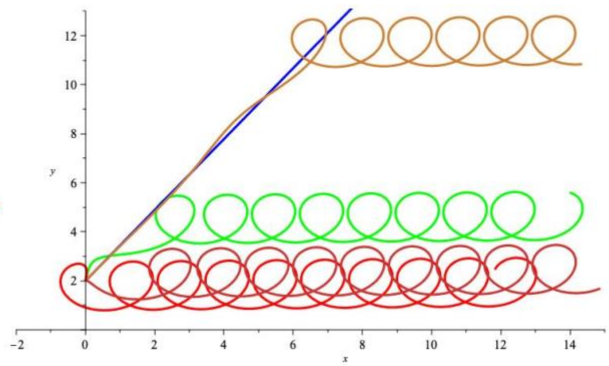
$$\dot{\theta} = \frac{v^2 - \cos\theta}{v}, \dot{v} = -\sin\theta - 0.4v^2 + 1$$
$$v^2 = \cos\theta, \ \sin\theta + 0.4v^2 = 1$$
$$\sin\theta + 0.4\cos\theta = 1, \Longrightarrow \theta_1 = \frac{\pi}{2}$$
$$\text{but } v^2 = \cos\left(\frac{\pi}{2}\right) = 0, \Longrightarrow \dot{\theta}, \dot{v} \ are\ undefined.$$

Trying to get around this through the analysis of the defined Vector Field, multiplied by v (always positive in our domain) to get rid of the singularity while preserving the solution curves, one can find $\left(\theta_1 = \frac{\pi}{2}, v_1 = 0\right)$, although evaluating this in the defined Jacobian, constructed through taking $\{F_1 = v^2 - \cos(\theta), F_2 = -v\sin(\theta) - 0.4v^3 + 1 * v\}$, $\begin{bmatrix} \sin\theta & 2v \\ -v\cos\theta & -\sin\theta - 1.2v^2 + 1 \end{bmatrix}$ produces the interesting result $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ with eigenvalues 0, 1. At the fixed point, only $\dot{\theta}$ moves in the direction of the unit vector at the same magnitude.

However, at the other angular solution for the fixed point, θ_2 is equal to $arctan\left(\frac{21}{20}\right) \approx 0.8098, v_2 = 0.8304$. This one can be easily computed, and the eigenvalues corresponding to it are $0.104 \pm 0.888i$. Therefore, the previous fixed point for $0.9373 < k < 1.0$ is still retained as a spiral source.

Things work out better at k > 1.0, for which the emerging fixed point is now defined. The two fixed points are now approximately $(1.415, 0.394), (0.966, 0.754)$. The former is a saddle point: it has one positive and one negative real eigenvalue, while the latter persists as a spiral source; in fact, it will persist up until around $k = 1.075$, when the eigenvalues lose their imaginary part. From there, it is a simple source up until approximately $k = 1.077$, when the two fixed points intersect and become complex. The images will demonstrate $k = 1.05$.

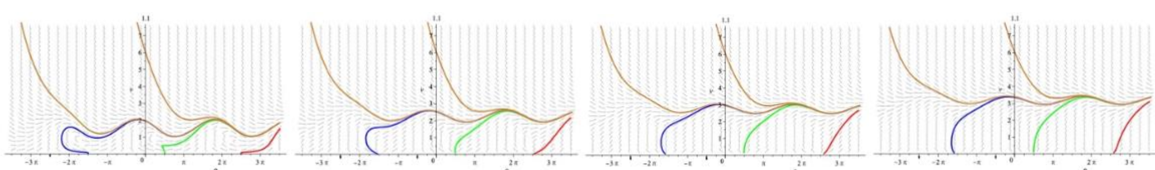




The source point is still a spiral as can be seen by the golden solution curve rotating around it before escaping towards the periodic orbit: in relation to the flight path, the behavior is very similar to the constant solution up until enough time passes that the local spiraling is no longer dominating. However, by the saddle-point one can observe just how swiftly the angle changes, two points barely a decimal apart had such drastic changes to their path behavior.

$k > 1.077$

The behavior for k above this range is topologically equivalent, as no more real fixed points remain. The behavior is altered only by the distance of the periodic solution to $v = 0$. The greater the value for k, the further up on the phase plane it is located. All solution curves converge to this periodic solution, and no real fixed points exist. All planes will now simply begin to loop after a set amount of time passes. Attached are four snapshots of increasing k, at 1.1, 2.1, 3.1, and 4.1. Only at k=1.1 is there a remnant of the previous fixed point due to the contortion of the imaginary fixed points. However, it soon gets overwhelmed.



Readers are invited to tweak the slider in the Maple document to see the evolution as well.

Conclusion

A wide variety of propulsion magnitudes affect the trajectory of the planes which use them. As has been seen in the write-up as well as the document itself, information can be visualized in plot details, but through the mathematics one can understand why the qualitative features are this way or that. It is the key to enter the warm home, as opposed to looking through the window from outside.